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Guo, L.Z., Billings, S.A. and Coca, D. (2009) Multiscale modelling and identification of a class of lattice dynamical systems. Research Report. ACSE Research Report no. 987 . Automatic Control and Systems Engineering, University of Sheffield

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# **Multiscale modelling and identification of a class of lattice dynamical systems**

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Research Report No. 987  
February 2009

# Multiscale modelling and identification of a class of lattice dynamical systems

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## Abstract

A new multiscale modelling framework is introduced to describe a class of lattice dynamical systems (LDS), which can be used to model natural systems involving multiphysics and the multi-resolution facets of a single spatio-temporal dynamical system. The emphasis of the paper is on the multi-resolution facets, with respect to the spatial domain, of a single spatio-temporal dynamical system by using a Haar wavelet decomposition technique. A multiscale identification method for such systems is then proposed, which can be considered as a dual of the multigrid method. The proposed identification method involves three steps: the system dynamics at some specific scale of interest are identified using a recursive least-squares algorithm; the residual is then projected onto coarser scales using Haar wavelets and the parameter estimation errors are minimized; and finally a coarse correction procedure is applied to the original scale. An outstanding advantage of the proposed identification method is a saving on the computational costs. Numerical examples are provided to demonstrate the application of the proposed new approach.

## 1 Introduction

Lattice dynamical systems (LDS) and the subset coupled map lattices (CML), are systems of differential equations, indexed by points in a homogeneous lattice such as the  $d$ -dimensional integer lattice  $Z^d$ , which incorporates some aspect of the spatial structure of the lattice. These systems arise as models in many applications including chemical, material and biological sciences and can exhibit many interesting dynamical behaviours such as spatio-temporal chaos (Kaneko 1993). Recently, LDS's have been extended to investigate physical phenomena occurring in heterogeneous media such as CML on a fractal lattice (Cosenza and Kapral 1994), on Cayley trees (Gade, Cerdeira, and Romaswamy 1995), and on hierarchical lattices (Cosenza and Tucci 2000). Apart from these results, there are also extensions which are obtained by introducing a

hierarchical structure on homogeneous lattices (Hilgers and Beck 1999). All these studies have significantly enriched the theory and applications of lattice dynamical systems and motivated an investigation of such systems from a multiscale point of view which is described in this paper.

As a generalisation to the above models, a multiscale modelling framework is introduced in this paper to describe a class of lattice dynamical systems, which can be used to model both systems involving multiphysics in nature and the multi-resolution facets of a single spatio-temporal dynamical system. Actually multiscale methods have been extensively studied for the past two decades with applications in the field of applied sciences (Glimm and Sharp 1997, Krumhansl 2000, and Li and Kwauk 2003). This has been mainly driven by the requirements of the analysis of the systems which involve natural multiscale effects such as molecular dynamics and by the progress of computational capability both in software and hardware. Apart from these multiscale methods, it is also interesting to investigate a single spatio-temporal dynamical system from a multiscale or multi-resolution point of view. There are some advantages of doing this for a single dynamical system: i) it may reduce the computational cost if a coarser scale can be used; ii) it can separate high frequency components from low frequency components so as to be able to deal with these separately. Some multiscale modelling methods are described in the literature, including a multiscale theory for linear dynamic processes (Stephanopoulos, Karligil, and Dyer 2008), the heterogeneous multiscale method (E, Engquist, Li, Ren, and Vanden-Eijnden 2007), the multiscale modelling of cancer (Martins, Ferreira Jr., and Vilela 2007), the variational multiscale method (Hughes, Frijoo, Mazzwi, and Quincy 1998), Bayesian multiscale modelling (Ferreira and Lee 2007), and multiscale autoregressive models (Daoudi, Frakt, and Willsky 1999). All of these methods made important contributions to multiscale theory and applications from different individual perspectives. The emphasis of the current paper is on the multi-resolution facets, with respect to the spatial domain, of a single spatio-temporal dynamical system and on multiscale parameter estimation using a Haar wavelet decomposition technique. The multiscale identification method for such systems can be considered as a dual of the multigrid method. The proposed identification method involves three steps: the system dynamics at some specific scale of interest are identified using a recursive least-squares algorithm; the residual is then projected onto coarser scales using Haar wavelets and the parameter estimation errors are minimized; and finally a coarse correction procedure is applied to the original scale. An outstanding advantage of the proposed identification method is a saving of the computational costs.

The paper begins in section 2 with a multiscale modelling framework for a class of lattice dynamical systems, in which a detailed discussion is given to the connections with Haar wavelet multi-resolution techniques. In section 3, a multiscale parameter estimation method is presented by using a Haar wavelet decomposition and a recursive least squares algorithm. Section 4 illustrates the proposed approach using examples. Finally conclusions are drawn in section 5.

## 2 Multiscale lattice dynamical systems

### 2.1 Lattice dynamical systems (LDS)

In this section, lattice dynamical systems are introduced and definitions and terminology used later are summarised. Let  $\Lambda \in \mathbb{Z}^d$  be an  $d$ -dimensional integral periodic lattice of points. Lattice dynamical systems defined over the  $d$ -dimensional integral lattice  $\Lambda$  can be described by the following system

$$u(t) = \mathcal{F}(u(t-1)), t = 1, 2, \dots \quad (1)$$

for the discrete-time case, or

$$\dot{u}(t) = \mathcal{F}(u(t)), t \geq 0 \quad (2)$$

for a continuous-time description, in which  $u = \{u_i\}_{i \in \Lambda} \in H$  ( $i$  will be a multi-indexed array when  $d > 1$ ) is the state of the system and  $t$  represents the time instant.  $H$ , the phase space of the system, can be the Banach space or Hilbert space  $l^\infty = \{u : \|u\|_\infty = \sup_{i \in \Lambda} |u_i| < +\infty\}$  or  $l^2 = \{u : \|u\|_2 = (\sum_{i \in \Lambda} |u_i|^2)^{1/2} < +\infty\}$  and  $\mathcal{F}$  is an operator over  $H$ . Equation (2) actually represents an infinite system of (coupled) differential equations, one equation at each point of the lattice representing the local dynamics at that specific point interacting with other lattice points. In this paper we consider the case of systems with finite-range interaction, and which are invariant under translations of the lattice. Let a finite subset  $Q \in \Lambda$  of lattice points be the fixed neighbours which will describe the range of the coupling. In this case, equation (2) can be given more specifically at each lattice point, by defining a map  $\mathcal{F} : H \rightarrow H$ ,  $\mathcal{F}(u)_i = F(\{u_{i+i'}\}_{i' \in Q})$ , as follows

$$u(i, t) = u(t)_i = \mathcal{F}(u(t-1))_i = F(\{u(i+i', t-1)\}_{i' \in Q}), t = 1, 2, \dots, i \in \Lambda \quad (3)$$

for discrete-time case, or

$$\dot{u}(i, t) = \dot{u}(t)_i = \mathcal{F}(u(t))_i = F(\{u(i+i', t)\}_{i' \in Q}), t \geq 0, i \in \Lambda \quad (4)$$

for continuous-time case, where the map  $F$  can be in  $C^1$ ,  $C^p$ , or  $C^\infty$  depending on the problems concerned. Various properties of solutions for lattice dynamical systems can be found in the literature under a variety of assumptions on the structure and parameters of the systems including linear and nonlinear cases.

