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The Parametric Characteristics of Frequency Response Functions for Nonlinear Systems

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Abstract: The characteristics of the frequency response functions of nonlinear systems can be revealed and analyzed through the analysis of the parametric characteristics of these functions. To achieve these objectives, a new operator is defined, and several fundamental and important results about the parametric characteristics of the frequency response functions of nonlinear systems are developed. These theoretical results provide a significant and novel insight into the frequency domain characteristics of nonlinear systems and circumvent a large amount of complicated integral and symbolic calculations which have previously been required to perform nonlinear system frequency domain analysis. Several new results for the analysis and synthesis of nonlinear systems are also developed. Examples are included to illustrate potential applications of the new results.

1 Introductions

Nonlinear systems are far more complex than linear systems, and can exhibit harmonics, complex inter-modulations and even chaos (Pearson 1994). In order to understand and unravel these complicated phenomena, many authors have studied the analysis of nonlinear systems in both the time domain and the frequency domain (Graham and McRuer 1961, Sastry 1999, Chua and Ng 1979, Rugh 1981).

The studies of nonlinear systems in the frequency domain are based on the concept of the generalized frequency response functions (GFRFs) (George 1959) which are defined as the multidimensional Fourier transformations of the kernel functions in the Volterra series. Many non-parametric algorithms have been derived to estimate the GFRFs of unknown non-linear systems from input output data (Brilliant 1958, Kim and Powers 1988, Bendat 1990, Nam and Powers 1994). Peyton-Jones and Billings (1989) derived a recursive algorithm to compute the GFRFs of discrete time nonlinear systems described by NARX (Nonlinear AutoRegressive model with eXogenous input) models. A similar result was developed in Billings and Peyton-Jones (1990) for continuous time nonlinear systems described by integro-differential equations. Swain and Billings (2001) extended these results to the case of MIMO nonlinear systems. The derivation of the GFRFs of nonlinear systems with mean level or DC terms was studied in Zhang *et al*. (1995). Based on these results, some important characteristics of the frequency response functions of nonlinear systems were developed (Yue *et al.* 2005). In Lang, and Billings (1996) and Lang, and Billings (1997), the output frequency response function and the corresponding characteristics of nonlinear systems were studied, respectively. The bound characteristics of the frequency response functions and energy transfer characteristics have also been studied and discussed (Zhang and Billings 1996, Billings and Lang 1996, Lang and Billings 2005).

Although significant results have been achieved, many problems remain unsolved regarding the characteristics of the GFRFs and the system output frequency response function, including how the frequency response functions are influenced by the parameters of the underlying system, and the connection to complex non-linear behaviours. The GFRFs are actually a sequence of multivariable functions defined in a high dimensional frequency space. The evaluation of the values of the GFRFs higher than fourth or fifth order can become hard due to the large amount of algebra or symbolic manipulations that are involved (Yue *et al.* 2005). Moreover, existing recursive algorithms for the computation of the GFRFs do not explicitly and simply reveal the analytical relationship between the time domain system model parameters and the system frequency response functions in a clear and straightforward manner. These inhibit the practical application of the existing theoretical results to a certain extent. Therefore, the development of new methods to circumvent the computational complexity and to more clearly reveal the characteristics of the frequency response functions is of great importance for the analysis and synthesis of nonlinear systems.

Previous results (Peyton-Jones and Billings 1989) show that for the NARX model the recursive computation procedure for the GFRFs involves the contribution of different model parameters to the system nonlinear characteristics of different orders. Hence, an alternative approach to analyse the characteristics of the frequency response functions and the effects of different types of nonlinearities on the system would involve study of the characteristics of the system model parameters on the frequency response functions. Therefore, a novel and effective coefficient extraction operator is defined in this paper. Using this new operator, several fundamental results relating to the parametric characteristics of the frequency response functions for nonlinear systems are derived. The parametric characteristics of the GFRFs and the system output spectrum can easily be achieved using a recursive algorithm which uses the system time domain model parameters without the need to determine individual GFRFs. The results reveal the explicit relationship between the system time domain model parameters and the system frequency response functions, and provide a significant and novel insight into the frequency domain characteristics of nonlinear systems. Important characteristics of the frequency response functions of nonlinear systems are also revealed and analyzed based on the new parametric method. The new results derived in the present study provide a new and effective approach which should be very useful for the analysis and synthesis of nonlinear systems in the frequency domain.

2 Frequency response functions of nonlinear systems

Considering the class of nonlinear systems which are stable at the zero equilibrium point and which can be approximated in the neighbourhood of the equilibrium point by the Volterra series

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \qquad (1)$$

where N is the maximum order of the series, and $h_n(\tau_1,\cdots,\tau_n)$ is a real valued function of $\tau_1,\cdots,\tau_n$ which is referred to as the nth order Volterra kernel. The frequency domain input-output description of the system can be obtained as (Lang and Billings 1996)

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \qquad (2)$$

where,

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \qquad (3)$$

is known as the nth order Generalized Frequency Response Function(GFRF).

When the system input is a multi-tone function described by

$$u(t)=\sum_{i=1}^{K}\left|F_i\right|\cos(\omega_i t+\angle F_i) \qquad (4)$$

The system output frequency response function can be described as (Lang and Billings, 1996):

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \qquad (5)$$

where, $F(\omega)=\begin{cases}\left|F_i\right|e^{j\angle F_i} & \text{if } \omega\in\{\omega_k, k=\pm1,\cdots,\pm K\}\\ 0 & \text{else}\end{cases}$.

The GFRFs in (3) for specific and simple nonlinear models can be derived by using the probing method in Rugh (1981). Consider the nonlinear systems which can be described by the Nonlinear AutoRegressive model with eXogenous input (NARX)

$$\begin{aligned}y(t)&=\sum_{m=1}^{M}y_m(t)\\ y_m(t)&=\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=1}^{K}c_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p}y(t-k_i)\prod_{i=p+1}^{p+q}u(t-k_i)\end{aligned} \qquad (6)$$

where, $y_m(t)$ is the mth-order output of the system, and $p+q=m$, k_i=1,…, K, $\sum_{k_1,k_{p+q}=1}^{K}(\cdot)=\sum_{k_1=1}^{K}(\cdot)\cdots\sum_{k_{p+q}=1}^{K}(\cdot)$.

Nonlinear system (6) represents a wide class of nonlinear systems and includes several well known nonlinear input-output models as special cases (Chen and Billings 1989). In this model, the parameters such as $c_{0,1}(.)$ and $c_{1,0}(.)$ represent the linear system parameters corresponding to the linear terms in the model such as $y(t$-1) or $u(t$-2) *etc*, and all other parameters represent the nonlinear system parameters. $p+q$ is referred to as the nonlinear degree of the nonlinear parameter $c_{pq}(\cdot)$. A nonlinear parameter $c_{pq}(\cdot)$ corresponds to a nonlinear term in the model of the $p+q$th degree of the form $\prod_{i=1}^{p}y(t-k_i)\prod_{i=p+1}^{p+q}u(t-k_i)$, *e.g.*,

$y(t-1)^p u(t-2)^q$. Obviously, when $p+q=1$, the corresponding parameters are associated with the linear model terms. Let

$$C(M,K)=\left(c_{pq}(k_1,\cdots,k_{p+q}) \middle| \begin{array}{c} p=0\cdots m,\ p+q=m, \\ m=1\cdots M \\ k_i=1\cdots K,\ i=1\cdots p+q \end{array} \right) \tag{7}$$

which includes all the parameters of the model (6).

