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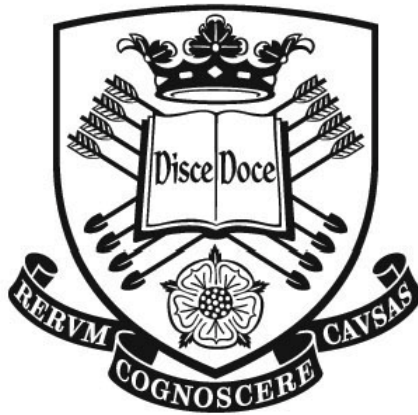
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# Model Term Selection for Spatio-temporal System Identification using Mutual Information

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# Model Term Selection for Spatio-temporal System Identification using Mutual Information

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## Abstract

A new mutual information based algorithm is introduced for term selection in spatio-temporal models. A generalised cross validation procedure is also introduced for model length determination and examples based on cellular automata, coupled map lattice and partial differential equations are described.

## 1 Introduction

Spatio-temporal systems represent a class of complex dynamic systems, which contain both time and space information. The study of spatio-temporal systems may help to decipher many spatio-temporal phenomena and behaviours that appear in nature and to better understand and possibly control the formation of spatio-temporal patterns[20][4][21].

One of the key concerns in the analysis of spatio-temporal systems is system identification, the reverse problem of pattern formation, which is still an open problem. One of main tasks in spatio-temporal system identification is model structure selection which enables construction of a mathematical model from experimental data. The Orthogonal Forward Regression (OFR) algorithm is one of the effective methods for the identification for spatio-temporal systems. Given a large number of candidate model terms in an initial model, this algorithm can be used to determine which terms or regressors are significant and should be included in the model based on the Error Reduction Ratio (ERR)[7][9]. However, when applied to some spatio-temporal data sets the OFR algorithm can occasionally select some spurious model terms, which can then result in a comparatively more complex model with some possible insignificant or redundant model terms.

In this paper, a new method, called the OFR-MI (Orthogonal Forward Regression using Mutual Information) algorithm is introduced for spatio-temporal system identification. Using mutual information as the criterion for detecting important terms, the OFR-MI algorithm can effectively avoid the high ERR value problems which seems to occur for some spatio-temporal systems. The new algorithm is tested on several benchmark spatio-temporal models including Cellular Automata (CA), Coupled Map Lattice (CML) and Partial Differential Equation (PDE) models.

## 2 Spatio-temporal model description

This study considers three main types of spatio-temporal models: Cellular Automata or simply CA, Coupled Map Lattices (CML), and Partial Differential Equations (PDE). CA are systems that have finite values at each cell site and the rules are usually represented by a combination of different Boolean rules. The class of systems that have continuous state cell values at each site can be described by Partial Differential Equations (PDE), or when based on a discrete lattice space as Lattice Dynamical System (LSD) or Coupled Map Lattices (CML). However, LSD, CML and CA models have the property of discrete space, but PDE models not only have a continuous state, but also continuous time and space.

### 2.1 Cellular Automata

Cellular Automata (CA) were initially introduced by von Neumann in the early 1950's. CA are dynamical systems in which space and time are both discrete. Each cell which is arranged in the form of a regular lattice structure has a finite number of states. All the states in the cells are updated synchronously by a specific transition rule based on the information of the individual states and of cells in a neighbourhood at past times.

An  $n$ -dimensional cellular automata is defined on a lattice structure. The typical and widely used lattice type is a square lattice, which is represented as  $S^d$ , where  $d = 2r + 1$ , and  $r$  is a finite integer which determines the size of the neighbourhood.  $S$  is a finite set of states of all cells in the lattice. The dynamics are described by a neighbourhood function  $f : S^d \rightarrow S$ . Thus, the output of each cell is produced by following the rule  $f$ . The transition function  $f$  shows the interaction of cells which can be listed in a finite look-up table. A shift operator then upgrades the cells as time passes.