### 2.2 Multiscale lattice dynamical systems (MLDS)

In this section, a multiscale modelling framework for lattice dynamical systems will be proposed, which will be called a multiscale lattice dynamical system in this paper. This class of spatio-temporal processes can generally be described in terms of the following three factors to determine

the spatio-temporal dynamical evolution at each site within a multiscale environment: a term describing the reaction behaviour, a term describing the coupling with other lattice points within the same spatial scale, and a term describing the coupling between different scales.

Similar to the previous discussion, the  $d$ -dimensional integral lattice  $Z^d$  is considered as the basic spatial structure. A multiscale lattice dynamical system with  $M$  scales is defined here as a  $d$ -dimensional lattice  $Z^d$  where each site evolves at scale  $m, m = 0, 1, \dots, M$  in time through a map of the form:

$$u^m(i, t) = R^m(u^m(i, t-1)) + D_1^m(\{u^m(i+i', t-1)\}_{i' \in Q}) + D_2^m(\{u^{m'}(i+i', t-1)\}_{i' \in Q', m' \in S'}) \quad (5)$$

$m = 0, 1, \dots, M, m' \neq m \in S' \subset \{0, 1, 2, \dots, M\}, t = 1, 2, \dots$ , for the discrete-time case, and

$$\dot{u}^m(i, t) = R^m(u^m(i, t)) + D_1^m(\{u^m(i+i', t)\}_{i' \in Q}) + D_2^m(\{u^{m'}(i+i', t)\}_{i' \in Q', m' \in S'}) \quad (6)$$

$m = 0, 1, \dots, M, m' \neq m \in S' \subset \{0, 1, 2, \dots, M\}$ , for the continuous-time case, where  $u^m(i, t)$  denotes the field value at site  $i$ , at the indicated time  $t$ , and at scale  $m$ . The three terms in the right hand side of eqn. (6) determine the dynamics of the system: (a) the first term  $R^m(\cdot)$  accounts for the local reaction at its own spatial level  $m$ , (b) the second term  $D_1^m(\cdot)$  represents the diffusive coupling within the same spatial level  $m$  with neighbour  $Q$ , and (c) the last term  $D_2^m(\cdot)$  indicates the diffusive coupling between different spatial scales, where  $S'$  denotes the levels involved. The multiscale lattice dynamical system model (6) describes a large class of spatio-temporal dynamical systems. The following are two important examples in the literature regarding multiscale modelling of spatio-temporal systems.

- Hierarchical CML (Hilgers and Bech 1999). Hierarchical CML was proposed to model hydrodynamical turbulence by Higers and Bech. They used these lattice dynamical systems to improve the efficiency of the simulation of the spatio-temporal stochastic processes. Interestingly, the hierarchical structure proposed by them is not towards the lattice itself but rather the levels of the energy cascade. The hierarchical CML can be treated as a special class of the proposed multiscale lattice dynamical systems, where

$$\begin{aligned} R(u^m(i, t)) &= \lambda_m u^m(i, t) \\ D_1^m(u^m(i, t)) &= \sum_{\text{nearest neighbours } \sigma} u^m(i + \sigma, t) \\ D_2^m(u^m(i, t)) &= \xi^m(t) u^{m-1}(i, t) \end{aligned} \quad (7)$$

- Coupled maps on hierarchical lattices (Cosenza and Tucci, 2000). These models were developed to describe those processes occurring in heterogeneous media with tree-like structures. This application is related to our multiscale lattice dynamical system framework, where

$$\begin{aligned}
F^m(u^m(i, t)) &= (1 - k\gamma)u^m(i, t) \\
D_1^m(u^m(i, t)) &= 0 \\
D_2^m(u^m(i, t)) &= \gamma \sum_{\text{daughters } d} u^{m+1}(i_d, t) + \gamma \sum_{\text{parents } p} u^{m-1}(i_p, t)
\end{aligned} \tag{8}$$

## 2.3 Haar wavelet multiresolution as multiscale lattice dynamical systems

In this section, we will consider how to obtain a multiscale LDS model from a linear LDS model through a Haar wavelet multiresolution analysis. As a special class of multiscale lattice dynamical systems, the multiscale models obtained in this way are essentially multiresolution facets, with respect to the spatial domain, of a single lattice dynamical system model. The Haar wavelets and multiresolution analysis theory can be found in Chui (1992). For the purpose of clarity and simplicity and without loss of generalisation, a one-dimensional linear lattice dynamical system with the two nearest neighbours is considered as an example system in this paper

$$u(i, t) = a_0u(i, t-1) + a_1u(i+1, t-1) + a_{-1}u(i-1, t-1) + b_0f(i, t-1), t = 1, 2, \dots, i \in Z \tag{9}$$

where the coefficients  $a_0$ ,  $a_1$ , and  $a_{-1}$  can be interpreted as associated with the reaction and the coupling parameters, and  $b_0$  is the control parameter for the external input  $f$ .

### 2.3.1 Multi-resolution representation

From a multiscale point of view, the LDS (9) can be considered as an approximation of an underlying spatio-temporal system in a space  $V_{2^m}$ , which we will consider as at its finest scale available. It follows that (9) can be rewritten explicitly as

$$u_a^m(i, t) = a_0u_a^m(i, t-1) + a_1u_a^m(i+1, t-1) + a_{-1}u_a^m(i-1, t-1) + b_0f_a^m(i, t-1), t = 1, 2, \dots, i \in Z \tag{10}$$