For the NARX model in (6), the following recursive algorithm was derived by Peyton-Jones and Billings (1989) to compute the GFRFs:

$$\begin{aligned} & L(n)\cdot H_n(j\omega_1,\cdots,j\omega_n) \\ &= \sum_{k_1,k_n=1}^{K} c_{0,n}(k_1,\cdots,k_n)\exp(-j(\omega_1 k_1+\cdots+\omega_n k_n)) \\ &+ \sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=1}^{K} c_{p,q}(k_1,\cdots,k_{p+q})\exp(-j(\omega_{n-q+1}k_{n-q+1}+\cdots+\omega_{p+q}k_{p+q}))H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \\ &+ \sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K} c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n) \end{aligned} \tag{8}$$

$$H_{n,p}(\cdot)=\sum_{i=1}^{n-p+1} H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_i)k_p) \tag{9}$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n)=H_n(j\omega_1,\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_n)k_1) \tag{10}$$

where $L(n)=1-\sum_{k_1=1}^{K} c_{1,0}(k_1)\exp(-j(\omega_1+\cdots+\omega_n)k_1)$. Moreover, $H_{n,p}(j\omega_1,\cdots,j\omega_n)$ in (9) can also be rewritten as

$$H_{n,p}(j\omega_1,\cdots,j\omega_n)=\sum_{\substack{r_1\cdots r_p=1 \\ \sum r_i=n}}^{n-p+1}\prod_{i=1}^{p} H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})\exp(-j(\omega_{r_{X+1}}+\cdots+j\omega_{r_{X+r_i}})k_i),\ \text{where } X=\sum_{x=1}^{i-1} r_x \tag{11}$$

There are three types of nonlinearities in model (6): pure input nonlinearities corresponding to the nonlinear parameters $c_{0,n}(.)$, which lead to the first term in the frequency response functions in equation (8); pure output nonlinearities corresponding to the nonlinear parameters $c_{n,0}(.)$, which lead to the last term of equation (8); and input-output cross nonlinearities corresponding to the nonlinear parameters $c_{p,q}(.)$, which lead to the second term in (8).

It should also be noted from the recursive algorithm for the *n*th-order GFRF given in (8-11) that the nonlinear parameters are separable from the system complex valued functions, thus each term in the *n*th-order GFRF has the form $\prod_{p,q,i,k_1,k_{p+q},m}\left(c_{p,q}(k_1,\cdots,k_{p+q})\right)^i\cdot\tilde{f}(j\omega_1,\cdots,j\omega_{n-q})$ after recursive computations, where $\prod_{p,q,m,k_1,k_{p+q},i}\left(c_{p,q}(k_1,\cdots,k_{p+q})\right)^i$ denotes a multiplication of some nonlinear parameters in (7) which will be determined later, and $\tilde{f}(j\omega_1,\cdots,j\omega_{n-q})$ denotes an appropriate complex valued function and is independent of the nonlinear parameters. This was further shown in Lang *et al* (2006). The separable property of the frequency response functions for nonlinear system (6) provides the basis of the study in this paper.

3 Parametric characteristics

Note that the nonlinear parameters of different nonlinear degrees correspond to different degrees of nonlinearities in the system model and the recursive algorithm for the GFRFs. Hence, the characteristics of the frequency response functions and the effect of different parameters on the system nonlinear behaviour can be studied through the characteristics of the corresponding time domain nonlinear parameters in the frequency response functions. The focus of this section is to analyse the parametric characteristics of the GFRFs and output frequency response functions of nonlinear system (6), and to study how these frequency response functions are determined by the nonlinear system time domain parameters. For this purpose, a powerful coefficient extraction operator will be defined, and then the parametric characteristics of the system GFRFs and the output spectrum will be investigated using the new operator to reveal important relationships between the frequency response functions and the system nonlinear parameters ($c_{pq}(.)$ for $p+q>1$).

3.1 Coefficient extraction operator

In order to analyze the parametric characteristics of the frequency response functions, a useful operator will be defined as follows.

Consider a series which can be written as

$$H_{CF} = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

where the coefficients c_i for i=1,…,n are real or complex numbers, and f_i for i=1,…,n are real or complex valued functions. Let C=[$c_1, c_2, \ldots, c_n$], F=[$f_1, f_2, \ldots, f_n$]^T.

Define a **Coefficient Extraction** operator CE: $\mathcal{C} \to \mathcal{C}^n$ such that for any

$$H_{CF} = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n \in \mathcal{C}$$

then $CE(H_{CF}) = [c_1, c_2, \cdots, c_n] = C \in \mathcal{C}^n$, where $\mathcal{C}$ denote all the complex numbers, and $\mathcal{C}^n$ is the n-dimensional complex vector space. This operator has the following properties which also act as operator rules:

(1) Reduced vectorized sum "$\oplus$".
$CE(H_{C_1F_1} + H_{C_2F_2}) = CE(H_{C_1F_1}) \oplus CE(H_{C_2F_2}) = C_1 \oplus C_2 = [C_1, C_2']$, where each element of C_2' belongs to C_2 but not to C_1, *i.e.*, $\forall i, j, C_2'(i) \neq C_1(j), C_2' \subseteq C_2'$.

(2) Reduced Kronecker product "$\otimes$".
$CE(H_{C_1F_1} \cdot H_{C_2F_2}) = CE(H_{C_1F_1}) \otimes CE(H_{C_2F_2}) = C_1 \otimes C_2$, and "reduced" here means that there are no repetitive components in $C_1 \otimes C_2$.

(3) Invariant. (a) $CE(\alpha \cdot H) = CE(H) \quad \forall \alpha \in \mathcal{C}$ but is not a concerned parameter;
(b) $CE(H_{CF_1} + H_{CF_2}) = CE(H_{C(F_1+F_2)}) = C$

(4) Unitary. $\forall H$ is not a function of c_i for i=1…n, $CE(H) = 1$.

Obviously, when there is a unitary 1 in $CE(H)$, there is a constant term in the corresponding series H_{CF} which has no relation with the coefficients c_i (for i=1…n).

(5) Inverse. $CE^{-1}(C)=H_{CF}$.

Moreover, notice that $H_{CF} = c_1 f_1 + c_2 f_2 + c_3 f_3 + \cdots + c_n f_n = c_3 f_3 + c_2 f_2 + \cdots + c_n f_n + c_1 f_1$, that is, the order of $c_i f_i$ in the summation has no effect on the value of H_{CF}. Thus the CE operator is also commutative and associative in this sense. It should also be noted that the coefficient extractor CE is a coefficient oriented operator. That is, not all the coefficients involved in a series are extracted after applying the operator, but only the coefficients of concern are extracted.

In what follows, the CE operator will be used to study the parametric characteristics of the frequency response functions of system (6). For convenience, let $\underset{(*)}{\otimes}(\cdot)$ and $\underset{(*)}{\oplus}(\cdot)$ denote multiplication and addition in the sense of the reduced Kronecker product"$\otimes$" and vectorized sum"$\oplus$", respectively, for the series (.) under the condition (*); without confusion, write $\overset{k}{\underset{i=1}{\otimes}} C_{pq} = C_{pq} \otimes \cdots \otimes C_{pq}$ simply as C_{pq}^k, and the operators "$\otimes$" and "$\oplus$" simply as "$\circ$" and "+", respectively. Moreover, define the $p+q$th degree parameter vector $C_{p,q} = [c_{p,q}(1,\cdots,1), c_{p,q}(1,\cdots,2), \cdots, c_{p,q}(\underbrace{K,\cdots,K}_{p+q=m})]$, which includes all the nonlinear parameters of the form $c_{p,q}(.)$ with nonlinearity degree $p+q$ in (7). Note that C_{pq} can also be regarded as a set of the $(p+q)$th degree nonlinear parameters of the form $c_{p,q}(.)$, which is a subset of (7).