Consider a one-dimensional 3-site CA model, the spatio-temporal patterns generated by 100 time steps evolution are shown in Figure 1. In the simulation, 100 random data valued 1 or 0 are set as initialization. The neighbourhood was set as  $\{c(j-1;t), c(j;t), c(j+1;t)\}$ , where  $c(j, t)$  indicates the cell state at the position  $j$  and the time instant  $t$ . The transition rule described by the Boolean equivalent is

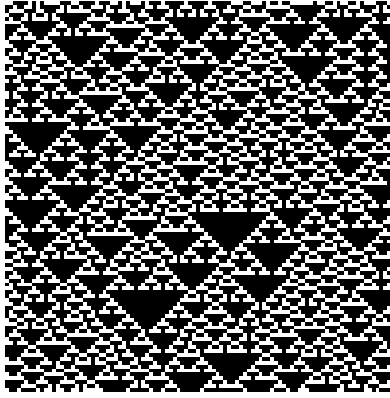
$$c(j; t + 1) = c(j; t) \vee (c(j - 1;t) \wedge c(j + 1;t)) \quad (1)$$

where ' $\vee$ ' denotes the *OR* operation and ' $\wedge$ ' denotes the *AND* operation. It has been shown in [6] that CA rules can be expressed in a polynomial form for the model in Equation (1), gives

$$\begin{aligned} c(j,t + 1) = & -2.0c(j - 1;t)c(j;t) - 2.0c(j;t)c(j + 1;t) + c(j;t) \\ & -2.0c(j - 1;t)c(j + 1;t) + c(j - 1;t) \\ & +3.0c(j - 1;t)c(j;t)c(j + 1;t) + c(j + 1;t) \end{aligned} \quad (2)$$

### 2.2 Coupled Map Lattice

CML models were introduced in a model simulation by K. Kaneko in the 1980s [13]. The model used by Kaneko consisted of a continuous sequence of logistic maps coupled to their neighbours and with parameters chosen to produce a chaotic behaviour. The system of coupled mappings was regarded as a CML model.



**Figure 1:** The CA model simulation for equation (2)

The CML model is a typical model of extended dynamical systems with discrete time and space, but with continuous state variables. The CML model lies somewhere between CA and PDE models. CML models are convenient for computer simulations especially of physical systems, for the interpretation of experimental results, and for mathematical analysis.

There are four key parts in CML models, that is, a lattice structure, the neighbourhood, the lattice states and a dynamical process. The process of a CML model can be defined by the following parts [12]:

- *A lattice architecture.*

Let  $x$  be a cell in a lattice  $X$ , so that  $x \in X$ . The neighbourhood can be described as

$$Nhd(x) = \{x, y_x^{p(x)}\} \quad (3)$$

where  $y$  represents the neighbouring cells corresponding to the cell  $x$ .  $p(x)$  represents the selection of neighbourhood cells and also specifies the size of the neighbourhood.

- *Lattice state description.*

The mapping for the state of lattice is  $\eta : X \rightarrow A$ , where  $A$  describes the states in the lattice. Thus, the state of a cell  $x$  in a lattice can be described as  $\eta(x)$ .

