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KERNEL OPERATORS WITH VARIABLE INTERVALS OF INTEGRATION IN LEBESGUE SPACES AND APPLICATIONS

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Dedicated to Professor Josip Pečarić on the occasion of his 60th birthday

(*Communicated by L.-E. Persson*)

Abstract. New criteria of $L_p - L_q$ boundedness of Hardy-Steklov type operator (1.1) with both increasing on $(0,\infty)$ boundary functions $a(x)$ and $b(x)$ are obtained for $1 < p \leqslant q < \infty$ and $0 < q < p < \infty$, $p > 1$. This result is applied for two-weighted $L_p - L_q$ characterization of the corresponding geometric Steklov operator (1.3) and other related problems.

1. Introduction

Let $0 < p < \infty$, $\|f\|_p := \left(\int_0^\infty |f(x)|^p dx\right)^{1/p}$ and L_p denotes the Lebesgue space of all measurable functions on $\mathbb{R}^+ := [0,\infty)$ such that $\|f\|_p < \infty$.

Assume $w(x)$ and $v(y)$ be locally integrable and almost everywhere positive functions (weights). We study the $L_p - L_q$ boundedness of the Hardy-Steklov operator of the form

$$\mathscr{H} f(x) := w(x) \int_{a(x)}^{b(x)} f(y) v(y) dy, \tag{1.1}$$

where the boundaries $a(x)$ and $b(x)$ satisfy the following conditions:

$$\begin{array}{ll} \text{(i)} & a(x) \text{ and } b(x) \text{ are differentiable and strictly increasing on } (0,\infty); \\ \text{(ii)} & a(0) = b(0) = 0,\ a(x) < b(x) \text{ for } 0 < x < \infty,\ a(\infty) = b(\infty) = \infty. \end{array} \tag{1.2}$$

In the limiting cases $a(x) = 0$ or $b(x) = \infty$ the operator $\mathscr{H}$ is reduced to the Hardy-type operators with variable upper or lower bound and this relation stands behind of the so-called *block-diagonal method* for investigation of $\mathscr{H}$ by a suitable decomposition into a sequence of Hardy-type operators with non-overlapping domains.

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However, apart from the limiting cases some properties of $\mathscr{H}$ could be rather different. For instance, $\mathscr{H}$ is a self-adjoint operator in L_2 for $w = v$ and $a(x) = b^{-1}(x)$ and to find the spectrum of $\mathscr{H}$ in this case is an interesting problem.

We give two alternative pairs of criteria for the $L_p - L_q$ boundedness of $\mathscr{H}$ (§ 4). The first pair is a complete analog of Tomaselli-Muckenhoupt-Bradley and Mazya-Rozin conditions for the Hardy-type operators in the cases $1 < p \leqslant q < \infty$ and $0 < q < p < \infty, p > 1$, respectively. The second pair is new and allows to characterize the weighted $L_p - L_q$ boundedness of the geometric Steklov operator (§ 5)

$$\mathscr{G}f(x) := \exp\left(\frac{1}{b(x)-a(x)}\int_{a(x)}^{b(x)} \log f(y)dy\right), \qquad f(y) \geqslant 0. \tag{1.3}$$

Both pairs of criteria involve a notion of the *fairway* - a curve between the graphs of $a(x)$ and $b(x)$ with such an equilibrium property which allows to squeeze up discrete portions produced by the block-diagonal method into one piece.

We demonstrate the block-diagonal method (§ 3) for an even more general operator

$$\mathscr{K}f(x) := w(x)\int_{a(x)}^{b(x)} k(x,y)f(y)v(y)dy, \tag{1.4}$$

where the kernel $k(x,y) \geqslant 0$ satisfies the *Oinarov-type condition* of the form

$$k(x,y) \approx k(x,b(z)) + k(z,y), \qquad z \leqslant x, \ a(x) \leqslant y \leqslant b(z). \tag{1.5}$$

The same method works for a formally dual operator

$$\mathscr{K}^*f(x) := w(x)\int_{a(x)}^{b(x)} k(y,x)f(y)v(y)dy \tag{1.6}$$

with the kernel $k(y,x) \geqslant 0$ satisfying

$$k(y,x) \approx k(y,z) + k(a(z),x), \qquad x \leqslant z, \ a(z) \leqslant y \leqslant b(x).$$

However, in both cases two-sided estimates of the norms have discrete forms, which are rather inconvenient for further applications. When $p \leqslant q$ the forms can be refined up to "continuous" ones, but with a double supremum. As for the case $q < p$ is concerned the attempts to find out the integral form of criteria, analogous to the Hardy-type case, met some difficulties. Nevertheless, it gives a solution for the $L_p - L_q$ boundedness of $\mathscr{H}$ narrowed on the cone of monotone functions (§ 6.2).

Our next observation is that $L_p - L_q$ boundedness of $\mathscr{H}$ is equivalent to the validity of the differential inequality

$$\|Fw\|_q \leqslant C\|F'/v\|_p \tag{1.7}$$

restricted to a non-linear class of absolutely continuous function defined by the border functions $a(x)$ and $b(x)$, which in turn is closely related to the embedding of the weighted Sobolev space to the weighted Lebesgue space to hold (§ 6.1), that is to inequalities of the form

$$\|Fw\|_q \leqslant C\left(\|Fu\|_s + \|F'/v\|_p\right). \tag{1.8}$$

We start the paper with the Preliminaries (§ 2) contained auxiliary results and conclude by the Bibliographical remarks.

Throughout of the paper products of the form $0\cdot\infty$ are taken to be equal to 0. Relations $A \ll B$ mean $A \leqslant cB$ with some constants c depending only on parameters of summations and, possibly, on the constants of equivalence in the inequalities of the type (1.5). We write $A \approx B$ instead of $A \ll B \ll A$ or $A = cB$. $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of all integers and all positive integers, respectively. χ_E stands for a characteristic function (indicator) of a subset $E \subset \mathbb{R}^+$. Also we make use of marks $:=$ and $=:$ for introducing new quantities and denote $p' := p/(p-1)$ for $0 < p < \infty$, $p \neq 1$ and $r := pq/(p-q)$ for $0 < q < p < \infty$. In Section 5 we denote $L_{p,v}$ the weighted Lebesgue space with the norm $\|f\|_{p,v} := \|fv\|_p$.

2. Preliminaries

2.1. Hardy and Hardy type operators

Here we collect some known results for Hardy operator

$$Hf(x) := w(x)\int_c^x f(y)v(y)dy, \qquad 0 \leqslant c \leqslant x \leqslant d \leqslant \infty, \tag{2.1}$$

and Hardy type operator of the form

$$Kf(x) := w(x)\int_c^x k(x,y)f(y)v(y)dy, \qquad 0 \leqslant c \leqslant x \leqslant d \leqslant \infty, \tag{2.2}$$

with a non-negative kernel $k(x,y)$ from Oinarov's class $\mathscr{O}$.

DEFINITION 2.1. Let $k(x,y) \geqslant 0$, $k(x,y) \in \mathscr{O}$ if there exists a constant $D \geqslant 1$ such that

$$D^{-1}k(x,y) \leqslant k(x,z) + k(z,y) \leqslant Dk(x,y), \qquad 0 \leqslant c \leqslant y \leqslant z \leqslant x \leqslant d \leqslant \infty. \tag{2.3}$$

THEOREM 2.1. *Let the operator $H : L_p(c,d) \to L_q(c,d)$ be defined by* (2.1).
(a) *If $1 < p \leqslant q < \infty$, then $\|H\|_{L_p(c,d)\to L_q(c,d)} \approx A_M \approx A_T$, where*

$$A_M := \sup_{c\leqslant t\leqslant d}\left(\int_t^d w^q(x)dx\right)^{\frac{1}{q}}\left(\int_c^t v^{p'}(y)dy\right)^{\frac{1}{p'}}, \tag{2.4}$$

$$A_T := \sup_{c\leqslant t\leqslant d}\left(\int_c^t \left[\int_c^x v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}}\left(\int_c^t v^{p'}(y)dy\right)^{-\frac{1}{p}}. \tag{2.5}$$

(b) *Let $0 < q < p < \infty$, $p > 1$. Then $\|H\|_{L_p(c,d)\to L_q(c,d)} \approx B_{MR} \approx B_{PS}$, where*

$$B_{MR} := \left(\int_c^d \left[\int_t^d w^q(x)dx\right]^{\frac{r}{p}}\left[\int_c^t v^{p'}(y)dy\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}}, \tag{2.6}$$

$$B_{PS} := \left(\int_c^d \left[\int_c^t \left\{\int_c^x v^{p'}(y)dy\right\}^q w^q(x)dx\right]^{\frac{r}{p}} \left[\int_c^t v^{p'}(y)dy\right]^{q-\frac{r}{p}} w^q(t)dt\right)^{\frac{1}{r}}. \quad (2.7)$$

REMARK 2.1. Since $\|H\|_{L_p(c,d)\to L_q(c,d)} = \|H^*\|_{L_{q'}(c,d)\to L_{p'}(c,d)}$ for $1 < p,q < \infty$, where

$$H^*g(y) := v(y)\int_y^d g(x)w(x)dx, \qquad 0 \leqslant c \leqslant y \leqslant d \leqslant \infty, \quad (2.8)$$

the above equivalences ought to be supplemented by

$$A_T^* := \sup_{c\leqslant t\leqslant d}\left(\int_t^d \left[\int_y^d w^q(x)dx\right]^{p'} v^{p'}(y)dy\right)^{\frac{1}{p'}} \left(\int_t^d w^q(x)dx\right)^{-\frac{1}{q'}}, \quad (2.9)$$

$$B_{PS}^* := \left(\int_c^d \left[\int_t^d \left\{\int_y^d w^q(x)dx\right\}^{p'} v^{p'}(y)dy\right]^{\frac{r}{q'}} \left[\int_t^d w^q(x)dx\right]^{p'-\frac{r}{q'}} v^{p'}(t)dt\right)^{\frac{1}{r}}. \quad (2.10)$$

For the Hardy type operator we have the following.

THEOREM 2.2. *Let the operator $K : L_p(c,d) \to L_q(c,d)$ be defined by* (2.2) *with $k(x,y) \in \mathscr{O}$.*
(a) *If $1 < p \leqslant q < \infty$, then $\|K\|_{L_p(c,d)\to L_q(c,d)} \approx A_0 + A_1$, where*

$$A_0 := \sup_{c\leqslant t\leqslant d}\left(\int_t^d k^q(x,t)w^q(x)dx\right)^{\frac{1}{q}} \left(\int_c^t v^{p'}(y)dy\right)^{\frac{1}{p'}}, \quad (2.11)$$

$$A_1 := \sup_{c\leqslant t\leqslant d}\left(\int_t^d w^q(x)dx\right)^{\frac{1}{q}} \left(\int_c^t k^{p'}(t,y)v^{p'}(y)dy\right)^{\frac{1}{p'}}. \quad (2.12)$$

(b) *Let $1 < q < p < \infty$. Then $\|K\|_{L_p(c,d)\to L_q(c,d)} \approx B_0 + B_1$, where*

$$B_0 := \left(\int_c^d \left[\int_t^d k^q(x,t)w^q(x)dx\right]^{\frac{r}{q}} \left[\int_c^t v^{p'}(y)dy\right]^{\frac{r}{q'}} v^{p'}(t)dt\right)^{\frac{1}{r}}, \quad (2.13)$$

$$B_1 := \left(\int_c^d \left[\int_t^d w^q(x)dx\right]^{\frac{r}{p}} \left[\int_c^t k^{p'}(t,y)v^{p'}(y)dy\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}}. \quad (2.14)$$

Let $b : [c,d] \to [0,\infty)$ be a strictly increasing differentiable function and let $K_b : L_p(b(c),b(d)) \to L_q(c,d)$ be an operator of the form

$$K_bf(x) := w(x)\int_{b(c)}^{b(x)} k(x,y)f(y)v(y)dy, \qquad 0 \leqslant c \leqslant x \leqslant d \leqslant \infty, \quad (2.15)$$

where a non-negative kernel $k(x,y)$ satisfies the following definition.

DEFINITION 2.2. $k(x,y) \in \mathscr{O}_b$ if there exists a constant $D \geqslant 1$ such that

$$D^{-1}k(x,y) \leqslant k(x,b(z)) + k(z,y) \leqslant Dk(x,y), \qquad \begin{cases} 0 \leqslant c \leqslant z \leqslant x \leqslant d \leqslant \infty, \\ 0 \leqslant b(c) \leqslant y \leqslant b(z). \end{cases} \tag{2.16}$$

COROLLARY 2.1. *Let the operator* K_b *be an operator given by* (2.15) *with a strictly increasing differentiable function* $b(x) \geqslant 0$ *and* $k(x,y) \in \mathscr{O}_b$.
(a) *If* $1 < p \leqslant q < \infty$, *then*

$$\|K_b\|_{L_p(b(c),b(d)) \to L_q(c,d)} \approx A_{b,0} + A_{b,1}, \tag{2.17}$$

where

$$A_{b,0} := \sup_{c \leqslant t \leqslant d} \left(\int_t^d k^q(x,b(t)) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{b(c)}^{b(t)} v^{p'}(y) dy \right)^{\frac{1}{p'}}, \tag{2.18}$$

$$A_{b,1} := \sup_{c \leqslant t \leqslant d} \left(\int_t^d w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{b(c)}^{b(t)} k^{p'}(t,y) v^{p'}(y) dy \right)^{\frac{1}{p'}}. \tag{2.19}$$

(b) *If* $1 < q < p < \infty$, *then*

$$\|K_b\|_{L_p(b(c),b(d)) \to L_q(c,d)} \approx B_{b,0} + B_{b,1}, \tag{2.20}$$

where

$$B_{b,0} := \left(\int_{b(c)}^{b(d)} \left[\int_{b^{-1}(t)}^d k^q(x,t) w^q(x) dx \right]^{\frac{r}{q}} \left[\int_{b(c)}^t v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}}, \tag{2.21}$$

$$B_{b,1} := \left(\int_c^d \left[\int_t^d w^q(x) dx \right]^{\frac{r}{p}} \left[\int_{b(c)}^{b(t)} k^{p'}(t,y) v^{p'}(y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}}. \tag{2.22}$$

Proof. By the substitution $\tau = b^{-1}(y)$ we see that the inequality

$$\left(\int_c^d (K_b f)^q(x) dx \right)^{\frac{1}{q}} \leqslant C \left(\int_{b(c)}^{b(d)} f^p(y) dy \right)^{\frac{1}{p}} \tag{2.23}$$

is equivalent to the inequality

$$\left(\int_c^d \left[\int_c^x \widetilde{k}(x,\tau) f(\tau) \widetilde{v}(\tau) d\tau \right]^q w^q(x) dx \right)^{\frac{1}{q}} \leqslant C \left(\int_c^d f^p(\tau) d\tau \right)^{\frac{1}{p}} \tag{2.24}$$

with $\widetilde{v}(\tau) = v(b(\tau))[b'(\tau)]^{1/p'}$, $\widetilde{k}(x,\tau) = k(x,b(\tau)) \in \mathscr{O}$. The result follows from Theorem 2.2.

REMARK 2.1. If $k(x,y) \equiv 1$ the result of Corollary 2.1(b) is true for $0 < q < p < \infty,\ p > 1$.

Similar characterization is valid for the operator $K_a : L_p(a(c),a(d)) \to L_q(c,d)$ with a lower variable limit of integration of the form

$$K_a f(x) := w(x) \int_{a(x)}^{a(d)} k(y,x) f(y) v(y) dy, \qquad 0 \leqslant c \leqslant x \leqslant d \leqslant \infty, \tag{2.25}$$

with a non-negative strictly increasing differentiable function $a(x)$ and a non-negative kernel $k(y,x)$ from Oinarov's type class $\mathscr{O}_a$ defined as follows.

DEFINITION 2.3. $k(y,x) \in \mathscr{O}_a$, if there exists a constant $D \geqslant 1$ such that

$$D^{-1} k(y,x) \leqslant k(y,z) + k(a(z),x) \leqslant D k(y,x), \qquad \begin{cases} 0 \leqslant c \leqslant x \leqslant z \leqslant d \leqslant \infty, \\ 0 \leqslant a(z) \leqslant y \leqslant a(d). \end{cases} \tag{2.26}$$

COROLLARY 2.2. (a) *If* $1 < p \leqslant q < \infty$, *then*

$$\|K_a\|_{L_p(a(c),a(d)) \to L_q(c,d)} \approx A_{a,0} + A_{a,1}, \tag{2.27}$$

where

$$A_{a,0} := \sup_{c \leqslant t \leqslant d} \left(\int_c^t k^q(a(t),x) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(t)}^{a(d)} v^{p'}(y) dy \right)^{\frac{1}{p'}}, \tag{2.28}$$

$$A_{a,1} := \sup_{c \leqslant t \leqslant d} \left(\int_c^t w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(t)}^{a(d)} k^{p'}(y,t) v^{p'}(y) dy \right)^{\frac{1}{p'}}. \tag{2.29}$$

(b) *If* $1 < q < p < \infty$, *then*

$$\|K_a\|_{L_p(a(c),a(d)) \to L_q(c,d)} \approx B_{a,0} + B_{a,1}, \tag{2.30}$$

where

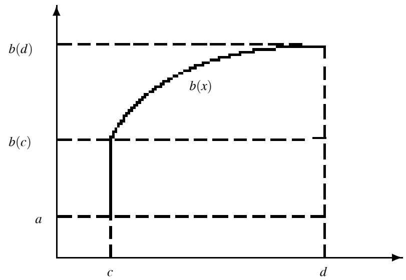
$$B_{a,0} := \left(\int_{a(c)}^{a(d)} \left[\int_c^{a^{-1}(t)} k^q(t,x) w^q(x) dx \right]^{\frac{r}{q}} \left[\int_t^{a(d)} v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}}, \tag{2.31}$$

$$B_{a,1} := \left(\int_c^d \left[\int_c^t w^q(x) dx \right]^{\frac{r}{p}} \left[\int_{a(t)}^{a(d)} k^{p'}(y,t) v^{p'}(y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}}. \tag{2.32}$$

REMARK 2.2. In the case $k(y,x) \equiv 1$ the part (b) holds for $0 < q < p < \infty,\ p > 1$.

In lemmas 2.1–2.4 we state norm estimates for certain Hardy type operators with only one variable limit. Such estimates are given above in a less general form (see corollaries 2.1-2.2 with $k \equiv 1$), but the authors find it difficult to give precise references. Therefore, we state these results here together with sketches of the proof for some of them. We start from the case $1 < p \leqslant q < \infty$.

LEMMA 2.1. *Let* $1 < p \leqslant q < \infty$ *and let* $0 \leqslant c < d \leqslant \infty$, $0 \leqslant a < \infty$. *Suppose that the function* $b(x)$ *is differentiable, strictly increasing and such that* $a \leqslant b(x) < \infty$, $x \in [c,d)$, *and let*



$$Sf(x) := w(x)\int_a^{b(x)} f(y)v(y)dy. \tag{2.33}$$

Then for the norm of S we have the following two-sided estimates with coefficients of equivalence depending on p and q only:

$$\|S\|_{L_p(a,b(d))\to L_q(c,d)} \approx A_b := \sup_{c\leqslant t\leqslant d}\left(\int_t^d w^q(x)dx\right)^{\frac{1}{q}}\left(\int_a^{b(t)} v^{p'}(y)dy\right)^{\frac{1}{p'}}, \tag{2.34}$$

$$\|S\|_{L_p(a,b(d))\to L_q(c,d)} \approx \mathbb{A}_b := \sup_{c\leqslant t\leqslant d}\left(\int_c^t\left[\int_a^{b(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}}\left(\int_a^{b(t)} v^{p'}(y)dy\right)^{-\frac{1}{p}}. \tag{2.35}$$

Proof. Necessity for both (2.34) and (2.35) follows from applying the test function $f_t(y) = [v(y)]^{p'-1}\ \chi_{[a,b(t)]}(y)$ to the inequality

$$\left(\int_c^d (Sf)^q(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_a^{b(d)} f^p(y)dy\right)^{\frac{1}{p}}. \tag{2.36}$$

To prove sufficiency in (2.34) we note that the least possible constant $C \geqslant 0$ of the inequality (2.36) coinciding with the operator norm $C = \|S\|_{L_p(a,b(d))\to L_q(c,d)}$ is equivalent to the sum of the least possible constants $C \approx C_1 + C_2$ of the inequalities

$$\int_a^{b(c)} f(y)v(y)dy\left(\int_c^d w^q(x)dx\right)^{\frac{1}{q}} \leqslant C_1\left(\int_a^{b(d)} f^p(y)dy\right)^{\frac{1}{p}},$$

$$\left(\int_c^d\left[\int_{b(c)}^{b(x)} f(y)v(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}} \leqslant C_2\left(\int_a^{b(d)} f^p(y)dy\right)^{\frac{1}{p}},$$

which, in their turn, are equivalent to the following two inequalities:

$$\int_a^{b(c)} f(y)v(y)dy\left(\int_c^d w^q(x)dx\right)^{\frac{1}{q}} \leqslant C_3\left(\int_a^{b(c)} f^p(y)dy\right)^{\frac{1}{p}}, \tag{2.37}$$

$$\left(\int_c^d \left[\int_{b(c)}^{b(x)} f(y)v(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}} \leqslant C_4 \left(\int_{b(c)}^{b(d)} f^p(y)dy\right)^{\frac{1}{p}}, \qquad (2.38)$$

respectively, therefore, $C \approx C_3 + C_4$. From (2.37) by the reverse Hölder inequality we find

$$C_3 = \left(\int_c^d w^q(x)dx\right)^{\frac{1}{q}} \left(\int_a^{b(c)} v^{p'}(y)dy\right)^{\frac{1}{p'}} \leqslant A_b$$

and for (2.38) by Corollary 2.1 (a) for $k(x,y) = 1$ we have

$$C_4 \approx \sup_{c \leqslant t \leqslant d} \left(\int_t^d w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{b(c)}^{b(t)} v^{p'}(y)dy\right)^{\frac{1}{p'}} \leqslant A_b$$

and the sufficiency of (2.34) follows.

For an upper estimate in (2.35) we write the inequality

$$I^{\frac{1}{p'}} := \left(\int_a^{b(d)} \left[\int_{b_0^{-1}(y)}^d g(x)w(x)dx\right]^{p'} dV(y)\right)^{\frac{1}{p'}} \leqslant C \left(\int_c^d g^{q'}(x)dx\right)^{\frac{1}{q'}},$$

dual to (2.36), where $V(y) := \int_a^y v^{p'}(z)dz$ and $b_0^{-1}(y) = \max\{c, b^{-1}(y)\}$. Integrating by parts and applying Hölder's inequality we find

$$\begin{aligned} I &= p' \int_{b(c)}^{b(d)} \left(\int_{b^{-1}(y)}^d g(x)w(x)dx\right)^{p'-1} V(y)g(b^{-1}(y))w(b^{-1}(y))db^{-1}(y) \\ &\leqslant p' \left(\int_{b(c)}^{b(d)} [g(b^{-1}(y))]^{q'} db^{-1}(y)\right)^{\frac{1}{q'}} \\ &\quad \times \left(\int_{b(c)}^{b(d)} \left[\int_{b^{-1}(y)}^d g(x)w(x)dx\right]^{\frac{q}{p-1}} V^q(y)w^q(b^{-1}(y))db^{-1}(y)\right)^{\frac{1}{q}} \\ &=: p' \left(\int_c^d g^{q'}(x)dx\right)^{\frac{1}{q'}} I_1^{\frac{1}{q}}. \end{aligned}$$

Now

$$\begin{aligned} I_1 &= \int_{b(c)}^{b(d)} \left\{\int_y^{b(d)} d\left[-\left(\int_{b^{-1}(t)}^d g(x)w(x)dx\right)^{\frac{q}{p-1}}\right]\right\} V^q(y)w^q(b^{-1}(y))db^{-1}(y) \\ &= \int_{b(c)}^{b(d)} \left(\int_{b(c)}^t V^q(y)w^q(b^{-1}(y))db^{-1}(y)\right) d\left[-\left(\int_{b^{-1}(t)}^d g(x)w(x)dx\right)^{\frac{q}{p-1}}\right] \\ &= \int_{b(c)}^{b(d)} \left(\int_c^{b^{-1}(t)} [V(b(x))]^q w^q(x)dx\right) d\left[-\left(\int_{b^{-1}(t)}^d g(x)w(x)dx\right)^{\frac{q}{p-1}}\right] \\ &= \int_c^d \left(\int_c^s [V(b(x))]^q w^q(x)dx\right) d\left[-\left(\int_s^d g(x)w(x)dx\right)^{\frac{q}{p-1}}\right] \end{aligned}$$

$$\leqslant \mathbb{A}_b^q \int_c^d \left(\int_a^{b(s)} v^{p'}(y)dy\right)^{\frac{q}{p}} d\left[-\left(\int_s^d g(x)w(x)dx\right)^{\frac{q}{p-1}}\right].$$

Furthermore, by Minkowskii's inequality

$$\int_c^d \left(\int_a^{b(s)} v^{p'}(y)dy\right)^{\frac{q}{p}} d\left[-\left(\int_s^d g(x)w(x)dx\right)^{\frac{q}{p-1}}\right]$$
$$\ll \left(\int_c^d g(x)w(x)dx\right)^{\frac{q}{p-1}} \left(\int_a^{b(c)} v^{p'}(y)dy\right)^{\frac{q}{p}}$$
$$+\left(\int_{b(c)}^{b(d)} \left\{\int_{b^{-1}(y)}^d d\left[-\left(\int_s^d g(x)w(x)dx\right)^{\frac{q}{p-1}}\right]\right\}^{\frac{p}{q}} dV(y)\right)^{\frac{q}{p}}$$
$$= \left(\int_c^d g(x)w(x)dx\right)^{\frac{q}{p-1}} \left(\int_a^{b(c)} v^{p'}(y)dy\right)^{\frac{q}{p}}$$
$$+\left(\int_{b(c)}^{b(d)} \left[\int_{b^{-1}(y)}^d g(x)w(x)dx\right]^{p'} dV(y)\right)^{\frac{q}{p}} \approx I^{\frac{q}{p}}.$$

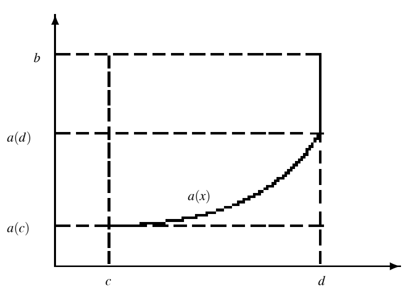
Thus,

$$I \ll \mathbb{A}_b \left(\int_c^d g^{q'}(x)dx\right)^{\frac{1}{q'}} I^{\frac{1}{p}}$$

and the proof of the upper bound $C \ll \mathbb{A}_b$ is completed.

