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# Square-Free Perfect Graphs

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## Abstract

We prove that square-free perfect graphs are bipartite graphs or line graphs of bipartite graphs or have a 2-join or a star cutset.

## 1 Introduction

In this paper, all graphs are simple. A *hole* is a chordless cycle with at least four nodes. A  $k$ -*hole* is a hole with  $k$  nodes and a 4-hole is also called a *square*. Our main result is a decomposition theorem for graphs that contain no odd hole and no 4-hole. (We say that a graph  $G$  contains  $H$  if  $H$  is an induced subgraph of  $G$ ). Partial results in this direction were obtained by Xue [31], [32], Fouquet [18] and recently by Linhares-Sales and Maffray [21].

A graph  $G$  is *perfect* if, in every induced subgraph, the size of a largest clique equals the chromatic number. A graph  $G$  is *minimally imperfect* if  $G$  is not perfect, but every proper induced subgraph of  $G$  is perfect. The Strong Perfect Graph Conjecture (SPGC), due to Berge [1], states that if  $G$  is a minimally imperfect graph, then either  $G$  or the complement of  $G$  is an odd hole. This conjecture was proved recently by Chudnovsky, Robertson, Seymour and Thomas [2]. In the 70's and 80's, the SPGC was proved for  $H$ -free graphs for quite a number of small graphs  $H$ . In fact, the class of square-free graphs was the last class of  $H$ -free graphs with  $|V(H)| = 4$  for which the SPGC was not known: Seinsche [26] proved it in 1974 for  $P_4$ -free graphs, Meyniel [22], [23] in 1976 for paw-free graphs (see also Olariu [24]), Parthasarathy and Ravindra [25] in 1976 for claw-free graphs, Tucker [28] in 1977 for  $K_4$ -free graphs, Conforti and Rao [13], [14], Fonlupt and Zemirline [17] and Tucker [29] in 1987 for diamond-free graphs (see also [4]).

Given a graph  $G$  and a subset  $S$  of its nodes,  $G \setminus S$  denotes the subgraph of  $G$  obtained by removing the nodes in  $S$  and the edges with at least one node in  $S$ . A node set  $S$  is a *cutset* of  $G$  if the graph  $G \setminus S$  has more connected components than  $G$ .

A node set  $S$  is a *star* if it consists of a node  $x$  and neighbors of  $x$ . Chvátal [3] showed that a minimally imperfect graph cannot contain a star cutset.

A graph  $G$  has a *2-join*, denoted by  $H_1|H_2$ , with special sets  $A_1, B_1, A_2, B_2$  that are nonempty and disjoint, if the nodes of  $G$  can be partitioned into sets  $H_1$  and  $H_2$  so that  $A_1, B_1 \subseteq H_1$ ,  $A_2, B_2 \subseteq H_2$ , all nodes of  $A_1$  are adjacent to all nodes of  $A_2$ , all nodes of  $B_1$  are adjacent to all nodes of  $B_2$  and these are the only adjacencies between  $H_1$  and  $H_2$ . Also, for  $i = 1, 2$ ,  $H_i$  has at least one path from  $A_i$  to  $B_i$  and if  $A_i$  and  $B_i$  are both of cardinality 1, then the graph induced by  $H_i$  is not a chordless path. Cornuéjols and Cunningham [16] showed that a minimally imperfect graph cannot contain a 2-join.

The *line graph* of a graph  $G$  is the graph  $H$  whose nodes are the edges of  $G$  and two nodes  $r, s$  of  $H$  are adjacent in  $H$  if and only if the edges  $r, s$  of  $G$  are incident to a common node of  $G$ . It is well known and easy to prove that that bipartite graphs and line graphs of bipartite graphs are perfect.

The main result of this paper is the following.

**Theorem 1.1** *A square-free graph that contains no odd hole is bipartite or the line graph of a bipartite graph or has a star cutset or a 2-join.*

Theorem 1.1 can be used to prove the validity of the SPGC for square-free graphs.

**Theorem 1.2** *Let  $G$  be a minimally imperfect square-free graph. Then  $G$  is an odd hole.*

*Proof:* Let  $G$  be a minimally imperfect graph. Chvátal [3] showed that  $G$  cannot contain a star cutset and Cornuéjols and Cunningham [16] showed that  $G$  cannot contain a 2-join (see also Cornuéjols [15]). Now if  $G$  is a square-free graph, by Theorem 1.1,  $G$  must be an odd hole.  $\square$

In fact, we prove the following result which is more general than Theorem 1.1. We *sign* a graph by assigning 0,1 weights to its edges. A graph  $G$  is *even-signable* if there exists a signing such that, in every triangle the sum of the weights of its edges is odd and in every hole the sum of the weights is even. It is obvious that  $G$  contains no odd holes if and only if  $G$  is even-signable and the above property is satisfied by assigning to all the edges of  $G$  the weight 1.

**Theorem 1.3** *A square-free even-signable graph is triangle-free or the line graph of a triangle-free graph or has a star cutset or a 2-join.*

In the first draft of this paper (February 2001) we made the following observations: “The approach followed in this paper to decompose square-free perfect graphs might be used as a template for the decomposition of general perfect graphs in the same way as the work of Conforti and Rao [12] on linear balanced matrices was a template for the decomposition of general balanced matrices [10] and [8], [9]. For general perfect graphs, it is natural to consider as basic not only the bipartite graphs and the line graphs of bipartite graphs, but also their complements. In terms of decompositions, star cutsets need to be replaced by more general cutsets such as T-cutsets and U-cutsets (Hoàng [20]) or skew partitions (Chvátal [3]).” Interestingly, Chudnovsky, Robertson, Seymour and Thomas [2] followed such an approach to prove the SPGC.

## 1.1 Even-signable graphs

A *wheel*, denoted by  $(H, x)$ , is a graph induced by a hole  $H$  and a node  $x \notin V(H)$  having at least three neighbors in  $H$ . Node  $x$  is the *center* of the wheel. If  $x$  has  $k$  neighbors in  $H$ , the wheel is called a  $k$ -wheel. A wheel  $(H, x)$  is *odd* if it contains an odd number of triangles and is *even* if it contains an even number of triangles: That is,  $(H, x)$  is odd (even) if the set of edges of  $H$  having both endnodes adjacent to  $x$  has odd (even) cardinality.

A  $3PC(x_1x_2x_3, y)$  is a graph induced by three chordless paths  $P^1 = x_1, \dots, y$ ,  $P^2 = x_2, \dots, y$  and  $P^3 = x_3, \dots, y$ , having no common nodes other than  $y$  and such that the only adjacencies between nodes of  $P^i \setminus y$  and  $P^j \setminus y$ , for  $i, j \in \{1, 2, 3\}$  distinct, are the edges of the triangle induced by  $\{x_1, x_2, x_3\}$ . Also, at most one of the paths  $P^1, P^2, P^3$  has length 1. We say that a graph  $G$  contains a  $3PC(\Delta, .)$  if it contains a  $3PC(x_1x_2x_3, y)$  for some  $x_1, x_2, x_3, y \in V(G)$ .

It is immediate to check that if  $G$  contains an odd wheel or a  $3PC(\Delta, .)$ ,  $G$  is not even-signable and hence  $G$  contains an odd hole. Our proofs use the following characterization of even-signable graphs, which can be derived from a theorem of Truemper [27]. (See also [11]).

**Theorem 1.4** [6] *A graph is even-signable if and only if it does not contain an odd wheel nor a  $3PC(\Delta, .)$ .*

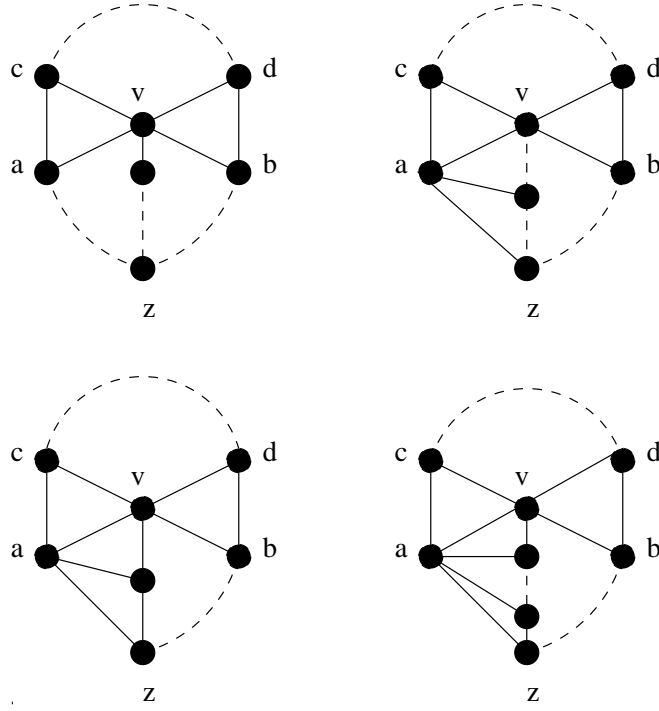


Figure 1: L-parachutes

## 2 WP-Free Graphs

In this section, we introduce a result proven in [5].

A *line wheel* is a 4-wheel  $(H, v)$  that contains exactly two triangles and these two triangles have only the center  $v$  in common. Note that if  $G$  is a line wheel, then  $G$  is the line graph of a triangle-free graph. A *twin wheel* is a 3-wheel containing exactly two triangles. A *universal wheel* is a wheel  $(H, v)$  where the center  $v$  is adjacent to all the nodes of  $H$ . A *triangle-free wheel* is a wheel containing no triangle. A *proper wheel* is a wheel that is not any of the above four types.

**Definition 2.1** An L-parachute  $LP(ca, db, v, z)$  is a graph induced by a line wheel  $(H, v)$  where  $H = a, \dots, z, \dots, b, d, \dots, c, a$ , where  $a, b, c, d$  are the neighbors of  $v$  in  $H$ , together with a chordless path  $P = v, \dots, z$  of length greater than one. No node of  $H \setminus \{z, a, b\}$  and at most one node among  $a, b$ , may be adjacent to an interior node of  $P$ .

**Definition 2.2** A T-parachute  $TP(t, v, a, b, z)$  is a graph induced by a twin wheel  $(H, v)$  where  $H = a, t, b, \dots, z, \dots, a$ , where  $t, a, b$  are the neighbors of  $v$  in  $H$ , together with a chordless path  $P = v, \dots, z$  of length greater than one. No node of  $H \setminus \{z, a, b\}$  and at most one node among  $a, b$  may be adjacent to an interior node of  $P$ .

A *parachute* is either an L-parachute or a T-parachute. A graph  $G$  is *WP-free* if it contains neither a proper wheel nor a parachute. The results in [5] imply the following:

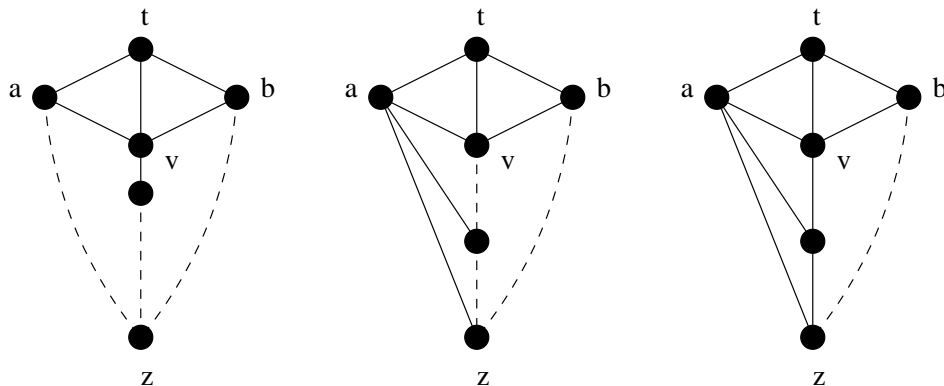


Figure 2: T-parachutes

**Theorem 2.3** *Let  $G$  be a square-free even-signable graph. If  $G$  is WP-free, then  $G$  is a triangle-free graph, or the line graph of a triangle-free graph, or  $G$  has a star cutset or a 2-join.*

### 3 Outline of the proof

Two graphs play an important role in our proof of Theorem 1.3:

A  $3PC(a_1a_2a_3, b_1b_2b_3)$  is the graph containing node disjoint triangles  $a_1, a_2, a_3$  and  $b_1, b_2, b_3$ , plus three chordless paths,  $P^1 = a_1, \dots, b_1$ ,  $P^2 = a_2, \dots, b_2$  and  $P^3 = a_3, \dots, b_3$ , having no common nodes and such that the only adjacencies between the nodes of distinct paths are the edges of the two triangles. A  $3PC(a_1a_2a_3, b_1b_2b_3)$  is also referred to as a  $3PC(\Delta, \Delta)$ .

A *beetle* is a 4-wheel  $(H, v)$  containing exactly two triangles and these triangles have two nodes in common, the center  $v$  and a node of  $H$ .

Now as a consequence of Theorem 2.3, in the proof of Theorem 1.3, we may assume that  $G$  is a square-free graph that contains a proper wheel or a parachute.

-We first prove the theorem when  $G$  contains a proper wheel that is not a beetle (Theorem 3.6), and show that if  $G$  contains a beetle, then  $G$  contains a  $3PC(\Delta, \Delta)$  plus additional nodes adjacent to it (Section 4, Theorem 3.4).

-We then prove the theorem when  $G$  contains an L-parachute (Section 5, Theorem 4.3).

In all the results that we obtain from Section 6 on, make use of the fact that the theorem holds whenever  $G$  contains an L-parachute, or a proper wheel that is not a beetle.

- In Section 7 we show that if  $G$  contains a beetle or a T-parachute, then  $G$  contains a  $3PC(\Delta, \Delta)$  plus some additional nodes adjacent to it.

- From Section 8 on, we show that the theorem holds when  $G$  contains a  $3PC(\Delta, \Delta)$ .

### 4 Proper Wheels

Given a subgraph  $H$  of a graph  $G$ , a node  $v \notin V(H)$  is *strongly adjacent* to  $H$  if  $|N(v) \cap V(H)| \geq 2$ , where  $N(v)$  denotes the set of neighbors of  $x$  in  $G$ .

Following the terminology of West [30], a *path*  $P$  is a sequence of distinct nodes  $x_1, \dots, x_n$ ,  $n \geq 1$ , such that  $x_i x_{i+1}$  is an edge, for all  $1 \leq i < n$ . For  $i \leq l$ , the path  $x_i, x_{i+1}, \dots, x_l$  is called the  $x_i x_l$ -*subpath* of  $P$  and is denoted by  $P_{x_i x_l}$ . The *length* of a path or a cycle is its number of edges.

Let  $A, B, C$  be three disjoint node sets such that no node of  $A$  is adjacent to a node of  $B$ . A path  $P = x_1, \dots, x_n$  *connects*  $A$  and  $B$  if either  $n = 1$  and  $x_1$  has neighbors in  $A$  and  $B$ , or  $n > 1$  and one of the two endnodes of  $P$  is adjacent to at least one node in  $A$  and the other is adjacent to at least one node in  $B$ . The path  $P$  is a *direct connection between*  $A$  and  $B$  if, in the subgraph induced by the node set  $V(P) \cup A \cup B$ , no path connecting  $A$  and  $B$  is shorter than  $P$ . A direct connection  $P$  between  $A$  and  $B$  *avoids*  $C$  if  $V(P) \cap C = \emptyset$ . The direct connection  $P$  is said to be *from*  $A$  *to*  $B$  if  $x_1$  is adjacent to some node in  $A$  and  $x_n$  to some node in  $B$ .

**Lemma 4.1** *Let  $(H, x)$  be a proper even wheel and let  $y$  be a node that is not adjacent to  $x$  but has at least 2 neighbors in  $N(x) \cap H$ . Then  $y$  has exactly 2 neighbors in  $H$  and they are adjacent.*

*Proof:* Since  $G$  is square-free and  $y$  is not adjacent to  $x$ , all the neighbors of  $y$  in  $N(x)$  are adjacent. So  $y$  has exactly two neighbors in  $N(x) \cap H$ , say  $y_1, y_2$  and  $y_1, y_2$  are adjacent. It remains to show that  $y$  has no other neighbor in  $H$ .

**Claim 1:** *Let  $Q$  be a subpath of  $H$  whose endnodes are in  $N(x)$  at least one of which is distinct from  $y_1, y_2$  and no interior node of  $Q$  is in  $N(x)$ . Then  $Q \cup \{y\}$  contains an even number of triangles.*

*Proof of Claim 1:* Assume  $H$  contains a subpath  $Q$  with the above properties such that  $Q \cup \{y\}$  contains an odd number of triangles. Then this number must be 1 and  $y$  has exactly two neighbors in  $Q$ , else  $Q \cup \{x, y\}$  induces an odd wheel with center  $y$ . Let  $x_i, x_j$  be the endnodes of  $Q$  and  $y_i, y_j$  be the adjacent neighbors of  $y$  in  $Q$ , closest to  $x_i, x_j$  respectively. Since  $G$  is square-free and the pair  $y_i, y_j$  is distinct from  $y_1, y_2$ , we may assume w.l.o.g. that  $x_i$  and  $y_i$  are distinct.

Let  $x'_i, x'_j$  be the neighbors of  $x_i, x_j$  in  $H \setminus Q$ . Since  $(H, x)$  is a proper even wheel,  $x$  has a neighbor in  $H \setminus \{x'_i, x_i, x_j, x'_j\}$ . So if  $y$  also has a neighbor in  $H \setminus (Q \cup \{x'_i, x'_j\})$ , we have a  $3PC(y y_i y_j, x)$ . Therefore  $N(y) \cap H \subseteq Q \cup \{x'_i, x'_j\}$ . Since  $(H, y)$  has a positive even number of triangles,  $y_i, y_j$  are the only two neighbors of  $y$  in  $Q$  and  $x_i \neq y_i$ , it follows that  $x_j = y_j$ . So both  $x$  and  $y$  are adjacent to  $x'_j$ . Let  $H'$  be the hole  $x_i, x, x'_j, y, y_i, Q_{y_i x_i}$ . Then  $(H', x_j)$  is an odd wheel. This concludes Claim 1.

Since  $y$  is adjacent to  $y_1, y_2$ , it follows by Claim 1 that if  $y$  has at least three neighbors in  $H$ , then  $(H, y)$  contains an odd number of triangles and is an odd wheel.  $\square$

Let  $(H, x)$  be a wheel that contains at least one triangle. A *sector* of  $(H, x)$  is a maximal subpath  $Q$  of  $H$  such that  $Q \cup \{x\}$  is a triangle-free graph. Two sectors  $Q_i$  and  $Q_j$  are *adjacent* if  $Q_i \cup Q_j \cup \{x\}$  contains at least one triangle. Note that a sector may have length zero.

A *bicoloring* of a proper even wheel  $(H, x)$  is a partition of the nodes in  $H$  into nonempty sets  $R$  and  $B$  (the red and blue nodes) so that nodes in the same sector have the same color while nodes in adjacent sectors have distinct colors.

**Lemma 4.2** *Let  $(H, x)$  be a proper even wheel that is bicolored and let  $y$  be a node that is not adjacent to  $x$  but has at least two neighbors in  $H$  painted with distinct colors. Then  $y$  has exactly 2 neighbors in  $H$  and these nodes are endnodes of adjacent sectors of  $(H, x)$ .*

*Proof:* Let  $y_1, y_2$  be neighbors of  $y$  in  $H$  having distinct colors, such that one of the  $y_1y_2$ -subpaths of  $H$ , say  $Q$ , contains no other neighbor of  $y$ . Since  $y_1, y_2$  have distinct colors, then  $Q \cup \{x\}$  contains an odd number of triangles. This number must be 1 and  $x$  has exactly two neighbors in  $Q$ , else  $Q \cup \{x, y\}$  induces an odd wheel with center  $x$ . Let  $x_1, x_2$  be the neighbors of  $x$  in  $Q$ , closest to  $y_1, y_2$  respectively.

If  $x$  and  $y$  have two common neighbors in  $H$ , we are done by Lemma 4.1. So we assume w.l.o.g. that  $y_2 \neq x_2$ . Let  $y'_1, y'_2$  be the neighbors of  $y_1, y_2$  in  $H \setminus Q$ .

**Claim 1:** *Node  $x$  has a neighbor in  $H \setminus (Q \cup \{y'_1, y'_2\})$ .*

*Proof of Claim 1:* Assume not. Since  $(H, x)$  is a proper even wheel, then  $y_1 = x_1$ , node  $x$  is adjacent to both  $y'_1, y'_2$  and  $y'_1y'_2$  is not an edge. Node  $y$  has at least three neighbors in  $H$ , else we have an odd wheel with center  $x$ , but  $y$  is not adjacent to  $y'_1$  or  $y'_2$  else we are done by Lemma 4.1. Let  $y_3$  be the neighbor of  $y$  in  $H \setminus Q$  closest to  $y_1$  and let  $y''_2$  be the neighbor of  $y'_2$  in  $H$ , distinct from  $y_2$ .

If  $y_3 \neq y''_2$ , there is an odd wheel  $(H', y_1)$  where  $H' = y, y_2, y'_2, x, H_{y'_1y_3}, y$ . So  $y_3 = y''_2$  and we have a  $3PC(y'_1y_1x, y''_2)$ . This completes the proof of Claim 1.

If  $y$  has a neighbor in  $H \setminus (Q \cup \{y'_1, y'_2\})$ , by Claim 1, we have a  $3PC(x_1x_2x, y)$ . So all the neighbors of  $y$  in  $H$  are included in  $\{y'_1, y_1, y_2, y'_2\}$ . Now  $y_1 = x_1$  and both  $x$  and  $y$  are adjacent to  $y'_1$ , else we have an odd wheel with center  $x$ . But now we are done by Lemma 4.1.  $\square$

**Definition 4.3** *A diamond is a graph with four nodes and five edges. Connected diamonds  $D(a_1a_2a_3a_4, b_1b_2b_3b_4)$  consist of two node disjoint sets  $\{a_1, \dots, a_4\}$  and  $\{b_1, \dots, b_4\}$  each of which induces a diamond such that  $a_1a_4$  and  $b_1b_4$  are not edges, together with four paths  $P^1, \dots, P^4$  such that for  $i = 1, \dots, 4$ ,  $P^i$  is an  $a_ib_i$ -path. Paths  $P^1, \dots, P^4$  are node disjoint and the only adjacencies between them are the edges of the two diamonds.*

**Lemma 4.4** *Let  $R, B$  be a bicoloring of a proper even wheel  $(H, x)$  and let  $P = y_1, \dots, y_n$ ,  $n > 1$ , be a chordless path with the following properties:*

1)  $y_1$  is adjacent to a node in  $B$  and to no node in  $R$ ,  $y_n$  is adjacent to a node in  $R$  and no node in  $B$ , and  $y_1, y_n$  have nonadjacent neighbors in  $H$ .

2) No node of  $P$  is adjacent to  $x$  and no interior node of  $P$  is adjacent to a node of  $H$ .

Then  $(H, x)$  is a beetle and  $(H, x) \cup P$  induces connected diamonds.

*Proof:* Let  $t_1, t_2$  be blue and red neighbors of  $y_1, y_n$  in  $H$  such that one of the  $t_1t_2$ -subpaths, say  $Q_{12}$  of  $H$  is shortest. Then  $Q_{12} \cup \{x\}$  contains an odd number of triangles and therefore

$x$  has exactly two neighbors in  $Q_{12}$ , else there is an odd wheel. Let  $x_1, x_2$  be these two neighbors, closest to  $t_1, t_2$  in  $Q_{12}$  respectively. Since  $(H, x)$  is a proper wheel and  $y_1, y_n$  have nonadjacent neighbors in  $H$ , then  $y_1$  or  $y_n$  has at least one other neighbor in  $H$ , else there is an odd wheel with center  $x$ .

Assume w.l.o.g. that  $y_1$  has at least one other neighbor and let  $t_3$  be such node, closest to  $t_2$  in  $H \setminus Q_{12}$ . Let  $t_4$  be the neighbor of  $y_n$ , closest to  $t_3$  in the  $t_3 t_2$ -subpath of  $H \setminus t_1$ . (Possibly  $t_4 = t_2$ ). Let  $Q_{34}$  be the  $t_3 t_4$ -subpath of  $H \setminus t_1$ . By the above argument,  $x$  has exactly two neighbors in  $Q_{34}$ . Let  $x_3, x_4$  be these two neighbors, closest to  $t_3, t_4$  in  $Q_{34}$  respectively. We use the notation  $H = t_1, Q_{12}, t_2, Q_{24}, t_4, Q_{34}, t_3, Q_{13}, t_1$ . If  $t_2$  and  $t_4$  coincide,  $Q_{24}$  has length 0.

Now either  $t_1$  and  $t_3$  are adjacent or  $t_2 = t_4 = x_4$ , else we have a  $3PC(x_1 x_2 x, y_1)$ . Assume the second case does not hold: So  $t_1$  and  $t_3$  are adjacent. Since  $t_1, t_3$  are both blue nodes,  $x$  cannot be adjacent to both. So, since  $(H, x)$  is not a line wheel,  $x$  has at least one neighbor  $x^*$  in  $H \setminus \{x_1, x_2, x_3, x_4\}$  and by construction  $x^*$  is an interior node of  $Q_{24}$ . This implies that  $t_2$  and  $t_4$  are distinct and nonadjacent and in  $H \cup P$  there is a  $3PC(y_1 t_1 t_3, y_n)$ .

So the second case must hold, i.e.  $t_2 = t_4 = x_4$ . Furthermore since  $x_1, x_2$  are the unique neighbors of  $x$  in  $Q_{12}$ , we must have  $x_2 = t_2 = t_4 = x_4$ .

Since  $(H, x)$  is not a twin wheel,  $x$  has at least one other neighbor, say  $x^*$ , and by construction  $x^*$  is an interior node of  $Q_{13}$ . Now  $x^*$  must be adjacent to both  $t_1, t_3$ , else there is a  $3PC(x_1 x_2 x, y_1)$  or a  $3PC(x_3 x_4 x, y_1)$ . Finally  $t_1 \neq x_1$  and  $t_3 \neq x_3$ , else  $(H, x)$  is either an odd wheel or an universal wheel. So  $(H, x)$  is a beetle. Finally  $x^*$  is adjacent to  $y_1$ , else we have a  $3PC(x_1 x_2 x, t_1)$ . Therefore  $(H, x) \cup P$  induces connected diamonds.  $\square$

**Theorem 4.5** *Let  $G$  be a square-free even-signable graph that contains a proper even wheel  $(H, x)$ . Furthermore if  $(H, x)$  is a beetle, assume that no connected diamonds contain  $(H, x)$ . Let  $R, B$  be a bicoloring of the nodes in  $H$  and assume w.l.o.g. that  $B \setminus N(x)$  is nonempty. Then  $(N(x) \cup x) \setminus R$  is a star cutset of  $G$ , separating  $R$  from  $B \setminus N(x)$ .*

*Proof:* Assume not and let  $P = y_1, \dots, y_n$  be a direct connection from  $R$  to  $B \setminus N(x)$  in  $G \setminus (N(x) \cup x \cup H)$ . Let  $y_k$  be the node of lowest index with a neighbor in  $B$  that is not adjacent to all the neighbors of  $y_1$  in  $R$ . (Possibly  $k = n$ ). By Lemma 4.2,  $k > 1$ . So  $y_k$  has no neighbor in  $R$ . If no node of  $P$  with index smaller than  $k$  is adjacent to a node in  $B$ , the theorem holds as a consequence of Lemma 4.4. Let  $r \in R$  be adjacent to  $y_1$  and let  $y_j, j < k$ , be the node of lowest index adjacent to a node  $b \in B$ . Since  $j < k$ ,  $b$  and  $r$  are adjacent. If  $j = 1$ , by Lemma 4.2,  $b$  and  $r$  are the unique neighbors of  $y_1$  in  $H$  and if  $j > 1$ , since  $G$  is square-free and  $b$  is adjacent to all the neighbors of  $y_1$  in  $H$ ,  $r$  is the unique neighbor of  $y_1$  in  $H$ . Let  $z$  be the neighbor of  $r$  in  $H \setminus \{b\}$ .

Since  $(H, x)$  is a proper even wheel and  $B \setminus N(x)$  is nonempty,  $x$  has a neighbor  $b_1 \in B$  that is adjacent to neither  $b$  nor  $z$ . Let  $b_2$  be a neighbor of  $y_k$  in  $H \setminus \{z, r, b\}$ , so that the  $b_1 b_2$ -path  $T$  in  $H \setminus \{z, r, b\}$  is shortest and let  $Q = y_k, b_2, T, b_1, x$ . Then  $H' = x, r, y_1, P_{y_1 y_k}, y_k, Q, x$  is a hole.

**Claim 1:** *If  $z \in B$ , then  $z$  has no neighbor in  $H' \setminus \{x, r\}$ .*

*Proof of Claim 1:* Assume not. Then both  $(H', b)$  and  $(H', z)$  are wheels. If  $(H', b)$  is a proper wheel, the claim follows by Lemma 4.2 applied to  $z$  and a bicoloring of  $(H', b)$ . Furthermore

if  $(H', z)$  is a proper wheel, then  $b$  contradicts Lemma 4.2. So  $(H', b)$  and  $(H', z)$  are either line wheels or twin wheels. There are three possibilities depending on whether both are line wheels, both are twin wheels or one is a line wheel and the other is a twin wheel. Since  $b$  and  $z$  are not adjacent to  $b_1$ , it can be verified that in all three cases there is an odd wheel or a square and this proves Claim 1.

Let  $b'$  be the first neighbor of  $y_k$ , encountered when traversing  $H \setminus \{r\}$  from  $z$  to  $b$  and let  $S$  be the  $rb'$ -subpath of  $H \setminus \{b\}$ . By Claim 1,  $z$  has no neighbor in  $P_{y_1 y_k}$ , so  $r, y_1, P_{y_1 y_k}, y_k, b', S, r$  is a hole. Since  $b' \in B$  and  $r \in R$ ,  $S \cup \{x\}$  contains an odd number of triangles. Therefore  $x$  is adjacent to  $r, z$  and to no other node of  $S$ , else there is an odd wheel with center  $x$ . In particular, by Claim 1,  $b' \neq z$ , so  $b' \notin N(x)$  and  $y_k = y_n$ .

Let  $x'$  be the first neighbor of  $x$ , encountered when traversing  $H \setminus \{r\}$  from  $b'$  to  $b$  and let  $T$  be the  $b'x'$ -subpath of  $H \setminus \{b\}$ . If  $b'$  is the unique neighbor of  $y_k$  in  $T$ , there is a  $3PC(xrz, b')$ . If  $y_k$  has exactly two neighbors in  $T$ , say  $b', b''$ , and  $b', b''$  are adjacent, there is a  $3PC(y_k b' b'', x)$ . Finally if  $y_k$  has two nonadjacent neighbors in  $T$ , there is a  $3PC(xrz, y_k)$ .  $\square$

The following result is an immediate consequence of the above theorem.

**Theorem 4.6** *Let  $G$  be a square-free even-signable graph. If  $G$  contains a proper wheel that is not a beetle, then  $G$  has a star cutset.*

## 5 L-Parachutes

In this section, we assume that  $G$  is a square-free even-signable graph that contains an L-parachute  $\Pi = LP(ca, db, v, z)$ .

We use the following notation. The two triangles are  $acv$  and  $bdv$ . The *top path* is  $P_{cd}$  (the  $cd$ -path of  $H \setminus b$ ) and we indicate with  $c', d'$  the neighbors of  $c$  and  $d$  in  $P_{cd}$ . The *bottom path* is  $P_{ab}$  (the  $ab$ -path of  $H \setminus d$ ). The *middle path* is  $P_{mz}$ , where  $z$  is an interior node in  $P_{ab}$  and  $m$  is a neighbor of  $v$  in  $P$ .