Let  $u_a^m(k, t)$  and  $u_d^m(k, t)$  be the Haar wavelet approximation and detail coefficients at the  $m$ th scale and  $k$ th shift with respect to spatial variable  $i$ , and at time instant  $t$ . Then the wavelet decomposition and reconstruction formula between scales  $m$  and  $m-1$  for the signal  $u(i, t)$  are

$$\begin{aligned}
u_a^{m-1}(k, t) &= \frac{1}{\sqrt{2}}u_a^m(2k, t) + \frac{1}{\sqrt{2}}u_a^m(2k+1, t) \\
u_d^{m-1}(k, t) &= \frac{1}{\sqrt{2}}u_a^m(2k, t) - \frac{1}{\sqrt{2}}u_a^m(2k+1, t)
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
u_a^m(2k, t) &= \frac{1}{\sqrt{2}}u_a^{m-1}(k, t) + \frac{1}{\sqrt{2}}u_d^{m-1}(k, t) \\
u_a^m(2k+1, t) &= \frac{1}{\sqrt{2}}u_a^{m-1}(k, t) - \frac{1}{\sqrt{2}}u_d^{m-1}(k, t)
\end{aligned} \tag{12}$$

Note that the above approximation equation can be equivalently written as the following two equations for  $k$  an even number

$$\begin{aligned}
u_a^{m-1}\left(\frac{k}{2}, t\right) &= \frac{1}{\sqrt{2}}u_a^m(k, t) + \frac{1}{\sqrt{2}}u_a^m(k+1, t) \\
u_a^{m-1}\left(\frac{k}{2}+1, t\right) &= \frac{1}{\sqrt{2}}u_a^m(k+2, t) + \frac{1}{\sqrt{2}}u_a^m(k+3, t)
\end{aligned} \tag{13}$$

This relationship is shown in the tree in Fig.1.

From eqns. (13) and (10), many different multiscale relationships including coarse-to-fine, fine-to-coarse, and the lattice dynamical systems at different scales can be derived depending on the applications. Some of these will be derived and shown in the next two sections.

### 2.3.2 Lattice dynamical systems without external inputs

Given a linear lattice dynamical system (9) and its fine-scale approximation (10) with  $f = 0$ , the following multiscale dynamical relations can be obtained

a). Coarse-to-fine equations (assume that  $k$  is an even number)

$$\begin{aligned}
u_a^m(k, t) &= (a_0 - a_1)u_a^m(k, t-1) + a_{-1}u_a^m(k-1, t-1) + \sqrt{2}a_1u_a^{m-1}\left(\frac{k}{2}, t-1\right) \\
u_a^m(k+1, t) &= (a_0 - a_{-1})u_a^m(k+1, t-1) + a_1u_a^m(k+2, t-1) + \sqrt{2}a_{-1}u_a^{m-1}\left(\frac{k}{2}, t-1\right)
\end{aligned} \tag{14}$$

b). Fine-to-coarse equations (assume that  $k$  is an even number)

$$\begin{aligned}
u_a^{m-1}\left(\frac{k}{2}, t\right) &= a_0u_a^{m-1}\left(\frac{k}{2}, t-1\right) + \frac{\sqrt{2}}{2}(a_1u_a^m(k+2, t-1) + a_1u_a^m(k+1, t-1) \\
&\quad + a_{-1}u_a^m(k, t-1) + a_{-1}u_a^m(k-1, t-1)) \\
u_a^{m-1}\left(\frac{k}{2}+1, t\right) &= a_0u_a^{m-1}\left(\frac{k}{2}+1, t-1\right) + \frac{\sqrt{2}}{2}(a_1u_a^m(k+4, t-1) + a_1u_a^m(k+3, t-1) \\
&\quad + a_{-1}u_a^m(k+2, t-1) + a_{-1}u_a^m(k+1, t-1))
\end{aligned} \tag{15}$$

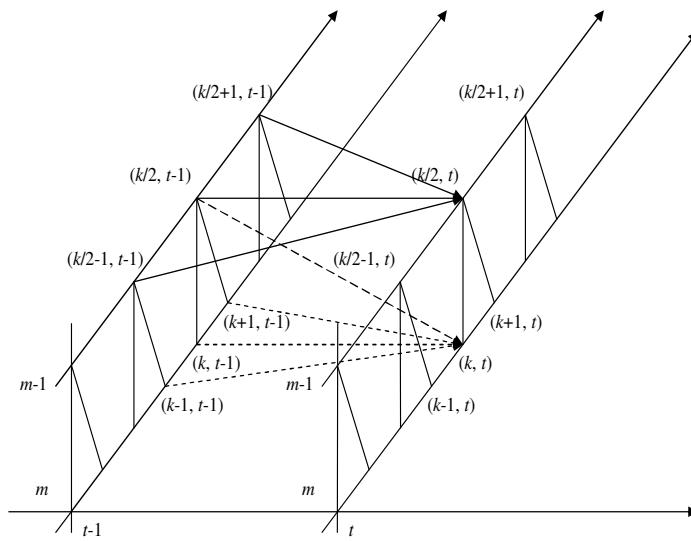


Figure 1: The tree structure of the proposed Haar wavelet based multiscale lattice dynamical systems

c). Lattice dynamical system equations at scale  $m - 1$

$$\begin{aligned} u_a^{m-1}\left(\frac{k}{2}, t\right) &= 2a_0 u_a^{m-1}\left(\frac{k}{2}, t-1\right) + \left(-\frac{a_0^2}{2} + 2a_1 a_{-1}\right) u_a^{m-1}\left(\frac{k}{2}, t-2\right) + a_{-1}^2 u_a^{m-1}\left(\frac{k}{2} - 1, t-2\right) \\ &\quad + a_1^2 u_a^{m-1}\left(\frac{k}{2} + 1, t-2\right) \end{aligned} \quad (16)$$

The coarse and fine relationships (14) in a) and (15) in b) can be easily derived as follows. Because of the nature of the tree structure by Haar wavelets, that is,  $(k/2, t)$  is the average of  $(k, t)$  and  $(k+1, t)$  while  $(k/2 + 1, t)$  is the average of  $(k+2, t)$  and  $(k+3, t)$  (see Fig. 1), there is a need to distinguish the case where  $k$  is an even number or an odd number when considering the lattice dynamics (10) at  $(k, t)$  and  $(k+1, t)$ . Rewriting (10) for  $k$  and  $k+1$  ( $k$  is assumed to be even) as follows

$$\begin{aligned} u_a^m(k, t) &= a_0 u_a^m(k, t-1) + a_1 u_a^m(k+1, t-1) + a_{-1} u_a^m(k-1, t-1) \\ u_a^m(k+1, t) &= a_0 u_a^m(k+1, t-1) + a_1 u_a^m(k+2, t-1) + a_{-1} u_a^m(k, t-1) \end{aligned} \quad (17)$$

Inserting (13) yields (14) while adding the two equations in (17) together and using (13) again yields (15).