For the Hardy type operator with a lower variable limit of integration the similar norm estimates can be obtained analogously.

LEMMA 2.2. *Let* $1 < p \leqslant q < \infty$ *and let* $0 \leqslant c < d \leqslant \infty$, $0 < b \leqslant \infty$. *Suppose that the function* $a(x)$ *is differentiable, strictly increasing and such that* $0 < a(x) \leqslant b$, $x \in (c,d]$, *and*



$$Tf(x) := w(x)\int_{a(x)}^b f(y)v(y)dy. \qquad (2.39)$$

Then

$$\|T\|_{L_p(a(c),b)\to L_q(c,d)} \approx A_a := \sup_{c\leqslant t\leqslant d} \left(\int_c^t w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{a(t)}^b v^{p'}(y)dy\right)^{\frac{1}{p'}}, \qquad (2.40)$$

$$\|T\|_{L_p(a(c),b)\to L_q(c,d)} \approx \mathbb{A}_a := \sup_{c\leqslant t\leqslant d} \left(\int_t^d \left[\int_{a(x)}^b v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{a(t)}^b v^{p'}(y)dy\right)^{-\frac{1}{p}}. \qquad (2.41)$$

Next two lemmas are concerned to the case $0<q<p<\infty,\ p>1$.

LEMMA 2.3. *Let* $0<q<p<\infty,\ p>1,\ 1/r=1/q-1/p$ *and let* $0\leqslant c<d\leqslant\infty$, $0\leqslant a<\infty$. *Suppose that the function* $b(x)$ *is differentiable, strictly increasing and such that* $a\leqslant b(x)<\infty,\ x\in[c,d)$, *and let the operator* S *be defined by* (2.33). *Then*

$$\|S\|^r_{L_p(a,b(d))\to L_q(c,d)} \approx B_b^r := \int_c^d \left(\int_t^d w^q(x)dx\right)^{\frac{r}{p}} \left(\int_a^{b(t)} v^{p'}(y)dy\right)^{\frac{r}{p'}} w^q(t)dt \quad (2.42)$$

$$\approx \left(\int_c^d w^q(x)dx\right)^{\frac{r}{q}} \left(\int_a^{b(c)} v^{p'}(y)dy\right)^{\frac{r}{p'}} \quad (2.43)$$

$$+\int_c^d \left(\int_t^d w^q(x)dx\right)^{\frac{r}{q}} d\left(\int_a^{b(t)} v^{p'}(y)dy\right)^{\frac{r}{p'}} =: \widehat{B}_b^r,$$

$$\|S\|^r_{L_p(a,b(d))\to L_q(c,d)}$$

$$\approx \mathbb{B}_b^r := \int_c^d \left(\int_c^t \left[\int_a^{b(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{r}{p}} \left(\int_a^{b(t)} v^{p'}(y)dy\right)^{q-\frac{r}{p}} w^q(t)dt \quad (2.44)$$

$$\approx \left(\int_c^d \left[\int_a^{b(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{r}{q}} \left(\int_a^{b(d)} v^{p'}(y)dy\right)^{-\frac{r}{p}} \quad (2.45)$$

$$+\int_c^d \left(\int_c^t \left[\int_a^{b(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{r}{q}} d\left[-\left(\int_a^{b(t)} v^{p'}(y)dy\right)^{-\frac{r}{p}}\right].$$

Proof. We start as in the proof of Lemma 2.1 and by the reverse Hölder inequality

$$E_0:\ = \left(\int_c^d w^q(x)dx\right)^{\frac{1}{q}} \left(\int_a^{b(c)} v^{p'}(y)dy\right)^{\frac{1}{p'}} = C_3,$$

and by Corollary 2.1 (b) with taking into account Remark 2.1 we obtain

$$E_1:\ = \left(\int_c^d \left[\int_t^d w^q(x)dx\right]^{\frac{r}{p}} \left[\int_{b(c)}^{b(t)} v^{p'}(y)dy\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}} \approx C_4,$$

and we conclude that

$$C \approx E_0+E_1$$

$$\approx \left(\int_c^d \left\{\int_t^d w^q(x)dx\right\}^{\frac{r}{p}} \left[\left(\int_a^{b(c)} v^{p'}(y)dy\right)^{\frac{r}{p'}} + \left(\int_{b(c)}^{b(t)} v^{p'}(y)dy\right)^{\frac{r}{p'}}\right] w^q(t)dt\right)^{\frac{1}{r}}$$

$$\approx B_b.$$

For the proof of the equivalence (2.42) to the sum (2.43) suppose first that $B_b^r < \infty$. Then

$$\int_t^d w^q(x)dx < \infty \quad \text{for any} \quad t \in (c,d] \tag{2.46}$$

and, therefore,

$$\begin{aligned} 0 &= \lim_{s\to d} \int_s^d \left(\int_a^{b(t)} v^{p'}(y)dy\right)^{\frac{r}{p'}} d\left[-\left(\int_t^d w^q(x)dx\right)^{\frac{r}{q}}\right] \\ &\geqslant \lim_{s\to d} \left(\int_a^{b(s)} v^{p'}(y)dy\right)^{\frac{r}{p'}} \int_s^d d\left[-\left(\int_t^d w^q(x)dx\right)^{\frac{r}{q}}\right] \\ &= \lim_{s\to d} \left(\int_a^{b(s)} v^{p'}(y)dy\right)^{\frac{r}{p'}} \left(\int_s^d w^q(x)dx\right)^{\frac{r}{q}}. \end{aligned} \tag{2.47}$$

It implies by integration by parts that $\infty > B_b^r \approx \widehat{B}_b^r$. In the reverse direction let $\widehat{B}_b^r =: I_1 + I_2 < \infty$, then

$$\begin{aligned} \infty > I_1 + I_2 = I_1 &+ \left(\int_t^d w^q(x)dx\right)^{\frac{r}{q}} \left(\int_a^{b(t)} v^{p'}(y)dy\right)^{\frac{r}{p'}}\Bigg|_c^d \\ &+ \int_c^d \left(\int_a^{b(t)} v^{p'}(y)dy\right)^{\frac{r}{p'}} d\left[-\left(\int_t^d w^q(x)dx\right)^{\frac{r}{q}}\right] \\ &\geqslant \int_c^d \left(\int_a^{b(t)} v^{p'}(y)dy\right)^{\frac{r}{p'}} d\left[-\left(\int_t^d w^q(x)dx\right)^{\frac{r}{q}}\right] \approx B_b^r. \end{aligned}$$

Hence, $B_b^r < \infty$, therefore, by (2.46) – (2.47) we have $B_b^r \approx \widehat{B}_b^r$.

To prove the second part of the lemma we put $V_a(t) := \int_a^{b(t)} v^{p'}(y)dy$ and let

$$\mathbb{B}_{PS}^r := \int_c^d \left(\int_c^t V_a^q(x) w^q(x)dx\right)^{\frac{r}{q}} d\left(-[V_a(t)]^{-r/p}\right).$$

It was proved in [18, Theorem 3] that

$$\mathbb{B}_b \approx \mathbb{B}_{PS}, \qquad \text{if} \quad V_a(d) = \infty \tag{2.48}$$

and

$$\mathbb{B}_b^r \approx \left(\int_c^d V_a^q(x) w^q(x)dx\right)^{\frac{r}{q}} [V_a(d)]^{-r/p} + \mathbb{B}_{PS}^r, \qquad \text{if} \quad 0 < V_a(d) < \infty. \tag{2.49}$$

For the upper estimate $C \ll \mathbb{B}_b$ we assume that $\mathbb{B}_b < \infty$ and suppose first $V_a(d) = \infty$. Then by Hölder's inequality with the exponents r/q and p/q we have

$$\begin{aligned} J :&= \int_c^d \left(\int_a^{b(x)} f(y)v(y)dy\right)^q w^q(x)dx \\ &= \int_c^d \left(\int_a^{b(x)} f(y)v(y)dy\right)^q w^q(x) V_a^q(x) V_a^{-q}(x)dx \end{aligned}$$

$$
\begin{aligned}
&= q\int_c^d \left(\int_a^{b(x)} f(y)v(y)dy\right)^q w^q(x)V_a^q(x)\left(\int_x^d [V_a(s)]^{-q-1}dV_a(s)\right)dx =: J_0\\
&= q\int_c^d [V_a(s)]^{-q-1}\left[\int_c^s \left(\int_a^{b(x)} f(y)v(y)dy\right)^q w^q(x)V_a^q(x)dx\right]dV_a(s)\\
&\leqslant q\int_c^d \left\{\left(\int_a^{b(s)} f(y)v(y)dy\right)^q V_a^{-q}(s)\right\}\left\{\left(\int_c^s w^q(x)V_a^q(x)dx\right)V_a^{-1}(s)\right\}dV_a(s)\\
&\ll \left(\int_c^d \left[\int_a^{b(s)} f(y)v(y)dy\right]^p V_a^{-p}(s)dV_a(s)\right)^{\frac{q}{p}} \mathbb{B}_{PS}^q.
\end{aligned}
$$

Hence, by using the estimate (2.34) from Lemma 2.1 we find

$$
J \ll \mathbb{B}_{PS}^q \left(\int_a^{b(d)} f^p(y)dy\right)^{\frac{q}{p}}.
$$

Therefore, in view of (2.48) the inequality $C \ll \mathbb{B}_b$ holds. If $V_a(d) < \infty$, then

$$
V_a^{-q}(x) = V_a^{-q}(d) + q\int_x^d [V_a(s)]^{-q-1}dV_a(s)
$$

and analogously we find that

$$
J = J_0 + V_a^{-q}(d)\int_c^d \left(\int_a^{b(x)} f(y)v(y)dy\right)^q w^q(x)V_a^q(x)dx =: J_0 + J_1.
$$

From the above estimate for J_0 and (2.49) it follows that

$$
J_0 \ll \mathbb{B}_b^q \left(\int_a^{b(d)} f^p(y)dy\right)^{\frac{q}{p}}.
$$

To estimate J_1 let $\{x_k\}_{k\in\mathbb{Z}}$ be such a sequence that $\int_{b(c)}^{b(x_k)} f(y)v(y)dy = 2^k,\ k \leqslant N$. Then

$$
\begin{aligned}
V_a^q(d)J_1 &= \int_c^d \left[\int_a^{b(c)} f(y)v(y)dy + \int_{b(c)}^{b(x)} f(y)v(y)dy\right]^q w^q(x)V_a^q(x)dx\\
&\approx \left(\int_a^{b(c)} f(y)v(y)dy\right)^q \int_c^d w^q(x)V_a^q(x)dx\\
&\quad + \sum_{k\leqslant N}\int_{x_k}^{x_{k+1}} \left(\int_{b(c)}^{b(x)} f(y)v(y)dy\right)^q w^q(x)V_a^q(x)dx =: J_{1,1} + J_{1,2}.
\end{aligned}
$$

By applying Hölder's inequality and (2.49) we find that

$$
J_{1,1} \leqslant \left(\int_a^{b(c)} f^p(y)dy\right)^{\frac{q}{p}} [V_a(c)]^{q/p'} \int_c^d w^q(x)V_a^q(x)dx \leqslant V_a^q(d)\mathbb{B}_b^q \left(\int_a^{b(c)} f^p(y)dy\right)^{\frac{q}{p}}.
$$

For the second term $J_{1,2}$ we have by Hölder's inequality with the exponents r/q and p/q

$$\begin{aligned}
J_{1,2} &\leqslant \sum_{k\leqslant N} 2^{q(k+1)} \int_{x_k}^{x_{k+1}} w^q(x)V_a^q(x)dx \\
&\leqslant 4^q \sum_{k\leqslant N} \left(\int_{b(x_{k-1})}^{b(x_k)} f(y)v(y)dy\right)^q \int_{x_k}^{x_{k+1}} w^q(x)V_a^q(x)dx \\
&\leqslant 4^q \sum_{k\leqslant N} \left(\int_{b(x_{k-1})}^{b(x_k)} f^p(y)dy\right)^{\frac{q}{p}} \left(\int_{b(x_{k-1})}^{b(x_k)} v^{p'}(y)dy\right)^{\frac{q}{p'}} \int_{x_k}^{x_{k+1}} w^q(x)V_a^q(x)dx \\
&\ll \left(\int_{b(c)}^{b(d)} f^p(y)dy\right)^{\frac{q}{p}} \left(\sum_{k\leqslant N} V_a(x_k)^{\frac{r}{p'}} \left(\int_{x_k}^{x_{k+1}} w^q(x)V_a^q(x)dx\right)^{\frac{r}{q}}\right)^{\frac{q}{r}} \\
&\ll \left(\int_{b(c)}^{b(d)} f^p(y)dy\right)^{\frac{q}{p}} \left(\int_c^d \left(\int_c^t w^q(x)V_a^q(x)dx\right)^{\frac{r}{p}} w^q(t)[V_a(t)]^{q+\frac{r}{p'}}dt\right)^{\frac{q}{r}} \\
&\ll V_a^q(d)\mathbb{B}_b^q \left(\int_{b(c)}^{b(d)} f^p(y)dy\right)^{\frac{q}{p}}.
\end{aligned}$$

By combining the above estimates we obtain the upper bound $C \ll \mathbb{B}_b$.

For the lower estimate we suppose that $C < \infty$ and note that in view of (2.42) the inequality $C \gg B_b$ holds. Under the condition $V_a(d) = \infty$ let us prove first that $B_b \gg \mathbb{B}_b$. We have

$$\begin{aligned}
\int_c^t w^q(x)V_a^q(x)dx &= \int_c^t V_a^q(x)d\left(-\int_x^t w^q(s)ds\right) \\
&= V_a^q(c)\int_c^t w^q(x)dx + q\int_c^t \left(\int_x^t w^q(s)ds\right)[V_a(x)]^{q-1}dV_a(x).
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{B}_b^r \approx \mathbb{B}_{PS}^r &\approx V_a^r(c)\int_c^d \left(\int_c^t w^q(s)ds\right)^{\frac{r}{q}} d\left(-[V_a(t)]^{-r/p}\right) \\
&+ \int_c^d \left\{\int_c^t \left(\int_x^t w^q(s)ds\right)[V_a(x)]^{q-1}dV_a(x)\right\}^{\frac{r}{q}} d\left(-[V_a(t)]^{-r/p}\right) =: I_1 + I_2.
\end{aligned}$$

Obviously,

$$I_1 \leqslant [V_a(c)]^{r/p'}\left(\int_c^d w^q(s)ds\right)^{\frac{r}{q}} \ll B_b^r.$$

To estimate I_2 we write

$$\begin{aligned}
&\int_c^t \left(\int_x^t w^q(s)ds\right)[V_a(x)]^{q-1}dV_a(x) \\
&\quad = \int_c^t \left\{\left(\int_x^t w^q(s)ds\right)[V_a(x)]^{q-1+q/2p}\right\}[V_a(x)]^{-q/2p}dV_a(x)
\end{aligned}$$

$$\leqslant \left\{ \int_c^t \left(\int_x^t w^q(s)ds \right)^{\frac{r}{q}} [V_a(x)]^{(q-1+q/2p)r/q} dV_a(x) \right\}^{\frac{q}{r}} \left(\int_c^t [V_a(x)]^{-1/2} dV_a(x) \right)^{\frac{q}{p}}$$

$$\ll [V_a(t)]^{q/2p} \left(\int_c^t \left(\int_x^t w^q(s)ds \right)^{\frac{r}{q}} [V_a(x)]^{(q-1+q/2p)r/q} dV_a(x) \right)^{\frac{q}{r}}.$$

By applying this estimate and the equivalence (2.42) and (2.43) we find that

$$I_2 \ll \int_c^d \left\{ \int_c^t \left(\int_x^t w^q(s)ds \right)^{\frac{r}{q}} [V_a(x)]^{(q-1+q/2p)r/q} dV_a(x) \right\} [V_a(t)]^{r/2p} d\left(-[V_a(t)]^{-r/p} \right)$$

$$\ll \int_c^d \left(\int_x^d w^q(s)ds \right)^{\frac{r}{q}} [V_a(x)]^{(q-1+q/2p)r/q} \left(\int_x^d [V_a(t)]^{r/2p-r/p-1} dV_a(t) \right) dV_a(x)$$

$$\ll \int_c^d \left(\int_x^d w^q(s)ds \right)^{\frac{r}{q}} [V_a(x)]^{r/q'} dV_a(x) \ll B_b^r.$$

Thus, since $V_a(d) = \infty$, we get that $\mathbb{B}_b^r \approx I_1 + I_2 \ll B_b^r \ll C^r$. If $0 < V_a(d) < \infty$, it follows from (2.36) with $f(x) = v^{p'-1}(x)$ that

$$C \geqslant [V_a(d)]^{-1/p} \left(\int_c^d w^q(x) V_a^q(x) dx \right)^{\frac{1}{q}}.$$

By combining this estimate and the previous one we find in view of (2.49) that $\mathbb{B}_b \ll C$. Now (2.44) is proved. The equivalence (2.44) and (2.45) follows from (2.49).

LEMMA 2.4. *Let $0 < q < p < \infty$, $p > 1$, $1/r = 1/q - 1/p$ and let $0 \leqslant c < d \leqslant \infty$, $0 < b \leqslant \infty$. Suppose that the function $a(x)$ is differentiable, strictly increasing and such that $0 < a(x) \leqslant b$, $x \in (c,d]$, and let the operator T be defined by* (2.39)*. Then*

$$\|T\|^r_{L_p(a(c),b)\to L_q(c,d)} \approx B_a := \int_c^d \left(\int_c^t w^q(x)dx \right)^{\frac{r}{p}} \left(\int_{a(t)}^b v^{p'}(y)dy \right)^{\frac{r}{p'}} w^q(t)dt \quad (2.50)$$

$$\approx \left(\int_c^d w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{a(d)}^b v^{p'}(y)dy \right)^{\frac{r}{p'}} \quad (2.51)$$

$$+ \int_c^d \left(\int_c^t w^q(x)dx \right)^{\frac{r}{q}} d\left[-\left(\int_{a(t)}^b v^{p'}(y)dy \right)^{\frac{r}{p'}} \right],$$

$$\|T\|^r_{L_p(a(c),b)\to L_q(c,d)}$$

$$\approx \mathbb{B}_a := \int_c^d \left(\int_t^d \left[\int_{a(x)}^b v^{p'}(y)dy \right]^q w^q(x)dx \right)^{\frac{r}{p}} \left(\int_{a(t)}^b v^{p'}(y)dy \right)^{q-\frac{r}{p}} w^q(t)dt \quad (2.52)$$

$$\approx \left(\int_c^d \left[\int_{a(x)}^b v^{p'}(y)dy \right]^q w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{a(c)}^b v^{p'}(y)dy \right)^{-\frac{r}{p}} \quad (2.53)$$

$$+ \int_c^d \left(\int_t^d \left[\int_{a(x)}^b v^{p'}(y)dy \right]^q w^q(x)dx \right)^{\frac{r}{q}} d\left(\int_{a(t)}^b v^{p'}(y)dy \right)^{-\frac{r}{p}}.$$

2.2. Fairway-function and technical lemmas.

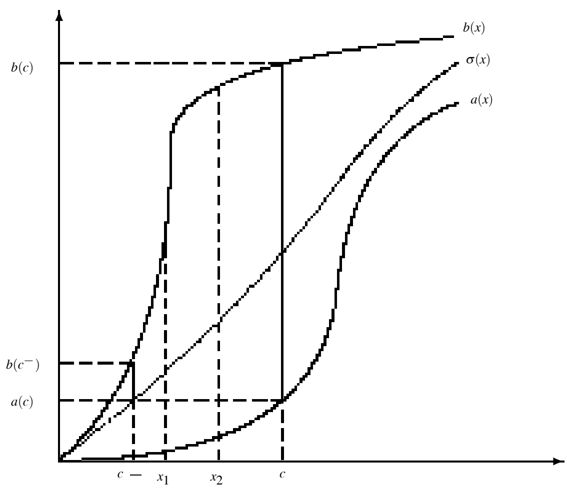
We start with the following definition.

DEFINITION 2.4. Given boundary functions $a(x)$ and $b(x)$, satisfying the conditions (1.2), a number $p \in (1,\infty)$ and a weight function $v(x)$ such that $0 < v(x) < \infty$ a.e. $x \in \mathbb{R}^+$ and $v^{p'}(x)$ is locally integrable on $\mathbb{R}^+$, we define the *fairway-function* $\sigma(x)$ such that $a(x) < \sigma(x) < b(x)$ and

$$\int_{a(x)}^{\sigma(x)} v^{p'}(y)dy = \int_{\sigma(x)}^{b(x)} v^{p'}(y)dy \quad \text{for all } x \in \mathbb{R}^+. \tag{2.54}$$

It is possible to prove that the fairway–function is differentiable and strictly increasing.