Recalling the separable property of the GFRFs noted at the end of the last section, consider the nth order GFRF in (8) as a series, and the nonlinear parameters in (7) as the coefficients of the series. Thus the CE operator can be applied to the frequency response functions to reveal the dependence of these functions on the nonlinear parameters. In order to illustrate the use of the CE operator, the following example is given.

Example 1. Apply the CE operator to the GFRFs in (8) up to the 3rd order.
For n=1 in (8),

$$L(1)H_1(j\omega_1) = \sum_{k_1,k_n=1}^{K} c_{0,1}(k_1)\exp(-j\omega_1 k_1)$$

Applying the CE operator to the 1st order GFRF for the nonlinear model parameters (recalling the properties of the CE), yields

$$CE(H_1(j\omega_1)) = CE(L(1)H_1(j\omega_1)) = CE(\sum_{k_1,k_n=1}^{K} c_{0,1}(k_1)\exp(-j\omega_1 k_1)) = 1 \tag{12}$$

For n=2, it is known from (8-11) that

$$
\begin{aligned}
&L(2)H_2(j\omega_1, j\omega_2)\\
&= \sum_{k_1,k_n=1}^{K} c_{0,2}(k_1,k_2)\exp(-j(\omega_1 k_1 + \omega_2 k_2)) + \sum_{k_1,k_2=1}^{K} c_{1,1}(k_1,k_2)\exp(-j(\omega_2 k_2))H_{1,1}(j\omega_1)\\
&\quad + \sum_{k_1,k_2=1}^{K} c_{2,0}(k_1,k_2)H_{2,2}(j\omega_1, j\omega_2)\\
&= \sum_{k_1,k_2=1}^{K} c_{0,2}(k_1,k_2)\exp(-j(\omega_1 k_1 + \omega_2 k_2)) + \sum_{k_1,k_2=1}^{K} c_{1,1}(k_1,k_2)\exp(-j(\omega_2 k_2))H_1(j\omega_1)\exp(-j(\omega_1 k_1))\\
&\quad + \sum_{k_1,k_2=1}^{K} c_{2,0}(k_1,k_2)H_1(j\omega_2)H_1(j\omega_1)\exp(-j(\omega_1 k_1 + \omega_2 k_2))
\end{aligned}
$$

Applying the CE operator to the 2nd order GFRF for the nonlinear model parameters, yields

$$
\begin{aligned}
&CE\big(H_2(j\omega_1, j\omega_2)\big) = CE\big(L(2)H_2(j\omega_1, j\omega_2)\big)\\
&= CE\left(\begin{aligned}&\sum_{k_1,k_2=1}^{K} c_{0,2}(k_1,k_2)\exp(-j(\omega_1 k_1 + \omega_2 k_2)) + \sum_{k_1,k_2=1}^{K} c_{1,1}(k_1,k_2)\exp(-j(\omega_2 k_2))H_1(j\omega_1)\exp(-j(\omega_1 k_1))\\ &\quad + \sum_{k_1,k_2=1}^{K} c_{2,0}(k_1,k_2)H_1(j\omega_2)H_1(j\omega_1)\exp(-j(\omega_1 k_1 + \omega_2 k_2))\end{aligned}\right)\\
&= C_{0,2} \oplus C_{1,1} \oplus C_{2,0} = C_{0,2} + C_{1,1} + C_{2,0}
\end{aligned} \tag{13}
$$

For n=3, it can be shown from (8-11) that

$$
\begin{aligned}
&L(3)H_3(j\omega_1,\cdots, j\omega_3)\\
&= \sum_{k_1,k_3=1}^{K} c_{0,3}(k_1,\cdots,k_3)\exp(-j(\omega_1 k_1 + \cdots + \omega_3 k_3))\\
&\quad + \sum_{q=1}^{2}\sum_{p=1}^{3-q}\sum_{k_1,k_3=1}^{K} c_{p,q}(k_1,\cdots,k_{p+q})\exp(-j(\omega_{3-q+1}k_{3-q+1} + \cdots + \omega_{p+q}k_{p+q}))H_{3-q,p}(j\omega_1,\cdots, j\omega_{3-q})\\
&\quad + \sum_{p=2}^{3}\sum_{k_1,k_p=1}^{K} c_{p,0}(k_1,\cdots,k_p)H_{3,p}(j\omega_1,\cdots, j\omega_3)
\end{aligned}
$$

Applying the CE operator to the 3rd order GFRF for the nonlinear model parameters, yields

$$
\begin{aligned}
&CE\big(L(3)H_3(j\omega_1,\cdots, j\omega_3)\big)\\
&= CE\left(\begin{aligned}&\sum_{k_1,k_3=1}^{K} c_{0,3}(k_1,\cdots,k_3)\exp(-j(\omega_1 k_1 + \cdots + \omega_3 k_3))\\ &+ \sum_{q=1}^{2}\sum_{p=1}^{3-q}\sum_{k_1,k_3=1}^{K} c_{p,q}(k_1,\cdots,k_{p+q})\exp(-j(\omega_{3-q+1}k_{3-q+1} + \cdots + \omega_{p+q}k_{p+q}))H_{3-q,p}(j\omega_1,\cdots, j\omega_{3-q})\\ &+ \sum_{p=2}^{3}\sum_{k_1,k_p=1}^{K} c_{p,0}(k_1,\cdots,k_p)H_{3,p}(j\omega_1,\cdots, j\omega_3)\end{aligned}\right) \qquad (14)\\
&= C_{0,3} \oplus \left(\mathop{\oplus}_{q=1}^{2}\mathop{\oplus}_{p=1}^{3-q}\big(C_{p,q} \otimes CE\big(H_{3-q,p}(j\omega_1,\cdots, j\omega_{3-q})\big)\big)\right) \oplus \left(\mathop{\oplus}_{p=2}^{3}\big(C_{p,0} \otimes CE\big(H_{3,p}(j\omega_1,\cdots, j\omega_3)\big)\big)\right)
\end{aligned}
$$

From (11),

$$\mathop{\oplus}_{q=1}^{2}\mathop{\oplus}_{p=1}^{3-q}\left(C_{p,q}\otimes CE\left(H_{3-q,p}(j\omega_1,\cdots,j\omega_{3-q})\right)\right)$$
$$=\mathop{\oplus}_{p=1}^{2}\left(C_{p,1}\otimes CE\left(H_{2,p}(j\omega_1,\cdots,j\omega_2)\right)\right)\oplus\mathop{\oplus}_{p=1}^{1}\left(C_{p,2}\otimes CE\left(H_{1,p}(j\omega_1)\right)\right)$$
$$=C_{1,1}\otimes CE\left(H_{2,1}(j\omega_1,\cdots,j\omega_2)\right)\oplus C_{2,1}\otimes CE\left(H_{2,2}(j\omega_1,\cdots,j\omega_2)\right)\oplus C_{1,2}\otimes CE\left(H_{1,1}(j\omega_1)\right)$$
$$=C_{1,1}\otimes CE\left(H_2(j\omega_1,\cdots,j\omega_2)\right)\oplus C_{2,1}\otimes CE\left(H_1(j\omega_1)H_1(j\omega_2)\right)\oplus C_{1,2}\otimes CE\left(H_1(j\omega_1)\right)$$
$$=C_{1,1}\otimes CE\left(H_2(j\omega_1,\cdots,j\omega_2)\right)\oplus C_{2,1}\otimes 1\oplus C_{1,2}\otimes 1$$
$$=C_{1,1}\otimes CE\left(H_2(j\omega_1,\cdots,j\omega_2)\right)\oplus C_{2,1}\oplus C_{1,2}$$

$$\mathop{\oplus}_{p=2}^{3}\left(C_{p,0}\otimes CE\left(H_{3,p}(j\omega_1,\cdots,j\omega_3)\right)\right)$$
$$=C_{2,0}\otimes CE\left(H_{3,2}(j\omega_1,\cdots,j\omega_3)\right)\oplus C_{3,0}\otimes CE\left(H_{3,3}(j\omega_1,\cdots,j\omega_3)\right)$$
$$=C_{2,0}\otimes CE\left(H_1(j\omega_1)H_2(j\omega_2,j\omega_3)+H_1(j\omega_3)H_2(j\omega_2,j\omega_1)\right)$$
$$\oplus C_{3,0}\otimes CE\left(H_1(j\omega_1)H_1(j\omega_2)H_1(j\omega_3)\right)$$
$$=C_{2,0}\otimes CE\left(H_2(j\omega_1,j\omega_2)\right)\oplus C_{3,0}$$

