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Instabilities of periodic orbits with spatio-temporal symmetries

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Abstract. Motivated by recent analytical and numerical work on two- and three-dimensional convection with imposed spatial periodicity, we analyse three examples of bifurcations from a continuous group orbit of spatio-temporally symmetric periodic solutions of partial differential equations. Our approach is based on centre manifold reduction for maps, and is in the spirit of earlier work by Iooss (1986) on bifurcations of group orbits of spatially symmetric equilibria. Two examples, two-dimensional pulsating waves (PW) and three-dimensional alternating pulsating waves (APW), have discrete spatio-temporal symmetries characterized by the cyclic groups Z_n , $n = 2$ (PW) and $n = 4$ (APW). These symmetries force the Poincaré return map \mathcal{M} to be the n^{th} iterate of a map $\tilde{\mathcal{G}}$: $\mathcal{M} = \tilde{\mathcal{G}}^n$. The group orbits of PW and APW are generated by translations in the horizontal directions and correspond to a circle and a two-torus, respectively. An instability of pulsating waves can lead to solutions that drift along the group orbit, while bifurcations with Floquet multiplier $+1$ of alternating pulsating waves do not lead to drifting solutions. The third example we consider, alternating rolls, has the spatio-temporal symmetry of alternating pulsating waves as well as being invariant under reflections in two vertical planes. When the bifurcation breaks these reflections, the map $\tilde{\mathcal{G}}$ has a “two-symmetry,” as analysed by Lamb (1996). This leads to a doubling of the marginal Floquet multiplier and the possibility of bifurcation to two distinct types of drifting solutions.

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1. Introduction

Techniques for analysing symmetry-breaking bifurcations of Γ -invariant equilibria of Γ -equivariant differential equations are well-developed in the case of compact Lie groups Γ (Golubitsky *et al* 1988). The motivation for developing these methods comes, in large part, from problems of pattern formation in fluid dynamics (see, for example, Crawford & Knobloch 1991). In the simplest cases, the symmetry-breaking bifurcation corresponds to a pattern-forming instability of a basic state that is both time-independent and fully symmetric, for example, a spatially uniform equilibrium solution of the governing equations. A symmetry-breaking Hopf bifurcation of this spatially uniform state often leads to time-periodic solutions that break the translation invariance of the governing equations and that have spatio-temporal and spatial symmetries. In this paper we address bifurcations of such periodic orbits, which have broken the translation invariance but have retained a discrete group of spatio-temporal symmetries.

We consider problems posed with periodic boundary conditions, for which there is an S^1 symmetry associated with each direction of imposed periodicity. If this symmetry is broken by an equilibrium solution, then the solution is not isolated; there is a continuous family of equilibria related through the translations. An instability of this solution can excite the neutral translations modes(s) and lead to new solutions that drift along the translation group orbit. This is the case, for example, in the ‘‘parity-breaking bifurcation’’: a reflection-symmetric steady state undergoes a symmetry-breaking bifurcation to a uniformly translating solution. Another example of a bifurcation leading to drift has been observed in two-dimensional convection: when the vertical mirror plane of symmetry that separates steady counter-rotating rolls is broken in a Hopf bifurcation, the resulting solution, called a direction-reversing travelling wave or pulsating wave (PW), drifts to and fro (Landsberg & Knobloch 1991; Matthews *et al* 1993). This periodic orbit is invariant under the combination of advance of half the period in time with a reflection; any drift in one direction in the first half of the oscillation is exactly balanced by a drift in the other direction in the second half, so there is no net drift during the oscillation. Similarly in three-dimensional convection with spatial periodicity imposed, for example, on a square lattice, a symmetry-breaking Hopf bifurcation from steady convection in a square pattern can lead to alternating pulsating waves (APW), which are invariant under the combination of advance of one quarter the period and rotation by 90° (Rucklidge 1997). These solutions drift alternately along the two horizontal coordinate directions, but again have no net drift over the whole period of the oscillation.

There have been a number of studies of bifurcations of compact group orbits of (relative) equilibria. Iooss (1986) developed an approach based on centre manifold reduction to investigate bifurcations of Taylor vortices in the Taylor–Couette problem. Specifically, he analysed bifurcations in directions orthogonal to the tangent space to the group orbit of equilibria, with the neutral translation mode incorporated explicitly in the bifurcation problem. Krupa (1990) provided a general setting for investigating bifurcations of relative equilibria that focuses on the local dynamics in directions orthogonal to the tangent space to the group orbit. He shows that the resulting bifurcation problem is Σ -equivariant, where Σ is the isotropy subgroup of symmetries of the relative equilibrium, and, building on work of Field (1980), provides a group theoretic method for determining whether or not the bifurcating solutions drift. Aston *et al* (1992), and Amdjadi *et al* (1997) develop a technique for numerically investigating bifurcations of relative equilibria in $O(2)$ -equivariant partial differential equations, and apply their method to the Kuramoto–Sivashinsky equation. Their approach isolates one solution on a group orbit, while still keeping track of any constant drift along the group orbit.

In this paper we investigate bifurcations of time-periodic solutions that are not isolated as they have broken the translation invariance, but that do possess a discrete group of spatio-temporal symmetries. Our approach is similar to that of Iooss (1986). However, we are

interested in instabilities of periodic solutions, so we use centre manifold reduction for Poincaré maps. We are particularly interested in determining whether the symmetries of the basic state place any restrictions on the types of bifurcations that occur, and whether the bifurcating solutions drift along the underlying group orbit or not. We consider three examples that are motivated by numerical studies of convection with periodic boundary conditions in the horizontal direction(s). First we investigate bifurcations of the pulsating waves and alternating pulsating waves described above. These solutions have discrete spatio-temporal symmetries Z_2 and Z_4 , respectively. The group orbit of the pulsating waves is S^1 , while the group orbit of the alternating pulsating waves is a two-torus, due to imposed periodicity in two horizontal directions. The third example we treat in this paper is alternating rolls (AR), which have the same spatio-temporal symmetry as APW but are also invariant under reflection in two orthogonal vertical planes (Silber & Knobloch 1991).

The Z_n ($n = 2, 4$) spatio-temporal symmetry of the basic state places restrictions on the Poincaré return map \mathcal{M} ; specifically, we show that it is the n^{th} iterate of a map $\tilde{\mathcal{G}}$. A direct consequence of this is that period-doubling bifurcations are nongeneric (Swift & Wiesenfeld 1984). Throughout the paper we restrict our analysis to bifurcation with Floquet multiplier $+1$; we do not consider Hopf bifurcations. We also restrict attention to bifurcations that preserve the spatial-periodicity of the basic state.

Our paper is organized as follows. In the next section we lay the framework for our analysis in the setting of a simple example, namely bifurcation of pulsating waves. We show how the spatio-temporal symmetry is manifest in the Poincaré return map. Section 3 considers bifurcation of the three-dimensional analogue of pulsating waves, namely alternating pulsating waves. Section 4 considers bifurcations of alternating rolls. For this problem we need to consider six different cases, which we classify by the degree to which the spatial, and spatio-temporal symmetries are broken. In the case that the spatial reflection symmetries are fully broken by the neutral modes, the Floquet multiplier $+1$ is forced to have multiplicity two, and more than one solution branch bifurcates from the basic AR state. In one case we find a bifurcation of the AR state leading to two distinct drifting solutions. We present an example of one of the drifting patterns that is obtained by numerically integrating the equations of three-dimensional compressible magnetoconvection. In the course of the analysis of bifurcations of alternating rolls, we make contact with the work on k -symmetries of Lamb & Quispel (1994) and Lamb (1996). Section 5 contains a summary and indicates some directions for future work.

2. Two dimensions: pulsating waves

We write the partial differential equations (PDEs) for two-dimensional convection symbolically as:

$$\frac{dU}{dt} = \mathcal{F}(U; \mu), \quad (1)$$

where U represents velocity, temperature, density, etc. as functions of the horizontal coordinate x , the vertical coordinate z and time t ; μ represents a parameter of the problem; and \mathcal{F} is a nonlinear operator between suitably chosen function spaces. We assume periodic boundary conditions, with spatial period ℓ , in the x -direction.

The symmetry group of the problem is $O(2)$, which is the semi-direct product of Z_2 , generated by a reflection κ_x , and an $SO(2)$ group of translations τ_a , which act as

$$\kappa_x: x \rightarrow -x, \quad \tau_a: x \rightarrow x + a \pmod{\ell}, \quad (2)$$

where τ_ℓ is the identity and $\tau_a \kappa_x = \kappa_x \tau_{-a}$. The PDEs (1) are equivariant under the action of these symmetry operators, so $\mathcal{F}(\tau_a U; \mu) = \tau_a \mathcal{F}(U; \mu)$ and $\mathcal{F}(\kappa_x U; \mu) = \kappa_x \mathcal{F}(U; \mu)$, where τ_a

and κ_x act on the functions as follows:

$$\tau_a U(x, z, t) \equiv U(x - a, z, t), \quad \kappa_x U(x, z, t) \equiv M_{\kappa_x} U(-x, z, t). \quad (3)$$

Here M_{κ_x} is a matrix representing κ_x ; it reverses the sign of the horizontal component of velocity and leaves all other fields in U unchanged.