The relationship (16) at scale  $m-1$  in c) can be done by introducing a spatial translation operator  $S^p : S^p u(k, t) = u(k+p, t)$  and a temporal translation operator  $T^q : T^q u(k, t) = u(k, t+q)$ . Then it follows that they have the following simple properties

- $S^0 u(k, t) = u(k, t)$  and  $T^0 u(k, t) = u(k, t)$ ;
- $S^{p_1} S^{p_2} = S^{p_1+p_2}$ ,  $T^{q_1} T^{q_2} = T^{q_1+q_2}$ ,  $p_1, p_2, q_1, q_2 \in R$ ;
- $S^p T^q = T^q S^p$  with  $T^q S^p u(k, t) = u(k+p, t+q)$ ,  $p, q \in R$ ;
- $S^p a u(k, t) = a u(k+p, t)$  and  $T^q a u(k, t) = a u(k, t+q)$ ,  $p, q \in R, a \in R$ .

To derive the lattice dynamical model at scale  $m-1$  from scale  $m$ , will require the reconstruction equations. Rewriting the reconstruction equations (12) for  $k$  even as follows

$$\begin{aligned} u_a^m(k, t) &= \frac{1}{\sqrt{2}} u_a^{m-1}\left(\frac{k}{2}, t\right) + \frac{1}{\sqrt{2}} u_d^{m-1}\left(\frac{k}{2}, t\right) \\ u_a^m(k+1, t) &= \frac{1}{\sqrt{2}} u_a^{m-1}\left(\frac{k}{2}, t\right) - \frac{1}{\sqrt{2}} u_d^{m-1}\left(\frac{k}{2}, t\right) \\ u_a^m(k+2, t) &= \frac{1}{\sqrt{2}} u_a^{m-1}\left(\frac{k}{2} + 1, t\right) + \frac{1}{\sqrt{2}} u_d^{m-1}\left(\frac{k}{2} + 1, t\right) \\ u_a^m(k+3, t) &= \frac{1}{\sqrt{2}} u_a^{m-1}\left(\frac{k}{2} + 1, t\right) - \frac{1}{\sqrt{2}} u_d^{m-1}\left(\frac{k}{2} + 1, t\right) \end{aligned} \quad (18)$$

Substituting  $u_a^m(k, t)$ ,  $u_a^m(k + 1, t - 1)$ ,  $u_a^m(k - 1, t - 1)$  given directly or indirectly by (18) into (10) with  $k$  an even number yields

$$\begin{aligned}
\frac{1}{\sqrt{2}}u_a^{m-1}\left(\frac{k}{2}, t\right) + \frac{1}{\sqrt{2}}u_d^{m-1}\left(\frac{k}{2}, t\right) &= a_0\left(\frac{1}{\sqrt{2}}u_a^{m-1}\left(\frac{k}{2}, t-1\right) + \frac{1}{\sqrt{2}}u_d^{m-1}\left(\frac{k}{2}, t-1\right)\right) \\
&+ a_1\left(\frac{1}{\sqrt{2}}u_a^{m-1}\left(\frac{k}{2}, t-1\right) - \frac{1}{\sqrt{2}}u_d^{m-1}\left(\frac{k}{2}, t-1\right)\right) \\
&+ a_{-1}\left(\frac{1}{\sqrt{2}}u_a^{m-1}\left(\frac{k}{2}-1, t-1\right) - \frac{1}{\sqrt{2}}u_d^{m-1}\left(\frac{k}{2}-1, t-1\right)\right)
\end{aligned} \tag{19}$$

Rearranging gives

$$\begin{aligned}
&(u_a^{m-1}\left(\frac{k}{2}, t\right) - (a_0 + a_1)u_a^{m-1}\left(\frac{k}{2}, t-1\right) - a_{-1}u_a^{m-1}\left(\frac{k}{2}-1, t-1\right)) \\
&+ (u_d^{m-1}\left(\frac{k}{2}, t\right) - (a_0 - a_1)u_d^{m-1}\left(\frac{k}{2}, t-1\right) + a_{-1}u_d^{m-1}\left(\frac{k}{2}-1, t-1\right)) = 0
\end{aligned} \tag{20}$$

and with the spatial and temporal operators

$$(1 - (a_0 + a_1)T^{-1} - a_{-1}T^{-1}S^{-1})u_a^{m-1}\left(\frac{k}{2}, t\right) + (1 - (a_0 - a_1)T^{-1} + a_{-1}T^{-1}S^{-1})u_d^{m-1}\left(\frac{k}{2}, t\right) = 0 \tag{21}$$

By repeating the same procedure, the following equation for the case of  $k + 1$  can be given

$$(1 - (a_0 + a_{-1})T^{-1} - a_1T^{-1}S^1)u_a^{m-1}\left(\frac{k}{2}, t\right) + (-1 + (a_0 - a_{-1})T^{-1} - a_1T^{-1}S^1)u_d^{m-1}\left(\frac{k}{2}, t\right) = 0 \tag{22}$$

By solving the equations (21) and (22) through eliminating the details  $u_d$ , yields

$$(-2 + 4a_0T^{-1} + (-a_0^2 + 4a_1a_{-1})T^{-2} + 2a_{-1}^2T^{-2}S^{-1} + 2a_1^2T^{-2}S^1)u_a^{m-1}\left(\frac{k}{2}, t\right) = 0 \tag{23}$$

which gives the lattice dynamical model at scale  $m - 1$  (16) by expressing the system in its conventional way without the operators.

### 2.3.3 Lattice dynamical systems with external inputs

For the linear lattice dynamical systems with external input  $f$ , the following multiscale dynamical relations can be obtained

a). Coarse-to-fine equations (assume that  $k$  is an even number)

$$\begin{aligned}
u_a^m(k, t) &= (a_0 - a_1)u_a^m(k, t-1) + a_{-1}u_a^m(k-1, t-1) + \sqrt{2}a_1u_a^{m-1}\left(\frac{k}{2}, t-1\right) \\
&\quad + b_0f_a^m(k, t-1) \\
u_a^m(k+1, t) &= (a_0 - a_{-1})u_a^m(k+1, t-1) + a_1u_a^m(k+2, t-1) + \sqrt{2}a_{-1}u_a^{m-1}\left(\frac{k}{2}, t-1\right) \\
&\quad + b_0f_a^m(k+1, t-1)
\end{aligned} \tag{24}$$

b). Fine-to-coarse equations (assume that  $k$  is an even number)