Substituting the above two equations and (13) into (14), yields

$$CE\left(H_3(j\omega_1,\cdots,j\omega_3)\right)=CE\left(L(3)H_3(j\omega_1,\cdots,j\omega_3)\right)$$
$$=C_{0,3}\oplus\left(C_{1,1}\otimes CE\left(H_2(j\omega_1,\cdots,j\omega_2)\right)\oplus C_{2,1}\oplus C_{1,2}\right)\oplus\left(C_{2,0}\otimes CE\left(H_2(j\omega_1,j\omega_2)\right)\oplus C_{3,0}\right)$$
$$=C_{03}\oplus C_{11}\otimes C_{02}\oplus C_{11}^2\oplus C_{11}\otimes C_{20}\oplus C_{21}\oplus C_{12}\oplus C_{20}\otimes C_{02}\oplus C_{20}\otimes C_{11}\oplus C_{20}^2\oplus C_{30}$$
$$=C_{03}+C_{21}+C_{12}+C_{30}+C_{11}^2+C_{11}\circ C_{20}+C_{20}\circ C_{02}+C_{11}\circ C_{02}+C_{20}^2 \tag{15}$$

From equations (13) and (15), it can be readily seen that how the nonlinear parameters of nonlinear degree 2 and 3 take a role in the composition of the 2nd and 3rd order GFRFs. In equation (13), different types and degrees of nonlinear parameters make independent contributions to the 2nd GFRF. However, in equation (15) there are cross multiplication terms between different types of nonlinear parameters, thus there may be some special nonlinear behaviour corresponding to these terms in the frequency response function. Note that the first four terms in equation (15) are nonlinear parameters of nonlinear degree 3, and the rest all come from the second nonlinear degree of the model parameters with cross multiplications. Thus equation (15) clearly reveals which and how the different nonlinear parameters take a role in the generation of the 3rd order GFRF. □

The example above demonstrates that the nonlinear model parameters in a series or a polynomial can be effectively extracted by the CE operator, and thus the characteristics that different non-repetitive parameters generate in a series or a polynomial can be revealed by neglecting the corresponding multiplied functions. Therefore, the CE operator provides a useful tool for the analysis of the effects of the nonlinear model parameters on the GFRFs and the system output spectrum.

3.2 Parametric characteristics of the GFRFs

In this section, the parametric characteristics of the GFRFs are derived and analyzed. The results are summarized by the following propositions.

Proposition 1. The parametric characteristics of the *n*th-order GFRF can be obtained as follows:

$$CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)=C_{0,n}\oplus\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}\left(C_{p,q}\otimes\left(\mathop{\oplus}_{\substack{r_1\cdots r_p=1\\ \sum r_i=n-q}}^{n-q-p+1}\mathop{\otimes}_{i=1}^{p}CE\big(H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})\big)\right)\right)$$
$$\oplus\mathop{\oplus}_{p=2}^{n}\left(C_{p,0}\otimes\left(\mathop{\oplus}_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\mathop{\otimes}_{i=1}^{p}CE\big(H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})\big)\right)\right) \tag{16}$$

PROOF OF PROPOSITION 1: The result can be obtained by directly applying the CE operator to equations (8) and (11). Applying the CE operator to equation (8), yields

$$CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)=CE\big(L(n)\cdot H_n(j\omega_1,\cdots,j\omega_n)\big)$$
$$=CE\left(\begin{array}{l}\displaystyle\sum_{k_1,k_n=1}^{K}c_{0,n}(k_1,\cdots,k_n)\exp(-j(\omega_1k_1+\cdots+\omega_nk_n))\\ \displaystyle+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=1}^{K}c_{p,q}(k_1,\cdots,k_{p+q})\exp(-j(\omega_{n-q+1}k_{n-q+1}+\cdots+\omega_{p+q}k_{p+q}))H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})\\ \displaystyle+\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K}c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n)\end{array}\right)$$
$$=C_{0,n}\oplus\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}\big(C_{p,q}\otimes CE\big(H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})\big)\big)\oplus\mathop{\oplus}_{p=2}^{n}\big(C_{p,0}\otimes CE\big(H_{n,p}(j\omega_1,\cdots,j\omega_n)\big)\big) \tag{17a}$$

Applying the CE operator to equation (11), yields

$$CE\big(H_{n,p}(j\omega_1,\cdots,j\omega_n)\big)=CE\left(\sum_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\prod_{i=1}^{p}H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})\exp(-j(\omega_{r_{X+1}}+\cdots+j\omega_{r_{X+r_i}})k_i)\right)$$
$$=\mathop{\oplus}_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\mathop{\otimes}_{i=1}^{p}CE\big(H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})\big) \tag{17b}$$

Substituting equation (17b) into equation (17a), the proposition follows immediately. This completes the proof. □

Proposition 1 provides an explicit expression for the parametric characteristics of the *n*th-order GFRF, which reveals clearly which type of nonlinear model parameters in (7) are included in the descriptions of the GFRF $H_n(j\omega_1,\cdots,j\omega_n)$ and how the GFRFs are determined by these nonlinear model parameters. The result indicates that the form of the separable nonlinear parameters in the GFRF representations can now be described clearly based on equation (16). Note that there are many repetitive terms in equation (16), which can also be seen from the derivation of (15). In order to make the parametric characteristics from Proposition 1 more comprehensible and to provide clear insight, the following results can be applied.

Proposition 2. CE($H_n(j\omega_1,\cdots,j\omega_n)$) includes the nonlinear parameter C_{0n} and all the non-repetitive monomial functions of the nonlinear parameters in (7) of the form $C_{pq}\otimes C_{p_1q_1}\otimes C_{p_2q_2}\otimes\cdots\otimes C_{p_kq_k}$, where the subscripts satisfy $p+q+\sum_{i=1}^{k}(p_i+q_i)=n+k$, $2\le p_i+q_i\le n$, $0\le k\le n-2$, $2\le p+q\le n$ and $1\le p\le n$. That is, the set of all the subscript combinations of the form $(p,q,p_1,q_1\cdots p_k,q_k)$ corresponding to the nonlinear parameter monomials of the form $C_{pq}\otimes C_{p_1q_1}\otimes C_{p_2q_2}\otimes\cdots\otimes C_{p_kq_k}$ which are included in CE($H_n(j\omega_1,\cdots,j\omega_n)$), are

$$\left\{\begin{array}{l|c}(p,q) & p+q+\sum_{i=1}^{k}(p_i+q_i)=n+k \\ (p,q,p_1,q_1) & 1\le p\le n \\ \vdots & 2\le p+q\le n \\ (p,q,p_1,q_1\cdots p_{n-2},q_{n-2}) & 2\le p_i+q_i\le n\end{array}\right\}\cup(0,n)$$