**Lemma 5.1** *Let  $y$  be a node not adjacent to  $v$  but adjacent to at least two nodes in  $\{a, b, c, d\}$ . Then  $N(y) \cap \Pi$  is either  $\{a, c\}$  or  $\{b, d\}$ .*

*Proof:* Since  $G$  is square-free,  $N(y) \cap \{a, b, c, d\}$  is either  $\{a, c\}$  or  $\{b, d\}$  and we assume w.l.o.g. that  $N(y) \cap \{a, b, c, d\} = \{a, c\}$ .

**Claim 1:**  $P_{cd} \cup \{y\}$  contains an even number of triangles.

*Proof of Claim 1:* Assume not. If  $y$  contains a neighbor in  $P_{cd} \setminus \{c, c'\}$ , since  $y$  is adjacent to  $c$ ,  $P_{cd} \cup \{v, y\}$  induces an odd wheel with center  $y$ . So  $c, c'$  are the only neighbors of  $y$  in  $P_{cd}$  and  $P_{cd} \cup \{v, a, y\}$  induces an odd wheel with center  $c$ . This proves Claim 1.

The argument used in Claim 1 also shows that  $P_{ab} \cup y$  contains an even number of triangles. So  $a, c$  are the only neighbors of  $y$  in  $H = P_{cd} \cup P_{ab}$ , else  $(H, y)$  is an odd wheel. Suppose  $y$  has a neighbor in  $P_{mz}$ . Note that  $y$  is not adjacent to  $m$  since otherwise  $v, c, y, m$  induces a square. Let  $R$  be a shortest path from  $y$  to  $b$  in  $y \cup P_{mz} \cup P_{bz} \setminus m$ . Then  $(R \cup P_{cd} \cup y, v)$  is an odd wheel.  $\square$

**Lemma 5.2** *Let  $y$  be a node not adjacent to  $v$  but adjacent to a node in  $P_{cd}$  and to a node in  $P_{ab} \cup P_{mz}$ . Then  $N(y) \cap \Pi$  is either  $\{a, c\}$  or  $\{b, d\}$ .*

*Proof:* By Lemma 5.1, it suffices to assume that  $y$  has at most one neighbor in  $\{a, b, c, d\}$  and derive a contradiction.

**Claim 1:** *Node  $y$  has exactly two neighbors in  $P_{cd}$  and they are adjacent.*

*Proof of Claim 1:* Assume not, then  $y$  has either 1 neighbor or 2 nonadjacent neighbors in  $P_{cd}$ .

If  $y_1$  is the unique neighbor of  $y$  in  $P_{cd}$ , assume w.l.o.g. that  $y_1 \neq c$ . When  $y$  has a neighbor in  $P_{ab} \cup P_{mz} \setminus \{b, m\}$  there is a  $3PC(acv, y_1)$ . Since  $G$  is square-free,  $y$  cannot be adjacent to both  $b$  and  $m$ . If  $y$  is adjacent to  $b$  then it is not adjacent to  $d$  and there is a  $3PC(acv, b)$ . So  $m$  is the unique neighbor of  $y$  in  $P_{ab} \cup P_{mz}$ . If  $y_1 = d$  then  $v, m, d, y$  induces a square. So  $y_1 \neq d$ . W.l.o.g.  $a$  does not have a neighbor in  $P_{mz} \setminus z$ . But then there is a  $3PC(acv, m)$ .

Assume  $y$  has nonadjacent neighbors in  $P_{cd}$ . Since  $y$  has at most one neighbor in  $\{a, b, c, d\}$ , there is a  $3PC(acv, y)$  or a  $3PC(bdv, y)$  whenever  $y$  has a neighbor in  $P_{ab} \cup P_{mz} \setminus \{m\}$ . So  $m$  is the unique neighbor of  $y$  in  $P_{ab} \cup P_{mz}$  and assume w.l.o.g. that  $a$  does not have a neighbor in  $P_{mz} \setminus z$ . Then there is a  $3PC(acv, m)$  and this completes Claim 1.

So  $P_{cd} \cup \{y\}$  contains a unique triangle, say  $yy_1y_2$ . If  $y$  has a neighbor in  $P_{ab} \cup P_{mz} \setminus \{a, b\}$ , there is a  $3PC(yy_1y_2, v)$ . So  $y$  is adjacent to  $a$  or  $b$  and no other node of  $P_{ab}$ . Since  $y$  has at most one neighbor in  $\{a, b, c, d\}$ ,  $P_{cd} \cup P_{ab} \cup y$  induces an odd wheel with center  $y$ .  $\square$

**Theorem 5.3** *Let  $G$  be a square-free even-signable graph. If  $G$  contains an  $L$ -parachute, then  $G$  has a star cutset.*

*Proof:* We show that, if  $G$  contains an  $L$ -parachute  $\Pi = LP(ca, db, v, z)$  then  $S = v \cup N(v) \setminus \{a, b, m\}$  is a star cutset of  $G$ , separating  $P_{cd} \setminus \{c, d\}$  from  $P_{ab} \cup P_{zm}$ .

Assume not and let  $P = y_1, \dots, y_n$  be a direct connection from  $P_{cd} \setminus \{c, d\}$  to  $P_{ab} \cup P_{mz}$  in  $G \setminus S$ . We assume w.l.o.g. that  $\Pi$  and  $P$  are chosen so that the cardinality of the node set of  $\Pi \cup P$  is minimized. By Lemma 5.2  $n > 1$ . It also follows from our assumption that at most one of  $c, d$  has a neighbor in  $P \setminus \{y_n\}$  and if  $c$  is adjacent to a node in  $P \setminus \{y_n\}$ , then  $c'$  and possibly  $c$  are the only neighbors of  $y_1$  in  $\Pi$ . From now on, we assume that  $d$  has no neighbor in  $P \setminus \{y_n\}$ .

**Claim 1:** *No node of  $P \setminus \{y_n\}$  is adjacent to  $c$ .*

*Proof of Claim 1:* Assume not. Then  $c'$  and possibly  $c$  are the only neighbors of  $y_1$  in  $\Pi$ .

**Case 1:** Node  $y_1$  is the only neighbor of  $c$  in  $P \setminus \{y_n\}$ .

If  $y_n$  has a neighbor in  $P_{ab} \cup P_{zm} \setminus \{a, b\}$ , by Lemma 5.2,  $y_n$  is not adjacent to  $c$  or  $d$  and there is a  $3PC(y_1cc', v)$ . If  $y_n$  is adjacent to  $a$ , possibly  $c$  and to no other node of  $\Pi$  there is an odd wheel with center  $c$ . If  $y_n$  is adjacent to  $b, d$  and to no other node of  $\Pi$  there is a  $3PC(y_1cc', d)$ . Finally, if  $y_n$  is adjacent to  $b$  and to no other node of  $\Pi$  there is a  $3PC(y_1cc', b)$ . By Lemma 5.2, these are all the possibilities.

**Case 2:** Node  $c$  has a neighbor in  $P \setminus \{y_1, y_n\}$ .

If  $y_n$  has a neighbor in  $P_{ab} \cup P_{zm} \setminus \{m, b\}$ , by Lemma 5.2,  $y_n$  is not adjacent to  $d$ . Then  $P_{ab} \cup P_{zm} \cup \{y_n\} \setminus \{m, b\}$  contains a  $y_n a$ -path  $Q_1$  and  $P_{ab} \cup P_{zm} \cup \{y_n\}$  contains a  $y_n b$ -path  $Q_2$  such that  $H_1 = d, P_{dc'}, c', y_1, P, y_n, Q_1, a, v, d$  and  $H_2 = d, P_{dc'}, c', y_1, P, y_n, Q_2, b, d$  are both holes. Now one of  $(H_1, c)$ ,  $(H_2, c)$  is an odd wheel. If  $y_n$  is adjacent to  $b$  and has no neighbor in  $P_{ab} \cup P_{zm} \setminus \{m, b\}$  there is a  $3PC(acv, b)$ . Finally if  $m$  is the only neighbor of  $y_n$  in  $P_{ab} \cup P_{zm}$  there is a  $3PC(acv, m)$  or a  $3PC(bdv, m)$ . This completes Claim 1.

By the minimality of  $\Pi \cup P$ ,  $y_1$  has either a unique neighbor or two adjacent neighbors in  $P_{cd}$ .

Assume that  $y_1$  has a unique neighbor, say  $y^*$ , in  $P_{cd}$ . If  $y_n$  is adjacent to  $c$  or  $d$ , say  $d$ , by Lemma 5.2,  $y_n$  is adjacent to  $b$ ,  $d$  and to no other node of  $\Pi$  and there is a  $3PC(y_n b d, y^*)$ . If  $y_n$  has a neighbor in  $P_{ab} \cup P_{zm} \setminus \{m\}$  there is a  $3PC(acv, y^*)$  or a  $3PC(bdv, y^*)$ . So  $m$  is the unique neighbor of  $y_n$  in  $\Pi$  and there is a  $3PC(acv, m)$  or a  $3PC(bdv, m)$ .

So  $y_1$  has two neighbors, say  $y', y''$  in  $P_{cd}$  and  $y', y''$  are adjacent. By Claim 1,  $y_1$  is not adjacent to  $c$  or  $d$ . If  $y_n$  has a neighbor in  $P_{ab} \cup P_{zm} \setminus \{a, b\}$ , by Lemma 5.2,  $y_n$  is not adjacent to  $c$  or  $d$  and hence there is a  $3PC(y' y'' y_1, v)$ . By Lemma 5.2  $y_n$  is not adjacent to both  $a$  and  $b$ . So w.l.o.g.  $a$  is the unique neighbor of  $y_n$  in  $P_{ab} \cup P_{zm}$ . By Lemma 5.2,  $y_n$  is not adjacent to  $d$ . If  $y_n$  is not adjacent to  $c$  there is a  $3PC(y' y'' y_1, a)$  and otherwise there is a  $3PC(y' y'' y_1, c)$ .  $\square$

## 6 Nodes Adjacent to a $3PC(\Delta, \Delta)$

We denote by  $\Sigma$  a  $3PC(a_1 a_2 a_3, b_1 b_2 b_3)$  with the three paths  $P^1 = P_{a_1 b_1}$ ,  $P^2 = P_{a_2 b_2}$  and  $P^3 = P_{a_3 b_3}$ . For  $i = 1, 2, 3$ , we denote by  $a'_i$  the neighbor of  $a_i$  in  $P^i$  and by  $b'_i$  the neighbor of  $b_i$  in  $P^i$ . For distinct  $i, j \in \{1, 2, 3\}$ , we denote by  $H_{ij}$  the hole induced by  $P^i \cup P^j$ .

**Lemma 6.1** *Let  $G$  be an even-signable graph and let  $\Sigma$  be a  $3PC(\Delta, \Delta)$ . If node  $u$  is adjacent to  $\Sigma$ , then it is one of the following types.*

**Type t $_j$  for  $j = 1, 2, 3$ :** *Node  $u$  has exactly  $j$  neighbors in  $\Sigma$  and they are either all contained in  $\{a_1, a_2, a_3\}$  or all in  $\{b_1, b_2, b_3\}$ .*

**Type p1:** *Node  $u$  has exactly one neighbor in  $\Sigma$  and  $u$  is not of Type t1.*

**Type p2:** *Node  $u$  has exactly two neighbors in  $\Sigma$ , which are furthermore adjacent and contained in  $P^i$  for some  $i \in \{1, 2, 3\}$ .*

**Type p3:** *Node  $u$  has at least two nonadjacent neighbors in  $\Sigma$ , and all the neighbors of  $u$  in  $\Sigma$  are contained in  $P^i$ , for some  $i \in \{1, 2, 3\}$ .*

**Type p4:** *Node  $u$  has exactly four neighbors in  $\Sigma$ ,  $u_1, u_2, u_3$  and  $u_4$ , where  $u_1 u_2$  is an edge that belongs to some  $P^i$ ,  $i \in \{1, 2, 3\}$ , and  $u_3 u_4$  is an edge that belongs to some  $P^j$ ,  $j \in \{1, 2, 3\} \setminus \{i\}$ . Furthermore,  $u$  is not adjacent to both  $a_i$  and  $a_j$ , and it is not adjacent to both  $b_i$  and  $b_j$ .*

**Type t2p:** For distinct indices  $i, j, k \in \{1, 2, 3\}$  and for  $z \in \{a, b\}$ ,  $u$  is adjacent to  $z_i$  and  $z_j$ , it has at least one neighbor in  $P^k \setminus \{z_k\}$ , and is not adjacent to any node in  $(P^i \cup P^j \cup \{z_k\}) \setminus \{z_i, z_j\}$ .

**Type t3p:** Node  $u$  has at least four neighbors in  $\Sigma$ . For some  $z \in \{a, b\}$ ,  $u$  is adjacent to  $z_1, z_2$  and  $z_3$ , and all the other neighbors of  $u$  in  $\Sigma$  belong to  $P^i$  for some  $i \in \{1, 2, 3\}$ .

**Type t $j$  for  $j = 4, 5, 6$ :** Node  $u$  is adjacent to  $j$  nodes in  $\{a_1, a_2, a_3, b_1, b_2, b_3\}$  and possibly other nodes of  $\Sigma$ . Furthermore, if  $u$  is of Type t4, then  $u$  has two neighbors in  $\{a_1, a_2, a_3\}$  and two in  $\{b_1, b_2, b_3\}$ .

*Proof:* Assume that  $u$  is not of Type p2 or p3. Then, w.l.o.g.  $u$  has neighbors in both  $P^1$  and  $P^2$ .

**Case 1:**  $u$  does not have a neighbor in  $P^3$ .

First assume that  $u$  has a unique neighbor in  $P^1$  or  $P^2$ , say  $P^1$ . Let  $u_1$  be the neighbor of  $u$  in  $P^1$ , and w.l.o.g. assume that  $u_1 \neq a_1$ . Let  $u_2$  be the neighbor of  $u$  in  $P^2$  that is closest to  $a_2$ . If  $u_2 \neq b_2$ , then the node set  $P^1 \cup P^2_{a_2 u_2} \cup P^3 \cup \{u\}$  induces a  $3PC(a_1 a_2 a_3, u_1)$ . If  $u_2 = b_2$ , then either  $u$  is of Type t2 or the node set  $P^1_{a_1 u_1} \cup P^2 \cup P^3 \cup \{u\}$  induces a  $3PC(a_1 a_2 a_3, u_2)$ .

Now assume that  $u$  has at least two neighbors in both  $P^1$  and  $P^2$ . Let  $u_1$  (resp.  $v_1$ ) be the neighbor of  $u$  in  $P^1$  that is closest to  $a_1$  (resp.  $b_1$ ). Let  $u_2$  (resp.  $v_2$ ) be the neighbor of  $u$  in  $P^2$  that is closest to  $a_2$  (resp.  $b_2$ ). First suppose that both  $u_1 v_1$  and  $u_2 v_2$  are edges. If  $u$  is adjacent to both  $a_1$  and  $a_2$ , then  $P^2 \cup P^3 \cup \{u, a_1\}$  induces an odd wheel with center  $a_2$ . So  $u$  is not adjacent to both  $a_1$  and  $a_2$ , and similarly  $u$  is not adjacent to both  $b_1$  and  $b_2$ . Hence  $u$  is of Type p4. Now assume w.l.o.g. that  $u_1 v_1$  is not an edge. If  $u$  is not adjacent to all four of the nodes  $a_1, a_2, b_1$  and  $b_2$ , then either  $P^1_{a_1 u_1} \cup P^1_{v_1 b_1} \cup P^2_{a_2 u_2} \cup P^3 \cup \{u\}$  or  $P^1_{a_1 u_1} \cup P^1_{v_1 b_1} \cup P^2_{v_2 b_2} \cup P^3 \cup \{u\}$  induces a  $3PC(\Delta, u)$ . So  $u$  is adjacent to  $a_1, a_2, b_1$  and  $b_2$ , and hence it is of Type t4.

**Case 2:**  $u$  has a neighbor in  $P^3$ .

For  $i \in \{1, 2, 3\}$ , let  $u_i$  (resp.  $v_i$ ) be the neighbor of  $u$  in  $P^i$  that is closest to  $a_i$  (resp.  $b_i$ ). If  $u$  is adjacent to at most one node in  $\{a_1, a_2, a_3\}$  and at most one node in  $\{b_1, b_2, b_3\}$ , then the node set  $P^1_{v_1 b_1} \cup P^2_{v_2 b_2} \cup P^3_{v_3 b_3} \cup \{u\}$  induces a  $3PC(b_1 b_2 b_3, u)$ . So assume w.l.o.g. that  $u$  is adjacent to  $b_1$  and  $b_2$ . If  $u$  does not have a neighbor in  $(P^1 \cup P^2) \setminus \{b_1, b_2\}$ , then  $u$  is of Type t2p, t3 or t3p. So assume w.l.o.g. that  $u_1 \neq b_1$ . Suppose  $u$  is not of Type t4, t5 or t6. Then  $u$  is adjacent to at most one node of  $\{a_1, a_2, a_3\}$ . If  $u_2 = b_2$  and  $u_3 = b_3$ , then  $u$  is of Type t3p. Otherwise,  $P^1_{a_1 u_1} \cup P^2_{a_2 u_2} \cup P^3_{a_3 u_3} \cup \{u\}$  induces a  $3PC(a_1 a_2 a_3, u)$ .  $\square$

Nodes adjacent to  $\Sigma$  are further classified as follows.

**Type t6a:** A node  $u$  that is of Type t6 w.r.t.  $\Sigma$ , such that none of the paths of  $\Sigma$  is an edge and  $u$  has no neighbors in the interior of any of the paths of  $\Sigma$ .

**Type t6b:** A node  $u$  that is of Type t6 w.r.t.  $\Sigma$  but is not of Type t6a w.r.t.  $\Sigma$ .

**Type t4d:** For distinct  $i, j \in \{1, 2, 3\}$ ,  $N(u) \cap \{a_1, a_2, a_3, b_1, b_2, b_3\} = \{a_1, a_2, a_3, b_1, b_2, b_3\} \setminus \{a_i, b_j\}$ .

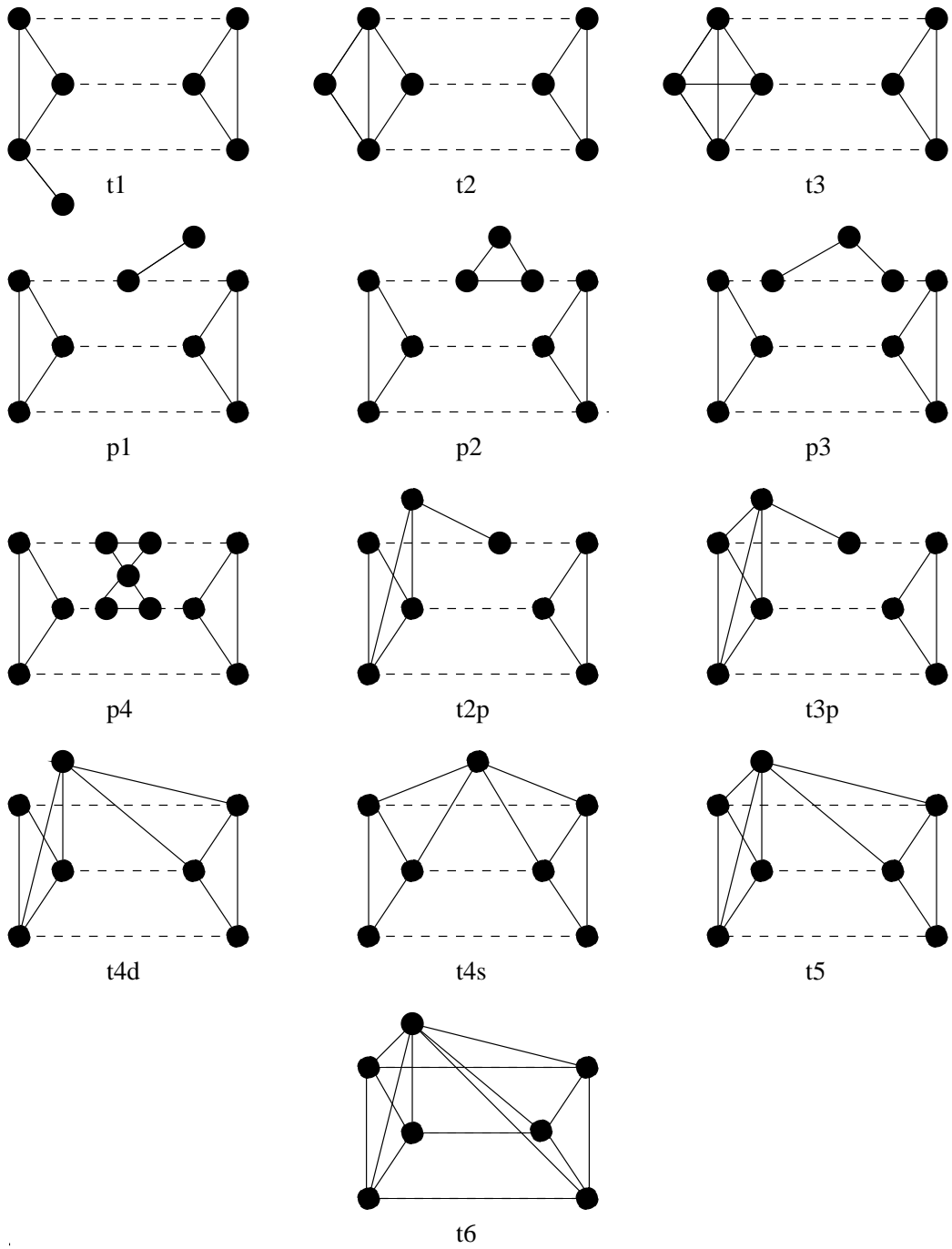


Figure 3: Strongly Adjacent Nodes to a  $3PC(\Delta, \Delta)$

**Type t4s:** For some  $i \in \{1, 2, 3\}$ ,  $N(u) \cap \{a_1, a_2, a_3, b_1, b_2, b_3\} = \{a_1, a_2, a_3, b_1, b_2, b_3\} \setminus \{a_i, b_i\}$ .

**Lemma 6.2** *Let  $G$  be a square-free even-signable graph with no star cutset. If  $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$  then the following holds.*

- (i) *If  $u$  is of Type t6b w.r.t.  $\Sigma$ , then  $u$  is adjacent to all nodes of  $\Sigma$ .*
- (ii) *If  $u$  is of Type t5 w.r.t.  $\Sigma$ , say not adjacent to  $a_3$ , then none of the paths of  $\Sigma$  is an edge and  $N(u) \cap \Sigma = \{a_1, a_2, b_1, b_2, b_3, b'_3\}$ .*
- (iii) *If  $u$  is of Type t4d w.r.t.  $\Sigma$ , say not adjacent to  $a_3$  and  $b_2$ , and  $a_1b_1$  is not an edge, then  $N(u) \cap \Sigma = \{a_1, a_2, a'_2, b_1, b_3, b'_3\}$ ,  $\{a_1, a_2, b_1, b_3, b'_1\}$  or  $\{a_1, a_2, a'_1, b_1, b_3\}$ .*
- (iv) *If  $u$  is of Type t4s w.r.t.  $\Sigma$ , say not adjacent to  $a_3$  and  $b_3$ , then  $a_1b_1$  and  $a_2b_2$  are not edges.*

*Proof:* Let  $u$  be of Type t6 w.r.t.  $\Sigma$ . For  $i, j \in \{1, 2, 3\}$ ,  $(H_{ij}, u)$  must be a line wheel or a universal wheel. If  $(H_{12}, u)$  is a line wheel, then so is  $(H_{13}, u)$ , and hence none of the paths of  $\Sigma$  is an edge and  $u$  has no neighbors in the interior of any of the paths of  $\Sigma$ , i.e.  $u$  is of Type t6a w.r.t.  $\Sigma$ . If  $(H_{12}, u)$  is a universal wheel, then so is  $(H_{13}, u)$ , and hence  $u$  is adjacent to all nodes of  $\Sigma$ .

Let  $u$  be of Type t5 w.r.t.  $\Sigma$ , say not adjacent to  $a_3$ . Suppose that  $(H_{12}, u)$  is a universal wheel. Since  $G$  is square-free,  $a_1b_1$  and  $a_2b_2$  cannot both be edges. W.l.o.g.  $a_1b_1$  is not an edge. But then  $(H_{13}, u)$  is a proper wheel that is not a beetle, contradicting Theorem 4.6. So  $(H_{12}, u)$  cannot be a universal wheel, and hence it must be a line wheel. But then  $(H_{13}, u)$  must be a beetle, and so (ii) holds.

Let  $u$  be of Type t4d w.r.t.  $\Sigma$ , say not adjacent to  $a_3$  and  $b_2$ . Suppose  $a_1b_1$  is not an edge.  $(H_{12}, u)$  must be a beetle or a line wheel. Suppose  $(H_{12}, u)$  is a line wheel. Then  $N(u) \cap (P^1 \cup P^2) = \{a_1, a_2, b_1, b'_1\}$ . If  $u$  has a neighbor in  $P^3 \setminus b_3$ , then  $(H_{13}, u)$  is a proper wheel that is not a beetle, contradicting Theorem 4.6. So now assume that  $(H_{12}, u)$  is a beetle. Suppose that  $u$  is adjacent to  $a'_1$ . Then  $N(u) \cap (P^1 \cup P^2) = \{a_1, a_2, a'_1, b_1\}$ . If  $u$  has a neighbor on  $P^3 \setminus b_3$ , then  $(H_{13}, u)$  is a proper wheel that is not a beetle, contradicting Theorem 4.6. Finally suppose that  $u$  is adjacent to  $a'_2$ . Then  $N(u) \cap (P^1 \cup P^2) = \{a_1, a_2, a'_2, b_1\}$ . But then  $(H_{23}, u)$  must be a line wheel and hence  $N(u) \cap P^3 = \{b_3, b'_3\}$ .

Let  $u$  be of Type t4s w.r.t.  $\Sigma$ , say not adjacent to  $a_3$  and  $b_3$ . Suppose  $a_1b_1$  is an edge. Since  $G$  is square-free,  $a_2b_2$  is not an edge. Node  $u$  must have a neighbor in  $P^3$ , since otherwise  $P^3 \cup \{a_1, a_2, b_1, u\}$  induces an odd wheel with center  $a_1$ . So  $(H_{13}, u)$  must be a line wheel. But then  $(H_{23}, u)$  is a proper wheel that is not a beetle, contradicting Theorem 4.6. So  $a_1b_1$  is not an edge, and similarly neither is  $a_2b_2$ .  $\square$

If node  $u$  is of Type p3, t2p or t3p w.r.t.  $\Sigma$ , then a subset of the node set  $\Sigma \cup \{u\}$  induces a  $\Sigma' = 3PC(\Delta, \Delta)$  that contains  $u$ . We say that  $\Sigma'$  is obtained by *substituting  $u$  into  $\Sigma$* . If  $u$  is of Type t2p or t3p w.r.t.  $\Sigma$ , and for some  $z \in \{a, b\}$  and  $i \in \{1, 2, 3\}$ ,  $\Sigma'$  does not contain  $z_i$ , then we say that  $u$  is a *sibling* of  $z_i$ .

## 7 Beetles and T-Parachutes

To prove the main result of this section, we need the following lemma, whose proof appears in [5].

**Lemma 7.1** *Let  $G$  be a T-parachute  $TP(t, v, a, b, z)$  that is not an L-parachute and such that no proper subgraph of  $G$  is a parachute or a proper wheel. Then  $G$  is one of the following graphs, see Figure 2.*

**Type a:** *No interior node of  $P$  is adjacent to  $a$  or  $b$ .*

**Type b:** *An interior node of  $P$  is adjacent to  $a$  or  $b$ , say  $a$ ,  $(C_b, a)$  is a triangle-free wheel and  $a$  is adjacent to  $z$ .*

**Type c:** *An interior node of  $P$  is adjacent to  $a$  or  $b$ , say  $a$ ,  $(C_b, a)$  is a twin wheel and  $a$  is adjacent to  $z$ .*

**Theorem 7.2** *Let  $G$  be a square-free even-signable graph. Assume that  $G$  has no  $3PC(\Delta, \Delta)$  with a Type  $t_2$ ,  $t_2p$  or  $t_4$  node. Then  $G$  is a triangle-free graph, or the line graph of a triangle-free graph, or  $G$  has a star cutset or a 2-join.*

*Proof:* By Theorem 2.3, the result holds for WP-free graphs. By Theorem 4.6, the result holds when  $G$  contains a proper wheel that is not a beetle. By Theorem 5.3, the result holds when  $G$  contains an L-parachute. So we may assume that  $G$  contains a beetle or a T-parachute.

Let  $\Sigma$  be a beetle or a T-parachute  $TP(t, v, a, b, z)$ . If  $\Sigma$  is a T-parachute, by Lemma 7.2, we assume that  $\Sigma$  is of Type a, b or c of Figure 2. If  $\Sigma$  is a beetle  $(H, v)$ , denote the neighbors of  $v$  on the hole by  $a, t, b, z$  where  $atv$  and  $tbv$  are the triangles. If  $\Sigma$  is a T-parachute, denote by  $(H, v)$  its twin wheel with center  $v$ . We assume w.l.o.g. that if  $\Sigma$  is a T-parachute of Type c, then  $G$  contains no beetle and no T-parachute of Type a or b. We denote by  $P$  the path of  $\Sigma$  from  $v$  to  $z$  that uses no edge of  $H$  and by  $H_{za}$  and  $H_{zb}$  the subpaths of  $H$  from  $z$  to  $a$  and from  $z$  to  $b$  that do not contain node  $t$ . W.l.o.g.  $b$  has no neighbor in the interior of  $P$ . Let  $C$  be the hole of  $\Sigma$  containing  $b, v, z$ . Consider the star  $S = (v \cup N(v)) \setminus \{t, m\}$  where  $m$  is the neighbor of  $v$  in  $\Sigma$  distinct from  $a, b, t$ . Assume that  $S$  is not a cutset separating  $t$  from  $B = V(\Sigma) \setminus \{a, b, v, t\}$  and let  $Q = x_1, \dots, x_n$  be a direct connection from  $t$  to  $B$  in  $G \setminus S$ . No node of  $Q$  is adjacent to both  $a$  and  $b$  since  $G$  is square-free.

**Case 1:**  $n = 1$ , or  $n > 1$  and no node of  $Q_{x_1 x_{n-1}}$  is adjacent to  $a$  or  $b$ .

Node  $x_n$  has at least one neighbor in  $C$  if  $\Sigma$  is a T-parachute of Type b or c and, by symmetry, we assume w.l.o.g. that this is also the case when  $\Sigma$  is a beetle or a T-parachute of Type a.

If  $x_n$  has exactly one neighbor  $p$  in  $C$ , then there is a  $3PC(bvt, p)$  since  $p$  is distinct from  $b$  and  $v$ .

Assume  $x_n$  has exactly two adjacent neighbors in  $C$ . Assume first that one of these neighbors is  $b$  and the other is  $b'$  adjacent to  $b$  in  $H_{zb}$ . If  $n = 1$ , there is an odd wheel with center  $b$ . So  $n > 1$ . If  $x_n$  has no neighbor in  $H_{za} \setminus z$ , there is a  $3PC(x_n bb', t)$ . Let  $z'$  be the

neighbor of  $z$  in  $H_{za}$ . If  $x_n$  has a neighbor in  $H_{za} \setminus \{z, z'\}$ , then there is a  $3PC(x_n bb', v)$ . Therefore  $x_n$  is adjacent to  $z'$ . If  $b' \neq z$ ,  $(H, x_n)$  is an odd wheel. So  $b' = z$  and  $(H, x_n)$  is a twin wheel. Now  $Q \cup \Sigma \setminus b$  induces a  $3PC(\Delta, \Delta)$  and  $b$  is a Type t4s node with respect to it, so the result holds.

Now assume that both neighbors of  $x_n$  in  $C$  are distinct from  $b$ . Then there is a  $3PC(\Delta, \Delta)$  with a node of Type t2 (when  $\Sigma$  is a beetle or of Type a) or of Type t2p (when  $\Sigma$  is of Type b) or of Type t4d (when  $\Sigma$  is of Type c). Therefore the result holds.