$$\begin{aligned}
u_a^{m-1}\left(\frac{k}{2}, t\right) &= a_0u_a^{m-1}\left(\frac{k}{2}, t-1\right) + \frac{\sqrt{2}}{2}(a_1u_a^m(k+2, t-1) + a_{-1}u_a^m(k+1, t-1) \\
&\quad + a_{-1}u_a^m(k, t-1) + a_{-1}u_a^m(k-1, t-1)) + b_0f_a^m(k, t-1) \\
u_a^{m-1}\left(\frac{k}{2}+1, t\right) &= a_0u_a^{m-1}\left(\frac{k}{2}+1, t-1\right) + \frac{\sqrt{2}}{2}(a_1u_a^m(k+4, t-1) + a_{-1}u_a^m(k+3, t-1) \\
&\quad + a_{-1}u_a^m(k+2, t-1) + a_{-1}u_a^m(k+1, t-1)) + b_0f_a^m(k+1, t-1)
\end{aligned} \tag{25}$$

c). Lattice dynamical system equations at scale  $m-1$

$$\begin{aligned}
u_a^{m-1}\left(\frac{k}{2}, t\right) &= 2a_0u_a^{m-1}\left(\frac{k}{2}, t-1\right) + \left(-\frac{a_0^2}{2} + 2a_1a_{-1}\right)u_a^{m-1}\left(\frac{k}{2}, t-2\right) + a_{-1}^2u_a^{m-1}\left(\frac{k}{2}-1, t-2\right) \\
&\quad + a_1^2u_a^{m-1}\left(\frac{k}{2}+1, t-2\right) + b_0f_a^{m-1}\left(\frac{k}{2}, t-1\right) - \frac{1}{2}(2a_0 - a - 1 - a_{-1})b_0f_a^{m-1}\left(\frac{k}{2}, t-2\right) \\
&\quad + \frac{a_1b_0}{2}f_a^{m-1}\left(\frac{k}{2}+1, t-2\right) + \frac{a_{-1}b_0}{2}f_a^{m-1}\left(\frac{k}{2}-1, t-2\right) - \frac{1}{2}(a_1 - a_{-1})b_0f_a^{m-1}\left(\frac{k}{2}, t-2\right) \\
&\quad + \frac{a_1b_0}{2}f_a^{m-1}\left(\frac{k}{2}+1, t-2\right) - \frac{a_2b_0}{2}f_a^{m-1}\left(\frac{k}{2}-1, t-2\right)
\end{aligned} \tag{26}$$

The coarse and fine relationships (14) in a) and (15) in b) can be obtained in the similar way as above. The relationship (16) at scale  $m-1$  in c) can be treated in a similar way but with an extra term representing the forces, where the equations to be solved are

$$\begin{aligned}
(1 - (a_0 + a_1)T^{-1} - a_{-1}T^{-1}S^{-1})u_a^{m-1}\left(\frac{k}{2}, t\right) &+ (1 - (a_0 - a_1)T^{-1} + a_{-1}T^{-1}S^{-1})u_d^{m-1}\left(\frac{k}{2}, t\right) \\
&= b_0T^{-1}(f_a^{m-1}\left(\frac{k}{2}, t\right) + f_d^{m-1}\left(\frac{k}{2}, t\right)) \\
(1 - (a_0 + a_{-1})T^{-1} - a_1T^{-1}S^1)u_a^{m-1}\left(\frac{k}{2}, t\right) &+ (-1 + (a_0 - a_{-1})T^{-1} - a_1T^{-1}S^1)u_d^{m-1}\left(\frac{k}{2}, t\right) \\
&= b_0T^{-1}(f_a^{m-1}\left(\frac{k}{2}, t\right) - f_d^{m-1}\left(\frac{k}{2}, t\right))
\end{aligned} \tag{27}$$

### 3 Multiscale identification of the linear lattice dynamical model

In this section, a least squares based multiscale method is proposed for the identification of the linear lattice dynamical system (10), that is

$$u(i, t) = a_0 u(i, t-1) + a_1 u(i+1, t-1) + a_{-1} u(i-1, t-1), t = 1, 2, \dots, i \in Z \quad (28)$$

#### 3.1 Recursive least squares estimates of the parameters at scale $m$

The approximation in finite dimensional space  $V_{2^m}$  can be written as

$$u_a^m(i, t) = a_0 u_a^m(i, t-1) + a_1 u_a^m(i+1, t-1) + a_{-1} u_a^m(i-1, t-1) + w^m(i, t) \quad (29)$$

where  $t = 1, 2, \dots$  and  $i = 0, 1, \dots, 2^m - 1$  or in matrix form as

$$y_m(t) = \phi_m(t-1)^T \theta_m + w_m(t) \quad (30)$$

where  $y_m(t) = (u_a^m(0, t), u_a^m(1, t), \dots, u_a^m(2^m - 1, t))^T$ ,  $\theta_m = (a_{-1}, a_0, a_1)^T$ , and the modelling error vector or noise  $w_m(t) = (w_m(0, t), w_m(1, t), \dots, w_m(2^m - 1, t))^T$ , and

$$\phi_m(t-1)^T = \begin{pmatrix} u_a^m(-1, t-1) & u_a^m(0, t-1) & u_a^m(1, t-1) \\ u_a^m(0, t-1) & u_a^m(1, t-1) & u_a^m(2, t-1) \\ & \vdots & \\ u_a^m(2^m - 2, t-1) & u_a^m(2^m - 1, t-1) & u_a^m(2^m, t-1) \end{pmatrix} \quad (31)$$

in which the  $u_a^m(-1, t-1)$  and  $u_a^m(2^m, t-1)$  are the boundary conditions. A standard least squares estimate to the problem (30) is given by

$$\hat{\theta}_m^n = (R_m^n)^{-1} \sum_{t=1}^n \phi_m(t-1) y_m(t) \quad (32)$$

with the least squares criterion

$$J_m^n(\theta) = \frac{1}{2} \sum_{t=1}^n \|y_m(t) - \phi_m(t-1)^T \theta\|^2 \quad (33)$$

where  $R_m^n = \sum_{t=1}^n \phi_m(t-1) \phi_m(t-1)^T$  and  $n$  the number of time samplings. The recursive estimation equation is

$$\hat{\theta}_m^n = \hat{\theta}_m^{n-1} + (R_m^n)^{-1} \phi_m(t-1) (y_m(n) - \phi_m(t-1)^T \hat{\theta}_m^{n-1}) \quad (34)$$

