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Abstract: The output frequency response function (OFRF) of nonlinear systems is a new concept, which defines an analytical relationship between the output spectrum and the parameters of nonlinear systems. In the present study, the parametric characteristics of the OFRF for nonlinear systems described by a polynomial form differential equation model are investigated based on the introduction of a novel coefficient extraction operator. Important theoretical results are established, which allow the explicit structure of the OFRF for this class of nonlinear systems to be readily determined, and reveal clearly how each of the model nonlinear parameters has its effect on the system output frequency response. Examples are provided to demonstrate how the theoretical results are used for the determination of the detailed structure of the OFRF. Simulation studies verify the effectiveness and illustrate the potential of these new results for the analysis and synthesis of nonlinear systems in the frequency domain.

Keywords:
Parametric characteristics, Output frequency response function, Nonlinear systems

1 Introduction

Analysis and synthesis of nonlinear systems have been studied for many years both in the time and frequency domain (Khalil 2002, Sastry 1999, Rugh 1981). The frequency domain analysis can often provide a physically meaningful insight into system behaviors. Consequently the frequency domain methods have always been important approaches in control and signal processing fields. The study of nonlinear systems in the frequency domain is based on the Volterra series method (Corduneanu and Sandberg 2000, Rugh 1981, Bedrosian and Rice 1971). As shown in Boyd and Chua (1985), any nonlinear system which is time invariant, causal and of fading memory can be approximated by a Volterra series of a sufficient high order. By defining the multidimensional Fourier transformation of the kernel functions of the Volterra series, George (1959) proposed the concept of the generalized frequency response functions (GFRFs). Based on this concept, many results and techniques were developed in order to estimate or compute the GFRFs of nonlinear systems (Bendat 1990, Nam and Powers 1994, Kim and Powers 1988). In Peyton-Jones and Billings (1989), Billings and Peyton-Jones (1990), and Swain and Billings (2001), the recursive algorithms for the computation of the GFRFs of discrete time, continuous time and multi-input multi-output nonlinear systems were developed, respectively. Based on these results, the output frequency characteristics of nonlinear systems were studied by Lang and Billings (1996), and some other frequency characteristics of nonlinear systems are also studied and discussed (Lang and Billings 2005, Zhang and Billings 1996). These results provide an important basis for further study of the analysis and synthesis of nonlinear systems in the frequency domain.

In Lang *et al.* (2006), an expression for the output frequency response, which defines an analytical relationship between the output spectrum and the system parameters, was derived for nonlinear systems described by a polynomial form differential equation model. The result is referred to as the output frequency response function (OFRF) of nonlinear Volterra systems. The new OFRF concept reveals that for a wide class of nonlinear systems there exists a simple polynomial relationship between the output spectrum and the system parameters which define the system nonlinearities. This is an important extension of the well-known linear frequency domain relationship that the output spectrum equals to the input spectrum times the frequency response function, and provides an important basis to extend the linear frequency domain analysis and synthesis methods to the nonlinear case.

In order to determine the structure of the OFRF so as to reveal what model nonlinear parameters are available in the analytical system output frequency response description and how each of those parameters has its effect on the system output spectrum, a symbolic computation procedure was adopted for the computation of the OFRF in Lang et al. (2006). Although the symbolic computation is effective when the degrees of the system nonlinearity involved in the OFRF is not high, it is very computationally demanding and especially difficult to be used when the maximum degree of system nonlinearity considered in the OFRF is higher than 5. In order to circumvent these problems, in the present study, the OFRF of nonlinear systems is studied by a new approach referred to as the parametric characteristic analysis proposed in Jing et al (2006b), which is to study what model parameters affect a specific system response function and how those parameters have their effect on this function. By introducing a novel coefficient extraction operator, the parametric characteristics of the GFRFs are first derived and studied. Then important results about the parametric characteristics of the OFRF are developed, which explicitly reveal the analytical polynomial relationship between the system nonlinear parameters and the output spectrum, and allow the detailed structure of the OFRF to be readily determined up to any high orders without complicated symbolic computations. Examples are provided to demonstrate the application of the new results. Simulation studies verify the effectiveness and illustrate the potential of using these new results for the analysis and synthesis of nonlinear systems in the frequency domain.

2 The output frequency response function of nonlinear systems

Consider the following nonlinear differential equation model

$$\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p}\frac{d^{k_i}y(t)}{dt^{k_i}}\prod_{i=p+1}^{p+q}\frac{d^{k_i}u(t)}{dt^{k_i}}=0 \tag{1}$$

where $\left.\frac{d^k x(t)}{dt^k}\right|_{k=0}=x(t)$, $p+q=m$, $\sum_{k_1,k_{p+q}=0}^{K}(\cdot)=\sum_{k_1=0}^{K}(\cdot)\cdots\sum_{k_{p+q}=0}^{K}(\cdot)$, M is the maximum degree of nonlinearity in terms of $y(t)$ and $u(t)$, and K is the maximum order of the derivative. In this model, the parameters such as $c_{0,1}(.)$ and $c_{1,0}(.)$ are linear parameters, which correspond to the linear terms in the model, *i.e.*, $\frac{d^k y(t)}{dt^k}$ and $\frac{d^k u(t)}{dt^k}$ for k=0,1,…,L, and $c_{pq}(\cdot)$ for $p+q>1$ are nonlinear parameters corresponding to nonlinear terms in the model of the form $\prod_{i=1}^{p}\frac{d^{k_i}y(t)}{dt^{k_i}}\prod_{i=p+1}^{p+q}\frac{d^{k_i}u(t)}{dt^{k_i}}$, *e.g.*, $y(t)^p u(t)^q$. $p+q$ is called the nonlinear degree of the nonlinear parameter $c_{pq}(\cdot)$.

Under the condition that system (1) is stable at zero equilibrium, the input output relationship of the system can be approximated in the neighbourhood of the equilibrium by a Volterra series up to maximum order N

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{2}$$

where $h_n(\tau_1,\cdots,\tau_n)$ is a real valued function of $\tau_1,\cdots,\tau_n$ called the nth order Volterra kernel. The output spectrum of the system when subject to a general input can be described as (Lang and Billings 1996)

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \tag{3}$$

where,

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \tag{4}$$

is the nth order GFRF. When the system is subject to a multi-tone input described by

$$u(t)=\sum_{i=1}^{K}|F_i|\cos(\omega_i t+\angle F_i) \tag{5}$$

the system output spectrum can be written as (Lang and Billings, 1996):

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{6}$$

where $F(\omega)=\begin{cases}|F_i|e^{j\angle F_i} & \text{if } \omega\in\{\omega_k, k=\pm1,\cdots,\pm K\}\\ 0 & \text{else}\end{cases}$.