Assume  $x_n$  has two nonadjacent neighbors in  $C$ . Then there is a  $3PC(bvt, x_n)$  if  $n > 1$  or if  $x_n$  is not adjacent to  $b$ . So assume  $n = 1$  and  $x_1$  is adjacent to  $b$ . If  $x_1$  has a neighbor in  $H_{za} \setminus z$ , there is an odd wheel with center  $t$ . So all the neighbors of  $x_1$  in  $\Sigma$  are in  $C \cup t$ . If  $\Sigma$  is a beetle, there is a  $3PC(amt, z)$ . So  $\Sigma$  is a T-parachute. Assume first that  $\Sigma$  is of Type a. If  $x_1$  has no neighbor in  $P$ , there is a  $3PC(amt, z)$ . If  $x_1$  has a unique neighbor  $p$  in  $P$ , there is a  $3PC(amt, p)$ . Therefore  $x_1$  has several neighbors in  $P$ . Node  $z$  is adjacent to  $b$  since, otherwise, there is an odd wheel with center  $t$ . Let  $p$  be the neighbor of  $x_1$  in  $P$  that is closest to  $z$ . Then  $H_{za} \cup P_{zp} \cup \{t, x_1\}$  induces a hole  $H_1$  and, if  $p \neq z$ ,  $(H_1, b)$  is an odd wheel. So  $p = z$ . Let  $z'$  be the neighbor of  $z$  in  $P$ . If  $x_1$  has a neighbor in  $P \setminus \{z, z'\}$ , there is a  $3PC(bzx_1, v)$ . So the neighbors of  $x_1$  in  $P$  are exactly  $z$  and  $z'$  and therefore  $(C, x_1)$  is a twin wheel. Then  $(\Sigma \setminus b) \cup x_1$  induces a  $3PC(\Delta, \Delta)$  and  $b$  is a Type t4d strongly adjacent node with respect to it, so the result holds. Now assume that  $\Sigma$  is a T-parachute of Type b or c. Node  $x_1$  is not adjacent to  $m$  since, otherwise,  $m, v, t, x_1$  would be a square. Since  $x_1$  has a neighbor in  $C$  distinct from  $b$  and its neighbor in  $C$ , there is an odd wheel with center  $t$ .

**Case 2:**  $n > 1$  and some node of  $Q_{x_1 x_{n-1}}$  is adjacent to  $a$  or  $b$ .

Let  $x_j$  be such a node with lowest index.

Assume first that  $j \geq 2$ . Denote by  $d$  the node among  $a, b$  that is adjacent to  $x_j$ . Let  $R$  be a shortest path from  $x_n$  to  $v$  in  $\Sigma \setminus \{a, b, t\}$  and let  $R'$  be a shortest path in  $\Sigma \setminus \{t, v, d\}$  from  $x_n$  to the node  $d'$  among  $a, b$  that is distinct from  $d$ . Let  $H_1$  be the hole induced by  $R \cup Q \cup t$  and  $H_2$  the hole induced by  $R' \cup Q \cup t$ . We will show that the wheel  $(H_1, d)$  is a proper wheel that is not a beetle.  $(H_1, d)$  is not a universal wheel since  $d$  is not adjacent to  $x_1$  nor a triangle-free wheel since  $d$  is adjacent to  $v$  and  $t$ , nor a twin wheel since  $x_j$  is adjacent to  $d$  but not to  $t$  or  $v$ . Suppose  $(H_1, d)$  is a line wheel. Then  $(H_2, d)$  is a proper wheel. Suppose  $(H_2, d)$  is a beetle. Then  $j = n - 1$  and  $z$  is adjacent to both  $x_n$  and  $d$ , and  $x_n$  has a neighbor in  $P_{vz} \setminus z$ . Note that if  $d = b$ , then  $az$  cannot be an edge (since  $bz$  is an edge and  $H$  cannot be a square) and hence  $\Sigma$  cannot be a T-parachute of Type a or b. If  $x_n$  has a neighbor in  $P \setminus \{z, z'\}$  (where  $z'$  is the neighbor of  $z$  in  $P$ ), then there is a  $3PC(tvd', x_n)$ . So  $z$  and  $z'$  are the only neighbors of  $x_n$  in  $P$ , and hence there is a  $3PC(\Delta, \Delta)$  with a Type t4d node and the result holds. Finally, suppose  $(H_1, d)$  is a beetle. Then  $d = a$ ,  $\Sigma$  is of Type c and  $x_n$  is adjacent to  $m$ . If  $x_n$  has no other neighbor in  $\Sigma$ , there is a  $3PC(tvd', m)$ . If  $x_n$  has a neighbor distinct from  $m$  and  $z$  in  $C$ , there is a  $3PC(tvd', x_n)$ . If  $x_n$  has exactly  $m$  and  $z$  as neighbors, there is a  $3PC(\Delta, \Delta)$  with node  $d$  being of Type t4d relative to it. So the result holds. Therefore  $(H_1, d)$  is a proper wheel that is not a beetle. So the result holds by Theorem 4.6.

Assume now that  $j = 1$ . Denote by  $d$  the node among  $a, b$  that is adjacent to  $x_1$ .

**Case 2.1:** No node of  $Q_{x_2 x_n}$  is adjacent to  $a$  or  $b$ .

If every chordless path from  $x_n$  to  $t$  in  $\Sigma \setminus \{d, v\}$  contains  $m$ , then let  $R_1$  be such a path and let  $H_3$  be the hole  $R_1 \cup Q$ . Then  $(H_3, v)$  is an odd wheel unless  $d = b$  and  $\Sigma$  is a T-parachute of Type c. But then  $Q \cup \Sigma \setminus v$  induces a  $3PC(\Delta, \Delta)$  and  $v$  is a strongly adjacent node of Type t4d with respect to it. So the result holds.

Now assume some chordless path  $R_2$  from  $x_n$  to  $t$  in  $\Sigma \setminus \{d, v\}$  does not contain  $m$ . If  $R_2$  does not contain neighbors of  $d$ , then let  $H_4$  be the hole  $(R_2 \setminus t) \cup Q \cup \{v, d\}$ . Then  $(H_4, t)$  is an odd wheel. So  $R_2$  contains a neighbor of  $d$ . Let  $H_5$  be the hole  $R_2 \cup Q$ . Then  $(H_5, d)$  is an odd wheel.

**Case 2.2:** Some node of  $Q_{x_2x_n}$  is adjacent to  $a$  or  $b$ .

Let  $x_k$  be such a node with lowest index. If  $x_k$  is not adjacent to  $d$ , then  $Q_{x_1x_k} \cup \{a, v, b\}$  induces a hole  $H_6$  and  $(H_6, t)$  is an odd wheel. So  $x_k$  is adjacent to  $d$ . Let  $H_7$  be a hole in  $Q \cup \Sigma \setminus \{a, b\}$  that contains  $v$ . Since  $d$  is adjacent to  $v, t, x_1$  and  $x_k$ , the wheel  $(H_7, d)$  is a proper wheel or a universal wheel.

**Case 2.2.1:**  $(H_7, d)$  is a proper wheel.

If  $(H_7, d)$  is not a beetle, the result holds by Theorem 4.6. So assume  $(H_7, d)$  is a beetle. If  $d = a$  then  $\Sigma$  cannot be a T-parachute of Type c. The node in  $\{a, b\} \setminus \{d\}$  has no neighbor in  $Q$ , since otherwise a subpath of  $Q$  together with  $a, t$  and  $b$  induces an odd wheel with center  $d$ . Let  $R$  be a chordless path from  $x_n$  to  $t$  in  $\Sigma \setminus \{d, v\}$ . Some interior node of  $R$  must be adjacent to  $d$  since, otherwise, there is an odd wheel with center  $d$ .  $x_n$  must have a neighbor in  $\Sigma$  distinct from the neighbor  $d'$  of  $d$  in  $H_{zd}$  since, if  $d'$  were the only neighbor of  $x_n$  in  $\Sigma$ , the assumption that  $(H_7, d)$  is a beetle would be contradicted. This implies that  $d' = z$  and that  $x_n$  has all its neighbors in  $P$ . So  $\Sigma$  is not a beetle. Let  $H_8$  be the hole  $R \cup Q$ .  $(H_8, d)$  is a wheel. Since  $d$  is adjacent to  $t$  and  $x_1$  but not its neighbors on  $H_8$ , it is not a beetle. So if  $(H_8, d)$  is a proper wheel, the result holds by Theorem 4.6. If  $(H_8, d)$  is not a proper wheel, it must be a line wheel. This implies that  $k = n$  and that  $x_n$  is adjacent to  $z$ . If  $d = b$  then, since  $bz$  is an edge,  $az$  cannot be an edge (else  $H$  is a square) and so  $\Sigma$  cannot be a T-parachute of Type c.  $\Sigma$  cannot be a T-parachute of Type b since, otherwise,  $d = a$  and there is a  $3PC(azx_n, v)$ . So  $\Sigma$  is a T-parachute of Type a. Since  $(H_7, d)$  is a beetle,  $x_n$  has a neighbor in  $P \setminus z$ . If  $x_n$  has a neighbor in  $P \setminus z$  distinct from the neighbor  $z'$  of  $z$ , there is a  $3PC(dx_n, v)$ . So  $x_n$  has exactly three neighbors in  $\Sigma$ , namely  $d, z$  and  $z'$ . In this case,  $Q \cup \Sigma \setminus d$  induces a  $3PC(\Delta, \Delta)$  and the node  $d$  is a strongly adjacent node of Type t4d with respect to it. So the result holds.

**Case 2.2.2:**  $(H_7, d)$  is a universal wheel.

Then  $\Sigma$  is a T-parachute of Type c and  $d = a$ . Let  $H_9$  be the hole in  $Q \cup \Sigma \setminus \{a, v\}$  that contains  $b$ . Then  $(H_9, a)$  is a proper wheel that is not a beetle unless  $n = 2$  and  $x_n$  has a neighbor in  $C \setminus \{z, m, v, b\}$ . Since  $n = 2$ ,  $x_2$  is not adjacent to  $m$  (otherwise there is a 5-wheel with center  $a$ ) but  $x_2$  is adjacent to  $z$  since  $(H_7, a)$  is a universal wheel. Let  $z'$  be the neighbor of  $z$  on  $C$  distinct from  $m$  and  $p$  the neighbor of  $x_2$  closest to  $b$  in  $H \setminus \{a, z\}$ . Suppose  $p \neq z'$  and let  $H_{10}$  be the hole induced by  $H_{bp} \cup \{v, m, z, x_2\}$ . Then  $(H_{10}, a)$  is an odd wheel. So  $p = z'$ . But now  $\Sigma \setminus a \cup \{x_1, x_2\}$  induces a  $3PC(\Delta, \Delta)$  and node  $a$  is a Type t4s node. So the result holds.  $\square$



or p1 w.r.t.  $\Sigma$  and similarly that  $x_n$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ . If  $x_n$  is of Type p2 w.r.t.  $\Sigma$ , then the node set  $P^1 \cup P^2 \cup P$  induces a  $3PC(\Delta, \cdot)$ . Hence  $x_n$  is also of Type t1 or p1 w.r.t.  $\Sigma$ . Let  $u_1$  (resp.  $u_2$ ) be the unique neighbor of  $x_1$  (resp.  $x_n$ ) in  $\Sigma$ . W.l.o.g.  $u_1 \neq a_1$ . If  $u_1 = b_1$  and  $u_2 = b_2$ , then  $P$  is a weak connection, and otherwise the node set  $P^1 \cup P_{a_2 u_2}^2 \cup P^3 \cup P$  induces a  $3PC(a_1 a_2 a_3, u_1)$ .

Now assume that  $G$  is square-free and has no star cutset. Suppose  $\Sigma = 3PC(a_1 a_2 a_3, b_1 b_2 b_3)$  has a weak connection  $P = x_1, \dots, x_n$  from  $a_1$  to  $a_2$ . Let  $S = (N(a_1) \cup a_1) \setminus \{x_1, a'_1\}$ . Since  $S$  is not a star cutset, there exists a direct connection  $Q = y_1, \dots, y_m$  from  $P$  to  $\Sigma \setminus S$  in  $G \setminus S$ . The nodes of  $P \cup P^1 \cup P^3 \cup a_2$  induce a Mickey Mouse as defined in [6]. By the Mickey Mouse theorem [6], some node of  $Q$  is adjacent to both  $a_2$  and  $a_3$ . Choose  $P, Q$  such that  $Q$  is as short as possible subject to the condition that  $P$  is a weak connection with one endnode adjacent to  $a_1$  and the other adjacent to either  $a_2$  or  $a_3$ . Let  $y_k$  be the node of lowest index adjacent to both  $a_2$  and  $a_3$ .

**Claim 1:** *If  $k \geq 2$ , no node  $y_1, \dots, y_{k-1}$  is adjacent to  $a_2$  or  $a_3$ .*

*Proof of Claim 1:* Suppose not and let  $y_j$  be the node of lowest index adjacent to  $a_2$  or  $a_3$  (note that  $j = 1$  is possible). Node  $y_1$  has a unique neighbor on  $P$  and this neighbor is  $x_n$  since, otherwise, our choice of  $P, Q$  would be violated. If  $y_j$  is adjacent to  $a_3$ , there is a  $3PC(a_1 a_2 a_3, x_n)$ . So  $y_j$  is adjacent to  $a_2$ . Let  $y_i, j < i \leq k$ , be the node of lowest index adjacent to  $a_3$ . Then  $W = (Q_{y_1 y_i} a_3 a_1 P, a_2)$  is a wheel that is not triangle-free, universal, a twin wheel or a beetle. Suppose it is a line wheel with triangles  $a_2 x_n y_1$  and  $a_1 a_2 a_3$ . Then  $i < k$  and therefore there is an L-parachute with middle path  $a_2, y_k, y_{k-1}, \dots, y_i$ . But then  $G$  has a star cutset by Theorem 5.3, a contradiction. So  $W$  is a proper wheel that is not a beetle. But then  $G$  has a star cutset by Theorem 4.6, a contradiction. This completes the proof of Claim 1.

Let  $S' = (N(a_2) \cup a_2) \setminus \{x_n, a'_2\}$ . Since  $G$  has no star cutset, there is a direct connection  $Q' = z_1, \dots, z_p$  from  $P$  to  $\Sigma \setminus S'$  in  $G \setminus S'$ . By the Mickey Mouse theorem applied to the Mickey Mouse induced by the nodes  $P \cup P^2 \cup P^3 \cup a_1$ , there is node of  $Q'$  adjacent to both  $a_1$  and  $a_3$ . Let  $z_l$  be the node of lowest index in  $Q'$  adjacent to both  $a_1$  and  $a_3$ . Let  $z_t$  be the node of lowest index in  $Q'$  that is adjacent to  $a_1$  or  $a_3$ .

**Case 1:**  $t < l$

Suppose first that  $z_t$  is adjacent to  $a_3$ . Then  $z_1$  is adjacent to  $a_1, P$  in a triangle since, otherwise, there is a  $3PC(a_1 a_2 a_3, \cdot)$ . Let  $u, v$  be the neighbors of  $z_1$  on  $P$ . If  $u, v \neq a_1$ , then there is a  $3PC(z_1 uv, a_1)$ . So  $u$  or  $v$  coincides with  $a_1$ . But then there is an L-parachute induced by a subpath of  $P \cup Q'_{z_1 z_t} \cup \{a_1, a_2, a_3\}$ . So, by Theorem 5.3, there is a star cutset, a contradiction.

So  $z_t$  is adjacent to  $a_1$ . Let  $z_r$  be the node of  $Q'$  with lowest index adjacent to  $a_3$ . Clearly  $t < r \leq l$ . Let  $H$  be the hole passing through  $a_3$  in  $Q' \cup P \cup a_3$ . Then  $(H, a_1)$  is a wheel that is not triangle-free, universal or a twin wheel. If  $r < l$ , then the wheel  $(H, a_1)$  is not a beetle. Suppose it is a line wheel with triangles  $a_1 x_1 z_t$  and  $a_1 a_2 a_3$ . Then there is an L-parachute with middle path being a subpath of  $a_1, z_l, z_{l-1}, \dots, z_t$ . If  $(H, a_1)$  is a line wheel with triangles  $a_1 z_t z_{t+1}$  and  $a_1 a_2 a_3$ , then there is an L-parachute with middle path being a subpath of  $a_1, z_l, \dots, z_r$ . But then  $G$  has a star cutset by Theorem 5.3, a contradiction. So

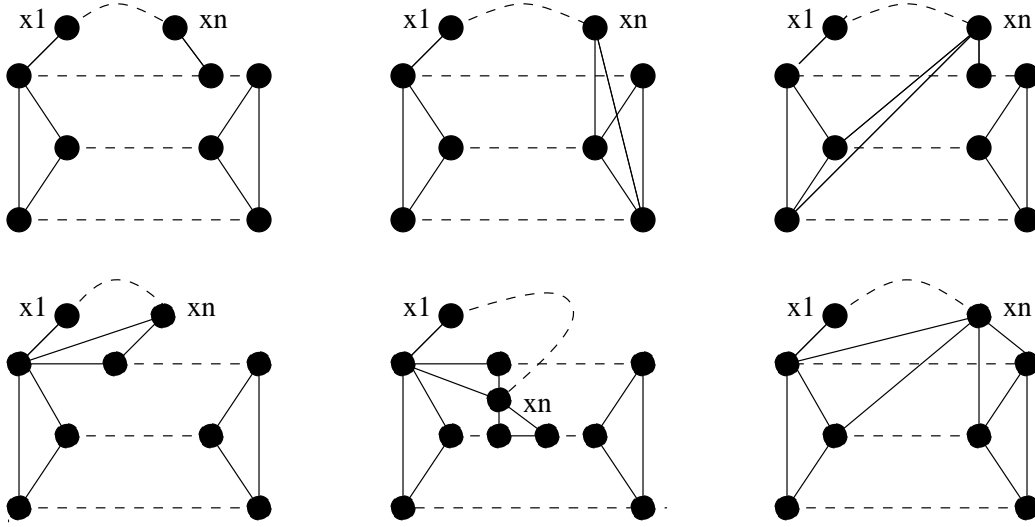


Figure 5: Paths from a Type t1 node

$W$  is a proper wheel that is not a beetle. But then  $G$  has a star cutset by Theorem 4.6, a contradiction. Therefore  $r = l$ . Now  $(H, a_1)$  is not an L-wheel. Since  $G$  does not have a star cutset,  $(H, a_1)$  cannot be a proper wheel by Theorem 4.6, so it must be a beetle. Note that  $Q$  is a path from the top of the beetle to the bottom that does not induce connected diamonds with the nodes of the beetle (since  $y_k$  is adjacent to  $a_2$ ), contradicting Lemma 4.4.

**Case 2:**  $t = l$

Let  $R$  be a shortest path from  $y_k$  to  $z_l$  in  $P \cup Q \cup Q'$ . If  $R$  contains neither  $x_1$  nor  $x_n$ , then  $R \cup \{a_1, a_2, a_3\}$  induces an odd wheel with center  $a_3$ . So  $R$  contains  $x_1$  or  $x_n$  and  $Q, Q'$  have no adjacent nodes. W.l.o.g. assume that  $y_1$  is adjacent to  $x_1$  or  $x_n$ . Then there is a  $3PC(a_1 a_3 z_l, x_1)$  or a  $3PC(a_2 a_3 y_k, x_n)$ .  $\square$

**Lemma 8.4** *Let  $P = x_1, \dots, x_n$  be a chordless path in  $G \setminus \Sigma$  such that  $x_1$  is of Type t1 w.r.t.  $\Sigma$ , say adjacent to  $a_1$ ,  $x_n$  has a neighbor in  $\Sigma \setminus \{a_1, a_2, a_3\}$ , and no interior node of  $P$  has a neighbor in  $\Sigma$ . Then one of the following holds.*

- (i)  $x_n$  is of Type t1, p1 or p3 w.r.t.  $\Sigma$ , with a neighbor in  $P^1$ .
- (ii)  $x_n$  is of Type t2, t2p or t3p w.r.t.  $\Sigma$  and  $N(x_n) \cap (P^2 \cup P^3) = \{b_2, b_3\}$ .
- (iii)  $x_n$  is of Type t2p or t3p w.r.t.  $\Sigma$  with at least two neighbors in  $\{a_1, a_2, a_3\}$ .
- (iv)  $x_n$  is of Type p2 or p4 w.r.t.  $\Sigma$  and it is adjacent to  $a_1$ .
- (v)  $x_n$  is of Type t4, t5 or t6 w.r.t.  $\Sigma$ .

*Proof:* Suppose  $x_n$  is of Type t1 or p1 with a unique neighbor  $u$  in  $\Sigma$ . If  $u$  is not in  $P^1$ , then  $P^2 \cup P^3 \cup P \cup x$  induces a  $3PC(a_1 a_2 a_3, u)$ . Similarly, if  $x_n$  is of Type p3, then it must satisfy

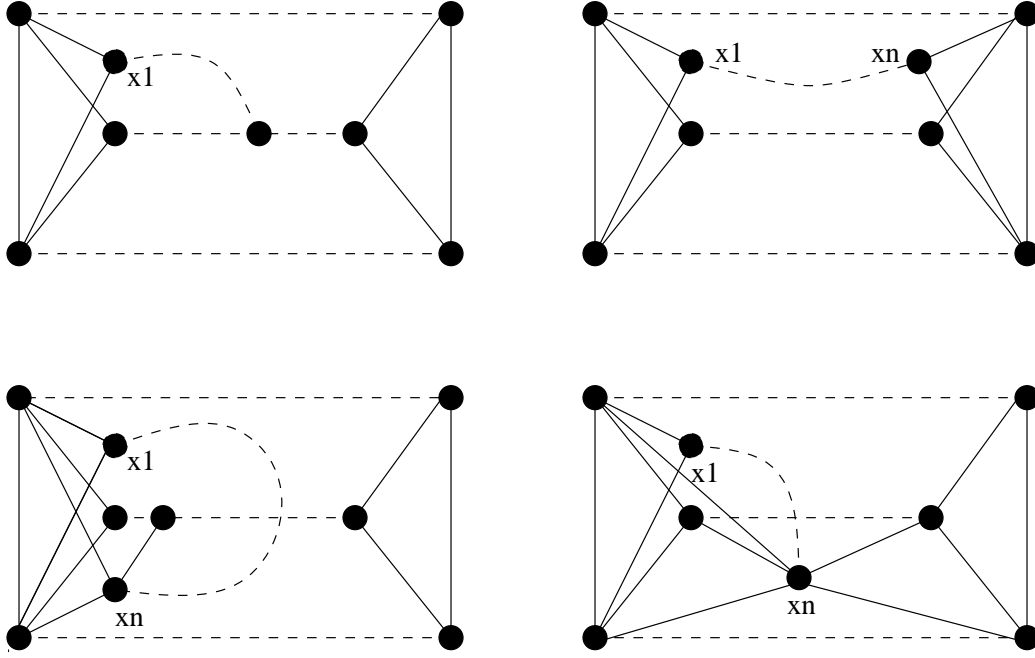


Figure 6: Paths from a Type t2 node

(i), else there is a  $3PC(a_1a_2a_3, x_n)$ . Suppose  $x_n$  is of Type p2, with neighbors  $u$  and  $v$  in  $\Sigma$ , and w.l.o.g. assume that  $u$  and  $v$  are not in  $P^3$ . If  $x_n$  does not satisfy (iv), then  $P^1 \cup P^2 \cup P$  induces a  $3PC(uvx_n, a_1)$ . If  $x_n$  is of Type t2 and it does not satisfy (ii), then w.l.o.g. we may assume that it is adjacent to  $b_1$  and  $b_3$ , and hence  $P \cup P^2 \cup P^3$  induces a  $3PC(a_1a_2a_3, b_3)$ . Suppose  $x_n$  is of Type t2p or t3p and it does not satisfy (ii) or (iii). W.l.o.g.  $x_n$  is adjacent to  $b_1, b_3$  and it has a neighbor in  $P^2 \setminus b_2$ . Then  $(P \cup P^1 \cup P^2) \setminus b_2$  contains a  $3PC(a_1a_2a_3, x_n)$ . If  $x_n$  is of Type t3, then it is adjacent to  $b_1, b_2, b_3$  and  $P \cup P^1 \cup P^2$  induces a  $3PC(b_1b_2x_n, a_1)$ . Finally assume that  $x_n$  is of Type p4. If the neighbors of  $x_n$  in  $\Sigma$  are contained in  $P^2 \cup P^3$ , then  $((P^2 \cup P^3) \setminus \{b_2, b_3\}) \cup P \cup a_1$  contains a  $3PC(a_1a_2a_3, x_n)$ . Else, we may assume w.l.o.g. that the neighbors of  $x_n$  in  $\Sigma$  are contained in  $P^1 \cup P^2$ . If  $x_n$  does not satisfy (v), then  $(\Sigma \setminus \{b_2, a'_1\}) \cup P$  contains a  $3PC(a_1a_2a_3, x_n)$ .  $\square$

**Lemma 8.5** *Let  $P = x_1, \dots, x_n$  be a chordless path in  $G \setminus \Sigma$  such that  $x_1$  is of Type t2 w.r.t.  $\Sigma$ , say adjacent to  $a_1$  and  $a_3$ ,  $x_n$  has a neighbor in  $\Sigma \setminus \{a_1, a_2, a_3\}$ , and no interior node of  $P$  has a neighbor in  $\Sigma$ . Then one of the following holds.*

- (i)  $x_n$  is of Type t1, p1 or p3 w.r.t.  $\Sigma$ , with a neighbor in  $P^2$ .
- (ii)  $x_n$  is of Type t2, t2p or t3p w.r.t.  $\Sigma$  and  $N(x_n) \cap (P^1 \cup P^3) = \{b_1, b_3\}$ .
- (iii)  $x_n$  is of Type t2p or t3p w.r.t.  $\Sigma$  with at least two neighbors in  $\{a_1, a_2, a_3\}$ .
- (iv)  $x_n$  is of Type t4, t5 or t6 w.r.t.  $\Sigma$

*Proof:* Suppose  $x_n$  is of Type t1 or p1 with  $u$  being its unique neighbor in  $\Sigma$  and (i) does not hold. Then w.l.o.g.  $u$  is in  $P^1$  and hence  $P^1 \cup P^3 \cup P$  induces a  $3PC(x_1 a_1 a_3, u)$ . Suppose  $x_n$  is of Type p3 and it does not satisfy (i). W.l.o.g. the neighbors of  $x_n$  in  $\Sigma$  are in  $P^1$ . If  $n = 2$  and  $x_n$  is adjacent to  $a_1$ , then  $P^1 \cup P^2 \cup P \cup a_3$  contains an odd wheel with center  $a_1$ , and otherwise  $P^1 \cup P^3 \cup P$  contains a  $3PC(x_1 a_1 a_3, x_n)$ . Suppose  $x_n$  is of Type p2, with neighbors  $u$  and  $v$  in  $\Sigma$ , and w.l.o.g. assume that  $u$  and  $v$  are not in  $P^3$ . If  $x_n$  is not adjacent to  $a_1$ , then  $P^1 \cup P^2 \cup P$  induces a  $3PC(x_n uv, a_1)$ . So  $x_n$  is adjacent to  $a_1$ , and hence  $P^1 \cup P^2 \cup P \cup a_3$  induces an odd wheel with center  $a_1$  when  $n > 2$  and  $P^1 \cup P^3 \cup P$  induces an odd wheel with center  $a_1$  when  $n = 2$ . If  $x_n$  is of Type t2, adjacent to  $b_2$  and say  $b_1$ , then  $P^1 \cup P^2 \cup P$  induces a  $3PC(x_n b_1 b_2, a_1)$ . So, if  $x_n$  is of Type t2, then it satisfies (ii). Similarly, if  $x_n$  is of Type t3, then there is a  $3PC(x_n b_1 b_2, a_1)$ . If  $x_n$  is of Type t2p or t3p, and it does not satisfy (ii) or (iii), then w.l.o.g. we may assume that  $x_n$  is adjacent to  $b_1, b_2$  and a node of  $P^3 \setminus b_3$ . But then  $P^1 \cup P^2 \cup P$  induces a  $3PC(x_n b_1 b_2, a_1)$ . Finally assume that  $x_n$  is of Type p4. If the neighbors of  $x_n$  in  $\Sigma$  are contained in  $P^1 \cup P^3$ , then w.l.o.g.  $x_n$  is not adjacent to  $a_3$  and hence  $(\Sigma \setminus \{a_1, a'_3\}) \cup P$  contains a  $3PC(b_1 b_2 b_3, x_n)$ . Otherwise, w.l.o.g. we may assume that the neighbors of  $x_n$  in  $\Sigma$  are contained in  $P^1 \cup P^2$ , and hence  $(\Sigma \setminus \{a_1, a_2\}) \cup P$  contains a  $3PC(b_1 b_2 b_3, x_n)$   $\square$

**Lemma 8.6** *Let  $P = x_1, \dots, x_n$  be a chordless path in  $G \setminus \Sigma$  such that  $x_1$  is of Type t3 w.r.t.  $\Sigma$ , say adjacent to  $a_1, a_2$  and  $a_3$ ,  $x_n$  has a neighbor in  $\Sigma \setminus \{a_1, a_2, a_3\}$ , and no interior node of  $P$  has a neighbor in  $\Sigma$ . Then one of the following holds.*

- (i)  $x_n$  is of Type p2 or t3 w.r.t.  $\Sigma$ .
- (ii)  $n = 2$ ,  $x_n$  is of Type t2p or t3p w.r.t.  $\Sigma$  with at least two neighbors in  $\{b_1, b_2, b_3\}$  and  $x_n$  is adjacent to  $a_1, a_2$  or  $a_3$ .
- (iii)  $x_n$  is of Type t2p or t3p w.r.t.  $\Sigma$  with at least two neighbors in  $\{a_1, a_2, a_3\}$ .
- (iv)  $n = 2$ ,  $x_n$  is of Type p3 w.r.t.  $\Sigma$  adjacent to  $a_1, a_2$  or  $a_3$ .
- (v)  $x_n$  is of Type t4, t5 or t6 w.r.t.  $\Sigma$ .

*Proof:* If  $x_n$  is of Type t1 or p1 with its neighbor  $u$  in say  $P_1$ , then there is a  $3PC(a_1 a_2 x_1, u)$ . If  $x_n$  is of Type p3, with neighbors in say  $P^1$ , and (iv) does not hold, then  $P^2 \cup P^3 \cup P$  contains a  $3PC(x_1 a_1 a_2, x_n)$ . By Lemma 8.5,  $x_n$  cannot be of Type t2 w.r.t.  $\Sigma$ . Suppose  $x_n$  is of Type t2p or t3p, but (ii) and (iii) do not hold. Then w.l.o.g.  $x_n$  is a sibling of  $b_1$ , and hence  $(P^1 \cup P^2 \cup P) \setminus b_1$  contains a  $3PC(x_1 a_1 a_2, x_n)$ . Finally assume that  $x_n$  is of Type p4, with neighbors w.l.o.g. in  $P^2 \cup P^3$ . Then  $(\Sigma \cup P) \setminus \{a_1, a_3\}$  contains a  $3PC(b_1 b_2 b_3, x_n)$ .  $\square$

## 9 Type t4, t5 and t6b Nodes

**Theorem 9.1** *Let  $G$  be a square-free even-signable graph. If  $G$  contains a  $\Sigma = 3PC(\Delta, \Delta)$  with a Type t4, t5 or t6b node then  $G$  has a star cutset.*

*Proof:* Assume  $G$  has no star cutset. Then by Theorem 4.6,  $G$  contains no proper wheel that is not a beetle.

Let  $\mathcal{C}$  be the set of all ordered pairs  $\Sigma, u$  such that  $\Sigma = 3PC(\Delta, \Delta)$  and  $u$  is of Type t4, t5 or t6b w.r.t.  $\Sigma$ .

