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# Blind Frequency-Offset Estimator for OFDM Systems Transmitting Constant-Modulus Symbols

Mounir Ghogho, *Member, IEEE,* and Ananthram Swami, *Senior Member, IEEE*

***Abstract*—We address the problem of carrier frequency offset (CFO) synchronization in OFDM communications systems in the context of frequency-selective fading channels. We consider the case where the transmitted symbols have *constant modulus*, i.e., PSK constellations. A novel blind CFO estimation algorithm is developed. The new algorithm is shown to greatly outperform a recently published blind technique that exploits the fact that practical OFDM systems are not fully loaded. Further, the proposed algorithm is consistent even when the system is fully loaded. Finally, the proposed CFO estimator is obtained via a one-dimensional search, the same as with the existing virtual subcarrier-based estimator, but achieves a substantial gain in performance (10-dB SNR or one order of magnitude in CFO MSE).**

***Index Terms*—Dispersive channels, frequency offset, OFDM.**

## I. Introduction

THE presence of carrier frequency offset (CFO) causes loss of orthogonality between the subcarriers in an OFDM scheme and leads to increased bit error rate. Consequently, there has been considerable work done in the area of CFO estimation. A number of pilot-assisted CFO synchronization techniques are available in the literature. Blind CFO synchronization is attractive because it saves bandwidth, i.e., no training pilots are required. A blind CFO estimator was recently proposed in [1] (see also [2] for more details); this estimator, designed to work with dispersive channels, exploits the fact that practical OFDM systems are not fully loaded, i.e., the number of information-bearing subcarriers is smaller than the size of the FFT block.

Here, we consider the case where the transmitted symbols have constant-modulus (as in various European ACTS projects, the 6–18-Mbps rates in the IEEE 802.11 5 GHz and HIPERLAN2 standards; [4]). Judicious use of this side information leads to a simple CFO estimator which significantly outperforms the estimators in [1], [2].

## II. Signal Model

OFDM modulation consists of $N$ (usually a power of 2) subcarriers, equispaced at a separation of $\Delta f = B/N$, where $B$ is the total system bandwidth. The subcarriers are mutually orthogonal over a time interval of length $T = 1/\Delta f$. Each subcarrier is modulated independently with symbols having constant modulus (CM), e.g., PSK constellations. Each OFDM block is preceded by a cyclic prefix whose duration is longer than the delay spread of the overall propagation channel (including transmit and receive filters), so that inter-block interference can be eliminated at the receiver, without affecting the orthogonality of the sub-carriers. Practical OFDM systems are, in general, not fully loaded in order to avoid aliasing. In this case, some of the sub-carriers at the edges of the OFDM block are not modulated; these subcarriers are referred to as virtual subcarriers (VSCs). Their number is dictated by system design requirements and is, in general, about 10% of $N$. Let $\mathcal{N} = \{-N/2+1, \ldots, N/2\}$ denote the entire set of subcarriers; also let $\mathcal{N}_a = \{-(N_a-1)/2, \ldots, (N_a-1)/2\}$ denote the $N_a$-element set of active or modulated subcarriers ($N_a < N$ and $N_a$ odd) and let $\mathcal{N}_v = \mathcal{N} - \mathcal{N}_a$ denote the set of $N_v = N - N_a$ VSCs.[1]

At the receiver, the output of the matched filter is sampled with period $T_s = T/N$. After discarding the cyclic prefix, the complex envelope of the baseband received signal in an OFDM block can be described as

$$x(k) = e^{j2\pi k\xi_o/N} \sum_{n\in\mathcal{N}_a} H_n s_n e^{j2\pi kn/N} + v(k) \qquad (1)$$

for $k = 0, \ldots, N-1$, where $\{s_n\}$ is a CM data sequence, $|s_n| = 1$, $\forall n$; $\xi_o$ (a real number, $|\xi_o| < N/2$) is the CFO normalized to $1/T$; $v(k)$ is additive noise which is assumed to be zero-mean, uncorrelated, circularly symmetric and Gaussian with variance $\sigma_v^2 = E\{|v(k)|^2\}$; $H_n$ is the complex channel response at the $n$th subcarrier frequency

$$H_n = \sum_{l=0}^{L} h_l e^{-j2\pi ln/N} \qquad (2)$$

where $\{h_l\}$ are the coefficients of the $(L+1)$-tap channel.

