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Time-varying model Identification for Time-Frequency Feature Extraction from EEG data

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Abstract A novel modelling scheme that can be used to estimate and track time-varying properties of nonstationary signals is investigated. This scheme is based on a class of time-varying AutoRegressive with an eXogenous input (ARX) models where the associated time-varying parameters are represented by multi-wavelet basis functions. The orthogonal least square (OLS) algorithm is then applied to refine the model parameter estimates of the time-varying ARX model. The main features of the multi-wavelet approach is that it enables smooth trends to be tracked but also to capture sharp changes in the time-varying process parameters. Simulation studies and applications to real EEG data show that the proposed algorithm can provide important transient information on the inherent dynamics of nonstationary processes.

Keywords: TVARX model, wavelet basis functions, model structure detection, recursive least squares (RLS), orthogonal least square (OLS), time-dependent spectra, EEG.

I. INTRODUCTION

Time-varying processes encountered in different engineering applications such as biomedical signal processing can be characterised by parametric representations [1]-[4]. Thus, the need to identify TV systems has naturally led to a growing interest in these areas. Parameter identification and modelling is now established based on the AutoRegressive with an eXogenous input (ARX) model. The ARX model, which can match the structure of many real-world processes, is one of the most widely applied linear dynamic models. The popularity and wide application of the ARX model comes mainly from its easy-to-compute parameters [5]-[7].

Many approaches have been proposed to identify time-varying Autoregressive with eXogenous input (TVARX)

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models. One of the most popular techniques to deal with TV modelling problems is to adopt an adaptive algorithm such as the Kalman filter and RLS approaches [8]-[10]. Another method for TV system identification is to expand the TV parameters onto a linear or nonlinear combination of a set of basis functions. Consequently, the unknown time-varying parameters are then converted into a set of constant coefficients of the basis functions. Hence, the initial TV identification and modelling problem is simplified and becomes a deterministic regression selection and parameter estimation problem.

The choice of basis functions can significantly affect the performance of the parameter estimates. Conventionally, the basis functions have been chosen to be Chebyshev and Legendre polynomials, prolate spheroidal sequences which are the best approximation to bandlimited functions [2], [4], [12]-[13] and wavelet basis that have a distinctive property of multi-resolution in both the time and frequency domains [3], [14]-[15]. Basis expansion methods have been widely applied to solve various engineering problems. For example, a TVAR model can be expanded over a Fourier–Bessel (FB) series to constitute a feature vector for segmentation of the EEG signal, and then to find a simple model for the parametric representation of EEG signals [16]. A good choice of the basis functions should allow abruptly or rapidly changing parameters to be tracked.

Wavelets have distinctive approximation properties and are well suited for approximating general nonstationary signals [2]-[3], [17]-[20], and thus have been successfully applied to many areas including nonlinear signal processing and parametric identification [21]-[25]. However, to our knowledge, not much work has been done to exploit the inherent approximation properties of wavelets to identify TV coefficient parameter estimation. The objective of this study is to present a novel TVARX modelling approach, where the time-dependent coefficients are expanded using a finite set of multi-wavelet basis functions. Based on a multi-wavelet expansion scheme, a new method for time-dependent parameter estimation is then proposed. The term 'multi-wavelet' here has a twofold meaning. Firstly, the TV coefficients of the ARX model are approximated using several types of wavelet basis functions (i.e. the TV parameter estimation involves multiple wavelets). Secondly, these wavelet basis functions are combined in a form of multi-resolution wavelet decomposition. The advantage of the proposed method, compared with a method involving only a single type of wavelets, is that the multi-wavelet expansion scheme is much more flexible in that it exploits the excellent properties of both non-smooth and smooth wavelet basis functions and thus can effectively track both rapid and slow variations of TV coefficients. In addition, the expansion of TV parameters onto multi-wavelet basis functions is more accurate and effective for dealing with nonstationary signal modelling than traditional power spectral estimation approaches and classical time-invariant parameter models.