For  $\Sigma, u \in \mathcal{C}$ , we assume w.l.o.g. that if  $u$  is of Type t5 w.r.t.  $\Sigma$  then  $u$  is not adjacent to  $a_3$ , if  $u$  is of Type t4d w.r.t.  $\Sigma$  then  $u$  is not adjacent to  $a_3$  and  $b_2$ , and if  $u$  is of Type t4s w.r.t.  $\Sigma$  then  $u$  is not adjacent to  $a_3$  and  $b_3$ .

For  $\Sigma, u \in \mathcal{C}$  define the corresponding sets  $S$  as follows. If  $u$  is of Type t5 or t6 w.r.t.  $\Sigma$ , then let  $S = (N(u) \cup u) \setminus (\Sigma \setminus \{a_1, a_2, b_2, b_3\})$ . If  $u$  is of Type t4d w.r.t.  $\Sigma$ , then let  $S = (N(u) \cup u) \setminus (\Sigma \setminus \{a_1, a_2, b_1, b_3\})$ . If  $u$  is of Type t4s w.r.t.  $\Sigma$ , then let  $S = (N(u) \cup u) \setminus (\Sigma \setminus \{a_1, a_2, b_1, b_2\})$ . Since  $S$  is not a star cutset, there exists a direct connection  $P = x_1, \dots, x_n$  in  $G \setminus S$  from  $P^1 \cup P^2$  to  $P^3$ . Let  $\Sigma, u$  be chosen from  $\mathcal{C}$  so that the cardinality of  $N(u) \cap \Sigma$  is minimized and, subject to this, the size of the corresponding  $P$  is minimized.

**Claim 1:** *No node of  $P$  is of Type t4, t5 or t6 w.r.t.  $\Sigma$ .*

*Proof of Claim 1:* It is enough to show that if  $v$  and  $w$  are both of Type t4, t5 or t6 w.r.t.  $\Sigma$ , then  $vw$  is an edge. Suppose not. Since  $v$  and  $w$  both have at least two neighbors in each of the sets  $\{a_1, a_2, a_3\}$  and  $\{b_1, b_2, b_3\}$ , for some  $i, j \in \{1, 2, 3\}$ ,  $a_i$  and  $b_j$  are common neighbors of  $v$  and  $w$ . Since  $\{a_i, b_j, u, v\}$  cannot induce a square,  $a_i b_j$  is an edge. W.l.o.g.  $i = j = 2$ ,  $N(v) \cap \{a_1, a_2, a_3\} = \{a_1, a_2\}$  and  $N(w) \cap \{a_1, a_2, a_3\} = \{a_2, a_3\}$ . But then  $\{a_1, a_2, a_3, b_2, v, w\}$  induces an odd wheel with center  $a_2$ . This completes the proof of Claim 1.

**Claim 2:** *No node of  $P$  is of Type p3 w.r.t.  $\Sigma$ .*

*Proof of Claim 2:* Suppose  $x_i$  is of Type p3 w.r.t.  $\Sigma$ . Let  $\Sigma'$  be obtained from  $\Sigma$  by substituting  $x_i$  into  $\Sigma$ . Since  $x_i \in G \setminus S$ , node  $x_i$  is not adjacent to  $u$ . Therefore  $|N(u) \cap \Sigma'| \leq |N(u) \cap \Sigma|$ . Therefore  $\Sigma', u$  and  $P'$ , where  $P' = P_{x_1 x_{i-1}}$  or  $P' = P_{x_{i+1} x_n}$ , contradict our choice of  $\Sigma, u$  and  $P$ . This completes the proof of Claim 2.

**Claim 3:** *No node of  $P$  is of Type t2p or t3p w.r.t.  $\Sigma$ .*

*Proof of Claim 3:* Suppose that  $x_i$  is of Type t2p or t3p w.r.t.  $\Sigma$  and let  $\Sigma'$  be obtained from  $\Sigma$  by substituting  $x_i$  for its sibling. Note that  $u$  cannot be of Type t6 w.r.t.  $\Sigma$ , since otherwise  $u$  is of Type t5 w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . In particular,  $u$  is not adjacent to  $a_3$ .

Suppose  $x_i$  is a sibling of  $a_1$ . Since  $u$  is adjacent to  $a_2, b_1$  and at least one of  $b_2, b_3$ , and it is not adjacent to  $x_i$  and  $a_3$ , node  $u$  must be of Type t2p or t3p w.r.t.  $\Sigma'$ , adjacent to  $b_3$ . In particular,  $N(u) \cap P^3 = \{b_3\}$  and  $b_1$  is the unique neighbor of  $u$  in the  $x_i b_1$ -path of  $\Sigma'$ . Node  $x_i$  is not adjacent to  $b_1$ , else  $\{a_2, b_1, x_i, u\}$  induces a square. Hence  $(H_{13}, u)$  must be a line wheel with  $u$  adjacent to  $a'_1$ . But then  $(P^1 \setminus \{a_1, b_1\}) \cup P^3 \cup \{a_2, u, x_i\}$  contains a  $3PC(x_i a_2 a_3, u)$ .

Suppose  $x_i$  is a sibling of  $a_2$ . Since  $u$  is adjacent to  $a_1, b_1$  and at least one of  $b_2, b_3$ , and it is not adjacent to  $a_3$  and  $x_i$ , it must be of Type t3p w.r.t.  $\Sigma'$ , and hence of Type t5 w.r.t.  $\Sigma$ . Since  $u$  is of Type t3p w.r.t.  $\Sigma'$ ,  $N(u) \cap P^3 = \{b_3\}$ . But this contradicts Lemma 6.2 applied to  $\Sigma$  and  $u$ .

Suppose  $x_i$  is a sibling of  $a_3$ . Then  $i > 1$  and  $u$  is of the same type w.r.t.  $\Sigma'$  as it is w.r.t.  $\Sigma$ . But then  $\Sigma', u$  and  $P_{x_1 x_{i-1}}$  contradict our choice of  $\Sigma, u$  and  $P$ .

Suppose  $x_i$  is a sibling of  $b_1$ . Then  $u$  is of Type t2p or t4d w.r.t.  $\Sigma'$ , adjacent to  $b_3$ . If  $u$  is of Type t4d w.r.t.  $\Sigma'$ , then  $\Sigma', u$  contradict our choice of  $\Sigma, u$ . So  $u$  is of Type t2p w.r.t.  $\Sigma'$ . In particular,  $N(u) \cap P^2 = \{a_2\}$ . So  $u$  must be of Type t4d w.r.t.  $\Sigma$ . Node  $x_i$  cannot be adjacent to  $a_1$ , else  $\{a_1, b_3, x_i, u\}$  induces a square. So  $a_1 b_1$  is not an edge. By Lemma 6.2,  $N(u) \cap P^3 = \{b_3\}$  and  $u$  has a neighbor in  $P^1 \setminus \{a_1, b_1\}$ . So there is a subpath  $P'$  of  $P^1 \setminus \{a_1, b_1\}$  such that one endnode of  $P'$  is adjacent to  $u$ , the other to  $x_i$  and no proper subpath of  $P'$  has this property. But then  $P^2 \cup P' \cup \{b_3, x_i, u\}$  induces a  $3PC(b_2 b_3 x_i, u)$ .

Suppose  $x_i$  is a sibling of  $b_2$ . Then  $u$  must be of Type t4d w.r.t.  $\Sigma'$ . Node  $x_i$  is not adjacent to  $a_2$ , else  $\{a_2, b_3, x_i, u\}$  induces a square. So  $i = 1$ . Since  $u$  is adjacent to  $b_3$ , node  $b_3$  is in  $S$  and hence  $n > 1$ . But then  $\Sigma', u$  and  $P_{x_2 x_n}$  contradict our choice of  $\Sigma, u$  and  $P$ .

Finally suppose that  $x_i$  is a sibling of  $b_3$ . Then  $u$  is of Type t4s w.r.t.  $\Sigma'$ . If  $u$  is of Type t5 w.r.t.  $\Sigma$ , then  $\Sigma', u$  contradict our choice of  $\Sigma, u$ . So  $u$  is of Type t4s w.r.t.  $\Sigma$ . Hence  $i = n$  and  $n > 1$ . But then  $\Sigma', u$  and  $P_{x_1 x_{i-1}}$  contradict our choice of  $\Sigma, u$  and  $P$ . This completes the proof of Claim 3.

**Claim 4:** *If  $x_i$  is of Type p4 w.r.t.  $\Sigma$ , then  $i = 1$  and the neighbors of  $x_i$  in  $\Sigma$  are contained in  $P^1 \cup P^2$ .*

*Proof of Claim 4:* Suppose  $x_i$  is of Type p4 w.r.t.  $\Sigma$ . If the neighbors of  $x_i$  in  $\Sigma$  are contained in  $P^1 \cup P^2$  then  $i = 1$ .

Suppose that the neighbors of  $x_i$  in  $\Sigma$  are contained in  $P^k \cup P^3$  for  $k = 1$  or  $2$ . For  $j = k, 3$ , let  $u_j$  (resp.  $v_j$ ) be the neighbor of  $x_i$  in  $P^j$  that is closest to  $a_j$  (resp.  $b_j$ ). If  $u$  is of Type t6b w.r.t.  $\Sigma$ , then by Lemma 6.2,  $u$  is adjacent to all nodes of  $\Sigma$  and hence  $\{u, x_i, v_k, u_3\}$  induces a square. Therefore,  $u$  is not adjacent to  $a_3$ .

First suppose that  $x_i$  is adjacent to  $a_3$ . Then  $x_i$  is not adjacent to  $a_k$  and so  $(\Sigma \cup x_i) \setminus P_{v_3 b_3}^3$  induces a  $\Sigma' = 3PC(a_1 a_2 a_3, u_k v_k x_i)$ . Since  $u$  is adjacent to  $a_1, a_2, b_1$  and it is not adjacent to  $a_3, x_i$ , it must be of Type t4s w.r.t.  $\Sigma'$ . Hence  $u$  is adjacent to  $u_k$  and  $v_k$ . Node  $u$  cannot have neighbors in  $P^3$ , since otherwise  $\Sigma', u$  would contradict the choice of  $\Sigma, u$ . So  $u$  is of Type t4s w.r.t.  $\Sigma$ . Since  $u$  is adjacent to  $u_k \neq a_k$ ,  $(H_{12}, u)$  must be a universal wheel. Then by Lemma 6.2,  $a_1 b_1$  and  $a_2 b_2$  are not edges and so both  $(H_{13}, u)$  and  $(H_{23}, u)$  must be twin wheels. Hence both  $P^1$  and  $P^2$  are of length 2. But then  $\{a_k, a_3, u_k, x_i\}$  induces a square. Therefore  $x_i$  is not adjacent to  $a_3$ .

Let  $\Sigma' = 3PC(a_1 a_2 a_3, x_i v_3 u_3)$  induced by  $(\Sigma \cup x_i) \setminus P_{v_k b_k}^k$ . Since  $u$  is adjacent to  $a_1, a_2$  and at least one of  $b_2, b_3$  (i.e. it has a neighbor in the  $a_3 v_{3-k}$ -path of  $\Sigma'$ ), and it is not adjacent to  $a_3$  and  $x_i$ , it must be of Type t4d w.r.t.  $\Sigma'$ . If  $k = 1$ ,  $\Sigma', u$  contradicts our choice of  $\Sigma, u$  since  $b_1$  is adjacent to  $u$  and belongs to  $\Sigma \setminus \Sigma'$ . So  $k = 2$ . If  $a_1 b_1$  is an edge, then  $\{u, x_i, u_3, b_1\}$  induces a square. But then by Lemma 6.2,  $v_3 = b_3$  and  $N(u) \cap \Sigma = \{a_1, a_2, b_1, b_3, b'_3\}$ . Hence  $P^2 \cup \{u, x_i, b_1, b'_3\}$  induces a  $3PC(u_2 v_2 x_i, u)$ . This completes the proof of Claim 4.

**Claim 5:**  *$n > 1$ ,  $x_1$  is of Type t1, p1, p2, t2, t3 or p4 w.r.t.  $\Sigma$ , and  $x_n$  is of Type t1, p1,*

$p2$ ,  $t2$  or  $t3$  w.r.t.  $\Sigma$ .

*Proof of Claim 5:* Follows from Claims 1, 2, 3 and 4.

**Claim 6:** If  $a_1$  (resp.  $a_2$ ) has a neighbor in the interior of  $P$ , then  $b_2$  and  $b_3$  (resp.  $b_1$  and  $b_3$ ) do not.

*Proof of Claim 6:* Suppose that  $x_i$  and  $x_j$  are nodes of the interior of  $P$  such that  $x_i$  is adjacent to  $a_1$ ,  $x_j$  is adjacent to  $b_2$  or  $b_3$ , and no proper subpath of  $P_{x_i x_j}$  has this property.

First suppose that  $x_j$  is adjacent to both  $b_2$  and  $b_3$ . Then, by the definition of  $S$ ,  $b_1$  has no neighbor in the interior of  $P$  and  $u$  is of Type t5 or t6 w.r.t.  $\Sigma$ . By Claims 1 and 3,  $x_j$  is of Type t2 w.r.t.  $\Sigma$ . By Lemma 8.5 applied to a subpath of  $P_{x_i x_j}$ ,  $a_2$  has no neighbor in  $P_{x_i x_j}$ . Then  $P^2 \cup P^3 \cup P_{x_i x_j} \cup a_1$  induces a  $\Sigma' = 3PC(a_1 a_2 a_3, x_j b_2 b_3)$ . If  $u$  is of Type t6 (resp. t5) w.r.t.  $\Sigma$  then it is of Type t5 (resp. t4d) w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ .

Next suppose that  $x_j$  is adjacent to  $b_2$  and not to  $b_3$ . If  $b_1$  does not have a neighbor in  $P_{x_i x_j}$ , then  $P^1 \cup P^3 \cup P_{x_i x_j} \cup b_2$  induces a  $3PC(b_1 b_2 b_3, a_1)$ . If  $a_2$  does not have a neighbor in  $P_{x_i x_j}$ , then  $P^2 \cup P^3 \cup P_{x_i x_j} \cup a_1$  induces a  $3PC(a_1 a_2 a_3, b_2)$ . So both  $a_2$  and  $b_1$  have a neighbor in  $P_{x_i x_j}$ . Let  $x_k$  and  $x_l$  be nodes of  $P_{x_i x_j}$  such that  $x_k$  is adjacent to  $a_2$ ,  $x_l$  to  $b_1$  and no proper subpath of  $P_{x_k x_l}$  has this property. If  $j \neq k, l$  then  $P^2 \cup P^3 \cup P_{x_k x_l} \cup b_1$  induces a  $3PC(b_1 b_2 b_3, a_2)$ . If  $i \neq k, l$  then  $P^1 \cup P^3 \cup P_{x_k x_l} \cup a_2$  induces a  $3PC(a_1 a_2 a_3, b_1)$ . By Claim 1,  $i \neq j$ . If  $j = k$  and  $i = l$ , then by Claim 2,  $\{a_1, a_2, b_1, b_2\}$  induces a square. So  $j = l$  and  $i = k$ . Hence  $P^1 \cup P^2 \cup P_{x_i x_j}$  induces a  $3PC(a_1 a_2 x_i, b_1 b_2 x_j)$ . Since  $b_1, b_2 \in S$ ,  $u$  is of Type t4s w.r.t.  $\Sigma$  and hence w.r.t.  $\Sigma'$  as well. If  $i < j$  let  $P' = P_{x_1 x_{i-1}}$  and otherwise let  $P' = P_{x_1 x_{j-1}}$ . Then  $\Sigma', u$  and  $P'$  contradict our choice of  $\Sigma, u$  and  $P$ .

Finally suppose that  $x_j$  is adjacent to  $b_3$  and not to  $b_2$ . So  $b_2$  has no neighbor in  $P_{x_i x_j}$ . If  $a_2$  has no neighbor in  $P_{x_i x_j}$ , then  $P^2 \cup P^3 \cup P_{x_i x_j} \cup a_1$  induces a  $3PC(a_1 a_2 a_3, b_3)$ . So  $a_2$  has a neighbor in  $P_{x_i x_j}$ . Let  $x_k$  be such a neighbor that is closest to  $x_j$ . By Claims 1 and 3,  $i \neq j$ . Suppose  $i \neq k$ . Then by Lemma 8.4 applied to a subpath of  $P_{x_k x_j}$ ,  $b_1$  has a unique neighbor  $x_j$  in  $P_{x_k x_j}$ . Since  $b_1, b_3 \in S$ ,  $u$  is of Type t4d w.r.t.  $\Sigma$ . Let  $\Sigma' = 3PC(a_1 a_2 a_3, b_1 x_j b_3)$  induced by  $P^1 \cup P^3 \cup P_{x_k x_j} \cup a_2$ . Then  $u$  is of Type t4d w.r.t.  $\Sigma'$  as well. If  $i > j$  let  $P' = P_{x_{k+1} x_n}$  and if  $i < j$  let  $P' = P_{x_{j+1} x_n}$ . Then  $\Sigma', u$  and  $P'$  contradict our choice of  $\Sigma, u$  and  $P$ . So  $i = k$  and hence  $x_i$  is of Type t2 w.r.t.  $\Sigma$ . By Lemma 8.5 applied to a subpath of  $P_{x_i x_j}$ ,  $b_1$  does not have a neighbor in  $P_{x_i x_j}$ . Hence  $P^1 \cup P^2 \cup P_{x_i x_j} \cup b_3$  induces a  $\Sigma' = 3PC(a_1 a_2 x_i, b_1 b_2 b_3)$ . Note that  $u$  is of Type t5 or t4d w.r.t.  $\Sigma'$ . If  $i < j$  let  $P' = P_{x_1 x_{i-1}}$  and otherwise let  $P' = P_{x_1 x_{j-1}}$ . Then  $\Sigma', u$  and  $P'$  contradict our choice of  $\Sigma, u$  and  $P$ .

Therefore, if  $a_1$  has a neighbor in the interior of  $P$ , then  $b_2$  and  $b_3$  do not.

Now suppose that  $a_2$  and at least one of  $b_1, b_3$  has a neighbor in the interior of  $P$ . Let  $x_i$  and  $x_j$  be nodes of the interior of  $P$  such that  $x_i$  is adjacent to  $a_2$ ,  $x_j$  is adjacent to  $b_1$  or  $b_3$ , and no proper subpath of  $P_{x_i x_j}$  has this property.

First suppose that  $x_j$  is adjacent to both  $b_1$  and  $b_3$ . Then  $a_1$  has no neighbor in the interior of  $P$  by the first part of Claim 6 and  $u$  is of Type t4d w.r.t.  $\Sigma$ . Let  $\Sigma' = 3PC(a_1 a_2 a_3, b_1 x_j b_3)$  induced by  $P^1 \cup P^3 \cup P_{x_i x_j} \cup a_2$ . Then  $u$  is of Type t4d w.r.t.  $\Sigma'$ . If  $i < j$  let  $P' = P_{x_{j+1} x_n}$  and otherwise let  $P' = P_{x_{i+1} x_n}$ . Then  $\Sigma', u$  and  $P'$  contradict our choice of  $\Sigma, u$  and  $P$ .

Next suppose that  $x_j$  is adjacent to  $b_1$  and not to  $b_3$ . If  $a_1$  does not have a neighbor in  $P_{x_i x_j}$ , then  $P^1 \cup P^3 \cup P_{x_i x_j} \cup a_2$  induces a  $3PC(a_1 a_2 a_3, b_1)$ . So  $a_1$  has a neighbor in  $P_{x_i x_j}$ , and hence  $b_2$  does not by the first part of Claim 6. But then  $P^2 \cup P^3 \cup P_{x_i x_j} \cup b_1$  induces a

$3PC(b_1b_2b_3, a_2)$ .

Finally suppose that  $x_j$  is adjacent to  $b_3$  and not to  $b_1$ . Then  $a_1$  does not have a neighbor in the interior of  $P$  by the first part of Claim 6, and hence  $P^1 \cup P^3 \cup P_{x_i x_j} \cup a_2$  induces a  $3PC(a_1a_2a_3, b_3)$ . This completes the proof of Claim 6.

**Claim 7:** *If  $a_1$  has a neighbor in the interior of  $P$ , then  $b_1$  does not.*

*Proof of Claim 7:* Suppose both  $a_1$  and  $b_1$  have a neighbor in the interior of  $P$ . Then, by Claim 6,  $a_2, b_2$  and  $b_3$  do not. Since  $b_1 \in S$ ,  $u$  is of Type t4 w.r.t.  $\Sigma$ .

Suppose  $a_1b_1$  is not an edge. Let  $x_i$  and  $x_j$  be nodes of the interior of  $P$  such that  $x_i$  is adjacent to  $a_1$ ,  $x_j$  to  $b_1$ , and no proper subpath of  $P_{x_i x_j}$  has this property. Let  $\Sigma' = 3PC(a_1a_2a_3, b_1b_2b_3)$  induced by  $P^2 \cup P^3 \cup P_{x_i x_j} \cup \{a_1, b_1\}$ . If  $i > j$  let  $P' = P_{x_{i+1}x_n}$  and otherwise let  $P' = P_{x_{j+1}x_n}$ . Note that  $u$  is of the same type w.r.t.  $\Sigma'$  as it is w.r.t.  $\Sigma$ . But then  $\Sigma', u$  and  $P'$  contradict our choice of  $\Sigma, u$  and  $P$ .

Therefore,  $a_1b_1$  is an edge. By Lemma 6.2,  $u$  is of Type t4d w.r.t.  $\Sigma$  and hence  $x_n$  has a neighbor in  $P^3 \setminus b_3$ . Let  $x_j$  (resp.  $x_i$ ) be the node of the interior of  $P$  with highest index adjacent to  $b_1$  (resp.  $a_1$ ). Suppose  $x_n$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ . By Theorem 8.3 applied to  $P_{x_i x_n}$  or  $P_{x_j x_n}$ ,  $i = j$  and  $x_n$  is of Type p2 w.r.t.  $\Sigma$ . Let  $u_3$  (resp.  $v_3$ ) be the neighbor of  $x_n$  in  $P^3$  that is closest to  $a_3$  (resp.  $b_3$ ). Let  $\Sigma' = 3PC(a_1b_1x_i, u_3v_3x_n)$  induced by  $P^1 \cup P^3 \cup P_{x_i x_n}$ . Since  $u$  is adjacent to  $a_1, b_1$  and  $b_3$ , and to no node of  $P_{x_i x_n}$ , it must be of Type t4s w.r.t.  $\Sigma'$ , adjacent to  $u_3$  and  $v_3$ . But then  $\Sigma', u$  contradicts the choice of  $\Sigma, u$  since  $a_2$  is a neighbor of  $u$  in  $\Sigma$  but not  $\Sigma'$ . Therefore,  $x_n$  is of Type t2 or t3 w.r.t.  $\Sigma$ , adjacent to  $a_3$ . If  $x_n$  is not adjacent to  $a_2$ , then  $P^2 \cup P^3 \cup P_{x_j x_n} \cup b_1$  induces a  $3PC(b_1b_2b_3, a_3)$ . So  $x_n$  is adjacent to  $a_2$ .

Suppose that  $x_1$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ . Since  $a_1b_1$  is an edge and  $a_1, b_1 \in S$ , the neighbors of  $x_1$  in  $\Sigma$  are contained in  $P^2$ . If  $x_1$  is of Type t1 or p1, then  $P^2 \cup P^3 \cup P$  induces a  $3PC(a_2a_3x_n, \cdot)$ . So  $x_1$  is of Type p2. Let  $u_2$  (resp.  $v_2$ ) be the neighbor of  $x_1$  in  $P^2$  that is closest to  $a_2$  (resp.  $b_2$ ). Let  $x_k$  be the node of the interior of  $P$  with lowest index adjacent to  $a_1$  or  $b_1$ . By Theorem 8.3 applied to  $P_{x_1 x_k}$ ,  $x_k$  is adjacent to both  $a_1$  and  $b_1$ . But then  $P^1 \cup P^2 \cup P_{x_1 x_k}$  induces a  $\Sigma' = 3PC(a_1b_1x_k, u_2v_2x_1)$ . Since  $u$  is adjacent to  $a_1, b_1$  and  $a_2$ , and no node of  $P_{x_1 x_k}$ , it must be of Type t4s w.r.t.  $\Sigma'$ . But then  $\Sigma', u$  contradicts the choice of  $\Sigma, u$  since  $b_3$  is a neighbor of  $u$  in  $\Sigma$  but not  $\Sigma'$ .

Suppose that  $x_1$  is of Type p4 w.r.t.  $\Sigma$ . Define  $u_2$  and  $v_2$  as before. Then  $u_2 \neq a_2$ , and hence  $P^2 \cup P^3 \cup P$  induces a  $\Sigma' = 3PC(a_1a_3x_n, u_2v_2x_1)$ . Since  $u$  is adjacent to  $a_2$  and  $b_3$ , and to no node of  $P \cup a_3$ , it must be of Type p4 w.r.t.  $\Sigma'$ . In particular,  $u$  is adjacent to  $a'_2$ . But then  $(H_{12}, u)$  is a proper wheel that is not a beetle.

Therefore,  $x_1$  is of Type t2 or t3 w.r.t.  $\Sigma$ . Since  $b_1, b_3 \in S$ ,  $x_1$  is adjacent to  $b_2$ . Node  $x_1$  must be adjacent to  $b_3$ , else  $P^2 \cup P^3 \cup P$  induces a  $3PC(a_2a_3x_n, b_2)$ . Then  $P^2 \cup P^3 \cup P$  induces a  $\Sigma' = 3PC(a_2a_3x_n, b_2b_3x_1)$ . Since  $u$  is adjacent to  $a_2$  and  $b_3$  and to no node of  $P \cup \{a_3, b_2\}$ , it must be of Type p4 w.r.t.  $\Sigma'$ . In particular,  $u$  is adjacent to  $a'_2$ . But then  $(H_{12}, u)$  is a proper wheel that is not a beetle. This completes the proof of Claim 7.

**Claim 8:** *If  $a_2$  has a neighbor in the interior of  $P$ , then  $b_2$  does not.*

*Proof of Claim 8:* Suppose that both  $a_2$  and  $b_2$  have a neighbor in the interior of  $P$ . Then, by Claim 6,  $a_1, b_1$  and  $b_3$  do not. Since  $b_2 \in S$ ,  $u$  cannot be of Type t4d w.r.t.  $\Sigma$ . By an

analogous argument as in Claim 7,  $a_2b_2$  is an edge. Hence, by Lemma 6.2,  $u$  is of Type t6 w.r.t.  $\Sigma$  adjacent to all nodes of  $\Sigma$ . Then  $b_3 \in S$  and so  $b_3$  cannot be the unique neighbor of  $x_n$  in  $P^3$ . Let  $x_i$  (resp.  $x_j$ ) be the node of the interior of  $P$  with highest index adjacent to  $a_2$  (resp.  $b_2$ ).

Suppose that  $x_n$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ . By Theorem 8.3 applied to  $P_{x_ix_n}$  or  $P_{x_jx_n}$ ,  $i = j$  and  $x_n$  is of Type p2 w.r.t.  $\Sigma$ . Hence  $P^2 \cup P^3 \cup P_{x_ix_n}$  induces a  $\Sigma' = 3PC(a_2b_2x_i, u_3v_3x_n)$ , where  $u_3$  and  $v_3$  are the neighbors of  $x_n$  in  $P^3$ . Since  $u$  is adjacent to all nodes of  $\Sigma$  and no node of  $P_{x_ix_n}$ , it must be of Type t4s w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ .

Therefore,  $x_n$  is of Type t2 or t3 w.r.t.  $\Sigma$ , adjacent to  $a_3$ . Suppose  $x_n$  is adjacent to  $a_1$ . Then  $P^1 \cup P^3 \cup P_{x_jx_n} \cup b_2$  induces a  $\Sigma' = 3PC(a_1x_n a_3, b_1b_2b_3)$ . Since  $u$  is of Type t6 w.r.t.  $\Sigma$ , it is of Type t5 w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . So  $x_n$  is not adjacent to  $a_1$ . But then  $P^1 \cup P^3 \cup P_{x_jx_n} \cup b_2$  induces a  $3PC(b_1b_2b_3, a_3)$ . This completes the proof of Claim 8.

**Claim 9:** *If  $a_1$  or  $a_2$  has a neighbor in the interior of  $P$ , then  $N(x_n) \cap \Sigma \subseteq \{a_1, a_2, a_3\}$ .*

*Proof of Claim 9:* Let  $x_i$  be the node of the interior of  $P$  with highest index adjacent to  $a_1$  or  $a_2$ , and suppose that  $x_n$  has a neighbor in  $\Sigma \setminus \{a_1, a_2, a_3\}$ . By Claims 6, 7 and 8,  $b_1, b_2$  and  $b_3$  do not have neighbors in the interior of  $P$ .

First suppose that  $x_i$  is adjacent to both  $a_1$  and  $a_2$ . By Lemma 8.5 applied to  $P_{x_ix_n}$ , and since  $x_n$  has a neighbor in  $P^3$  and is of Type t1, p1, p2, t2, or t3 by Claim 5,  $x_n$  must be of Type t1 or p1 w.r.t.  $\Sigma$ . Let  $\Sigma' = 3PC(a_1a_2x_i, b_1b_2b_3)$  contained in  $(\Sigma \setminus a_3) \cup P_{x_ix_n}$ . If  $u$  is of Type t6 w.r.t.  $\Sigma$ , then it is of Type t5 w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . So  $u$  is of the same type w.r.t.  $\Sigma'$  as it is w.r.t.  $\Sigma$ . But then  $\Sigma', u$  and  $P_{x_1x_{i-1}}$  contradict our choice of  $\Sigma, u$  and  $P$ .

Next suppose that  $x_i$  is adjacent to  $a_2$  and not to  $a_1$ . By Lemma 8.4 applied to  $P_{x_ix_n}$ ,  $x_n$  is of Type t2 w.r.t.  $\Sigma$ , adjacent to  $b_1$  and  $b_3$ . Let  $\Sigma' = 3PC(a_1a_2a_3, b_1x_nb_3)$  contained in  $(\Sigma \setminus b_2) \cup P_{x_ix_n}$ . By definition of  $S$ ,  $x_n$  must be of Type t4s w.r.t.  $\Sigma$ . But then  $x_n$  is a strongly adjacent node relative to  $\Sigma'$  that violates Lemma 6.1.

Finally suppose that  $x_n$  is adjacent to  $a_1$  and not to  $a_2$ . By Lemma 8.4 applied to  $P_{x_ix_n}$ ,  $x_n$  is of Type t2 w.r.t.  $\Sigma$ , adjacent to  $b_2$  and  $b_3$ . Then  $u$  is of Type t4s w.r.t.  $\Sigma$ . Let  $\Sigma' = 3PC(a_1a_2a_3, x_nb_2b_3)$  induced by  $P^2 \cup P^3 \cup P_{x_ix_n} \cup a_1$ . Then  $u$  violates Lemma 6.1 w.r.t.  $\Sigma'$ . This completes the proof of Claim 9.

**Claim 10:** *If  $b_1$  or  $b_2$  has a neighbor in the interior of  $P$ , then  $N(x_n) \cap \Sigma \subseteq \{b_1, b_2, b_3\}$ .*

*Proof of Claim 10:* Let  $x_i$  be the node of the interior of  $P$  with highest index adjacent to  $b_1$  or  $b_2$ , and suppose that  $x_n$  has a neighbor in  $\Sigma \setminus \{b_1, b_2, b_3\}$ . By Claims 6, 7 and 8,  $a_1$  and  $a_2$  do not have neighbors in the interior of  $P$ .

First suppose that  $x_i$  is adjacent to both  $b_1$  and  $b_2$ . Then  $u$  is of Type t4s w.r.t.  $\Sigma$ , and hence  $b_3$  does not have a neighbor in the interior of  $P$ . By Lemma 8.5 applied to  $P_{x_ix_n}$ ,  $x_n$  is of Type t1 or p1 w.r.t.  $\Sigma$ . Then  $(\Sigma \setminus b_3) \cup P_{x_ix_n}$  contains a  $\Sigma' = 3PC(a_1a_2a_3, b_1b_2x_i)$ . Node  $u$  is of Type t4s w.r.t.  $\Sigma'$ , and hence  $\Sigma', u$  and  $P_{x_1x_{i-1}}$  contradict our choice of  $\Sigma, u$  and  $P$ .