Suppose that when the parameter $\mu = 0$, there is a known pulsating wave solution $U_0(x, z, t)$ of (1) with temporal period T and spatial period $\lambda = \ell/N$, where N specifies the number of PWs that fit into the periodic box. The symmetries of U_0 are summarized as follows:

$$U_0(x, z, t) = \kappa_x U_0(x, z, t + \frac{1}{2}T) = U_0(x, z, t + T) = \tau_\lambda U_0(x, z, t). \quad (4)$$

There is a continuous group orbit of PWs generated by translations: $U_a = \tau_a U_0$. We are interested in bifurcations from this group orbit. Following the approach developed by Iooss (1986) and Chossat & Iooss (1994) for studying instabilities of continuous group orbits of steady solutions, we expand about the group orbit of periodic solutions as follows:

$$U(x, z, t) = \tau_{c(t)}(U_0(x, z, t) + A(x, z, t)). \quad (5)$$

Here translation along the group orbit is given by $\tau_{c(t)}$, where c is a coordinate parameterizing the group orbit. Small perturbations, orthogonal to the tangent direction of the group orbit, are specified by $A(x, z, t)$. The expansion (5) is substituted into the PDEs (1) and, after suitable projection that separates translations along the group orbit from the evolution of the perturbation orthogonal to it, we obtain equations of the form (see Chossat & Iooss (1994)):

$$\frac{dA}{dt} = \mathcal{G}(A, U_0; \mu), \quad \frac{dc}{dt} = h(A, U_0; \mu), \quad (6)$$

where \mathcal{G} and h satisfy $\mathcal{G}(0, U_0; 0) = 0$ and $h(0, U_0; 0) = 0$. An important consequence of the translation invariance of the original PDEs is that \mathcal{G} and h do not depend on the position c along the group orbit; the equation for the drift c is decoupled from the equation for the amplitude of the perturbation A . Here we find it convenient to keep track of the explicit time dependence of \mathcal{G} and h , which enters through their dependence on the basic state U_0 , by listing U_0 as one of the arguments of \mathcal{G} and h . We determine how the spatio-temporal reflection symmetry of U_0 is manifest in the equations for c and A by noting that if $\tau_{c(t)}(U_0(x, z, t) + A(x, z, t))$ is a solution of the PDEs (1), then so is

$$\kappa_x \tau_{c(t)}(U_0(x, z, t) + A(x, z, t)) = \tau_{-c(t)}(\kappa_x U_0(x, z, t) + \kappa_x A(x, z, t)). \quad (7)$$

Hence

$$\begin{aligned} \mathcal{G}(\kappa_x A, \kappa_x U_0; \mu) &= \kappa_x \mathcal{G}(A, U_0; \mu), \\ h(\kappa_x A, \kappa_x U_0; \mu) &= -h(A, U_0; \mu). \end{aligned} \quad (8)$$

Since our basic state U_0 is T -periodic, we seek a map that gives the perturbation A at time $t = T$ given a perturbation $A(0)$ at some initial time $t = 0$. Specifically, we define a time advance map \mathcal{M}_0^t acting on the perturbation A by $A(t) = \mathcal{M}_0^t(A(0))$. We adopt the approach of Swift & Wiesenfeld (1984) and split the time interval from 0 to T into two stages using the symmetry property of the underlying pulsating waves. Specifically, since $\kappa_x A(t)$ satisfies $\frac{d(\kappa_x A)}{dt} = \mathcal{G}(\kappa_x A, \kappa_x U_0; \mu)$ and $\kappa_x U_0(x, z, t) = U_0(x, z, t + \frac{T}{2})$, we have $\kappa_x A(t) = \mathcal{M}_{T/2}^{t+T/2}(\kappa_x A(0))$; hence

$$\kappa_x \mathcal{M}_0^t = \mathcal{M}_{T/2}^{T/2+t} \kappa_x. \quad (9)$$

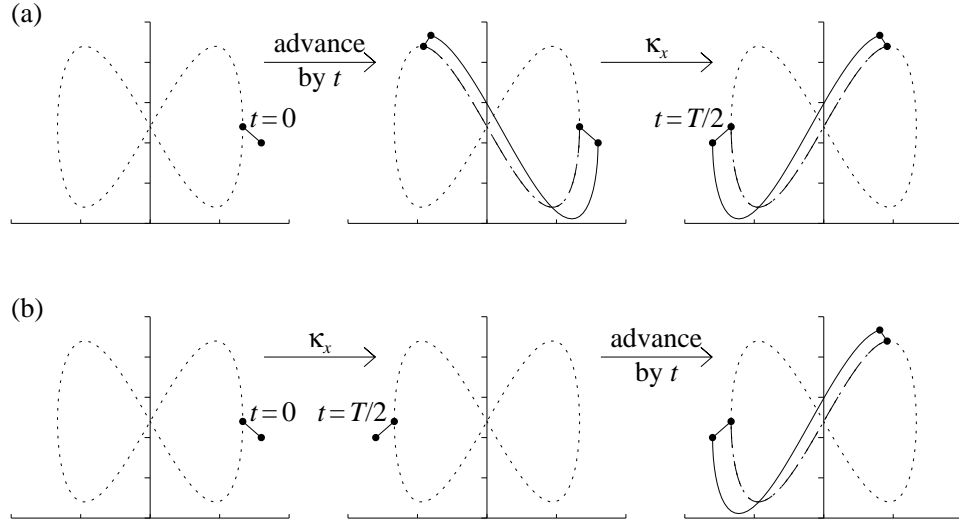


Figure 1. Illustration of $\kappa_x \mathcal{M}_0^t = \mathcal{M}_{T/2}^{T/2+t} \kappa_x$. In this example, the reflection κ_x changes the sign of the horizontal coordinate. The PW periodic solution is shown as a dotted line. (a) A perturbation at $t = 0$ is advanced in time by an amount t (the solid line, which stays close to the broken line on the periodic orbit), then the system is reflected. (b) We arrive at the same final position if we reflect (so now the perturbation is about the PW at $t = \frac{1}{2}T$) and then advance in time by the same amount.

Advancing the perturbation by a time t starting from time 0 and then reflecting the whole system is equivalent to reflecting the whole system then advancing by a time t starting from time $\frac{1}{2}T$ (see Figure 1). It follows immediately that the full period map \mathcal{M}_0^T can be written as the second iterate of a map $\tilde{\mathcal{G}}$:

$$\mathcal{M}_0^T = \mathcal{M}_{T/2}^T \kappa_x^2 \mathcal{M}_0^{T/2} = \left(\kappa_x \mathcal{M}_0^{T/2} \right)^2 \equiv \tilde{\mathcal{G}}^2. \quad (10)$$

Rather than consider the full period map \mathcal{M}_0^T , we will consider the map $\tilde{\mathcal{G}} \equiv \kappa_x \mathcal{M}_0^{T/2}$. The map $\tilde{\mathcal{G}}$ has no special property under reflections, but it commutes with translations τ_λ , which leave the underlying pulsating waves invariant: $\tilde{\mathcal{G}}\tau_\lambda = \tau_\lambda\tilde{\mathcal{G}}$.

The dynamics of the perturbation is now given by the map $\tilde{\mathcal{G}}$: $\mathcal{A}_{n+1} = \tilde{\mathcal{G}}(\mathcal{A}_n; \mu)$, where each iterate corresponds to advancing in time by $\frac{1}{2}T$ and reflecting; thus $A(\frac{1}{2}T) = \kappa_x \mathcal{A}_1$, starting from \mathcal{A}_0 at time 0. In order to compute the drift c_1 of the solution at time $\frac{1}{2}T$, we integrate the dc/dt equation (6) for a time $\frac{1}{2}T$, starting at a position c_0 and with initial perturbation $A(0) = \mathcal{A}_0$:

$$c_1 = c_0 + \int_0^{T/2} h(\mathcal{M}_0^t(\mathcal{A}_0), U_0(t); \mu) dt \equiv c_0 + \tilde{h}(\mathcal{A}_0; \mu). \quad (11)$$

Then, after a second half-period,

$$\begin{aligned} c_2 &= c_1 + \int_{T/2}^T h(\mathcal{M}_{T/2}^t(A(\frac{1}{2}T)), U_0(t); \mu) dt \\ &= c_1 + \int_{T/2}^T h(\mathcal{M}_{T/2}^t(\kappa_x \mathcal{A}_1), U_0(t); \mu) dt \\ &= c_1 + \int_{T/2}^T h(\kappa_x \mathcal{M}_0^{t-T/2}(\mathcal{A}_1), \kappa_x U_0(t - T/2); \mu) dt \end{aligned}$$

$$\begin{aligned}
&= c_1 - \int_0^{T/2} h(\mathcal{M}_0^{t'}(\mathcal{A}_1), U_0(t'); \mu) dt' \\
&= c_1 - \tilde{h}(\mathcal{A}_1; \mu).
\end{aligned} \tag{12}$$

Thus the combined dynamics of the perturbation and translation can be written as

$$\mathcal{A}_{n+1} = \tilde{\mathcal{G}}(\mathcal{A}_n; \mu), \quad c_{n+1} = c_n + (-1)^n \tilde{h}(\mathcal{A}_n; \mu). \tag{13}$$

Since the unperturbed PW is a non-drifting solution of the problem at $\mu = 0$ we have $\tilde{\mathcal{G}}(0; 0) = 0$ and $\tilde{h}(0; 0) = 0$. Moreover, the spatial periodicity of U_0 places some symmetry restrictions on $\tilde{\mathcal{G}}$ and \tilde{h} ; specifically, $\tilde{\mathcal{G}}(\tau_\lambda \mathcal{A}; \mu) = \tau_\lambda \tilde{\mathcal{G}}(\mathcal{A}; \mu)$ and $\tilde{h}(\tau_\lambda \mathcal{A}; \mu) = \tilde{h}(\mathcal{A}; \mu)$.