In order to study the system output spectrum from (3) or (6), the GFRFs should first be determined. A recursive algorithm can be utilized to compute the GFRFs of system (1) as follows (Billings and Peyton-Jones 1990):

$$L(n)\cdot H_n(j\omega_1,\cdots,j\omega_n)=\sum_{k_1,k_n=1}^{K}c_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n}$$

$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}c_{p,q}(k_1,\cdots,k_{p+q})(j\omega_{n-q+1})^{k_{n-q+1}}\cdots(j\omega_{p+q})^{k_{p+q}}H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \tag{7}$$

$$+\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K}c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n)$$

$$H_{n,p}(\cdot)=\sum_{i=1}^{n-p+1}H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{k_p} \tag{8a}$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n)=H_n(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{k_1} \tag{8b}$$

$H_{n,p}(j\omega_1,\cdots,j\omega_n)$ in (8a) can also be rewritten as

$$H_{n,p}(j\omega_1,\cdots,j\omega_n)=\sum_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\prod_{i=1}^{p}H_{r_i}(j\omega_{r_X+1},\cdots,j\omega_{r_X+r_i})(j\omega_{r_X+1}+\cdots+j\omega_{r_X+r_i})^{k_i} \tag{9}$$

where $L(n)=1-\sum_{k_1=0}^{K}c_{1,0}(k_1)(j\omega_1+\cdots+j\omega_n)^{k_1}$, $X=\sum_{x=1}^{i-1}r_x$.

From equations (3)(6-9), it can be seen that the direct computation of the system output spectrum involves very complicated integral and symbolic operations. Consequently the analytical relationship between the model parameters and the output frequency response can not be readily revealed from these results. In order to solve this problem, in Lang et al. (2006) an analytical relationship between the output spectrum and the system parameters was derived from these equations. The result is a polynomial function of the system nonlinear parameters as given by

$$Y(j\omega)=\sum_{j_1\cdots j_{s_N}}\gamma_{j_1\cdots j_{s_N}}(\omega)x_1^{j_1}\cdots x_{s_N}^{j_{s_N}} \tag{10}$$

where $x_1\cdots x_{s_N}$ are the elements in a set consisting of all the system nonlinear parameters, $j_1\cdots j_{s_N}$ are nonnegative integers, and $\gamma_{j_1\cdots j_{s_N}}(\omega)$ represents the coefficient of the term $x_1^{j_1}\cdots x_{s_N}^{j_{s_N}}$ which is a function of frequency variable and depends on the system linear parameters. Equation (10) was referred to as the output frequency response function (OFRF) of system (1). In order to conduct an OFRF based nonlinear system analysis and design, the first step is to determine the detailed structure of the OFRF to reveal which of the system nonlinear parameters is actually in the polynomial form expression (10) and how these parameters literally consists of the terms of the polynomial. To solve this basic problem more effectively, the OFRF is studied by using the parametric characteristic analysis in the next section.

3 Parametric characteristic analysis of the OFRF

In this section, some notations and a novel operator are first introduced. Then the parametric characteristic analysis is performed for the GFRFs. Finally, the parametric characteristics of the OFRF for system (1) are established, which provide an effective approach for the determination of the OFRF.

3.1 Notations and a novel coefficient extraction operator

Define the $p+q$th degree nonlinear parameter vector as

$$C_{p,q} = [c_{p,q}(0,\cdots,0), c_{p,q}(0,\cdots,1), \cdots, c_{p,q}(\underbrace{K,\cdots,K}_{p+q=m})]$$

which includes all the nonlinear parameters of the form $c_{p,q}(.)$ with nonlinear degree $p+q$ in equation (1). Note that C_{pq} can also be regarded as a set of the $(p+q)$th degree nonlinear parameters of the form $c_{p,q}(.)$.

Consider a series which can be written as

$$H_{CF} = c_1 f_1 + c_2 f_2 + \cdots + c_{|C|} f_{|C|}$$

where the coefficients $c_i \in \Re$ (i=1,…, $|C|$), $\Re$ denotes all the real numbers, C=[c_1,c_2,…, $c_{|C|}$], $|C|$ denotes the dimension of vector C, $f_i \in \psi$ (i=1,…,$|C|$) are real or complex valued functions, ψ denotes a set of real or complex valued scalar functions, and F=[f_1,f_2,…, $f_{|C|}$]$^{\mathrm{T}}$.

Define a **Coefficient Extraction** operator $CE: \psi \to \Re^{|C|}$ such that for any

$$H_{CF} = c_1 f_1 + c_2 f_2 + \cdots + c_{|C|} f_{|C|} \in \psi$$

$CE(H_{CF}) = C \in \Re^{|C|}$, where $\Re^{|C|}$ is the $|C|$-dimensional real vector space. This operator has the following properties acting as operation rules:

(1) Reduced vectorized sum "$\oplus$".
$CE(H_{C_1F_1} + H_{C_2F_2}) = CE(H_{C_1F_1}) \oplus CE(H_{C_2F_2}) = C_1 \oplus C_2 = [C_1, C_2']$, where C_2' is a reduced vector of C_2 which include all the elements in C_2 except the same elements as those in C_1.

(2) Reduced Kronecker product "$\otimes$".
$CE(H_{C_1F_1} \cdot H_{C_2F_2}) = CE(H_{C_1F_1}) \otimes CE(H_{C_2F_2}) = C_1 \otimes C_2$, and "reduced" here means that there are no repetitive components in $C_1 \otimes C_2$.

(3) Invariant.
(a) $CE(\alpha \cdot H_{CF}) = CE(H_{CF})$, $\forall \alpha \in \Re$ which is not a parameter of concern;
(b) $CE(H_{CF_1} + H_{CF_2}) = CE(H_{C(F_1+F_2)}) = C$

(4) Unitary. $\forall H_{CF}$ is not a function of c_i for i=1…n, $CE(H_{CF}) = 1$.
Obviously, when there is a unitary 1 in $CE(H_{CF})$, there is a constant term in the corresponding series H_{CF} which has no relation with the coefficients c_i (for i=1…n).

(5) Inverse. $CE^{-1}(C)=H_{CF}$.

(6) $CE(H_{C_1F_1}) \approx CE(H_{C_2F_2})$ if the elements of C_1 are the same as those of C_2, where "$\approx$" means equivalence, *i.e.*, both series are in fact the same result.

From property (6), it is known that the CE operator is also commutative and associative considering the order of $c_i f_i$ in the series has no effect on the value of a function series H_{CF}, for instance, $CE(H_{C_1F_1} + H_{C_2F_2}) = C_1 \oplus C_2 \approx CE(H_{C_2F_2} + H_{C_1F_1}) = C_2 \oplus C_1$. It should be emphasized that the coefficient extractor CE is a coefficient oriented operator. That is, only the concerned coefficients involved in a series are extracted after applying the operator. Hence, the operation result is different for different purposes. Moreover, for convenience, let $\underset{(*)}{\otimes}(\cdot)$ and $\underset{(*)}{\oplus}(\cdot)$ denote the multiplication and addition in terms of the reduced Kronecker product "$\otimes$" and vectorized sum "$\oplus$" for the non-repetitive (.)'s satisfying (*), respectively; and denote $\overset{k}{\underset{i=1}{\otimes}} C_{pq} = C_{pq} \otimes \cdots \otimes C_{pq}$ simply as C_{pq}^k.