Next suppose that  $x_i$  is adjacent to  $b_1$  and not to  $b_2$ . If  $x_n$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ , then  $P^1 \cup P^2 \cup (P^3 \setminus b_3) \cup P_{x_ix_n}$  contains a  $3PC(a_1a_2a_3, b_1)$ . So  $x_n$  is of Type t2 or t3 w.r.t.  $\Sigma$ , adjacent to  $a_3$ . Node  $x_n$  cannot be adjacent to both  $a_1$  and  $a_2$ , else  $P^1 \cup P^2 \cup P_{x_ix_n}$

induces a  $3PC(a_1a_2x_n, b_1)$ . Suppose  $x_n$  is adjacent to  $a_1$ . Then  $x_n$  is of Type t2 w.r.t.  $\Sigma$ , and so by Lemma 8.5,  $x_i$  is the unique neighbor of  $b_3$  in  $P_{x_ix_n}$ . Hence  $P^1 \cup P^3 \cup P_{x_ix_n}$  induces a  $\Sigma' = 3PC(a_1x_n a_3, b_1x_ib_3)$ . Since  $b_1, b_3 \in S$ ,  $u$  is of Type t4d w.r.t.  $\Sigma$ , and hence it violates Lemma 6.1 applied to  $\Sigma'$ . Hence  $x_n$  is of Type t2 w.r.t.  $\Sigma$ , adjacent to  $a_2$  and  $a_3$ . If  $b_3$  has a neighbor in  $P_{x_ix_n}$ , then a subpath of  $P_{x_ix_n}$  contradicts Lemma 8.5 applied to  $\Sigma$ . So  $b_3$  does not have a neighbor in  $P_{x_ix_n}$  and hence  $P^2 \cup P^3 \cup P_{x_ix_n} \cup b_1$  induces a  $\Sigma' = 3PC(x_n a_2 a_3, b_1 b_2 b_3)$ . If  $u$  is of Type t6 w.r.t.  $\Sigma$  then it is of Type t5 w.r.t.  $\Sigma'$ , and hence  $\Sigma', u$  contradicts our choice of  $\Sigma, u$ . If  $u$  is of Type t4s or t5 w.r.t.  $\Sigma$ , then by Lemma 6.2,  $u$  and  $\Sigma'$  violate Lemma 6.1. So  $u$  is of Type t4d w.r.t.  $\Sigma$ , and hence of Type t2p w.r.t.  $\Sigma'$ . In particular,  $N(u) \cap P^3 = \{b_3\}$ . So  $b_2$  has no neighbors in the interior of  $P$ . If  $b_3$  has a neighbor in the interior of  $P$ , then  $P^2 \cup P^3 \cup P_{x_2x_n}$  contains a  $3PC(x_n a_2 a_3, b_3)$ . So  $b_3$  has no neighbors in the interior of  $P$ . Suppose  $a_1 b_1$  is not an edge. Then by Lemma 6.2 (iii),  $P^1 \cup P_{x_ix_n} \{a_3, u\}$  induces an odd wheel with center  $u$ . So  $a_1 b_1$  is an edge, and hence  $x_1$  has a neighbor in  $P^2 \setminus a_2$ . Let  $x_j$  be the node of the interior of  $P$  with lowest index adjacent to  $b_1$ . If  $x_1$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ , then  $P_{x_1x_j}$  contradicts Theorem 8.3. If  $x_1$  is of Type p4 w.r.t.  $\Sigma$ , then  $(P^2 \setminus b_2) \cup P^3 \cup \{x_1, u\}$  contains an odd wheel with center  $u$ . So  $x_1$  is of Type t2 or t3 w.r.t.  $\Sigma$  adjacent to  $b_2$ . If  $x_1$  is not adjacent to  $b_3$  then  $P \cup P^2 \cup P^3$  induces a  $3PC(x_n a_2 a_3, b_2)$ . So  $x_1$  is adjacent to  $b_3$ , and hence  $P \cup P^2 \cup P^3$  induces a  $\Sigma'' = 3PC(x_n a_2 a_3, x_1 b_2 b_3)$ . But then  $u$  and  $\Sigma''$  violate Lemma 6.1.

Finally suppose that  $x_i$  is adjacent to  $b_2$  and not to  $b_1$ . If  $x_n$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ , then  $(\Sigma \setminus b_3) \cup P_{x_ix_n}$  contains a  $3PC(a_1 a_2 a_3, b_2)$ . So  $x_n$  is of Type t2 or t3 w.r.t.  $\Sigma$ , adjacent to  $a_3$ . Node  $x_n$  cannot be adjacent to both  $a_1$  and  $a_2$ , else  $P^1 \cup P^2 \cup P_{x_ix_n}$  induces a  $3PC(a_1 a_2 x_n, b_2)$ . Suppose  $x_n$  is adjacent to  $a_2$ . Then  $x_n$  is of Type t2 w.r.t.  $\Sigma$  and so by Lemma 8.5,  $x_i$  is the unique neighbor of  $b_3$  in  $P_{x_ix_n}$ . Let  $\Sigma' = 3PC(a_2 a_3 x_n, b_2 b_3 x_1)$  induced by  $P^2 \cup P^3 \cup P_{x_ix_n}$ . Since  $b_2, b_3 \in S$ ,  $u$  is of Type t5 or t6 w.r.t.  $\Sigma$ . If  $u$  is of Type t6 w.r.t.  $\Sigma$ , then  $u$  is of Type t5 w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . If  $u$  is of Type t5 w.r.t.  $\Sigma$ , then  $u$  violates Lemma 6.1 applied to  $\Sigma'$ . Therefore  $x_n$  is adjacent to  $a_1$  and not to  $a_2$ . By Lemma 8.5,  $b_3$  cannot have a neighbor in  $P_{x_ix_n}$ . Then  $P^1 \cup P^3 \cup P_{x_ix_n} \cup b_2$  induces a  $\Sigma' = 3PC(a_1 x_n a_3, b_1 b_2 b_3)$ . If  $u$  is of Type t6 w.r.t.  $\Sigma$ , then  $u$  is of Type t5 w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . If  $u$  is of Type t5 w.r.t.  $\Sigma$ , then by Lemma 6.2,  $u$  is adjacent to  $b'_3$  and hence it violates Lemma 6.1 applied to  $\Sigma'$ . If  $u$  is of Type t4 w.r.t.  $\Sigma$ , then it violates Lemma 6.1 applied to  $\Sigma'$ . This completes the proof of Claim 10.

By Claim 5, we now consider the following cases.

**Case 1:**  $x_n$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ .

**Case 1.1:**  $x_1$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ .

We first show that  $a_1$  and  $a_2$  do not have a neighbor in the interior of  $P$ . Assume not. Then by Claims 6, 7 and 8,  $b_1, b_2$  and  $b_3$  do not. By Claim 9,  $N(x_n) \cap \Sigma = \{a_3\}$ . W.l.o.g. we may assume that the neighbors of  $x_1$  in  $\Sigma$  are contained in  $P^1$ . If  $a_2$  does not have a neighbor in the interior of  $P$ , then  $(\Sigma \setminus a_1) \cup P$  contains a  $3PC(b_1 b_2 b_3, a_3)$ . Let  $x_i$  be the node of  $P$  with lowest index adjacent to  $a_2$ . Then  $(\Sigma \setminus a_1) \cup P_{x_1x_i}$  contains a  $3PC(b_1 b_2 b_3, a_2)$ .

Suppose that  $b_3$  is the unique neighbor of  $x_n$  in  $\Sigma$ . Then  $u$  is of Type t4s w.r.t.  $\Sigma$  and so  $x_1$  has a neighbor in  $(P^1 \cup P^2) \setminus \{a_1, a_2, b_1, b_2\}$ . We may assume w.l.o.g. that the neighbors of  $x_1$  in  $\Sigma$  are contained in  $P^2$ . Then  $(\Sigma \setminus b_2) \cup P$  contains either a  $3PC(a_1 a_2 a_3, b_3)$  (if  $b_1$  has

no neighbors in the interior of  $P$ ) or a  $3PC(a_1a_2a_3, b_1)$  (otherwise). So  $b_3$  is not the unique neighbor of  $x_n$  in  $\Sigma$ , and hence by Claim 10,  $b_1$  and  $b_2$  do not have neighbors in the interior of  $P$ .

Suppose  $b_3$  has a neighbor in the interior of  $P$ , and let  $x_i$  be such a neighbor with lowest index. Then  $P_{x_1x_i}$  contradicts Theorem 8.3. So  $b_3$  does not have a neighbor in the interior of  $P$ . By Theorem 8.3 applied to  $P$ ,  $x_1$  and  $x_n$  must both be of Type p2 w.r.t.  $\Sigma$ .

Suppose that the neighbors of  $x_1$  in  $\Sigma$  are contained in  $P^2$ . Let  $u_2$  (resp.  $v_2$ ) be the neighbor of  $x_1$  in  $P^2$  that is closest to  $a_2$  (resp.  $b_2$ ). Let  $u_3$  (resp.  $v_3$ ) be the neighbor of  $x_n$  in  $P^3$  that is closest to  $a_3$  (resp.  $b_3$ ). Let  $\Sigma'$  be the  $3PC(u_2v_2x_1, u_3v_3x_n)$  induced by  $P^2 \cup P^3 \cup P$ . Let  $P'_{u_2u_3}$  be the  $u_2u_3$ -path of  $\Sigma'$  and similarly define  $P'_{v_2v_3}$ . Since  $u$  is adjacent to  $a_2$ , it has a neighbor in  $P'_{u_2u_3} \setminus u_3$ . Since  $u$  is adjacent to  $b_2$  or  $b_3$ , it has a neighbor in  $P'_{v_2v_3}$ . Note that  $u$  cannot be of Type t2 w.r.t.  $\Sigma'$  since then  $u$  is of Type t4s w.r.t.  $\Sigma$  and  $a_2b_2$  is not an edge by Lemma 6.2, a contradiction. If  $u$  is of Type t4s w.r.t.  $\Sigma'$ , then our choice of  $\Sigma, u$  is contradicted. Therefore,  $u$  is of Type p4 w.r.t.  $\Sigma'$ . By Lemma 6.2,  $u$  is not of Type t6 w.r.t.  $\Sigma$ , and hence  $u$  is not adjacent to  $a_3$ . So the neighbors of  $u$  in  $P'_{u_2u_3}$  are  $a_2$  and  $a'_2$ . But then  $P'_{u_2u_3} \cup P \cup \{u, a_1\}$  induces an odd wheel with center  $a_2$ .

Analogous argument holds when the neighbors of  $x_1$  in  $\Sigma$  are contained in  $P^1$ .

**Case 1.2:**  $x_1$  is of Type t2 or t3 w.r.t.  $\Sigma$ .

Then  $x_1$  is adjacent to  $b_1$  or  $b_2$ , and  $u$  is not of Type t4s w.r.t.  $\Sigma$ . So  $b_3 \in S$ .

Suppose that  $a_1$  or  $a_2$  has a neighbor in the interior of  $P$ . Then by Claims 6, 7 and 8,  $b_1, b_2$  and  $b_3$  do not. By Claim 9,  $N(x_n) \cap \Sigma = \{a_3\}$ . Let  $x_i$  (resp.  $x_j$ ) be the node of  $P$  with lowest index adjacent to  $a_1$  (resp.  $a_2$ ). Suppose  $x_1$  is adjacent to  $b_3$ , and say  $b_2$ . If  $a_2$  has no neighbor in the interior of  $P$  then  $P^2 \cup P^3 \cup P$  induces a  $3PC(b_2b_3x_1, a_3)$ , and otherwise  $P^2 \cup P^3 \cup P_{x_1x_j}$  induces a  $3PC(b_2b_3x_1, a_2)$ . Hence  $x_1$  is not adjacent to  $b_3$ , and so it is adjacent to  $b_1$  and  $b_2$ . So  $x_1$  is of Type t2 w.r.t.  $\Sigma$  and hence by Lemma 8.5,  $i = j$ . Let  $\Sigma' = 3PC(a_1a_2x_i, b_1b_2x_1)$  induced by  $P^1 \cup P^2 \cup P_{x_1x_i}$ . If  $u$  is of Type t5 or t6 w.r.t.  $\Sigma$ , then  $u$  is of Type t4s w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . If  $u$  is of Type t4d w.r.t.  $\Sigma$ , then  $u$  is a strongly adjacent node to  $\Sigma'$  violating Lemma 6.1. Therefore  $a_1$  and  $a_2$  do not have neighbors in the interior of  $P$ .

Since  $b_3 \in S$ ,  $b_3$  cannot be the unique neighbor of  $x_n$  in  $\Sigma$ , and so by Claim 10,  $b_1$  and  $b_2$  do not have a neighbor in the interior of  $P$ . Node  $x_1$  must be adjacent to both  $b_1$  and  $b_2$ , else  $(\Sigma \setminus b_3) \cup P$  contains a  $3PC(a_1a_2a_3, b_1)$  or a  $3PC(a_1a_2a_3, b_2)$ . Hence,  $(\Sigma \setminus b_3) \cup P$  contains a  $\Sigma' = 3PC(a_1a_2a_3, b_1b_2x_1)$ . If  $u$  is of Type t5 or t6 w.r.t.  $\Sigma$ , then  $u$  is of Type t4s or t5 respectively w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . If  $u$  is of Type t4d w.r.t.  $\Sigma$ , then  $u$  is a strongly adjacent node to  $\Sigma'$  that violates Lemma 6.1.

**Case 1.3:**  $x_1$  is of Type p4 w.r.t.  $\Sigma$ .

Suppose that  $a_1$  or  $a_2$  has a neighbor in the interior of  $P$ . Then by Claims 6, 7 and 8,  $b_1, b_2$  and  $b_3$  do not. By Claim 9,  $N(x_n) \cap \Sigma = \{a_3\}$ . But then  $(\Sigma \setminus \{a_1, a_2\}) \cup P$  contains a  $3PC(b_1b_2b_3, x_1)$ . Therefore  $a_1$  and  $a_2$  do not have neighbors in the interior of  $P$ .

Then  $(\Sigma \cup P) \setminus \{b_1, b_2\}$  contains a  $3PC(a_1a_2a_3, x_1)$ .

**Case 2:**  $x_n$  is of Type t2 or t3 w.r.t.  $\Sigma$ , adjacent to  $a_3$ .

Then by Claim 10,  $b_1$  and  $b_2$  do not have neighbors in the interior of  $P$ . Suppose  $b_3$  has a neighbor in the interior of  $P$  and let  $x_i$  be such a neighbor with highest index. By Claims 6

and 7,  $a_1$  and  $a_2$  do not have neighbors in the interior of  $P$ . Then  $P_{x_i x_n}$  contradicts Lemma 8.4. Therefore,  $b_3$  does not have a neighbor in the interior of  $P$ .

**Case 2.1:**  $x_1$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ .

Suppose that the neighbors of  $x_1$  in  $\Sigma$  are contained in  $P^1$ . If  $a_2$  does not have a neighbor in  $P_{x_2 x_n}$ , then  $(\Sigma \setminus a_1) \cup P$  contains a  $3PC(b_1 b_2 b_3, a_3)$ . Let  $x_i$  be the node of  $P_{x_2 x_n}$  with lowest index adjacent to  $a_2$ . If  $i \neq n$  then  $(\Sigma \setminus a_1) \cup P_{x_1 x_i}$  contains a  $3PC(b_1 b_2 b_3, a_2)$ . So  $i = n$  and hence  $(\Sigma \setminus a_1) \cup P$  contains a  $\Sigma' = 3PC(x_n a_2 a_3, b_1 b_2 b_3)$ . If  $u$  is of Type t6 w.r.t.  $\Sigma$ , then it is of Type t5 w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . If  $u$  is of Type t5 w.r.t.  $\Sigma$ , then by Lemma 6.2,  $N(u) \cap \Sigma = \{a_1, a_2, b_1, b_2, b_3, b'_3\}$ , and hence  $u$  is a strongly adjacent node to  $\Sigma'$  that violates Lemma 6.1. If  $u$  is of Type t4s w.r.t.  $\Sigma$ , then  $u$  violates Lemma 6.1 w.r.t.  $\Sigma'$ . So  $u$  is of Type t4d w.r.t.  $\Sigma$ , and hence it must be of Type t2p w.r.t.  $\Sigma'$ . In particular,  $u$  has no neighbor in  $P^3 \setminus b_3$ . Since  $x_1$  has a neighbor in  $P^1 \setminus S$ ,  $a_1 b_1$  is not an edge. Hence, by Lemma 6.2,  $P^3 \cup P \cup \{u, a_2\} \cup P^1 \setminus \{a_1\}$  contains a  $3PC(x_n a_2 a_3, u)$ .

Now suppose that the neighbors of  $x_1$  in  $\Sigma$  are contained in  $P^2$ . If  $a_1$  does not have a neighbor in  $P_{x_2 x_n}$ , then  $(\Sigma \setminus a_2) \cup P$  contains a  $3PC(b_1 b_2 b_3, a_3)$ . Let  $x_i$  be the node of  $P_{x_2 x_n}$  with lowest index adjacent to  $a_1$ . If  $i \neq n$  then  $(\Sigma \setminus a_2) \cup P_{x_1 x_i}$  contains a  $3PC(b_1 b_2 b_3, a_1)$ . So  $i = n$  and hence  $(\Sigma \setminus a_2) \cup P$  contains a  $\Sigma' = 3PC(a_1 x_n a_3, b_1 b_2 b_3)$ . If  $u$  is of Type t6 w.r.t.  $\Sigma$ , then it is of Type t5 w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . If  $u$  is of Type t5 w.r.t.  $\Sigma$  then by Lemma 6.2,  $u$  is adjacent to  $b'_3$  and hence it violates Lemma 6.1 w.r.t.  $\Sigma'$ . If  $u$  is of Type t4 w.r.t.  $\Sigma$ , then it violates Lemma 6.1 w.r.t.  $\Sigma'$ .

**Case 2.2:**  $x_1$  is of Type t2 or t3 w.r.t.  $\Sigma$ .

Then the neighbors of  $x_1$  in  $\Sigma$  are contained in  $\{b_1, b_2, b_3\}$ , and  $u$  is not of Type t4s w.r.t.  $\Sigma$ . Let  $x_i$  be the node of  $P_{x_2 x_n}$  with lowest index adjacent to  $a_1$  or  $a_2$ .

Suppose  $x_1$  is adjacent to  $b_1$  and  $b_2$ . Node  $x_i$  must be adjacent to both  $a_1$  and  $a_2$ , else  $P^1 \cup P^2 \cup P_{x_1 x_i}$  induces a  $3PC(b_1 b_2 x_1, \cdot)$ . Then  $P^1 \cup P^2 \cup P_{x_1 x_i}$  induces a  $\Sigma' = 3PC(a_1 a_2 x_n, b_1 b_2 x_1)$ . If  $u$  is of Type t5 or t6 w.r.t.  $\Sigma$ , then it is of Type t4s w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . If  $u$  is of Type t4d w.r.t.  $\Sigma$ , then it violates Lemma 6.1 w.r.t.  $\Sigma'$ . Therefore,  $x_1$  is not adjacent to both  $b_1$  and  $b_2$ , and hence it is adjacent to  $b_3$ .

Suppose that  $x_1$  is adjacent to  $b_1$ . Not both  $a_1$  and  $a_2$  can be adjacent to  $x_i$ , else  $P^1 \cup P^2 \cup P_{x_1 x_i}$  induces a  $3PC(a_1 a_2 x_i, b_1)$ . Suppose  $a_1$  is adjacent to  $x_i$ . By Lemma 8.5,  $i = n$ , and hence  $P^1 \cup P^3 \cup P$  induces a  $\Sigma' = 3PC(a_1 x_n a_3, b_1 x_1 b_3)$ . If  $u$  is of Type t6 w.r.t.  $\Sigma$ , then it is of Type t4s w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . If  $u$  is of Type t5 w.r.t.  $\Sigma$ , then it violates Lemma 6.1 w.r.t.  $\Sigma'$ . Since  $N(x_1) \cap \Sigma = \{b_1, b_3\}$ ,  $u$  cannot be of Type t4d w.r.t.  $\Sigma$ . Therefore  $a_1$  is not adjacent to  $x_i$ , and so  $a_2$  is. By Lemma 8.5,  $i \neq n$ , and hence  $P^1 \cup P^3 \cup P_{x_1 x_i}$  induces a  $\Sigma' = 3PC(a_1 a_2 a_3, b_1 x_1 b_3)$ . If  $u$  is of Type t6 (resp. t5) w.r.t.  $\Sigma$ , then it is of Type t5 (resp. t4d) w.r.t.  $\Sigma'$ , contradicting our choice of  $\Sigma, u$ . Since  $N(x_1) \cap \Sigma = \{b_1, b_3\}$ ,  $u$  cannot be of Type t4d w.r.t.  $\Sigma$ .

Therefore  $N(x_1) \cap \Sigma = \{b_2, b_3\}$ . Hence  $u$  must be of Type t4d w.r.t.  $\Sigma$ . If  $a_2$  is not adjacent to a node of  $P_{x_2 x_n}$ , then  $P^2 \cup P^3 \cup P$  induces a  $3PC(x_1 b_2 b_3, a_3)$ . Let  $x_j$  be the node of  $P_{x_2 x_n}$  with lowest index adjacent to  $a_2$ . If  $j \neq n$  then  $P^2 \cup P^3 \cup P_{x_1 x_j}$  induces a  $3PC(x_1 b_2 b_3, a_2)$ . So  $j = n$  and hence  $x_n$  is the unique neighbor of  $a_2$  in  $P$ . Then  $P^2 \cup P^3 \cup P$  induces a  $\Sigma' = 3PC(x_n a_2 a_3, x_1 b_2 b_3)$ . Since  $u$  is of Type t4d w.r.t.  $\Sigma$ , it must be of Type p4 w.r.t.  $\Sigma'$ , adjacent to  $a'_2$  and  $b'_3$ . Note that  $a_1 b_1$  cannot be an edge, else  $(H_{12}, u)$  induces an

odd wheel. So, by Lemma 6.2,  $N(u) \cap \Sigma = \{a_1, a_2, a'_2, b_1, b_3, b'_3\}$ . If  $i \neq n$ , then  $(H, u)$ , where  $H$  is the hole induced by  $P^3 \cup P_{x_1 x_i} \cup a_1$ , is an odd wheel. So  $i = n$ . If  $a_1$  is adjacent to  $x_n$ , then  $P^1 \cup P^3 \cup P$  induces a  $3PC(a_1 a_3 x_n, b_3)$ . Hence  $a_1$  has no neighbor in  $P$ . Let  $H$  be the hole induced by  $P \cup \{a_1, a_3, b_1, b_2, u\}$ . Then  $(H, a_2)$  is an odd wheel.

**Case 2.3:**  $x_1$  is of Type p4 w.r.t.  $\Sigma$ .

Then  $(\Sigma \cup P) \setminus \{a_1, a_2\}$  contains a  $3PC(b_1 b_2 b_3, a_3)$ .

**Case 3:**  $x_n$  is of Type t2 or t3 w.r.t.  $\Sigma$ , adjacent to  $b_3$ .

Then  $b_3 \notin S$  and hence  $u$  is of Type t4s w.r.t.  $\Sigma$ . By Claim 9,  $a_1$  and  $a_2$  do not have neighbors in the interior of  $P$ .

Suppose that  $x_1$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ . We may assume w.l.o.g. that the neighbors of  $x_1$  in  $\Sigma$  are contained in  $P^1$ . Suppose  $b_2$  has a neighbor in the interior of  $P$ , and let  $x_i$  be the node of  $P$  with lowest index adjacent to  $b_2$ . Then  $(\Sigma \setminus b_1) \cup P_{x_1 x_i}$  contains a  $3PC(a_1 a_2 a_3, b_2)$ . Hence  $b_2$  has no neighbors in the interior of  $P$ . If  $b_2$  is not adjacent to  $x_n$ , then  $(\Sigma \setminus b_1) \cup P$  contains a  $3PC(a_1 a_2 a_3, b_3)$ . Therefore  $b_2$  is adjacent to  $x_n$  and hence  $(\Sigma \setminus b_1) \cup P$  contains a  $\Sigma' = 3PC(a_1 a_2 a_3, x_n b_2 b_3)$ . Since  $u$  is of Type t4s w.r.t.  $\Sigma$ , it violates Lemma 6.1 w.r.t.  $\Sigma'$ .

Since  $u$  is of Type t4s w.r.t.  $\Sigma$ ,  $a_1, a_2, b_1, b_2 \in S$  and hence  $x_1$  cannot be of Type t2 or t3 w.r.t.  $\Sigma$ . So  $x_1$  is of Type p4 w.r.t.  $\Sigma$ . But then  $(\Sigma \cup P) \setminus \{b_1, b_2\}$  contains a  $3PC(a_1 a_2 a_3, x_1)$ .  $\square$

## 10 Attachments

In this section, we assume that  $G$  is a square-free even-signable graph. Furthermore, we assume that  $G$  has no star cutset. So by Theorem 9.1, there are no Type t4, t5 and t6b nodes.

**Definition 10.1** Let  $\Sigma = 3PC(a_1 a_2 a_3, b_1 b_2 b_3)$  and let  $u$  be of Type t1 w.r.t.  $\Sigma$ , adjacent to say  $a_3$ . A chordless path  $P = y_1, \dots, y_m$  in  $G \setminus (\Sigma \cup u)$  is an attachment of  $u$  to  $\Sigma$  if  $u$  is adjacent to  $y_1$  and to no other node of  $P$ , no node of  $P \setminus y_m$  has a neighbor in  $\Sigma \setminus a'_3$  and one of the following holds.

- (i)  $y_m$  is of Type t1, p1 or p3 w.r.t.  $\Sigma$ , it is not adjacent to  $a_3$  and it has a neighbor in  $P^3 \setminus \{a_3, a'_3\}$ .
- (ii)  $y_m$  is of Type t2, t2p or t3p w.r.t.  $\Sigma$ , adjacent to  $b_1, b_2$  and no node of  $(P^1 \setminus b_1) \cup (P^2 \setminus b_2) \cup a_3$ .
- (iii)  $y_m$  is of Type p2 w.r.t.  $\Sigma$ ,  $N(a'_3) \cap P = \{y_{m-1}, y_m\}$  and  $y_m$  is not adjacent to  $a_3$ .

**Definition 10.2** Let  $\Sigma = 3PC(a_1 a_2 a_3, b_1 b_2 b_3)$  and let  $u$  be of Type t1 w.r.t.  $\Sigma$ , adjacent to say  $a_3$ . A chordless path  $P = y_1, \dots, y_m$  in  $G \setminus (\Sigma \cup u)$  is a bad connection of  $u$  to  $\Sigma$  if  $u$  is adjacent to  $y_1$  and to no other node of  $P$ , no node of  $P \setminus y_m$  has a neighbor in  $\Sigma \setminus a'_3$  and  $y_m$  is of Type t2 or t2p w.r.t.  $\Sigma$ , adjacent to  $a_1$  and  $a_2$ .

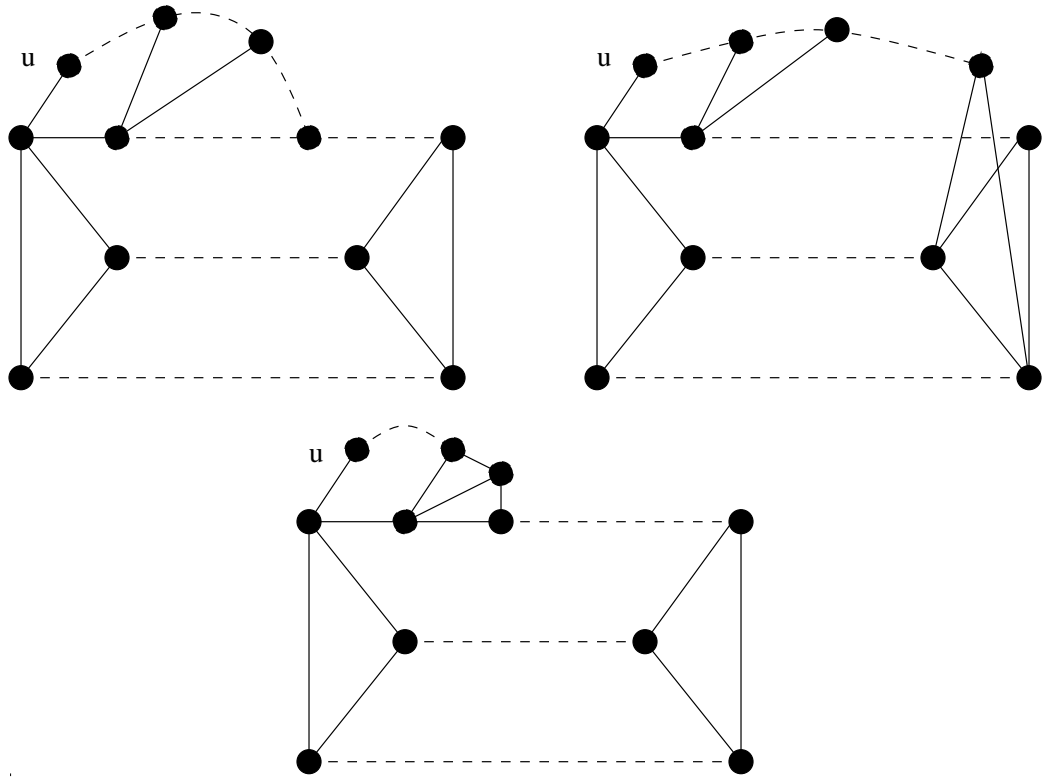


Figure 7: Attachments of a node of Type  $t_1$

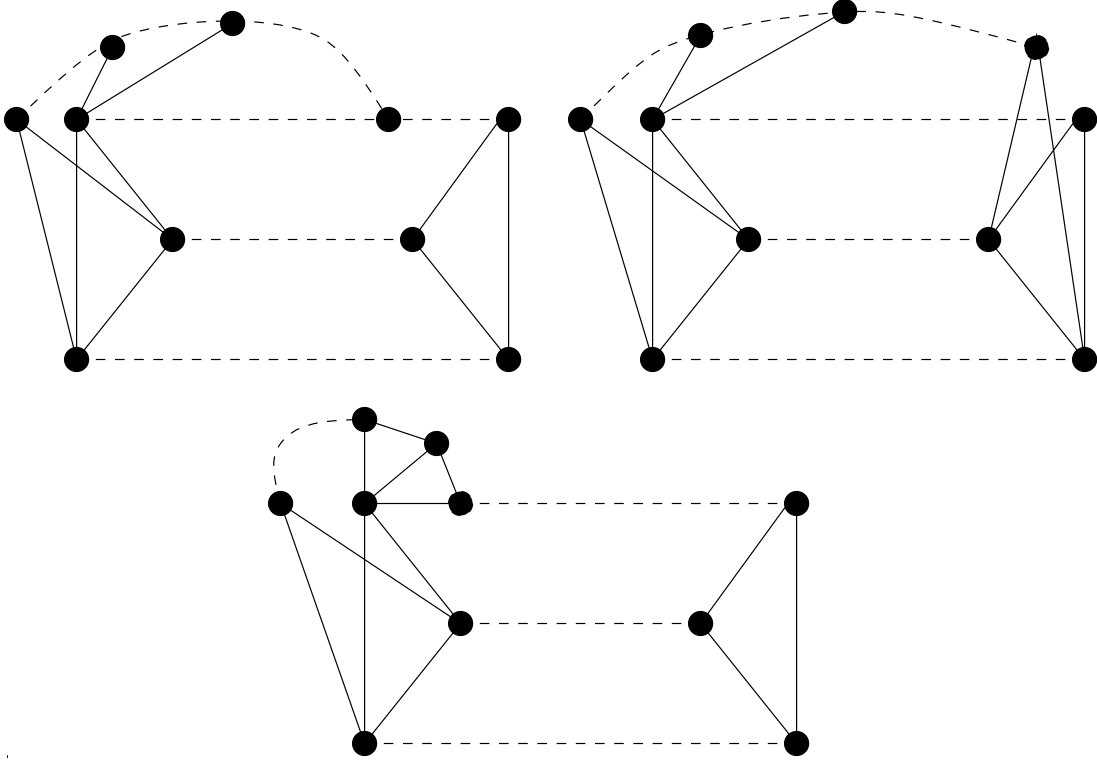


Figure 8: Attachments of a node of Type t2

**Definition 10.3** Let  $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$  and let  $u$  be of Type t2 w.r.t.  $\Sigma$ , adjacent to say  $a_2$  and  $a_3$ . A chordless path  $P = y_1, \dots, y_m$  in  $G \setminus (\Sigma \cup u)$  is an attachment of  $u$  to  $\Sigma$  if  $u$  is adjacent to  $y_1$  and to no other node of  $P$ , no node of  $P \setminus y_m$  has a neighbor in  $\Sigma \setminus a_1$  and one of the following holds.