Proof of Proposition 2: $C_{0,n}$ is the first term in equation (16). Consider the last term of equation (16). Note that $\underset{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}{\overset{n-p+1}{\oplus}}\overset{p}{\underset{i=1}{\otimes}}CE\left(H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})\right)$ includes all the combinations of ($r_1,r_2,\ldots,r_p$) satisfying $\sum_{i=1}^{p}r_i=n$, $1\le r_i\le n-p+1$, and $2\le p\le n$. Also note that CE($H_1(j\omega_{r_x})$)=1 since there are no nonlinear parameters, and any repetitive combinations make no contribution. Hence, it can be seen that $\underset{\substack{r_1\cdots r_p=2\\ \sum r_i=n}}{\overset{n-p+1}{\oplus}}\overset{p}{\underset{i=1}{\otimes}}CE\left(H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})\right)$ should include all the possible non-repetitive combinations of ($r_1,r_2,\ldots,r_k$) satisfying $\sum_{i=1}^{k}r_i=n-p+k$, $2\le r_i\le n-p+1$ and $1\le k\le p$. Similarly for CE($H_n(j\omega_1,\cdots,j\omega_n)$). Each of the subscript combinations corresponds to a monomial of the involved nonlinear parameters. Thus, including the term $C_{p,0}$, CE($H_n(j\omega_1,\cdots,j\omega_n)$) includes all the possible non-repetitive monomial functions of the nonlinear parameters of the form $C_{p0}\otimes C_{r_10}\otimes C_{r_20}\otimes\cdots\otimes C_{r_k0}$ satisfying $p+\sum_{i=1}^{k}r_i=n+k$, $2\le r_i\le n$, $0\le k\le n-2$ and $2\le p\le n$. Regarding the second term of equation (16), the nonlinear parameters appear in the form $C_{pq}\otimes C_{p_1q_1}\otimes C_{p_2q_2}\otimes\cdots\otimes C_{p_kq_k}$ in this case, and the results are similar to the above. Hence, the proposition follows. □

It should be noted that repetitive monomials are not considered in Proposition 2. For instance, the subscript combinations (1,1,2,0) and (2,0,1,1) correspond to the nonlinear parameter monomials $C_{1,1}\circ C_{2,0}$ and $C_{2,0}\circ C_{1,1}$, respectively. Both are the same monomial, thus only one is counted. Following Proposition 2 for the special case: when $p=n$, then $p+q+\sum_{i=1}^{k}(p_i+q_i)=n+q+\sum_{i=1}^{k}(p_i+q_i)=n+k$; note that $2\le p_i+q_i\le n$, $0\le k\le n-2$, $2\le p+q\le n$ and $1\le p\le n$, thus $q=k=p_i=q_i=0$. Therefore, the biggest nonlinear degree of nonlinear parameters included in *CE*($H_n(j\omega_1,\cdots,j\omega_n)$) is n corresponding to the nonlinear parameters C_{pq} with $p+q=n$. This can be verified by Example 1. In order to further illustrate the result in Proposition 2, the following example is provided.

Example 2. Consider the 3rd order GFRF. Then $p+q+\sum_{i=1}^{k}(p_i+q_i)=3+k$.

When k=0, the involved nonlinear parameters are $C_{0,3}$, $C_{1,2}$, $C_{2,1}$, $C_{3,0}$;
When k=1, $p+q+p_1+q_1=3+1=4$, which has the following non-repetitive combinations

(p,q,p_1,q_1):(1,1,2,0), (1,1,1,1),(1,1,0,2),(2,0,0,2),(2,0,2,0)

then the involved nonlinear parameter monomials are:

$C_{1,1}\circ C_{2,0}$, $C_{1,1}\circ C_{1,1}$, $C_{1,1}\circ C_{0,2}$, $C_{2,0}\circ C_{0,2}$, $C_{2,0}\circ C_{2,0}$

Note that $0\le k\le n-2=1$, thus the calculations stop at k=1. The result is consistent with Equation (15).

Proposition 3. $CE(H_{n,p}(j\omega_1,\cdots,j\omega_n))=CE\big(H_{n-p+1}(j\omega_p,\cdots,j\omega_n)\big)$.

PROOF OF PROPOSITION 3: According to Proposition 2, $CE\big(H_{n-p+1}(j\omega_p,\cdots,j\omega_n)\big)$ includes all the monomials $C_{p_1q_1}\otimes C_{p_2q_2}\otimes\cdots\otimes C_{p_kq_k}$ satisfying $\sum_{i=1}^{k}(p_i+q_i)=n-p+1+k-1=n-p+k$, $2\le p_i+q_i\le n-p+1$, and $0\le k\le n-p-1$. Equation (17b) can be rewritten as

$$CE\big(H_{n,p}(j\omega_1,\cdots,j\omega_n)\big)=\mathop{\oplus}_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\mathop{\otimes}_{i=1}^{p}CE\big(H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})\big)$$

$$=CE\big(H_{n-p+1}(j\omega_p,\cdots,j\omega_n)\big)\oplus\left(\mathop{\oplus}_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p}\mathop{\otimes}_{i=1}^{p}CE\big(H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})\big)\right)$$

Following the same idea in the proof of Proposition 2, the second term on the right of the equality in this equation can be written as

$$\mathop{\oplus}_{\substack{r_1\cdots r_p=2\\ \sum r_i=n-p+q'}}^{n-p}\Big(CE(H_{r_1}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}}))\otimes CE(H_{r_2}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}}))\otimes\cdots\otimes CE(H_{r_{q'}}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}}))\Big)\quad\text{(A1)}$$

That is, all the terms in (A1) satisfy $\sum_{i=1}^{q'}r_i=n-p+1+q'-1=n-p+q'$, $2\le r_i\le n-p+1$ and $0\le q'\le p$, *i.e.*, $\sum_{i=1}^{q'}(p_i+q_i)=n-p+q'$, $2\le p_i+q_i\le n-p+1$, and $0\le q'\le n-p-1$ corresponding to the subscripts of the nonlinear parameter monomials. Hence, the terms in (A1) are included in $CE\big(H_{n-p+1}(j\omega_p,\cdots,j\omega_n)\big)$. The proposition is proved. □

Based on Proposition 3, many repetitive terms in the expression of the parametric characteristics of $H_n(j\omega_1,\cdots,j\omega_n)$ in (16) can be cancelled since the parametric characteristics of $H_{n,p}(j\omega_1,\cdots,j\omega_n)$ are the same as those of $H_{n-p+1}(j\omega_1,\cdots,j\omega_n)$. The following result can be further obtained directly from Propositions 2 and 3.

Proposition 4. (1) $\mathop{\otimes}_{i=1}^{k}CE(H_{r_i}(\cdot))\subseteq CE(H_Z(\cdot))$, where $Z=\sum_{i=1}^{k}r_i-k+1$, r_i>1;

(2) $\mathop{\otimes}_{i=1}^{k}C_{p_iq_i}\subseteq CE(H_Z(\cdot))$, where $Z=\sum_{i=1}^{k}(p_i+q_i)-k+1$, p_i+q_i>1, and at least one p_i>0 when k>1.

Note that $a\subseteq b$ denotes all the elements in a are elements in b.

From Proposition 4, it is easy to determine which nonlinear parameters are included in a specific GFRF of any order and how a specific nonlinear parameter appears in different orders of the GFRFs. For instance, consider a nonlinear parameter $c_{2,3}(.)$, which corresponds to the nonlinear term $\prod_{i=1}^{2} y(t-k_i)\prod_{i=3}^{5} u(t-k_i)$. According to Proposition 4(2), it follows Z=(2+3)-1+1=5. That is, this nonlinear term has only an independent contribution in 5th order GFRF $H_5(.)$, and has no effect on the GFRFs less than the 5th order. Note that for a convergent Volterra series, the magnitude of $H_5(.)$ may be very small. Hence, the effect of the nonlinear term $\prod_{i=1}^{2} y(t-k_i)\prod_{i=3}^{5} u(t-k_i)$ on the system behaviour is sure to be very small unless $c_{2,3}(.)$ is properly designed or the magnitude of the system input is very large.