LEMMA 2.5. *Let the functions* $a(x)$ *and* $b(x)$ *satisfy the conditions* (1.2) *and* $\sigma(x)$ *is the fairway-function. For any* $c \in (0,\infty)$ *put* $c^- = \sigma^{-1}(a(c))$ *and let* $[\mathscr{N}^-]$ *be the integral part of the number*



$$\mathscr{N}^- := \log_2 \frac{\int_{a(c)}^{b(c)} v^{p'}(y)dy}{\int_{a(c)}^{b(c^-)} v^{p'}(y)dy},$$

that is

$$2^{[\mathscr{N}^-]} \leqslant \frac{\int_{a(c)}^{b(c)} v^{p'}(y)dy}{\int_{a(c)}^{b(c^-)} v^{p'}(y)dy} < 2^{[\mathscr{N}^-]+1}.$$

Let the point sequence $\{x_j\}_{j=0}^{j_b}$, *where* $j_b = \begin{cases} [\mathscr{N}^-], & if \;\; \mathscr{N}^- = [\mathscr{N}^-] \\ [\mathscr{N}^-]+1, & if \;\; \mathscr{N}^- > [\mathscr{N}^-] \end{cases}$, *be defined by:*

(1) $x_0 = c^-, \quad x_{j_b} = c;$

(2) *if* $[\mathscr{N}^-] = 0$ *or* $\mathscr{N}^- = [\mathscr{N}^-] = 1$, *then* $j_b = 1$;

(3) *if* $\mathscr{N}^- > 1$, *then the points* x_j *for* $1 \leqslant j \leqslant [\mathscr{N}^-]$ *are taken so that*

$$\int_{a(c)}^{b(x_j)} v^{p'}(y)dy = 2\int_{a(c)}^{b(x_{j-1})} v^{p'}(y)dy. \tag{2.55}$$

Then for any $t \in [x_j, x_{j+1}],\ 0 \leqslant j \leqslant j_b - 1$, *we have that*

$$\int_{a(t)}^{b(t)} v^{p'}(y)dy \approx \int_{a(x_j)}^{b(x_{j+1})} v^{p'}(y)dy, \tag{2.56}$$

and for all $t \in [c^-, c]$

$$\int_{a(t)}^{b(t)} v^{p'}(y)dy < \int_{a(c^-)}^{b(c)} v^{p'}(y)dy \approx \int_{a(c)}^{b(c)} v^{p'}(y)dy. \tag{2.57}$$

Moreover, if $j_b \geqslant 2$ *and* $0 \leqslant j \leqslant j_b - 2$ *it holds that*

$$\int_{a(c)}^{b(x_{j+l})} v^{p'}(y)dy = 2^l \int_{a(c)}^{b(x_j)} v^{p'}(y)dy \quad \text{for all} \quad l \in \{1, \dots, j_b - j - 1\}. \tag{2.58}$$

Proof. Clearly

$$\int_{a(x_{j+1})}^{b(x_j)} v^{p'}(y)dy < \int_{a(t)}^{b(t)} v^{p'}(y)dy < \int_{a(x_j)}^{b(x_{j+1})} v^{p'}(y)dy \tag{2.59}$$

is trivial for any $t \in [x_j, x_{j+1}],\ 0 \leqslant j \leqslant j_b - 1$. First we prove (2.56) for $t = x_j$. From the definition x_{j+1} we have

$$\int_{a(x_j)}^{b(x_{j+1})} v^{p'}(y)dy \overset{(2.55)}{=} \int_{a(x_j)}^{a(c)} v^{p'}(y)dy + 2\int_{a(c)}^{b(x_j)} v^{p'}(y)dy \leqslant 3\int_{a(x_j)}^{b(x_j)} v^{p'}(y)dy. \tag{2.60}$$

Now (2.59) and (2.60) yield (2.56) for $t = x_j$. If $t = x_{j+1}$, then in view of Lemma's condition

$$\begin{aligned}\int_{a(x_{j+1})}^{b(x_{j+1})} v^{p'}(y)dy &\geqslant \int_{a(c)}^{b(x_{j+1})} v^{p'}(y)dy = \int_{\sigma(c^-)}^{b(x_{j+1})} v^{p'}(y)dy \\ &\geqslant \int_{\sigma(x_j)}^{b(x_j)} v^{p'}(y)dy + \int_{b(x_j)}^{b(x_{j+1})} v^{p'}(y)dy = \frac{1}{2}\int_{a(x_j)}^{b(x_j)} v^{p'}(y)dy \\ &\quad + \int_{b(x_j)}^{b(x_{j+1})} v^{p'}(y)dy \geqslant \frac{1}{2}\int_{a(x_j)}^{b(x_{j+1})} v^{p'}(y)dy.\end{aligned} \tag{2.61}$$

Thus, (2.61) and (2.59) imply (2.56) for $t = x_{j+1}$. For $t \in (x_j, x_{j+1})$ we write

$$\begin{aligned}\int_{a(t)}^{b(t)} v^{p'}(y)dy &\geqslant \int_{a(c)}^{b(x_j)} v^{p'}(y)dy \overset{(2.55)}{=} \frac{1}{2}\int_{a(c)}^{b(x_{j+1})} v^{p'}(y)dy = \frac{1}{2}\int_{\sigma(c^-)}^{b(x_{j+1})} v^{p'}(y)dy \\ &\geqslant \frac{1}{2}\int_{\sigma(x_{j+1})}^{b(x_{j+1})} v^{p'}(y)dy = \frac{1}{4}\int_{a(x_{j+1})}^{b(x_{j+1})} v^{p'}(y)dy \overset{(2.61)}{\geqslant} \frac{1}{8}\int_{a(x_j)}^{b(x_{j+1})} v^{p'}(y)dy\end{aligned}$$

and in view of (2.59) the proof of (2.56) for $t \in (x_j, x_{j+1})$ is completed.

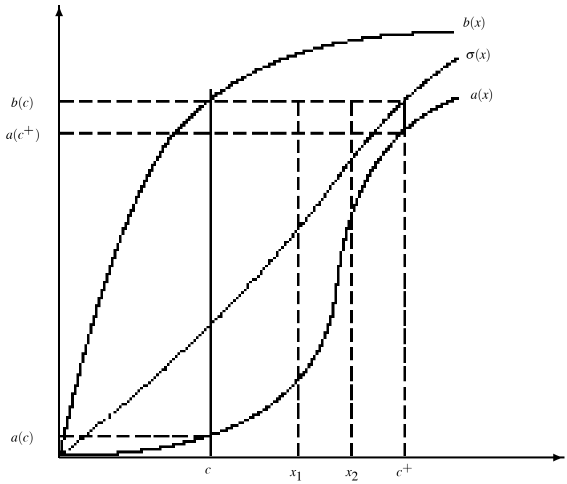
The left inequality in (2.57) is obvious. Since $\int_{a(c)}^{b(c^-)} v^{p'}(y)dy = \int_{a(c^-)}^{a(c)} v^{p'}(y)dy$, then

$$\int_{a(c^-)}^{b(c)} v^{p'}(y)dy = \int_{a(c)}^{b(c)} v^{p'}(y)dy + \int_{a(c)}^{b(c^-)} v^{p'}(y)dy \leqslant 2\int_{a(c)}^{b(c)} v^{p'}(y)dy \tag{2.62}$$

and (2.57) follows. The relation (2.58) is a simple consequence of (2.55).

The proof of the next lemma is analogous.

LEMMA 2.6. *Let the functions* $a(x)$ *and* $b(x)$ *satisfy the conditions* (1.2) *and* $\sigma(x)$ *is the fairway-function. For any* $c \in (0,\infty)$ *put* $c^+ = \sigma^{-1}(b(c))$ *and let* $[\mathcal{N}^+]$ *be the integral part of the number*



$$\mathcal{N}^+ := \log_2 \frac{\int_{a(c)}^{b(c)} v^{p'}(y)dy}{\int_{a(c^+)}^{b(c)} v^{p'}(y)dy},$$

that is

$$2^{[\mathcal{N}^+]} \leqslant \frac{\int_{a(c)}^{b(c)} v^{p'}(y)dy}{\int_{a(c^+)}^{b(c)} v^{p'}(y)dy} < 2^{[\mathcal{N}^+]+1}.$$

Let the point sequence $\{x_j\}_{j=0}^{j_a}$, *where* $j_a = \begin{cases} [\mathcal{N}^+], & if\ \ \mathcal{N}^+ = [\mathcal{N}^+] \\ [\mathcal{N}^+]+1, & if\ \ \mathcal{N}^+ > [\mathcal{N}^+] \end{cases}$, *be defined by:*

(1) $x_0 = c, \quad x_{j_a} = c^+$;

(2) *if* $[\mathcal{N}^+] = 0$ *or* $\mathcal{N}^+ = [\mathcal{N}^+] = 1$ *then* $j_a = 1$;

(3) *if* $\mathcal{N}^+ > 1$ *then the points* x_j *for* $1 \leqslant j \leqslant [\mathcal{N}^+]$ *are taken so that*

$$\int_{a(x_j)}^{b(c)} v^{p'}(y)dy = \frac{1}{2}\int_{a(x_{j-1})}^{b(c)} v^{p'}(y)dy. \tag{2.63}$$

Then for any $t \in [x_j, x_{j+1}]$, $0 \leqslant j \leqslant j_a - 1$, *we have* (2.56)*, and for all* $t \in [c, c^+]$

$$\int_{a(t)}^{b(t)} v^{p'}(y)dy \leqslant \int_{a(c)}^{b(c^+)} v^{p'}(y)dy \approx \int_{a(c)}^{b(c)} v^{p'}(y)dy. \tag{2.64}$$

Moreover, if $j_a \geqslant 2$ *and* $0 \leqslant j \leqslant j_a - 2$ *it holds that*

$$2^l \int_{a(x_{j+l})}^{b(c)} v^{p'}(y)dy = \int_{a(x_j)}^{b(c)} v^{p'}(y)dy \quad \text{for all } l \in \{1,\dots,j_a - j - 1\}. \qquad (2.65)$$

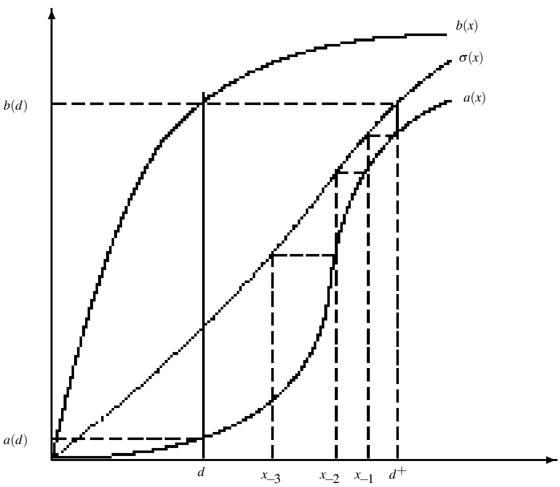
In conclusion of the section we provide two lemmas with a different kind of decomposition than in the previous two lemmas.

LEMMA 2.7. *Let the functions* $a(x)$ *and* $b(x)$ *satisfy the conditions* (1.2) *and* $\sigma(x)$ *is the fairway-function. For any* $d \in (0,\infty)$ *put* $d^- = \sigma^{-1}(a(d))$, $d^+ = \sigma^{-1}(b(d))$ *and let the point sequence* $\{x_j\}_{i=-j_a}^0$ *be defined by:*

(1) $x_{-j_a} = d$, $x_0 = d^+$;

(2) *if* $(d^+)^- \leqslant d$, *then* $j_a = 1$;

(3) *if* $(d^+)^- > d$, *then* $j_a > 1$ *and* $x_{j-1} = (x_j)^-$, *where* $(x_j)^- > d$ *and* $-j_a + 2 \leqslant j \leqslant 0$.



Then for any $t \in [x_j, x_{j+1}]$, $-j_a \leqslant j \leqslant -1$, *we have* (2.56)*. Moreover, if* $d \leqslant x^- \leqslant t \leqslant x \leqslant d^+$,

$$\int_{a(t)}^{b(t)} v^{p'}(y)dy \approx \int_{a(x^-)}^{b(x)} v^{p'}(y)dy \approx \int_{a(x)}^{b(x)} v^{p'}(y)dy. \qquad (2.66)$$

Proof. We start with the proof of (2.66). Since $\int_{a(x^-)}^{a(x)} v^{p'}(y)dy = \int_{a(x)}^{b(x^-)} v^{p'}(y)dy$ and $d \leqslant x^- \leqslant t \leqslant x \leqslant d^+$, then

$$\begin{aligned}\int_{a(t)}^{b(t)} v^{p'}(y)dy &\leqslant \int_{a(x^-)}^{b(x)} v^{p'}(y)dy = \int_{a(x)}^{b(x^-)} v^{p'}(y)dy + \int_{a(x)}^{b(x)} v^{p'}(y)dy \\ &\leqslant 2\int_{a(x)}^{b(x)} v^{p'}(y)dy \leqslant 2\int_{a(x^-)}^{b(x)} v^{p'}(y)dy, \qquad (2.67)\end{aligned}$$

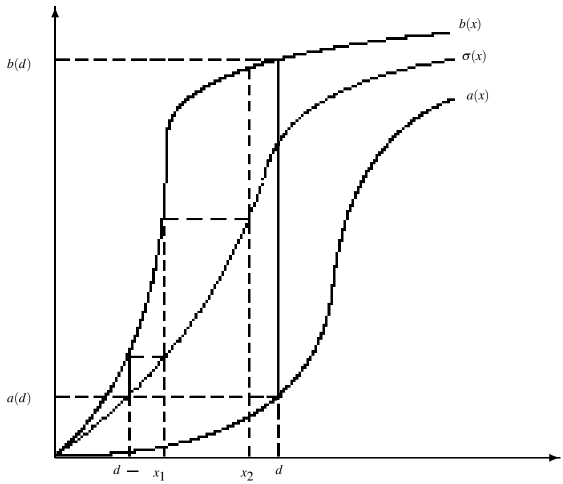
and the second equivalence in (2.66) is proved. On the other hand, in view of $d \leqslant x^- \leqslant t \leqslant x \leqslant d^+$ and Lemma's condition we have that

$$\int_{a(t)}^{b(t)} v^{p'}(y)dy \geqslant \int_{a(x)}^{b(d)} v^{p'}(y)dy = \int_{a(x)}^{\sigma(d^+)} v^{p'}(y)dy \geqslant \int_{a(x)}^{\sigma(x)} v^{p'}(y)dy$$

$$= \frac{1}{2}\int_{a(x)}^{b(x)} v^{p'}(y)dy \overset{(2.67)}{\geqslant} \frac{1}{4}\int_{a(x^-)}^{b(x)} v^{p'}(y)dy, \tag{2.68}$$

and now (2.66) is proved. Since $d \leqslant x_j = (x_{j+1})^- \leqslant t \leqslant x_{j+1} \leqslant d^+$, (2.56) follows from (2.66).

LEMMA 2.8. *Let the functions $a(x)$ and $b(x)$ satisfy the conditions* (1.2) *and $\sigma(x)$ is the fairway-function. For any $d \in (0,\infty)$ put $d^- = \sigma^{-1}(a(d))$, $d^+ = \sigma^{-1}(b(d))$ and let the point sequence $\{x_j\}_{j=0}^{j_b}$ be given by:*



(1) $x_0 = d^-$, $x_{j_b} = d$;

(2) *if* $(d^-)^+ \geqslant d$, *then* $j_b = 1$;

(3) *if* $(d^-)^+ < d$, *then* $j_b > 1$ *and* $x_{j+1} = (x_j)^+$, *where* $(x_j)^+ < d$ *and* $0 \leqslant j \leqslant j_b - 2$.

Then for any $t \in [x_j, x_{j+1}]$, $0 \leqslant j \leqslant j_b - 1$, we have (2.56). *Moreover, if $d^- \leqslant x \leqslant t \leqslant x^+ \leqslant d$,*

$$\int_{a(t)}^{b(t)} v^{p'}(y)dy \approx \int_{a(x)}^{b(x^+)} v^{p'}(y)dy \approx \int_{a(x)}^{b(x)} v^{p'}(y)dy. \tag{2.69}$$

3. Block-diagonal method

3.1. Key lemma.

We need the following assertion about a block-diagonal operator.

LEMMA 3.1. *Let $U = \bigsqcup_k U_k$ and $V = \bigsqcup_k V_k$ be unions of non-overlapping measurable sets and $T = \sum_k T_k$, where $T_k\colon\ L_p(U_k) \to L_q(V_k)$. Then*

$$\|T\|_{L_p(U)\to L_q(V)} = \sup_k \|T_k\|_{L_p(U_k)\to L_q(V_k)}, \qquad 0 < p \leqslant q < \infty \tag{3.1}$$

and

$$\|T\|_{L_p(U)\to L_q(V)} \approx \left(\sum_k \|T_k\|^r_{L_p(U_k)\to L_q(V_k)}\right)^{1/r}, \quad 0 < q < p < \infty, \quad 1/r = 1/q - 1/p. \tag{3.2}$$

Proof. Let $0 < p \leqslant q < \infty$. Since $\|Tf\|_{L_q(V)} \geqslant \|T_k f\|_{L_q(V_k)}$ for all k, the lower estimate in (3.1) is trivial. For the upper bound, applying Jensen's inequality, we write

$$\begin{aligned} \|Tf\|^q_{L_q(V)} &= \sum_k \|T_k f\|^q_{L_q(V_k)} \leqslant \left(\sup_k \|T_k\|_{L_p(U_k)\to L_q(V_k)}\right)^q \sum_k \|f\chi_k\|^q_{L_p(U_k)} \\ &\leqslant \left(\sup_k \|T_k\|_{L_p(U_k)\to L_q(V_k)}\right)^q \|f\|^q_{L_p(U)}. \end{aligned} \tag{3.3}$$

Let $0 < q < p < \infty$. The upper bound in (3.2) follows similar to (3.3) by application of Hölder's inequality. For the lower bound let $0 < \lambda < 1$ and the functions $f_k \in L_p(U_k)$ are such that

$$\lambda \|T_k\|_{L_p(U_k)\to L_q(V_k)} \|f_k\|_{L_p(U_k)} \leqslant \|T_k f_k\|_{L_p(U_k)}.$$

Because of homogeneity we can choose f_k so that

$$\|f_k\|_{L_p(U_k)} = \|T_k\|^{r/p}_{L_p(U_k)\to L_q(V_k)}.$$

If we put $f = \sum_k \chi_{U_k} f_k$, then in view of $r/q = r/p + 1$

$$\begin{aligned} \lambda^q \sum_k \|T_k\|^r_{L_p(U_k)\to L_q(V_k)} &= \lambda^q \sum_k \left(\|T_k\|_{L_p(U_k)\to L_q(V_k)} \|f_k\|_{L_p(U_k)}\right)^q \\ &\leqslant \sum_k \|T_k f_k\|^q_{L_q(V_k)} = \|Tf\|^q_{L_q(V)} \leqslant \|T\|^q_{L_p(U)\to L_q(V)} \|f\|^q_{L_p(U)} \\ &= \|T\|^q_{L_p(U)\to L_q(V)} \left(\sum_k \|T_k\|^r_{L_p(U_k)\to L_q(V_k)}\right)^{q/p} \end{aligned}$$

and the lower bound in (3.2) follows by tending $\lambda \to 1$.

3.2. Integral operators with Oinarov type kernels.

We demonstrate the block-diagonal method for the operator (1.6)

$$\mathcal{K}^* f(x) := w(x) \int_{a(x)}^{b(x)} k(y,x) f(y) v(y) dy$$

with a kernel $k(y,x) \geqslant 0$ satisfying

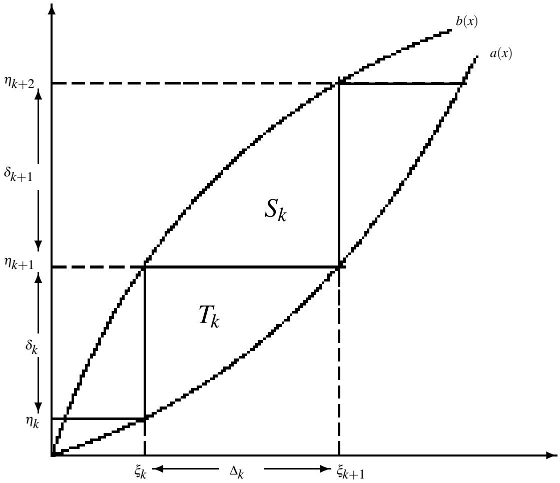
$$k(y,x) \approx k(y,z) + k(a(z),x), \qquad x \leqslant z, \qquad a(z) \leqslant y \leqslant b(x). \tag{3.4}$$

Given functions $a(x)$ and $b(x)$ satisfying (1.2) we take a sequence of points $\{\xi_k\}_{k\in\mathbb{Z}} \subset (0,\infty)$ such that

$$\xi_0 = 1, \qquad \xi_k = (a^{-1} \circ b)^k (1), \qquad k \in \mathbb{Z}, \tag{3.5}$$

and put

$$\eta_k = a(\xi_k) = b(\xi_{k-1}), \qquad \Delta_k = [\xi_k, \xi_{k+1}), \qquad \delta_k = [\eta_k, \eta_{k+1}), \qquad k \in \mathbb{Z}. \tag{3.6}$$



Breaking the semiaxis $(0,\infty)$ into intervals by points of the sequence $\{\xi_k\}_{k\in\mathbb{Z}}$, we decompose the operator $\mathscr{K}^*$ into the sum

$$\mathscr{K}^* = \mathscr{T} + \mathscr{S} \tag{3.7}$$

of block-diagonal operators $\mathscr{T}$ and $\mathscr{S}$ such that

$$\mathscr{T} = \sum_{k\in\mathbb{Z}} T_k, \qquad \mathscr{S} = \sum_{k\in\mathbb{Z}} S_k, \tag{3.8}$$

where

$$T_k f(x) = w(x)\int_{a(x)}^{a(\xi_{k+1})} k(y,x)f(y)v(y)dy, \qquad T_k : L_p(\delta_k) \to L_q(\Delta_k),$$

$$S_k f(x) = w(x)\int_{b(\xi_k)}^{b(x)} k(y,x)f(y)v(y)dy, \qquad S_k : L_p(\delta_{k+1}) \to L_q(\Delta_k).$$

$\mathscr{K}^*$, $\mathscr{T}$ and $\mathscr{S}$ are integral operators with non-negative kernels, then

$$\|\mathscr{K}^*\|_{L_p\to L_q} \approx \|\mathscr{T}\|_{L_p\to L_q} + \|\mathscr{S}\|_{L_p\to L_q}.$$

Since $\bigsqcup \Delta_k = \bigsqcup \delta_k = (0,\infty)$ by Lemma 3.1 we can estimate the norms of $\mathscr{T}$ and $\mathscr{S}$ via the norms of T_k and S_k. Moreover, kernels $k(y,x)$ of the operators T_k and S_k satisfy the condition (3.4) for $x \leqslant z,\ x \in \Delta_k$ and

$$a(z) \leqslant y \leqslant a(\xi_{k+1}), \quad b(\xi_k) \leqslant y \leqslant b(x), \tag{3.9}$$

respectively. It allows us to apply preliminary results (see Section 2).

More precisely we have the following theorem.

THEOREM 3.1. *Let* $1 < p \leqslant q < \infty$. *Then*

$$\|\mathscr{K}^*\|_{L_p\to L_q} \approx \mathscr{A}^* :\ = \mathscr{A}_0^* + \mathscr{A}_1^*, \tag{3.10}$$

where

$$\mathscr{A}_0^* := \sup_{s>0} \mathscr{A}_0^*(s) = \sup_{s>0} \sup_{s\leqslant t\leqslant a^{-1}(b(s))} \left(\int_s^t k^q(a(t),x)w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} v^{p'}(y)dy\right)^{\frac{1}{p'}},$$

$$\mathscr{A}_1^* := \sup_{s>0} \mathscr{A}_1^*(s) = \sup_{s>0} \sup_{s\leqslant t\leqslant a^{-1}(b(s))} \left(\int_s^t w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} k^{p'}(y,t)v^{p'}(y)dy\right)^{\frac{1}{p'}}.$$

Moreover, $\mathscr{K}^*$ *is compact if and only if* $\mathscr{A}^* < \infty$ *and* $\lim_{s\to 0} \mathscr{A}_i^*(s) = \lim_{s\to\infty} \mathscr{A}_i^*(s) = 0,\ i = 0,1.$

If $1 < q < p < \infty$, *then*

$$\|\mathscr{K}^*\|_{L_p\to L_q} \approx \mathscr{B}^* :\ = \left(\sum_k \left[\left(\mathscr{B}_{k,1}^*\right)^r + \left(\mathscr{B}_{k,2}^*\right)^r + \left(\mathscr{B}_{k,3}^*\right)^r + \left(\mathscr{B}_{k,4}^*\right)^r\right]\right)^{\frac{1}{r}}, \quad (3.11)$$

where

$$\mathscr{B}_{k,1}^* := \left(\int_{a(\xi_k)}^{a(\xi_{k+1})} \left[\int_{\xi_k}^{a^{-1}(t)} k^q(t,x)w^q(x)dx\right]^{\frac{r}{q}} \left[\int_t^{a(\xi_{k+1})} v^{p'}(y)dy\right]^{\frac{r}{q'}} v^{p'}(t)dt\right)^{\frac{1}{r}},$$

$$\mathscr{B}_{k,2}^* := \left(\int_{\xi_k}^{\xi_{k+1}} \left[\int_{\xi_k}^{t} w^q(x)dx\right]^{\frac{r}{p}} \left[\int_{a(t)}^{a(\xi_{k+1})} k^{p'}(y,t)v^{p'}(y)dy\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}},$$

$$\mathscr{B}_{k,3}^* := \left(\int_{b(\xi_k)}^{b(\xi_{k+1})} \left[\int_{b^{-1}(t)}^{\xi_{k+1}} k^q(a(\xi_{k+1}),x)w^q(x)dx\right]^{\frac{r}{q}} \left[\int_{b(\xi_k)}^{t} v^{p'}(y)dy\right]^{\frac{r}{q'}} v^{p'}(t)dt\right)^{\frac{1}{r}},$$

$$\mathscr{B}_{k,4}^* := \left(\int_{\xi_k}^{\xi_{k+1}} \left[\int_{t}^{\xi_{k+1}} w^q(x)dx\right]^{\frac{r}{p}} \left[\int_{b(\xi_k)}^{b(t)} k^{p'}(y,\xi_{k+1})v^{p'}(y)dy\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}},$$

and $\mathscr{K}^*$ *is compact if and only if* $\mathscr{B}^* < \infty$.