Since $\{s_n\}$ is a CM sequence, $H_n s_n$ in (1) may be rewritten as $H_n s_n = |H_n| e^{j\theta_n}$ where $\theta_n$ is the angle of $H_n s_n$, $|\theta_n| \le \pi$. Note that the $N_a$ unknown parameters, $\{|H_n|\}$ in (1), are parameterized by only $(L+1)$ complex parameters—the channel coefficients $\{h_l\}$. This property is the core of the proposed algorithm.

*Notation:* Let $\mathcal{R}[z], \mathcal{I}[z]$ and $\arg\{z\}$ denote the real part, the imaginary part and the argument of the complex variable $z$. Let $X(f)$ denote the discrete-time Fourier transform of $\{x(k)\}$ at frequency $f/N$

$$X(f) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) e^{-j2\pi kf/N}.$$

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M. Ghogho is with the School of Electronic and Electrical Engineering, University of Leeds, Leeds LS2 9JT, U.K. (e-mail: m.ghogho@ieee.org).

A. Swami is with the Army Research Lab, Adelphi, MD 20783 USA (e-mail: a.swami@ieee.org).

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[1]The assumptions that $N$ is even and $N_a$ is odd are not critical.

## III. Frequency-Offset Estimation

We consider the parameters $\{|H_n|\}$ and $\{\theta_n\}$ as unknown deterministic parameters. Since the additive noise is white, circularly symmetric and Gaussian, the maximum-likelihood estimates (MLEs) of $\xi_o$, $|\mathbf{H}| = \{|H_n|, n \in \mathcal{N}_a\}$ and $\boldsymbol{\theta} = \{\theta_n, n \in \mathcal{N}_a\}$ are obtained by minimizing the $L_2$ norm

$$\begin{aligned} J(\xi, |\mathbf{H}|, \boldsymbol{\theta}) &= \sum_{k=0}^{N-1} \left| x(k) - e^{j2\pi k\xi/N} \sum_{n\in\mathcal{N}_a} |H_n|\, e^{j\theta_n} e^{j2\pi kn/N} \right|^2 \\ &= \sum_{k=0}^{N-1} |x(k)|^2 + N \sum_{n\in\mathcal{N}_a} |H_n|^2 \\ &\quad - 2N\mathcal{R}\left[ \sum_{n\in\mathcal{N}_a} |H_n|\, X(n+\xi) e^{-j\theta_n} \right] \end{aligned} \tag{3}$$

Setting $\partial J/\partial\theta_n = 0$ and assuming that $|H_n| \neq 0$,[2] the MLE $\hat{\theta}_n$ satisfies

$$\begin{aligned} \mathcal{I}\left[X(n+\xi)e^{-j\hat{\theta}_n}\right] &= 0,\ n \in \mathcal{N}_a, \\ \hookrightarrow \hat{\theta}_n &= \arg\{X(n+\xi)\}. \end{aligned} \tag{4}$$

Substituting $\hat{\theta}_n$ into (3), the cost function becomes

$$J(\xi, |\mathbf{H}|) = J_{VSC}(\xi) + J_A(\xi, |\mathbf{H}|) \tag{5}$$

$$J_{VSC}(\xi) = \sum_{n\in\mathcal{N}_v} |X(n+\xi)|^2 \tag{6}$$

$$J_A(\xi, |\mathbf{H}|) = \sum_{n\in\mathcal{N}_a} \left(|X(n+\xi)| - |H_n|\right)^2 \tag{7}$$

where we have used Parseval's theorem and the fact that $H_n s_n = 0$ for $n \notin \mathcal{N}_a$. The linear relations in (2) should be taken into account in the minimization of the above criterion.