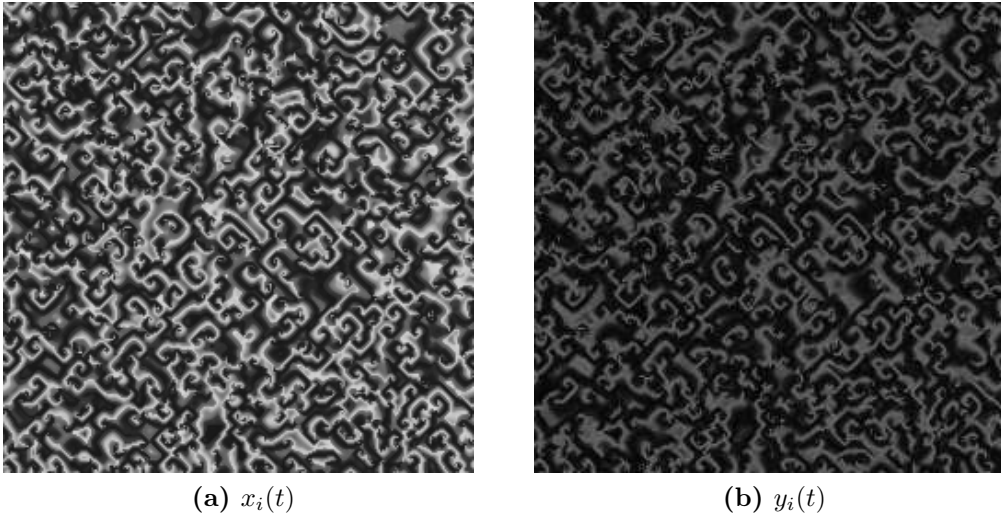
- *Dynamic process.*

There are two basic processes involved: isolated local processes and interaction processes. For an isolated local mapping  $f_x : A \rightarrow M$ ,  $M$  is the set of all possible values of cells in a lattice, and  $f_x(a)$  shows the output value at point  $x$  when the input value is  $a$ . Unlike the isolated process, the interaction process couples the states generated from the cells in the neighbourhood. This is represented by  $g_x : M^{p(x)} \rightarrow A$ . A global description of such a dynamic process was expressed by Holden [12],  $V_x : T \times [X \rightarrow A] \rightarrow A$ , where  $T$  is a time delay matrix.  $V_x(0, \eta) = \eta(x)$  when  $t = 0$ , so for  $t > 0$ ,

$$V_x(t + 1, \eta) = g_x(f_x(V_x(0, \eta)), f_{y_{x,1}}(V_{y_{x,1}}(t, \eta)), \dots, f_{y_{x,p(x)-1}}(V_{y_{x,p(x)-1}}(t, \eta))) \quad (4)$$

Consider an example of the CML model proposed by Sole and Valls [23]:

$$x_i(t) = \mu x_i(t - 1)[1 - x_i(t - 1)] \exp[-\beta y_i(t - 1)] + D_1 \nabla^2 x_i(t - 1) \quad (5)$$



**Figure 2:** Original CML model simulated patterns for Eqns (5) and (6).

$$y_i(t) = x_i(t-1)\{1 - \exp[-\beta y_i(t-1)]\} + D_2 \nabla^2 y_i(t-1) \quad (6)$$

where  $i = (i_1, i_2) \in \mathbb{Z}^2$ , describes the cell location in the lattice.  $\nabla^2$  is the Laplace operator and can be given by

$$\nabla^2 x_{i_1, i_2}(t-1) = x_{i_1-1, i_2}(t-1) + x_{i_1+1, i_2}(t-1) + x_{i_1, i_2-1}(t-1) + x_{i_1, i_2+1}(t-1) - 4x_{i_1, i_2}(t-1) \quad (7)$$

This system with  $\mu = 4$ ,  $\beta = 5$ ,  $D_1 = 0.001$ ,  $D_2 = 0.2$  was simulated on a lattice of  $256 \times 256$  with 50 random initial seeds of values between 0.3 to 0.4 [23] and periodic boundary conditions. The simulation results for 2000 time steps are show in Figure 2.

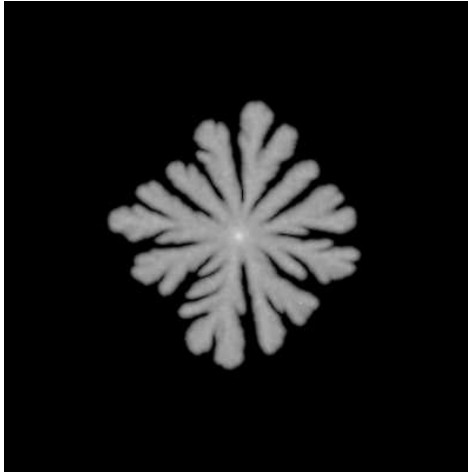
### 2.3 Partial Differential Equations Models