II. PROBLEM FORMULATION

A. Time-Varying ARX Model and Multi-wavelet Coefficient Expansions

The TVARX(p, q) model for a single-input/output system can be represented as

$$y(t)=\sum_{i=1}^{p} a_i(t)\,y(t-i)+\sum_{n=1}^{q} b_n(t)\,u(t-n)+e(t)\,, \tag{1}$$

where t is the time instant or sampling index of the signal $y(t)$, $y(t-i)$ and $u(t-n)$ are the measured response, respectively. $a_i(t)$ and $b_n(t)$ are the TV coefficient functions to be determined in the model; the term $e(t)$ is the residual error accommodating the effects of measurement noise, and modelling noise that can be viewed as a stationary white noise sequence with zero mean and variance σ_e^2. The proposed method is to expand the TV parameters $a_i(t)$ and $b_n(t)$ onto multi-wavelet families cardinal B-splines basis functions, $\left\{\phi_k^{(m)}:\ m=3,4,5;\ k\in\Gamma_m\right\}$, such that the following expression hold:

$$\begin{aligned} a_i(t)&=\sum_{k\in\Gamma_\eta}\alpha_{i,k}^{(\eta)}\phi_k^{(\eta)}\left(\frac{t}{N}\right)+\sum_{k\in\Gamma_r}\alpha_{i,k}^{(r)}\phi_k^{(r)}\left(\frac{t}{N}\right)+\sum_{k\in\Gamma_s}\alpha_{i,k}^{(s)}\phi_k^{(s)}\left(\frac{t}{N}\right)\\ b_n(t)&=\sum_{k\in\Gamma_\eta}\beta_{n,k}^{(\eta)}\phi_k^{(\eta)}\left(\frac{t}{N}\right)+\sum_{k\in\Gamma_r}\beta_{n,k}^{(r)}\phi_k^{(r)}\left(\frac{t}{N}\right)+\sum_{k\in\Gamma_s}\beta_{n,k}^{(s)}\phi_k^{(s)}\left(\frac{t}{N}\right) \end{aligned} \tag{2}$$

where $\alpha_{i,k}$ and $\beta_{n,k}$ represent the expansion parameters, $\Gamma_m=\left\{k:-m\le k\le 2^j-1\right\}$ for $m=3,4,5$, $j=3$ is the wavelet scale, $\eta=3$, $r=4$, and $s=5$, $t=1,2,\cdots,N$, and N is the number of observations of the measurement data, respectively. Substituting (2) into (1), it yields,

$$y(t)=\sum_{i=1}^{p}\left\{\sum_{k\in\Gamma_m}\alpha_{i,k}^{(m)}\left(\phi_k^{(m)}\left(\frac{t}{N}\right)y(t-i)\right)\right\}+\sum_{n=1}^{q}\left\{\sum_{k\in\Gamma_m}\beta_{n,k}^{(m)}\left(\phi_k^{(m)}\left(\frac{t}{N}\right)u(t-n)\right)\right\}, \tag{3}$$

From (3), the original TVARX model in Eq. (1) has now been converted into a time invariant (LTI) regression model with respect to the time invariant coefficients $\alpha_{i,k}^{(m)}$ and $\beta_{n,k}^{(m)}$. In this study, cardinal B-splines wavelets, which have been proved to have a lot of excellent properties, are considered and will be employed for time-varying parameter expansion. Detailed discussions about how to build the associated multi-wavelet model using B-splines can be found in [2] and [26].

B. Time-Dependent Spectrum Estimation

Equation (3) can be solved by using linear least squares algorithms. Let $\hat{a}_i(t)$, $\hat{b}_n(t)$ be the estimates of $a_i(t)$ and $b_n(t)$, and $\hat{\sigma}_e^2$ is the estimate of σ_e^2. The time-dependent spectral function associated to the TVARX model in Eq. (1) is defined as,

$$H(f,t)=\left|\frac{\sum_{n=1}^{q}\hat{b}_n(t)e^{-j2\pi nf/f_s}}{1-\sum_{i=1}^{p}\hat{a}_i(t)e^{-j2\pi if/f_s}}\right|^2, \tag{4}$$

where $j=\sqrt{-1}$, and f_s is the sampling frequency. Note that the spectral function (4) is continuous with respect to the frequency f and thus can be used to produce spectral estimates at any desired frequency up to the Nyquist frequency $f_s/2$. The frequency resolution is not infinite, but is determined by the underlying model order and the associated parameters.