According to the definition of the CE operator, the parametric characteristics of the GFRFs can be obtained by directly replacing the operations "+" and "." in the recursive algorithm (7-9) with "$\oplus$" and "$\otimes$"respectively, and neglecting the corresponding complex valued functions of frequency which are independent of the nonlinear parameters. In order to derive the parametric characteristics of the OFRF, the GFRFs are first investigated by the parametric characteristic analysis in the following section.

3.2 Parametric characteristics of the GFRFs

This is the first step to derive the parametric characteristics of the OFRF.

3.2.1 Derivation of the GFRFs' parametric characteristics

The parametric characteristics of the GFRFs of equation (1) can be obtained by directly applying the CE operator as follows.

Applying the CE operator to equation (7) for the nonlinear parameters yields

$$
\begin{aligned}
CE(H_n(j\omega_1,\cdots,j\omega_n)) &= CE(L(n)\cdot H_n(j\omega_1,\cdots,j\omega_n)) = CE\left(\sum_{k_1,k_n=0}^{K} c_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n}\right) \\
&+ CE\left(\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}(k_1,\cdots,k_{p+q})(j\omega_{n-q+1})^{k_{n-q+1}}\cdots(j\omega_{p+q})^{k_{p+q}} H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})\right) \\
&+ CE\left(\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K} c_{p,0}(k_1,\cdots,k_p) H_{n,p}(j\omega_1,\cdots,j\omega_n)\right) \\
&= \underset{k_1,k_n=0}{\overset{K}{\oplus}} c_{0,n}(k_1,\cdots,k_n) \oplus \underset{q=1}{\overset{n-1}{\oplus}}\underset{p=1}{\overset{n-q}{\oplus}}\underset{k_1,k_{p+q}=0}{\overset{K}{\oplus}} c_{p,q}(k_1,\cdots,k_{p+q}) \otimes CE(H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})) \\
&\quad \oplus \underset{p=2}{\overset{n}{\oplus}}\underset{k_1,k_p=0}{\overset{K}{\oplus}} c_{p,0}(k_1,\cdots,k_p) \otimes CE(H_{n,p}(j\omega_1,\cdots,j\omega_n)) \\
&= C_{0,n} \oplus \underset{q=1}{\overset{n-1}{\oplus}}\underset{p=1}{\overset{n-q}{\oplus}} C_{p,q} \otimes CE(H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})) \oplus \underset{p=2}{\overset{n}{\oplus}} C_{p,0} \otimes CE(H_{n,p}(j\omega_1,\cdots,j\omega_n))
\end{aligned}
$$

which can be rewritten as

$$
CE(H_n(j\omega_1,\cdots,j\omega_n)) = C_{0,n} \oplus \left(\underset{q=1}{\overset{n-1}{\oplus}}\underset{p=1}{\overset{n-q}{\oplus}} C_{p,q} \otimes CE(H_{n-q,p}(\cdot))\right) \oplus \left(\underset{p=2}{\overset{n}{\oplus}} C_{p,0} \otimes CE(H_{n,p}(\cdot))\right) \quad (11a)
$$

Applying the CE operator to equation (8) yields

$$
\begin{aligned}
CE(H_{n,p}(j\omega_1,\cdots,j\omega_n)) &= CE\left(\sum_{i=1}^{n-p+1} H_i(j\omega_1,\cdots,j\omega_i) H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{k_p}\right) \\
&= \underset{i=1}{\overset{n-p+1}{\oplus}} CE(H_i(j\omega_1,\cdots,j\omega_i)) \otimes CE(H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n))
\end{aligned}
$$

which can be written as

$$CE(H_{n,p}(\cdot)) = \mathop{\oplus}_{i=1}^{n-p+1} CE(H_i(\cdot)) \otimes CE(H_{n-i,p-1}(\cdot)) \tag{11b}$$

Applying the CE operator to equation (9) yields

$$\begin{aligned} CE(H_{n,1}(j\omega_1,\cdots,j\omega_n)) &= CE(H_n(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{k_1}) \\ &= CE(H_n(j\omega_1,\cdots,j\omega_n) \end{aligned}$$

which can be written as

$$CE(H_{n,1}(\cdot)) = CE(H_n(\cdot))) \tag{11c}$$

Moreover, by applying the CE operator to equation (9), (11b) can also be written as

$$CE(H_{n,p}(\cdot)) = \mathop{\oplus}_{\substack{r_1\cdots r_p=1 \\ \sum r_i=n}}^{n-p+1} \mathop{\otimes}_{i=1}^{p} CE(H_{r_i}(\cdot)) \tag{11d}$$

Note that when n=1, $H_1(j\omega_1)$ has no relation with the nonlinear parameters of equation (1), thus it follows from the definition of the CE operator that $CE(H_1(\cdot))=1$. From equations (11a-d), the nonlinear parameters involved in the nth order GFRF $H_n(j\omega_1,\cdots,j\omega_n)$ can be determined recursively. The following example is provided to make an illustration of the results above.

Example 1. Computation of the parametric characteristics of the GFRFs of system (1) up to 3rd order according to (11a-d). For n=2,

$$\begin{aligned} CE(H_2(j\omega_1,j\omega_2)) &= C_{0,2} \oplus \mathop{\oplus}_{q=1}^{1} \mathop{\oplus}_{p=1}^{2-q} C_{p,q} \otimes CE(H_{2-q,p}(\cdot)) \oplus \mathop{\oplus}_{p=2}^{2} C_{p,0} \otimes CE(H_{2,p}(\cdot)) \\ &= C_{0,2} \oplus C_{1,1} \otimes CE(H_{1,1}(\cdot)) \oplus C_{2,0} \otimes CE(H_{2,2}(\cdot)) \\ &= C_{0,2} \oplus C_{1,1} \otimes 1 \oplus C_{2,0} \otimes \left(\mathop{\oplus}_{\substack{r_1\cdots r_2=1 \\ \sum r_i=2}}^{1} \mathop{\otimes}_{i=1}^{2} CE(H_{r_i}(\cdot)) \right) \\ &= C_{0,2} \oplus C_{1,1} \otimes 1 \oplus C_{2,0} \otimes \left(CE(H_1(\cdot)) \otimes CE(H_1(\cdot))\right) \\ &= C_{0,2} \oplus C_{1,1} \oplus C_{2,0} \end{aligned} \tag{12}$$