PDE models are continuous in both the time and space domains, and also continuous in the state at each point. With these continuous properties, PDE models provide an effective tool to understand and reconstruct continuous spatio-temporal systems in the real world. It is because these models may be related to previously derived analytical PDE models so that there is the potential to provide a clear physical explanation of the underlying system properties. PDE models demonstrate a relation between an unknown function with several independent variables and the associated partial derivatives. The general form of a PDE model based on a function  $u(x_1, x_2, \dots, x_n)$  is

$$F(x_1, x_2, \dots, x_n, u, u_{x_1}, u_{x_2}, \dots, u_{x_{11}}, \dots) = 0 \quad (8)$$

where  $x_1, x_2, \dots, x_n$  are the independent variables,  $u$  is the unknown function, and  $u_{x_i}$  represent the partial derivatives  $\frac{\partial u}{\partial x_i}$ . Generally, additional conditions such as initial conditions and boundary conditions are included. One example of a PDE model is the model of the streaming movement in a colonization period of bacterial cells [14]

$$\frac{\partial n}{\partial t} = \nabla^2 n - bn \quad (9)$$



**Figure 3:** The simulation of bacterial population  $b$  in PDE models for Eqns (9) and (10).

$$\frac{\partial b}{\partial t} = \nabla \cdot \{\sigma n b \nabla b\} + n b \quad (10)$$

where  $n$  and  $b$  represent the concentration of the nutrient and the population density of the bacterial cells respectively. Here  $\sigma = \sigma_0(1 + \Delta)$ ,  $\sigma$  is random and normally distributed with the mean  $\sigma_0$ . Computer simulations of equations (9) and (10) were applied on the space domain  $(0,1) \times (0,1)$  and over a lattice with the size of  $400 \times 400$  and no-flux boundary conditions. For the initialization, the bacterial cells were distributed in a round-shaped area in the center. The initial distribution can be described as below,

$$b_i(0) = \beta_{i_1, i_2}(0) = \beta_M \exp\{-(i_1^2 + i_2^2)/6.25\} \quad (11)$$

where  $\beta_M$  represents the maximum density. The nutrient was evenly distributed at a level  $n_0$ .

The simulation results for 8000 steps with  $\beta_M = 0.71$ ,  $n_0 = 0.35$ ,  $\sigma_0 = 4$  are show in Figure 3.

### 3 Identification of Spatio-temporal models using the OLS algorithm

Many non-linear dynamic systems can be represented by the NARMAX model, (Non-linear Auto Regressive Moving Average with eXogenous inputs)[15], which is defined as

$$y(t) = F(y(t-1), \dots, y(t-n_y), u(t-1), \dots, u(t-n_u), e(t-1), \dots, e(t-n_e)) + e(t) \quad (12)$$

where  $y(t)$ ,  $u(t)$  and  $e(t)$  represent the output, input and noise sequences respectively. When this model is extended to the MIMO case with  $m$  variables in the system output and  $r$  variables in the input, the variables can be written as vectors  $y(t) = [y_1(t), y_2(t), \dots, y_m(t)]^T$ ,  $u(t) = [u_1(t), u_2(t), \dots, u_r(t)]^T$  and  $e(t) = (e_1(t), e_2(t), \dots, e_m(t))^T$ .  $n_y$ ,  $n_u$  and  $n_e$  are the maximum time delay,  $e(t)$  is a zero mean independent sequence, and  $F$  is some non-linear function. The

objective of system identification is to find a proper approximation with respect to  $F$ . A common choice is to describe  $F$  by a polynomial representation with a given degree  $l$ ,

$$\begin{aligned}
y(t) = & \theta_0 + \sum_{i_1=1}^n \theta_{i_1} T_{i_1}(t) + \sum_{i_1=1}^n \sum_{i_2=i_1}^n \theta_{i_1, i_2} T_{i_1}(t) T_{i_2}(t) + \cdots \\
& + \sum_{i_1=1}^n \cdots \sum_{i_l=i_{l-1}}^n \theta_{i_1, \dots, i_l} T_{i_1}(t) \cdots T_{i_l}(t) + e(t)
\end{aligned} \tag{13}$$

where  $n = n_y + n_u + n_e$  and  $T(t)$  represents  $y$ ,  $u$  or  $e$  with time lags. Equation (13) can be written as a linear in the parameters regression model

$$y(t) = \sum_{i=1}^M \theta_i x_i(t) + \xi(t), \quad t = 1, \dots, N \tag{14}$$

where  $N$  is the data length.  $\theta_i$  are unknown parameters to be estimated.  $x_i(t)$  are model terms from the combination of  $T(t)$  up to degree  $l$ ,  $M$  is the number of terms involved in this system, and  $\xi(t)$  is the modelling error.