- (i)  $y_m$  is of Type t1, p1 or p3 w.r.t.  $\Sigma$  and it has a neighbor in  $P^1 \setminus a_1$ .
- (ii)  $y_m$  is of Type t2, t2p or t3p w.r.t.  $\Sigma$ , adjacent to  $b_2, b_3$  and no node of  $(P^2 \cup P^3) \setminus \{b_2, b_3\}$ .
- (iii)  $y_m$  is of Type p2 w.r.t.  $\Sigma$  and  $N(a_1) \cap P = \{y_{m-1}, y_m\}$ .

Suppose  $u$  is of Type t1 or t2 w.r.t.  $\Sigma$ . If there exists an attachment of  $u$  to  $\Sigma$ , we say that  $u$  is attached to  $\Sigma$ . If  $P$  is an attachment of  $u$  to  $\Sigma$ , then a subset of  $\Sigma \cup P \cup u$  induces a  $\Sigma' = 3PC(\Delta, \Delta)$  that contains  $u$  and two of the paths of  $\Sigma$ . We say that  $\Sigma'$  is obtained from  $\Sigma$  by substituting  $u$  and  $P$  into  $\Sigma$ .

**Lemma 10.4** Every Type t1 node w.r.t.  $\Sigma = 3PC(\Delta, \Delta)$  is either attached to  $\Sigma$  or it has a bad connection to  $\Sigma$ .

*Proof:* Let  $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$  and assume that  $u$  is of Type t1 w.r.t.  $\Sigma$ , say adjacent to  $a_3$ . Let  $S = (N(a_3) \cup a_3) \setminus u$  and let  $x_1, \dots, x_n$  be a direct connection from  $u$  to  $\Sigma \setminus S$

in  $G \setminus S$ . Let  $x_0 = u$  and  $P = x_0, x_1, \dots, x_n$ . By definition of  $S$ ,  $x_n$  cannot be of Type t6a w.r.t.  $\Sigma$  and the only nodes of  $\Sigma$  that can have a neighbor in  $P \setminus \{x_0, x_n\}$  are  $a_1, a_2$  and  $a'_3$ .

First suppose that  $a_1$  or  $a_2$  has a neighbor in  $P \setminus x_n$ . Let  $x_i$  be the node of  $P \setminus x_n$  with lowest index adjacent to  $a_1$  or  $a_2$ . Since  $G$  is square-free,  $x_i$  is not adjacent to  $a'_3$ . If  $x_i$  is adjacent to exactly one of  $a_1, a_2$ , then Theorem 8.3 is contradicted. So  $x_i$  is adjacent to both  $a_1$  and  $a_2$ . Thus  $P_{x_1 x_i}$  is a bad connection of  $u$  to  $\Sigma$ .

So now we may assume that  $a_1$  and  $a_2$  do not have neighbors in  $P \setminus x_n$ . If  $a'_3$  does not have a neighbor in  $P \setminus x_n$ , then the result follows from Lemma 8.4 applied to  $P$ . So we may assume that  $a'_3$  has a neighbor in  $P \setminus x_n$ . Let  $x_i$  be such a neighbor with highest index.

If  $x_n$  is of Type t1, p1, p2 or p3 with neighbors in  $\Sigma$  contained in  $P^1 \cup P^2$ , then  $P_{x_i x_n}$  contradicts Theorem 8.3. If  $x_n$  is of Type t1, p1 or p3 with neighbors in  $\Sigma$  contained in  $P^3$ , then the result follows. Suppose  $x_n$  is of Type p2 and its neighbors in  $\Sigma$  are contained in  $P^3$ . Then  $x_n$  is adjacent to  $a'_3$ , else  $P^1 \cup P^3 \cup P_{x_i x_n}$  induces a  $3PC(\Delta, a'_3)$ . But then  $P^1 \cup P^3 \cup P$  must induce a beetle with center  $a'_3$ , and hence the result follows.

Suppose  $x_n$  is of Type t2, t2p, t3 or t3p w.r.t.  $\Sigma$  and the result does not hold. Then we may assume w.l.o.g. that  $x_n$  is adjacent to  $b_1, b_3$  and to no node of  $(P^1 \cup P^3) \setminus \{b_1, b_3\}$ . But then  $P^1 \cup P^3 \cup P_{x_i x_n}$  induces a  $3PC(b_1 b_3 x_n, a'_3)$ .

Suppose  $x_n$  is of Type p4 w.r.t.  $\Sigma$ . If the neighbors of  $x_n$  in  $\Sigma$  are contained in  $P^1 \cup P^2$ , then  $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup P_{x_i x_n}$  contains a  $3PC(b_1 b_2 b_3, x_n)$ . So we may assume w.l.o.g. that the neighbors of  $x_n$  in  $\Sigma$  are contained in  $P^1 \cup P^3$ . But then  $(\Sigma \setminus \{a_1, a'_3\}) \cup P$  contains a  $3PC(b_1 b_2 b_3, x_n)$ .  $\square$

**Lemma 10.5** *Every Type t2 node w.r.t.  $\Sigma = 3PC(\Delta, \Delta)$  is attached to  $\Sigma$ . Furthermore, let  $\Sigma = 3PC(a_1 a_2 a_3, b_1 b_2 b_3)$  and  $u$  be a Type t2 node adjacent to  $a_2$  and  $a_3$ . Then every direct connection from  $u$  to  $\Sigma \setminus \{a_1, a_2, a_3\}$  that contains no neighbor of  $a_3$  is an attachment.*

*Proof:* Let  $\Sigma = 3PC(a_1 a_2 a_3, b_1 b_2 b_3)$  and let  $u$  be a node of Type t2 w.r.t.  $\Sigma$ , say adjacent to  $a_2$  and  $a_3$ . Let  $S = (N(a_3) \cup a_3) \setminus \{u, a'_3\}$  and let  $x_1, \dots, x_n$  be a direct connection from  $u$  to  $\Sigma \setminus S$  in  $G \setminus S$ . Let  $x_0 = u$  and  $P = x_0, x_1, \dots, x_n$ . By definition of  $S$ ,  $x_n$  cannot be of Type t6a w.r.t.  $\Sigma$  and the only nodes of  $\Sigma$  that can have a neighbor in  $P \setminus \{x_0, x_n\}$  are  $a_1$  and  $a_2$ . Let  $x_j$  be the node of  $P \setminus x_n$  with highest index adjacent to  $a_2$ . If  $a_1$  has a neighbor in  $P \setminus x_n$ , then let  $x_i$  be such a neighbor with highest index.

**Case 1:**  $j > 0$

We first show that  $a_2$  is adjacent to  $x_1$ . Suppose not and let  $x_k$  be the node of  $P \setminus u$  with lowest index adjacent to  $a_2$ . Let  $H$  be the hole in  $(\Sigma \cup P) \setminus \{a_1, a_2\}$  that contains  $u$ . Since  $j > 0$ ,  $W = (H, a_2)$  is a wheel. Since  $a_2$  is adjacent to  $a_3$  and  $u$  but not to  $a'_3$  and  $x_1$ ,  $W$  is either a proper wheel that is not a beetle, or a line wheel. In the former case, Theorem 4.6 is contradicted. So assume  $W$  is a line wheel. Then  $(W, a_2)$  belongs to an L-parachute with center path obtained by taking the shortest path from  $a_2$  to  $W$  in  $\Sigma \setminus \{a_1, a_3\}$ . So Theorem 5.3 is contradicted. Therefore  $a_2$  is adjacent to  $x_1$ .

**Case 1.1:**  $a_1$  has a neighbor in  $P \setminus x_n$ .

Let  $x_k$  be such a neighbor with lowest index. If  $k = 1$  then  $\{a_1, a_3, x_0, x_1\}$  induces a square. So  $k > 1$ .  $P_{x_0 x_k} \cup \{a_1, a_2, a_3\}$  must be a universal wheel with center  $a_2$ . In particular,  $a_2$  is adjacent to  $x_2$ .

Let  $H$  be a hole that contains  $u$  in  $P \cup \Sigma \cup \{a_1, a_2\}$ . By construction,  $H$  contains  $a'_3$ . So  $(H, a_2)$  is a proper wheel that is not a beetle.

**Case 1.2:**  $a_1$  does not have a neighbor in  $P \setminus x_n$ .

By Lemma 8.4 applied to  $P_{x_j x_n}$ ,  $x_n$  is either of Type t1, p1 or p3 w.r.t.  $\Sigma$  with a neighbor in  $P^2 \setminus a_2$ , or of Type t2, t2p or t3p w.r.t.  $\Sigma$  adjacent to  $b_1, b_3$  and no node of  $(P^1 \cup P^3) \setminus \{b_1, b_3\}$ , or of Type t2p w.r.t.  $\Sigma$  adjacent to  $a_1, a_2$  and a node of  $P^3 \setminus a_3$ , or of Type p2 or p4 w.r.t.  $\Sigma$  adjacent to  $a_2$ . If  $x_n$  is of Type t1, p1, p2 or p3, then  $(\Sigma \setminus a_2) \cup P$  contains a  $3PC(b_1 b_2 b_3, a_3)$ . If  $x_n$  is of Type t2, t2p or t3p adjacent to  $b_1$  and  $b_3$ , then  $P^1 \cup P^3 \cup P$  induces a  $3PC(b_1 x_n b_3, a_3)$ . If  $x_n$  is of Type p4 with a neighbor in  $P^1$ , let  $u_1$  be such a neighbor closest to  $a_1$ . Then  $P_{a_1 u_1}^1 \cup P \cup \{a_2, a_3\}$  induces a proper wheel with center  $a_2$  that is not a beetle. If  $x_n$  is of Type p4 with a neighbor in  $P^3$ , then  $P^1 \cup P^2 \cup P \cup a_3$  induces a proper wheel with center  $a_2$  that is not a beetle. So  $x_n$  is of Type t2p adjacent to  $a_1$  and  $a_2$ . Note that  $n > 1$  since otherwise  $a_1, a_3, u, x_1$  induces a square. Let  $H$  be the hole induced by  $P \cup \{a_1, a_3\}$ .  $(H, a_2)$  must be a universal wheel. In particular,  $a_2$  is adjacent to all nodes of  $P$ . Let  $v$  be the neighbor of  $x_n$  in  $P^3$  that is closest to  $a_3$ . Let  $H'$  be the hole induced by  $P_{a_3 v}^3 \cup P$ . Then  $(H', a_2)$  is a proper wheel that is not a beetle.

**Case 2:**  $j = 0$

Suppose  $a_1$  does not have a neighbor in  $P \setminus x_n$ . By Lemma 8.5 applied to  $P$ ,  $P_{x_1 x_n}$  is either an attachment of  $u$  to  $\Sigma$ , or  $x_n$  is of Type t2p w.r.t.  $\Sigma$  adjacent to  $a_1$  and  $a_2$ . Suppose  $x_n$  is of Type t2p adjacent to  $a_1$  and  $a_2$ . If  $n = 1$  then  $\{a_1, a_3, x_0, x_1\}$  induces a square. So  $n > 1$ . But then  $P \cup \{a_1, a_2, a_3\}$  induces an odd wheel with center  $a_2$ .

So we may assume that  $a_1$  has a neighbor in  $P \setminus x_n$ . By Lemma 8.4 applied to  $P_{x_1 x_n}$ ,  $x_n$  is either of Type t1, p1 or p3 w.r.t.  $\Sigma$  adjacent to a node of  $P^1 \setminus a_1$ , or of Type p2 or p4 w.r.t.  $\Sigma$  adjacent to  $a_1$ , or of Type t2, t2p or t3p w.r.t.  $\Sigma$  adjacent to  $b_2, b_3$  and no node of  $(P^2 \cup P^3) \setminus \{b_2, b_3\}$ , or of Type t2p w.r.t.  $\Sigma$  adjacent to  $a_1$  and  $a_2$ . If  $x_n$  is of Type t1, p1 or p3, or of Type t2, t2p or t3p adjacent to  $b_2$  and  $b_3$ , then  $P_{x_1 x_n}$  is an attachment of  $u$  to  $\Sigma$ . If  $x_n$  is of Type p4, then w.l.o.g.  $x_n$  does not have a neighbor in  $P^2$ , and hence  $(\Sigma \setminus \{a_1, a_3\}) \cup P$  contains a  $3PC(b_1 b_2 b_3, x_n)$ . If  $x_n$  is of Type p2, then  $P^1 \cup P^2 \cup P$  must induce a beetle with center  $a_1$ , and hence  $P_{x_1 x_n}$  is an attachment of  $u$  to  $\Sigma$ . So we may assume that  $x_n$  is of Type t2p w.r.t.  $\Sigma$ , adjacent to  $a_1$  and  $a_2$ . Let  $u_3$  be the neighbor of  $x_n$  in  $P_3$  that is closest to  $a_3$ . Then  $P_{a_3 u_3} \cup P$  induces a hole  $H$  and  $(H, a_2)$  is an odd wheel.  $\square$

**Lemma 10.6** *Let  $\Sigma = 3PC(a_1 a_2 a_3, b_1 b_2 b_3)$  and let  $y$  be a Type t2 or t2p node w.r.t.  $\Sigma$ , adjacent to say  $b_2$  and  $b_3$ . Then*

- (i) *there cannot exist a node  $x$  that is of Type t1 w.r.t.  $\Sigma$  adjacent to  $b_3$  and  $y$ ;*
- (ii) *every node  $x$  of Type t2 or t2p w.r.t.  $\Sigma$  adjacent to  $b_1, b_2$  is adjacent to  $y$ . Every sibling  $x$  of  $b_3$  of Type t3p w.r.t.  $\Sigma$  is adjacent to  $y$ .*

*Proof:* We first prove (i). By Lemma 10.5 let  $P^y = y_1, \dots, y_m$  be an attachment of  $y$  to  $\Sigma$ , and let  $\Sigma^y$  be obtained by substituting  $y$  and  $P^y$  into  $\Sigma$ . Assume there is a node  $x$  of Type t1 w.r.t.  $\Sigma$ , adjacent to  $b_3$  and to  $y$ . By Lemma 6.1 applied to  $\Sigma^y$ ,  $x$  is of Type t2 w.r.t.  $\Sigma^y$ . By Lemma 10.5, let  $P^x = x_1, \dots, x_n$  be an attachment of  $x$  to  $\Sigma^y$ .

First we show that no node of  $P^1$  is adjacent to or coincident with a node of  $P^x \setminus x_n$ . Assume not and let  $x_i$  be the node of  $P^x \setminus x_n$  with lowest index that is adjacent to a node of  $P^1$ . If  $b_2$  has no neighbor in  $P_{x_1 x_i}^x$ , then  $x, P_{x_1 x_i}^x$  contradicts Theorem 8.3 applied to  $\Sigma$ . So  $b_2$  has a neighbor in  $P_{x_1 x_i}^x$ , and let  $x_j$  be such a neighbor with lowest index. By Theorem 8.3 applied to  $x, P_{x_1 x_j}^x$  and  $\Sigma$ ,  $i = j$ . By Lemma 6.1 applied to  $x_i$  and  $\Sigma$ ,  $x_i$  is of Type t2 w.r.t.  $\Sigma$ , adjacent to  $b_1$  and  $b_2$ . Let  $R$  be the shortest path from  $b_1$  to  $y$  in  $P^y \cup P^1 \cup \{y, a_3\}$ . Then  $R \cup P_{x_1 x_i}^x \cup \{b_2, x\}$  induces an odd wheel with center  $b_2$ . Therefore, no node of  $P^1$  is adjacent to or coincident with a node of  $P^x \setminus x_n$ .

If  $b_2$  has a neighbor in  $P^x \setminus x_n$ , then  $P_{x_1 x_i}^x$  (where  $x_i$  is the neighbor of  $b_2$  in  $P^x \setminus x_n$  with lowest index) contradicts Theorem 8.3 applied to  $\Sigma$ . So  $b_2$  has no neighbor in  $P^x \setminus x_n$ . In particular,  $x_n$  is not of Type p2 w.r.t.  $\Sigma^y$  and therefore the attachment  $P^x$  of  $x$  to  $\Sigma^y$  satisfies Definition 10.3(i) or (ii). Suppose that  $x_n$  is of type t1, p1 or p3 w.r.t.  $\Sigma^y$ . Then its neighbors in  $\Sigma^y$  are contained in  $P^2$ . By Lemma 8.4 applied to  $x, P^x$  and  $\Sigma$ ,  $x_n$  is of Type t2 w.r.t.  $\Sigma$ , adjacent to  $a_1$  and  $a_2$  and  $y$  is a Type t2 node w.r.t.  $\Sigma$  with attachment  $P^y$  satisfying Definition 10.3(ii). But then  $P^1 \cup P^x \cup \{x, y, b_2, b_3\}$  induces an odd wheel with center  $b_3$ . So  $x_n$  is of Type t2, t2p or t3p w.r.t.  $\Sigma^y$ . So  $x_n$  is adjacent to  $a_3$ , and if it is of Type t2p or t3p w.r.t.  $\Sigma^y$  then it has a neighbor in  $P^2 \setminus a_2$ . If  $x_n$  is adjacent to  $a_1$ , then by Lemma 6.1  $x_n$  is of Type t2, t2p or t3p w.r.t.  $\Sigma$ , and hence  $x, P^x$  contradicts Lemma 8.4 applied to  $\Sigma$ . So  $x_n$  is not adjacent to  $a_1$ . By Lemma 6.1,  $x_n$  is of Type t1 w.r.t.  $\Sigma$  and  $y$  is of Type t2 w.r.t.  $\Sigma$ . But then  $P^1 \cup P^x \cup \{x, y, b_2, b_3, a_3\}$  induces an odd wheel with center  $b_3$ .

Now we prove (ii). If  $x$  is of Type t2p or t3p w.r.t.  $\Sigma$ , let  $\Sigma'$  be obtained from  $\Sigma$  by substituting  $x$  for its sibling  $b_3$ . If  $x$  is of Type t2 w.r.t.  $\Sigma$ , then by Lemma 10.5, there is an attachment  $Q = x_1, \dots, x_n$  of  $x$  to  $\Sigma$ . In this case, let  $\Sigma'$  be obtained by substituting  $x$  and its attachment  $Q$  into  $\Sigma$ . Note that  $P^1 \cup P^2 \subseteq \Sigma'$ . Let  $P_x^3$  be the path of  $\Sigma' \setminus (P^1 \cup P^2)$ . Suppose that  $y$  is not adjacent to  $x$ . Then by Lemma 6.1 applied to  $\Sigma'$ ,  $y$  is of Type t1 w.r.t.  $\Sigma'$  and hence of Type t2 w.r.t.  $\Sigma$ . By Lemma 10.5, there is an attachment  $P^y = y_1, \dots, y_m$  of  $y$  to  $\Sigma$ . Let  $\Sigma^y$  be obtained by substituting  $y$  and  $P^y$  into  $\Sigma$ . If  $x$  is of Type t2p or t3p in  $\Sigma$ , then  $x$  violates Lemma 6.1 in  $\Sigma^y$ . So  $x$  is of Type t2 in  $\Sigma$ .

Let  $R$  be a shortest path from  $x$  to  $y$  in  $P^y \cup \Sigma' \setminus (P^2 \cup \{b_1, b_3\})$ . Then  $R \cup b_2$  induces a hole  $H'$ . If  $b_1$  has a neighbor in  $R \setminus x$ , then  $W = (H', b_1)$  is a wheel that is not a twin wheel, a triangle-free wheel, a universal wheel or a beetle. Suppose  $W$  is a line wheel. Then there is an L-parachute with center path included in  $P^1 \cup P^y \cup a_3$ . But then, by Theorem 5.3, there is a star cutset. So  $W$  is a proper wheel that is not a beetle and by Theorem 4.6 there is a star cutset. So  $b_1$  has no neighbor in  $R \setminus x$ . Similarly  $b_3$  has no neighbor in  $R \setminus x$ . But then  $(Rb_3b_1, b_2)$  is an odd wheel.  $\square$

## 11 Type t6a Nodes

**Theorem 11.1** *Let  $G$  be a square-free even-signable graph. Let  $\Sigma = 3PC(a_1 a_2 a_3, b_1 b_2 b_3)$  and let  $u$  be a Type t6a node w.r.t.  $\Sigma$ . If for some  $i \in \{1, 2, 3\}$ , there is no  $P^i$ -crosspath w.r.t.  $\Sigma$ , then  $G$  has a star cutset.*

*Proof:* Assume there is no  $P^3$ -crosspath w.r.t.  $\Sigma$ . Suppose there is no star cutset. Then by Theorem 9.1, no node is of Type t4, t5 or t6b w.r.t. a  $3PC(\Delta, \Delta)$ . Let  $S = (N(u) \cup u) \setminus (\Sigma \setminus$

$\{a_1, a_2, b_3\}$ ) and let  $P = x_1, \dots, x_n$  be a direct connection from  $P^1 \cup P^2$  to  $P^3$  in  $G \setminus S$ .

**Claim 1:** *No node of  $P$  is of Type t2, t2p, t3p or t6a w.r.t.  $\Sigma$ .*

*Proof of Claim 1:* If  $x_i$  is of Type t2p or t3p w.r.t.  $\Sigma$ , then let  $\Sigma'$  be obtained from  $\Sigma$  by substituting  $x_i$  for its sibling. If  $x_i$  is of Type t2 w.r.t.  $\Sigma$ , then by Lemma 10.5,  $x_i$  is attached to  $\Sigma$ . Let  $\Sigma'$  be obtained from  $\Sigma$  by substituting  $x_i$  and its attachment into  $\Sigma$ . Since  $u$  is not adjacent to  $x_i$ , it is of Type t5 or t4s w.r.t.  $\Sigma'$ , a contradiction. Since  $G$  is square-free, every node of Type t6a w.r.t.  $\Sigma$  is adjacent to  $u$ . It follows from the definition of  $S$  that no node of  $P$  is of Type t6a w.r.t.  $\Sigma$ . This completes the proof of Claim 1.

**Claim 2:**  *$n > 1$ ,  $x_1$  is of Type t1, p1, p2, p3 or p4 w.r.t.  $\Sigma$  with neighbors contained in  $P^1 \cup P^2$ , or of Type t3 w.r.t.  $\Sigma$  adjacent to  $b_1, b_2$  and  $b_3$ , and  $x_n$  is of Type t1, p1, p2 or p3 w.r.t.  $\Sigma$  with neighbors contained in  $P^3$ , or of Type t3 w.r.t.  $\Sigma$  adjacent to  $a_1, a_2$  and  $a_3$ .*

*Proof of Claim 2:* Since there is no  $P^3$ -crosspath, if a node of  $P$  is of Type p4 w.r.t.  $\Sigma$ , then its neighbors are contained in  $P^1 \cup P^2$ . Now the result follows from Claim 1. This completes the proof of Claim 2.

**Claim 3:** *If  $a_1$  or  $a_2$  has a neighbor in the interior of  $P$ , then  $b_3$  has no neighbor in the interior of  $P$ .*

*Proof of Claim 3:* Let  $x_i$  and  $x_j$  be nodes in the interior of  $P$  adjacent to  $a_1$  or  $a_2$ , and to  $b_3$  respectively so that the  $P_{x_i x_j}$  subpath is shortest possible. W.l.o.g. assume that  $x_i$  is adjacent to  $a_1$ . By Claim 1,  $x_i$  is of Type t1 w.r.t.  $\Sigma$ , and hence  $P_{x_i x_j} \cup P^2 \cup P^3$  induces a  $3PC(a_1 a_2 a_3, b_3)$ . This completes the proof of Claim 3.

**Claim 4:** *No interior node of  $P$  is adjacent to  $a_1, a_2$  or  $b_3$ .*

*Proof of Claim 4:* Assume first that  $b_3$  has a neighbor in the interior of  $P$ . Let  $x_i$  be a node with highest index in  $P_{x_2 x_{n-1}}$  adjacent to  $b_3$ . By Claim 3, no interior node of  $P$  is adjacent to  $a_1$  or  $a_2$ . So, by Lemma 8.4 applied to  $P_{x_i x_n}$ ,  $x_n$  is of Type t1, p1 or p3 with neighbors in  $P^3$  or of Type p2 adjacent to  $b_3$ . Similarly by Theorem 8.3 and Lemma 8.4,  $x_1$  is of Type p4 with neighbors in  $P^1 \cup P^2$ , or of Type t3 adjacent to  $b_1, b_2, b_3$ . If  $x_1$  is of Type p4, there is a  $3PC(b_1 b_2 b_3, x_1)$  contained in  $(P \cup \Sigma) \setminus \{a_1, a_2, a_3, x_n\}$ . If  $x_1$  is of Type t3 adjacent to  $b_1, b_2, b_3$ , then  $(P \cup \Sigma) \setminus b_3$  contains a  $3PC(a_1 a_2 a_3, b_1 b_2 x_1)$  and  $u$  is of Type t5 w.r.t. it, a contradiction.

Assume now that some interior node of  $P$  is adjacent to  $a_1$  or  $a_2$ . Let  $x_i$  be the node with lowest index in  $P_{x_2 x_{n-1}}$  adjacent to  $a_1$  or  $a_2$ , say  $a_1$ . By Claim 1,  $x_i$  is of Type t1 w.r.t.  $\Sigma$ . By Lemma 8.4 applied to  $P_{x_1 x_i}$ ,  $x_1$  is of Type t1, p1 or p3 with neighbors in  $P^1$  or of Type p2 or p4 adjacent to  $a_1$ . Suppose  $a_2$  has a neighbor in  $P_{x_{i+1} x_{n-1}}$  and let  $x_j$  be such a neighbor with lowest index. If  $x_1$  is of Type t1, p1, p2 or p3 w.r.t.  $\Sigma$ , then  $(\Sigma \cup P_{x_1 x_j}) \setminus a_1$  contains a  $3PC(b_1 b_2 b_3, a_2)$ . If  $x_1$  is of Type p4 w.r.t.  $\Sigma$ , then  $P^1 \cup P^3 \cup P_{x_1 x_j} \cup a_2$  induces a proper wheel with center  $a_1$  that is not a beetle, contradicting Theorem 4.6. Therefore  $a_2$  has no neighbor in  $P_{x_2 x_{n-1}}$ . Then, by Theorem 8.3,  $x_n$  is of Type t3 adjacent to  $a_1, a_2, a_3$ . If  $x_1$  is of Type p4 w.r.t.  $\Sigma$ , then  $(\Sigma \cup P) \setminus \{a_1, a_2\}$  contains a  $3PC(b_1 b_2 b_3, x_1)$ . So  $x_1$  is not of Type p4 w.r.t.  $\Sigma$  and hence by a symmetric argument as in the previous paragraph, we

obtain a contradiction. This completes the proof of Claim 4.

**Case 1:**  $x_n$  is of Type t1, p1, p2 or p3 w.r.t.  $\Sigma$ .

If  $x_1$  is of Type t1, p1, p2 or p3 w.r.t.  $\Sigma$ , by Theorem 8.3,  $P$  is a  $P^3$ -crosspath, a contradiction. Suppose  $x_1$  is of Type t3 w.r.t.  $\Sigma$ . Let  $\Sigma' = 3PC(a_1a_2a_3, b_1b_2x_1)$  contained in  $(\Sigma \setminus b_3) \cup P$ . Then  $u$  is of Type t5 w.r.t.  $\Sigma'$ , a contradiction. So  $x_1$  is of Type p4 w.r.t.  $\Sigma$ . But then  $(\Sigma \setminus \{b_1, b_2, b_3\}) \cup P$  contains a  $3PC(a_1a_2a_3, x_1)$ .

**Case 2:**  $x_n$  is of Type t3 w.r.t.  $\Sigma$

Suppose  $x_1$  is of Type t1, p1, p2 or p3 w.r.t.  $\Sigma$ , with neighbors w.l.o.g. in  $P^2$ .  $(\Sigma \setminus a_2) \cup P$  contains a  $\Sigma' = 3PC(a_1x_na_3, b_1b_2b_3)$ . But then  $u$  is of Type t5 w.r.t.  $\Sigma'$ , a contradiction.

Suppose  $x_1$  is of Type t3 w.r.t.  $\Sigma$ . Then  $P^1 \cup P^3 \cup P$  induces a  $\Sigma' = 3PC(a_1x_na_3, b_1x_1b_3)$ . Then  $u$  is of Type t4s w.r.t.  $\Sigma'$ , a contradiction. So  $x_1$  is of Type p4 w.r.t.  $\Sigma$ . But then  $(\Sigma \setminus \{a_1, a_2\}) \cup P$  contains a  $3PC(b_1b_2b_3, x_1)$ .  $\square$

## 12 Type t2 and t2p Nodes

In this section, we assume that  $G$  is a square-free even-signable graph. Furthermore, we assume that  $G$  has no star cutset. So by Theorem 9.1, there are no Type t4, t5 and t6b nodes.

**Definition 12.1** *A  $3PC(a_1a_2a_3, b_1b_2b_3) = \Sigma$  in  $G$  is decomposable if there exists a node of Type t2 or t2p w.r.t.  $\Sigma$ , say adjacent to  $a_2$  and  $a_3$ , but there is no  $P^1$ -crosspath w.r.t.  $\Sigma$ . Path  $P^1$  of  $\Sigma$  is called the middle path.*

Denote by  $H$  the graph induced by a decomposable  $3PC(a_1a_2a_3, b_1b_2b_3)$  together with a node  $a_4$  of Type t2 or t2p adjacent to  $a_2, a_3$ . Let  $H_1 = P^1 \cup a_4$  and  $H_2 = P^2 \cup P^3$ . Then  $H_1|H_2$  is a 2-join of  $H$  with special sets  $A_1 = \{a_1, a_4\}$ ,  $B_1 = \{b_1\}$ ,  $A_2 = \{a_2, a_3\}$  and  $B_2 = \{b_2, b_3\}$ . In this section, we show that the 2-join  $H_1|H_2$  of  $H$  extends to a 2-join of  $G$ . First, we prove the following result.