where equations (11cd) are used in the third and fourth equalities. From the definition of the CE operator, equation (12) implies that there exist complex valued functions of frequencies $f_{2i}(j\omega_1,j\omega_2)$, i=1,2,3,…,$D_2$, where D_2 is the dimension of $CE(H_2(j\omega_1,j\omega_2))$, such that

$$\begin{aligned} H_2(j\omega_1,j\omega_2) &= CE(H_2(j\omega_1,j\omega_2)) \cdot \begin{bmatrix} f_{21}(\cdot) & f_{21}(\cdot) & \cdots & f_{2D_2}(\cdot) \end{bmatrix}^T \\ &= \left(C_{0,2} \oplus C_{1,1} \oplus C_{2,0}\right) \cdot f_2(j\omega_1,j\omega_2) \end{aligned} \tag{13}$$

where $f_2(j\omega_1,j\omega_2) = \begin{bmatrix} f_{21}(\cdot) & f_{21}(\cdot) & \cdots & f_{2D_2}(\cdot) \end{bmatrix}^T$. Similarly, for n=3

$$CE(H_3(j\omega_1,\cdots,j\omega_3)) = C_{0,3} \oplus \mathop{\oplus}_{q=1}^{2}\mathop{\oplus}_{p=1}^{3-q} C_{p,q} \otimes CE(H_{3-q,p}(\cdot)) \oplus \mathop{\oplus}_{p=2}^{3} C_{p,0} \otimes CE(H_{3,p}(\cdot))$$
$$= C_{0,3} \oplus \mathop{\oplus}_{q=1}^{2}\mathop{\oplus}_{p=1}^{3-q} C_{p,q} \otimes CE(H_{3-q,p}(\cdot)) \oplus \mathop{\oplus}_{p=2}^{3} C_{p,0} \otimes CE(H_{3,p}(\cdot))$$
$$= C_{0,3} \oplus \left(C_{1,1} \otimes CE(H_{2,1}(\cdot)) \oplus C_{2,1} \otimes CE(H_{2,2}(\cdot)) \oplus C_{1,2} \otimes CE(H_{1,1}(\cdot))\right)$$
$$\oplus \left(C_{2,0} \otimes CE(H_{3,2}(\cdot)) \oplus C_{3,0} \otimes CE(H_{3,3}(\cdot))\right)$$
$$= C_{0,3} \oplus \left(C_{1,1} \otimes CE(H_2(\cdot)) \oplus C_{2,1} \otimes CE(H_1(\cdot)) \otimes CE(H_1(\cdot)) \oplus C_{1,2} \otimes CE(H_1(\cdot))\right)$$
$$\oplus \left(C_{2,0} \otimes CE(H_2(\cdot)) \otimes CE(H_1(\cdot)) \oplus C_{3,0} \otimes CE(H_1(\cdot)) \otimes CE(H_1(\cdot)) \otimes CE(H_1(\cdot))\right)$$
$$= C_{0,3} \oplus \left(C_{1,1} \otimes CE(H_2(\cdot)) \oplus C_{2,1} \oplus C_{1,2}\right) \oplus \left(C_{2,0} \otimes CE(H_2(\cdot)) \oplus C_{3,0}\right)$$
$$= C_{0,3} \oplus \left(C_{1,1} \otimes (C_{0,2} \oplus C_{1,1} \oplus C_{2,0}) \oplus C_{2,1} \oplus C_{1,2}\right) \oplus \left(C_{2,0} \otimes (C_{0,2} \oplus C_{1,1} \oplus C_{2,0}) \oplus C_{3,0}\right)$$
$$= C_{0,3} \oplus C_{1,1} \otimes C_{0,2} \oplus C_{1,1}{}^2 \oplus C_{1,1} \otimes C_{2,0} \oplus C_{2,1} \oplus C_{1,2} \oplus C_{2,0} \otimes C_{0,2} \oplus C_{2,0}{}^2 \oplus C_{3,0} \quad (14)$$

In the derivation of equation (14), equations (11cd) and (12) and the properties of CE are used. Obviously, there also exists a complex valued function vector $f_3(j\omega_1, j\omega_2, j\omega_3) = \left[f_{31}(\cdot) \quad f_{31}(\cdot) \quad \cdots \quad f_{3D_3}(\cdot)\right]^T$ of frequencies such that

$$H_3(j\omega_1, j\omega_2, j\omega_3) = CE(H_3(j\omega_1, j\omega_2, j\omega_3)) \cdot f_2(j\omega_1, j\omega_2, j\omega_3) \quad (15)$$

Equations (12) and (14) show clearly that which of the nonlinear parameters in equation (1) contribute to the 2nd and 3rd order GFRFs and how the contributions are made. In addition, equations (13) and (15) indicate that the GFRFs can be expressed as an explicit polynomial function of the system nonlinear parameters. ■

From the definition of the CE operator and the parameter characteristics of the GFRFs in (11), the following proposition can be established following the discussions in Example 1.

Proposition 1. There exists a complex valued function vector $f_n(j\omega_1,\cdots,j\omega_n)$ with an appropriate dimension, such that

$$H_n(j\omega_1,\cdots,j\omega_n) = CE\left(H_n(j\omega_1,\cdots,j\omega_n)\right) \cdot f_n(j\omega_1,\cdots,j\omega_n) \quad (16)$$

where $CE\left(H_n(j\omega_1,\cdots,j\omega_n)\right)$ can be recursively determined from (11a)-(11d). ■

Remark 1. It should be noted that $f_n(j\omega_1,\cdots,j\omega_n)$ in equation (16) is a function of frequency variables $\omega_1,\cdots,\omega_n$ and the linear parameters $c_{10}(.)$ and $c_{01}(.)$, but is independent of the nonlinear parameters in $CE\left(H_n(j\omega_1,\cdots,j\omega_n)\right)$. ■

Proposition 1 provides an explicit analytical relationship between the nonlinear parameters and the GFRFs of system (1) by the parametric characteristic analysis of the GFRF. From this explicit analytical expression of the GFRFs in Equation (16), the parametric characteristics of the OFRF of system (1) can be obtained.

3.2.2 Some further results

It can be noted from Example 1 that there are many repetitive terms and computations in equations (11a-d). In order to have a much deeper understanding of the parametric characteristics of the GFRFs and to simplify the computation of $CE(H_n(j\omega_1,\cdots,j\omega_n))$, the following results are established.