We turn now to the codimension-one bifurcations of the PW, which are the trivial fixed points $\mathcal{A} = 0$, $c = c_0$ of (13) when $\mu = 0$. The map (13) always has one Floquet multiplier (FM) equal to one because of the translation invariance of the c part of the map. Bifurcations occur when a FM of the linearization of $\tilde{\mathcal{G}}$ crosses the unit circle: either a FM = 1, or a FM = -1, or there is a pair of complex conjugate FMs with unit modulus. Because we have assumed periodic boundary conditions in the original PDEs, we expect the spectrum of the linearization to be discrete and the centre manifold theorem for maps to apply. (See Chossat & Iooss (1994) for a discussion of the centre manifold reduction in the similar problem of bifurcations from Taylor vortices.) Let ζ be the eigenfunction associated with the critical FM, so that on the centre manifold, we can write

$$\mathcal{A}_n = a_n \zeta + \Phi(a_n), \tag{14}$$

where Φ is the graph of the centre manifold. The unfolded dynamics takes the form

$$a_{n+1} = \hat{g}(a_n; \mu), \quad c_{n+1} = c_n + (-1)^n \hat{h}(a_n; \mu), \tag{15}$$

where \hat{g} and \hat{h} are the maps $\tilde{\mathcal{G}}$ and \tilde{h} reduced to the centre manifold; \hat{g} and \hat{h} share the same symmetry properties as $\tilde{\mathcal{G}}$ and \tilde{h} .

In this paper, we only consider the case where τ_λ acts trivially. We therefore expect only generic bifurcations in the map \hat{g} : saddle-node when FM = 1, period-doubling when FM = -1 and Hopf when there are a pair of complex FMs. The FMs for the full period map \mathcal{M}_0^T , which are the squares of the FMs of \hat{g} , will generically be either one or come in complex conjugate pairs. In particular, we do not expect \mathcal{M}_0^T to have a FM = -1; this mechanism for suppressing period-doubling bifurcations was discussed by Swift & Wiesenfeld (1984).

Here we consider only the cases where \hat{g} has a FM = +1 or -1. The normal form in the case FM = 1 is:

$$a_{n+1} = \mu + a_n - a_n^2, \quad c_{n+1} = c_n + (-1)^n \hat{h}(a_n; \mu), \tag{16}$$

to within a rescaling and a change of sign. The parameter μ is zero at the bifurcation point, and the fixed points of the a part of the map are $a = \pm\sqrt{\mu}$ when μ is positive. The spatial translations are

$$c_0, \quad c_1 = c_0 + \hat{h}(a; \mu), \quad c_2 = c_0, \quad \dots \tag{17}$$

We therefore have a c_0 -parameterized family of solutions that vanish, in pairs, as μ is decreased through $\mu = 0$. We interpret this bifurcation by considering the solutions with $c_0 = -\frac{1}{2}\hat{h}(a; \mu)$, $a = \pm\sqrt{\mu}$. In this case, we have a pair of pulsating wave solutions, translated with respect to the original PW by $\frac{1}{2}\hat{h}(\pm\sqrt{\mu}; \mu)$, which collide in a saddle-node bifurcation at $\mu = 0$. The remainder of the family of solutions is obtained by translating this pair.

The case $\text{FM} = -1$ is more interesting. The normal form in the supercritical case is

$$a_{n+1} = (-1 + \mu)a_n - a_n^3, \quad c_{n+1} = c_n + (-1)^n \hat{h}(a_n; \mu), \quad (18)$$

with a fixed point $a = 0$ and a period-two orbit $a_n = (-1)^n \sqrt{\mu}$. The dynamics of the spatial translations are

$$\begin{aligned} c_0, \quad c_1 &= c_0 + \hat{h}(a_0; \mu), \quad c_2 = c_0 + \hat{h}(a_0; \mu) - \hat{h}(-a_0; \mu), \\ c_3 &= c_0 + 2\hat{h}(a_0; \mu) - \hat{h}(-a_0; \mu), \quad \dots \end{aligned} \quad (19)$$

Since $\hat{h}(0; 0) = 0$, and generically $\frac{\partial \hat{h}}{\partial a}(0; 0) \neq 0$, $\hat{h}(a_0; \mu)$ and $\hat{h}(-a_0; \mu)$ have opposite sign for small μ ; this represents a symmetry-breaking bifurcation that leads to a solution that drifts along the group orbit of the PW.

The main points of interest in this section are the approach that we have taken in analysing the instabilities of the group orbit of the spatio-temporally symmetric periodic orbit, and the observation that an instability of the pulsating wave with $\text{FM} = 1$ in the full-period map can lead to drifting solutions or not. Whether solutions drift can only be determined by examining the half-period map. In the next two sections, we apply our method to three-dimensional alternating pulsating waves and to alternating rolls, the latter having spatial as well as spatio-temporal symmetries.

3. Three dimensions: alternating pulsating waves

Alternating pulsating waves (APW) are the simplest three-dimensional analogue of the pulsating waves discussed in the previous section. These periodic oscillations have been observed in numerical simulations of three-dimensional compressible magnetoconvection with periodic boundary conditions in the two horizontal directions (Matthews *et al* 1995). They appear either after a series of global bifurcations (Rucklidge & Matthews 1995; Matthews *et al* 1996) or in a Hopf bifurcation from convection in a square pattern (Rucklidge 1997), and are invariant under the combined operation of advancing one quarter period in time and rotating 90° in space.

The full symmetry group of the problem is the semi-direct product of the D_4 symmetry group of the square lattice and a two-torus T^2 of translations in the two horizontal directions, x and y . D_4 is generated by a reflection κ_x , a clockwise rotation by 90° ρ ,

$$\begin{aligned} \kappa_x: (x, y) &\rightarrow (-x, y), \quad \rho: (x, y) \rightarrow (y, -x), \\ \tau_{a,b}: (x, y) &\rightarrow (x + a \pmod{\ell}, y + b \pmod{\ell}), \end{aligned} \quad (20)$$

where $\rho\tau_{a,b} = \tau_{b,-a}\rho$.

As before, we assume that at $\mu = 0$, we have a known APW solution $U_{0,0}(x, y, z, t)$ with spatial period λ in each direction and temporal period T ; then $U_{0,0}$ satisfies:

$$\begin{aligned} U_{0,0}(x, y, z, t) &= \rho U_{0,0}(x, y, z, t + \tfrac{1}{4}T) = U_{0,0}(x, y, z, t + T) \\ &= \tau_{\lambda,0} U_{0,0}(x, y, z, t) = \tau_{0,\lambda} U_{0,0}(x, y, z, t). \end{aligned} \quad (21)$$

There is a two-parameter continuous group orbit of APWs generated by translations: $U_{a,b} = \tau_{a,b} U_{0,0}$. We expand about this group orbit:

$$U(x, y, z, t) = \tau_{c_x(t), c_y(t)}(U_{0,0}(x, y, z, t) + A(x, y, z, t)), \quad (22)$$

where (c_x, c_y) is a time-dependent translation around the group orbit and A is the perturbation orthogonal to the tangent plane to the group orbit. As before, we separate the evolution of the translations from that of the perturbation:

$$\frac{dA}{dt} = \mathcal{G}(A, U_{0,0}; \mu), \quad \frac{dc_x}{dt} = h_x(A, U_{0,0}; \mu), \quad \frac{dc_y}{dt} = h_y(A, U_{0,0}; \mu), \quad (23)$$

where we keep track of the explicit time-dependence of \mathcal{G} , h_x , and h_y through the argument $U_{0,0}$. The spatio-temporal symmetry of the basic state $U_{0,0}$ is manifest in \mathcal{G} , h_x and h_y as follows:

$$\begin{aligned} \mathcal{G}(\rho A, \rho U_{0,0}; \mu) &= \rho \mathcal{G}(A, U_{0,0}; \mu), \\ h_x(\rho A, \rho U_{0,0}; \mu) &= h_x(A, U_{0,0}; \mu), \\ h_y(\rho A, \rho U_{0,0}; \mu) &= -h_x(A, U_{0,0}; \mu), \end{aligned} \quad (24)$$

where $\rho U_{0,0}(t + \frac{1}{4}T) = U_{0,0}(t)$. It is convenient to introduce a complex translation $c \equiv c_x + ic_y$ and a corresponding $h \equiv h_x + ih_y$, so $\rho\tau_c = \tau_{-ic}\rho$.