**Theorem 12.2** *If  $G$  contains a  $3PC(\Delta, \Delta)$  with a Type t2 or t2p node, then  $G$  contains a decomposable  $3PC(\Delta, \Delta)$ .*

*Proof:* First suppose that  $G$  contains a connected diamonds  $D(a_1a_2a_3a_4, b_1b_2b_3b_4)$ . Let  $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$  (resp.  $\Sigma' = 3PC(a_4a_2a_3, b_4b_2b_3)$ ) induced by paths  $P^1, P^2$  and  $P^3$  (resp.  $P^4, P^2$  and  $P^3$ ) of  $D$ . Suppose that  $P = x_1, \dots, x_n$  is a  $P^1$ -crosspath w.r.t.  $\Sigma$ . W.l.o.g.  $x_1$  has a neighbor in  $P^1$  and  $x_n$  has a neighbor in  $P^2$ . Let  $u_1$  and  $v_1$  be the neighbors of  $x_1$  in  $P^1$ . If no node of  $P^4$  has a neighbor in  $P$ , then  $P^1 \cup (P^2 \setminus b_2) \cup b_3 \cup P^4 \cup P$  contains a  $3PC(x_1u_1v_1, a_2)$ . So a node of  $P^4$  has a neighbor in  $P$ . Let  $x_i$  be such a neighbor with highest index. Let  $v$  be the neighbor of  $x_i$  in  $P^4$  that is closest to  $a_4$ . By Lemma 6.1 and Theorem 8.3 applied to  $P_{x_ix_n}$  and  $\Sigma'$ ,  $P_{x_ix_n}$  is a  $P^4$ -crosspath w.r.t.  $\Sigma'$ . Hence  $v \neq b_4$ . If  $i \neq 1$  then the path  $P_{a_4v}^4, P_{x_ix_n}$  contradicts Lemma 8.5 applied to  $\Sigma$ . So  $i = 1$  and hence the path  $P_{a_4v}^4, x_1$  contradicts Lemma 8.5 applied to  $\Sigma$ . Therefore, there is no  $P^1$ -crosspath w.r.t.  $\Sigma$ , and hence  $\Sigma$  is a decomposable  $3PC(\Delta, \Delta)$ .

Now we may assume that  $G$  does not contain connected diamonds.

Let  $\Sigma$  be a  $3PC(\Delta, \Delta)$  with a Type t2 or t2p node that has shortest middle path. Let  $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$  and w.l.o.g. let  $a_4$  be a node of Type t2 or t2p adjacent to  $a_2$  and  $a_3$ . Suppose  $\Sigma$  is not decomposable and let  $P = x_1, \dots, x_n$  be a  $P^1$ -crosspath w.r.t.  $\Sigma$ . W.l.o.g.  $x_1$  has a neighbor in  $P^1$  and  $x_n$  in  $P^2$ . Let  $u_1$  (resp.  $v_1$ ) be the neighbor of  $x_1$  in  $P^1$  that is closest to  $a_1$  (resp.  $b_1$ ).

First suppose that  $u_1 \neq a_1$ . Let  $\Sigma' = 3PC(a_1a_2a_3, u_1x_1v_1)$  contained in  $(\Sigma \cup P) \setminus b_2$ . Since  $a_4$  cannot be of Type t4 or t5 w.r.t.  $\Sigma'$ , it must be of Type t2 or t2p w.r.t.  $\Sigma'$ . Since  $\Sigma'$  has a shorter middle path than  $\Sigma$ , this contradicts our choice of  $\Sigma$ . Therefore,  $u_1 = a_1$ .

Suppose  $a_4$  is of Type t2p w.r.t.  $\Sigma$ . We show that  $a_4$  has no neighbor in  $P$ . Suppose it does. Let  $H$  be the hole contained in  $(\Sigma \cup P) \setminus \{a_1, b_2\}$ . Consider the wheel  $(H, a_4)$ . Since  $a_4$  is adjacent to  $a_2, a_3$ ,  $(H, a_4)$  is not triangle-free. Since  $a_4$  has a neighbor in  $P_1$ , it is not a twin wheel. Since  $a_4$  is not adjacent to  $b_3$ , it is not a universal wheel.  $(H, a_4)$  is not a line wheel since, if  $a_4$  were adjacent to  $a'_1$  and  $x_1$ , then  $a_1, a_2, a_4, a'_1$  would induce a square.  $(H, a_4)$  is not a beetle since if  $x_n$  were adjacent to  $a_2$  and to  $a_4$ , there would be an odd wheel with center  $a_2$  induced by  $P^2 \cup P^3 \cup \{a_4, x_n\}$ . Therefore  $(H, a_4)$  is a proper wheel that is not a beetle and by Theorem 4.6,  $G$  has a star cutset, a contradiction. Therefore,  $a_4$  has no neighbor in  $P$ . Let  $H$  be the hole contained in  $P \cup (P^1 \setminus a_1) \cup (P^2 \setminus b_2) \cup a_4$ . Then  $(H, a_1)$  is an odd wheel.

Therefore,  $a_4$  is of Type t2 w.r.t.  $\Sigma$ . By Lemma 10.5, let  $Q = y_1, \dots, y_m$  be an attachment of  $a_4$  to  $\Sigma$ . Let  $\Sigma'$  be obtained by substituting  $a_4$  and  $Q$  into  $\Sigma$ . Suppose  $a_4$  has a neighbor in  $P$  and let  $x_i$  be its neighbor in  $P$  with highest index. If  $i = 1$  then  $a_4, x_1$  contradicts Lemma 8.5 applied to  $\Sigma$  and otherwise  $a_4, P_{x_i x_n}$  contradicts Lemma 8.5 applied to  $\Sigma$ . So  $a_4$  does not have a neighbor in  $P$ . Next we show that no node of  $Q \setminus y_m$  is adjacent to or coincident with a node of  $P$ . Suppose not and let  $y_i$  be the node of  $Q$  with lowest index adjacent to a node of  $P$ , and let  $x_j$  be the node of  $P$  with highest index adjacent to  $y_i$ . If  $j = 1$ , then  $a_4, Q_{y_1 y_i}, x_1$  must satisfy Lemma 10.5. Therefore  $y_i$  is adjacent to  $a_1$ . But then there is a  $3PC(a_1 y_i x_1, a_2)$  contained in  $Q_{y_1 y_i} \cup (P^2 \setminus b_2) \cup P \cup \{a_1, a_4\}$ . If  $j > 1$ , then  $a_4, Q_{y_1 y_i}, P_{x_j x_n}$  violates Lemma 10.5 w.r.t.  $\Sigma$ . So no node of  $Q \setminus y_m$  is adjacent to or coincident with a node of  $P$ .

Assume  $y_m$  is of Type p2 w.r.t.  $\Sigma$ . Node  $y_m$  cannot be adjacent to any node of  $P \setminus x_1$  since, otherwise, there is a  $3PC(a_1 y_{m-1} y_m, a_2)$ . Now  $y_m$  is adjacent to  $x_1$  since, otherwise, there is an odd wheel with center  $a_1$  contained in  $Q \cup (P^2 \setminus b_2) \cup P \cup \{a_1, a_4\}$ . But then there is a  $3PC(a_2 a_3 a_4, x_1 a'_1 y_m)$  with  $a_1$  a strongly adjacent node of Type t5, a contradiction.

Assume  $y_m$  is of Type t1, p1 or p3 w.r.t.  $\Sigma$ . We show that  $y_m$  does not have a neighbor in  $P$ . Suppose not and let  $x_i$  be the neighbor of  $y_m$  in  $P$  with highest index. Then  $P_{x_i x_n}$  contradicts Theorem 8.3 applied to  $\Sigma'$  unless  $i = 1$  and  $y_m$  is of Type p1 adjacent to  $a'_1$ . But then there is a  $3PC(a_2 a_3 a_4, x_1 a'_1 y_m)$  and  $a_1$  is a strongly adjacent node of Type t4s relative to it, a contradiction. Therefore  $y_m$  does not have a neighbor in  $P$ . Let  $v$  be the neighbor of  $y_m$  in  $P^1 \setminus a_1$  that is closest to  $a'_1$ . Let  $H$  be the hole contained in  $P \cup P_{a'_1 v}^1 \cup (P^2 \setminus b_2) \cup Q \cup a_4$ . Then  $(H, a_1)$  is a proper wheel that is not a beetle unless  $y_m$  is a strongly adjacent node of Type p3 adjacent to  $a_1, a'_1$  and at least one other node of  $P^1$ . Since  $P \cup a'_1$  is a crosspath w.r.t.  $\Sigma'$ , the only other node of  $P^1$  adjacent to  $y_m$  is the neighbor  $a''_1$  of  $a'_1$ . But then  $a'_1$  is the center of an odd wheel with hole contained in  $P \cup \{a_1, y_m\} \cup P_{a''_1 b_1} \cup P^2$ .

Therefore,  $y_m$  is of Type t2, t2p or t3p w.r.t.  $\Sigma$ . We show that  $y_m$  does not have a neighbor in  $P$ . Assume not and let  $x_i$  be the neighbor of  $y_m$  in  $P$  with largest index. If

$i = n$  and  $x_n$  is adjacent to  $b_2$ , then  $P^2 \cup P^3 \cup \{x_n, y_m\}$  induces an odd wheel with center  $b_2$ . Otherwise,  $P^2 \cup P_{x_i x_n} \cup Q \cup a_4$  induces a  $3PC(\Delta, y_m)$ . So  $y_m$  does not have a neighbor in  $P$ . If  $y_m$  is of Type t2 and  $a_1$  has no neighbor in  $Q$ ,  $\Sigma \cup Q$  induces connected diamonds and the result holds. If  $y_m$  is of Type t2 and  $a_1$  has at least one neighbor in  $Q$ , there is a  $3PC(b_2 b_3 y_m, a_1)$  contained in  $(P \setminus a_2) \cup P^3 \cup P \cup Q \cup a_1$ . So  $y_m$  is of Type t2p or t3p w.r.t.  $\Sigma$  and hence it has a neighbor in  $P^1 \setminus b_1$ . If  $y_m$  is adjacent to  $a_1$ , then  $(P^2 \setminus a_2) \cup P^3 \cup P \cup \{a_1, y_m\}$  induces a  $3PC(b_2 b_3 y_m, a_1)$ . So  $y_m$  is not adjacent to  $a_1$ . Let  $v$  be the neighbor of  $y_m$  in  $P^1$  that is closest to  $a_1$ . Then  $P_{a_1 v}^1 \cup (P^2 \setminus b_2) \cup P \cup Q \cup a_4$  contains a proper wheel with center  $a_1$  that is not a beetle.  $\square$

## 12.1 2-Joins and Blocking Sequences

In this section, we consider an induced subgraph  $H$  of  $G$  that contains a 2-join  $H_1|H_2$ . We say that a 2-join  $H_1|H_2$  *extends* to  $G$  if there exists a 2-join of  $G$ ,  $H'_1|H'_2$  with  $H_1 \subseteq H'_1$  and  $H_2 \subseteq H'_2$ . We characterize the situation in which the 2-join of  $H$  does not extend to a 2-join of  $G$ .

**Definition 12.3** A blocking sequence for a 2-join  $H_1|H_2$  of a subgraph  $H$  of  $G$  is a sequence of distinct nodes  $x_1, \dots, x_n$  in  $G \setminus H$  with the following properties:

1. *i)*  $H_1|H_2 \cup x_1$  is not a 2-join of  $H \cup x_1$ ,  
*ii)*  $H_1 \cup x_n|H_2$  is not a 2-join of  $H \cup x_n$ , and  
*iii)* if  $n > 1$  then, for  $i = 1, \dots, n-1$ ,  $H_1 \cup x_i|H_2 \cup x_{i+1}$  is not a 2-join of  $H \cup \{x_i, x_{i+1}\}$ .
2.  $x_1, \dots, x_n$  is minimal with respect to Property 1, in the sense that no sequence  $x_{j_1}, \dots, x_{j_k}$  with  $\{x_{j_1}, \dots, x_{j_k}\} \subset \{x_1, \dots, x_n\}$ , satisfies Property 1.

Blocking sequences with respect to a 1-join were introduced and studied by Geelen in [19]. Blocking sequences with respect to a 2-join were introduced in [7], where the following results are obtained.

Let  $H$  be an induced subgraph of  $G$  with 2-join  $H_1|H_2$  and special sets  $A_1, B_1, A_2, B_2$ .

In the following remarks and lemmas, we let  $S = x_1, \dots, x_n$  be a blocking sequence for the 2-join  $H_1|H_2$  of a subgraph  $H$  of  $G$ .

**Remark 12.4**  $H_1|H_2 \cup u$  is a 2-join in  $H \cup u$  if and only if  $N(u) \cap H_1 = \emptyset, A_1$  or  $B_1$ . Similarly  $H_1 \cup u|H_2$  is a 2-join in  $H \cup u$  if and only if  $N(u) \cap H_2 = \emptyset, A_2$  or  $B_2$ .

**Lemma 12.5** If  $n > 1$  then, for every node  $x_j$ ,  $j \in \{1, \dots, n-1\}$ ,  $N(x_j) \cap H_2 = \emptyset, A_2$  or  $B_2$ , and for every node  $x_j$ ,  $j \in \{2, \dots, n\}$ ,  $N(x_j) \cap H_1 = \emptyset, A_1$  or  $B_1$ .

**Lemma 12.6** If  $n > 1$  and  $x_i x_{i+1}$  is not an edge, where  $i \in \{1, \dots, n-1\}$ , then either  $N(x_i) \cap H_2 = A_2$  and  $N(x_{i+1}) \cap H_1 = A_1$ , or  $N(x_i) \cap H_2 = B_2$  and  $N(x_{i+1}) \cap H_1 = B_1$ .

**Theorem 12.7** Let  $H$  be an induced subgraph of graph  $G$  that contains a 2-join  $H_1|H_2$ . The 2-join  $H_1|H_2$  of  $H$  extends to a 2-join of  $G$  if and only if there exists no blocking sequence for  $H_1|H_2$  in  $G$ .

**Lemma 12.8** For  $1 < i < n$ ,  $H_1 \cup \{x_1, \dots, x_{i-1}\} | H_2 \cup \{x_{i+1}, \dots, x_n\}$  is a 2-join in  $H \cup (S \setminus \{x_i\})$ .

**Lemma 12.9** If  $x_i x_k$ ,  $n \geq k > i + 1 \geq 2$ , is an edge then either  $N(x_i) \cap H_2 = A_2$  and  $N(x_k) \cap H_1 = A_1$ , or  $N(x_i) \cap H_2 = B_2$  and  $N(x_k) \cap H_1 = B_1$ .

**Lemma 12.10** If  $x_j$  is the node of lowest index adjacent to a node in  $H_2$ , then  $x_1, \dots, x_j$  is a chordless path. Similarly, if  $x_j$  is the node of highest index adjacent to a node in  $H_1$ , then  $x_j, \dots, x_n$  is a chordless path.

**Theorem 12.11** Let  $G$  be a graph and  $H$  an induced subgraph of  $G$  with 2-join  $H_1 | H_2$  and special sets  $A_1, B_1, A_2, B_2$ . Let  $H'$  be an induced subgraph of  $G$  with 2-join  $H'_1 | H_2$  and special sets  $A'_1, B'_1, A_2, B_2$  such that  $A'_1 \cap A_1 \neq \emptyset$  and  $B'_1 \cap B_1 \neq \emptyset$ . If  $S$  is a blocking sequence for  $H_1 | H_2$  and  $H'_1 \cap S \neq \emptyset$ , then a proper subset of  $S$  is a blocking sequence for  $H'_1 | H_2$ .

## 12.2 Decomposable $3PC(\Delta, \Delta)$ and 2-Joins

**Lemma 12.12** Let  $\Sigma$  be a decomposable  $3PC(a_1 a_2 a_3, b_1 b_2 b_3)$  and  $d$  a Type t2 or t2p node adjacent to  $a_2$  and  $a_3$ , or to  $b_2$  and  $b_3$ . Let  $H_1 | H_2$  be the 2-join of  $H = \Sigma, d$  and let  $A_1, B_1, A_2, B_2$  be its special sets. If  $u$  is of Type p3, t2 or t2p w.r.t.  $\Sigma$ , of Type t3p w.r.t.  $\Sigma$  being a sibling of  $a_2, a_3, b_2$  or  $b_3$ , or of Type t1 w.r.t.  $\Sigma$  adjacent to a node in  $\{a_2, a_3, b_2, b_3\}$  and being attached to  $\Sigma$ , then there exists a decomposable  $\Sigma' = 3PC(\Delta, \Delta)$  and a Type t2 or t2p node  $d'$  such that  $H' = \Sigma', d'$  satisfies the following properties:  $H_i \subseteq H'$ , for some  $i \in \{1, 2\}$ ,  $u \in H'_j = H' \setminus H_i$  where  $j = 3 - i$ , and  $H_i | H'_j$  is a 2-join of  $H'$  with special sets  $A_i, B_i, A'_j$  and  $B'_j$ , where  $A_j \cap A'_j \neq \emptyset$  and  $B_j \cap B'_j \neq \emptyset$ .

*Proof:* We consider the following cases.

**Case 1:**  $u$  is of Type p3 w.r.t.  $\Sigma$ .

Let  $\Sigma'$  be obtained by substituting  $u$  into  $\Sigma$ . By Lemma 6.1,  $d$  is of Type t2 or t2p w.r.t.  $\Sigma'$ .

Suppose  $u$  has a neighbor in  $P^1$ . Let  $P'$  be the  $a_1 b_1$ -path of  $\Sigma'$ . Suppose  $P = x_1, \dots, x_n$  is a  $P'$ -crosspath w.r.t.  $\Sigma'$ . W.l.o.g.  $x_1$  has a neighbor in  $P'$  and  $x_n$  in  $P^3$ . Then  $x_1$  has a neighbor in  $P^1$ . Let  $x_i$  be the node of  $P$  with highest index that has a neighbor in  $P^1$ . By Theorem 8.3,  $P_{x_i x_n}$  is a  $P^1$ -crosspath w.r.t.  $\Sigma$ , a contradiction. Therefore, there is no  $P'$ -crosspath w.r.t.  $\Sigma'$ .

If  $u$  has a neighbor in  $P^2 \cup P^3$ , then by analogous argument, there is no  $P^1$ -crosspath w.r.t.  $\Sigma'$ . Hence,  $\Sigma'$  is the desired decomposable  $3PC(\Delta, \Delta)$ .

**Case 2:**  $u$  is of Type t2 or t2p w.r.t.  $\Sigma$ , or of Type t3p w.r.t.  $\Sigma$  being a sibling of  $a_2, a_3, b_2$  or  $b_3$ .

If  $u$  is of Type t2 or t2p w.r.t.  $\Sigma$  adjacent to  $a_2$  and  $a_3$  or to  $b_2$  and  $b_3$ , then  $(H \setminus d) \cup u$  is the desired decomposable  $3PC(\Delta, \Delta)$ . So by symmetry, it is enough to consider the case when  $u$  is adjacent to  $a_1$  and  $a_2$ , and it is of Type t2, t2p or t3p w.r.t.  $\Sigma$ . If  $u$  is of Type t2p or t3p w.r.t.  $\Sigma$ , let  $\Sigma'$  be obtained from  $\Sigma$  by substituting  $u$  for its sibling. If  $u$  is of Type t2 w.r.t.  $\Sigma$ , then by Lemma 10.5, there is an attachment  $Q = y_1, \dots, y_m$  of  $u$  to  $\Sigma$ . In this

case, let  $\Sigma'$  be obtained by substituting  $u$  and  $Q$  into  $\Sigma$ . Note that  $P^1 \cup P^2 \subseteq \Sigma'$ . Let  $P_u^3$  be the path of  $\Sigma' \setminus (P^1 \cup P^2)$ .

We first show that there is no  $P^1$ -crosspath w.r.t.  $\Sigma'$ . Suppose not and let  $P = x_1, \dots, x_n$  be a  $P^1$ -crosspath w.r.t.  $\Sigma'$ . W.l.o.g.  $x_1$  has a neighbor in  $P^1$ . If a node of  $P^3$  has a neighbor in  $P \setminus x_n$ , then by Lemma 6.1 and Theorem 8.3, a subpath of  $P$  is a  $P^1$ -crosspath w.r.t.  $\Sigma$ , a contradiction. So no node of  $P^3$  is adjacent to or coincident with a node of  $P \setminus x_n$ . Suppose that  $x_n$  has a neighbor in  $P^2$ . Then by Lemma 6.1,  $x_n$  is of Type p2 or p4 w.r.t.  $\Sigma$ . Since  $P$  cannot be a  $P^1$ -crosspath w.r.t.  $\Sigma$ ,  $n > 1$ ,  $x_n$  is of Type p4 w.r.t.  $\Sigma$ , and  $N(x_n) \cap \Sigma \subseteq P^2 \cup P^3$ . But then  $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup P$  contains a  $3PC(b_1 b_2 b_3, x_n)$ . So  $x_n$  does not have a neighbor in  $P^2$ , and hence it has a neighbor in  $P_u^3$ . If  $x_n$  has a neighbor in  $P^3$ , then by Lemma 6.1 and Theorem 8.3,  $P$  is a  $P^1$ -crosspath w.r.t.  $\Sigma$ . So  $x_n$  does not have a neighbor in  $P^3$ , and hence the neighbors of  $x_n$  in  $P_u^3$  are contained in  $P_u^3 \setminus P^3$ . Since  $x_n$  is of Type p2 or p4 w.r.t.  $\Sigma'$ ,  $x_n$  has a neighbor in  $Q$ . Let  $y_i$  be such a neighbor with highest index. Node  $a_3$  does not have a neighbor in  $Q_{y_i y_{m-1}}$ , since otherwise  $P, Q_{y_i y_j}$  (where  $y_j$  is the neighbor of  $a_3$  in  $Q_{y_i y_{m-1}}$  with lowest index) contradicts Theorem 8.3 applied to  $\Sigma$ . If  $y_m$  is of Type t1, p1, p2 or p3 w.r.t.  $\Sigma$ , then by Theorem 8.3,  $P, Q_{y_i y_m}$  is a  $P^1$ -crosspath w.r.t.  $\Sigma$ . So  $y_m$  is of Type t2, t2p or t3p w.r.t.  $\Sigma$ , adjacent to  $b_1, b_2$  and no node of  $(P^1 \cup P^2) \setminus \{b_1, b_2\}$ . If  $y_m$  is of Type t2 w.r.t.  $\Sigma$ , then  $P, Q_{y_i y_m}$  contradicts Lemma 8.5 applied to  $\Sigma$ . So  $y_m$  is of Type t2p or t3p w.r.t.  $\Sigma$ . Let  $\Sigma''$  be obtained by substituting  $y_m$  into  $\Sigma$ . But then  $P, Q_{y_i y_{m-1}}$  contradicts Theorem 8.3 applied to  $\Sigma''$ . Therefore, there is no  $P^1$ -crosspath w.r.t.  $\Sigma'$ .

If  $d$  is adjacent to  $a_2, a_3$  then, by Lemma 10.6(ii),  $d$  is adjacent to  $u$ . Therefore, by Lemma 6.1,  $d$  is of Type t2 or t2p w.r.t.  $\Sigma'$ , and hence  $\Sigma'$  is the desired decomposable  $3PC(\Delta, \Delta)$ . Now consider the case where  $d$  is adjacent to  $b_2, b_3$ . If  $y_m$  is of Type t1, p1, p2 or p3,  $\Sigma'$  is the desired decomposable  $3PC(\Delta, \Delta)$ . If  $y_m$  is of Type t2, t2p or t3p then  $y_m$  is adjacent to  $d$  by Lemma 10.6(ii), and therefore  $d$  is of Type t2 or t2p w.r.t.  $\Sigma'$ . Hence  $\Sigma'$  is the desired decomposable  $3PC(\Delta, \Delta)$ .

**Case 3:**  $u$  is of Type t1 w.r.t.  $\Sigma$  adjacent to a node in  $\{a_2, a_3, b_2, b_3\}$  and it is attached to  $\Sigma$ .

W.l.o.g.  $u$  is adjacent to  $a_3$ . Let  $Q = y_1, \dots, y_m$  be an attachment of  $u$  to  $\Sigma$ , and let  $\Sigma'$  be obtained by substituting  $u$  and  $Q$  into  $\Sigma$ . Note that  $P^1 \cup P^2 \subseteq \Sigma'$ . Let  $P_u^3$  be the path of  $\Sigma' \setminus (P^1 \cup P^2)$ .

We first show that there is no  $P^1$ -crosspath w.r.t.  $\Sigma'$ . Suppose not and let  $P = x_1, \dots, x_n$  be a  $P^1$ -crosspath w.r.t.  $\Sigma'$ . W.l.o.g.  $x_1$  has a neighbor in  $P^1$ . By the same argument as in Case 2, no node of  $P^3$  is adjacent to or coincident with a node of  $P \setminus x_n$  and  $x_n$  does not have a neighbor in  $P^2 \cup P^3$ . So  $x_n$  has a neighbor in  $P_u^3$ , and its neighbors in  $P_u^3$  are contained in  $u, Q$ . Let  $y_i$  be the neighbor of  $x_n$  in  $Q$  with highest index. If  $a'_3$  has a neighbor in  $Q_{y_i y_{m-1}}$  then  $Q_{y_i y_j}, P$  (where  $y_j$  is its neighbor in  $Q_{y_i y_{m-1}}$  with lowest index) contradicts Theorem 8.3 applied to  $\Sigma$ . So  $a'_3$  does not have a neighbor in  $Q_{y_i y_{m-1}}$ . If  $y_m$  is of Type t1, p1, p2 or p3 w.r.t.  $\Sigma$ , then by Theorem 8.3 applied to  $\Sigma$ ,  $P, Q_{y_i y_m}$  is a  $P^1$ -crosspath w.r.t.  $\Sigma$ . If  $y_m$  is of Type t2 w.r.t.  $\Sigma$ , then  $P, Q_{y_i y_m}$  contradicts Lemma 8.5 applied to  $\Sigma$ . So  $y_m$  is of Type t2p or t3p w.r.t.  $\Sigma$ . Let  $\Sigma''$  be obtained by substituting  $y_m$  into  $\Sigma$ . Then  $P, Q_{y_i y_{m-1}}$  contradicts Lemma 8.4 applied to  $\Sigma''$ . Therefore, there is no  $P^1$ -crosspath w.r.t.  $\Sigma'$ .

Assume  $d$  is adjacent to  $a_2, a_3$ . By Lemma 6.1,  $d$  is of Type t2 or t2p w.r.t.  $\Sigma'$  and hence  $\Sigma'$  is the desired decomposable  $3PC(\Delta, \Delta)$ . Assume  $d$  is adjacent to  $b_2, b_3$ . If  $y_m$  is of Type t1, p1, p2 or p3, then  $d$  is of Type t2 or t2p w.r.t.  $\Sigma'$ , and hence  $\Sigma'$  is the desired

decomposable  $3PC(\Delta, \Delta)$ . If  $y_m$  is of Type t2, t2p or t3p, then, by Lemma 10.6(ii),  $y_m$  is adjacent to  $d$  and therefore  $d$  is of Type t2 or t2p w.r.t.  $\Sigma'$  and hence  $\Sigma'$  is the desired decomposable  $3PC(\Delta, \Delta)$ .  $\square$

**Theorem 12.13** *Let  $G$  be a square-free even-signable graph. If  $G$  contains a  $3PC(\Delta, \Delta)$  with a Type t2 or t2p node, then  $G$  has a star cutset or a 2-join.*

*Proof:* Assume  $G$  has no star cutset. By Theorems 4.6 and 5.3,  $G$  contains neither a proper wheel that is not a beetle nor an L-parachute. By Theorem 9.1, there is no node of Type t4, t5 or t6b w.r.t. a  $3PC(\Delta, \Delta)$ .

Assume  $G$  contains a  $3PC(\Delta, \Delta)$  with a Type t2 or t2p node. Then by Theorem 12.2,  $G$  contains a decomposable  $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$  together with a node  $d$  of Type t2 or t2p adjacent to  $a_2, a_3$  or to  $b_2, b_3$ . Suppose that the 2-join  $H_1|H_2$  of  $H = \Sigma, d$  does not extend to a 2-join of  $G$ . By Theorem 12.7, there is a blocking sequence  $S = x_1, \dots, x_n$ . W.l.o.g. assume that  $H$  and  $S$  are chosen so that the size of  $S$  is minimized. Since  $\Sigma$  has no  $P^1$ -crosspath, no node is of Type t6a w.r.t.  $\Sigma$  by Theorem 11.1. By Lemma 12.12 and Theorem 12.11, no node of  $S$  is of Type p3, t2 or t2p w.r.t.  $\Sigma$ , of Type t3p w.r.t.  $\Sigma$  being a sibling of  $a_2, a_3, b_2$  or  $b_3$ , or of Type t1 w.r.t.  $\Sigma$  adjacent to a node in  $\{a_2, a_3, b_2, b_3\}$  and being attached to  $\Sigma$ .

**Claim 1:** *If  $x_i$  is of Type p4 w.r.t.  $\Sigma$ , then  $N(x_i) \cap H \subseteq P^2 \cup P^3$ . If  $x_i$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$  and  $N(x_i) \cap \Sigma \subseteq P^2 \cup P^3$ , then  $N(x_i) \cap H \subseteq P^2 \cup P^3$ .*

*Proof of Claim 1:* W.l.o.g. assume that  $d$  is adjacent to  $a_2, a_3$ . Suppose  $x_i$  is of Type p4 w.r.t.  $\Sigma$ . Since there is no  $P^1$ -crosspath w.r.t.  $\Sigma$ ,  $N(x_i) \cap \Sigma \subseteq P^2 \cup P^3$ . Suppose  $x_i$  is adjacent to  $d$ . If  $d$  is of Type t2 w.r.t.  $\Sigma$ , then  $d, x_i$  contradicts Lemma 8.5. So  $d$  is of Type t2p w.r.t.  $\Sigma$ , and hence  $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup \{d, x_i\}$  contains a  $3PC(b_1b_2b_3, x_i)$ . So  $x_i$  is not adjacent to  $d$ .

Now suppose that  $x_i$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ , with neighbors in  $\Sigma$  contained in say  $P^3$ . It is enough to show that  $x_i$  is not adjacent to  $d$ . Suppose  $x_i$  is adjacent to  $d$ . Suppose  $x_i$  has a neighbor in  $P^3 \setminus a_3$ . If  $d$  is of Type t2 w.r.t.  $\Sigma$ , then  $d, x_i$  contradicts Lemma 8.5. If  $d$  is of Type t2p w.r.t.  $\Sigma$ , then  $(\Sigma \setminus \{a_1, a_3\}) \cup \{d, x_i\}$  contains a  $3PC(b_1b_2b_3, d)$ . So  $x_i$  is of Type t1 w.r.t.  $\Sigma$  adjacent to  $a_3$ . We have seen above that  $x_i$  cannot be attached to  $\Sigma$ . So, by Lemma 10.4, there is a bad connection  $Q = y_1, \dots, y_m$  of  $x_i$  to  $\Sigma$ , where  $y_m$  is of Type t2 or t2p w.r.t.  $\Sigma$ , adjacent to  $a_1$  and  $a_2$ . If  $d$  has a neighbor in  $Q$ , let  $y_j$  be such a neighbor with highest index. Then  $Q_{y_j y_m} \cup \{a_1, a_2, a_3, d\}$  either contains a square or induces an odd wheel with center  $a_2$ . So  $d$  does not have a neighbor in  $Q$ . If  $d$  is of Type t2p w.r.t.  $\Sigma$ , then  $Q \cup P^1 \cup \{x_i, a_1, d\}$  contains a  $3PC(x_i a_3 d, a_1)$ . So  $d$  is of Type t2 w.r.t.  $\Sigma$ . By Lemma 10.5 there is an attachment  $P = u_1, \dots, u_l$  of  $d$  to  $\Sigma$ . Let  $\Sigma' = 3PC(\Delta, \Delta)$  obtained by substituting  $d$  and its attachment  $P$  into  $\Sigma$ . By Lemma 6.1 applied to  $\Sigma'$ ,  $x_i$  does not have a neighbor in  $P$ . If  $(P \setminus u_1) \cup (Q \setminus y_m) \cup P^1 \cup \{x_i, b_3\}$  contains a path from  $x_i$  to  $a_1$ , then a shortest such path together with  $a_2, a_3$  and  $d$  induces a proper wheel with center  $a_3$  that is not a beetle. Therefore no such path exists and hence no node of  $Q \setminus y_m$  is adjacent to or coincident with a node of  $P \setminus u_1$ . By Lemma 6.1 applied to  $\Sigma'$ ,  $y_m$  does not have a neighbor in  $P$ . Suppose  $u_1$  has a neighbor in  $Q$  and let  $y_j$  be such a neighbor with lowest index. If  $a'_3$  has a neighbor in  $Q_{y_1 y_j}$  then a subpath of  $Q_{y_1 y_j}$  contradicts Lemma 6.1 or Theorem 8.3 applied to  $\Sigma'$ . So  $a'_3$  does not have a neighbor in  $Q_{y_1 y_j}$  and hence  $x_i, Q_{y_1 y_j}$

contradicts Lemma 8.5 applied to  $\Sigma'$ . Therefore  $u_1$  does not have a neighbor in  $Q$  and so no node of  $Q$  is adjacent to or coincident with a node of  $P$ . Let  $R$  be a shortest path from  $d$  to  $a_1$  in  $P \cup P^1 \cup \{d, b_3\}$ . Then  $R \cup Q \cup \{x_i, a_3\}$  induces a proper wheel with center  $a_3$  that is not a beetle. This completes the proof of Claim 1.