An optimized identified model should contain all the significant terms in equation (14), and all the redundant terms should have been removed from the model. The orthogonal least squares (OLS), also known as the orthogonal forward regression (OFR) algorithm with error reduction ratio (ERR) has proved to be one of the most efficient methods for term selection and parameter estimation in nonlinear temporal system identification [7][9][8][1][2]. This method was initially applied to single-input single-output (SISO) systems, but it has been widely extended to many multi-input multi-output (MIMO) systems.

The Orthogonal forward regression (OFR) algorithm is based on the orthogonalization of regressors which are the terms in models. The classical orthogonal forward regression algorithm results in a particularly simple estimation procedure which is described by the following steps:

1. Orthogonalize all the regressors in a model so that the correlations between all the terms are removed.
2. Determine significant terms using the error reduction ratio (ERR).
3. Estimate the corresponding parameters with respect to the selected terms.

## 4 The New OFR-MI algorithm

### 4.1 Mutual Information

Mutual Information (MI) which was initially proposed by Shannon in 1948 [22], is one of the effective measurements of the similarity between two variables. If two variables are strictly independent, the MI between the two variables should be zero.

Consider  $X$  and  $Y$  are two stochastic sequences with marginal probability density functions  $p(x)$  and  $p(y)$  respectively.  $p(x, y)$  is the joint probability density function. The mutual information  $I(x, y)$  is defined as

$$I(x, y) = \sum_{x \in X} \sum_{y \in Y} p(x, y) \log\left(\frac{p(x, y)}{p(x)p(y)}\right) \quad (15)$$

For the specific example of the model in Equation (14), where  $y$  is the output and  $x_i$  is one of the orthogonal regressors in the model, the mutual information  $I(x_i, y)$  between  $x_i$  and  $y$  measures how a knowledge of  $x_i$  reduces the uncertainty about  $y$ , or the information that  $x_i$  and  $y$  share. Hence, the regressor  $x_i$  with the biggest MI value may make the most contribution to the model. Thus, Mutual Information incorporated with an orthogonalisation procedure can be used as an alternative to the ERR term selection procedure in the classical OFR algorithm to aid the selection of significant model terms.

Several algorithms have been developed to estimate mutual information from observed data, including the approach using a histogram based technique [10][16], methods based on kernel density estimators [18], and parametric methods [11]. In this work, the adaptive histogram-based method proposed in [10] is employed, because this method is applicable to any distribution and appears to be asymptotically unbiased and efficient[25].

## 4.2 The New OFR-MI algorithm

Mutual information will be added into the OLS algorithm's orthogonalisation procedure as a criterion to decide the significance of model terms instead of using the ERRs [26]. According to Equation (14), The algorithm can be described as follows.

1. (a) *Step 1.* All the model terms  $X_1 = x_i(t)$ ,  $i = 1, \dots, M$  are candidates for the important term  $w_1(t)$ . For  $i = 1, \dots, M$ ,

$$w_1^{(i)}(t) = x_i(t), [MI]_1^{(i)}(y(t), x_i(t)) = \sum_{y \in Y} \sum_{x_i \in X} p(y, x_i) \log\left(\frac{p(y, x_i)}{p(y)p(x_i)}\right)$$

where  $Y = y(t)$ . Find the maximum of  $[MI]_1^{(i)}$ , say,  $[MI]_1^{(j)} = \max\{[MI]_1^{(i)}, 1 \leq i \leq M\}$ . The first significant terms can be  $w_1(t) = w_1^{(j)}(t)$ ,  $x_j(t)$  is selected with

$$y_1(t) = y(t) - \frac{y(t)w_1(t)}{w_1^2(t)}w_1(t), \alpha_{11} = 1, \hat{g}_1 = \frac{w_1(t)y(t)}{w_1^2(t)}, MI_1 = [MI]_1^{(j)}$$

and the error-to-signal ratio (ESR), which is used as the criterion to terminate the search procedure, is  $\|r_1\|^2 = \frac{\|y_1\|^2}{\|y\|^2} = (\|y\|^2 - \frac{(yw_1)^2}{w_1^2})/\|y\|^2$ .