Proof. Put

$$\|\mathscr{K}^*\| = \|\mathscr{K}^*\|_{L_p\to L_q}, \qquad \|T_k\| = \|T_k\|_{L_p(\delta_k)\to L_q(\Delta_k)}, \qquad \|S_k\| = \|S_k\|_{L_p(\delta_{k+1})\to L_q(\Delta_k)}.$$

Let $1 < p \leqslant q < \infty$. It follows from (3.7) – (3.8) and Lemma 3.1 that

$$\|\mathscr{K}^*\| \approx \sup_k \|T_k\| + \sup_k \|S_k\|. \quad (3.12)$$

The norm of the operator T_k is characterized by Corollary 2.2 (a):

$$\|T_k\| \approx \sup_{t\in\Delta_k} \left(\int_{\xi_k}^{t} k^q(a(t),x)w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{a(t)}^{a(\xi_{k+1})} v^{p'}(y)dy\right)^{\frac{1}{p'}}$$
$$+ \sup_{t\in\Delta_k} \left(\int_{\xi_k}^{t} w^q(y)dy\right)^{\frac{1}{q}} \left(\int_{a(t)}^{a(\xi_{k+1})} k^{p'}(y,t)v^{p'}(y)dy\right)^{\frac{1}{p'}}.$$

For the kernel of the operator S_k it follows from (3.4) provided $z=\xi_{k+1},\ x\in\Delta_k$ and (3.9) that

$$k(y,x)\approx k(y,\xi_{k+1})+k(a(\xi_{k+1}),x).$$

Therefore, $S_kf(x)\approx S_{k,1}f(x)+S_{k,2}f(x),$ where

$$S_{k,1}f(x)=w(x)\int_{b(\xi_k)}^{b(x)}k(y,\xi_{k+1})f(y)v(y)dy, \qquad x\in\Delta_k,$$

$$S_{k,2}f(x)=w(x)k(a(\xi_{k+1}),x)\int_{b(\xi_k)}^{b(x)}f(y)v(y)dy, \qquad x\in\Delta_k.$$

The norm estimates for $S_{k,1}$ and $S_{k,2}$ follow from Lemma 2.1:

$$\|S_k\|\approx\sup_{t\in\Delta_k}\left(\int_t^{\xi_{k+1}}k^q(a(\xi_{k+1}),x)w^q(x)dx\right)^{\frac{1}{q}}\left(\int_{b(\xi_k)}^{b(t)}v^{p'}(y)dy\right)^{\frac{1}{p'}}$$
$$+\sup_{t\in\Delta_k}\left(\int_t^{\xi_{k+1}}w^q(x)dx\right)^{\frac{1}{q}}\left(\int_{b(\xi_k)}^{b(t)}k^{p'}(y,\xi_{k+1})v^{p'}(y)dy\right)^{\frac{1}{p'}}.$$

Define the following functions

$$\mathscr{A}_0(s,t)=\left(\int_s^t k^q(a(t),x)w^q(x)dx\right)^{\frac{1}{q}}\left(\int_{a(t)}^{b(s)}v^{p'}(y)dy\right)^{\frac{1}{p'}},$$

$$\mathscr{A}_1(s,t)=\left(\int_s^t w^q(x)dx\right)^{\frac{1}{q}}\left(\int_{a(t)}^{b(s)}k^{p'}(y,t)v^{p'}(y)dy\right)^{\frac{1}{p'}},$$

where $s<t<a^{-1}(b(s))$. Then

$$\|T_k\|\approx\sup_{t\in\Delta_k}\mathscr{A}_0(\xi_k,t)+\sup_{t\in\Delta_k}\mathscr{A}_1(\xi_k,t), \tag{3.13}$$

$$\|S_k\|\approx\sup_{t\in\Delta_k}\mathscr{A}_0(t,\xi_{k+1})+\sup_{t\in\Delta_k}\mathscr{A}_1(t,\xi_{k+1}) \tag{3.14}$$

and

$$\mathscr{A}_0^*=\sup_{s>0}\sup_{s\leqslant t\leqslant a^{-1}(b(s))}\mathscr{A}_0(s,t), \qquad \mathscr{A}_1^*=\sup_{s>0}\sup_{s\leqslant t\leqslant a^{-1}(b(s))}\mathscr{A}_1(s,t),$$

$$\sup_{t\in\Delta_k}\mathscr{A}_i(\xi_k,t)\leqslant\sup_{\xi_k\leqslant t\leqslant a^{-1}(b(\xi_k))}\mathscr{A}_i(\xi_k,t)\leqslant\mathscr{A}_i^*, \qquad i=0,1,$$

$$\sup_{t\in\Delta_k}\mathscr{A}_i(t,\xi_{k+1})\leqslant\sup_{t\leqslant\xi_{k+1}\leqslant a^{-1}(b(t))}\mathscr{A}_i(t,\xi_{k+1})\leqslant\mathscr{A}_i^*, \qquad i=0,1.$$

Now it follows from (3.12) – (3.14) that

$$\|\mathscr{K}^*\|\ll\mathscr{A}_0^*+\mathscr{A}_1^*.$$

For the lower bound in (3.10) we suppose that $\|\mathscr{K}^*\| < \infty$ and $s < x < t < a^{-1}(b(s))$. If

$$f(y) = \chi_{[a(t),b(s)]}(y)v^{p'-1}(y),$$

then

$$\mathscr{K}^* f(x) \geqslant w(x)\int_{a(t)}^{b(s)} k(y,x)v^{p'}(y)dy.$$

The condition (3.4) with $z = t$ implies $k(y,x) \gg k(a(t),x)$, therefore

$$\mathscr{K}^* f(x) \gg w(x)k(a(t),x)\int_{a(t)}^{b(s)} v^{p'}(y)dy.$$

Hence,

$$\|\mathscr{K}^*\| \geqslant \frac{\|\mathscr{K}^* f\|_q}{\|f\|_p} \geqslant \left(\int_s^t k^q(a(t),x)w^q(x)dx\right)^{\frac{1}{q}}\left(\int_{a(t)}^{b(s)} v^{p'}(y)dy\right)^{\frac{1}{p'}}.$$

Therefore, $\|\mathscr{K}^*\| \geqslant \mathscr{A}_0(s,t)$ for all $s < t < a^{-1}(b(s))$. Consequently, $\|\mathscr{K}^*\| \geqslant \mathscr{A}_0^*$. The inequality $\|\mathscr{K}^*\| \geqslant \mathscr{A}_1^*$ can be proved analogously by applying a test function

$$g(x) = \chi_{[s,t]}(x)w^{q-1}(x)$$

into the inequality for the operator dual to $\mathscr{K}^*$.

To prove (3.11) we note that by Lemma 3.1

$$\|\mathscr{K}^*\| \approx \left(\sum_k \|T_k\|^r\right)^{\frac{1}{r}} + \left(\sum_k \|S_k\|^r\right)^{\frac{1}{r}}. \tag{3.15}$$

Norms of the operators T_k and S_k are estimated with the help of Corollary 2.2 (b) and Lemma 2.3 by the following way:

$$\|T_k\| \approx \mathscr{B}_{k,1}^* + \mathscr{B}_{k,2}^*, \qquad \|S_k\| \approx \mathscr{B}_{k,3}^* + \mathscr{B}_{k,4}^* \tag{3.16}$$

and the required result follows.

The proof of compactness assertion of the theorem for $1 < p \leqslant q < \infty$ follows from representation of the operator by the sum of a compact operator and an operator with a small norm. For $1 < q < p < \infty$ the required result follows by applying Ando's theorem (see [11], [16] and [27]).

Using decomposition (3.5) for the operator $\mathscr{K}$ defined by (1.4) with the kernel $k(x,y) \geqslant 0$ satisfying the condition (1.5) we obtain the analogous result for $\mathscr{K}$.

THEOREM 3.2. *If* $1 < p \leqslant q < \infty$, *then*

$$\|\mathscr{K}\|_{L_p \to L_q} \approx \mathscr{A} := \mathscr{A}_0 + \mathscr{A}_1,$$

where

$$\mathscr{A}_0 := \sup_{t>0} \mathscr{A}_0(t) = \sup_{t>0} \sup_{b^{-1}(a(t)) \leqslant s \leqslant t} \left(\int_s^t k^q(x,b(s)) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} v^{p'}(y) dy \right)^{\frac{1}{p'}},$$

$$\mathscr{A}_1 := \sup_{t>0} \mathscr{A}_1(t) = \sup_{t>0} \sup_{b^{-1}(a(t)) \leqslant s \leqslant t} \left(\int_s^t w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} k^{p'}(s,y) v^{p'}(y) dy \right)^{\frac{1}{p'}}.$$

Moreover, $\mathscr{K}$ *is compact if and only if* $\mathscr{A} < \infty$ *and* $\lim_{t\to 0} \mathscr{A}_i(t) = \lim_{t\to\infty} \mathscr{A}_i(t) = 0, i = 0,1$.

If $1 < q < p < \infty$*, then*

$$\|\mathscr{K}\|_{L_p \to L_q} \approx \mathscr{B} := \left(\sum_k \left[\mathscr{B}_{k,1}^r + \mathscr{B}_{k,2}^r + \mathscr{B}_{k,3}^r + \mathscr{B}_{k,4}^r \right] \right)^{\frac{1}{r}},$$

where

$$\mathscr{B}_{k,1} := \left(\int_{a(\xi_k)}^{a(\xi_{k+1})} \left[\int_{\xi_k}^{a^{-1}(t)} k^q(x,b(\xi_k)) w^q(x) dx \right]^{\frac{r}{q}} \left[\int_t^{a(\xi_{k+1})} v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}},$$

$$\mathscr{B}_{k,2} := \left(\int_{\xi_k}^{\xi_{k+1}} \left[\int_{\xi_k}^t w^q(x) dx \right]^{\frac{r}{p}} \left[\int_{a(t)}^{a(\xi_{k+1})} k^{p'}(\xi_k,y) v^{p'}(y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}},$$

$$\mathscr{B}_{k,3} := \left(\int_{b(\xi_k)}^{b(\xi_{k+1})} \left[\int_{b^{-1}(t)}^{\xi_{k+1}} k^q(x,t) w^q(x) dx \right]^{\frac{r}{q}} \left[\int_{b(\xi_k)}^t v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}},$$

$$\mathscr{B}_{k,4} := \left(\int_{\xi_k}^{\xi_{k+1}} \left[\int_t^{\xi_{k+1}} w^q(x) dx \right]^{\frac{r}{p}} \left[\int_{b(\xi_k)}^{b(t)} k^{p'}(t,y) v^{p'}(y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}},$$

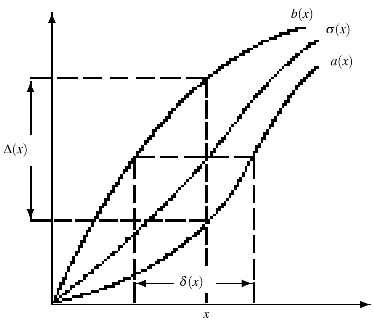
and the operator $\mathscr{K}$ *is compact if and only if* $\mathscr{B} < \infty$.

REMARK 3.1. It is curious to note, that in spite of duality of Theorems 3.1 and 3.2 the conditions (3.4) and (1.5) are independent in general, that is one of them can hold whereas the other is broken. In practice the operators (1.4) and (1.6) are almost indistinguishable and, therefore, a form of criterion entirely depends on which of conditions (3.4) or (1.5) holds. This phenomenon is caused by the following. The condition (3.4) extends (2.3) for the Volterra operator with variable lower limit, but (1.5) generalizes (2.3) for the operator with variable upper limit. Losing Volterra's form the operator ((1.4) or (1.6)) forgets the origin, but its kernel remembers.

4. Hardy-Steklov operator

4.1. Muckenhoupt and Mazya-Rosin type criteria

In this section we give complete analogs of the conditions (2.4) and (2.6) for the operator (1.1) basing on the fairway–function conception.



Put

$$\Delta(x) = [a(x), b(x)],$$
$$\delta(x) = [b^{-1}(\sigma(x)), a^{-1}(\sigma(x))],$$

where $a^{-1}(y)$ and $b^{-1}(y)$ are the functions converse to $y = a(x)$ and $y = b(x)$, respectively.

THEOREM 4.1. *Let the operator $\mathscr{H}$ of the form* (1.1) *be given with the boundary functions $a(x)$ and $b(x)$ satisfying the conditions* (1.2)*. Then for the norm of $\mathscr{H}$ and $1 < p \leqslant q < \infty$ the estimate*

$$\|\mathscr{H}\|_{L_p \to L_q} \approx \mathscr{A}_M, \tag{4.1}$$

holds, where

$$\mathscr{A}_M := \sup_{t>0} \mathscr{A}_M(t) = \sup_{t>0} \left(\int_{\delta(t)} w^q(x)dx \right)^{\frac{1}{q}} \left(\int_{\Delta(t)} v^{p'}(y)dy \right)^{\frac{1}{p'}}. \tag{4.2}$$

Moreover, $\mathscr{H} : L_p \to L_q$ is compact if and only if $\mathscr{A}_M < \infty$ and $\lim_{t\to 0} \mathscr{A}_M(t) = \lim_{t\to\infty} \mathscr{A}_M(t) = 0$.

If $0 < q < p < \infty,\ p > 1,\ 1/r = 1/q - 1/p$, then

$$\|\mathscr{H}\|_{L_p \to L_q} \approx \mathscr{B}_{MR}, \tag{4.3}$$

where

$$\mathscr{B}_{MR} := \left(\int_0^\infty \left[\int_{\delta(t)} w^q(x)dx \right]^{\frac{r}{p}} \left[\int_{\Delta(t)} v^{p'}(y)dy \right]^{\frac{r}{p'}} w^q(t)dt \right)^{\frac{1}{r}} \tag{4.4}$$

and $\mathscr{H} : L_p \to L_q$ is compact if and only if $\mathscr{B}_{MR} < \infty$.

Proof. First we consider the case $1 < p \leqslant q < \infty$. It follows from Theorem 3.1 with $k(y,x) = 1$ that $\mathscr{A}_0^* = \mathscr{A}_1^*$ and by (3.10)

$$\|\mathscr{H}\| := \|\mathscr{H}\|_{L_p \to L_q} \approx \mathbb{A} := \sup_{s>0} \sup_{s \leqslant t \leqslant a^{-1}(b(s))} \mathbb{A}(s,t), \tag{4.5}$$

where

$$\mathbb{A}(s,t) = \left(\int_s^t w^q(x)dx \right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} v^{p'}(y)dy \right)^{\frac{1}{p'}}.$$

Let $\mathbb{A}^* = \sup_{s>0} \sup_{b^{-1}(a(s))\leqslant t\leqslant s} \mathbb{A}(t,s)$. Then $\mathbb{A} = \mathbb{A}^*$ and using (2.54) we find for all $t > 0$

$$\begin{aligned}\mathscr{A}_M(t) &\approx \left(\int_{b^{-1}(\sigma(t))}^{t} w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{a(t)}^{\sigma(t)} v^{p'}(y)dy\right)^{\frac{1}{p'}}\\ &+ \left(\int_{t}^{a^{-1}(\sigma(t))} w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{\sigma(t)}^{b(t)} v^{p'}(y)dy\right)^{\frac{1}{p'}}\\ &= \mathbb{A}(b^{-1}(\sigma(t)),t) + \mathbb{A}^*(t,a^{-1}(\sigma(t))) \leqslant 2\mathbb{A}.\end{aligned}$$

It implies $\mathscr{A}_M \ll \mathbb{A}$. For the proof of the opposite inequality we put $\tau = \sigma^{-1}(b(s))$ and write

$$\mathbb{A} \leqslant \sup_{s\leqslant t\leqslant \tau} \mathbb{A}(s,t) + \sup_{\tau\leqslant t\leqslant a^{-1}(b(s))} \mathbb{A}(s,t) \ll \mathscr{A}_M.$$

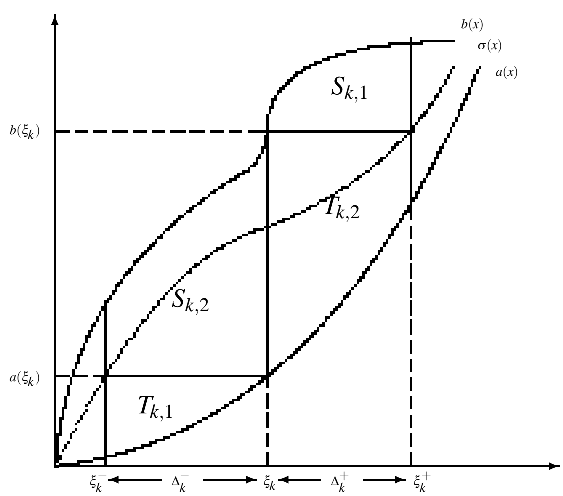
Indeed, if $s \leqslant t \leqslant \tau$, then $(s,t) \subset (b^{-1}(\sigma(t)), a^{-1}(\sigma(t)))$, $(a(t),b(s)) \subset (a(t),b(t))$ and $\sup_{s\leqslant t\leqslant\tau} \mathbb{A}(s,t) \leqslant \mathscr{A}_M$. If $s \leqslant \tau \leqslant t \leqslant a^{-1}(b(s))$, then $(s,t) \subset (s,a^{-1}(b(s))) = (b^{-1}(\sigma(\tau)),\ a^{-1}(\sigma(\tau)))$, $(a(t),b(s)) \subset (a(\tau),b(\tau))$ and $\sup_{\tau<t<a^{-1}(b(s))} \mathbb{A}(s,t) \leqslant \mathscr{A}_M$. Therefore, $\mathscr{A}_M \approx \mathbb{A}$ and (4.1) follows from (4.5). The criterion of compactness of the operator $\mathscr{H}$ follows from Theorem 3.1.

Now we consider the case $0 < q < p < \infty$, $p > 1$. For the proof (4.3) we show first that $\|\mathscr{H}\| \ll \mathscr{B}_{MR}$. To this end we introduce some notations:

$$\xi_k^- = \sigma^{-1}(a(\xi_k)), \qquad \xi_k^+ = \sigma^{-1}(b(\xi_k)), \qquad \Delta_k = [\xi_k^-, \xi_k^+] = \Delta_k^- \cup \Delta_k^+,$$
$$\Delta_k^- = [\xi_k^-, \xi_k], \qquad \Delta_k^+ = [\xi_k, \xi_k^+], \qquad k \in \mathbb{Z},$$

where $\{\xi_k\}_{k\in\mathbb{Z}}$ are given by relation (3.5). The operator $\mathscr{H}$ with $x \in \Delta_k$ splits into the sum of four operators

$$\mathscr{H} f(x) = T_{k,1} f(x) + T_{k,2} f(x) + S_{k,1} f(x) + S_{k,2} f(x), \qquad x \in \Delta_k,$$



where

$$T_{k,1}f(x) = w(x)\int_{a(x)}^{a(\xi_k)} f(y)v(y)dy, \quad x \in \Delta_k^-,$$

$$T_{k,2}f(x) = w(x)\int_{a(x)}^{b(\xi_k)} f(y)v(y)dy, \quad x \in \Delta_k^+,$$

$$S_{k,1}f(x) = w(x)\int_{b(\xi_k)}^{b(x)} f(y)v(y)dy, \quad x \in \Delta_k^+,$$

$$S_{k,2}f(x) = w(x)\int_{a(\xi_k)}^{b(x)} f(y)v(y)dy, \quad x \in \Delta_k^-.$$

Applying known estimates of these operators (see lemmas 2.1 – 2.4) we receive

$$\begin{aligned}\|T_{k,1}\|^r &\approx \int_{\xi_k^-}^{\xi_k}\left(\int_{\xi_k^-}^{t} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{a(t)}^{a(\xi_k)} v^{p'}(y)dy\right)^{\frac{r}{p'}} w^q(t)dt \\ &\leqslant \frac{q}{r}\left(\int_{a(\xi_k^-)}^{\sigma(\xi_k^-)} v^{p'}(y)dy\right)^{\frac{r}{p'}}\left(\int_{\xi_k^-}^{\xi_k} w^q(x)dx\right)^{\frac{r}{q}} = \\ &\overset{(2.54)}{=} \left(\int_{\sigma(\xi_k^-)}^{b(\xi_k^-)} v^{p'}(y)dy\right)^{\frac{r}{p'}}\left(\int_{\xi_k^-}^{\xi_k}\left(\int_{t}^{\xi_k} w^q(x)dx\right)^{\frac{r}{p}} w^q(t)dt\right).\end{aligned}$$

For $t \in [\xi_k^-, \xi_k]$ we have $a(t) \leqslant \sigma(\xi_k^-)$, $b(\xi_k^-) \leqslant b(t)$, $\xi_k \leqslant a^{-1}(\sigma(t))$, then $[\sigma(\xi_k^-), b(\xi_k^-)] \subseteq \Delta(t)$, $[t,\xi_k] \subset [t, a^{-1}(\sigma(t))] \subset \delta(t)$, therefore

$$\|T_{k,1}\|^r \ll \int_{\xi_k^-}^{\xi_k}\left(\int_{\delta(t)} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{\Delta(t)} v^{p'}(y)dy\right)^{\frac{r}{p'}} w^q(t)dt =: J_{k,1}. \tag{4.6}$$

Analogously, we find

$$\|T_{k,2}\|^r \approx \int_{\xi_k}^{\xi_k^+}\left(\int_{\xi_k}^{t} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{a(t)}^{b(\xi_k)} v^{p'}(y)dy\right)^{\frac{r}{p'}} w^q(t)dt.$$

Since for $t \in [\xi_k, \xi_k^+]$ we have $\xi_k \geqslant b^{-1}(\sigma(t))$, $b(\xi_k) \leqslant b(t)$, then $[\xi_k, t] \subset (b^{-1}(\sigma(t)), t) \subset \delta(t)$, $[a(t), b(\xi_k)] \subset \Delta(t)$, hence

$$\|T_{k,2}\|^r \ll \int_{\xi_k}^{\xi_k^+}\left(\int_{\delta(t)} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{\Delta(t)} v^{p'}(y)dy\right)^{\frac{r}{p'}} w^q(t)dt =: J_{k,2}. \tag{4.7}$$

Similarly, we prove that

$$\|S_{k,1}\|^r \ll J_{k,2}, \qquad \|S_{k,2}\|^r \ll J_{k,1}. \tag{4.8}$$

Using a decomposition of operator $\mathscr{H}$ and the estimates (4.6) – (4.8) we obtain the required inequality $\|\mathscr{H}\| \ll \mathscr{B}_{MR}$.

For the proof of the estimate $\|\mathscr{H}\| \gg \mathscr{B}_{MR}$ we suppose first that

$$\sigma(x) = x, \tag{4.9}$$

i.e. fairway is the bisectrix of the first quadrant. In this case we denote

$$x^- = a(x), \qquad x^+ = b(x), \qquad \Delta(x) = [x^-, x^+] = \Delta^-(x) \cup \Delta^+(x),$$

$$\Delta^-(x) = [x^-, x], \qquad \Delta^+(x) = [x, x^+]$$

and we name *basic* an interval of the form $[x^-, x^+]$. We need the following

DEFINITION 4.1. Let $p \in (1,\infty)$ and the functions $a(x)$, $b(x)$ and weight function $v(x)$ satisfy the conditions of Definition 2.4 provided (4.9) is fulfilled. Define $\mathscr{L}$ as the set of all absolutely continuous functions F on $(0,\infty)$ such that $\|F'/v\|_p < \infty$ and if $F \in \mathscr{L}$, then there exist mutually disjoint intervals $I_k = (\alpha_k, \beta_k) \subset (0,\infty)$ and basic intervals $J_k = [c_k^-, c_k^+]$ such that $I_k \subset J_k$, $\operatorname{supp} F \subset \bigcup_k I_k$ and $F(\alpha_k) = F(\beta_k) = 0$ for all k.