Notice that $J_{VSC}$ is equivalent to the cost function used in an existing blind CFO estimator [1], which exploits the virtual sub-carriers ($\mathcal{N}_v$). The extra term $J_A$ exploits the CM property of the transmitted symbols. Let $\xi_o = m + \nu_o$ where $m$ is an integer and $\nu_o \in [0, 1)$ is a fraction of the sub-carrier spacing. The estimator based solely on $J_A$ cannot determine $m$; only $\nu_o$ can be estimated. The identifiability conditions of our new estimator are, therefore, the same as those of the VSC-based estimator (see [3] and references therein).

### A. Non-Dispersive Channels

In this case, $L = 0$ and $H_n = h_o$, $\forall n \in \mathcal{N}_a$. Using (5), the MLE of $\xi_o$ is the solution to the $L_1$-norm problem

$$\hat{\xi}_o = \arg\max_{\xi} \sum_{n\in\mathcal{N}_a} |X(n+\xi)|. \tag{8}$$

Note that the VSC-based estimator is equivalently obtained by maximizing the $L_2$-norm of the same vector as above. Indeed, using Parseval's theorem, we have that

$$\arg\min_{\xi} J_{VSC}(\xi) = \arg\max_{\xi} \sum_{n\in\mathcal{N}_a} |X(n+\xi)|^2.$$

[2]If $|H_n| = 0$, $\theta_n$ becomes nonidentifiable. A channel with $L+1$ taps can have at most $L$ zeros that coincide with the subcarriers; clearly $N_a > L$ ensures that $H_n \not\equiv 0, \forall n \in \mathcal{N}_a$. With $N_a > L$, all the $H_n$'s cannot be zero; hence having $H_n = 0$ for some $n \in \mathcal{N}_a$ will not affect the final CFO estimator.

### B. Dispersive Channels

First, we note that $J_{\mathrm{VSC}}$ is not a function of $|\mathbf{H}|$. Therefore, only $J_A$ needs to be considered in the minimization of $J$ with respect to $|\mathbf{H}|$. This should be carried out under the constraint

$$|H_n|^2 = \sum_{l,p=0}^{L} h_l h_p^* e^{-j2\pi(l-p)n/N}$$

which follows from (2). The above constraint states that $|H_n|^2$, $n \in \mathcal{N}_a$, are quadratic forms of the $(L+1)$ channel coefficients. We modify $J_A$ as follows

$$J_A'(\xi, |\mathbf{H}|) = \sum_{n\in\mathcal{N}_a} \left(|X(n+\xi)|^2 - |H_n|^2\right)^2. \tag{9}$$

The motivation for using this modified criterion is given next. Since our goal is to estimate the CFO by using a 1-D optimization procedure, we need to eliminate the $|H_n|$'s from the criterion in (7). Toward this objective, we use $|H_n|^2$ instead of $|H_n|$ since the former can be re-parameterized as follows:

$$|H_n|^2 = \boldsymbol{c}_n^T \boldsymbol{\lambda}$$

where

$$\begin{aligned} \boldsymbol{c}_n &= \left[1, \sqrt{2}\cos\left(\frac{2\pi n}{N}\right), \ldots, \sqrt{2}\cos\left(\frac{2\pi nL}{N}\right),\right. \\ &\qquad \left.\sqrt{2}\sin\left(\frac{2\pi n}{N}\right), \ldots, \sqrt{2}\sin\left(\frac{2\pi nL}{N}\right)\right]^T \\ \boldsymbol{\lambda} &= \left[g_0, \sqrt{2}\mathcal{R}[g_1], \ldots, \sqrt{2}\mathcal{R}[g_L],\right. \\ &\qquad \left.\sqrt{2}\mathcal{I}[g_1], \ldots, \sqrt{2}\mathcal{I}[g_L]\right]^T \\ g_i &= \sum_{l=0}^{L-i} h_l^* h_{l+i}. \end{aligned}$$