- (b) *Step 2.* All the rest of the terms  $X_2 = x_i(t)$ ,  $i = 1, \dots, M$ ,  $i \neq j$  form the candidate terms for  $w_2(t)$ , For  $i = 1, \dots, M$ ,  $i \neq j$ ,

$$w_2^{(i)}(t) = x_i(t) - \alpha_{12}^{(i)}w_1(t),$$

$$[MI]_2^{(i)}(y_1(t), x_i(t)) = \sum_{y_1 \in Y_1} \sum_{x_i \in X_1} p(y_1, x_i) \log\left(\frac{p(y_1, x_i)}{p(y_1)p(x_i)}\right)$$

where

$$\alpha_{12}^{(i)} = \frac{w_1(t)x_i(t)}{w_1^2(t)}$$

Find the maximum of  $[MI]_2^{(i)}$ ,  $[MI]_2^{(k)} = \max\{[MI]_2^{(i)}, 1 \leq i \leq M, i \neq j\}$ . Then the second basis  $w_2(t) = w_2^{(k)}(t)$ ,  $x_k(t)$  is selected with

$$y_2(t) = y_1(t) - \frac{y_1(t)w_2(t)}{w_2^2(t)}w_2(t), \quad a_{22} = 1, \quad a_{12} = a_{12}^{(k)},$$

$$\hat{g}_2 = \frac{w_2^{(k)}(t)y(t)}{\left(w_2^{(k)}(t)\right)^2}, \quad MI_2 = [MI]_2^{(k)}$$

and the ESR is  $\|r_2\|^2 = \frac{\|y_2\|^2}{\|y\|^2} = (\|y_1\|^2 - \frac{(y_1 w_2)^2}{w_2^2})/\|y\|^2$ .

(c) This procedure is terminated at the  $M_s$ th step when either  $\|r_{M_s}\|^2 < \rho$  or  $M_s = M$ , where  $\rho$  is a desired stopping tolerance.

2. Compute the estimated parameters  $\hat{\theta}_i$

$$\hat{\theta}_M = \hat{g}_{M_s};$$

$$\hat{\theta} = \hat{g}_i - \sum_{k=i+1}^{M_s} \alpha_{ik} \hat{\theta}_k, \quad i = M_s - 1, \dots, 1$$

### 4.3 Model length determination

In practice, an identified model from real data can be either overfitting or underfitting, which may cause the model lacks good generalization properties. Thus, the validation of selected model terms and the final model is important. One of the effective methods to refine the model is cross validation [24][8][1][2], a tool that can be used to determine model size. Generalised cross-validation (GCV) is one type of cross validation, that is commonly and widely used. The GCV criterion used for linear regression model [19][3] can be expressed

$$GCV(n) = \left(\frac{N}{N-n}\right)^2 MSE(n) \quad (16)$$

where  $N$  is the length of the test data set,  $n$  is the number of selected model terms and the Mean-Square-Error (MSE) is  $MSE(n) = \|r_n\|^2/N$  corresponding to a model with  $n$  terms [26][5]. GCV will have a minimum value when  $n$  is the effective number of model terms [17].

## 5 Examples

In this section, several identification examples for spatio-temporal systems using both OFR and OFR-MI algorithms will be described. It is shown that OFR-MI produces good results for selecting the correct terms for spatio-temporal model identification.

### 5.1 CA model Identification

The model described in Equation (2) was simulated. The data for the identification is from the simulation over 100 time steps, shown in Figure 1, so the data length is  $100 \times 100$ . Tables 1 and 2 show the identification results produced by both OFR and OFR-MI.

**Table 1:** Identified model structure for the CA model of Eqn.(2) using OFR algorithm

Terms	Parameters		ERR(%)	GCV
	Ture	Estimated		
1	0	-5.4401E-15	35.71	0.6431
$c(j-1;t)c(j;t)$	-2.0	-2.0	8.36	0.5595
$c(j-1;t)$	1.0	1.0	7.6	0.4836
$c(j-1;t)c(j+1;t)$	-2.0	-2.0	2.83	0.4554
$c(j+1;t)$	1.0	1.0	7.0	0.3854
$c(j-1;t)c(j;t)c(j+1;t)$	3.0	3.0	4.82	0.3373
$c(j;t)c(j+1;t)$	-2.0	-2.0	14.18	0.1954
$c(j;t)$	1.0	1.0	19.51	0.0