We consider the inequality

$$\|Fw\|_q \leqslant C\|F'/v\|_p, \qquad F \in \mathscr{L}, \tag{4.10}$$

with a constant C independent of $F \in \mathscr{L}$ and chosen as the least possible. We show that

$$C \leqslant \|\mathscr{H}\|. \tag{4.11}$$

To this end for any function $F \in \mathscr{L}$ we write

$$\int_0^\infty |F(x)|^q w^q(x)dx = \sum_k \int_{I_k} |F(x)|^q w^q(x)dx.$$

Since $I_k = (\alpha_k, \beta_k) \subset [c_k^-, c_k^+] = J_k$ the only three variants are possible:

$$\text{(i)} \quad \beta_k \leqslant c_k, \qquad \text{(ii)} \quad c_k \leqslant \alpha_k, \qquad \text{(iii)} \quad c_k \in I_k.$$

In the case (i) we have

$$\int_{I_k} |F(x)|^q w^q(x)dx \leqslant \int_{I_k} \left(\int_{\alpha_k}^x |F'(y)|dy\right)^q w^q(x)dx \leqslant \int_{I_k} \left(\int_{\Delta(x)} |F'(y)|dy\right)^q w^q(x)dx,$$

because $(\alpha_k, x) \subset (a(x), x) \subset \Delta(x)$ on the strength of $\alpha_k \geqslant c_k^- = a(c_k) \geqslant a(\beta_k) \geqslant a(x)$. Analogously, for the case (ii) we write

$$\int_{I_k} |F(x)|^q w^q(x)dx \leqslant \int_{I_k} \left(\int_x^{\beta_k} |F'(y)|dy\right)^q w^q(x)dx \leqslant \int_{I_k} \left(\int_{\Delta(x)} |F'(y)|dy\right)^q w^q(x)dx$$

since $(x, \beta_k) \subset (x, b(x)) \subset \Delta(x)$ which follows from the chain of inequalities $\beta_k \leqslant c_k^+ = b(c_k) \leqslant b(\alpha_k) \leqslant b(x)$. Thus, using the above arguments, we obtain for the case (iii)

$$\int_{I_k} |F(x)|^q w^q(x)dx \leqslant \int_{\alpha_k}^{c_k} \left(\int_{\alpha_k}^{x} |F'(y)|dy \right)^q w^q(x)dx + \int_{c_k}^{\beta_k} \left(\int_{x}^{\beta_k} |F'(y)|dy \right)^q w^q(x)dx$$
$$\ll \int_{I_k} \left(\int_{\Delta(x)} |F'(y)|dy \right)^q w^q(x)dx.$$

Consequently,

$$\int_0^\infty |F(x)|^q w^q(x)dx \leqslant \sum_k \int_{I_k} \left(\mathscr{H}|F'/v| \right)^q (x)dx \leqslant \|\mathscr{H}|F'/v|\|_q^q \leqslant \|\mathscr{H}\|^q \|F'/v\|_p^q$$

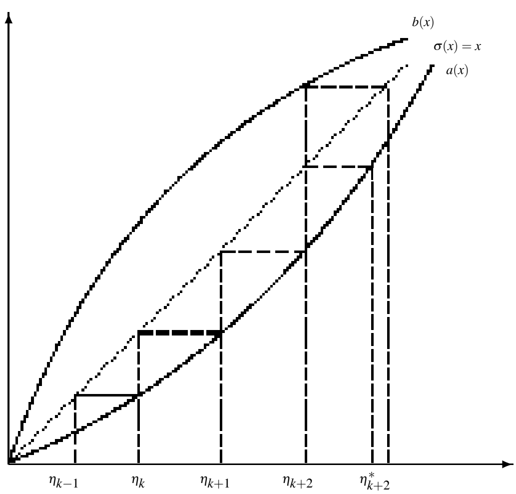
and the inequality (4.11) is proved. Thus, Theorem 4.1 will be established under the condition (4.9) if we show that

$$\mathscr{B}_{MR} \ll C. \tag{4.12}$$

For this purpose it is sufficient to prove that $\mathscr{B}_{MR}^{\pm} \ll C$, where

$$\mathscr{B}_{MR}^{\pm} = \left(\int_0^\infty \left[\int_{\delta^{\pm}(t)} w^q(x)dx \right]^{\frac{r}{p}} \left[\int_{\Delta(t)} v^{p'}(y)dy \right]^{\frac{r}{p'}} w^q(t)dt \right)^{\frac{1}{r}},$$

and $\delta^-(t) = [b^{-1}(t), t]$, $\delta^+(t) = [t, a^{-1}(t)]$. We prove the inequality $\mathscr{B}_{MR}^{+} \ll C$, arguments for $\mathscr{B}_{MR}^{-} \ll C$ are similar.



Let $\eta_0 = 1$ and the sequence $\{\eta_k\}_{k\in\mathbb{Z}} \subset (0,\infty)$ be defined by

$$\eta_{k+1} = a^{-1}(\eta_k), \qquad k \in \mathbb{Z}.$$

Put

$$\eta_k^* = \min(\eta_k^+, \eta_{k+1}).$$

For a fixed $k \in \mathbb{Z}$ we take five neighboring points η_{k-1}, η_k, η_{k+1}, η_{k+2}, η_{k+2}^* and let

$$g_k(t) = \chi_{[\eta_{k-1},\eta_{k+2}]}(t) \int_{\eta_{k-1}}^{t} \left(\int_s^{\eta_{k+2}} w^q(x)dx \right)^{\frac{r}{pq}} \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right)^{\frac{r}{pq'}} v^{p'}(s)ds.$$

On the interval $[\eta_{k-1}, \eta_{k+2}^*]$ we define the function

$$h_k(t) = \begin{cases} g_k(t), & t \notin [\eta_{k+2}, \eta_{k+2}^*], \\ g_k(\eta_{k+2})\Omega_{k+2}(t), & t \in [\eta_{k+2}, \eta_{k+2}^*], \end{cases}$$

where

$$\Omega_l(t) = \left(\int_{\eta_l}^{\eta_l^*} v^{p'}(y)dy \right)^{-1} \int_t^{\eta_l^*} v^{p'}(s)ds.$$

Put

$$\lambda_k := \int_{\eta_{k-1}}^{\eta_{k+2}} \left(\int_s^{\eta_{k+2}} w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right)^{\frac{r}{q'}} v^{p'}(s)ds. \tag{4.13}$$

and observe, that $\lambda_k \in (0,\infty)$ for all k. Now we find decomposition

$$h_k(t) = \sum_{i=1}^{3} h_{k,i}(t), \tag{4.14}$$

with $h_{k,i} \in \mathscr{L}$, $i = 1,2,3$. To this end we determine $h_{k,1}$ by

$$h_{k,1}(t) = \begin{cases} 0, & t \notin [\eta_{k-1}, \eta_k^*], \\ h_k(t), & t \in [\eta_{k-1}, \eta_k], \\ g_k(\eta_k)\Omega_k(t), & t \in [\eta_k, \eta_k^*]. \end{cases}$$

Since $\eta_k^- = \eta_{k-1}$, then $\operatorname{supp} h_{k,1} \subseteq [\eta_k^-, \eta_k^*] \subseteq [\eta_k^-, \eta_k^+]$, therefore $h_{k,1} \in \mathscr{L}$. Now, let

$$h_k^{(1)}(t) := h_k(t) - h_{k,1}(t).$$

Define

$$h_{k,2}(t) = \begin{cases} 0, & t \notin [\eta_k, \eta_{k+1}^*], \\ h_k^{(1)}(t), & t \in [\eta_k, \eta_{k+1}], \\ h_k^{(1)}(\eta_{k+1})\Omega_{k+1}(t), & t \in [\eta_{k+1}, \eta_{k+1}^*]. \end{cases}$$

Obviously, $h_{k,2} \in \mathscr{L}$ and $h_{k,3} \in \mathscr{L}$ too, where

$$h_{k,3}(t) = h_k^{(1)}(t) - h_{k,2}(t).$$

Now it is clear that (4.14) holds. We also need the following estimates

$$\|h'_{k,i}/v\|_p^p \ll \lambda_k, \qquad i = 1,2,3. \tag{4.15}$$

Write

$$\kappa_1 := \|h'_{k,1}/v\|_p^p = \int_{\eta_{k-1}}^{\eta_k} |h'_k(s)|^p v^{-p}(s)ds + g_k^p(\eta_k) \int_{\eta_k}^{\eta_k^*} |\Omega'_k(s)|^p v^{-p}(s)ds =: \kappa_{1,1} + \kappa_{1,2}.$$

Evidently,

$$\kappa_{1,1} = \int_{\eta_{k-1}}^{\eta_k} \left(\int_s^{\eta_{k+2}} w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right)^{\frac{r}{q'}} v^{p'}(s)ds \leqslant \lambda_k.$$

Hölder's inequality yields

$$g_k^p(\eta_k) \leqslant \int_{\eta_{k-1}}^{\eta_k} \left(\int_s^{\eta_{k+2}} w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right)^{\frac{r}{q'}} v^{p'}(s)ds \left(\int_{\eta_{k-1}}^{\eta_k} v^{p'}(y)dy \right)^{p-1}.$$

Since

$$\int_{\eta_k}^{\eta_k^*} |\Omega_k'(s)|^p v^{-p}(s)ds = \left(\int_{\eta_k}^{\eta_k^*} v^{p'}(y)dy\right)^{1-p},$$

then

$$\kappa_{1,2} \leqslant \lambda_k \left(\int_{\eta_k^-}^{\eta_k} v^{p'}(y)dy\right)^{p-1} \left(\int_{\eta_k}^{\eta_k^*} v^{p'}(y)dy\right)^{1-p}.$$

If $\eta_k^* = \eta_k^+$, then $\kappa_{1,2} \leqslant \lambda_k$ because of (2.54) and (4.9). If $\eta_k^* = \eta_{k+1}$, we observe that

$$\int_{\eta_k^-}^{\eta_k} v^{p'}(y)dy = \int_{\eta_k}^{\eta_k^+} v^{p'}(y)dy \leqslant \int_{\eta_{k+1}^-}^{\eta_{k+1}^+} v^{p'}(y)dy = 2\int_{\eta_k}^{\eta_{k+1}} v^{p'}(y)dy, \tag{4.16}$$

therefore, $\kappa_{1,2} \leqslant 2^{p-1}\lambda_k$. Thus, (4.15) is proved for $i=1$. Since $h_k^{(1)}(\eta_{k+1}) = h_k(\eta_{k+1})$, the other cases of (4.15) follow from the first. Now, on the strength of

$$\begin{aligned} g_k(t) &\geqslant \left(\int_t^{\eta_{k+2}} w^q(x)dx\right)^{\frac{r}{pq}} \int_{\eta_{k-1}}^t \left(\int_{\eta_{k-1}}^s v^{p'}(y)dy\right)^{\frac{r}{pq'}} v^{p'}(s)ds \\ &\approx \left(\int_t^{\eta_{k+2}} w^q(x)dx\right)^{\frac{r}{pq}} \left(\int_{\eta_{k-1}}^t v^{p'}(y)dy\right)^{\frac{r}{p'q}}, \qquad t \in [\eta_{k-1}, \eta_{k+2}], \end{aligned}$$

the lower bound

$$\begin{aligned} \|h_k w\|_q^q &\geqslant \int_{\eta_{k-1}}^{\eta_{k+2}} |g_k(t)|^q w^q(t)dt \\ &\gg \int_{\eta_{k-1}}^{\eta_{k+2}} \left(\int_t^{\eta_{k+2}} w^q(x)dx\right)^{\frac{r}{p}} \left(\int_{\eta_{k-1}}^t v^{p'}(y)dy\right)^{\frac{r}{p'}} w^q(t)dt \end{aligned}$$

holds and integration by parts brings

$$\|h_k w\|_q^q \gg \lambda_k. \tag{4.17}$$

Now we construct the functions

$$h_i = \sum_{|k|\leqslant N} h_{k,i} = \sum_{|k|\leqslant N} h_{2k,i} + \sum_{|k|\leqslant N} h_{2k+1,i} =: F_{1,i} + F_{2,i},$$

where $N \in \mathbb{N}$. For each $i = 1,2,3$ the supports of $h_{2k,i}$, $k \in \mathbb{Z}$, are mutually disjoint. Therefore, $F_{1,i} \in \mathscr{L}$ and by the same reason $F_{2,i} \in \mathscr{L}$. Observe that

$$1 \leqslant \sum_{|k|\leqslant N} \chi_{\operatorname{supp} h_k}(x) \leqslant 4, \qquad x \in \bigcup_{|k|\leqslant N} \operatorname{supp} h_k.$$

Letting

$$\Lambda_N := \sum_{|k|\leqslant N} \lambda_k \in (0,\infty)$$

and using (4.14) – (4.17) and (4.10) we find

$$\Lambda_N^{1/q} \ll \|\sum_{|k|\leqslant N} h_k w\|_q \leqslant \sum_{i=1}^{3}\sum_{j=1}^{2}\|F_{j,i}w\|_q \leqslant C\sum_{i=1}^{3}\sum_{j=1}^{2}\|F'_{j,i}/v\|_p$$
$$\ll C\left(\sum_{|k|\leqslant N,i}\|h'_{k,i}/v\|_p^p\right)^{\frac{1}{p}} \ll C\Lambda_N^{1/p}.$$

It implies $C \gg \Lambda_N^{1/r}$. Letting $N \to \infty$ we obtain $C \gg \sum_k \lambda_k$ and because of definition (4.13) we have

$$\lambda_k \gg \int_{\eta_k}^{\eta_{k+1}}\left(\int_s^{\eta_{k+2}} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy\right)^{\frac{r}{p'}} w^q(s)ds$$
$$\geqslant \int_{\eta_k}^{\eta_{k+1}}\left(\int_{\delta^+(s)} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{\Delta^-(s)} v^{p'}(y)dy\right)^{\frac{r}{p'}} w^q(s)ds.$$

Then the required estimate $C \gg \mathscr{B}_{MR}^+$ follows from (2.54) and (4.9). The inequality $C \gg \mathscr{B}_{MR}^-$ can be proved by a similar construction using the intervals formed by the upper boundary function $b(x)$. Thus, the estimate (4.12) is true. Consequently, inequality $\|\mathscr{H}\| \gg \mathscr{B}_{MR}$ is proved in the case when the fairway $\sigma(x) = x$.

Now we set free from this constraint. Let

$$\widetilde{f}(t) = f(\sigma(t))[\sigma'(t)]^{1/p}, \qquad \widetilde{a}(x) = \sigma^{-1}(a(x)), \qquad \widetilde{b}(x) = \sigma^{-1}(b(x)). \tag{4.18}$$

By change of variables in the left and right hand sides of

$$\|\mathscr{H}f\|_q \leqslant \|\mathscr{H}\|\|f\|_p \tag{4.19}$$

it follows from the inequality of the form

$$\left(\int_0^\infty w^q(x)\left|\int_{\widetilde{a}(x)}^{\widetilde{b}(x)} \widetilde{f}(y)\widetilde{v}(y)dy\right|^q dx\right)^{\frac{1}{q}} \leqslant \|\mathscr{H}\|\left(\int_0^\infty |\widetilde{f}(y)|^p dy\right)^{\frac{1}{p}}, \tag{4.20}$$

where $\widetilde{v}(y) = v(\sigma(y))[\sigma'(y)]^{1/p'}$. It is easy to see, that

$$\int_{\widetilde{a}(x)}^{x} \widetilde{v}^{p'}(y)dy = \int_{a(x)}^{\sigma(x)} v^{p'}(y)dy = \int_{\sigma(x)}^{b(x)} v^{p'}(y)dy = \int_x^{\widetilde{b}(x)} \widetilde{v}^{p'}(y)dy, \tag{4.21}$$

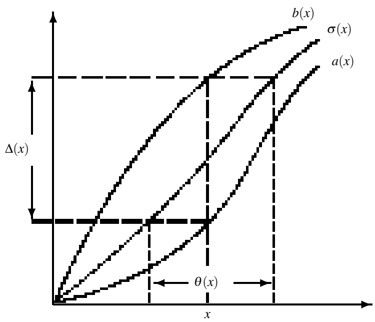
therefore the fairway for $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{v}$ is $\widetilde{\sigma}(x) = x$. It has already been proved above that $\widetilde{\mathscr{B}}_{MR} \ll \|\mathscr{H}\|$, where

$$\widetilde{\mathscr{B}}_{MR}^r = \int_0^\infty\left(\int_{\widetilde{\delta}(t)} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{\widetilde{\Delta}(t)} \widetilde{v}^{p'}(y)dy\right)^{\frac{r}{p'}} w^q(t)dt.$$

Because of $\widetilde{\Delta}(t){:=}[\widetilde{a}(t),\widetilde{b}(t)]$, $\widetilde{\delta}(t){:=}[\widetilde{b}^{-1}(\widetilde{\sigma}(t)),\widetilde{a}^{-1}(\widetilde{\sigma}(t))] = [b^{-1}(\sigma(t)),a^{-1}(\sigma(t))] = \delta(t)$ and $\int_{\widetilde{\Delta}(t)} \widetilde{v}^{p'}(y)dy = \int_{\Delta(t)} v^{p'}(y)dy$ the equality $\widetilde{\mathscr{B}}_{MR} = \mathscr{B}_{MR}$ follows. The assertion about compactness for $q < p$ is a direct corollary of obtained criterion of the boundedness and Ando's theorem.

4.2. Tomaselli and Persson-Stepanov type criteria

Now we give complete analogs of the alternative boundedness conditions for the operator (1.1) similar to (2.5) and (2.7).



Let $\sigma^{-1}(y)$ denote the inverse function to the fairway $\sigma(x)$ and put

$$\Delta(t) = (a(t), b(t)),$$

$$\theta(t) = (\sigma^{-1}(a(t)), \sigma^{-1}(b(t))).$$

THEOREM 4.2. *Let the hypothesies of Theorem* 4.1 *hold. If* $1 < p \leqslant q < \infty$, *then*

$$\|\mathscr{H}\|_{L_p \to L_q} \approx \mathscr{A}_T, \tag{4.22}$$

where

$$\mathscr{A}_T := \sup_{t>0} \left(\int_{\theta(t)} \left[\int_{\Delta(x)} v^{p'}(y) dy \right]^q w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{\Delta(t)} v^{p'}(y) dy \right)^{-\frac{1}{p}}.$$

If $0 < q < p < \infty,\ p > 1,\ 1/r = 1/q - 1/p$, *then*

$$\|\mathscr{H}\|_{L_p \to L_q} \approx \mathscr{B}_{PS}, \tag{4.23}$$

where

$$\mathscr{B}_{PS} := \left(\int_0^\infty \left[\int_{\theta(t)} \left\{ \int_{\Delta(x)} v^{p'}(y) dy \right\}^q w^q(x) dx \right]^{\frac{r}{p}} \left[\int_{\Delta(t)} v^{p'}(y) dy \right]^{q - \frac{r}{p}} w^q(t) dt \right)^{\frac{1}{r}}.$$

Proof. The upper bounds. Given boundary functions $a(x)$ and $b(x)$ satisfying the conditions (1.2) let a point sequence $\{\xi_k\}_{k \in \mathbb{Z}} \subset (0, \infty)$ be given by relation (3.5) and put as before

$$\xi_k^- = \sigma^{-1}(a(\xi_k)), \qquad \xi_k^+ = \sigma^{-1}(b(\xi_k)), \qquad \Delta_k = [\xi_k^-, \xi_k^+) = \Delta_k^- \cup \Delta_k^+,$$

$$\Delta_k^- = [\xi_k^-, \xi_k), \qquad \Delta_k^+ = [\xi_k, \xi_k^+), \qquad k \in \mathbb{Z}.$$

Note that in view of (2.54)

$$\frac{1}{2} \int_{\Delta(t)} v^{p'}(y) dy \leqslant \int_{a(\xi_k)}^{b(t)} v^{p'}(y) dy \leqslant \int_{\Delta(t)} v^{p'}(y) dy \quad \text{for} \quad t \in \Delta_k^- \tag{4.24}$$

and

$$\frac{1}{2}\int_{\Delta(t)} v^{p'}(y)dy \leqslant \int_{a(t)}^{b(\xi_k)} v^{p'}(y)dy \leqslant \int_{\Delta(t)} v^{p'}(y)dy \quad \text{for} \quad t\in\Delta_k^+. \tag{4.25}$$

As in Theorem 4.1 we split the operator $\mathscr{H}$ into the sum of four sequences of operators

$$\mathscr{H}f(x) = T_{k,1}f(x)+T_{k,2}f(x)+S_{k,1}f(x)+S_{k,2}f(x), \qquad x\in\Delta_k, \tag{4.26}$$

then

$$\begin{aligned}\|\mathscr{H}f\|_q^q = \sum_k \|\mathscr{H}f\|_{L_q(\Delta_k)}^q \approx \sum_k \left\|T_{k,1}f\right\|_{L_q(\Delta_k^-)}^q + \sum_k \left\|S_{k,2}f\right\|_{L_q(\Delta_k^-)}^q \\ + \sum_k \left\|T_{k,2}f\right\|_{L_q(\Delta_k^+)}^q + \sum_k \left\|S_{k,1}f\right\|_{L_q(\Delta_k^+)}^q.\end{aligned} \tag{4.27}$$

Let us start with trapeze-shaped operators $S_{k,2}$ and $T_{k,2}$. If $1<p\leqslant q<\infty$, we find by using the estimate (2.35) of Lemma 2.1

$$\begin{aligned}&\|S_{k,2}\|_{L_p(a(\xi_k),b(\xi_k))\to L_q(\Delta_k^-)}^q \\ &\quad\approx \sup_{t\in\Delta_k^-}\left(\int_{\xi_k^-}^t\left[\int_{a(\xi_k)}^{b(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)\left(\int_{a(\xi_k)}^{b(t)} v^{p'}(y)dy\right)^{-\frac{q}{p}}.\end{aligned}$$

For any $t\in[\xi_k^-,\xi_k]$ we have $(\xi_k^-,t)\subset\theta(t)$, and $(a(\xi_k),b(x))\subset\Delta(x)$ for $x\in(\xi_k^-,t)\subset(\xi_k^-,\xi_k)$. Therefore, in view of (4.24),

$$\begin{aligned}&\|S_{k,2}\|_{L_p(a(\xi_k),b(\xi_k))\to L_q(\Delta_k^-)}^q \\ &\quad\ll \sup_{t\in\Delta_k^-}\left(\int_{\theta(t)}\left[\int_{\Delta(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)\left(\int_{\Delta(t)} v^{p'}(y)dy\right)^{-\frac{q}{p}} \leqslant \mathscr{A}_T^q.\end{aligned} \tag{4.28}$$

Analogously, by applying (2.41) of Lemma 2.2 and (4.25) we obtain

$$\begin{aligned}&\|T_{k,2}\|_{L_p(a(\xi_k),b(\xi_k))\to L_q(\Delta_k^+)}^q \\ &\quad\ll \sup_{t\in\Delta_k^+}\left(\int_{\theta(t)}\left[\int_{\Delta(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)\left(\int_{\Delta(t)} v^{p'}(y)dy\right)^{-\frac{q}{p}} \ll \mathscr{A}_T^q.\end{aligned} \tag{4.29}$$

If $0<q<p<\infty$, $p>1$ we have by applying the estimate (2.44) of Lemma 2.3

$$\begin{aligned}&\|S_{k,2}\|_{L_p(a(\xi_k),b(\xi_k))\to L_q(\Delta_k^-)}^r \\ &\quad\approx \int_{\xi_k^-}^{\xi_k}\left(\int_{\xi_k^-}^t\left[\int_{a(\xi_k)}^{b(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{a(\xi_k)}^{b(t)} v^{p'}(y)dy\right)^{q-\frac{r}{p}} w^q(t)dt.\end{aligned}$$

For $t\in[\xi_k^-,\xi_k]$ it yields that $(\xi_k^-,t)\subset\theta(t)$, $(a(\xi_k),b(t))\subset\Delta(t)$ and also $(a(\xi_k),b(x))\subset\Delta(x)$ for any $x\in(\xi_k^-,t)\subset(\xi_k^-,\xi_k)$. Moreover, $a(\xi_k)\leqslant\sigma(t)$ for $t\in[\xi_k^-,\xi_k]$. Therefore, in view of (4.24)

$$\begin{aligned}\|S_{k,2}\|^r_{L_p(a(\xi_k),b(\xi_k))\to L_q(\Delta_k^-)} &\leqslant \int_{\xi_k^-}^{\xi_k}\left(\int_{\theta(t)}\left[\int_{\Delta(x)}v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{r}{p}}\\ &\quad\times\left(\int_{\Delta(t)}v^{p'}(y)dy\right)^q\left(\int_{\sigma(t)}^{b(t)}v^{p'}(y)dy\right)^{-\frac{r}{p}}w^q(t)dt\\ &\overset{(2.54)}{\leqslant} 2^{\frac{r}{p}}\int_{\xi_k^-}^{\xi_k}\left(\int_{\theta(t)}\left[\int_{\Delta(x)}v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{r}{p}}\\ &\quad\times\left(\int_{\Delta(t)}v^{p'}(y)dy\right)^{q-\frac{r}{p}}w^q(t)dt\\ &=:\left(\mathscr{B}_{\Delta_k^-}\right)^r. \end{aligned}\tag{4.30}$$

Analogously, by applying (2.52) of Lemma 2.4 and (4.25) we can get that

$$\begin{aligned}&\|T_{k,2}\|^r_{L_p(a(\xi_k),b(\xi_k))\to L_q(\Delta_k^+)}\\ &\ll\int_{\xi_k}^{\xi_k^+}\left(\int_{\theta(t)}\left[\int_{\Delta(x)}v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{\Delta(t)}v^{p'}(y)dy\right)^{q-\frac{r}{p}}w^q(t)dt\\ &=:\left(\mathscr{B}_{\Delta_k^+}\right)^r. \end{aligned}\tag{4.31}$$