The sequence $\{|H_n|^2\}$ is then described by linear combinations of the $(2L+1)$ elements of the parameter vector $\boldsymbol{\lambda}$. Note that both $\boldsymbol{c}_n$ and $\boldsymbol{\lambda}$ are real-valued vectors. The cost function $J_A'$ is no longer the ML cost, but it does lead to a consistent estimate of $\xi_o$

By assumption we have that $\boldsymbol{c}_n^T\boldsymbol{\lambda} \not\equiv 0$, $\forall n \in \mathcal{N}_a$. Setting $\partial J_A'/\partial\boldsymbol{\lambda} = 0$ yields the following closed-form estimator

$$\hat{\boldsymbol{\lambda}} = \mathbf{C_2}^\dagger \sum_{n\in\mathcal{N}_\mathbf{a}} |\mathbf{X}(n+\xi)|^2 \boldsymbol{c}_n, \tag{10}$$

where $\mathbf{C_2} := \sum_{\mathbf{n}\in\mathcal{N}_\mathbf{a}} \boldsymbol{c}_\mathbf{n}\boldsymbol{c}_\mathbf{n}^\mathbf{T}$ and $^\dagger$ denotes the pseudo-inverse. Note that the matrix $\mathbf{C_2}$ depends only upon $\mathcal{N}_a$; hence, its pseudo-inverse can be precomputed. Substituting $\widehat{|H_n|^2} = \boldsymbol{c}_n^T\hat{\boldsymbol{\lambda}}$ into (5)–(7), we obtain the following cost function:

$$\begin{aligned} J(\xi) &= J_{VSC}(\xi) + J_{CM}(\xi) \\ J_{VSC}(\xi) &= \sum_{n\in\mathcal{N}_v} |X(n+\xi)|^2 \\ J_{CM}(\xi) &= \sum_{n\in\mathcal{N}_a} \left(|X(n+\xi)| - \sqrt{Y(n;\xi)}\right)^2 \end{aligned}$$

where

$$Y(n;\xi) = \boldsymbol{c}_n^T \mathbf{C_2}^\dagger \sum_{k\in\mathcal{N}_\mathbf{a}} |\mathbf{X}(k+\xi)|^2 \boldsymbol{c}_k. \tag{11}$$

In the above criterion, $J_{CM}$ exploits the CM property of the symbol constellations. Using Parseval's theorem, the two criteria can be merged together; after dropping constant terms, the pseudo-MLE of the CFO is obtained as

$$\hat{\xi}_o = \arg\min_{\xi} \sum_{n \in \mathcal{N}_a} \left( Y(n;\xi) - 2|X(n+\xi)| \sqrt{Y(n;\xi)} \right). \quad (12)$$

We refer to the estimator minimizing $J_{\rm VSC}$ as the VSC-based estimator, which is equivalent to the estimator in [1] for a specific choice of $\mathcal{N}_v$. We refer to our estimator in (12) as the VSC&CM-based estimator.

We have, therefore, shown that the CM property can be exploited in blindly estimating $\xi_o$ without increasing the dimension of the optimization procedure. Indeed, the VSC&CM-based estimator is obtained by a one-dimensional search, the same as with the existing VSC-based estimator. The computation of the criterion in (12) is obviously more demanding than that of $J_{VSC}$. However, as we see next, the performance improvement of the VSC&CM-based estimator over the VSC-based estimator is more than one order of magnitude (i.e., more than 10 dB!) provided $N_a \gg L$ (typically $N_a > 2L$), which is usually the case.

If the system is fully loaded, then $N_a = N$ and the VSC-based estimator fails. Our estimator, which in this case should be referred to as the CM-based estimator, continues to perform well. Furthermore, in this case, $\mathbf{C_2} = N\mathbf{I}$ where $\mathbf{I}$ is the $N \times N$ identity matrix and hence no matrix inversion is required. However, in this case, the CFO can be unambiguously estimated only in the interval $[-0.5, 0.5)$.