**Table 2:** Identified model structure for the CA model of Eqn.(2) using OFR-MI algorithm

Terms	Parameters		MI	GCV
	Ture	Estimated		
$c(j-1;t)c(j;t)$	-2.0	-2.0	0.1383	0.3571
$c(j;t)c(j+1;t)$	-2.0	-2.0	0.2114	0.3572
$c(j;t)$	1.0	1.0	0.3108	0.2916
$c(j-1;t)c(j+1;t)$	-2.0	-2.0	1.5207	0.2830
$c(j-1;t)c(j;t)c(j+1;t)$	3.0	3.0	1.6856	0.2696
$c(j-1;t)$	1.0	1.0	0.3008	0.1348
$c(j+1;t)$	1.0	1.0	0.57	1.84E-16

Table 1 shows that a constant term is selected by the OFR algorithm with the highest ERR value. However, this term should not be in the model. Table 2 shows the results produced by the new OFR-MI algorithm. All the seven selected terms are exactly consistent with the true model terms. In addition, it shows that identified model enables the GCV value to be minimised.

## 5.2 CML model Identification

The CML model described in Section 2.2 was simulated. The identification was performed using data from eight points at locations (200,192), (200,193), (200,194), (200,195), (200,196), (200,197), (200,198), and (200,199) over 500 time steps. The data length is therefore  $8 \times 500$ . The final models identified from the data are detailed in Tables 3 and 4.

In Table 3, the ERR values for the sub-model of  $y(t)$  show the first three terms are significant, and GCV reaches a minimum value at the third term indicating that these three terms should be included in the true model. From the selected terms for the sub-model of  $x(t)$ , it is noticed that the first item has a very high ERR value, very close to 1.0, and the other terms therefore have very small ERR values, which suggests that only the first term can describe the true model with a very small and acceptable error. In temporal systems modelling a high initial ERR value often evaluates over sampled data because adjacent samples are then have

**Table 3:** Identified model structure for the CML model of Eqns (5) and (6) using OFR algorithm

Output	Terms	Parameters		ERR(%)	GCV
		Ture	Estimated		
$x(t)$	$x(t-1)[1-x(t-1)] \exp[-\beta y(t-1)]$	4.0	4.0	99.999988345	1.1661E-7
	$\nabla^2 x$	0.001	0.001	1.1655E-5	0.0
$y(t)$	$x(t-1)$	1.0	1.0	88.82	0.1119
	$x(t-1) \exp[-\beta y_i(t-1)]$	-1.0	-1.0	10.78	0.004
	$\nabla^2 y$	0.2	0.2	0.4	2.2238E-16
	$x(t-1) \exp[-\beta y_i(t-1)] \nabla^2 y$	0	3.2513E-14	4.0235E-30	2.2249E-16
	$\exp[-\beta y_i(t-1)]$	0	6.0457E-17	2.1667E-30	2.2260E-16
	$x(t-1) \nabla^2 y$	0	-1.5438E-14	2.7343E-30	2.2271E-16
	$\exp[-\beta y_i(t-1)] \nabla^2 y$	0	-7.7381E-15	8.4522E-30	2.2283E-16

**Table 4:** Identified model structure for the CML model of Eqns (5) and (6) using OFR-MI algorithm

Output	Terms	Parameters		MI	GCV
		Ture	Estimatied		
$x(t)$	$x(t-1)[1-x(t-1)] \exp[-\beta y(t-1)]$	4.0	4.0	7.1866	1.4796E-08
	$\nabla^2 x$	0.001	0.001	6.2765	0.0
$y(t)$	$x(t-1)$	1.0	1.0	2.6799	0.0039
	$x(t-1) \exp[-\beta y_i(t-1)]$	-1.0	-1.0	3.8430	1.3961E-04
	$\nabla^2 y$	0.2	0.2	5.4032	8.7180E-18

almost the same amplitude because of the high sampling. This problem exists in the models studied here but reducing the sampling was not found to be an effective solution.

However, the new OFR-MI algorithm can effectively avoid the problem of high initial ERR values. From Table 4, the estimated terms are identical to the true model terms.