Lemma 1. The elements of CE($H_n(j\omega_1,\cdots,j\omega_n)$) include the nonlinear parameters in C_{0n} and all the non-repetitive monomial functions (of the nonlinear parameters) in $C_{pq}\otimes C_{p_1q_1}\otimes C_{p_2q_2}\otimes\cdots\otimes C_{p_kq_k}$, where the subscripts satisfy $p+q+\sum_{i=1}^{k}(p_i+q_i)=n+k$, $2\le p_i+q_i\le n-k$, $0\le k\le n-2$, $2\le p+q\le n-k$ and $1\le p\le n-k$.
Proof. See Appendix. ■

Remark 2. Lemma 1 provides detailed information about what nonlinear parameters are involved in the analytical description of the GFRF $H_n(j\omega_1,\cdots,j\omega_n)$ and how these parameters have their effects on the GFRFs. It also provides a very effective method for the determination of $CE(H_n(j\omega_1,\cdots,j\omega_n))$. It is easy to directly write out the elements of $CE(H_n(j\omega_1,\cdots,j\omega_n))$ according to Lemma 1 without any recursive computations. This can be performed by a simple computer program which will be discussed in a future study. ■

To demonstrate the significance of Lemma 1, consider the following example.

Example 2. Consider the 2nd order GFRF. Then $p+q+\sum_{i=1}^{k}(p_i+q_i)=2+k$, $0\le k\le 2-2$ =0. Therefore, all the subscript combinations for k=0 is (p,q): (0,2);(1,1),(2,0), corresponding to the nonlinear parameters: $C_{0,2}$, $C_{1,1}$, $C_{2,0}$. Thus $CE(H_2(j\omega_1,j\omega_2))$= $C_{0,2}$+$C_{1,1}$+$C_{2,0}$. This is consistent with equation (12).

Consider the 3rd order GFRF, then $p+q+\sum_{i=1}^{k}(p_i+q_i)=3+k$, $0\le k\le 3-2=1$.
When k=0, the involved nonlinear parameters are $C_{0,3}$, $C_{1,2}$, $C_{2,1}$, $C_{3,0}$;
When k=1, $p+q+p_1+q_1=3+1=4$, which has the following non-repetitive combinations

(p,q,p_1,q_1):(1,1,2,0), (1,1,1,1),(1,1,0,2),(2,0,0,2),(2,0,2,0)

thus the involved nonlinear parameter monomials are:

$$C_{1,1}\circ C_{2,0},\ C_{1,1}\circ C_{1,1},\ C_{1,1}\circ C_{0,2},\ C_{2,0}\circ C_{0,2},\ C_{2,0}\circ C_{2,0}$$

Hence $CE(H_3(j\omega_1,\cdots,j\omega_3))$= $C_{0,3}+C_{1,2}+C_{2,1}+C_{3,0}+C_{1,1}\circ C_{2,0}+C_{1,1}\circ C_{1,1}+C_{1,1}\circ C_{0,2}$
$+C_{2,0}\circ C_{0,2}+C_{2,0}\circ C_{2,0}$

The result is consistent with Equation (14). ■

The following result follows from Lemma 1, which can be used to simplify equation (11a).

Lemma 2. $CE(H_{n,p}(\cdot))=CE(H_{n-p+1}(\cdot))$
Proof. See the Appendix. ■

Based on the discussions above, a simplified formula for the computation of the parametric characteristics of the nth-order GFRF can be obtained.

Proposition 2. For n>1,

$$CE(H_n(j\omega_1,\cdots,j\omega_n))$$
$$= C_{0,n} \oplus \left(\mathop{\oplus}_{q=1}^{n-1} \mathop{\oplus}_{p=1}^{n-q} C_{p,q} \otimes CE(H_{n-q-p+1}(\cdot)) \right) \oplus \left(C_{n,0} \oplus \mathop{\oplus}_{p=2}^{\lfloor n+1/2 \rfloor} C_{p,0} \otimes CE(H_{n-p+1}(\cdot)) \right) \quad (17)$$

where $\lfloor \cdot \rfloor$ is to take the integer part.

Proof. Using Lemma 2, equation (11a) now can be written as (n>1)

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = C_{0,n} \oplus \left(\mathop{\oplus}_{q=1}^{n-1} \mathop{\oplus}_{p=1}^{n-q} C_{p,q} \otimes CE(H_{n-q-p+1}(\cdot)) \right) \oplus \left(\mathop{\oplus}_{p=2}^{n} C_{p,0} \otimes CE(H_{n-p+1}(\cdot)) \right)$$

Considering the symmetry of the last term of this equation, Equation (17) follows. This completes the proof. ■

Obviously, it is much simpler to determine the parametric characteristics of the GFRFs by equation (17) than by equations (11a-d).

3.3 Parametric characteristics of the OFRF

From the parametric characteristics of the GFRFs above, the following results for the output spectra of system (1) can be achieved.

Proposition 3. Assume system (1) is stable at zero equilibrium and can be approximated by a Volterra series of a finite order. Then there exist a series of complex valued function vectors $\overline{F}_n(j\omega)$ (n=1,2,…,N) of frequency variable ω with appropriate dimensions such that the OFRF of system (1) can be expressed as

$$Y(j\omega) = \left(\mathop{\oplus}_{n=1}^{N} CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big) \right) \cdot \left[\overline{F}_1(j\omega)^T \quad \overline{F}_2(j\omega)^T \quad \cdots \quad \overline{F}_N(j\omega)^T \right]^T \quad (18)$$

where $\overline{F}_n(j\omega) = \frac{1}{\sqrt{n}(2\pi)^{n-1}} \int_{\omega_1+\cdots+\omega_n=\omega} f_n(j\omega_1,\cdots,j\omega_n) \cdot \prod_{i=1}^{n} U(j\omega_i) d\sigma_\omega$. If the input of system (1) is the multi-tone signal (5), then there exist a series of complex valued function vectors $\overline{\overline{F}}_n(j\omega)$ (n=1,2,…,N) of frequency variable ω with appropriate dimensions, such that the OFRF of system (1) can be expressed as

$$Y(j\omega) = \left(\mathop{\oplus}_{n=1}^{N} CE\big(H_n(j\omega_{k_1},\cdots,j\omega_{k_n})\big) \right) \cdot \left[\overline{\overline{F}}_1(j\omega)^T \quad \overline{\overline{F}}_2(j\omega)^T \quad \cdots \quad \overline{\overline{F}}_N(j\omega)^T \right]^T \quad (19)$$

where $\overline{\overline{F}}_n(j\omega) = \frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} f_n(j\omega_{k_1},\cdots,j\omega_{k_n}) \cdot F(\omega_{k_1})\cdots F(\omega_{k_n})$. The parametric characteristics of the OFRF is

$$CE(Y(j\omega)) = \mathop{\oplus}_{n=1}^{N} CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big) \quad (20)$$