Based on Proposition 3, the expression for the parameter characteristics of $H_n(j\omega_1,\cdots,j\omega_n)$ in (16) can now be simplified as

$$\begin{aligned}&CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\\&=C_{0,n}\oplus\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}\Big(C_{p,q}\otimes CE\big(H_{n-q-p+1}(j\omega_1,\cdots,j\omega_{n-q})\big)\Big)\oplus\mathop{\oplus}_{p=2}^{n}\Big(C_{p,0}\otimes CE\big(H_{n-p+1}(j\omega_1,\cdots,j\omega_n)\big)\Big)\end{aligned} \tag{18}$$

Considering the symmetry of the last term of equation (18), only half of the sum is enough to include all the possible monomial combinations except the new term C_{n0}. Hence, (18) can be further written as

$$\begin{aligned}&CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\\&=C_{0,n}\oplus\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}\Big(C_{p,q}\otimes CE\big(H_{n-q-p+1}(j\omega_1,\cdots,j\omega_{n-q})\big)\Big)\oplus\left(C_{n0}\oplus\mathop{\oplus}_{p=2}^{\lfloor n+1/2\rfloor}\Big(C_{p,0}\otimes CE\big(H_{n-p+1}(j\omega_1,\cdots,j\omega_n)\big)\Big)\right)\end{aligned} \tag{19}$$

where $\lfloor\cdot\rfloor$ means the integer part of (.).

Equation (19) is more concise than equation (16), and it is easy to recursively determine $CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)$ using a computer program. Propositions 2-4 and equation (19) demonstrate which and how the nonlinear parameters appear in the GFRFs. These results provide not only a clear insight into the relationship between the nonlinear parameters of the system time domain model and the system GFRFs, but they also provide a useful tool for analysing the characteristics of the GFRFs. Based on the results above, some further results can be obtained, especially for some special but frequently encountered cases, part of which will be studied in Section 4.

3.3 Parameter characteristics of the output frequency response functions

Based on the definition of the CE operator and the theoretical results achieved above, there exists a complex valued function vector with appropriate dimension $f_n(j\omega_1,\cdots,j\omega_n)$, which is a complex function of $j\omega_1,\cdots,j\omega_n$, such that

$$H_n(j\omega_1,\cdots,j\omega_n)=CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\cdot f_n(j\omega_1,\cdots,j\omega_n) \tag{20}$$

where $CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)$ is defined in (19).

Substituting (20) into equation (2), yields

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega} CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\cdot f_n(j\omega_1,\cdots,j\omega_n)\cdot\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega$$

$$=\sum_{n=1}^{N}CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\cdot\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega} f_n(j\omega_1,\cdots,j\omega_n)\cdot\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega$$

Let $\overline{F}_n(j\omega)=\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega} f_n(j\omega_1,\cdots,j\omega_n)\cdot\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega$, giving

$$Y(j\omega)=\sum_{n=1}^{N}CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\cdot\overline{F}_n(j\omega) \tag{21a}$$

Equation (21a) provides an explicit expression of the output frequency response function of the NARX model in (6) under a general input, which is described as a polynomial form in terms of the model nonlinear parameters. A similar result was also obtained in Lang *et al* (2006) using a different method. The present study produces this expression with much more detail, and reveals the relationship between the model parameters and the output frequency response function more clearly. From equation (21a), the parametric characteristics of the output spectrum under a general input is obviously

$$CE(Y(j\omega))=\mathop{\oplus}_{n=1}^{N}CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big) \tag{21b}$$

Similarly, when the system (6) is subject to a multi-tone input, the output frequency response function can be obtained from (5) and (20) as

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}CE\big(H_n(j\omega_{k_1},\cdots,j\omega_{k_n})\big)\cdot f_n(j\omega_{k_1},\cdots,j\omega_{k_n})\cdot F(\omega_{k_1})\cdots F(\omega_{k_n})$$

$$=\sum_{n=1}^{N}CE\big(H_n(j\omega_{k_1},\cdots,j\omega_{k_n})\big)\cdot\left(\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}f_n(j\omega_{k_1},\cdots,j\omega_{k_n})\cdot F(\omega_{k_1})\cdots F(\omega_{k_n})\right)$$

Let $\widetilde{F}_n(j\omega)=\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}f_n(j\omega_{k_1},\cdots,j\omega_{k_n})\cdot F(\omega_{k_1})\cdots F(\omega_{k_n})$, giving

$$Y(j\omega)=\sum_{n=1}^{N}CE\big(H_n(j\omega_{k_1},\cdots,j\omega_{k_n})\big)\cdot\widetilde{F}_n(\omega) \tag{22}$$

Obviously, the parametric characteristics of the output spectrum under a multi-tone input are the same as equation (21b).

Equations (20)-(22) give the parameter characteristics of the system output frequency response functions, which show an analytical relationship between the system model parameters and the system frequency response functions, and provide an important insight into the frequency domain characteristics of nonlinear systems.

4 Some further results

Some characteristics of the GFRFs and the output spectrum can be revealed and derived easily by analyzing the parametric characteristics of the frequency response functions. Based on the theoretical results developed in the previous section, some further results are provided for a special case of the NARX model (6) in the following to demonstrate potential applications of the theoretical results. More detailed studies on the application issues will be presented in other publications.

For the NARX model (6), if there are only pure output nonlinearities in the model, then

$$y(t)=\sum_{m=p=1}^{M}\left(\sum_{k_1,k_p=1}^{K}c_{p,0}(k_1,\ldots,k_p)\prod_{i=1}^{p}y(t-k_i)+\delta(m-1)\sum_{k_1=1}^{K}c_{0,1}(k_1)u(t-k_1)\right)\tag{23}$$

where $\delta(m)=\begin{cases}1, & m=0\\ 0, & \text{else}\end{cases}$. For many engineering applications, this model can be used to represent a nonlinear feedback control system, and consequently has significance in the analysis and synthesis of nonlinear feedback control systems in practice (Jing *et al* 2006).

4.1 The parameter characteristics of the GFRFs

Noting that only the nonlinear parameters $c_{p,0}(.)$ for $p>1$ are nonzero in this case, the GFRFs of model (23) can be written from (8) as

$$H_n(j\omega_1,\cdots,j\omega_n)=\frac{1}{L(n)}\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K}c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n)\tag{24}$$

From equation (19), the parameter characteristics of the nth-order GFRF is

$$CE\left(H_n(j\omega_1,\cdots,j\omega_n)\right)=C_{n0}\oplus\mathop{\oplus}_{p=2}^{\lfloor n+1/2\rfloor}\left(C_{p,0}\otimes CE\left(H_{n-p+1}(j\omega_1,\cdots,j\omega_n)\right)\right)\tag{25}$$

Equation (25) is just a special case of equation (19). In this case, taking the nonlinear parameters $c_{p,0}(.)$ for $2\le p\le n$, then Proposition 5 follows.

Proposition 5.

(1) $(C_{n0})^i$ appears in $H_m(.)$ from the *mth* order, where $m=1+(n-1)i$.

(2) If $C_{i0}=0$ for $2\le i\le n$ then $H_n(j\omega_1,\ldots,j\omega_n)=0$ for all $\omega_1,\ldots,\omega_n$. The inverse of this point does not hold.

PROOF OF PROPOSITION 5:

(1) According to Proposition 2 or Proposition 4, the term $(C_{n0})^i$ should be in the GFRF $H_m(.)$, where m is computed as $m+k=m+i-1=ni$. Hence we have $m=ni-i+1=1+(n-1)i$.