By Claim 1,  $n > 1$ . Since  $H_1|H_2 \cup x_1$  is not a 2-join of  $H \cup x_1$ ,  $x_1$  has a neighbor in  $P^1 \cup \{d\}$  and either (i)  $N(x_1) \cap H \subseteq P^1 \cup \{d\}$ , or (ii)  $x_1$  is of Type t3p w.r.t.  $\Sigma$  being a sibling of  $a_1$  or  $b_1$ , or (iii)  $x_1$  is of Type t3 w.r.t.  $\Sigma$  adjacent to, say,  $a_1, a_2$  and  $a_3$ ,  $x_1$  is not adjacent to  $d$ , and  $d$  is adjacent to  $a_2, a_3$ . Note that the case where  $x_1$  is of Type t3 adjacent to  $a_1, a_2, a_3$  and  $d$  where  $d$  is adjacent to  $b_2, b_3$  cannot occur since, in this case, there is a  $3PC(x_1 a_1 a_3, b_3)$ . Since  $H_1 \cup x_n | H_2$  is not a 2-join of  $H \cup x_n$ ,  $x_n$  has a neighbor in  $P^2 \cup P^3$ , and it is of Type t1, p1, p2 or p4 w.r.t.  $\Sigma$ . By Lemma 12.5, for  $i \in \{2, \dots, n-1\}$ ,  $x_i$  either has no neighbor in  $H$  or  $N(x_i) \cap \Sigma = \{a_1, a_2, a_3\}$  or  $\{b_1, b_2, b_3\}$  and, furthermore, if say  $N(x_i) \cap \Sigma = \{a_1, a_2, a_3\}$  then  $x_i$  is adjacent to  $d$  if  $d$  is adjacent to  $a_2, a_3$ , and  $x_i$  is not adjacent to  $d$  if  $d$  is adjacent to  $b_2, b_3$ . Let  $x_j$  be the node of  $S$  with highest index adjacent to a node of  $H_1$ . By Lemma 12.10,  $x_j, \dots, x_n$  is a chordless path. Note that nodes  $x_{j+1}, \dots, x_{n-1}$  have no neighbors in  $H$ .

**Claim 2:** *Let  $\Sigma$  be a  $3PC(a_1 a_2 a_3, b_1 b_2 b_3)$  with no  $P^1$ -crosspath. Suppose that  $x_j$  is of Type t3 w.r.t.  $\Sigma$ , say adjacent to  $b_1, b_2$  and  $b_3$ , and there is a  $\Sigma' = 3PC(a_1 a_2 t, b_1 b_2 x_j)$  that contains  $P^1 \cup P^2$  and such that  $t$  is not of Type t3 w.r.t.  $\Sigma$ . Then there is no  $P^1$ -crosspath w.r.t.  $\Sigma'$ .*

*Proof of Claim 2:* Let  $P'$  be the path of  $\Sigma' \setminus (P^1 \cup P^2)$ . Suppose  $P = y_1, \dots, y_m$  is a  $P^1$ -crosspath w.r.t.  $\Sigma'$ . W.l.o.g.  $y_1$  has a neighbor in  $P^1$ . Suppose  $y_m$  has a neighbor in  $P^2$ . Since  $P$  cannot be a  $P^1$ -crosspath w.r.t.  $\Sigma$ , a node of  $P$  has a neighbor in  $P^3$ . Let  $y_i$  be such a node with lowest index. If  $i \neq m$  then by Theorem 8.3,  $P_{y_1 y_i}$  is a  $P^1$ -crosspath w.r.t.  $\Sigma$ . So  $i = m$  and hence  $y_m$  is of Type p4 w.r.t.  $\Sigma$ . But then  $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup P$  contains a  $3PC(b_1 b_2 b_3, y_m)$ . So  $y_m$  has a neighbor in  $P'$ . Suppose that  $P \cup P^3 \cup P' \setminus \{x_j, t\}$  contains a path from  $y_1$  to  $P^3$ . Then by Theorem 8.3 applied to the shortest such path, there is a  $P^1$ -crosspath w.r.t.  $\Sigma$ . So no such path exists and hence  $t \neq a_3$  and no node of  $P^3$  is adjacent to or coincident with a node of  $P \cup P' \setminus \{x_j, t\}$ . So  $t$  is of Type t2, t2p or t3p w.r.t.  $\Sigma$ . Note that  $P \cup P' \setminus x_j$  contains a chordless path  $T$  from  $y_1$  to  $t$ . If  $t$  is of Type t2 w.r.t.  $\Sigma$ , then  $T$  contradicts Lemma 8.5 applied to  $\Sigma$ . So  $t$  is of Type t2p or t3p w.r.t.  $\Sigma$ . Let  $\Sigma''$  be obtained by substituting  $t$  into  $\Sigma$ . Then  $T \setminus t$  contradicts Theorem 8.3 applied to  $\Sigma''$ . This completes the proof of Claim 2.

We now consider the following cases.

**Case 1:**  $x_j$  is of Type t3 w.r.t.  $\Sigma$ .

If  $x_n$  is of Type p1 or p4 w.r.t.  $\Sigma$ , then  $x_j, \dots, x_n$  contradicts Lemma 8.6.

**Case 1.1:**  $x_n$  is of Type p2 w.r.t.  $\Sigma$ .

W.l.o.g.  $x_n$  has a neighbor in  $P^3$  and  $d$  is adjacent to  $a_2, a_3$ . Suppose  $x_j$  is adjacent to  $b_1, b_2$  and  $b_3$ . Then there is a  $\Sigma' = 3PC(a_1 a_2 a_3, b_1 b_2 x_j)$  contained in  $(\Sigma \setminus b_3) \cup \{x_j, \dots, x_n\}$ . By Claim 2, there is no  $P^1$ -crosspath w.r.t.  $\Sigma'$ . By Lemma 6.1,  $d$  is of Type t2 or t2p w.r.t.  $\Sigma'$  and hence  $\Sigma', d$  is a decomposable  $3PC(\Delta, \Delta)$ . But then, by Theorem 12.11, the minimality of  $S$  is contradicted. So  $x_j$  is adjacent to  $a_1, a_2$  and  $a_3$ . Let  $\Sigma' = 3PC(a_1 a_2 x_j, b_1 b_2 b_3)$  be

contained in  $(\Sigma \setminus a_3) \cup \{x_j, \dots, x_n\}$ . By Claim 2, there is no  $P^1$ -crosspath w.r.t.  $\Sigma'$ . If  $d$  is adjacent to  $x_j$ , then by Lemma 6.1 applied to  $\Sigma'$ ,  $x_j$  is of Type t2 or t2p w.r.t.  $\Sigma'$ , and hence  $\Sigma', d$  is a decomposable  $3PC(\Delta, \Delta)$  and the minimality of  $S$  is contradicted.

So  $d$  is not adjacent to  $x_j$ , and hence by Lemma 6.1,  $d$  is of Type t1 w.r.t.  $\Sigma'$  and of Type t2 w.r.t.  $\Sigma$ . By Lemma 10.5, let  $Q = y_1, \dots, y_m$  be an attachment of  $d$  to  $\Sigma$ .

First we show that no node of  $Q$  is adjacent to or coincident with a node of  $\{x_j, \dots, x_n\}$ . Suppose not and let  $y_k$  be the node of  $Q$  with highest index that has a neighbor in  $\{x_j, \dots, x_n\}$ . Let  $x_i$  be the neighbor of  $y_k$  in  $\{x_j, \dots, x_n\}$  with highest index.

Suppose  $i \neq j$ . If  $a_1$  has a neighbor in  $Q_{y_k y_{m-1}}$  then let  $y_l$  be such a neighbor with lowest index. Then  $Q_{y_k y_l, x_i, \dots, x_n}$  contradicts Lemma 8.4 applied to  $\Sigma$ . So  $a_1$  has no neighbor in  $Q_{y_k y_{m-1}}$ . If  $y_m$  is of Type t1, p1, p2 or p3 w.r.t.  $\Sigma$ , then by Theorem 8.3 applied to  $\Sigma$ ,  $Q_{y_k y_m, x_i, \dots, x_n}$  is a  $P^1$ -crosspath w.r.t.  $\Sigma$ . If  $y_m$  is of Type t2 w.r.t.  $\Sigma$ , then  $Q_{y_k y_m, x_i, \dots, x_n}$  contradicts Lemma 8.5 applied to  $\Sigma$ . So  $y_m$  is of Type t2p or t3p w.r.t.  $\Sigma$ . Let  $\Sigma''$  be obtained by substituting  $y_m$  into  $\Sigma$ . Then either  $Q_{y_{k+1} y_{m-1}, x_i, \dots, x_n}$  (if  $k \neq m$ ) or  $x_i, \dots, x_n$  (otherwise) contradicts Lemma 8.4 applied to  $\Sigma''$ . Therefore,  $i = j$ .

If  $x_j$  is adjacent to  $y_m$ , then  $y_m$  and  $\Sigma'$  contradict Lemma 6.1 (since by our assumption  $y_m$  cannot be of Type t4 or t5 w.r.t.  $\Sigma'$ ). So  $x_j$  is not adjacent to  $y_m$ , i.e.  $k < m$ . If  $a_1$  does not have a neighbor in  $Q_{y_k y_{m-1}}$ , then  $y_m$  is not of Type p2 w.r.t.  $\Sigma$  and hence  $x_j, Q_{y_k y_m}$  contradicts Lemma 8.6 applied to  $\Sigma$ . So  $a_1$  has a neighbor in  $Q_{y_k y_{m-1}}$ . If  $y_m$  is of Type t2, t2p or t3p w.r.t.  $\Sigma$ , then let  $H'$  be the hole induced by  $P^2 \cup Q_{y_k y_m} \cup x_j$ , and  $H''$  the hole contained in  $(P^3 \setminus a_3) \cup Q_{y_k y_m} \cup \{x_j, \dots, x_n\}$ . If  $y_m$  is of Type t1, p1, p2 or p3 w.r.t.  $\Sigma$ , then let  $v$  be the neighbor of  $y_m$  in  $P^1$  that is closest to  $b_1$ , let  $H'$  be the hole induced by  $P_{vb_1}^1 \cup P^2 \cup Q_{y_k y_m} \cup x_j$  and let  $H''$  be the hole contained in  $P_{vb_1}^1 \cup (P^3 \setminus a_3) \cup Q_{y_k y_m} \cup \{x_j, \dots, x_n\}$ . Since neither  $(H', a_1)$  nor  $(H'', a_1)$  can be an odd wheel,  $y_k$  is the unique neighbor of  $a_1$  in  $Q_{y_k y_m}$ . So  $y_k$  is of Type t2 w.r.t.  $\Sigma'$  and hence  $Q_{y_k y_m}$  contradicts Lemma 8.5 applied to  $\Sigma$ .

Therefore, no node of  $Q$  is adjacent to or coincident with a node of  $\{x_j, \dots, x_n\}$ . Let  $\Sigma'' = 3PC(a_2 a_3 d, \Delta)$  be obtained by substituting  $d$  and  $Q$  into  $\Sigma$ . Then  $x_j, \dots, x_n$  contradicts Lemma 8.5 applied to  $\Sigma''$ .

**Case 1.2:**  $x_n$  is of Type t1 w.r.t.  $\Sigma$ .

Assume w.l.o.g. that  $x_j$  is adjacent to  $b_1, b_2$  and  $b_3$ . If  $x_n$  is adjacent to  $a_3$ , then  $x_j, \dots, x_n$  contradicts Lemma 8.6. So  $x_n$  is adjacent to  $b_3$ .

**Claim 3:** *If there exists a  $\Sigma' = 3PC(a_1 a_2 t, b_1 b_2 x_j)$  that contains  $P^1 \cup P^2$ , then  $t$  is of Type t3 w.r.t.  $\Sigma$ .*

*Proof of Claim 3:* Suppose that there exists a  $\Sigma' = 3PC(a_1 a_2 t, b_1 b_2 x_j)$  that contains  $P^1 \cup P^2$  and such that  $t$  is not of Type t3 w.r.t.  $\Sigma$ . Then by Claim 2, there is no  $P^1$ -crosspath w.r.t.  $\Sigma'$ . Suppose  $d$  is of Type t2 or t2p w.r.t.  $\Sigma'$ . Then  $\Sigma', d$  is a decomposable  $3PC(\Delta, \Delta)$ . Now, by Theorem 12.11 applied to  $H = \Sigma, d$  and  $H' = \Sigma', d$ , the minimality of  $S$  is contradicted. Therefore  $d$  cannot be of Type t2 or t2p w.r.t.  $\Sigma'$ .

Assume first that  $d$  is adjacent to  $a_2$  and  $a_3$ . If  $t = a_3$  then  $d$  is of Type t2 or t2p w.r.t.  $\Sigma'$  by Lemma 6.1 applied to  $\Sigma'$ , a contradiction. So we may assume that  $t \neq a_3$ . Since  $t$  is adjacent to  $a_1, a_2$  and it is not of Type t3 w.r.t.  $\Sigma$ , it is of Type t2, t2p or t3p w.r.t.  $\Sigma$ . By Lemma 10.6(ii),  $d$  is adjacent to  $t$ , and hence by Lemma 6.1,  $d$  is of Type t2 or t2p w.r.t.  $\Sigma'$ , a contradiction.

Assume now that  $d$  is adjacent to  $b_2$  and  $b_3$ . If  $d$  is adjacent to  $x_j$ , then by Lemma 6.1,  $d$  is of Type t2 or t2p w.r.t.  $\Sigma'$ , a contradiction. So  $d$  is not adjacent to  $x_j$ , and hence it is of Type t1 w.r.t.  $\Sigma'$ . So  $d$  is of Type t2 w.r.t.  $\Sigma$ . By Lemma 10.5, let  $Q = y_1, \dots, y_m$  be an attachment of  $d$  to  $\Sigma$ . Let  $\Sigma^d$  be obtained by substituting  $d$  and  $Q$  into  $\Sigma$ . Let  $P'$  be the  $tx_j$ -path of  $\Sigma'$ .

We now show that  $x_j$  has a neighbor in  $Q$ . Suppose it does not. Suppose that a node of  $Q \setminus y_m$  is adjacent to a node of  $P'$ . Let  $y_i$  be such a node with lowest index. Let  $p$  be a neighbor of  $y_i$  in  $P'$  that is closest to  $x_j$ . If  $b_1$  does not have a neighbor in  $Q_{y_1 y_i}$ , then  $d, Q_{y_1 y_i}$  contradicts Lemma 8.4 applied to  $\Sigma'$ . So  $b_1$  has a neighbor in  $Q_{y_1 y_i}$ . Let  $y_b$  be such a neighbor with lowest index. If  $y_b = y_i$  then  $y_b$  contradicts Lemma 6.1 applied to  $\Sigma'$ . So  $y_b \neq y_i$  and hence  $d, Q_{y_1 y_b}$  contradicts Theorem 8.3 applied to  $\Sigma'$ . So no node of  $Q \setminus y_m$  is adjacent to or coincident with a node of  $P'$ . By the same argument as above,  $b_1$  does not have a neighbor in  $Q \setminus y_m$ . Then by Lemma 8.4 applied to  $d, Q$  and  $\Sigma'$ ,  $N(y_m) \cap (\Sigma \cup \Sigma') = \{a_2, a_3\}$  and  $t \neq a_3$ . If  $a_3$  has no neighbor in  $P'$ , then  $Q \cup P' \cup \{a_1, a_2, a_3, b_2, d\}$  induces an odd wheel with center  $a_2$ . Otherwise, let  $u$  be the neighbor of  $a_3$  in  $P'$  that is closest to  $x_j$ . Note that since  $t$  is not of Type t3 w.r.t.  $\Sigma$ ,  $u \neq t$ . But then  $Q \cup P^2 \cup P'_{ux_j} \cup \{a_3, d\}$  induces a  $3PC(a_2 a_3 y_m, b_2)$ . Therefore,  $x_j$  has a neighbor in  $Q$ .

Let  $y_i$  be the neighbor of  $x_j$  in  $Q$  with highest index. We now show that  $b_1$  is adjacent to  $y_i$ , it is not adjacent to  $y_{i+1}$  and it has at most two neighbors in  $Q_{y_i y_m}$ , and  $y_m$  is not of Type p2 w.r.t.  $\Sigma$ . Suppose that  $b_1$  has no neighbor in  $Q_{y_i y_m}$ . Since  $y_m$  is not adjacent to  $b_1, x_j, Q_{y_i y_m}$  and  $\Sigma$  violate Lemma 8.6. So  $b_1$  has a neighbor in  $Q_{y_i y_m}$ . Let  $R$  be a shortest path from  $y_m$  to  $b_2$  in  $(\Sigma \cup y_m) \setminus \{b_1, b_3\}$ . Let  $H$  be the hole induced by  $R \cup Q_{y_i y_m} \cup x_j$ . If  $y_m$  is of Type p2 w.r.t.  $\Sigma$ , then  $(H, b_1)$  is an odd wheel. So  $y_m$  is not of Type p2 w.r.t.  $\Sigma$ . If  $(H, b_1)$  is a line wheel, then  $H \cup b_1 \cup P^1$  contains an L-parachute, contradicting Theorem 5.3. So  $(H, b_1)$  is a twin wheel or a beetle, and hence the result holds.

Next we show that no node of  $Q_{y_i y_m}$  is adjacent to or coincident with a node of  $P' \setminus x_j$ . Assume not and let  $y_k$  be a node of  $Q_{y_i y_m}$  with lowest index adjacent to a node of  $P' \setminus x_j$ . If  $k \neq m$  then either  $Q_{y_i y_k}$  contradicts Lemma 8.5 applied to  $\Sigma'$  (if  $y_i$  is the unique neighbor of  $b_1$  in  $Q_{y_i y_k}$ ) or a subpath of  $Q_{y_{i+1} y_k}$  contradicts Theorem 8.3 applied to  $\Sigma'$  (otherwise). So  $k = m$ . By Lemma 6.1 applied to  $y_m$  and  $\Sigma'$ , either  $t = a_3$  and  $y_m$  is of Type t2, t2p or t3p w.r.t.  $\Sigma'$ , adjacent to  $a_2, a_3$  and possibly a node of  $P^1 \setminus a_1$ , or  $t \neq a_3$  and  $N(y_m) \cap (\Sigma \cup \Sigma') = \{a_1, t\}$ . In the second case,  $t$  and  $\Sigma^d$  violate Lemma 6.1. So  $t = a_3$ . If  $b_1$  has a neighbor in  $Q_{y_{i+1} y_m}$  then  $Q_{y_i y_m} \cup P' \cup b_1$  induces an odd wheel with center  $b_1$ . So  $b_1$  has no neighbor in  $Q_{y_{i+1} y_m}$ . But then  $Q_{y_i y_m}$  contradicts Lemma 8.5 applied to  $\Sigma'$ . Therefore, no node of  $Q_{y_i y_m}$  is adjacent to or coincident with a node of  $P' \setminus x_j$ .

Suppose  $y_m$  is of Type t1, p1 or p3 w.r.t.  $\Sigma$  and let  $y$  be the neighbor of  $y_m$  in  $P^1$  that is closest to  $a_1$ . If  $b_1$  has a neighbor in  $Q_{y_{i+1} y_m}$ , then either  $Q_{y_i y_m} \cup P^1_{a_1 y} \cup P' \cup b_1$  or  $Q_{y_i y_m} \cup P^1_{a_1 y} \cup P^2 \cup \{b_1, x_j\}$  induces an odd wheel with center  $b_1$ . So  $b_1$  has no neighbor in  $Q_{y_{i+1} y_m}$ . But then  $Q_{y_i y_m}$  contradicts Lemma 8.5 applied to  $\Sigma'$ . Hence  $y_m$  is of Type t2, t2p or t3p w.r.t.  $\Sigma$ , and since  $y_m$  is not adjacent to  $t$ ,  $t \neq a_3$ . If  $b_1$  has a neighbor in  $Q_{y_{i+1} y_m}$ , then  $Q_{y_i y_m} \cup P' \cup \{a_2, b_1\}$  induces an odd wheel with center  $b_1$ . So  $b_1$  has no neighbor in  $Q_{y_{i+1} y_m}$ . By Lemma 8.5 applied to  $Q_{y_i y_m}$  and  $\Sigma'$ ,  $y_m$  is of Type t1 w.r.t.  $\Sigma'$  adjacent to  $a_2$ . So  $y_m$  must be of Type t2 w.r.t.  $\Sigma$ , adjacent to  $a_2$  and  $a_3$ . If  $a_3$  has no neighbor in  $P'$  then  $Q_{y_i y_m} \cup P' \cup \{a_1, a_2, a_3\}$  induces an odd wheel with center  $a_2$ . So  $a_3$  has a neighbor in  $P'$ . Let  $u$  be its neighbor in  $P'$  that is closest to  $x_j$ . Since  $t$  is not of Type t3 w.r.t.  $\Sigma$ ,  $u \neq t$ . But

then  $P^1 \cup Q_{y_i y_m} \cup P'_{u x_j} \cup a_3$  induces a  $3PC(y_i b_1 x_j, a_3)$ . This completes the proof of Claim 3.

Let  $T = (N(b_1) \cup b_1) \setminus b'_1$  and let  $Q = y_1, \dots, y_m$  be a direct connection from  $x_n$  to  $\Sigma \setminus T$  in  $G \setminus T$ . Let  $y_0 = x_n$  and  $Q' = y_0, Q$ . Note that  $b_2$  and  $b_3$  are the only nodes of  $\Sigma$  that can have a neighbor in  $Q \setminus y_m$ . We first show that  $b_2$  cannot have a neighbor in  $Q \setminus y_m$ . Suppose otherwise and let  $y_k$  be the node of  $Q \setminus y_m$  with lowest index adjacent to  $b_2$ . If  $y_k$  is also adjacent to  $b_3$ , then  $y_k$  is of Type t2 w.r.t.  $\Sigma$  and therefore, by Lemma 10.5, it is attached. Let  $\Sigma'$  be obtained by substituting  $y_k$  and one of its attachment into  $\Sigma$ . Let  $y_i$  be the node of  $Q'_{y_0 y_{k-1}}$  with highest index adjacent to  $b_3$ . By Lemma 10.6,  $i \neq k-1$ . Let  $y_l$  be the node of  $Q'_{y_i y_{k-1}}$  with lowest index that has a neighbor in  $\Sigma' \setminus b_3$ . By Lemma 6.1,  $y_l$  is of Type t1 w.r.t.  $\Sigma'$ , and hence  $l \neq i$ . But then  $Q'_{y_l y_i}$  contradicts Theorem 8.3 applied to  $\Sigma'$ . So  $y_k$  is not adjacent to  $b_3$ . But then Theorem 8.3 is contradicted in  $\Sigma$  by the path  $x_n, y_1, \dots, y_k$  when  $b_3$  has no neighbor in  $Q_{y_1 y_k}$  or a subpath starting from  $y_k$  otherwise. Therefore  $b_2$  does not have a neighbor in  $Q \setminus y_m$ .

Suppose  $y_m$  is of Type p2 or p4 w.r.t.  $\Sigma$  adjacent to  $b_3$ . If  $y_m$  is of Type p2 w.r.t.  $\Sigma$ , then  $(\Sigma \setminus b_3) \cup \{x_j, x_n\}$  contains a  $3PC(a_1 a_2 a_3, b_1 b_2 x_j)$  contradicting Claim 3. So  $y_m$  is of Type p4 w.r.t.  $\Sigma$ . W.l.o.g.  $y_m$  does not have a neighbor in  $P^2$ . Let  $R$  be the shortest path from  $x_j$  to  $y_m$  in  $Q \cup \{x_j, \dots, x_n\}$ . If  $b_3$  is adjacent to a node of  $R \setminus \{x_j, y_m\}$ , then  $P^2 \cup P^3 \cup R$  induces a proper wheel with center  $b_3$  that is not a beetle. Otherwise,  $R$  contradicts Lemma 8.6 applied to  $\Sigma$ . Therefore,  $y_m$  is not of Type p2 or p4 w.r.t.  $\Sigma$  adjacent to  $b_3$ .

Let  $y_i$  be the node of  $Q' \setminus y_m$  with highest index adjacent to  $b_3$ . By Lemma 8.4 applied to  $Q'_{y_i y_m}$  and  $\Sigma$ , since  $y_m$  is not of Type p2 or p4 w.r.t.  $\Sigma$  adjacent to  $b_3$ ,  $y_m$  is either of Type t1, p1 or p3 w.r.t.  $\Sigma$  with a neighbor in  $P^3 \setminus b_3$ , or of Type t2, t2p or t3p w.r.t.  $\Sigma$  adjacent to  $a_1, a_2$  and no node of  $(P^1 \cup P^2) \setminus \{a_1, a_2\}$ . But then  $(\Sigma \setminus b_3) \cup Q \cup \{x_j, \dots, x_n\}$  contains a  $3PC(a_1 a_2 t, b_1 b_2 x_j)$  that contains  $P^1 \cup P^2$  and such that  $t$  is not of Type t3 w.r.t.  $\Sigma$ , contradicting Claim 3.

**Case 2:**  $j = 1$  and  $x_1$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ .

If  $x_n$  is of Type t1, p1 or p2 w.r.t.  $\Sigma$ , then by Theorem 8.3,  $x_1, \dots, x_n$  is a  $P^1$ -crosspath w.r.t.  $\Sigma$ . If  $x_n$  is of Type p4 w.r.t.  $\Sigma$ , then  $(\Sigma \setminus \{b_2, b_3\}) \cup \{x_1, \dots, x_n\}$  contains a  $3PC(a_1 a_2 a_3, x_n)$ .

**Case 3:**  $j = 1$  and  $d$  is the unique neighbor of  $x_1$  in  $H$ .

W.l.o.g. assume that  $d$  is adjacent to  $a_2, a_3$ . If  $d$  is of Type t2p w.r.t.  $\Sigma$ , let  $\Sigma'$  be obtained by substituting  $d$  into  $\Sigma$ . If  $x_n$  is not of Type t1 adjacent to  $a_2$  or  $a_3$ , then  $x_1, \dots, x_n$  contradicts Lemma 8.4 applied to  $\Sigma'$ . If  $x_n$  is of Type t1 adjacent to  $a_2$  or  $a_3$ , then  $x_1, \dots, x_n$  contradicts Theorem 8.3 applied to  $\Sigma'$ .

So  $d$  is of Type t2 w.r.t.  $\Sigma$ . If  $x_n$  is not of Type t1 w.r.t.  $\Sigma$  adjacent to  $a_2$  or  $a_3$ , then  $d, x_1, \dots, x_n$  contradicts Lemma 8.5. So  $x_n$  is of Type t1 w.r.t.  $\Sigma$ , w.l.o.g. adjacent to  $a_3$ . Let  $\Sigma'$  be a  $3PC(\Delta, \Delta)$  obtained by substituting  $d$  and an attachment of  $d$  to  $\Sigma$  into  $\Sigma$ . Now  $x_1, \dots, x_n$  or a subpath starting from  $x_n$  contradicts Theorem 8.3 applied to  $\Sigma'$ .

**Case 4:**  $j = 1$  and  $x_1$  is of Type t3p w.r.t.  $\Sigma$ .

W.l.o.g.  $x_1$  is a sibling of  $b_1$ . Let  $\Sigma'$  be obtained by substituting  $x_1$  into  $\Sigma$ . Let  $v$  be the neighbor of  $x_1$  in  $P^1$  that is closest to  $a_1$ . If  $x_n$  is not of Type t1 w.r.t.  $\Sigma$  adjacent to  $b_2$  or  $b_3$ , then  $x_2, \dots, x_n$  contradicts Lemma 8.4 applied to  $\Sigma'$ . So  $x_n$  is of Type t1 w.r.t.  $\Sigma$ , w.l.o.g.

adjacent to  $b_3$ . If  $n \neq 2$ , then  $x_1, \dots, x_n$  contradicts Theorem 8.3 in  $\Sigma'$ . So  $n = 2$ .

Let  $T = (N(b_1) \cup b_1) \setminus b'_1$  and let  $Q = y_1, \dots, y_m$  be a direct connection from  $x_n$  to  $\Sigma \setminus T$  in  $G \setminus T$ . As in Case 1.2,  $b_2$  does not have a neighbor in  $Q \setminus y_m$ . By Lemma 8.4 applied to  $\Sigma$  and  $x_n, Q$  (if  $b_3$  does not have a neighbor in  $Q \setminus y_m$ ) or a subpath of  $Q$  (otherwise),  $y_m$  is either of Type t1, p1 or p3 w.r.t.  $\Sigma$  with a neighbor in  $P^3 \setminus b_3$ , or of Type p2 or p4 w.r.t.  $\Sigma$  adjacent to  $b_3$ , or of Type t2, t2p or t3p w.r.t.  $\Sigma$  adjacent to  $a_1, a_2$  and no node of  $(P^1 \cup P^2) \setminus \{a_1, a_2\}$ .

Let  $x_n = y_0$  and  $Q' = y_0, Q$ . Let  $y_i$  be the node of  $Q'$  with highest index adjacent to  $x_1$ . By Lemma 6.1 applied to  $\Sigma'$  and since  $y_m$  cannot be of Type t4d w.r.t.  $\Sigma'$ ,  $i \neq m$ . If  $b_3$  does not have a neighbor in  $Q'_{y_i y_{m-1}}$ , then  $Q'_{y_i y_m}$  contradicts Lemma 8.4 applied to  $\Sigma'$ . So  $b_3$  has a neighbor in  $Q'_{y_i y_{m-1}}$ . Let  $y_l$  be such a node with lowest index. If  $l \neq i$ , then  $Q_{y_i y_l}$  contradicts Theorem 8.3 in  $\Sigma'$ . So  $b_3$  is adjacent to  $y_i$ . If  $b_3$  does not have a neighbor in  $Q'_{y_{i+1} y_{m-1}}$ , then  $Q'_{y_i y_m}$  contradicts Lemma 8.5 applied to  $\Sigma'$ . So  $b_3$  has a neighbor in  $Q'_{y_{i+1} y_{m-1}}$ . If  $y_m$  is of Type t2, t2p or t3p w.r.t.  $\Sigma$ , then let  $C$  be the hole induced by  $P^2 \cup Q'_{y_i y_m} \cup x_1$  and  $C'$  the hole induced by  $P^1_{a_1 v} \cup Q'_{y_i y_m} \cup x_1$ . Then  $(C, b_3)$  or  $(C', b_3)$  is an odd wheel. If  $y_m$  is of Type t1, p1, p2 or p3 w.r.t.  $\Sigma$ , then let  $C$  be the hole passing through  $x_1$  contained in  $P^2 \cup (P^3 \setminus b_3) \cup Q'_{y_i y_m} \cup x_1$  and  $C'$  the hole contained in  $P^1_{a_1 v} \cup (P^3 \setminus b_3) \cup Q'_{y_i y_m} \cup x_1$ . Then  $(C, b_3)$  or  $(C', b_3)$  is an odd wheel. So  $y_m$  is of Type p4 w.r.t.  $\Sigma$ . Since there is no  $P^1$ -crosspath w.r.t.  $\Sigma$ ,  $y_m$  does not have a neighbor in  $P^1$ . Hence,  $P^1_{a_1 v} \cup P^3 \cup Q'_{y_i y_m} \cup x_1$  induces a proper wheel with center  $b_3$  that is not a beetle.  $\square$

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