Proof. From Proposition 1, substituting (16) into (3) yields

$$
\begin{aligned}
Y(j\omega) &= \sum_{n=1}^{N} \frac{1}{\sqrt{n}(2\pi)^{n-1}} \int_{\omega_1+\cdots+\omega_n=\omega} CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\cdot f_n(j\omega_1,\cdots,j\omega_n)\cdot \prod_{i=1}^{n} U(j\omega_i)d\sigma_\omega \\
&= \sum_{n=1}^{N} CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\cdot \frac{1}{\sqrt{n}(2\pi)^{n-1}} \int_{\omega_1+\cdots+\omega_n=\omega} f_n(j\omega_1,\cdots,j\omega_n)\cdot \prod_{i=1}^{n} U(j\omega_i)d\sigma_\omega \\
&= \sum_{n=1}^{N} CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\cdot \overline{F}_n(j\omega) \\
&= \left(\mathop{\oplus}_{n=1}^{N} CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\right)\cdot \left[\overline{F}_1(j\omega)^T \quad \overline{F}_2(j\omega)^T \quad \cdots \quad \overline{F}_N(j\omega)^T\right]^T
\end{aligned}
$$

Following a similar procedure, equation (19) can be obtained for the OFRF of equation (1) actuated by a multi-tone input function in (5). Moreover, (20) is obvious from equations (18-19). This completes the proof. ■

Equations (18) and (19) show that the OFRF of system (1) can now be expressed as an explicit polynomial function of the nonlinear parameters, and the specific form of the OFRF of system (1) is completely defined by its parametric characteristics in (20). Because the parametric characteristics defined by equation (20) can be readily determined using Proposition 2 or Lemma 1, the parametric characteristic analysis for the OFRF of system (1) provides an effective approach to the determination of the OFRF.

In order to demonstrate the use of Proposition 3, consider the following example.

Example 3. Consider a nonlinear system,

$$240\ddot{x} = -16000x - 296\dot{x} - c_1\dot{x}^3 - c_2\dot{x}^2 x + u(t) \tag{21}$$

subject to a sinusoidal input $u(t) = 100\sin(8.1t)$. (21) is a simple case of system (1) with M=3, K=2, $c_{10}(2) = 240$, $c_{10}(1) = 296$, $C_{10}(0) = 16000$, $c_{30}(111) = c_1$, $c_{30}(110) = c_2$, $c_{01}(0) = -1$, and all other parameters are zero.

In system (21), only nonlinear parameters in C_{30} are not zero, *i.e.*,

$$C_{30} = [c_{30}(110) \quad c_{30}(111)] = [c_2 \quad c_1]$$

In this case, equation (17) can be rewritten as (n>1)

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = C_{n,0} \oplus \mathop{\oplus}_{p=2}^{\lfloor n+1/2 \rfloor} C_{p,0} \otimes CE(H_{n-p+1}(\cdot)) \tag{22}$$

Since only C_{30} is not zero, it can be shown from equation (22) that

$CE(H_{2k}(j\omega_1,\cdots,j\omega_n)) = 0$ and $CE(H_{2k+1}(j\omega_1,\cdots,j\omega_n)) = C_{30}^k$, for k=1,2, 3,…

From equation (20) in Proposition 3,

$$CE(X(j\omega)) = \mathop{\oplus}_{n=1}^{N} CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big) = 1 \oplus C_{30} \oplus C_{30}^2 \oplus C_{30}^3 \oplus \cdots \oplus C_{30}^{\lfloor N-1/2 \rfloor} \tag{23a}$$

so that

$$X(j\omega) = \left(1 \oplus C_{30} \oplus C_{30}^2 \oplus C_{30}^3 \oplus \cdots \oplus C_{30}^{\lfloor N-1/2 \rfloor}\right)\cdot \left[\overline{\overline{F}}_1(j\omega)^T \quad \overline{\overline{F}}_2(j\omega)^T \quad \cdots \quad \overline{\overline{F}}_N(j\omega)^T\right]^T \tag{23b}$$

where $\lfloor N-1/2 \rfloor$ is to take the integer part. From (23) $CE(X(j\omega))$ can be readily obtained. For instance, for N=5

$CE(X(j\omega)) = 1 \oplus C_{30} \oplus C_{30}^2 \oplus C_{30}^3 \oplus \cdots \oplus C_{30}^{\lfloor N-1/2 \rfloor} =$ [1,c_2,c_1,c_2^2,c_2c_1,c_1^2,c_2^3,$c_2^2c_1$,$c_2c_1^2$,c_1^3,c_2^4, $c_2^3c_1$,$c_2^2c_1^2$,$c_2c_1^3$,c_1^4,c_2^5, $c_2^4c_1$, $c_2^3c_1^2$,$c_2^2c_1^3$,$c_2c_1^4$,c_1^5]. (23c)

Therefore an explicit analytical expression for the OFRF $X(j\omega)$ in terms of the system nonlinear parameters c_1 and c_2 are obtained as given by (23b-c). ■

3.4 A numerical method for the determination of the OFRF

As discussed in the last section, the detailed polynomial structure of the OFRF can be determined definitely by the parametric characteristic analysis. Based on structure of the OFRF, a numerical method can be used to determine the OFRF of system (1).

Denote the dimension of $CE(Y(j\omega))$ as D_Y which can be known from the parametric characteristics of the OFRF. Let $\phi = [a_1 \quad a_2 \quad \cdots \quad a_{D_Y}] = CE(Y(j\omega))$ where a_i, i=1…D_Y, are the monomial functions of the system nonlinear parameters known from the OFRF structure using Proposition 3, and

$$h(j\omega) = \begin{bmatrix} h_1(j\omega) & h_2(j\omega) & \cdots & h_{D_Y}(j\omega) \end{bmatrix} = \begin{bmatrix} \widetilde{F}_1(j\omega)^T & \widetilde{F}_2(j\omega)^T & \cdots & \widetilde{F}_N(j\omega)^T \end{bmatrix}^T$$

where $\widetilde{F}_i(j\omega) = \overline{F}_i(j\omega)$ when the input is a general input and $\widetilde{F}_i(j\omega) = \overline{\overline{F}}_i(j\omega)$ when the input is a muli-tone signal. Then equation (18) can be rewritten as

$$Y(j\omega) = \phi \cdot h(j\omega) \tag{24}$$

In order to obtain the OFRF, $h(j\omega)$ in (24) is needed to be determined. This can be achieved by following a numerical method in Lang et al. (2006). The basic idea of this method is to perform simulations or experimental tests on the system under N_Y (>0) different sets of nonlinear parameters ϕ_1, ϕ_2, … ϕ_{N_Y} to obtain N_Y output frequency responses of the system $Y(j\omega)_1$, $Y(j\omega)_2$, … $Y(j\omega)_{N_Y}$ for a considered input excitation. Thus from (24), it is known that

$$\Phi \cdot h(j\omega) = Y_Y \tag{25}$$

where $\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N_Y} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{D_Y 1} \\ a_{12} & a_{22} & \cdots & a_{D_Y 2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1N_Y} & a_{1N_Y} & & a_{D_Y N_Y} \end{bmatrix}$, $Y_Y = \begin{bmatrix} Y(j\omega)_1 \\ Y(j\omega)_2 \\ \vdots \\ Y(j\omega)_{D_Y} \end{bmatrix}$. Consequently $h(j\omega)$ can be obtained as

$$h(j\omega) = \left(\Phi^T \cdot \Phi\right)^{-1} \cdot \Theta^T \cdot Y_Y \tag{26}$$

Therefore, the OFRF of system (1) can be well determined. From this explicit analytical expression of the system output frequency response, the analysis and synthesis of system (1) in the frequency domain can be conducted. These will be illustrated in a simulation study in the next section.