To estimate the norms $\|T_{k,1}\|^r_{L_p(a(\xi_k^-),a(\xi_k))\to L_q(\Delta_k^-)}$ and $\|S_{k,1}\|^r_{L_p(b(\xi_k),b(\xi_k^+))\to L_q(\Delta_k^+)}$ of triangle-shaped operators we introduce sequences $\{t_i\}_{i=0}^{i_b(k)-1}$ and $\{s_j\}_{j=0}^{j_a(k)-1}$ according to the constructions of Lemmas 2.5 and 2.6 with $c=\xi_k$. Then the operators $T_{k,1}f(x)$, $x\in\Delta_k^-$, and $S_{k,1}f(x)$, $x\in\Delta_k^+$, split into the following two sums:

$$\begin{aligned}T_{k,1}f(x)&=\sum_{i=0}^{i_b(k)-1}T_{k,1}^{(i)}f(x):=\sum_{i=0}^{i_b(k)-1}\left[T_{k,1}f(x)\chi_{(t_i,t_{i+1})}(x)\right],\\ S_{k,1}f(x)&=\sum_{j=0}^{j_a(k)-1}S_{k,1}^{(j)}f(x):=\sum_{j=0}^{j_a(k)-1}\left[S_{k,1}f(x)\chi_{(s_j,s_{j+1})}(x)\right].\end{aligned}\tag{4.32}$$

If $1<p\leqslant q<\infty$ we have for $T_{k,1}^{(i)}$, $0\leqslant i\leqslant i_b-1$, by applying the estimate (2.40) of Lemma 2.2

$$\begin{aligned}\|T_{k,1}^{(i)}\|^q_{L_p(a(t_i),a(\xi_k))\to L_q(t_i,t_{i+1})} &\approx \sup_{t_i\leqslant t\leqslant t_{i+1}}\left(\int_{t_i}^t w^q(x)dx\right)\left(\int_{a(t)}^{a(\xi_k)}v^{p'}(y)dy\right)^{\frac{q}{p'}}\\ &\leqslant \sup_{t_i\leqslant t\leqslant t_{i+1}}\left(\int_{t_i}^t w^q(x)dx\right)\left(\int_{a(\xi_k^-)}^{\sigma(\xi_k^-)}v^{p'}(y)dy\right)^{\frac{q}{p'}}\end{aligned}$$

$$\overset{(2.54)}{=} \left(\int_{t_i}^{t_{i+1}} w^q(x)dx\right)\left(\int_{a(\xi_k)}^{b(\xi_k^-)} v^{p'}(y)dy\right)^{\frac{q}{p'}}$$
$$\overset{(2.58)}{=} \left(\frac{1}{2^i}\right)^{\frac{q}{p'}}\left(\int_{t_i}^{t_{i+1}} w^q(x)dx\right)\left(\int_{a(\xi_k)}^{b(t_i)} v^{p'}(y)dy\right)^{\frac{q}{p'}}.$$

Since $q/p' = q - q/p$ and $(t_i,t) \subseteq (\xi_k^-,t) \subset \theta(t)$ for $t \in [\xi_k^-,\xi_k]$ we get in view of (2.55) that

$$\begin{aligned}
&\|T_{k,1}^{(i)}\|^q_{L_p(a(t_i),a(\xi_k))\to L_q(t_i,t_{i+1})}\\
&\quad\overset{(2.55)}{\ll} 2^{\frac{q}{p}}\left(\frac{1}{2^i}\right)^{\frac{q}{p'}}\left(\int_{t_i}^{t_{i+1}} w^q(x)dx\right)\left(\int_{a(\xi_k)}^{b(t_i)} v^{p'}(y)dy\right)^q\left(\int_{a(\xi_k)}^{b(t_{i+1})} v^{p'}(y)dy\right)^{-\frac{q}{p}}\\
&\quad\leqslant 2^{\frac{q}{p}}\left(\frac{1}{2^i}\right)^{\frac{q}{p'}}\sup_{t_i\leqslant t\leqslant t_{i+1}}\left(\int_{t_i}^{t}\left[\int_{a(\xi_k)}^{b(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)\left(\int_{a(\xi_k)}^{b(t)} v^{p'}(y)dy\right)^{-\frac{q}{p}}\\
&\quad\overset{(4.24)}{\ll} 2^{\frac{q}{p}}\left(\frac{1}{2^i}\right)^{\frac{q}{p'}}\sup_{t_i\leqslant t\leqslant t_{i+1}}\left(\int_{\theta(t)}\left[\int_{\Delta(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)\left(\int_{\Delta(t)} v^{p'}(y)dy\right)^{-\frac{q}{p}}\\
&\quad\ll \left(\frac{1}{2^i}\right)^{\frac{q}{p'}}\mathscr{A}_T^q. \qquad (4.33)
\end{aligned}$$

Hence, since $p \leqslant q$ it holds that

$$\begin{aligned}
\|T_{k,1}f\|^q_{L_q(\Delta_k^-)} &= \sum_{i=0}^{i_b-1}\|T_{k,1}^{(i)}f\|^q_{L_q(t_i,t_{i+1})} \leqslant \sum_{i=0}^{i_b-1}\|T_{k,1}^{(i)}\|^q_{L_p(a(t_i),a(\xi_k))\to L_q(t_i,t_{i+1})}\|f\|^q_{L_p(a(t_i),a(\xi_k))}\\
&\overset{(4.33)}{\ll} \mathscr{A}_T^q\|f\|^q_{L_p(a(\xi_k^-),a(\xi_k))}\sum_{i=0}^{i_b-1}\left(\frac{1}{2^i}\right)^{\frac{q}{p'}} \leqslant \frac{2^{q/p'}}{2^{q/p'}-1}\mathscr{A}_T^q\|f\|^q_{L_p(a(\xi_k^-),a(\xi_k))}. \qquad (4.34)
\end{aligned}$$

Analogously, from (2.34) of Lemma 2.1 by using (2.65), (2.63) and (4.25) we can get that

$$\|S_{k,1}f\|^q_{L_q(\Delta_k^+)} \ll \frac{2^{q/p'}}{2^{q/p'}-1}\mathscr{A}_T^q\|f\|^q_{L_p(b(\xi_k),b(\xi_k^+))}. \qquad (4.35)$$

If $0<q<p<\infty,\ p>1$ we have for $T_{k,1}^{(i)},\ 0\leqslant i\leqslant i_b-1$, by applying the estimate (2.50) of Lemma 2.4

$$\begin{aligned}
\|T_{k,1}^{(i)}\|^r_{L_p(a(t_i),a(\xi_k))\to L_q(t_i,t_{i+1})} &\approx \int_{t_i}^{t_{i+1}}\left(\int_{t_i}^{t} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{a(t)}^{a(\xi_k)} v^{p'}(y)dy\right)^{\frac{r}{p'}} w^q(t)dt\\
&\leqslant \left(\int_{a(\xi_k^-)}^{\sigma(\xi_k^-)} v^{p'}(y)dy\right)^{\frac{r}{p'}}\int_{t_i}^{t_{i+1}}\left(\int_{t_i}^{t} w^q(x)dx\right)^{\frac{r}{p}} w^q(t)dt
\end{aligned}$$

$$\overset{(2.54)}{=} \left(\int_{a(\xi_k)}^{b(\xi_k^-)} v^{p'}(y)dy\right)^{\frac{r}{p'}} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{t} w^q(x)dx\right)^{\frac{r}{p}} w^q(t)dt$$
$$\overset{(2.58)}{=} \left(\frac{1}{2^i}\right)^{\frac{r}{p'}} \left(\int_{a(\xi_k)}^{b(t_i)} v^{p'}(y)dy\right)^{\frac{r}{p'}} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{t} w^q(x)dx\right)^{\frac{r}{p}} w^q(t)dt.$$

By using the relation $r/p' = q\cdot r/p + q - r/p$ we write

$$\begin{aligned}
&\|T_{k,1}^{(i)}\|^r_{L_p(a(t_i),a(\xi_k))\to L_q(t_i,t_{i+1})} \\
&\ll \left(\frac{1}{2^i}\right)^{\frac{r}{p'}} \left(\int_{a(\xi_k)}^{b(t_i)} v^{p'}(y)dy\right)^{q\cdot\frac{r}{p}+q-\frac{r}{p}} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{t} w^q(x)dx\right)^{\frac{r}{p}} w^q(t)dt \\
&\overset{(2.55)}{=} 2^{\frac{r}{p}} \left(\frac{1}{2^i}\right)^{\frac{r}{p'}} \left(\int_{a(\xi_k)}^{b(t_i)} v^{p'}(y)dy\right)^{q\cdot\frac{r}{p}+q} \left(\int_{a(\xi_k)}^{b(t_{i+1})} v^{p'}(y)dy\right)^{-\frac{r}{p}} \\
&\quad\times \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{t} w^q(x)dx\right)^{\frac{r}{p}} w^q(t)dt \\
&\leqslant 2^{\frac{r}{p}} \left(\frac{1}{2^i}\right)^{\frac{r}{p'}} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{t} \left[\int_{a(\xi_k)}^{b(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{r}{p}} \left(\int_{a(\xi_k)}^{b(t)} v^{p'}(y)dy\right)^q \\
&\quad\times \left(\int_{a(\xi_k)}^{b(t)} v^{p'}(y)dy\right)^{-\frac{r}{p}} w^q(t)dt \\
&\overset{(4.24)}{\approx} \left(\frac{1}{2^i}\right)^{\frac{r}{p'}} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{t} \left[\int_{\Delta(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{r}{p}} \left(\int_{\Delta(t)} v^{p'}(y)dy\right)^{q-\frac{r}{p}} w^q(t)dt.
\end{aligned}$$

As before $(t_i,t) \subseteq (\xi_k^-,t) \subset \theta(t)$ for $t \in [\xi_k^-, \xi_k]$. Therefore, we have

$$\begin{aligned}
&\|T_{k,1}^{(i)}\|^r_{L_p(a(t_i),a(\xi_k))\to L_q(t_i,t_{i+1})} \\
&\ll \left(\frac{1}{2^i}\right)^{\frac{r}{p'}} \int_{t_i}^{t_{i+1}} \left(\int_{\theta(t)} \left[\int_{\Delta(x)} v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{r}{p}} \left(\int_{\Delta(t)} v^{p'}(y)dy\right)^{q-\frac{r}{p}} w^q(t)dt \\
&=: \left(\frac{1}{2^i}\right)^{\frac{r}{p'}} \left(\mathscr{B}^{(i)}_{\Delta_k^-}\right)^r. \qquad (4.36)
\end{aligned}$$

By applying Hölder's inequality with the powers r/q and p/q we get that

$$\begin{aligned}
\|T_{k,1}f\|^q_{L_q(\Delta_k^-)} &= \sum_{i=0}^{i_b-1} \|T_{k,1}^{(i)}f\|^q_{L_q(t_i,t_{i+1})} \\
&\leqslant \sum_{i=0}^{i_b-1} \|T_{k,1}^{(i)}\|^q_{L_p(a(t_i),a(\xi_k))\to L_q(t_i,t_{i+1})} \|f\|^q_{L_p(a(t_i),a(\xi_k))} \\
&\overset{(4.36)}{\ll} \|f\|^q_{L_p(a(\xi_k^-),a(\xi_k))} \sum_{i=0}^{i_b-1} \left(\frac{1}{2^i}\right)^{\frac{q}{p'}} \left(\mathscr{B}^{(i)}_{\Delta_k^-}\right)^q
\end{aligned}$$

$$\leqslant \|f\|^q_{L_p(a(\xi_k^-),a(\xi_k))}\left(\sum_{i=0}^{i_b-1}\left(\mathscr{B}^{(i)}_{\Delta_k^-}\right)^r\right)^{\frac{q}{r}}\left(\sum_{i=0}^{i_b-1}\left(\frac{1}{2^i}\right)^{\frac{p}{p'}}\right)^{\frac{q}{p}}$$

$$\leqslant \left(\frac{2^{p-1}}{2^{p-1}-1}\right)^{\frac{q}{p}}\left(\mathscr{B}_{\Delta_k^-}\right)^q\|f\|^q_{L_p(a(\xi_k^-),a(\xi_k))}. \tag{4.37}$$

Analogously, from (2.42) of Lemma 2.3 by using (4.25), (2.65) and (2.63) we can get that

$$\|S_{k,1}f\|^q_{L_q(\Delta_k^+)} \ll \left(\frac{2^{p-1}}{2^{p-1}-1}\right)^{\frac{q}{p}}\left(\mathscr{B}_{\Delta_k^+}\right)^q\|f\|^q_{L_p(b(\xi_k),b(\xi_k^+))}. \tag{4.38}$$

Since every series $\{T_{k,1}\}$, $\{T_{k,2}\}$, $\{S_{k,1}\}$ or $\{S_{k,2}\}$ is block-diagonal, we have the following pair of inequalities coming from Lemma 3.1 for each term in (4.27). Namely, if $1<p\leqslant q<\infty$ then

$$\sum_{k\in\mathbb{Z}}\|T_{k,1}f\|^q_{L_q(\Delta_k^-)} \overset{(4.34)}{\ll} \frac{2^{q/p'}}{2^{q/p'}-1}\mathscr{A}_T^q\sum_{k\in\mathbb{Z}}\|f\|^q_{L_p(a(\xi_k^-),a(\xi_k))} \leqslant \frac{2^{q/p'}}{2^{q/p'}-1}\mathscr{A}_T^q\|f\|^q_p.$$

If $0<q<p<\infty,\ p>1$ then

$$\sum_{k\in\mathbb{Z}}\|T_{k,1}f\|^q_{L_q(\Delta_k^-)} \overset{(4.37)}{\ll} \left(\sum_{k\in\mathbb{Z}}\left(\mathscr{B}_{\Delta_k^-}\right)^r\right)^{\frac{1}{r}}\|f\|_p \leqslant \mathscr{B}_{PS}\|f\|_p.$$

Analogous inequalities hold for $\{T_{k,2}\}$, $\{S_{k,1}\}$ and $\{S_{k,2}\}$ on the strength of (4.29), (4.35), (4.28) for $1<p\leqslant q<\infty$ and (4.31), (4.38), (4.30) in the case $0<q<p<\infty$, $p>1$, respectively. By combining these estimates with (4.27) we obtain the upper bounds in (4.22) and (4.23).

The lower bound. Let $1<p\leqslant q<\infty$.
First we prove that $\mathscr{A}_T\approx\mathscr{A}_{T,1}+\mathscr{A}_{T,2}$, where $\mathscr{A}_{T,1}:=\sup_{t>0}\sup_{b^{-1}(\sigma(t))\leqslant s\leqslant t}\mathscr{A}_{T,1}(s,t)$ and $\mathscr{A}_{T,2}:=\sup_{t>0}\sup_{t\leqslant s\leqslant a^{-1}(\sigma(t))}\mathscr{A}_{T,2}(t,s)$ with

$$\mathscr{A}_{T,1}(s,t):=\left(\int_s^t\left[\int_{\Delta(x)}v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}}\left(\int_{a(s)}^{\sigma(t)}v^{p'}(y)dy\right)^{-\frac{1}{p}}$$

and

$$\mathscr{A}_{T,2}(t,s):=\left(\int_t^s\left[\int_{\Delta(x)}v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}}\left(\int_{\sigma(t)}^{b(s)}v^{p'}(y)dy\right)^{-\frac{1}{p}}.$$

Indeed, in view of (2.54)

$$\frac{1}{2}\int_{a(s)}^{b(s)}v^{p'}(y)dy\leqslant\int_{a(s)}^{\sigma(t)}v^{p'}(y)dy\leqslant\int_{a(s)}^{b(s)}v^{p'}(y)dy, \qquad b^{-1}(\sigma(t))\leqslant s\leqslant t. \tag{4.39}$$

Therefore, it holds that

$$\mathscr{A}_{T,1}\approx\sup_{s>0}\left(\int_{a(s)}^{b(s)}v^{p'}(y)dy\right)^{-\frac{1}{p}}\sup_{s\leqslant t\leqslant\sigma^{-1}(b(s))}\left(\int_s^t\left[\int_{\Delta(x)}v^{p'}(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}}$$

$$= \sup_{s>0} \left(\int_s^{\sigma^{-1}(b(s))} \left[\int_{\Delta(x)} v^{p'}(y)dy \right]^q w^q(x)dx \right)^{\frac{1}{q}} \left(\int_{\Delta(s)} v^{p'}(y)dy \right)^{-\frac{1}{p}}.$$

Analogously, because of (2.54)

$$\frac{1}{2}\int_{a(s)}^{b(s)} v^{p'}(y)dy \leqslant \int_{\sigma(t)}^{b(s)} v^{p'}(y)dy \leqslant \int_{a(s)}^{b(s)} v^{p'}(y)dy, \qquad t \leqslant s \leqslant a^{-1}(\sigma(t)). \tag{4.40}$$

Thus,

$$\mathscr{A}_{T,2} \approx \sup_{s>0} \left(\int_{\sigma^{-1}(a(s))}^{s} \left[\int_{\Delta(x)} v^{p'}(y)dy \right]^q w^q(x)dx \right)^{\frac{1}{q}} \left(\int_{\Delta(s)} v^{p'}(y)dy \right)^{-\frac{1}{p}}.$$

Now suppose that $\|\mathscr{H}\| < \infty$ and insert the function $f_{s,t}(y) = v^{p'-1}(y)\chi_{[a(s),\sigma(t)]}(y)$, where $b^{-1}(\sigma(t)) \leqslant s \leqslant t$, into the inequality

$$\left(\int_0^\infty (\mathscr{H}f)^q(x)dx \right)^{\frac{1}{q}} \leqslant \|\mathscr{H}\| \left(\int_0^\infty f^p(y)dy \right)^{\frac{1}{p}}.$$

Since $a(s) \leqslant a(x)$ and $\sigma(x) \leqslant \sigma(t)$ for $s \leqslant x \leqslant t$, we have in view of (2.54) that

$$\|\mathscr{H}\| \geqslant \frac{1}{2} \left(\int_s^t \left[\int_{a(x)}^{b(x)} v^{p'}(y)dy \right]^q w^q(x)dx \right)^{\frac{1}{q}} \left(\int_{a(s)}^{\sigma(t)} v^{p'}(y)dy \right)^{-\frac{1}{p}} \approx \mathscr{A}_{T,1}(s,t).$$

Therefore, $\|\mathscr{H}\| \gg \mathscr{A}_{T,1}(s,t)$ for all $s \leqslant t$ and, thus, $\|\mathscr{H}\| \gg \mathscr{A}_{T,1}$. Analogously, by applying the function $f_{t,s}(y) = v^{p'-1}(y)\chi_{[\sigma(t),b(s)]}(y)$ with $t \leqslant s \leqslant a^{-1}(\sigma(t))$ we get that $\|\mathscr{H}\| \gg \mathscr{A}_{T,2}$.

Let $0 < q < p < \infty,\ p > 1$. First, similar to Theorem 4.1 we prove $\|\mathscr{H}\| \gg \mathscr{B}_{PS}$ under the condition (4.9). Let us remind that in this case we denote

$$x^- = a(x), \qquad x^+ = b(x), \qquad \Delta(x) = [x^-, x^+]$$

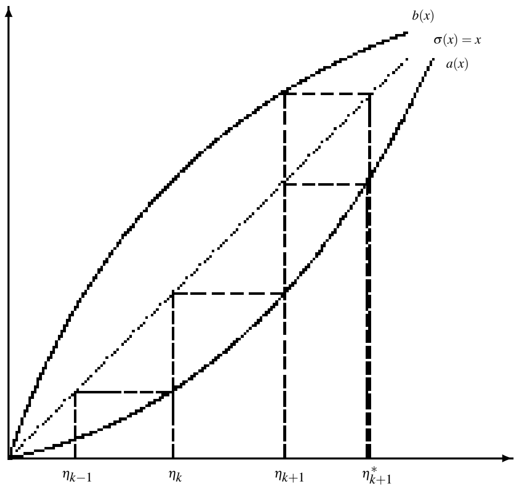
and establish the estimate

$$C \gg \mathscr{B}_{PS}, \tag{4.41}$$

where C is the least possible constant of the inequality (4.10) which holds for the function class from Definition 4.1. For this purpose it is sufficient to prove that $C \gg \mathscr{B}_{PS}^{\pm}$, where

$$\mathscr{B}_{PS}^{\pm} := \left(\int_0^\infty \left[\int_{\Delta^{\pm}(t)} \left\{ \int_{\Delta(x)} v^{p'}(y)dy \right\}^q w^q(x)dx \right]^{\frac{r}{p}} \left[\int_{\Delta(t)} v^{p'}(y)dy \right]^{q-\frac{r}{p}} w^q(t)dt \right)^{\frac{1}{r}}$$

and $\Delta^-(t) = (t^-, t),\ \Delta^+(t) = (t, t^+)$.



Further we utilize in part notations from the proof of Theorem 4.1 with sometimes a different meaning. To prove $C \gg \mathscr{B}^{-}_{PS}$ we put $\eta_0 = 1$ and define the sequence $\{\eta_k\}_{k\in\mathbb{Z}} \subset (0,\infty)$ by

$$\eta_{k+1} = a^{-1}(\eta_k), \qquad k \in \mathbb{Z}.$$

Let

$$\eta_k^* = \min(\eta_k^+, \eta_{k+1}).$$

For a fixed $k \in \mathbb{Z}$ we take four neighboring points η_{k-1}, η_k, η_{k+1}, η_{k+1}^* and let

$$f_k(t) = \chi_{[\eta_{k-1},\eta_{k+1}]}(t)\,[g_k(t) + h_k(t)],$$

where

$$g_k(t) = \int_{\eta_{k-1}}^{t} \left(\int_s^{\eta_{k+1}} w^q(x)dx\right)^{\frac{r}{pq}} \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy\right)^{\frac{r}{pq'}} v^{p'}(s)ds,$$

$$h_k(t) = \left(\int_{\eta_{k-1}}^{\eta_{k+1}} v^{p'}(y)dy\right)^{-\frac{r}{pq}} \left(\int_{\eta_{k-1}}^{\eta_{k+1}} \left[\int_{\eta_{k-1}}^{s} v^{p'}(y)dy\right]^q w^q(s)ds\right)^{\frac{r}{pq}} \left(\int_{\eta_{k-1}}^{t} v^{p'}(s)ds\right).$$

On the interval $[\eta_{k-1}, \eta_{k+1}^*]$ we define the function

$$\phi_k(t) = \begin{cases} f_k(t), & t \in [\eta_{k-1}, \eta_{k+1}], \\ f_k(\eta_{k+1})\Omega_{k+1}(t), & t \in [\eta_{k+1}, \eta_{k+1}^*], \end{cases}$$

where

$$\Omega_l(t) = \left(\int_{\eta_l}^{\eta_l^*} v^{p'}(y)dy\right)^{-1} \int_t^{\eta_l^*} v^{p'}(s)ds.$$

Put

$$\nu_k := \lambda_k + \mu_k,$$

where

$$\lambda_k := \int_{\eta_{k-1}}^{\eta_{k+1}} \left(\int_s^{\eta_{k+1}} w^q(x)dx\right)^{\frac{r}{q}} \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(s)ds,$$

$$\mu_k := \left(\int_{\eta_{k-1}}^{\eta_{k+1}} v^{p'}(y)dy\right)^{-\frac{r}{p}} \left(\int_{\eta_{k-1}}^{\eta_{k+1}} \left[\int_{\eta_{k-1}}^{s} v^{p'}(y)dy\right]^q w^q(s)ds\right)^{\frac{r}{q}}.$$

Now we find a decomposition

$$\phi_k(t) = \sum_{i=1}^{2} \phi_{k,i}(t) \tag{4.42}$$

with $\phi_{k,i} \in \mathscr{L},\ i=1,2.$ To this end we first determine $\phi_{k,1}(t)$ by

$$\phi_{k,1}(t) = \begin{cases} 0, & t \notin [\eta_{k-1}, \eta_k^*], \\ \phi_k(t), & t \in [\eta_{k-1}, \eta_k], \\ \phi_k(\eta_k)\Omega_k(t), & t \in [\eta_k, \eta_k^*]. \end{cases}$$