C. Model Identification and Parameter Estimation

In general, the estimation of LTI system parameters is formulated as an overdetermined problem. Then the least squares solution is the optimal estimate of the parameters in the sense of minimum residual error. However, if the parameters are time-varying, the problem of parameter estimation becomes underdetermined, and it is much more difficult to find the 'best' solution. Expanding the TV parameters onto a linear combination of a set of basis functions can solve the underdetermined problem. Consequently, the parameter estimation of unknown variables can be reduced to a set of constant coefficients of the basis functions. However, the multi-wavelet expansion model (3) involves a large number of candidate model terms that may be highly correlated. The resultant parameter estimates may be over-fitted. Experience suggests that most of the candidate model terms can be removed from the model, and that only a small number of significant model terms are needed to provide a satisfactory representation for most linear and nonlinear dynamical systems. Many approaches have been introduced to eliminate the possible linear dependency of candidate model terms by selecting best bases, for example, Kaipio and Karjalainen [27] introduced a principal-component-analysis (PCA)-type approximation scheme to select the 'optimal basis'. The mutual correlation of the coefficients is also taken into account in their approach.

In this work, TV coefficients are expanded by multi-resolution cardinal B-splines wavelet series, and then the forward orthogonal least squares (OLS) algorithm [28]-[31], which have been proven to be a very effective to deal with multiple dynamical regressions problems, is applied to determine the forms in model (3). Detailed discussions of the procedure of the forward OLS can be found in [23], [28]-[31]. The TV coefficients $a_i(t)$ and $b_n(t)$ in Eq. (1) can be recovered by the resultant estimates from model (3).

III. SIMULATION EXAMPLE

Consider a TVARX (2, 2) model below

$$y(t)=a_1(t)y(t-1)+a_2(t)y(t-2)+b_1(t)u(t-1)+b_2(t)u(t-2)+e(t) \tag{5}$$

where $e(t)$ is zero-mean Gaussian white noise. The TV parameters in Eq. (5) are given by:

$$a_1(t)=\begin{cases}0.32\cos\left(1.5-\cos\left(4\pi t/N+\pi\right)\right), & 1\le t\le N/4\\ 0.32\cos\left(3-\cos\left(4\pi t/N+\pi/2\right)\right), & N/4+1\le t\le 3N/4\\ 0.32\cos\left(1.5-\cos\left(4\pi t/N+\pi\right)\right), & 3N/4+1\le t\le N\end{cases},$$

$$a_2(t)=0.4\cos\left(4\pi t/N\right), 1\le t\le N\ ,$$

$$b_1(t)=\begin{cases}0.65, & 1\le t\le N/4\\ -0.5, & N/4+1\le t\le N/2\\ 0.65, & N/2+1\le t\le 3N/4\\ -0.5, & 3N/4+1\le t\le N\end{cases},\quad b_2(t)=0.6,\ 1\le t\le N, \tag{6}$$

where the length of data N is 512. Model (5) was simulated by setting the input $u(t)$ as a Pesudo-Random Binary Sequence (PRBS) [32]. The variance of the noise $e(t)$ was chosen to be 0.04, and this made the signal-to-noise ratio to be around 13 dB. Both the input and the associated output sequences were recorded and were used for subsequent model estimation.

Figure 1 compares three different methods, that is, the RLS algorithm, the RLS algorithm with B-spline basis functions, and the OLS algorithm with B-spline basis functions. Panel (a) shows the results using the RLS estimation algorithm (forgetting factor (ff) 0.92) [8]. Panel (b) gives the results of the RLS (ff: 0.998, using B-spline wavelets and selecting scale index $j=3$) algorithm and Panel (c) shows the OLS identification results (using B-spline wavelets and selecting scale index $j=3$). Obviously, The RLS approach attains smooth but relatively poor estimates that cannot track the rapidly changing TV parameters, the parameter estimates are underdetermined. The RLS approach with B-spline obtains irregular estimates with large variances (over-fitted), however, compared with RLS method, the resultant estimates from the RLS method with B-spline can track the sharp changes of the TV parameters. These interesting results have been verified by Li et al., [26]. The OLS method with B-spline appears to outperform the RLS approach and the RLS approach with B-spline. The results using the OLS approach with B-spline is impressive because it is able to track three quite different waveforms: the constant value, an abrupt change, and the sinusoidal waveform. The proposed method (the OLS with B-spline) can attain smooth estimates while providing rapid tracking. The mean absolute error (MAE), normalized root mean squared error (RMSE) and the standard deviations (Std) of the parameter estimates (with respect to the true parameters) are estimated and shown in Table 1.