5 Simulation studies

Consider the nonlinear system (21) again, but the output of interest in this simulation study is

$$y=16000x+296\dot{x}+c_1\dot{x}^3+c_2\dot{x}^2x \quad (27)$$

which is a more complicated function of the system states.

Since it can be shown that $CE(Y(j\omega))=CE(X(j\omega))$ (Jing, *et. al.* 2006a), the theoretical results for $X(j\omega)$ still hold for $Y(j\omega)$. Therefore, it follows from Example 3 that

$$CE(Y(j\omega))=\mathop{\oplus}_{n=1}^{N} CE\left(H_n(j\omega_1,\cdots,j\omega_n)\right)=1\oplus C_{30}\oplus C_{30}^2\oplus C_{30}^3\oplus\cdots\oplus C_{30}^{\lfloor N-1/2\rfloor} \quad (28a)$$

and

$$Y(j\omega)=\left(1\oplus C_{30}\oplus C_{30}^2\oplus C_{30}^3\oplus\cdots\oplus C_{30}^{\lfloor N-1/2\rfloor}\right)\cdot\left[\overline{\overline{F}}_1(j\omega)^T \quad \overline{\overline{F}}_2(j\omega)^T \quad \cdots \quad \overline{\overline{F}}_N(j\omega)^T\right]^T \quad (28b)$$

For clarity of illustration, consider the simpler case of c_2=0, *i.e.*, $C_{30}=c_1$. When N=21, it can be shown from (28ab) that

$$CE(Y(j\omega))=\begin{bmatrix}1 & c_1 & c_1^2 & c_1^3 & \cdots & c_1^{10}\end{bmatrix}$$

and

$$Y(j\omega)=\phi\cdot h(j\omega)=\begin{bmatrix}1 & c_1 & c_1^2 & c_1^3 & \cdots & c_1^{10}\end{bmatrix}\cdot\begin{bmatrix}h_1(j\omega) & h_2(j\omega) & \cdots & h_{11}(j\omega)\end{bmatrix}^T$$

In order to determine $h(j\omega)$ in the above equation, simulation studies are carried out for 11 different values of c_1 as c_1=0.5,50,100,500, 800,1200, 1800, 2600, 3500, 4500, 5000, to produce 11 corresponding output responses. The FFT results of these responses at the system driving frequency ω_0=8.1 rad/s were obtained as

Y_Y=[(3.355387229685395e+002)-9.144123368552089e+000i,
(3.311400634432650e+002)-8.791324203084603e+000i,
(3.270304131496312e+002)-8.453482697096458e+000i,
(3.020996757260479e+002)-6.232073185455284e+000i,
(2.889224705331136e+002)-4.937579404570077e+000i,
(2.753247618357106e+002)-3.513785421406298e+000i,
(2.599814606290563e+002)-1.799344961942028e+000i,
(2.449407272303421e+002)-7.146831574203648e-003i,
(2.322782654921158e+002)+1.587748875652816e+000i,
(2.213884644417550e+002)+3.022652971105967e+000i,
(2.168038059608033e+002)+3.644341792781596e+000i]

Then from (26), $h(j\omega_0)$ was determined as

$h(j\omega_0)$=[3.355850061999765e+002 +9.147787717329777e+000i,
-0.09260545518186 - 0.00733079515829i
7.802545290190465e-005 +4.196941358069068e-006i
-8.171412395831490e-008 -3.472552369765044e-009i
7.983194136013857e-011 +2.975659825236403e-012i
-6.014819558373321e-014 -2.095287675780629e-015i
3.139462445085954e-017 +1.055716258995395e-018i
-1.065920417366710e-020 -3.515136904764629e-022i
2.214834610655676e-024 +7.220197982843919e-026i
-2.536564081104798e-028 -8.209302192093296e-030i
1.219975622824295e-032 +3.929425356306088e-034i].

Consequently, the OFRF of nonlinear system (27) at frequency ω_0 was obtained as

$$Y(j\omega_0)=\begin{bmatrix}1 & c_1 & c_1^2 & c_1^3 & \cdots & c_1^{10}\end{bmatrix}\cdot\begin{bmatrix}h_1(j\omega_0) & h_2(j\omega_0) & \cdots & h_{11}(j\omega_0)\end{bmatrix}^T \quad (29)$$

From equation (29), the effect of the nonlinear parameter c_1 on the system output frequency response at frequency ω_0 can readily be analysed. Figure 1 shows a comparison of the magnitudes of the output spectrum evaluated by (29) and their real values under different values of the nonlinear parameter c_1. Clearly, an excellent match is achieved. Furthermore, the frequency domain analysis and design of system (21) to achieve a desired output response y(t) can now be conducted from (29). The idea is straightforward. Given a desired output spectrum Y^* at frequency ω_0, the nonlinear parameter c_1 can be optimized using (29) such that the difference $|Y(j\omega_0)-Y^*|$ can be made as small as possible. ■