For the deterministic case where  $w_m(t) = 0$  for all  $t$ , it follows that

$$R_m^n \tilde{\theta}_m^n = R_m^{n-1} \tilde{\theta}_m^{n-1} = \text{constant} \quad (35)$$

with  $\tilde{\theta}_m^n = \theta_m^* - \hat{\theta}_m^n$ . It follows that  $\lim_{n \rightarrow \infty} \tilde{\theta}_m^n = 0$  if  $\lim_{n \rightarrow \infty} \lambda_{\min} R_m^n = \infty$  where  $\lambda_{\min} R_m^n$  is the smallest eigenvalue of  $R_m^n$ . In the stochastic case, it has been shown that if the noise  $w_m(t)$  is a martingale difference sequence, then under certain assumptions (e.g. Lai and Wei 1982) the least squares estimate  $\hat{\theta}_m^n$  of the true parameter  $\theta_m^*$  satisfies

$$\|\hat{\theta}_m^n - \theta_m^*\|^2 = O\left(\frac{\log \lambda_{\max} R_m^n}{\lambda_{\min} R_m^n}\right) \quad (36)$$

provided that  $\lambda_{\min} R_m^n \rightarrow \infty$  as  $n \rightarrow \infty$ . Here  $\lambda_{\max} R_m^n$  and  $\lambda_{\min} R_m^n$  are the largest and smallest eigenvalues of the matrix  $R_m^n$ . Moreover, the least squares estimates are strongly consistent if  $\lambda_{\min} R_m^n \rightarrow \infty$  and the following excitation condition

$$\lim_{n \rightarrow \infty} \frac{\log \lambda_{\max} R_m^n}{\lambda_{\min} R_m^n} \rightarrow 0, \text{ a.s.} \quad (37)$$

is satisfied.

## 3.2 Multiscale view of the parameter estimation

In this section a coarser scale correction method to the least squares estimates obtained at certain scale  $m$  will be presented. The proposed method is similar to the coarse grid correction to the multigrid method but here this procedure is used in the parameter estimation process.

From the earlier discussion, the least squares estimates  $\hat{\theta}_m^n$  of the system parameters  $\theta_m^*$  at scale  $m$  leads to two possible errors: the first is the parameter estimate error  $\tilde{\theta}_m^n$  and the other is the residual  $e_m(t) = y_m(t) - \phi_m(t-1)^T \hat{\theta}_m^n$ . From (30), it follows that

$$e_m(t) = y_m(t) - \phi_m(t-1)^T \hat{\theta}_m^n = \phi_m(t-1)^T \theta_m^* + w_m(t) - \phi_m(t-1)^T \hat{\theta}_m^n = \phi_m(t-1)^T \tilde{\theta}_m^n + w_m(t) \quad (38)$$

which shows the relationship between the residual and estimate errors of the least squares method. Consider the following criterion

$$J_m^{m-1}(\eta_{m-1}) = \frac{1}{2} \langle \phi_m(t-1)^T (\hat{\theta}_m^n + \eta_{m-1}), \hat{\theta}_m^n + \eta_{m-1} \rangle - \langle y_m(t), \hat{\theta}_m^n + \eta_{m-1} \rangle \quad (39)$$

where  $\langle \cdot, \cdot \rangle$  denotes the bilinear form defined by

$$\langle u, v \rangle = \sum_{i=1}^{2^m} u_i v_i, u, v \in R^{2^m} \quad (40)$$

A direct solution to the minimisation of (39) can be obtained easily as

$$\phi_m(t-1)^T \eta_{m-1} = y_m(t) - \phi_m(t-1)^T \hat{\theta}_m^n = e_m(t) \quad (41)$$

which is in exactly the same form as the residual equation (38), which can be rewritten as

$$e^m(k, t) = \tilde{a}_0 u_a^m(k, t-1) + \tilde{a}_1 u_a^m(k+1, t-1) + \tilde{a}_{-1} u_a^m(k-1, t-1), i = 0, 1, \dots, 2^m - 1 \quad (42)$$

Projecting this equation from space  $V_{2^m}$  onto the coarser space  $V_{2^{m-1}}$  by multiplying by the  $2^{m-1} \times 2^m$  Haar wavelet transform matrix  $P$

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \quad (43)$$

on both sides of the eqn. (41) yields the following coarse relationship in coarse space  $V_{2^{m-1}}$

$$\phi_{m-1}(t-1)^T \eta_{m-1} = e_{m-1}(t) \quad (44)$$

where  $e_{m-1}(t) = P e_m(t) = (e_a^{m-1}(0, t), e_a^{m-1}(1, t), \dots, e_a^{m-1}(2^{m-1} - 1, t))^T$ ,  $\eta_{m-1} = (\tilde{a}_{-1}, \tilde{a}_0, \tilde{a}_1)^T$ , and

$$\phi_{m-1}(t-1)^T = P \phi_m(t-1)^T = \begin{pmatrix} \underline{u}_a^{m-1}(0, t-1) & u_a^{m-1}(0, t-1) & \underline{u}_a^{m-1}(1, t-1) \\ \underline{u}_a^{m-1}(1, t-1) & u_a^{m-1}(1, t-1) & \underline{u}_a^{m-1}(2, t-1) \\ & \vdots & \\ \underline{u}_a^{m-1}(2^{m-1} - 1, t-1) & u_a^{m-1}(2^{m-1} - 1, t-1) & \underline{u}_a^{m-1}(2^{m-1}, t-1) \end{pmatrix} \quad (45)$$

in which  $\underline{u}_a^{m-1}(k+1, t-1) = 1/\sqrt{2}(u_a^m(2k+1, t-1) + u_a^m(2k+2, t-1))$ ,  $k = 0, 1, \dots, 2^{m-1} - 1$ . Similar results can be achieved by using Haar wavelet detail transform  $Q$

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \quad (46)$$

Next let us see how the Haar wavelet transforms  $P$  and  $Q$  change the eigenvalues of the matrix  $\phi_m(t-1)\phi_m(t-1)^T$ . Firstly, notice that  $P^T P + Q^T Q = I$  so that

$$\begin{aligned} \phi_m(t-1)\phi_m(t-1)^T &= \phi_m(t-1)(P^T P + Q^T Q)\phi_m(t-1)^T \\ &= \phi_m(t-1)P^T P\phi_m(t-1)^T + \phi_m(t-1)Q^T Q\phi_m(t-1)^T \end{aligned} \quad (47)$$