As before, we define a time advance map acting on the perturbation so $A(t) = \mathcal{M}_0^t(A(0))$; this has the property

$$\mathcal{M}_0^t \rho = \rho \mathcal{M}_{T/4}^{T/4+t}, \quad \mathcal{M}_0^t \rho^2 = \rho^2 \mathcal{M}_{T/2}^{T/2+t}, \quad \mathcal{M}_0^t \rho^3 = \rho^3 \mathcal{M}_{3T/4}^{3T/4+t} \quad (25)$$

because of the underlying spatio-temporal symmetry of the APW. The full period map \mathcal{M}_0^T is then the fourth iterate of a map $\tilde{\mathcal{G}}$:

$$\mathcal{M}_0^T = \rho^4 \mathcal{M}_{3T/4}^T \mathcal{M}_0^{3T/4} = \rho \mathcal{M}_0^{T/4} \rho^3 \mathcal{M}_0^{3T/4} = \left(\rho \mathcal{M}_0^{T/4} \right)^4 \equiv \tilde{\mathcal{G}}^4. \quad (26)$$

Instead of \mathcal{M}_0^T , we consider $\tilde{\mathcal{G}} \equiv \rho \mathcal{M}_0^{T/4}$, which has no special properties under reflections and rotations, but does commute with $\tau_{\lambda,0}$ and $\tau_{0,\lambda}$, which leave the underlying APW invariant.

The dynamics of the perturbation is given by $\mathcal{A}_{n+1} = \tilde{\mathcal{G}}(\mathcal{A}_n)$, where $A(\frac{1}{4}T) = \rho^3 \mathcal{A}_1$, etc. Then

$$c_1 = c_0 + \int_0^{T/4} h(\mathcal{M}_0^t(\mathcal{A}_0), U_{0,0}(t); \mu) dt \equiv c_0 + \tilde{h}(\mathcal{A}_0; \mu), \quad (27)$$

where the map $\tilde{h} = \tilde{h}_x + i\tilde{h}_y$ is invariant under the translations $\tau_{\lambda,0}$ and $\tau_{0,\lambda}$. After the next quarter period, we find

$$\begin{aligned} c_2 &= c_1 + \int_{T/4}^{T/2} h(\mathcal{M}_{T/4}^t(A(\frac{1}{4}T)), U_{0,0}(t); \mu) dt \\ &= c_1 + \int_{T/4}^{T/2} h(\mathcal{M}_{T/4}^t(\rho^3 \mathcal{A}_1), U_{0,0}(t); \mu) dt \\ &= c_1 + \int_{T/4}^{T/2} h(\rho^3 \mathcal{M}_0^{t-T/4}(\mathcal{A}_1), \rho^3 U_{0,0}(t - T/4); \mu) dt \\ &= c_1 + i\tilde{h}(\mathcal{A}_1; \mu). \end{aligned} \quad (28)$$

So the combined dynamics of the perturbation and the translation can be written as

$$\mathcal{A}_{n+1} = \tilde{\mathcal{G}}(\mathcal{A}_n; \mu), \quad c_{n+1} = c_n + i^n \tilde{h}(\mathcal{A}_n; \mu), \quad (29)$$

where $\tilde{\mathcal{G}}(0; 0) = \tilde{h}(0; 0) = 0$.

We consider the bifurcations of (29) only in the case where $\tau_{\lambda,0}$ and $\tau_{0,\lambda}$ act trivially. Note that, as in the case of pulsating waves, the generic bifurcations of APW are either steady state (FM = +1) or Hopf, since $\mathcal{M}_0^T = \tilde{\mathcal{G}}^4$. We consider bifurcations with FM = +1 of \mathcal{M}_0^T only; generically, these occur when the linearization of $\tilde{\mathcal{G}}$ has a FM of +1 or -1. Near a bifurcation point we reduce the dynamics onto the centre manifold

$$a_{n+1} = \hat{g}(a_n; \mu), \quad c_{n+1} = c_n + i^n \hat{h}(a_n; \mu). \quad (30)$$

When a FM = 1, once again we have a saddle-node bifurcation, this time involving pairs of APWs that are translated relative to each other. If a FM is -1, we have $a_n = (-1)^n \sqrt{\mu}$, and the spatial translations are:

$$\begin{aligned} c_0, \quad c_1 = c_0 + \hat{h}(a_0; \mu), \quad c_2 = c_0 + \hat{h}(a_0; \mu) + i\hat{h}(-a_0; \mu), \\ c_3 = c_0 + i\hat{h}(-a_0; \mu), \quad c_4 = c_0, \quad \dots \end{aligned} \quad (31)$$

This solution has no net drift (unlike in the two-dimensional problem), but travels back and forth different amounts in the two horizontal directions since, generically, $\hat{h}_x(a_0; \mu) \neq \hat{h}_y(a_0; \mu)$. The solution remains invariant under advance of half its period in time combined with a rotation of 180°. To see this, we construct the solution $U(x, y, z, t)$ at $t = 0$ and $t = \frac{1}{2}T$ using the solution in the c_0 -parameterized family that satisfies $c_0 = -c_2$. Specifically, we insert the centre manifold solution $A(0) = \mathcal{A}_0 = a_0\zeta + \Phi(a_0)$, $A(\frac{1}{2}T) = \rho^2\mathcal{A}_2 = \rho^2(a_0\zeta + \Phi(a_0))$ in (22). We obtain

$$\begin{aligned} U(0) &= \tau_{c_0}(U_{0,0}(0) + a_0\zeta + \Phi(a_0)) \\ U(\frac{1}{2}T) &= \tau_{-c_0}(U_{0,0}(\frac{1}{2}T) + \rho^2 a_0\zeta + \rho^2 \Phi(a_0)) \\ &= \tau_{-c_0} \rho^2 (U_{0,0}(0) + a_0\zeta + \Phi(a_0)) \\ &= \rho^2 U(0), \end{aligned} \quad (32)$$

where we have suppressed the (x, y, z) -dependence of U , retaining only its t -dependence.

Thus, in the simple case of APW, we cannot get drifting solutions in a bifurcation with FM = 1 for the time- T return map. We next consider the same bifurcation for the more complicated example of alternating rolls. This solution has the same spatio-temporal symmetry as APW but has extra spatial reflection symmetries. We shall see that in this case a particular symmetry-breaking bifurcation leads to two distinct types of drifting solutions.

4. Additional spatial symmetries: alternating rolls

Alternating rolls (AR) are created in a primary Hopf bifurcation from a $D_4 \times T^2$ invariant trivial solution (Silber & Knobloch 1991). Like alternating pulsating waves, alternating rolls are invariant under the spatio-temporal symmetry of advancing one-quarter period in time and rotating 90° in space, but have the additional property of being invariant under reflections in two orthogonal vertical planes. Alternating rolls have been observed in three-dimensional incompressible and compressible magnetoconvection (Clune & Knobloch 1994; Matthews *et al* 1995).

For convenience in this section, we define $\tilde{\rho}$ to be the combined advance of one quarter period in time followed by a 90° clockwise rotation about the line $(x, y) = (0, 0)$. Reflecting in the planes $x = \frac{1}{4}\lambda$ or $y = \frac{1}{4}\lambda$ leaves alternating rolls unchanged at all times, so the sixteen-element group that leaves AR invariant is generated by κ'_x , κ'_y and $\tilde{\rho}$, where

$$\begin{aligned} \kappa'_x: (x, y, z, t) &\rightarrow (\frac{1}{2}\lambda - x, y, z, t), \\ \kappa'_y: (x, y, z, t) &\rightarrow (x, \frac{1}{2}\lambda - y, z, t), \\ \tilde{\rho}: (x, y, z, t) &\rightarrow (y, -x, z, t + \frac{1}{4}T). \end{aligned} \quad (33)$$

Table 1. Summary of six types of bifurcations of alternating rolls, distinguished by the action of κ'_x and κ'_y on the critical modes, and by the critical Floquet multipliers of \mathcal{G} . Isotropy subgroups (up to conjugacy) of the bifurcating solution branches are indicated, along with their order; in cases B+ and B-, there are two distinct solution branches.