Since $\eta_k^- = \eta_{k-1}$, it yields that $\operatorname{supp}\phi_{k,1} \subseteq [\eta_k^-, \eta_k^*] \subseteq [\eta_k^-, \eta_k^+]$ and, hence, $\phi_{k,1} \in \mathscr{L}$. Obviously that $\phi_{k,2}(t) = \phi_k(t) - \phi_{k,1}(t)$ is in $\mathscr{L}$. It is clear that (4.42) holds. Our next step is to prove that

$$\|\phi_{k,i}'/v\|_p^p \ll \nu_k, \qquad i=1,2. \tag{4.43}$$

If

$$\kappa_1 := \|\phi_{k,1}'/v\|_p^p = \int_{\eta_{k-1}}^{\eta_k} |\phi_k'(s)|^p v^{-p}(s)ds + f_k^p(\eta_k)\int_{\eta_k}^{\eta_k^*} |\Omega_k'(s)|^p v^{-p}(s)ds =: \kappa_{1,1} + \kappa_{1,2},$$

then

$$\begin{aligned} \kappa_{1,1} = & \int_{\eta_{k-1}}^{\eta_k} \left(\int_s^{\eta_{k+1}} w^q(x)dx\right)^{\frac{r}{q}} \left(\int_{\eta_{k-1}}^s v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(s)ds \\ & + \left(\int_{\eta_{k-1}}^{\eta_{k+1}} v^{p'}(y)dy\right)^{-\frac{r}{q}} \left(\int_{\eta_{k-1}}^{\eta_{k+1}} \left[\int_{\eta_{k-1}}^s v^{p'}(y)dy\right]^q w^q(s)ds\right)^{\frac{r}{q}} \\ & \times \left(\int_{\eta_{k-1}}^{\eta_k} v^{p'}(s)ds\right) \leqslant \lambda_k + \mu_k = \nu_k. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} f_k^p(\eta_k) \approx & \left(\int_{\eta_{k-1}}^{\eta_k} \left[\int_s^{\eta_{k+1}} w^q(x)dx\right]^{\frac{r}{pq}} \left[\int_{\eta_{k-1}}^s v^{p'}(y)dy\right]^{\frac{r}{pq'}} v^{p'}(s)ds\right)^p \\ & + \left(\int_{\eta_{k-1}}^{\eta_{k+1}} v^{p'}(y)dy\right)^{-\frac{r}{q}} \left(\int_{\eta_{k-1}}^{\eta_{k+1}} \left[\int_{\eta_{k-1}}^s v^{p'}(y)dy\right]^q w^q(s)ds\right)^{\frac{r}{q}} \left(\int_{\eta_{k-1}}^{\eta_k} v^{p'}(s)ds\right)^p \\ \leqslant & \int_{\eta_{k-1}}^{\eta_k} \left(\int_s^{\eta_{k+1}} w^q(x)dx\right)^{\frac{r}{q}} \left(\int_{\eta_{k-1}}^s v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(s)ds \left(\int_{\eta_{k-1}}^{\eta_k} v^{p'}(s)ds\right)^{p-1} \\ & + \mu_k \left(\int_{\eta_{k-1}}^{\eta_k} v^{p'}(s)ds\right)^{p-1} \\ \leqslant & (\lambda_k + \mu_k)\left(\int_{\eta_{k-1}}^{\eta_k} v^{p'}(s)ds\right)^{p-1} = \nu_k \left(\int_{\eta_{k-1}}^{\eta_k} v^{p'}(s)ds\right)^{p-1}. \end{aligned}$$

Since

$$\int_{\eta_k}^{\eta_k^*} |\Omega_k'(s)|^p v^{-p}(s)ds = \left(\int_{\eta_k}^{\eta_k^*} v^{p'}(s)ds\right)^{1-p},$$

we have that

$$\kappa_{1,2} \ll \nu_k \left(\int_{\eta_{k-1}}^{\eta_k} v^{p'}(s)ds\right)^{p-1} \left(\int_{\eta_k}^{\eta_k^*} v^{p'}(s)ds\right)^{1-p}$$

and in view of (2.54), (4.9) and (4.16) it holds that $\kappa_{1,2} \ll \nu_k$. Thus, (4.43) is proved for $i=1$. Since $\phi_{k,2}(\eta_{k+1}) = \phi_k(\eta_{k+1})$, we can write for the second case of (4.43) that

$$\begin{aligned}
\kappa_2 &:= \|\phi'_{k,2}/v\|_p^p = \int_{\eta_k}^{\eta_{k+1}} |\phi'_{k,2}(s)|^p v^{-p}(s)ds + \phi_k^p(\eta_{k+1}) \int_{\eta_{k+1}}^{\eta^*_{k+1}} |\Omega'_{k+1}(s)|^p v^{-p}(s)ds \\
&= \int_{\eta_k}^{\eta_{k+1}} |\phi'_k(s) - \phi'_{k,1}(s)|^p v^{-p}(s)ds + \phi_k^p(\eta_{k+1}) \int_{\eta_{k+1}}^{\eta^*_{k+1}} |\Omega'_{k+1}(s)|^p v^{-p}(s)ds \\
&\ll \int_{\eta_k}^{\eta_{k+1}} |\phi'_k(s)|^p v^{-p}(s)ds + f_k^p(\eta_k) \int_{\eta_k}^{\eta_{k+1}} \chi_{[\eta_k,\eta_k^*]}(s) |\Omega'_k(s)|^p v^{-p}(s)ds \\
&\quad + f_k^p(\eta_{k+1}) \int_{\eta_{k+1}}^{\eta^*_{k+1}} |\Omega'_{k+1}(s)|^p v^{-p}(s)ds =: \kappa_{2,1} + \kappa_{2,2} + \kappa_{2,3}.
\end{aligned}$$

Obviously, $\kappa_{2,2} = \kappa_{1,2} \ll \nu_k$. Similar to $\kappa_{1,1}$ and $\kappa_{1,2}$ we can prove that $\kappa_{2,1} + \kappa_{2,3} \ll \nu_k$. Therefore, (4.43) is true for $i=2$ also.

Now, since

$$\begin{aligned}
g_k(t) &\geqslant \left(\int_t^{\eta_{k+1}} w^q(x)dx\right)^{\frac{r}{pq}} \int_{\eta_{k-1}}^t \left(\int_{\eta_{k-1}}^s v^{p'}(y)dy\right)^{\frac{r}{pq'}} v^{p'}(s)ds \\
&\approx \left(\int_t^{\eta_{k+1}} w^q(x)dx\right)^{\frac{r}{pq}} \left(\int_{\eta_{k-1}}^t v^{p'}(y)dy\right)^{\frac{r}{p'q}}, \qquad t \in [\eta_{k-1}, \eta_{k+1}],
\end{aligned}$$

the lower bound

$$\|\phi_k w\|_q^q \gg \nu_k \tag{4.44}$$

follows. Next we construct the functions

$$F_i = \sum_{|k|\leqslant N} \phi_{k,i} = \sum_{|k|\leqslant N} \phi_{2k,i} + \sum_{|k|\leqslant N} \phi_{2k+1,i} =: F_{1,i} + F_{2,i},$$

where $N \in \mathbb{N}$. For every $i = 0,1$ the supports of $\phi_{2k,i}$, $k \in \mathbb{Z}$, are mutually disjoint. Therefore, $F_{1,i} \in \mathscr{L}$ and by the same reason $F_{2,i} \in \mathscr{L}$. Observe that

$$2 \leqslant \sum_{|k|\leqslant N} \chi_{\mathrm{supp}\phi_k}(x) \leqslant 3, \qquad x \in \bigcup_{|k|\leqslant N} \mathrm{supp}\ \phi_k.$$

On the strength of (4.42), (4.43) and (4.44) we find that

$$\begin{aligned}
\left(\sum_{|k|\leqslant N} \nu_k\right)^{1/q} &\ll \|\sum_{|k|\leqslant N} \phi_k w\|_q \leqslant \sum_{i=1}^2 \sum_{j=1}^2 \|F_{j,i} w\|_q \leqslant C \sum_{i=1}^2 \sum_{j=1}^2 \|F'_{j,i}/v\|_p \\
&\ll C \left(\sum_{|k|\leqslant N,i} \|\phi'_{k,i}/v\|_p^p\right)^{1/p} \ll C \left(\sum_{|k|\leqslant N} \nu_k\right)^{1/p}.
\end{aligned} \tag{4.45}$$

Therefore, by letting $N \to \infty$ we obtain $C \gg \left(\sum_{k\in\mathbb{Z}} \nu_k\right)^{1/r}$. Now we put

$$\widetilde{\lambda}_k := \int_{\eta_{k-1}}^{\eta_{k+1}} \left(\int_{\eta_{k-1}}^t \left[\int_{\eta_{k-1}}^s v^{p'}(y)dy\right]^q w^q(s)ds\right)^{\frac{r}{q}} \left(\int_{\eta_{k-1}}^t v^{p'}(y)dy\right)^{-\frac{r}{q}} v^{p'}(t)dt,$$

$$\lambda_k^* := \int_{\eta_{k-1}}^{\eta_{k+1}} \left(\int_{\eta_{k-1}}^{t} \left[\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right]^q w^q(s)ds \right)^{\frac{r}{p}} \left(\int_{\eta_{k-1}}^{t} v^{p'}(y)dy \right)^{q-\frac{r}{p}} w^q(t)dt.$$

Write

$$\widetilde{\lambda}_k = \int_{\eta_{k-1}}^{\eta_{k+1}} \left(\int_{\eta_{k-1}}^{t} \left[\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right]^q d\left[-\int_{s}^{\eta_{k+1}} w^q(x)dx \right] \right)^{\frac{r}{q}} \left(\int_{\eta_{k-1}}^{t} v^{p'}(y)dy \right)^{-\frac{r}{q}} v^{p'}(t)dt,$$

and note that

$$\begin{aligned}
&\int_{\eta_{k-1}}^{t} \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right)^q d\left(-\int_{s}^{\eta_{k+1}} w^q(x)dx \right) \\
&\quad \leqslant q \int_{\eta_{k-1}}^{t} \left(\int_{s}^{\eta_{k+1}} w^q(x)dx \right) \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right)^{q-1} v^{p'}(s)ds \\
&\quad \approx \int_{\eta_{k-1}}^{t} \left\{ \left(\int_{s}^{\eta_{k+1}} w^q(x)dx \right) \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right)^{q-1+\frac{q}{2p}} \right\} \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right)^{-\frac{q}{2p}} v^{p'}(s)ds
\end{aligned}$$

[applying Hölder's inequality with the powers r/q and p/q]

$$\begin{aligned}
&\leqslant \left(\int_{\eta_{k-1}}^{t} \left[\int_{s}^{\eta_{k+1}} w^q(x)dx \right]^{\frac{r}{q}} \left[\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right]^{\frac{r}{q'}+\frac{r}{2p}} v^{p'}(s)ds \right)^{\frac{q}{r}} \\
&\quad \times \left(\int_{\eta_{k-1}}^{t} \left[\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right]^{-\frac{1}{2}} v^{p'}(s)ds \right)^{\frac{q}{p}}.
\end{aligned}$$

This implies

$$\begin{aligned}
\widetilde{\lambda}_k &\ll \int_{\eta_{k-1}}^{\eta_{k+1}} \int_{\eta_{k-1}}^{t} \left(\int_{s}^{\eta_{k+1}} w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right)^{\frac{r}{q'}+\frac{r}{2p}} v^{p'}(s)ds \\
&\quad \times \left(\int_{\eta_{k-1}}^{t} v^{p'}(y)dy \right)^{\frac{r}{2p}-\frac{r}{q}} v^{p'}(t)dt \\
&= \int_{\eta_{k-1}}^{\eta_{k+1}} \left(\int_{s}^{\eta_{k+1}} w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right)^{\frac{r}{q'}+\frac{r}{2p}} v^{p'}(s)ds \\
&\quad \times \int_{s}^{\eta_{k+1}} \left(\int_{\eta_{k-1}}^{t} v^{p'}(y)dy \right)^{\frac{r}{2p}-\frac{r}{q}} v^{p'}(t)dt \ll \lambda_k.
\end{aligned}$$

Observe that

$$\begin{aligned}
\widetilde{\lambda}_k &= \frac{p}{r} \int_{\eta_{k-1}}^{\eta_{k+1}} \left(\int_{\eta_{k-1}}^{t} \left[\int_{\eta_{k-1}}^{s} v^{p'}(y)dy \right]^q w^q(s)ds \right)^{\frac{r}{q}} d\left(-\left[\int_{\eta_{k-1}}^{t} v^{p'}(y)dy \right]^{-\frac{r}{p}} \right) \\
&= -\frac{p}{r}\mu_k + \frac{p}{q}\lambda_k^*.
\end{aligned}$$

Therefore,

$$\lambda_k^* = \frac{q}{p}\widetilde{\lambda}_k + \frac{q}{r}\mu_k,$$

and, hence,

$$\lambda_k^* \ll \nu_k.$$

This implies $C \gg (\sum_{k\in\mathbb{Z}} \nu_k)^{1/r} \gg \left(\sum_{k\in\mathbb{Z}} \lambda_k^*\right)^{1/r}$. Note that

$$\int_{\eta_{k-1}}^{t} v^{p'}(y)dy \leqslant \int_{\eta_{k-1}}^{t^+} v^{p'}(y)dy \leqslant 2\int_{\eta_k}^{t^+} v^{p'}(y)dy \tag{4.46}$$

and

$$\frac{1}{2}\int_{\Delta(t)} v^{p'}(y)dy \leqslant \int_{\eta_k}^{t^+} v^{p'}(y)dy \leqslant \int_{\Delta(t)} v^{p'}(y)dy \tag{4.47}$$

for any $t \in [\eta_k, \eta_{k+1}]$. Therefore, in view of (2.54) and (4.9) we obtain that

$$\begin{aligned}
\lambda_k^* &\geqslant \int_{\eta_k}^{\eta_{k+1}} \left(\int_{\eta_{k-1}}^{t} \left[\int_{\eta_{k-1}}^{s} v^{p'}(y)dy\right]^q w^q(s)ds\right)^{\frac{r}{p}} \left(\int_{\eta_{k-1}}^{t} v^{p'}(y)dy\right)^{q-\frac{r}{p}} w^q(t)dt \\
&\overset{(4.46)}{\geqslant} 2^{-\frac{r}{p}} \int_{\eta_k}^{\eta_{k+1}} \left(\int_{\Delta^-(t)} \left[\int_{\Delta^-(s)} v^{p'}(y)dy\right]^q w^q(s)ds\right)^{\frac{r}{p}} \\
&\quad \times \left(\int_{\Delta^-(t)} v^{p'}(y)dy\right)^q \left(\int_{\eta_k}^{t^+} v^{p'}(y)dy\right)^{-\frac{r}{p}} w^q(t)dt \\
&\overset{(4.47)}{\gg} \int_{\eta_k}^{\eta_{k+1}} \left(\int_{\Delta^-(t)} \left[\int_{\Delta(s)} v^{p'}(y)dy\right]^q w^q(s)ds\right)^{\frac{r}{p}} \left(\int_{\Delta(t)} v^{p'}(y)dy\right)^{q-\frac{r}{p}} w^q(t)dt.
\end{aligned}$$

Hence, the required estimate $C \gg \mathscr{B}_{PS}^-$ is proved. The estimate $C \gg \mathscr{B}_{PS}^+$ can be proved similarly with intervals formed by the fairway $\sigma(x)$ and the upper boundary function $b(x)$. Thus, the estimate (4.41) holds and, therefore, the inequality $\|\mathscr{H}\| \gg \mathscr{B}_{PS}$ is proved in the case when the fairway $\sigma(x) = x$. The general case follows similar to the proof of Theorem 4.1. By changing variables (4.18) in both hand sides of the inequality (4.19) we arrive to the inequality (4.20). Since (4.21) is true the fairway–function for $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{v}$ is just $\widetilde{\sigma}(x) = x$. Therefore, in view of the obtained estimate $\widetilde{\mathscr{B}}_{PS} \ll \|\mathscr{H}\|$ with

$$\widetilde{\mathscr{B}}_{PS}^r = \int_0^\infty \left(\int_{\widetilde{\Delta}(t)} \left[\int_{\widetilde{\Delta}(s)} \widetilde{v}^{p'}(y)dy\right]^q w^q(s)ds\right)^{\frac{r}{p}} \left(\int_{\widetilde{\Delta}(t)} \widetilde{v}^{p'}(y)dy\right)^{q-\frac{r}{p}} w^q(t)dt$$

and because of $\widetilde{\Delta}(t) = [\widetilde{a}(t), \widetilde{b}(t)] = \theta(t)$, $\int_{\widetilde{\Delta}(t)} \widetilde{v}^{p'}(y)dy = \int_{\Delta(t)} v^{p'}(y)dy$ we get $\widetilde{\mathscr{B}}_{PS} = \mathscr{B}_{PS}$.

5. Geometric Steklov operator

In this section we study the geometric Steklov operator

$$\mathscr{G}f(x) := \exp\left(\frac{1}{b(x)-a(x)}\int_{a(x)}^{b(x)} \log f(y)dy\right), \qquad f(y) \geqslant 0,$$

acting in weighted Lebesgue spaces with the border functions $a(x)$ and $b(x)$ satisfying the conditions (1.2). This operator is closely related to the Hardy operator (1.1) because of Jensen's inequality

$$[\mathscr{G}(fv)](x)w(x) \leqslant \frac{\mathscr{H}f(x)}{b(x)-a(x)}, \tag{5.1}$$

from which the upper bound for the weighted $L_p - L_q$ boundedness of $\mathscr{G}$ easy follows if $L_p - L_q$ boundedness of $\mathscr{H}$ has a suitable characterization.

THEOREM 5.1. *Let $0 < p \leqslant q < \infty$ and the operator $\mathscr{G}$ be given by* (1.3) *with the boundary functions $a(x)$ and $b(x)$ satisfying the condition* (1.2)*. Then for the "norm" $\|\mathscr{G}\|_{L_{p,v}\to L_{q,w}} := \sup_{f\geqslant 0} \frac{\|(\mathscr{G}f)w\|_q}{\|fv\|_p}$ the estimate*

$$\|\mathscr{G}\|_{L_{p,v}\to L_{q,u}} \approx \sup_{t>0} \left(\int_{\sigma_0^{-1}(a(t))}^{\sigma_0^{-1}(b(t))} u^q(x)dx \right)^{\frac{1}{q}} [b(t)-a(t)]^{-\frac{1}{p}}, \tag{5.2}$$

holds.

If $0 < q < p < \infty$ then

$$\|\mathscr{G}\|_{L_{p,v}\to L_{q,w}} \approx \left(\int_0^\infty \left[\int_{\sigma_0^{-1}(a(t))}^{\sigma_0^{-1}(b(t))} u^q(s)ds \right]^{\frac{r}{p}} [b(t)-a(t)]^{-\frac{r}{p}} u^q(t)dt \right)^{\frac{1}{r}} =: \mathscr{B}_{\mathscr{G}}, \tag{5.3}$$

where

$$\sigma_0(t) := \frac{a(t)+b(t)}{2}, \qquad u(t) := \left(\mathscr{G}v^{-1}\right)(t)w(t). \tag{5.4}$$

Proof. The proof of the estimate (5.2) is due to L.-E. Persson and D.V. Prokhorov. We prove the estimate (5.3).

The upper bound. Let $0 < q < p < \infty$ and $\mathscr{B}_{\mathscr{G}} < \infty$. According to Jensen's inequality (5.1) we can get the upper bound in (5.3) by using Theorem 4.2 with suitable summation parameters and proper weight functions. Indeed, since $\mathscr{G}(f^s) = (\mathscr{G}f)^s$ and $\mathscr{G}(f\cdot g) = (\mathscr{G}f)\cdot(\mathscr{G}g)$ the equalities

$$\|\mathscr{G}\|_{L_{p,v}\to L_{q,w}} = \|\mathscr{G}\|_{L_{p,1}\to L_{q,u}} = \|\mathscr{G}\|^{\frac{1}{s}}_{L_{\frac{p}{s},1}\to L_{\frac{q}{s},u^s}} \tag{5.5}$$

hold with u defined by (5.4). Note that for $\mathscr{H}: L_{\bar{p},1} \to L_{\bar{q},\bar{u}}$ with $\bar{u} = u^s$, $\bar{p} = p/s$ and $\bar{q} = q/s$ the corresponding parameter $\bar{r}$ is equal to r/s and, respectively, $\bar{r}/\bar{p} = r/p$. Moreover, the condition $\bar{p} > 1$ of Theorem 4.2 is satisfied if we put $0 < s < q < p < \infty$. Therefore, according to (5.1) and Theorem 4.2 with $v = 1$ and $w = u^s$ we have for any $0 < 1 < q/s < p/s < \infty$ that

$$\|\mathscr{G}\|^{\frac{1}{s}}_{L_{\frac{p}{s},1}\to L_{\frac{q}{s},u^s}} \leqslant \|\mathscr{H}\|^{\frac{1}{s}}_{L_{\frac{p}{s}}\to L_{\frac{q}{s}}} \ll \mathscr{B}_{\mathscr{G}}. \tag{5.6}$$

Now, in view of (5.5) the upper bound in (5.3) is proved.

The lower bound. Suppose that $\|\mathscr{G}\|_{L_{p,v}\to L_{q,w}} = \|\mathscr{G}\|_{L_{p,1}\to L_{q,u}} = \|\mathscr{G}\|^{1/s}_{L_{\widetilde{p},1}\to L_{\widetilde{q},\widetilde{u}}} < \infty$, where $\widetilde{u} = u^s$ and $\widetilde{p} = p/s$, $\widetilde{q} = q/s$ for any $0 < q < p < s < \infty$. In other words the inequality

$$\|(\mathscr{G}f)\widetilde{u}\|_{\widetilde{q}} \leqslant C\|f\|_{\widetilde{p}} \tag{5.7}$$

holds for all $f \in L_{\widetilde{p}}$ and $0 < \widetilde{q} < \widetilde{p} < 1$ with $C < \infty$ independent on f. Now the fairway–function $\sigma(x)$ is such that $b(x) - \sigma(x) = \sigma(x) - a(x)$. Let us remind that as before we denote

$$x^- = \sigma^{-1}(a(x)), \qquad x^+ = \sigma^{-1}(b(x)), \qquad \theta(x) = [x^-, x^+].$$

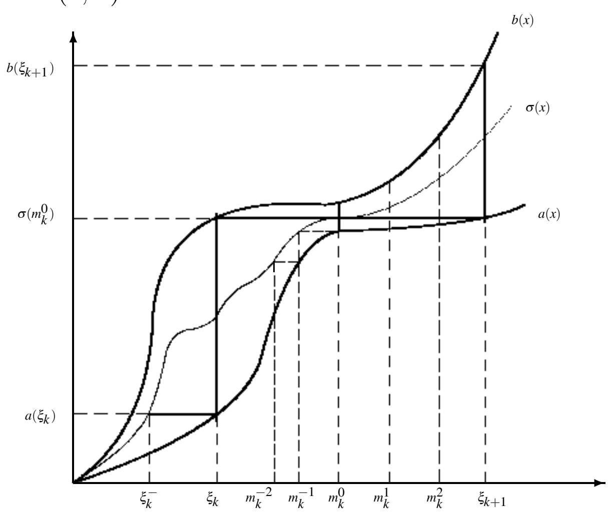
Similarly to the sections 4.1 and 4.2 we prove that

$$\left(\int_0^\infty \left[\int_{\theta^\pm(x)} \widetilde{u}^{\widetilde{q}}(s)ds\right]^{\frac{\widetilde{r}}{\widetilde{p}}} [b(x)-a(x)]^{-\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u}^{\widetilde{q}}(x)dx\right)^{\frac{1}{\widetilde{r}}} =: (\widetilde{\mathscr{B}}_{\mathscr{G}})^\pm \ll \|\mathscr{G}\|_{L_{\widetilde{p},1}\to L_{\widetilde{q},\widetilde{u}}},$$

where $\theta^-(x) = (x^-, x)$, $\theta^+(x) = (x, x^+)$ and $1/\widetilde{r} = 1/\widetilde{q} - 1/\widetilde{p}$. To prove the inequality with $(\widetilde{\mathscr{B}}_{\mathscr{G}})^-$ we put $\xi_0 = 1$ and define the sequence $\{\xi_k\} \subset (0,\infty)$ as before such that

$$\xi_k = (a^{-1}\circ b)^k(1), \qquad k \in \mathbb{Z}.$$

Then $\xi_k < \xi_k^+ = \xi_{k+1}^- < \xi_{k+1}$. Moreover, $\bigcup_k[\xi_k, \xi_{k+1}) = (0,\infty)$ and $\bigcup_k[\sigma(\xi_k), \sigma(\xi_{k+1})) = (0,\infty)$.