By adding up both side of (47), it follows that

$$R_m^n = R_{m-1}^n + H_{m-1}^n \quad (48)$$

where  $H_{m-1}^n = \sum_{t=1}^n \phi_m(t-1)Q^T Q \phi_m(t-1)^T$ . It is interesting to see that the covariance matrix  $R_m^n$  at scale  $m$  can be divided into a low frequency part  $R_{m-1}^n$  and a high frequency part  $H_{m-1}^n$  at scale  $m-1$ . It can be checked that the  $3 \times 3$  matrices  $\phi_m(t-1)P^T P \phi_m(t-1)^T$  and  $\phi_m(t-1)Q^T Q \phi_m(t-1)^T$  have the same eigenvalues:  $1/2(\lambda_1(t-1)^2 + \lambda_2(t-1)^2)$ ,  $1/2\lambda_3(t-1)^2$  assuming that  $\lambda_1(t-1)^2$ ,  $\lambda_2(t-1)^2$ ,  $\lambda_3(t-1)^2$  are the three eigenvalues of the matrix  $\phi_m(t-1)\phi_m(t-1)^T$ . This shows that the matrices  $\phi_m(t-1)P^T P \phi_m(t-1)^T$  and  $\phi_m(t-1)Q^T Q \phi_m(t-1)^T$  are only semi-positive definite but this does not affect the positive definiteness of the matrix  $R_{m-1}^l$  because  $R_{m-1}^l = \sum_{t=1}^l \phi_m(t-1)P^T P \phi_m(t-1)^T = \sum_{t=1}^l \phi_{m-1}(t-1)\phi_{m-1}(t-1)^T$  will be positive definite as long as  $\phi_{m-1}(0)\phi_{m-1}(0)^T$  is chosen to be positive definite.

### 3.3 Convergence analysis

The proposed multiscale method involves the following three steps

- Recursive least squares estimates for the parameters at scale  $m$ ;
- Recursive least squares estimates of the estimation errors of the parameters at scale  $m-1$
- Coarse correction from  $m-1$  to  $m$

The coarse correction is accomplished by simply adding the coarse estimates to the fine estimates as follows

$$\hat{\theta}_m^n \leftarrow \hat{\theta}_m^n + \hat{\eta}_{m-1}^l \quad (49)$$

Applying the standard least squares method to the problem (44) again gives the following estimate

$$\hat{\eta}_{m-1}^l = (R_{m-1}^l)^{-1} \sum_{t=1}^l \phi_{m-1}(t-1)e_{m-1}(t) \quad (50)$$

where the matrix  $R_{m-1}^l = \sum_{t=1}^l \phi_{m-1}(t-1)\phi_{m-1}(t-1)^T$  and  $l$  the number of time samplings. For the deterministic case, again the least squares solution to (44) satisfies

$$R_{m-1}^l \tilde{\eta}_{m-1}^l = R_{m-1}^{l-1} \tilde{\eta}_{m-1}^{l-1} = \text{constant} \quad (51)$$

with  $\tilde{\eta}_{m-1}^l = \eta_{m-1}^* - \hat{\eta}_{m-1}^l = \tilde{\theta}_m^n - \hat{\eta}_{m-1}^l$ . Now let  $\tilde{\theta}_m^{n+l} = \theta_m^* - (\hat{\theta}_m^n + \hat{\eta}_{m-1}^l)$ , then it follows that

$$\begin{aligned}
R_{m-1}^l \tilde{\theta}_m^{n+l} &= R_{m-1}^l (\theta_m^* - (\hat{\theta}_m^n + \hat{\eta}_{m-1}^l)) \\
&= R_{m-1}^l (\theta_m^* - \hat{\theta}_m^n) - R_{m-1}^l \hat{\eta}_{m-1}^l \\
&= R_{m-1}^l \tilde{\theta}_m^n - R_{m-1}^l \hat{\eta}_{m-1}^l \\
&= R_{m-1}^l \tilde{\eta}_{m-1}^l \\
&= R_{m-1}^{l-1} \tilde{\eta}_{m-1}^{l-1} \\
&= R_{m-1}^{l-1} \tilde{\theta}_m^{n+(l-1)}
\end{aligned} \tag{52}$$

so that

$$R_{m-1}^l \tilde{\theta}_m^{n+l} = R_{m-1}^{l-1} \tilde{\theta}_m^{n+(l-1)} = \text{constant} \tag{53}$$

Hence, since  $(R_{m-1}^l)^+$  is a sequence of decreasing semi-positive definite matrices, both  $(R_{m-1}^l)^+$  and  $\tilde{\theta}_m^{n+l}$  converge. This shows that  $\lim_{l \rightarrow \infty} \tilde{\theta}_m^{n+l} = 0$  if  $\lambda_{\min} R_{m-1}^l \rightarrow \infty$ . Moreover

$$(\tilde{\theta}_m^{n+l})^T R_{m-1}^l \tilde{\theta}_m^{n+l} \leq (\tilde{\theta}_m^{n+(l-1)})^T R_{m-1}^l \tilde{\theta}_m^{n+(l-1)} \leq \dots \leq (\tilde{\theta}_m^n)^T R_{m-1}^0 \tilde{\theta}_m^n \tag{54}$$

Since  $R_{m-1}^l = R_{m-1}^{l-1} + \phi_{m-1}(l-1)\phi_{m-1}(l-1)^T$ , it has

$$\lambda_{\min} R_{m-1}^l \geq \lambda_{\min} R_{m-1}^{l-1} \geq \dots \geq \lambda_{\min} R_{m-1}^0 \tag{55}$$

It follows that

$$\|\tilde{\theta}_m^{n+l}\|^2 \leq \kappa(R_{m-1}^{l-1}) \|\tilde{\theta}_m^{n+l-1}\|^2 \tag{56}$$

and

$$\|\tilde{\theta}_m^{n+l}\|^2 \leq \kappa(R_{m-1}^0) \|\tilde{\theta}_m^n\|^2 \tag{57}$$

where  $\kappa(R) = \lambda_{\max} R / \lambda_{\min} R$  is the condition number of the matrix  $R$ .

This shows that  $\lim_{l \rightarrow \infty} \tilde{\theta}_m^{n+l} = 0$  if  $\lim_{l \rightarrow \infty} \lambda_{\min} R_{m-1}^l = \infty$  when the system is deterministic, where  $\lambda_{\min} R_{m-1}^l$  is the smallest eigenvalue of  $R_{m-1}^l$ .

## 4 Numerical results

In this section, a numerical simulation is conducted to illustrate the proposed multiscale parameter estimation method. We will continue to use the example system (10), that is

$$u(i, t) = a_0 u(i, t-1) + a_1 u(i+1, t-1) + a_{-1} u(i-1, t-1), t = 1, 2, \dots \tag{58}$$

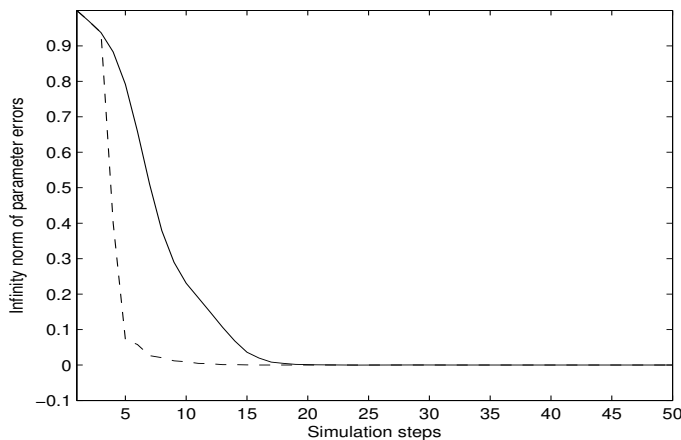


Figure 2: The parameter estimate errors for case 1 without noise (solid: LS at scale  $m$ , dashed: Multiscale method)

where  $i = 0, 1, \dots, 2^m - 1$  representing an approximation in finite dimensional space  $V_{2^m}$ . The numerical studies are carried out for four different cases, chosen to illustrate the properties of the new algorithm.