### 5.3 PDE models

For the models in Equations (9) and (10), the identification procedure was applied on the data from the first 1000 successive frames in the simulation. Eight successive points in each frame, located at (200,192), (200,193), (200,194), (200,195), (200,196), (200,197), (200,198), and (200,199), were selected to form the training data set. Therefore, the size of the data set is  $8 \times 1000$ . The results are illustrated in Tables 5 and 6.

**Table 5:** Identified model structure for the PDE model of Eqns (9) and (10) using OFR algorithm

Output	Terms	Parameters		ERR(%)	GCV
		Ture	Estimated		
$n(t)$	$n(t-1)$	1.0	1.0	99.96	3.7533E-4
	$b(t-1)n(t-1)$	-0.2	-0.2	2.5720E-2	1.1810E-4
	$\nabla^2 n$	0.2	0.2	1.1804E-2	1.4444E-15
$b(t)$	$b(t-1)$	1.0	1.0	99.99931422	6.8595E-6
	$\sigma b(t-1)n(t-1)$	0	-2.3822E-13	5.2959E-4	1.5627E-6
	$\sigma b(t-1)n(t-1)\nabla^2 b$	2.0	2.0	1.0804E-4	4.8185E-7
	$\nabla(\sigma n b)\nabla b$	2.0	2.0	4.1976E-5	6.1797E-8
	$b(t-1)n(t-1)$	2.0	2.0	6.1735E-6	0.0

**Table 6:** Identified model structure for the PDE model of Eqns (9) and (10) using OFR-MI algorithm

Output	Terms	Parameters		MI	GCV
		Ture	Estimated		
$n(t)$	$\nabla^2 n$	0.2	0.2	3.9504	0.0033
	$n(t-1)$	1	1	6.2936	1.0237E-6
	$b(t-1)n(t-1)$	-0.2	-0.2	13.3318	4.8288E-19
$b(t)$	$n(t-1)\nabla(\sigma n b)\nabla b$	0	-5.2225E-13	6.388	0.5180
	$\sigma n(t-1)$	0	-1.6653E-16	8.0544	0.5139
	$b(t-1)n(t-1)\nabla^2 b$	0	9.4502E-13	4.3195	0.5094
	$b(t-1)n(t-1)$	0.2	0.2	6.8711	0.4869
	$\sigma b(t-1)n(t-1)\nabla^2 b$	0.2	0.2	3.9573	0.4853
	$b(t-1)$	1.0	1.0	2.6855	2.9212E-8
	$\nabla(\sigma n b)\nabla b$	0.2	0.2	1.2248	0.0

From Table 5, the ERRs of the first terms,  $n_i(t-1)$  and  $b_i(t-1)$ , for both  $n(t)$  and  $b(t)$  are close to 1.0 using the OFR algorithm. As noted above this may be caused by the high sampling frequency, so that the output values at the time step  $t-1$  are almost identical to the ones at  $t$  step. Hence, the terms at  $t-1$  time step are selected as the first term every time. If the spurious terms are included in the model, poor estimations may be resulted. However, if the sampling frequency is reduced, the correct models may not be correctly detected. This problem appears to be important in spatio-temporal system modelling. The OFR-MI algorithm overcomes these problems and is applicable for spatio-temporal system identification. However, the OFR-MI algorithm can not always produce better results than the OFR algorithm. For example, the results in Table 6, for the model of  $n(t)$ , all the right terms have been detected. But for the model of  $b(t)$ , the first three terms are spurious.

## 6 Conclusions

The new OFR-MI algorithm provides an effective model term selection approach for spatio-temporal system identification. In some spatio-temporal system cases spurious terms may be detected using the classical OFR algorithm due to high initial ERR values. This means that the subsequent selection procedure based on ERR values can be affected by the spurious terms. However, by using the new OFR-MI algorithm, this problem can be overcome, because the mutual information is introduced as a criterion for the term selection, which works as a replacement of the ERR procedure in the OFR algorithm. The OFR-MI algorithm works well on spatio-temporal models including CA, CML and PDE models. The OFR-MI algorithm is therefore a complementary method for the OFR algorithm, rather than a substitute.

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