Case	Action of κ'_x, κ'_y on marginal modes	Floquet multiplier(s)	Bifurcation (drift or not)	Isotropy subgroup (order)
A+(+1)	$\kappa'_x \kappa'_y \zeta = \zeta$ $\kappa'_x \zeta = \kappa'_y \zeta = \zeta$	FM = +1	Saddle-node (no drift)	$\langle \kappa'_x, \kappa'_y, \tilde{\rho} \rangle$ (16)
A+(-1)	as A+(+1)	FM = -1	Symmetry-breaking (no drift)	$\langle \kappa'_x, \kappa'_y, \tilde{\rho}^2 \rangle$ (8)
A-(+1)	$\kappa'_x \kappa'_y \zeta = \zeta$ $\kappa'_x \zeta = \kappa'_y \zeta = -\zeta$	FM = +1	Symmetry-breaking (no drift)	$\langle \kappa'_x \kappa'_y, \tilde{\rho} \rangle$ (8)
A-(-1)	as A-(+1)	FM = -1	Symmetry-breaking (no drift)	$\langle \kappa'_x \kappa'_y, \kappa'_x \tilde{\rho} \rangle$ (8)
B+	$\kappa'_x \kappa'_y \zeta_{\pm} = -\zeta_{\pm}$ $\zeta_- = \kappa'_x \zeta_+ = -\kappa'_y \zeta_+$	FM = ± 1	Symmetry-breaking (no net drift)	$\langle \kappa'_x, \tilde{\rho}^2 \rangle$ (4) $\langle \tilde{\rho} \rangle$ (4)
B-	as B+	FM = $\pm i$	Symmetry-breaking (drift)	$\langle \kappa'_y \rangle$ (2) Trivial (1)

The basic AR solution $U_{0,0}(x, y, z, t)$ exists at $\mu = 0$ and satisfies

$$\begin{aligned}
U_{0,0}(x, y, z, t) &= \rho U_{0,0}(x, y, z, t + \frac{1}{4}T) = U_{0,0}(x, y, z, t + T) \\
&= \kappa'_x U_{0,0}(x, y, z, t) = \kappa'_y U_{0,0}(x, y, z, t) \\
&= \tau_{\lambda,0} U_{0,0}(x, y, z, t) = \tau_{0,\lambda} U_{0,0}(x, y, z, t).
\end{aligned} \tag{34}$$

As in section 3, we expand about this basic solution and recover the map (29). The presence of extra reflection symmetries of the underlying solution manifests itself in the following way:

$$\begin{aligned}
\tilde{\mathcal{G}}(\kappa'_x \mathcal{A}) &= \kappa'_y \tilde{\mathcal{G}}(\mathcal{A}), & \tilde{\mathcal{G}}(\kappa'_y \mathcal{A}) &= \kappa'_x \tilde{\mathcal{G}}(\mathcal{A}), \\
\tilde{h}_x(\kappa'_x \mathcal{A}) &= -\tilde{h}_x(\mathcal{A}), & \tilde{h}_x(\kappa'_y \mathcal{A}) &= \tilde{h}_x(\mathcal{A}), \\
\tilde{h}_y(\kappa'_x \mathcal{A}) &= \tilde{h}_y(\mathcal{A}), & \tilde{h}_y(\kappa'_y \mathcal{A}) &= -\tilde{h}_y(\mathcal{A}).
\end{aligned} \tag{35}$$

Note that the rotation in the definition of $\tilde{\mathcal{G}}$ (26) implies that reflecting with κ'_x then applying $\tilde{\mathcal{G}}$ is equivalent to applying $\tilde{\mathcal{G}}$ then reflecting with κ'_y , since $\rho \kappa'_x = \kappa'_y \rho$. In the terminology of Lamb & Quispel (1994), κ'_x and κ'_y are 2-symmetries of $\tilde{\mathcal{G}}$, that is, $\tilde{\mathcal{G}}^2(\kappa'_x \mathcal{A}) = \kappa'_x \tilde{\mathcal{G}}^2(\mathcal{A})$. In general, k -symmetries arise when the spatial part of the spatio-temporal symmetry of a time-periodic solution does not commute with its purely spatial symmetries (Lamb 1997).

The remainder of this section is devoted to the discussion of the codimension-one steady bifurcations of this problem. We do not consider bifurcations that break the spatial periodicity, so $\tau_{\lambda,0}$ and $\tau_{0,\lambda}$ act trivially, nor do we consider Hopf bifurcations. The results are summarised in Table 1.

We begin by noting that $\tilde{\mathcal{G}}(\kappa'_x \kappa'_y \mathcal{A}) = \kappa'_x \kappa'_y \tilde{\mathcal{G}}(\mathcal{A})$, so $\kappa'_x \kappa'_y$ commutes with the linearisation $\tilde{\mathcal{L}}$ of $\tilde{\mathcal{G}}$, whereas $\kappa'_x \tilde{\mathcal{L}} = \tilde{\mathcal{L}} \kappa'_y$. The eigenspaces of $\tilde{\mathcal{L}}$ are invariant under the reflection $\kappa'_x \kappa'_y$. We assume the generic situation of one-dimensional eigenspaces, then each eigenfunction ζ must be either even or odd under the reflection $\kappa'_x \kappa'_y$, i.e., $\kappa'_x \kappa'_y \zeta = \zeta$ (case A) or $\kappa'_x \kappa'_y \zeta = -\zeta$ (case B), since $(\kappa'_x \kappa'_y)^2$ is the identity. In case A, if $\zeta = \kappa'_x \kappa'_y \zeta$ is an eigenfunction of $\tilde{\mathcal{L}}$ with FM s , then $\kappa'_x \zeta = \kappa'_y \zeta$ has the same FM:

$$\tilde{\mathcal{L}} \kappa'_x \zeta = \kappa'_y \tilde{\mathcal{L}} \zeta = s \kappa'_y \zeta = s \kappa'_x \zeta. \tag{36}$$

Therefore, ζ and $\kappa'_x \zeta$ are linearly dependent; moreover, $\kappa'_x{}^2$ is the identity, so either $\kappa'_x \zeta = \zeta$ (case A+) or $\kappa'_x \zeta = -\zeta$ (case A-). Finally, these two cases are subdivided according to the value of the critical Floquet multiplier of $\tilde{\mathcal{L}}$, (either $+1$ or -1) at the bifurcation point.

Case B is rather different. Here we have $\kappa'_x \zeta = -\kappa'_y \zeta$, so

$$\tilde{\mathcal{L}}\kappa'_x \zeta = \kappa'_y \tilde{\mathcal{L}}\zeta = s\kappa'_y \zeta = -s\kappa'_x \zeta. \quad (37)$$

Thus $\kappa'_x \zeta$ has FM = $-s$ and is linearly independent of ζ , which has FM = s . We define ζ_+ to be the eigenfunction of s and ζ_- to be the eigenfunction of $-s$, with $\zeta_- = \kappa'_x \zeta_+ = -\kappa'_y \zeta_+$. There are two ways in which two Floquet multipliers s and $-s$ can cross the unit circle: either at $+1$ and -1 (case B+) or at $+i$ and $-i$ (case B-). Note that in the absence of the reflection symmetries these bifurcations would be codimension-two; here they occur as generic bifurcations. Since the FMs of the time- T map \mathcal{M}_0^T are the fourth power of the FMs of $\tilde{\mathcal{G}}$, the effect of the symmetry in case B is to force a repeated FM = $+1$ in the map \mathcal{M}_0^T .

In case A, we write

$$\mathcal{A}_n = a_n \zeta + \Phi(a_n), \quad (38)$$

near the bifurcation point, where Φ is the graph of the centre manifold. On the centre manifold we have $\mathcal{A} = \kappa'_x \kappa'_y \mathcal{A}$, so

$$\tilde{h}_x(\mathcal{A}) = \tilde{h}_x(\kappa'_x \kappa'_y \mathcal{A}) = -\tilde{h}_x(\kappa'_y \mathcal{A}) = -\tilde{h}_x(\mathcal{A}) = 0, \quad (39)$$

where we have used (35). Thus in case A, \tilde{h}_x and \tilde{h}_y are identically zero, and no bifurcation will lead to drift along the group orbit of alternating rolls.

The reflections κ'_x and κ'_y act trivially in case A+. A FM = $+1$ leads to a saddle-node bifurcation of alternating rolls. The normal form in the case FM = -1 gives $a_n = (-1)^n a_0$, from which the bifurcating solution $U(t)$ can be reconstructed. Choosing the initial translation c_0 to be zero, and suppressing the (x, y, z) -dependence of U , we have

$$\begin{aligned} U(0) &= U_{0,0}(0) + a_0 \zeta + \Phi(a_0), & U(\tfrac{1}{4}T) &= \rho^3(U_{0,0}(0) - a_0 \zeta + \Phi(-a_0)), \\ U(\tfrac{1}{2}T) &= \rho^2(U_{0,0}(0) + a_0 \zeta + \Phi(a_0)), & U(\tfrac{3}{4}T) &= \rho(U_{0,0}(0) - a_0 \zeta + \Phi(-a_0)). \end{aligned} \quad (40)$$

Here it should be recalled that $U_{0,0}(\tfrac{1}{4}T) = \rho^3 U_{0,0}(0)$, and that on the centre manifold

$$A(\tfrac{1}{4}T) = \rho^3 \mathcal{A}_1 = \rho^3(a_1 \zeta + \Phi(a_1)) = \rho^3(-a_0 \zeta + \Phi(-a_0)). \quad (41)$$

This solution satisfies

$$U(t) = \kappa'_x U(t) = \kappa'_y U(t) = \rho^2 U(t + \tfrac{1}{2}T), \quad (42)$$

and thus has the same symmetries as “standing cross-rolls”, described by Silber & Knobloch (1991).