Table 1 A comparison of the model performance for TVARX (2, 2) model with SNR 13 dB.

Approach	Estimated coefficient	MAE	RMSE	Std
RLS $(\mu = 0.92)$	$\hat{a}_1(t)$	0.0917	2.3104	0.1199
	$\hat{a}_2(t)$	0.1030	1.8667	0.1292
	$\hat{b}_1(t)$	0.1080	1.2315	0.2172
	$\hat{b}_2(t)$	0.0627	0.7260	0.0863
RLS with B-spline $(\mu = 0.9998)$	$\hat{a}_1(t)$	0.2045	3.0639	0.2623
	$\hat{a}_2(t)$	0.2047	1.9951	0.2746
	$\hat{b}_1(t)$	0.1411	1.3973	0.2084
	$\hat{b}_2(t)$	0.1803	1.0434	0.2900
OLS with B-spline	$\hat{a}_1(t)$	0.0893	2.1750	0.1112
	$\hat{a}_2(t)$	0.0614	1.3520	0.0865
	$\hat{b}_1(t)$	0.0642	0.8638	0.1133
	$\hat{b}_2(t)$	0.0246	0.2250	0.0305

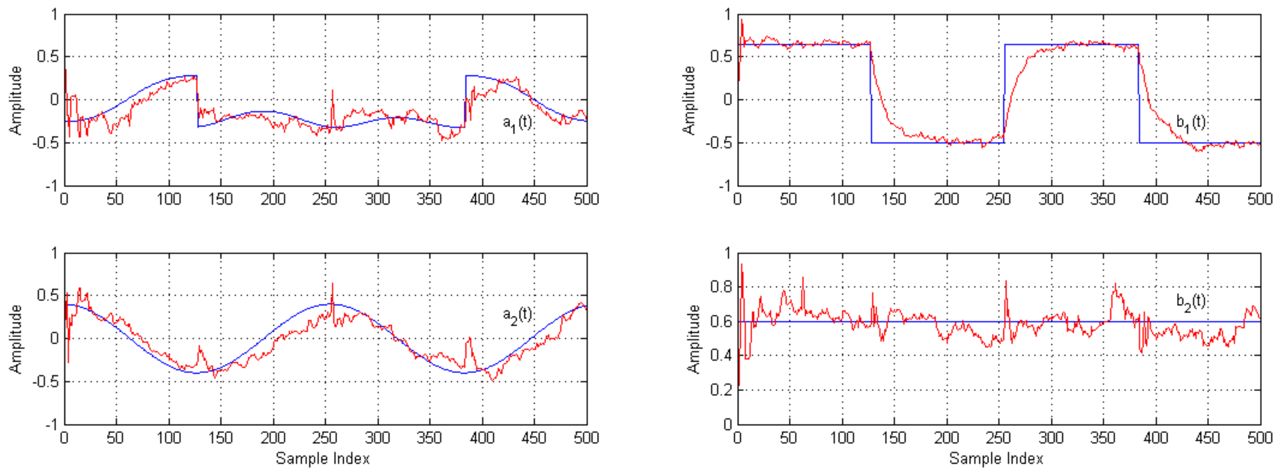
where μ represents the forgetting factor.