(2) From (25), $CE\left(H_n(j\omega_1,\cdots,j\omega_n)\right)$ includes all the nonlinear parameters in $C_{i0}=0$ for $2\le i\le n$, and no nonlinear parameters in $C_{i0}=0$ for i>n are included in $CE\left(H_n(j\omega_1,\cdots,j\omega_n)\right)$. Therefore, if $C_{i0}=0$ for $2\le i\le n$ then $CE\left(H_n(j\omega_1,\cdots,j\omega_n)\right)$=0, which further follows from (20) that $H_n(j\omega_1,\ldots,j\omega_n)=0$ for all $\omega_1,\ldots,\omega_n$. Note that $(C_{30})^i$ appears in $H_{1+2i}(.)$ for i=1,2,3,… from Proposition 5 (1), which implies $(C_{30})^i$ only contributes to the odd order of the GFRFs. Hence, even if certain even order GFRF is zero, $(C_{30})^i$ can still be nonzero. Thus the inverse of the statement above is not true. This completes the proof. □

From Proposition 5, if the 2nd and 3rd degree nonlinear parameters are all zero C_{20}=0 and C_{30}=0, then $H_2(.)$=0, and $H_3(.)$=0. However, even if C_{n0}=0 (for $n>3$) but there are nonzero terms in C_{20} or C_{30}, the nth order GFRF $H_n(.)$ may not be zero, because $(C_{20})^i$ appears in $H_{1+i}(.)$ for i=1,2,3,…, and $(C_{30})^i$ appears in $H_{1+2i}(.)$ for i=1,2,3,… from Proposition 5 (1). This implies that the nonlinear parameters in C_{20} and C_{30} take much greater roles in the GFRFs than other nonlinear parameters. That is, the higher the nonlinear degree of a particular model terms, the smaller the effect of the associated nonlinear parameters on the system becomes, since the higher degree (>3) nonlinear parameters only take a role in the higher order GFRFs. Therefore, in many cases C_{p0} from p=1 to 3 are likely to form the basis of the GFRFs and the system output spectrum. Moreover, the results obtained above also demonstrate that different types of nonlinearity have different effects on the system output spectrum, and nonlinear terms of the same nonlinear degree may have a similar effect on the output spectrum.

4.1.1 A special case

In this subsection, another case of the NARX model (23) is considered to demonstrate that some characteristics of the GFRFs can be studied by only analysing the parametric characteristics of the GFRFs without the need to completely evaluate the GFRFs. This special case includes the two different situations of $c_{2k,0}(.)=0$ or $c_{2k+1,0}(.)=0$ (for k=1,2,3,…). Some significant conclusions regarding the characteristics of the GFRFs in these two situations can be reached.

Proposition 6. For the GFRFs of nonlinear system (23),

(1) If $c_{2k,0}(.)=0$ and $c_{2k+1,0}(.)\neq 0$ for k=1,2,3,… in (23), then $H_{2k}(.)=0$ and $H_{2k+1}(.)\neq 0$.

(2) If $c_{2k,0}(.)\neq 0$ and $c_{2k+1,0}(.)=0$ for k=1,2,3,… in (23), then $H_{2k}(.)\neq 0$ and $H_{2k+1}(.)\neq 0$.

Proof of Proposition 6: From the definition of the CE operator and the theoretical results above, to prove $H_k(.)=0$ or $\neq 0$, we need only to prove $CE(H_k(.))=0$ or $\neq 0$. According to equation (25), we have

$$\begin{aligned} CE(H_{2k}(j\omega_1,\cdots,j\omega_{2k})) &= C_{2k,0} \oplus \mathop{\oplus}_{p=2}^{k}\left(C_{p,0}\otimes CE(H_{2k-p+1}(j\omega_1,\cdots,j\omega_{2k}))\right) \\ &= C_{2k,0} + C_{2,0}\circ CE(H_{2k-1}(.)) + C_{3,0}\circ CE(H_{2k-2}(.)) + \cdots + C_{k,0}\circ CE(H_{k+1}(.)) \end{aligned} \tag{26a}$$

$$\begin{aligned} CE(H_{2k+1}(j\omega_1,\cdots,j\omega_{2k+1})) &= C_{2k+1,0} \oplus \mathop{\oplus}_{p=2}^{k+1}\left(C_{p,0}\otimes CE(H_{2k+2-p}(j\omega_1,\cdots,j\omega_{2k+1}))\right) \\ &= C_{2k+1,0} + C_{2,0}\circ CE(H_{2k}(.)) + C_{3,0}\circ CE(H_{2k-1}(.)) + \cdots + C_{k+1,0}\otimes CE(H_{k+1}(.)) \end{aligned} \tag{26b}$$

(1) If $c_{2k,0}(.)=0$ for k=1,2,3,… in (23), then equation (26a) follows

$$CE(H_2(j\omega_1,\cdots,j\omega_2)) = C_{2,0} = 0$$

$$CE(H_{2k}(j\omega_1,\cdots,j\omega_{2k})) = C_{3,0}\circ CE(H_{2k-2}(.)) + C_{5,0}\circ CE(H_{2k-4}(.)) + \cdots + C_{k',0}\circ CE(H_{k'+1}(.)) \tag{27}$$

where k' is the largest odd number less than or equal to k. Note that 2k-2, 2k-4, …, $k'+1$ are all even numbers. Hence, from the recursive calculations of (27), it can be shown that $CE(H_{2k}(j\omega_1,\cdots,j\omega_{2k}))=0$ for all k=1,2,3,… Similarly, equation (26b) follows

$$CE(H_{2k+1}(j\omega_1,\cdots,j\omega_{2k+1})) = C_{2k+1,0} + C_{3,0}\circ CE(H_{2k-1}(.)) + \cdots + C_{k',0}\otimes CE(H_{k'}(.))$$

where k' is the largest odd number less than or equal to k+1. Note that $C_{2k+1,0}\neq 0$, thus $CE(H_{2k+1}(j\omega_1,\cdots,j\omega_{2k+1}))\neq 0$. This proves the first conclusion of the proposition.

(2) If $c_{2k+1,0}(.)=0$ for k=1,2,3,… in (23), then equation (26a) follows

$$CE\left(H_{2k}(j\omega_1,\cdots,j\omega_{2k})\right)=C_{2k,0}+C_{2,0}\circ CE\left(H_{2k-1}(.)\right)+C_{4,0}\circ CE\left(H_{2k-3}(.)\right)+\cdots+C_{k',0}\circ CE\left(H_{k'+1}(.)\right)$$

where k' is the largest even number less than or equal to k. Note that $C_{2k,0}\neq 0$, thus $CE\left(H_{2k}(j\omega_1,\cdots,j\omega_{2k})\right)\neq 0$. Similarly, equation (26b) follows

$$CE\left(H_2(j\omega_1,\cdots,j\omega_2)\right)=C_{2,0}\neq 0$$

$$CE\left(H_{2k+1}(j\omega_1,\cdots,j\omega_{2k+1})\right)=C_{2,0}\circ CE\left(H_{2k}(.)\right)+C_{4,0}\circ CE\left(H_{2k-2}(.)\right)+\cdots+C_{k',0}\otimes CE\left(H_{k'}(.)\right) \quad (28)$$

where k' is the largest even number less than or equal to k+1. Hence, from the recursive calculations of (28), it can be shown that $CE\left(H_{2k}(j\omega_1,\cdots,j\omega_{2k})\right)\neq 0$ for all k=1,2,3,… This completes the proof. □

Proposition 6 shows clearly that some characteristics of the GFRFs can be studied easily through the parametric characteristics of the corresponding GFRFs, and this provides a new and effective approach to the analysis of nonlinear systems in the frequency domain. Note that each GFRF defines the output spectrum of nonlinear systems over a corresponding output frequency range (Lang and Billings 1996). Thus $H_k(.)=0$ (for some k) implies that there are no output frequency spectra over the frequency range corresponding to $H_k(.)$. Therefore, Proposition 6 implies that, if $c_{2k,0}(.)\neq 0$ and $c_{2k+1,0}(.)=0$ for k=1,2,3,… in (23), then the output frequency response of the system is available over a wider frequency range than in the case where $c_{2k,0}(.)=0$ and $c_{2k+1,0}(.)\neq 0$ for k=1,2,3,… in (23). This comment can be verified by the following example.