In case A-, κ'_x and κ'_y act nontrivially, so the behaviour on the centre manifold is governed by a pitchfork normal form ($a_n = a_0$) when the FM = $+1$ and by a period-doubling normal form ($a_n = (-1)^n a_0$) when the FM = -1 . At leading order in a_0 , the bifurcating solutions $U(t)$ in the two cases are

$$\begin{aligned} U(0) &= U_{0,0}(0) + a_0 \zeta, & U(\tfrac{1}{4}T) &= \rho^3(U_{0,0}(0) \pm a_0 \zeta), \\ U(\tfrac{1}{2}T) &= \rho^2(U_{0,0}(0) + a_0 \zeta), & U(\tfrac{3}{4}T) &= \rho(U_{0,0}(0) \pm a_0 \zeta). \end{aligned} \quad (43)$$

These solutions are not invariant under κ'_x or κ'_y (since these change the sign of ζ), but are invariant under the product $\kappa'_x \kappa'_y$. In addition, $U(t) = \rho U(t + \tfrac{1}{4}T)$ in the case FM = $+1$ and $U(t) = \kappa'_x \rho U(t + \tfrac{1}{4}T)$ in the case FM = -1 .

Case B is more interesting. On the two-dimensional centre manifold, we write

$$\mathcal{A}_n = (-a_n + b_n)\zeta_+ + (a_n + b_n)\zeta_- + \Phi(a_n, b_n); \quad (44)$$

the form of this expression is chosen for later convenience. The map (29) reduces to

$$(a_{n+1}, b_{n+1}) = \hat{g}(a_n, b_n; \mu), \quad c_{n+1} = c_n + i^n (\hat{h}_x(a_n, b_n; \mu) + i\hat{h}_y(a_n, b_n; \mu)). \quad (45)$$

Since $\zeta_- = \kappa'_x \zeta_+ = -\kappa'_y \zeta_+$, we have

$$\begin{aligned} \kappa'_x \mathcal{A}_n &= (a_n + b_n)\zeta_+ + (-a_n + b_n)\zeta_- + \kappa'_x \Phi(a_n, b_n), \\ \kappa'_y \mathcal{A}_n &= (-a_n - b_n)\zeta_+ + (a_n - b_n)\zeta_- + \kappa'_y \Phi(a_n, b_n); \end{aligned} \quad (46)$$

thus

$$\kappa'_x(a_n, b_n) = (-a_n, b_n), \quad \kappa'_y(a_n, b_n) = (a_n, -b_n). \quad (47)$$

From this and from (35), we deduce that on the centre manifold

$$\begin{aligned} \hat{h}_x(a_n, b_n) &= -\hat{h}_x(-a_n, b_n) = \hat{h}_x(a_n, -b_n) \\ \hat{h}_y(a_n, b_n) &= -\hat{h}_y(a_n, -b_n) = \hat{h}_y(-a_n, b_n), \end{aligned} \quad (48)$$

implying that $\hat{h}_x(0, b; \mu) = 0$ and $\hat{h}_y(a, 0; \mu) = 0$. Moreover, \hat{g} inherits the symmetries (35) of $\tilde{\mathcal{G}}$:

$$\kappa'_y \hat{g}(a_n, b_n) = \hat{g}(\kappa'_x(a_n, b_n)), \quad \kappa'_x \hat{g}(a_n, b_n) = \hat{g}(\kappa'_y(a_n, b_n)). \quad (49)$$

Thus the linearisation $\hat{\mathcal{L}}$ of \hat{g} satisfies

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\mathcal{L}} = \hat{\mathcal{L}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (50)$$

which forces $\hat{\mathcal{L}}$ to be of the form

$$\hat{\mathcal{L}} = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \quad (51)$$

where a_n, b_n can be scaled so that $\alpha = 1$. There is a bifurcation when $\beta = +1$ or $\beta = -1$, yielding FMs ± 1 (case B+) or $\pm i$ (case B-), respectively.

In order to analyse the dynamics near the bifurcation point, we compute the normal form of the bifurcation problems, expanding \hat{g} as a Taylor series in a and b . The reflection symmetry $\kappa'_x \kappa'_y$ prohibits quadratic terms, and all but two of the cubic terms can be removed by near-identity transformations. We thus have the unfolded normal form, truncated at cubic order, in the two cases B+ and B-:

$$\begin{aligned} a_{n+1} &= b_n, \\ b_{n+1} &= \pm(1 + \mu)a_n + Pa_n^3 + Qa_nb_n^2, \\ c_{n+1} &= c_n + i^n (\hat{h}_x(a_n, b_n; \mu) + i\hat{h}_y(a_n, b_n; \mu)), \end{aligned} \quad (52)$$

where $\mu = 0$ at the bifurcation point and P and Q are constants.

Lamb (1996) deduced the (a, b) part of this normal form for a local bifurcation of a map with a $Z_2 \times Z_2$ 2-symmetry group, appropriate to the case under study here. We have chosen

our scalings and near-identity transformations to match Lamb's notation. Lamb described the period-one, two and four orbits that are created in the bifurcation at $\mu = 0$ and calculated their stability as a function of the constants P and Q ; we will interpret those results in terms of bifurcations from, and drift along, the group orbit of alternating rolls.

In case B+, there are three types of orbits created. The first is a period-two orbit $(a_0, 0) \leftrightarrow (0, a_0)$, with $0 = \mu + Pa_0^2$. From this and the symmetries (48) of \hat{h} , we deduce the drift of the solution at each iterate:

$$\begin{aligned} c_0, \quad c_1 = c_0 + \hat{h}_x(a_0, 0; \mu), \quad c_2 = c_0 + \hat{h}_x(a_0, 0; \mu) - \hat{h}_y(0, a_0; \mu), \\ c_3 = c_0 - \hat{h}_y(0, a_0; \mu), \quad c_4 = c_0, \quad \dots \end{aligned} \quad (53)$$

There is no net drift along the group orbit in this case. Moreover, the c_0 -parameterized family of solutions drift to and fro in the x direction only since $c_n - c_0$ is real. Consider $c_0 = \frac{1}{2}(-\hat{h}_x(a_0, 0; \mu) + \hat{h}_y(0, a_0; \mu)) = -c_2$, where c_0 is real and thus corresponds to a translation in the x -direction. The reconstructed solution $U(t)$, at leading order in a_0 , satisfies

$$\begin{aligned} U(0) = \tau_{c_0}(U_{0,0}(0) - a_0\zeta_+ + a_0\zeta_-), \quad U(\tfrac{1}{4}T) = \tau_{c_1}\rho^3(U_{0,0}(0) + a_0\zeta_+ + a_0\zeta_-), \\ U(\tfrac{1}{2}T) = \tau_{-c_0}\rho^2(U_{0,0}(0) - a_0\zeta_+ + a_0\zeta_-), \quad U(\tfrac{3}{4}T) = \tau_{-c_1}\rho(U_{0,0}(0) + a_0\zeta_+ + a_0\zeta_-), \end{aligned} \quad (54)$$

so we have $U(t) = \kappa'_y U(t) = \rho^2 U(t + \frac{1}{2}T)$. The conjugate orbit, $(0, a_0) \leftrightarrow (a_0, 0)$, has symmetry $\langle \kappa'_x, \tilde{\rho}^2 \rangle$ and does not drift at all in the x direction.

The second and third types of orbit created in case B+ are a period one orbit (a_0, a_0) and a period two orbit $(a_0, -a_0) \leftrightarrow (-a_0, a_0)$, with $0 = \mu + (P + Q)a_0^2$ in both cases. These orbits are mapped to each other by κ'_x or by κ'_y , so we consider only the orbit with period one. The translations at each iterate are

$$\begin{aligned} c_0, \quad c_1 = c_0 + \hat{h}(a_0, a_0; \mu), \quad c_2 = c_0 + \hat{h}(a_0, a_0; \mu) + i\hat{h}(a_0, a_0; \mu), \\ c_3 = c_0 + i\hat{h}(a_0, a_0; \mu), \quad c_4 = c_0, \quad \dots \end{aligned} \quad (55)$$

This orbit also has no net drift, and by choosing $c_0 = -\frac{1}{2}(1 + i)\hat{h}(a_0, a_0; \mu)$, we have $c_1 = ic_0$. The reconstructed solution $U(t)$, at leading order in a_0 , satisfies

$$\begin{aligned} U(0) = \tau_{c_0}(U_{0,0}(0) + 2a_0\zeta_-), \quad U(\tfrac{1}{4}T) = \tau_{ic_0}\rho^3(U_{0,0}(0) + 2a_0\zeta_-), \\ U(\tfrac{1}{2}T) = \tau_{-c_0}\rho^2(U_{0,0}(0) + 2a_0\zeta_-), \quad U(\tfrac{3}{4}T) = \tau_{-ic_0}\rho(U_{0,0}(0) + 2a_0\zeta_-), \end{aligned} \quad (56)$$

so $U(t) = \rho U(t + \frac{1}{4}T)$, and the isotropy subgroup is $\langle \tilde{\rho} \rangle$. This solution has the same symmetries as the alternating pulsating waves described in section 3, so alternating pulsating waves may be created in a symmetry-breaking bifurcation of alternating rolls. The period-two orbit $(a_0, -a_0) \leftrightarrow (-a_0, a_0)$ has the conjugate isotropy subgroup $\langle \kappa'_x \kappa'_y \tilde{\rho} \rangle$.