Compared with the RLS approach and the RLS approach with B-spline estimates, Table 1 statistically confirms that the MAE, RMSE and Std estimates produced by the OLS approach with B-spline yield smallest. The MAE and RMSE are both defined by

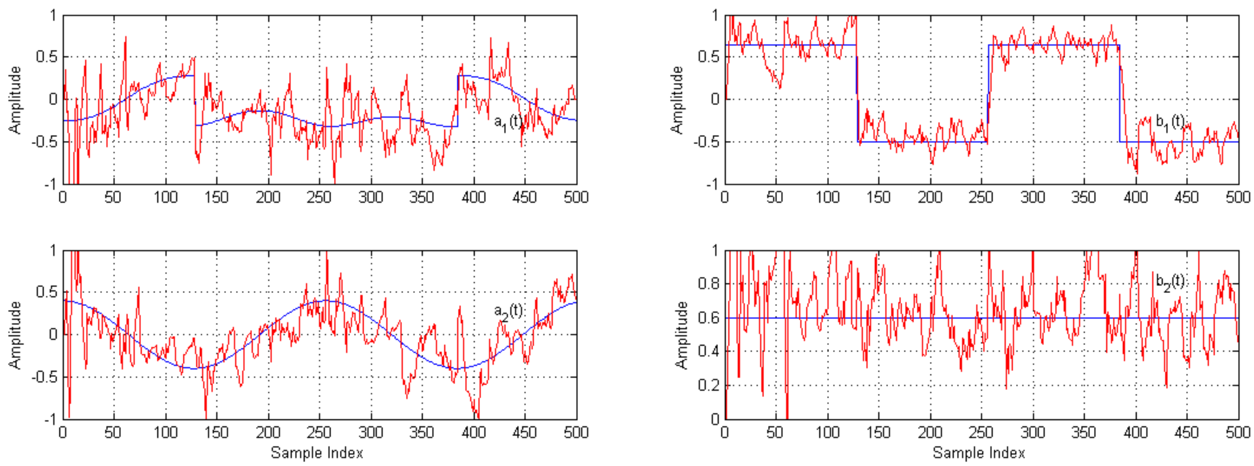
$$MAE = \frac{1}{N}\sum_{t=1}^{N}\left|\hat{a}(t) - a(t)\right|, \tag{7}$$

$$RMSE = \sqrt{\frac{1}{N}\sum_{t=1}^{N}\frac{\left\|\hat{a}(t) - a(t)\right\|^2}{\left\|a(t)\right\|^2}}, \tag{8}$$

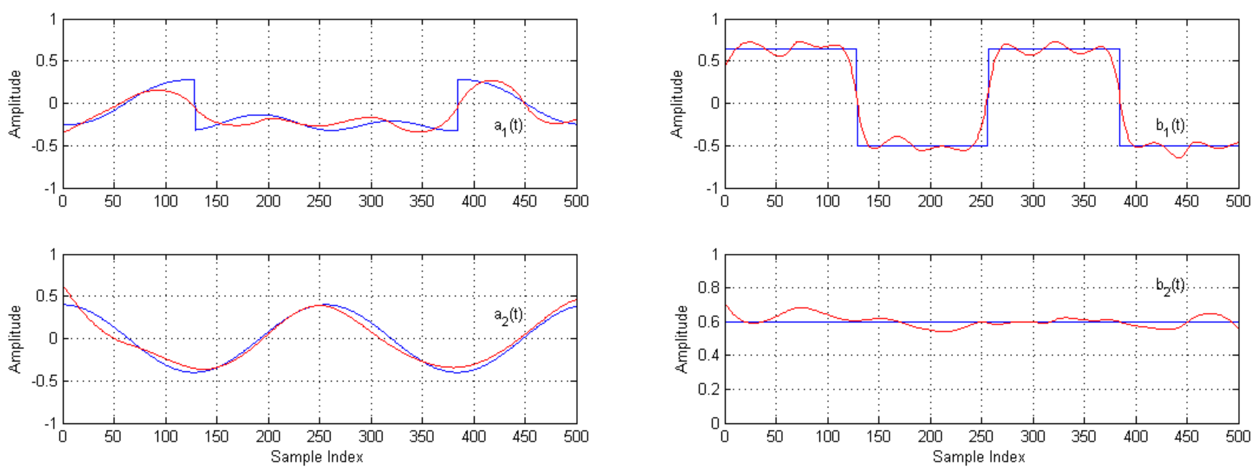
where $\hat{a}(t)$ represents the estimates of coefficients $a(t)$ in the TVARX model (3), and N is the length of the data set.