Case 1. In this simulation, the parameters were chosen as  $a_0 = 0.1, a_1 = 0.5, a_{-1} = -0.2$  and  $m = 6$ . The initial conditions of the lattice dynamical system were

$$u(i, 0) = \sin\left(\frac{2i\pi}{2^m}\right) + \frac{1}{2} \sin\left(\frac{16i\pi}{2^m}\right) + \frac{1}{2} \sin\left(\frac{32i\pi}{2^m}\right) \quad (59)$$

and Dirichlet boundary conditions were considered. The reason for the choice of this initial condition is to include different frequency components. The recursive least squares algorithm was run for three simulation steps at fine scale  $m = 6$  and then projected to coarse scale  $m = 5$  for the rest of the computation. Moreover, to test the performance of the proposed multiscale identification method, the same procedure was applied to the system with a white noise of zero mean and std = 0.3. Fig.2 and Fig.3 shows the infinity norm of the parameter estimate errors  $\|e(t)\|_\infty$  for the cases without noise and with noise. These results indicate that the multiscale identification method has a faster convergence rate than the conventional recursive least squares method.

Case 2. In this case, the parameters were still chosen as  $a_0 = 0.1, a_1 = 0.5, a_{-1} = -0.2$  and  $m = 6$  but the initial conditions of the lattice dynamical system were chosen as a Gaussian white noise with zero mean and std = 1. The same procedure as in case 1 was employed and the results are shown in Fig.4 and Fig.5.

From all these numerical results, it can be observed that

- 1) The proposed multiscale method can speed up the convergence of the parameter estimates

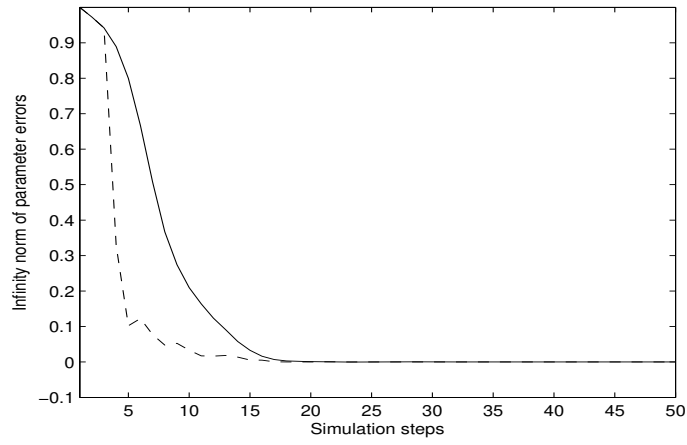


Figure 3: The parameter estimate errors for case 1 with noise (solid: LS at scale  $m$ , dashed: Multiscale method)

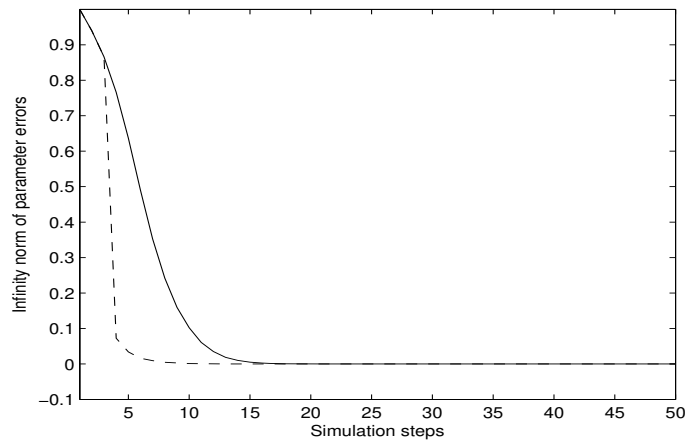


Figure 4: The parameter estimate errors for case 2 without noise (solid: LS at scale  $m$ , dashed: Multiscale method)

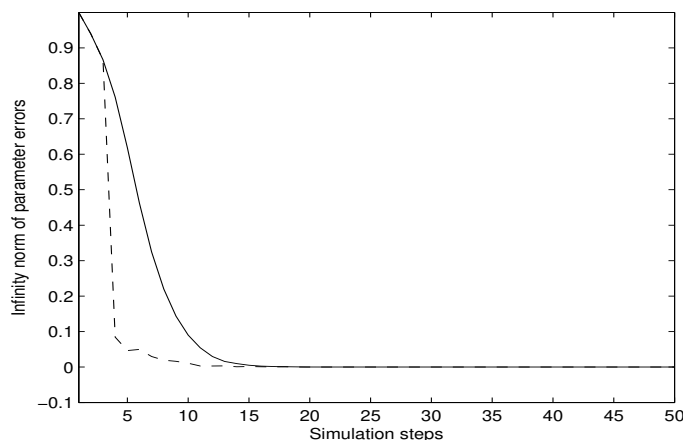


Figure 5: The parameter estimate errors for case 2 with noise (solid: LS at scale  $m$ , dashed: Multiscale method)

compared to applying the least squares method at a single scale.

2) Due to the projection from fine to coarse, the computational costs are reduced because of the reduction of the dimension of the problem.

## 5 Conclusions

A multiscale modelling framework for a class of lattice dynamical systems has been introduced. It has been demonstrated that the proposed framework not only includes many existing models as special cases but also connects the multiresolution analysis to the multiscale problem. Apart from that, a new multiscale identification method has been presented by using a Haar wavelet decomposition method and a recursive least squares algorithm. It has been shown by numerical studies that the convergence properties of the parameter estimation can be improved by the proposed multiscale method.

With the proposed framework, there are plenty of problems to be solved such as a detailed analysis of the convergence and consistency in the presence of noise and how to extend the proposed identification method to nonlinear lattice dynamical systems.

## 6 Acknowledgement

The authors gratefully acknowledge financial support from EPSRC (UK).

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