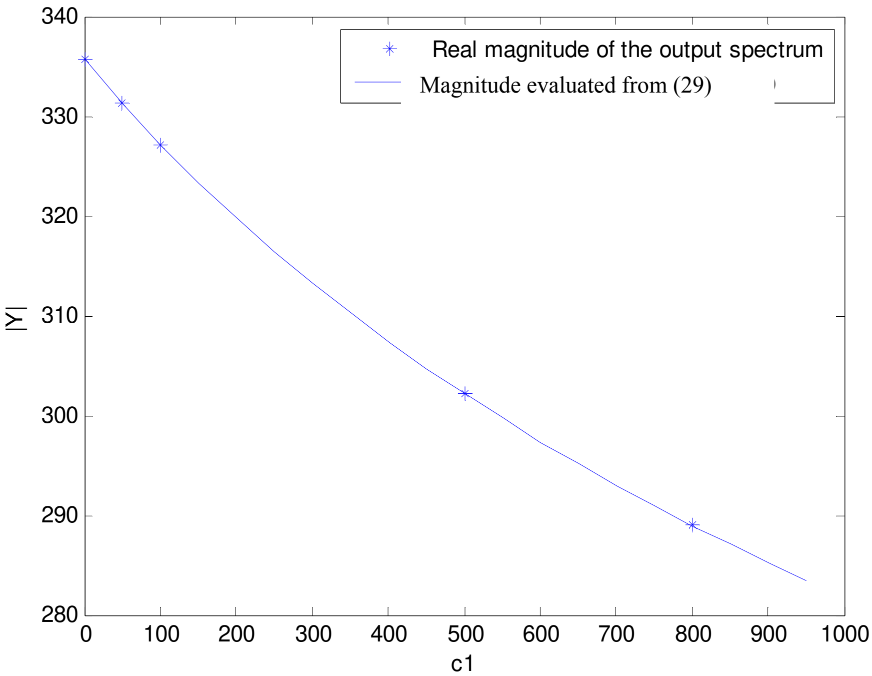

Figure 1 Relationship between the output spectrum and nonlinear parameter c_1

6 Conclusions

The parametric characteristics of the output frequency response function (OFRF) of nonlinear systems described by a polynomial form differential equation model have been established. Based on these results, the OFRF with its detailed structure for this class of nonlinear systems can explicitly be determined up to any high orders. The OFRF concept provides an important basis for the analysis and design of nonlinear systems in the frequency domain. The present study solves an important and basic problem associated with the OFRF based nonlinear system analysis and design. The results should be considerately significant for the frequency domain study of nonlinear systems, and for the application of the nonlinear system frequency domain approach in engineering practice.

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Appendix: Proofs

PROOF OF LEMMA 1:

$C_{0,n}$ is the first term in equation (11a). For clarity, consider a simpler case that there is only output nonlinearities in (11a), then (11a) is reduced to only the last term of equation (11a), *i.e.*, $\mathop{\oplus}\limits_{p=2}^{n} C_{p,0} \otimes CE(H_{n,p}(\cdot)) = \mathop{\oplus}\limits_{p=2}^{n} C_{p,0} \otimes \mathop{\oplus}\limits_{\substack{r_1 \cdots r_p = 1 \\ \sum r_i = n}}^{n-p+1} \mathop{\otimes}\limits_{i=1}^{p} CE(H_{r_i}(\cdot))$. Note that $\mathop{\oplus}\limits_{\substack{r_1 \cdots r_p = 1 \\ \sum r_i = n}}^{n-p+1} \mathop{\otimes}\limits_{i=1}^{p} CE(H_{r_i}(\cdot))$ includes all the combinations of $(r_1, r_2, \ldots, r_p)$ satisfying $\sum_{i=1}^{p} r_i = n$, $1 \le r_i \le n-p+1$, and $2 \le p \le n$. Moreover, CE($H_1(\cdot)$)=1 since there are no nonlinear parameters in it, and any repetitive combinations have no contribution. Hence, $\mathop{\oplus}\limits_{\substack{r_1 \cdots r_p = 1 \\ \sum r_i = n}}^{n-p+1} \mathop{\otimes}\limits_{i=1}^{p} CE(H_{r_i}(\cdot))$must include all the possible non-repetitive combinations of $(r_1, r_2, \ldots, r_k)$ satisfying $\sum_{i=1}^{k} r_i = n-p+k$, $2 \le r_i \le n-p+1$ and $1 \le k \le p$. So does CE($H_n(j\omega_1, \cdots, j\omega_n)$). Each of the subscript combinations corresponds to a monomial of the involved nonlinear parameters. Thus, including the term $C_{p,0}$ and considering the range of each variable (*i.e.*, r_i, p, and k), CE($H_n(j\omega_1, \cdots, j\omega_n)$) must include all the possible non-repetitive monomial functions of the nonlinear parameters of the form $C_{p0} \otimes C_{r_1 0} \otimes C_{r_2 0} \otimes \cdots \otimes C_{r_k 0}$ satisfying $p + \sum_{i=1}^{k} r_i = n+k$, $2 \le r_i \le n-k$, $0 \le k \le n-2$ and $2 \le p \le n-k$. When the other types of nonlinearities are considered, just extend the results above to a more general case that the nonlinear parameters appear in the form $C_{pq} \otimes C_{p_1 q_1} \otimes C_{p_2 q_2} \otimes \cdots \otimes C_{p_k q_k}$ and the subscripts satisfy $p + q + \sum_{i=1}^{k} (p_i + q_i) = n+k$, $2 \le p_i + q_i \le n-k$, $0 \le k \le n-2$, $2 \le p+q \le n-k$ and $1 \le p \le n-k$. Hence, the proposition is proved. ■

PROOF OF LEMMA 2:

According to Lemma 1, $CE(H_{n-p+1}(\cdot))$ includes all the terms $C_{p_1 q_1} \otimes C_{p_2 q_2} \otimes \cdots \otimes C_{p_k q_k}$ satisfying

$$\sum_{i=1}^{k} (p_i + q_i) = n-p+1+k-1 = n-p+k \, , \; 2 \le p_i + q_i \le n-p-k+2 \, , \; 0 \le k-1 \le n-p-1 \, ,$$

and at least one p_i>0. Equation (11b) can be rewritten as

$$CE\left(H_{n,p}(j\omega_1,\cdots,j\omega_n)\right)=\underset{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}{\overset{n-p+1}{\oplus}}\ \overset{p}{\underset{i=1}{\otimes}}CE\left(H_{r_i}(\cdot)\right)=CE\left(H_{n-p+1}(\cdot)\right)\oplus\left(\underset{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}{\overset{n-p}{\oplus}}\ \overset{p}{\underset{i=1}{\otimes}}CE\left(H_{r_i}(\cdot)\right)\right)$$

Considering all the possible effective combinations of $(r_1,r_2,\ldots,r_p)$ in the second term on the right of the second equality in this equation, which can be written as

$$\underset{\substack{r_1\cdots r_p=2\\ \sum r_i=n-p+q'}}{\overset{n-p}{\oplus}}\left(CE(H_{r_1}(\cdot))\otimes CE(H_{r_2}(\cdot))\otimes\cdots\otimes CE(H_{r_{q'}}(\cdot))\right) \tag{A1}$$

As shown in the proof of Lemma 1, all the terms in (A1) satisfy $\sum_{i=1}^{q'} r_i = n-p+1+q'-1=n-p+q'$, $2\le r_i\le n-p-(q'-1)+1$ and $0\le q'-1\le p$, *i.e.*, $\sum_{i=1}^{q'}(p_i+q_i)=n-p+q'$, $2\le p_i+q_i\le n-p-q'+2$, and $0\le q'-1\le n-p-1$ corresponding to the nonlinear parameter monomials $C_{p_1q_1}\otimes C_{p_2q_2}\otimes\cdots\otimes C_{p_{q'}q_{q'}}$. Moreover, it can be noted from equation (11a) that the variable $p>0$. This implies that there at least is one $p_i>0$ from the nature of the recursive computation of equation (11a). Hence, the terms in (A1) are included in $CE\left(H_{n-p+1}(\cdot)\right)$. This completes the proof. ■

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