(a) RLS method



(b) RLS method with B-splines



(c) OLS method with B-splines

Fig. 1. One implementation of the TVARX (2, 2) system identification results with a SNR of 13 dB using the different approaches. Blue curve represents the true value of the TV parameters; red curve indicates the estimation value of the TV parameters. (a) the RLS method; (b) the RLS method with B-spline; (c) the OLS method with B-spline.

IV. APPLICATION—EEG DATA MODELLING and ANALYSIS

The proposed TVARX modelling scheme has been applied to analyse dynamic relationships from EEG recordings to illustrate the application of the proposed multi-wavelet basis function method based on TV parametric modelling. Scalp EEG signals are synchronous discharges from cerebral neurons detected by electrodes attached to the scalp. The EEG signals discussed here were recorded with the same 32-channel amplifier system. An XLTEK 32 channel headbox (Excel-Tech Ltd) with the international 10-20 electrode placement system was used in the Sheffield Teaching Hospitals NHS Foundation Trust, Royal Hallamshire Hospital, UK. The sampling frequency of the device was 500 Hz. Andrzejak et al. [33] has discussed in detail dynamical properties of brain electrical activity from different extracranial and intracranial recording regions and from different physiological and pathological brain states. The central objective of this paper for the EEG signals is to propose an empirical and data-based modelling framework from model identification that can produce an accurate but simple description of the dynamical relationships between different recording regions during brain activity. This is a complicated black box system where the true model structure is unknown, and thus, needs to be identified from available experimental data. As an example, the symmetrical two channels (F3, located over the left superior frontal area of the brain, and F4 located over the same area on the right) of EEG recorded from a patient with an absence of seizure epileptic discharge was investigated. Channel F3 was treated as the input, denoted by $u(t)$, and Channel F4 was treated as the output, denoted by $y(t)$, note that Channel F3 is the signal input and Channel F4 is the signal output, the main reason is that the phase of Channel F4 is related to the phase of Channel F3, and other criteria including the change in the ERR distribution given in [28] can also verify the input-output relationship between Channel F3 and Channel F4. The objective is to learn, from the available Channel F3 and Channel F4 recordings, if an identified TVARX model is suitable to describe the dynamical characteristics from the time-dependent spectrum analysis approach. The input-output EEG signals of 3500 data points pairs of representing one seizure, with a sampling rate of 500 Hz, recorded during 7 seconds, were analysed.

Similar to the simulation example given in section III, the third, fourth and fifth order B-splines were adopted to establish TVARX models for the EEG recordings. Several TVARX models with different model orders were estimated using the OLS approach with B-spline, the classical generalized cross-validation (GCV) criteria [31] suggested that the model order can be chosen to be $p=4$ and $q=3$ when using the B-splines as building blocks to represent the time-varying coefficients in the TVARX model.

The time-varying coefficients estimated $a_i(t)$ with $i = 1, 2, \cdots, 4$ and $b_n(t)$ with $n = 1, 2, 3$ are depicted in Figure 3. Figure 4 shows the recovered signal, recovered by the TVARX model from the estimated time-varying coefficients $a_i(t)$ and $b_n(t)$. The topographical diagram of the time-dependent spectrum estimated from the TVARX (4, 3) model is shown in Figure 5, and the 2-D image diagram and the contour plot of the time-dependent spectrum produced from the 3-D topographical diagram are given in Figure 6.

From Figure 5 and Figure 6, the distribution scale of the power spectrum of the EEG signal considered here is mainly from zero to around 18 Hz. Two frequency bands can clearly be observed as: (a) the low frequency band (about the 3 Hz, namely, a spike at 3-Hz); (b) around 18 Hz represents the high frequency band component. The contour plot of the time-dependent spectrum given in Figure 6(b) clearly reflects the distribution of these frequency components along with the time course. It is clear that the variations of the time course signals can be observed from the contour diagram of the transient spectrum. For instance, the power spectrum is mainly distributed by a 3-Hz spike frequency component during the period from 5 to 6s, while the high frequency (around 18 Hz) activity is dominated by the time course from 0.2 to 0.3s. Any time-invariant parametric modelling framework such as the commonly applied ARX models cannot attain these properties which are only possessed by the TVARX model proposed.