Example 3. Consider a simple nonlinear system described as

$$y(t)=0.3y(t-1)+0.5y(t-2)+0.3u(t-1)+ay(t-1)^2+by(t-1)^3$$

which can be written in the form (23) with $c_{1,0}(1)=0.3, c_{1,0}(2)=0.5, c_{0,1}(1)=0.3,$ $c_{2,0}(1,1)=a, c_{3,0}(1,1,1)=b$ else $c_{p,q}(\cdot)=0$, and K=2, M=3. There are only pure output nonlinearities in this model. Let u(t)=10sin(50t). Consider two cases: (1) $c_{2k,0}(.)=0$ and $c_{2k+1,0}(.)\neq 0$, *i.e.*, a=0 and b=-0.005; (2) $c_{2k,0}(.)\neq 0$ and $c_{2k+1,0}(.)=0$, *i.e.*, a=-0.005 and b=0. In the two cases, the system output spectrum can be obtained by applying the FFT to the system time domain output. The results are shown in Figure 1 and 2. It can be seen from the two figures that, the output spectrum for the same input is quite different. As expected when a=-0.005 and b=0, there are more output frequency components (*e.g.*, at frequencies 0, 100, 200, 300,…,) than in the case where a=0 and b=-0.005.

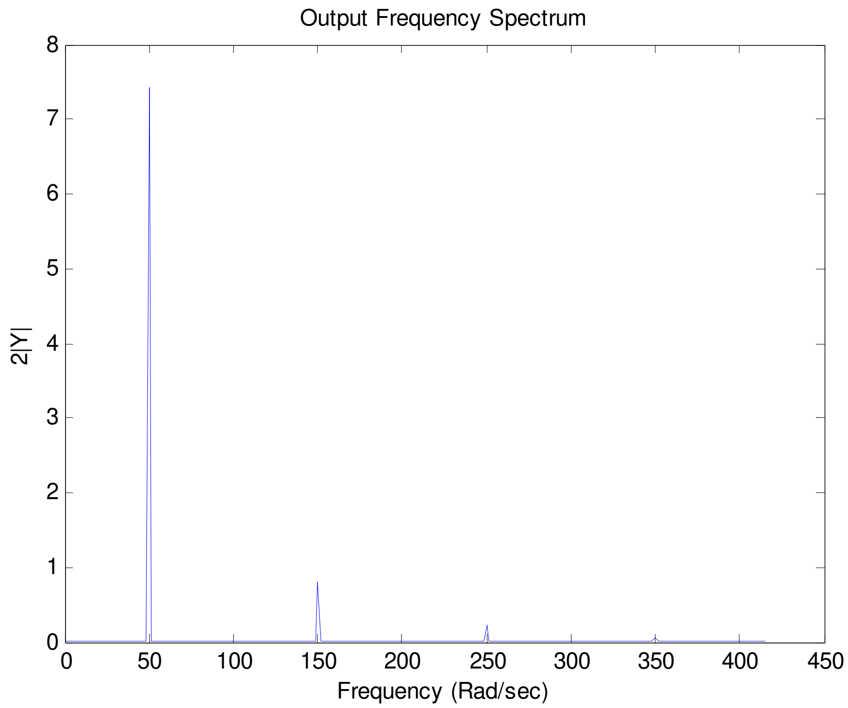

Figure 1. System output spectrum when a=0 and b=-0.005

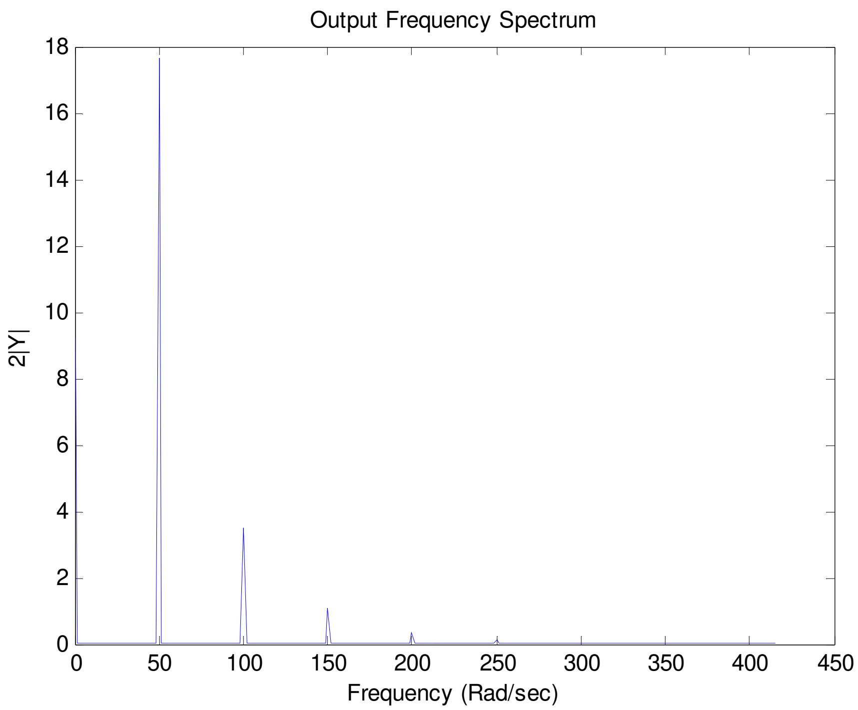

Figure 2. System output spectrum when a=-0.005 and b=0

4.2 Determination of the output frequency response function

From equations (21) and (22), the output spectrum of nonlinear system (23) can uniformly be rewritten as

$$\begin{aligned} Y(j\omega) &= \sum_{n=1}^{N} Y_n(j\omega) \\ Y_n(j\omega) &= CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)\cdot \hat{F}_n(j\omega) \end{aligned} \tag{29}$$

In (29), $CE\big(H_n(j\omega_1,\cdots,j\omega_n)\big)$ can be computed according to (25), and $\hat{F}_n(j\omega)$ can be determined using an effective data analysis based method proposed in Lang *et al.* (2006). Therefore, the output frequency response function of system (23) can be obtained without a large amount of recursive calculations of the GFRFs and the corresponding complicated integral terms in (2) or (5). Further studies will investigate employing the detailed information about the structure of equation (29) provided by the analysis of the parameter characteristics of the system GFRFs to directly determine an accurate description for the system output frequency response function.

5 Conclusions

Several fundamental and important results relating to the parametric characteristics of the GFRFs and the output frequency response functions of nonlinear systems described by a NARX model have been derived in this paper. These results effectively reveal the relationship between the system model parameters in the time domain and the system frequency response functions, and provide for the first time a significant and novel insight into the frequency characteristics of nonlinear systems. Further theoretical results and practical algorithms for the analysis and synthesis of nonlinear systems in the frequency domain can be developed based on these new results. Further research will focus on the application of the parametric characteristics of nonlinear systems derived in the present study to investigate the analysis and design of nonlinear systems in the frequency domain. The analysis will investigate the effects of the system model parameters on the GFRFs and the output spectrum to reveal how the model parameters affect the system behavior in the frequency domain. Designs of optimal values for some of the model parameters to achieve a desired system output frequency response will also be studied in later publications.

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