## IV. Performance Analysis

Here, we compare our estimator with the estimator of [1], [2] via Monte Carlo simulations. Only one OFDM block is used. We consider a total of $N = 64$ sub-carriers where only $N_a = 49$ sub-carriers are used, i.e., the number of virtual sub-carriers is $N_v = 15$. The transmitted symbols are drawn from equiprobable 8-PSK constellations. The channel coefficients are generated using an uncorrelated Rayleigh scattering model with exponential power delay profile, i.e., $E\{h_i{}^* h_j\} = \exp(-\beta i)\delta(i-j)$, with $\beta = 1/5$. The CFO is generated randomly between $-2$ and 2. Both the channel coefficients and the CFO are generated randomly for each Monte Carlo run. The signal-to-noise ratio is defined as $SNR = 10\log_{10} N_a \sum_{i=0}^{L} E\{|h_i|^2\}/\sigma_v^2$ where $\sigma_v^2$ is the variance of the noise. Mean-square error (MSE) was estimated empirically from 200 Monte Carlo runs.

### A. Performance Versus the SNR

Fig. 1 displays the MSE versus SNR for a seven-tap channel, i.e., $L = 6$. The performance of the proposed estimator is significantly better than that of the estimators in [1] and [2]. Fig. 1 indicates a 10-dB improvement.

### B. Unknown Channel Order

Here, we set the actual channel order to $L = 6$ and estimate the MSE when the assumed channel order varies from 0 to 25,

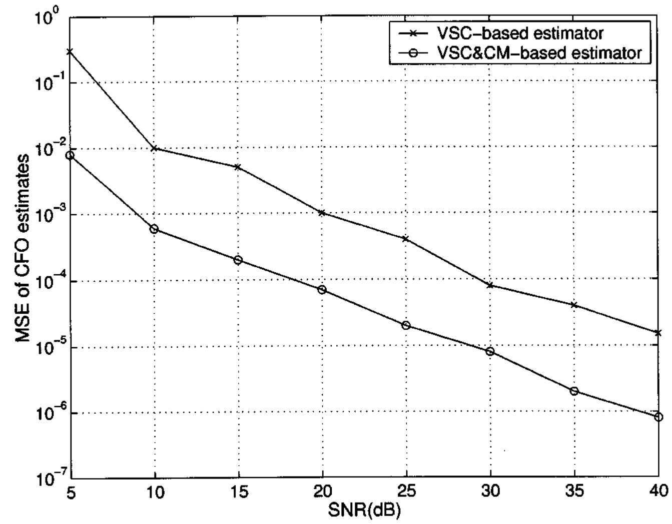


Fig. 1. MSE of CFO estimators versus SNR; $L = 6$.

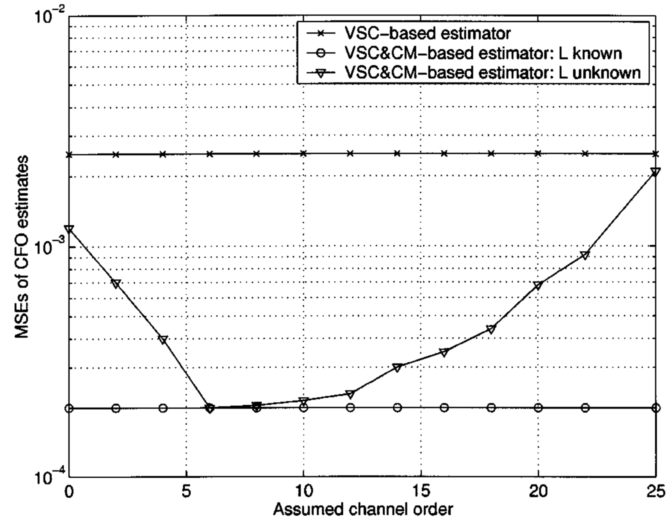


Fig. 2. MSE of CFO estimators vs. assumed channel order; the actual channel order is 6; $\mathrm{SNR} = 15$ dB.

with $\mathrm{SNR} = 15$ dB. The results are depicted in Fig. 2. It is seen that our estimator still performs well if the assumed channel order is (slightly) larger than the actual channel order. This implies that in practice, if $L$ is unknown, one only needs to use a reasonable upper limit on the possible values of $L$.

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