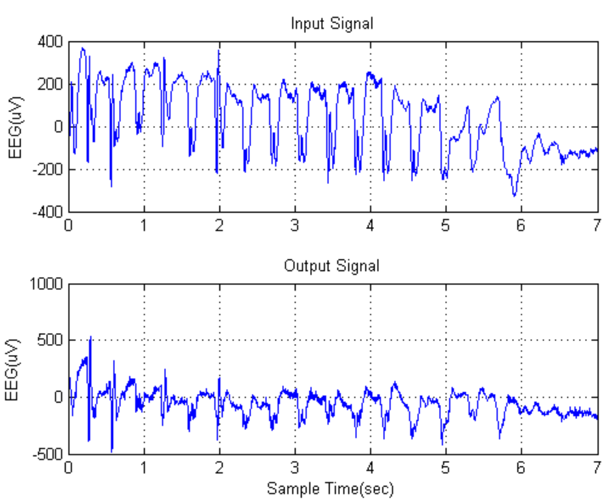


Fig. 2. The EEG recordings (F3 Channel: Input signal, F4 Channel: Output signal), for a seizure activity of a patient, recorded over 7 seconds, with a sampling rate of 500 Hz.

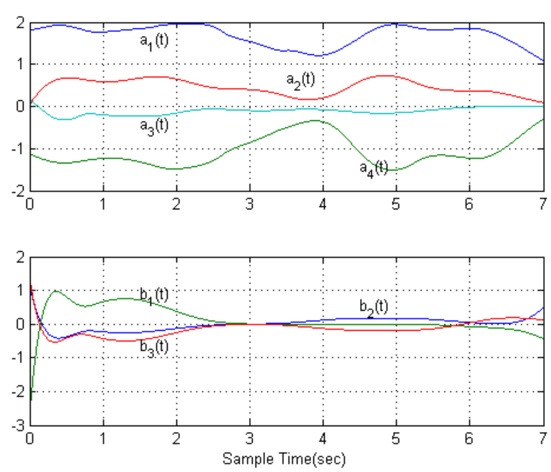


Fig. 3. Estimates of the time-varying coefficients $a_i(t)$ for $i = 1, 2, 3, 4$ and $b_n(t)$ for $n = 1, 2, 3$ for the EEG signal.

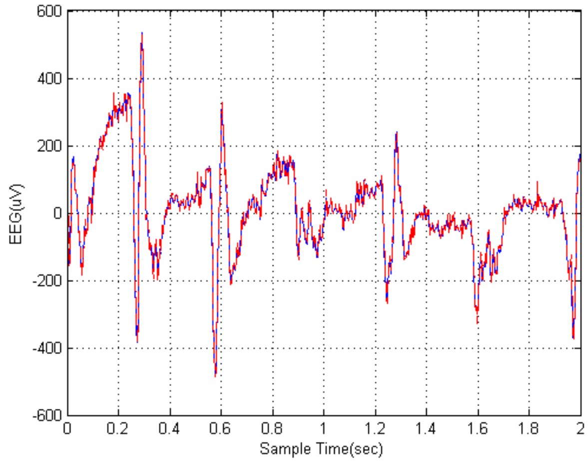


Fig. 4. A comparison between the recovered signal from the identified TVARX (4, 3) model and the original observations for the EEG signal. Solid (blue) line represents the observations and the dashed (red) line represents the signal recovered from the TVARX (4, 3) model. For a clear visualization only the data points of the period from 0 to 2 seconds are shown.