Finally, we turn to case B-. Here, there are two types of periodic orbit created in the bifurcation at $\mu = 0$, and in this case they are both of period four. The first orbit is $(a_0, 0) \rightarrow (0, -a_0) \rightarrow (-a_0, 0) \rightarrow (0, a_0)$, with $0 = -\mu + Pa_0^2$ in (52). The translations are

$$\begin{aligned} c_0, \quad c_1 = c_0 + \hat{h}_x(a_0, 0; \mu), \quad c_2 = c_0 + \hat{h}_x(a_0, 0; \mu) + \hat{h}_y(0, a_0; \mu), \\ c_3 = c_0 + 2\hat{h}_x(a_0, 0; \mu) + \hat{h}_y(0, a_0; \mu), \quad c_4 = c_0 + 2\hat{h}_x(a_0, 0; \mu) + 2\hat{h}_y(0, a_0; \mu), \quad \dots \end{aligned} \quad (57)$$

Note that $c_n - c_0$ is real so there is no drift at all in the y direction, but there is a systematic drift in the x direction. The reconstructed solution $U(t)$ satisfies

$$\begin{aligned} U(0) = \tau_{c_0}(U_{0,0}(0) - a_0\zeta_+ + a_0\zeta_-), \quad U(\tfrac{1}{4}T) = \tau_{c_1}\rho^3(U_{0,0}(0) - a_0\zeta_+ - a_0\zeta_-), \\ U(\tfrac{1}{2}T) = \tau_{c_2}\rho^2(U_{0,0}(0) + a_0\zeta_+ - a_0\zeta_-), \quad U(\tfrac{3}{4}T) = \tau_{c_3}\rho(U_{0,0}(0) + a_0\zeta_+ + a_0\zeta_-), \end{aligned} \quad (58)$$

so we have $U(t) = \kappa'_y U(t)$. A conjugate orbit, started a quarter period later, has isotropy subgroup $\langle \kappa'_x \rangle$ and drifts systematically in the y direction.

The second type of orbit created in case B– is $(a_0, a_0) \rightarrow (a_0, -a_0) \rightarrow (-a_0, -a_0) \rightarrow (-a_0, a_0)$, with $0 = -\mu + (P + Q)a_0^2$. The translations are

$$\begin{aligned} c_0, \quad c_1 &= c_0 + \hat{h}_x(a_0, a_0; \mu) + i\hat{h}_y(a_0, a_0; \mu), \\ c_2 &= c_0 + (1 + i)(\hat{h}_x(a_0, a_0; \mu) + \hat{h}_y(a_0, a_0; \mu)), \\ c_3 &= c_0 + (2 + i)\hat{h}_x(a_0, a_0; \mu) + (1 + 2i)\hat{h}_y(a_0, a_0; \mu), \\ c_4 &= c_0 + (2 + 2i)(\hat{h}_x(a, a; \mu) + \hat{h}_y(a_0, a_0; \mu)), \quad \dots \end{aligned} \tag{59}$$

This corresponds to a solution that drifts along the diagonal, with a wobble from side to side as it goes. At leading order in a_0 , the reconstructed solution $U(t)$ satisfies

$$\begin{aligned} U(0) &= \tau_{c_0}(U_{0,0}(0) + 2a_0\zeta_-), & U(\tfrac{1}{4}T) &= \tau_{c_1}\rho^3(U_{0,0}(0) - 2a_0\zeta_+), \\ U(\tfrac{1}{2}T) &= \tau_{c_2}\rho^2(U_{0,0}(0) - 2a_0\zeta_-), & U(\tfrac{3}{4}T) &= \tau_{c_3}\rho(U_{0,0}(0) + 2a_0\zeta_+), \end{aligned} \tag{60}$$

which has fully broken the spatial and spatio-temporal symmetries of the underlying alternating rolls solution.

We close our discussion of case B– by considering spatio-temporal symmetries of the drifting solutions that are obtained by moving to an appropriate travelling frame. These symmetries are not listed in Table 1. We first consider the solution (58) that drifts in the y direction; it has the following spatio-temporal symmetry

$$U(t) = \kappa'_x \rho^2 \tau_{c_0 - c_2} U(t + \tfrac{1}{2}T). \tag{61}$$

Next we consider the solution (60) that drifts along the diagonal; it has spatio-temporal symmetry

$$U(t) = \kappa'_y \rho \tau_{ic_0^* - c_1} U(t + \tfrac{1}{4}T). \tag{62}$$

In summary, we have examined the six different cases in which alternating rolls undergo a bifurcation with $FM = +1$ in the full period map. All six bifurcations preserve the underlying spatial periodicity of the alternating rolls, but may break the spatial and spatio-temporal symmetries. The 2-symmetry present in the B cases forces two Floquet multipliers to cross the unit circle together, and we find two branches of bifurcating solutions, with distinct symmetry properties. It is only in case B–, with Floquet multipliers $\pm i$ in the map $\tilde{\mathcal{G}}$, that the bifurcation leads to systematically drifting solutions: one solution drifts along a coordinate axis, while the other drifts along a diagonal.

We finish this section by presenting an example of a bifurcation from alternating rolls to a drifting pattern, which we interpret as an instance of a B– bifurcation. We have solved the PDEs for three-dimensional compressible magnetoconvection in a periodic $2 \times 2 \times 1$ box, using the code of Matthews *et al* (1995). The PDEs and description of the parameters and numerical method can be found in that paper. Figure 2 shows an example of an alternating roll at times approximately $0, \frac{1}{4}T, \frac{1}{2}T$ and $\frac{3}{4}T$; the two reflection symmetries κ'_x and κ'_y in planes $x = \frac{1}{4}\lambda$ and $y = \frac{1}{4}\lambda$ and the spatio-temporal symmetry of advancing a quarter period in time followed by a 90° rotation about the centre of the box are manifest. Increasing the controlling parameter, the temperature difference across the layer, leads to the solution in Figure 3: the data are shown at times $0, \frac{1}{4}T, \frac{1}{2}T, \frac{3}{4}T, T$ and $2T$. The only spatial symmetry remaining is the invariance under κ'_y , and the spatio-temporal symmetry has been broken. By comparing frames (a), (e) and (f) at times $0, T$ and $2T$, it can be seen that the solution is drifting slowly leftwards along the x -axis. Moreover, a drift symmetry $\kappa'_x \rho^2 \tau_{c_0 - c_2}$ conjugate to (61) can be seen by comparing frames (a) and (c). All evidence points to the bifurcation being of type B– (though we have not computed the Floquet multipliers and critical eigenfunctions in the PDEs).