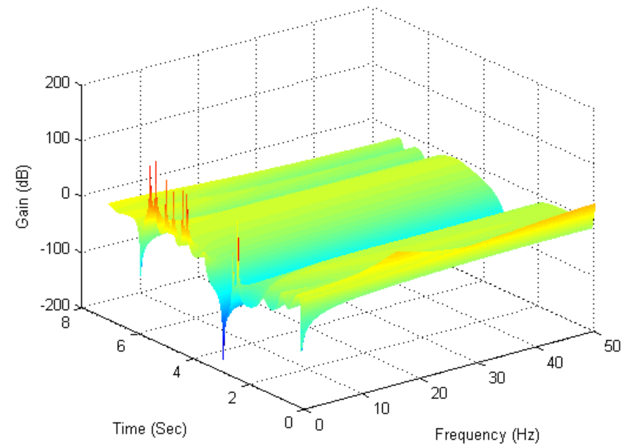
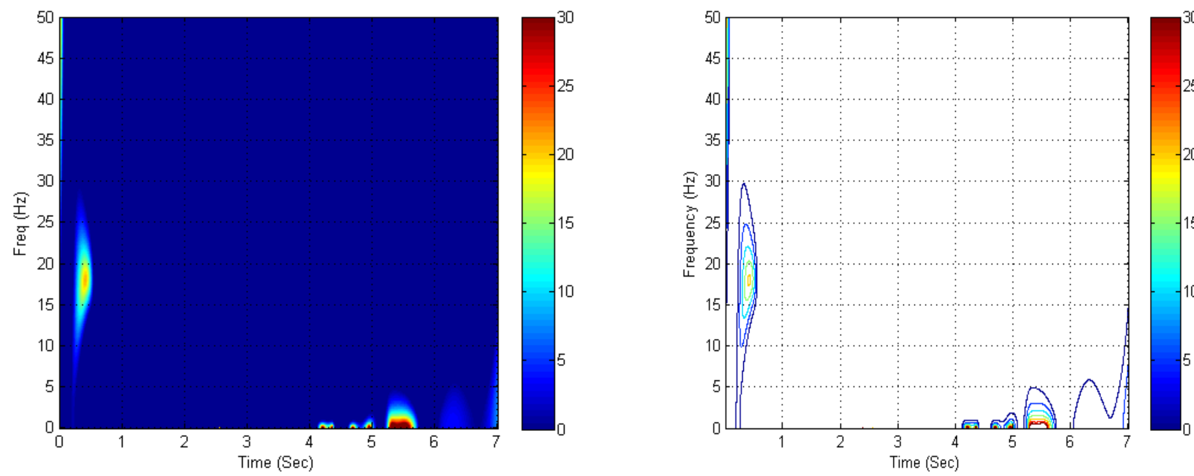


Fig. 5. The 3-D topographical map of the time-dependent spectrum estimated from the TVARX (4, 3) model for the EEG signal.



(a) (b)

Fig. 6. The 2-D image and the contour diagram of the time-dependent spectrum produced by the 3-D topographical map. (a) the 2-D image; (b) the contour diagram.

V. CONCLUSIONS

A novel TV parametric modelling approach has been presented based on time-dependent coefficients approximated multi-wavelet basis functions to account for the transient spectrum information. The TVARX model in this study is completely different from existing TV parametric models where the associated time-dependent coefficients are expanded by multi-wavelet basis functions. In most existing TV parametric models, the time-dependent coefficients represented by the basis functions are global, whereas in the new proposed modelling method, the basis functions involved are locally defined. Wavelets have been proved to show excellent approximation properties [18], therefore, the TV models established by the multi-wavelet basis function expansion scheme can be much more adaptable and flexible for tracking the sharp variations of nonstationary biomedical signals such as EEG recordings.

The time-dependent spectrum based on TVARX model, with multi-wavelet basis functions, can reflect the global frequency behaviour of the signal and to reveal the local variations of the signal along the time course. One advantage of the proposed model, compared with traditional time-invariant models, is that it can capture much more transient information of the inherent nonstationary dynamics of the associated processes.

A further study in this direction is that the authors are currently extending this novel multi-wavelet approach to extract more features of EEG signals based on TVARX and a nonlinear ARX modelling method, so that these results can be applied for EEG signal diagnostic tasks, classification, and phase synchronization. Accurate detection and analysis of the time-dependent spectrum for various types of seizure is a complicated problem requiring analysis of a large set of EEG recordings, this will be the subject of a future report.

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