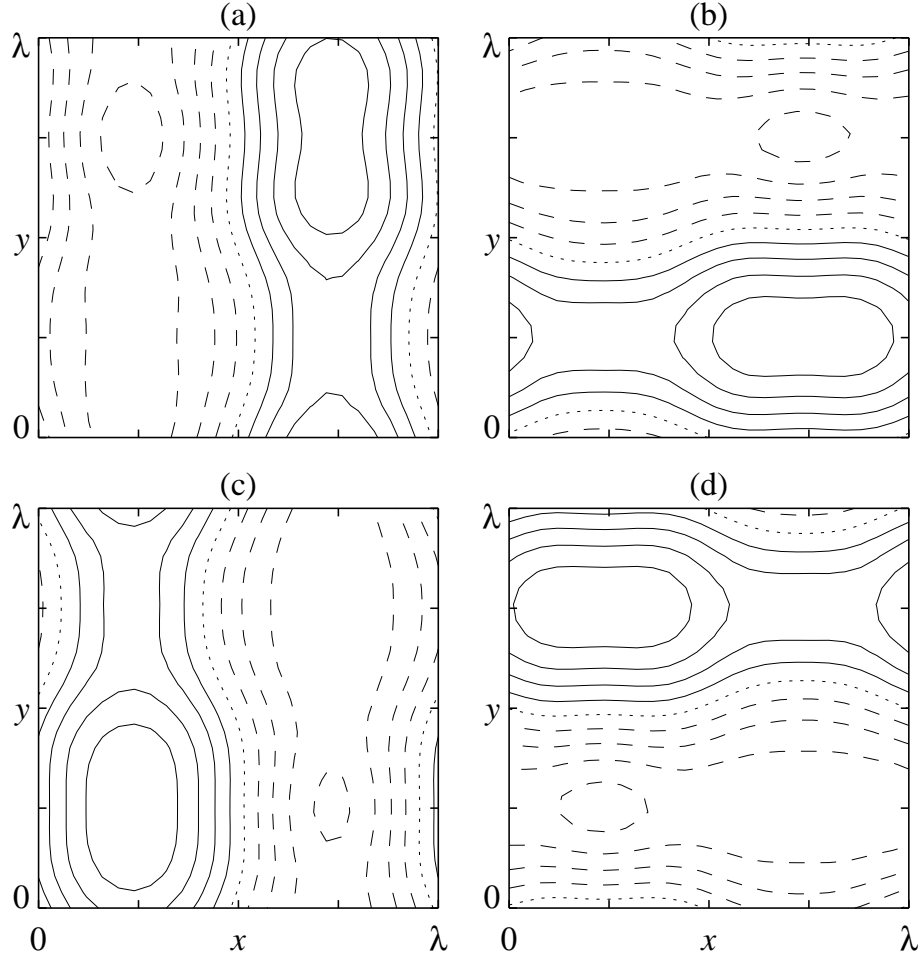


Figure 2. Alternating rolls in three-dimensional compressible magnetoconvection, starting with parameter values from Matthews *et al* (1995). The four frames are (approximately) at times (a) $t = 0$, (b) $t = \frac{1}{4}T$, (c) $t = \frac{1}{2}T$ and (d) $t = \frac{3}{4}T$. The frames show contours of the vertical velocity in a horizontal plane in the middle of the layer: solid lines denote fluid travelling upwards, dashed lines denote fluid travelling downwards, and the dotted line denotes zero vertical velocity. The spatial symmetries κ'_x and κ'_y are manifest, as is the spatio-temporal symmetry of advancing a quarter period in time followed by a 90° rotation (counterclockwise in this example). The dimensionless parameters are: the mid-layer Rayleigh number (proportional to the temperature difference across the layer) $R = 2324$; the Chandrasekhar number (proportional to the square of the imposed magnetic field) $Q = 1033$; the Prandtl number $\sigma = 0.1$; the mid-layer magnetic diffusivity ratio $\zeta = 0.1$; the adiabatic exponent $\gamma = 5/3$; the polytropic index $m = 1/4$; the thermal stratification $\theta = 6$; the mid-layer plasma beta $\beta = 32$; and the horizontal wavelengths $\lambda = 2$ in units of the layer depth).

5. Conclusion

We have developed a technique for investigating the possible instabilities from continuous group orbits of spatio-temporally symmetric time-periodic solutions of partial differential equations in periodic domains. Our approach is based on centre manifold reduction and symmetry arguments. It is in the spirit of earlier work by Iooss (1986) on bifurcations from continuous group orbits of spatially symmetric steady solutions of partial differential equations. We have treated three examples that arise in convection problems: pulsating waves in two dimensions, and alternating pulsating waves and alternating rolls in three dimensions.

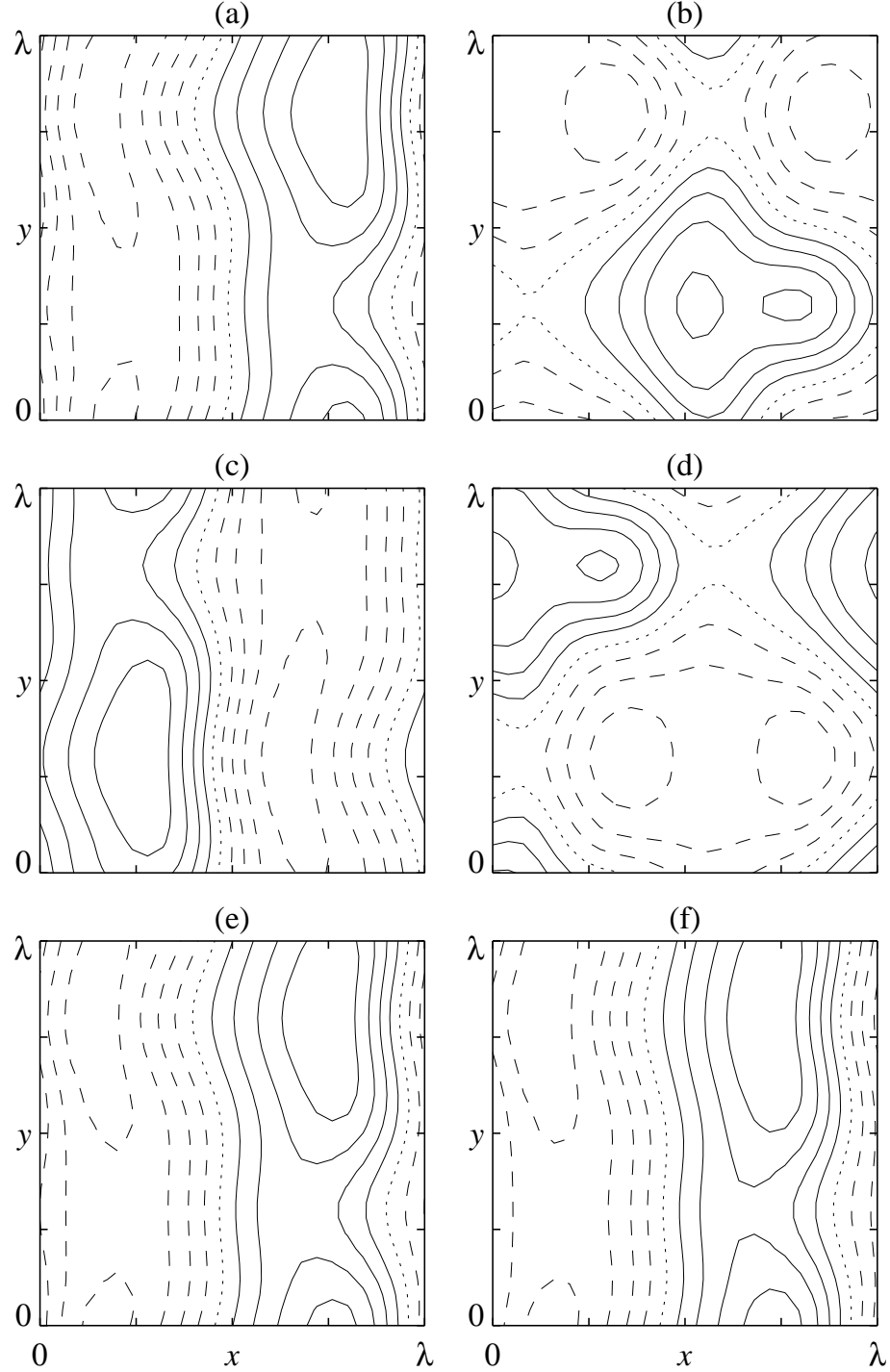


Figure 3. After a bifurcation of type B $^-$, the alternating rolls begin to drift. The parameter values are as in Figure 2, but with a higher thermal forcing: $R = 3000$ and $Q = 1333$. The frames are (approximately) at times (a) $t = 0$, (b) $t = \frac{1}{4}T$, (c) $t = \frac{1}{2}T$, (d) $t = \frac{3}{4}T$, (e) $t = T$ and (f) $t = 2T$. Note how all spatial and spatio-temporal symmetries have been broken, with the exception of κ'_y , a reflection in the plane $y = \frac{1}{4}\lambda$ (modulo a slight shift in the periodic box). The slow leftward drift of the pattern can be seen by comparing frames (a), (e) and (f). In addition, a drift symmetry $\kappa'_x \rho^2 \tau_{c_0 - c_2}$, conjugate to (61), can be seen by comparing frames (a) and (c) or (b) and (d).

A simple bifurcation can lead to drifting solutions in the case of pulsating waves but not alternating pulsating waves. The additional spatial symmetries of alternating rolls can force two Floquet multipliers to cross the unit circle together; this degeneracy can lead to drifting solutions, as in the numerical example presented in the previous section. We have related our work to the theory of k -symmetries developed by Lamb & Quispel (1994).

Our approach can readily be applied to other problems. In the future, we plan to tackle spatial period doubling and multiplying, where the τ_λ symmetries do not act trivially; such instabilities are relevant to simulations of convection carried out in larger boxes (Weiss *et al* 1996), and will be related to the study of the long-wavelength instabilities of alternating rolls (Hoyle 1994). We also plan to examine the case of the hexagonal lattice: a Hopf bifurcation on a hexagonal lattice leads to a wide variety of periodic orbits with different spatio-temporal symmetries (Roberts *et al* 1986). Finally, we plan to investigate the effect of including the extra Z_2 mid-layer reflection symmetry that arises when making the Boussinesq approximation for incompressible fluids.

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