Further, put $\xi_k^+ = \xi_{k+1}^- = m_k^0$ and construct on every segment $[\xi_k, \xi_{k+1}]$ the sequence $\{m_k^j\}$ with $-j_a(k) \leqslant j \leqslant j_b(k)$ by the following way: for $-j_a(k) \leqslant j \leqslant 0$ we use the construction of Lemma 2.7 with $v = 1$ and $d = \xi_k$, while for $0 \leqslant j \leqslant j_b(k)$ we use Lemma 2.5 with $v = 1$ and $c = \xi_{k+1}$. Actually we have the sequence $\{m_k^j\}$, $-j_a(k) \leqslant j \leqslant j_b(k)$, defined by:

(1) $m_k^{-j_a(k)} = \xi_k, \quad m_k^{j_b(k)} = \xi_{k+1};$

(2) if $(\xi_k^+)^- \leqslant \xi_k$ then $j_a(k)=1$; $j_b(k)=1$ if $[\mathscr{N}^-(k)]=0$ or $\mathscr{N}^-(k)=[\mathscr{N}^-(k)]=1$;

(3) if $(\xi_k^+)^- > \xi_k$, then $j_a(k) > 1$ and $m_k^{j-1} = (m_k^j)^-$, where $(m_k^j)^- > \xi_k$ and $-j_a(k)+2 \leqslant j \leqslant 0$;

(4) if $[\mathcal{N}^-(k)]>0$, then the points m_k^j for $1\leqslant j\leqslant[\mathcal{N}^-(k)]$ are taken so that

$$b(m_k^j)-a(\xi_{k+1})=2\,[b(m_k^{j-1})-a(\xi_{k+1})]. \tag{5.8}$$

Then we have $\bigcup_k\bigcup_{j=-j_a(k)}^{j_b(k)-1}[m_k^j,m_k^{j+1})=(0,\infty)$ and the following useful properties of $b(t)-a(t)$ coming from Lemmas 2.5 and 2.7:

1°) if $t\in(m_k^j,m_k^{j+1})$, then

$$b(t)-a(t)\approx b(m_k^{j+1})-a(m_k^j) \tag{5.9}$$

and

$$b(m_k^{j+1})-a(m_k^j)\approx b(m_k^j)-a(m_k^j)\approx b(m_k^{j+1})-a(m_k^{j+1}); \tag{5.10}$$

2°) if $t\in[x^-,x]$, then

$$b(t)-a(t)\approx b(x)-a(x^-)\approx b(x)-a(x)\qquad\text{for}\qquad \xi_k\leqslant x^-<t<x\leqslant\xi_k^+, \tag{5.11}$$

and

$$b(t)-a(t)<b(x)-a(x^-)\approx b(x)-a(x); \tag{5.12}$$

3°) if $0\leqslant j\leqslant j_b(k)-2$, then

$$b(m_k^{j+i})-a(\xi_{k+1})=2^i\,[b(m_k^j)-a(\xi_{k+1})]\ \text{ for some } i\in\{1,\dots,j_b(k)-j-1\}. \tag{5.13}$$

Now we apply in (5.7) a test function of the form $f(y)=\sum_{k\in\mathbb{Z}}\sum_{j=-j_a(k)}^{j_b(k)-1}f_{k,j}(y)$, where

$$f_{k,j}(y)=\chi_{(a(m_k^j),b(m_k^{j+1}))}(y)l_{k,j}$$

and

$$l_{k,j}:=\left(\int_{(m_k^j)^-}^{m_k^{j+1}}\widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}\widetilde{q}}}[b(m_k^{j+1})-a(m_k^j)]^{-\frac{\widetilde{r}}{\widetilde{p}\widetilde{q}}}.$$

Then we have

$$\begin{aligned}\|(\mathcal{G}f)\widetilde{u}\|_{\widetilde{q}}^{\widetilde{q}} &= \int_0^\infty\left(\exp\frac{1}{b(x)-a(x)}\int_{a(x)}^{b(x)}\log f(y)dy\right)^{\widetilde{q}}\widetilde{u}^{\widetilde{q}}(x)dx\\ &= \sum_{k\in\mathbb{Z}}\sum_{j=-j_a(k)}^{j_b(k)-1}\int_{m_k^j}^{m_k^{j+1}}\left(\exp\frac{1}{b(x)-a(x)}\int_{a(x)}^{b(x)}\log\left[\sum_{n\in\mathbb{Z}}\sum_{i=-j_a(n)}^{j_b(n)-1}f_{n,i}(y)\right]dy\right)^{\widetilde{q}}\\ &\quad\times\widetilde{u}^{\widetilde{q}}(x)dx\\ &\geqslant \sum_{k\in\mathbb{Z}}\sum_{j=-j_a(k)}^{j_b(k)-1}l_{k,j}^{\widetilde{q}}\int_{m_k^j}^{m_k^{j+1}}\widetilde{u}^{\widetilde{q}}(x)dx\end{aligned}$$

$$\stackrel{(5.9)}{\gg} \sum_{k\in\mathbb{Z}} \sum_{j=-j_a(k)}^{j_b(k)-1} \int_{m_k^j}^{m_k^{j+1}} \left(\int_{(m_k^j)^-}^{x} \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}} (b(x)-a(x))^{-\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u}^{\widetilde{q}}(x)dx$$

$$=: \sum_{k\in\mathbb{Z}} \sum_{j=-j_a(k)}^{j_b(k)-1} \gamma_{k,j}. \tag{5.14}$$

On the other hand in view of $\widetilde{p}<1$ we have that

$$\|f\|_{\widetilde{p}}^{\widetilde{p}} = \sum_{k\in\mathbb{Z}} \int_{\sigma(\xi_k)}^{\sigma(\xi_{k+1})} \left(\sum_{n\in\mathbb{Z}} \sum_{i=-j_a(n)}^{j_b(n)-1} f_{n,i}(y)\right)^{\widetilde{p}} dy \leqslant \sum_{k\in\mathbb{Z}} \int_{\sigma(\xi_k)}^{\sigma(\xi_{k+1})} \left(\sum_{n\in\mathbb{Z}} \sum_{i=-j_a(n)}^{j_b(n)-1} f_{n,i}^{\widetilde{p}}(y)\right) dy$$

$$= \sum_{k\in\mathbb{Z}} \int_{\sigma(\xi_k)}^{\sigma(\xi_{k+1})} \left(\sum_{n\in\mathbb{Z}} \sum_{i=-j_a(n)}^{j_b(n)-1} \chi_{(a(m_n^i),b(m_n^{i+1}))}(y)\, l_{n,i}^{\widetilde{p}}\right) dy.$$

Denote

$$M_{n,i}(k):\ = (\sigma(\xi_k),\sigma(\xi_{k+1}))\cap(a(m_n^i),b(m_n^{i+1}))$$

and observe that $M_{n,i}(k)=\varnothing$ if $n\leqslant k-2$ and $n\geqslant k+2$, while $M_{n,i}(k)\subseteq(a(m_n^i),b(m_n^{i+1}))$ for the rest $k-1\leqslant n\leqslant k+1$ and all corresponding them $-i_a(n)\leqslant i\leqslant i_b(n)-1$. Therefore,

$$\|f\|_{\widetilde{p}}^{\widetilde{p}} \leqslant \sum_{k\in\mathbb{Z}} \int_{\sigma(\xi_k)}^{\sigma(\xi_{k+1})} \left(\sum_{n=k-1}^{k+1} \sum_{i=-j_a(n)}^{j_b(n)-1} \chi_{M_{n,i}(k)}(y)\, l_{n,i}^{\widetilde{p}}\right) dy$$

$$= \sum_{k\in\mathbb{Z}} \int_{\sigma(\xi_k)}^{\sigma(\xi_{k+1})} \Bigg(\sum_{j=-j_a(k-1)}^{j_b(k-1)-1} \chi_{M_{k-1,j}(k)}(y)\, l_{k-1,j}^{\widetilde{p}} + \sum_{j=-j_a(k)}^{j_b(k)-1} \chi_{M_{k,j}(k)}(y)\, l_{k,j}^{\widetilde{p}}$$

$$+ \sum_{j=-j_a(k+1)}^{j_b(k+1)-1} \chi_{M_{k+1,j}(k)}(y)\, l_{k+1,j}^{\widetilde{p}} \Bigg) dy \leqslant \sum_{k\in\mathbb{Z}} \sum_{j=-j_a(k-1)}^{j_b(k-1)-1} l_{k-1,j}^{\widetilde{p}}[b(m_{k-1}^{j+1})-a(m_{k-1}^j)]$$

$$+\sum_{k\in\mathbb{Z}} \sum_{j=-j_a(k)}^{j_b(k)-1} l_{k,j}^{\widetilde{p}}[b(m_k^{j+1})-a(m_k^j)] + \sum_{k\in\mathbb{Z}} \sum_{j=-j_a(k+1)}^{j_b(k+1)-1} l_{k+1,j}^{\widetilde{p}}[b(m_{k+1}^{j+1})-a(m_{k+1}^j)]$$

$$= \sum_{k\in\mathbb{Z}} \left(\sum_{j=-j_a(k-1)}^{j_b(k-1)-1} \tau_{k-1,j} + \sum_{j=-j_a(k)}^{j_b(k)-1} \tau_{k,j} + \sum_{j=-j_a(k+1)}^{j_b(k+1)-1} \tau_{k+1,j} \right)$$

$$\leqslant 3 \sum_{k\in\mathbb{Z}} \sum_{j=-j_a(k)}^{j_b(k)-1} \tau_{k,j}, \tag{5.15}$$

where

$$\tau_{k,j}:\ = \left(\int_{(m_k^j)^-}^{m_k^{j+1}} \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{q}}} [b(m_k^{j+1})-a(m_k^j)]^{-\frac{\widetilde{r}}{\widetilde{p}}}$$

and

$$\tau_{k,j} \;=\; \frac{q}{r}[b(m_k^{j+1})-a(m_k^j)]^{-\frac{\widetilde{r}}{\widetilde{p}}} \int_{(m_k^j)^-}^{m_k^{j+1}} \left(\int_{(m_k^j)^-}^{t} \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u}^{\widetilde{q}}(t)dt$$

$$
\begin{aligned}
&= \frac{q}{r}[b(m_k^{j+1})-a(m_k^j)]^{-\frac{\widetilde{r}}{\widetilde{p}}}\int_{(m_k^j)^-}^{m_k^j}\left(\int_{(m_k^j)^-}^t \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}}\widetilde{u}^{\widetilde{q}}(t)dt \\
&\quad +\frac{q}{r}[b(m_k^{j+1})-a(m_k^j)]^{-\frac{\widetilde{r}}{\widetilde{p}}}\int_{m_k^j}^{m_k^{j+1}}\left(\int_{(m_k^j)^-}^t \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}}\widetilde{u}^{\widetilde{q}}(t)dt \\
&=: I_{k,j}+II_{k,j}. \qquad (5.16)
\end{aligned}
$$

On the strength of (5.9) $II_{k,j}\approx\gamma_{k,j}$ and we can write that

$$
\|f\|_{\widetilde{p}}^{\widetilde{p}} \ll \sum_{k\in\mathbb{Z}}\sum_{j=-j_a(k)}^{j_b(k)-1}\left[I_{k,j}+\gamma_{k,j}\right]. \qquad (5.17)
$$

Now we need to estimate $\sum_{k\in\mathbb{Z}}\sum_{j=-j_a(k)}^{j_b(k)-1} I_{k,j}$ by $\sum_{k\in\mathbb{Z}}\sum_{j=-j_a(k)}^{j_b(k)-1}\gamma_{k,j}$. Let $j=-j_a(k)$, then $(m_k^{-j_a(k)})^-=m_{k-1}^0$ and

$$
\begin{aligned}
I_{k,-j_a(k)} &= \left[b(m_k^{-j_a(k)+1})-a(m_k^{-j_a(k)})\right]^{-\frac{\widetilde{r}}{\widetilde{p}}}\int_{\xi_k^-}^{\xi_k}\left(\int_{\xi_k^-}^t \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}}\widetilde{u}^{\widetilde{q}}(t)dt \\
&\overset{(5.10)}{\approx} [b(\xi_k)-a(\xi_k)]^{-\frac{\widetilde{r}}{\widetilde{p}}}\int_{\xi_k^-}^{\xi_k}\left(\int_{\xi_k^-}^t \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}}\widetilde{u}^{\widetilde{q}}(t)dt \\
&\overset{(5.12)}{\ll} \int_{\xi_k^-}^{\xi_k}\left(\int_{\xi_k^-}^t \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}}[b(t)-a(t)]^{-\frac{\widetilde{r}}{\widetilde{p}}}\widetilde{u}^{\widetilde{q}}(t)dt \\
&\leqslant \sum_{j=0}^{j_b(k-1)-1}\int_{m_{k-1}^j}^{m_{k-1}^{j+1}}\left(\int_{(m_{k-1}^j)^-}^t \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}}[b(t)-a(t)]^{-\frac{\widetilde{r}}{\widetilde{p}}}\widetilde{u}^{\widetilde{q}}(t)dt \\
&= \sum_{j=0}^{j_b(k-1)-1}\gamma_{k-1,j}. \qquad (5.18)
\end{aligned}
$$

Let $j=0$, then we have if $j_a(k)=1$ that

$$
\begin{aligned}
I_{k,0} &= \left[b(m_k^1)-a(m_k^0)\right]^{-\frac{\widetilde{r}}{\widetilde{p}}}\int_{(m_k^0)^-}^{m_k^0}\left(\int_{(m_k^0)^-}^t \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}}\widetilde{u}^{\widetilde{q}}(t)dt \\
&\overset{(5.10),(5.12)}{\ll} \int_{(m_k^0)^-}^{m_k^0}\left(\int_{(m_k^0)^-}^t \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}}[b(t)-a(t)]^{-\frac{\widetilde{r}}{\widetilde{p}}}\widetilde{u}^{\widetilde{q}}(t)dt \\
&\leqslant \sum_{j=0}^{j_b(k-1)-1}\int_{m_{k-1}^j}^{m_{k-1}^{j+1}}\left(\int_{(m_{k-1}^j)^-}^t \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}}[b(t)-a(t)]^{-\frac{\widetilde{r}}{\widetilde{p}}}\widetilde{u}^{\widetilde{q}}(t)dt \\
&\quad +\int_{m_k^{-1}}^{m_k^0}\left(\int_{(m_k^{-1})^-}^t \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}}[b(t)-a(t)]^{-\frac{\widetilde{r}}{\widetilde{p}}}\widetilde{u}^{\widetilde{q}}(t)dt
\end{aligned}
$$

$$\leqslant \sum_{j=0}^{j_b(k-1)-1} \gamma_{k-1,j} + \gamma_{k,-1}. \tag{5.19}$$

If $j_a(k) > 1$ and $-j_a(k)+1 \leqslant j \leqslant 0$, then

$$\sum_{j=-j_a(k)+1}^{0} I_{k,j} = I_{k,-j_a(k)+1} + \sum_{j=-j_a(k)+2}^{0} I_{k,j}. \tag{5.20}$$

Analogously to (5.19),

$$I_{k,-j_a(k)+1} \ll \sum_{j=0}^{j_b(k-1)-1} \gamma_{k-1,j} + \gamma_{k,-j_a(k)}, \tag{5.21}$$

while

$$\sum_{j=-j_a(k)+2}^{0} I_{k,j} \leqslant \sum_{j=-j_a(k)+2}^{0} \gamma_{k,j-1} = \sum_{j=-j_a(k)+1}^{-1} \gamma_{k,j}. \tag{5.22}$$

Finally we get from (5.18) – (5.22) that

$$\sum_{j=-j_a(k)}^{0} I_{k,j} \ll 2\sum_{j=0}^{j_b(k-1)-1} \gamma_{k-1,j} + \sum_{j=-j_a(k)}^{-1} \gamma_{k,j} \leqslant 2\left(\sum_{j=0}^{j_b(k-1)-1} \gamma_{k-1,j} + \sum_{j=-j_a(k)}^{j_b(k)-1} \gamma_{k,j}\right). \tag{5.23}$$

Further, if $j_b(k) = 1$, the required estimate follows from (5.23). Let $j_b(k) > 1$ and $1 \leqslant j \leqslant j_b(k)-1$. Since $(m_k^0)^- < (m_k^j)^- < m_k^0 < m_k^j$ we have for $1 \leqslant j \leqslant j_b(k)-1$ that

$$\begin{aligned} I_{k,j} \leqslant & \left[b(m_k^{j+1}) - a(m_k^j)\right]^{-\frac{\widetilde{r}}{\widetilde{p}}} \int_{(m_k^0)^-}^{m_k^0} \left(\int_{(m_k^0)^-}^{t} \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u}^{\widetilde{q}}(t)dt \\ & + \left[b(m_k^{j+1}) - a(m_k^j)\right]^{-\frac{\widetilde{r}}{\widetilde{p}}} \sum_{i=1}^{j} \int_{m_k^{i-1}}^{m_k^i} \left(\int_{(m_k^j)^-}^{t} \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u}^{\widetilde{q}}(t)dt =: I'_{k,j} + I''_{k,j}. \end{aligned}$$

To estimate $I'_{k,j}$ observe that $b(m_k^j) - a(m_k^j) \approx b(m_k^j) - a(\xi_{k+1})$ and in view of (5.10) and (5.13)

$$b(m_k^{j+1}) - a(m_k^j) \approx b(m_k^j) - a(m_k^j) \approx b(m_k^j) - a(\xi_{k+1}) = 2^j\,[b(m_k^0) - \sigma(m_k^0)].$$

Analogously to (5.19) we have for each $1 \leqslant j \leqslant j_b(k)-1$ that

$$\begin{aligned} I'_{k,j} & \ll 2^{-\frac{j\widetilde{r}}{\widetilde{p}}} \left[(b(m_k^0) - \sigma(m_k^0)\right]^{-\frac{\widetilde{r}}{\widetilde{p}}} \int_{(m_k^0)^-}^{m_k^0} \left(\int_{(m_k^0)^-}^{t} \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u}^{\widetilde{q}}(t)dt \\ & \overset{(5.12)}{\ll} 2^{-\frac{j\widetilde{r}}{\widetilde{p}}} \int_{(m_k^0)^-}^{m_k^0} \left(\int_{(m_k^0)^-}^{t} \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}} [b(t) - a(t)]^{-\frac{\widetilde{r}}{p}} \widetilde{u}^{\widetilde{q}}(t)dt \end{aligned}$$

$$\leqslant 2^{-\frac{j\widetilde{r}}{\widetilde{p}}} \left[\sum_{i=0}^{j_b(k-1)-1} \int_{m_{k-1}^i}^{m_{k-1}^{i+1}} \left(\int_{(m_{k-1}^i)^-}^t \widetilde{u^q}(s)ds \right)^{\frac{\widetilde{r}}{\widetilde{p}}} [b(t)-a(t)]^{-\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u^q}(t)dt \right.$$

$$\left. + \sum_{n=-j_a(k)}^{-1} \int_{m_k^n}^{m_k^{n+1}} \left(\int_{(m_k^n)^-}^t \widetilde{u^q}(s)ds \right)^{\frac{\widetilde{r}}{\widetilde{p}}} [b(t)-a(t)]^{-\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u^q}(t)dt \right]$$

$$= 2^{-\frac{j\widetilde{r}}{\widetilde{p}}} \left[\sum_{i=0}^{j_b(k-1)-1} \gamma_{k-1,i} + \sum_{n=-j_a(k)}^{-1} \gamma_{k,n} \right]. \tag{5.24}$$

Therefore,

$$\sum_{j=1}^{j_b(k)-1} I'_{k,j} \ll \left[\sum_{j=1}^{j_b(k)-1} 2^{-\frac{j\widetilde{r}}{\widetilde{p}}} \right] \left[\sum_{i=0}^{j_b(k-1)-1} \gamma_{k-1,i} + \sum_{n=-j_a(k)}^{-1} \gamma_{k,n} \right]$$

$$\leqslant \frac{2^{\widetilde{r}/\widetilde{p}}}{2^{\widetilde{r}/\widetilde{p}}-1} \left[\sum_{i=0}^{j_b(k-1)-1} \gamma_{k-1,i} + \sum_{n=-j_a(k)}^{-1} \gamma_{k,n} \right]. \tag{5.25}$$

Further, in view of (5.10),

$$I''_{k,j} \ll \left[b(m_k^j) - a(m_k^j) \right]^{-\frac{\widetilde{r}}{\widetilde{p}}} \sum_{i=1}^{j} \int_{m_k^{i-1}}^{m_k^i} \left(\int_{(m_k^j)^-}^t \widetilde{u^q}(s)ds \right)^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u^q}(t)dt. \tag{5.26}$$

Therefore, on the strength of (5.13) and the equivalence $b(m_k^j) - a(m_k^j)$ to $b(m_k^j) - a(\xi_{k+1})$ we get

$$\sum_{j=1}^{j_b(k)-1} I''_{k,j} \ll \sum_{j=1}^{j_b(k)-1} \left[b(m_k^j) - a(\xi_{k+1}) \right]^{-\frac{\widetilde{r}}{\widetilde{p}}} \sum_{i=1}^{j} \int_{m_k^{i-1}}^{m_k^i} \left(\int_{(m_k^j)^-}^t \widetilde{u^q}(s)ds \right)^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u^q}(t)dt$$

$$= \left[b(m_k^1) - a(\xi_{k+1}) \right]^{-\frac{\widetilde{r}}{\widetilde{p}}} \int_{m_k^0}^{m_k^1} \left(\int_{(m_k^1)^-}^t \widetilde{u^q}(s)ds \right)^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u^q}(t)dt$$

$$+ \left[b(m_k^2) - a(\xi_{k+1}) \right]^{-\frac{\widetilde{r}}{\widetilde{p}}} \left(\left[\int_{m_k^0}^{m_k^1} + \int_{m_k^1}^{m_k^2} \right] \left[\int_{(m_k^2)^-}^t \widetilde{u^q}(s)ds \right]^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u^q}(t)dt \right) + \ldots$$

$$\ldots + \left[b(m_k^{j_b(k)-1}) - a(\xi_{k+1}) \right]^{-\frac{\widetilde{r}}{\widetilde{p}}}$$

$$\times \left(\left[\int_{m_k^0}^{m_k^1} + \ldots + \int_{m_k^{j_b(k)-2}}^{m_k^{j_b(k)-1}} \right] \left[\int_{(m_k^{j_b(k)-1})^-}^t \widetilde{u^q}(s)ds \right]^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u^q}(t)dt \right)$$

$$\overset{(5.13)}{\leqslant} \left[\sum_{j=0}^{j_b(k)-1} 2^{-\frac{j\widetilde{r}}{\widetilde{p}}} \right] \left[b(m_k^1) - a(\xi_{k+1}) \right]^{-\frac{\widetilde{r}}{\widetilde{p}}} \int_{m_k^0}^{m_k^1} \left(\int_{(m_k^1)^-}^t \widetilde{u^q}(s)ds \right)^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u^q}(t)dt$$

$$+ \left[\sum_{j=0}^{j_b(k)-1} 2^{-\frac{j\widetilde{r}}{\widetilde{p}}} \right] \left[b(m_k^2) - a(\xi_{k+1}) \right]^{-\frac{\widetilde{r}}{\widetilde{p}}} \int_{m_k^1}^{m_k^2} \left(\int_{(m_k^2)^-}^t \widetilde{u^q}(s)ds \right)^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u^q}(t)dt + \ldots$$

$$\dots + \left[\sum_{j=0}^{j_b(k)-1} 2^{-\frac{j\widetilde{r}}{\widetilde{p}}}\right] \left[b(m_k^{j_b(k)-1}) - a(\xi_{k+1})\right]^{-\frac{\widetilde{r}}{\widetilde{p}}} \int_{m_k^{j_b(k)-2}}^{m_k^{j_b(k)-1}}$$

$$\times \left(\int_{(m_k^{j_b(k)-1})^-}^{t} \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u}^{\widetilde{q}}(t)dt$$

$$\overset{(5.9),(5.10),(5.12)}{\ll} \frac{2^{\widetilde{r}/\widetilde{p}}}{2^{\widetilde{r}/\widetilde{p}}-1} \left[\int_{m_k^0}^{m_k^1} \left(\int_{(m_k^1)^-}^{t} \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}} [b(t)-a(t)]^{-\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u}^{\widetilde{q}}(t)dt + \dots\right.$$

$$\left.\dots + \int_{m_k^{j_b(k)-2}}^{m_k^{j_b(k)-1}} \left(\int_{(m_k^{j_b(k)-1})^-}^{t} \widetilde{u}^{\widetilde{q}}(s)ds\right)^{\frac{\widetilde{r}}{\widetilde{p}}} [b(t)-a(t)]^{-\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u}^{\widetilde{q}}(t)dt\right]$$

$$\leqslant \frac{2^{\widetilde{r}/\widetilde{p}}}{2^{\widetilde{r}/\widetilde{p}}-1} \sum_{j=0}^{j_b(k)-1} \gamma_{k,j}. \tag{5.27}$$

Now we have from (5.25) – (5.27) that

$$\sum_{j=1}^{j_b(k)-1} I_{k,j} \ll \frac{2^{\widetilde{r}/\widetilde{p}}}{2^{\widetilde{r}/\widetilde{p}}-1} \left[\sum_{j=0}^{j_b(k-1)-1} \gamma_{k-1,j} + \sum_{j=-j_a(k)}^{j_b(k)-1} \gamma_{k,j}\right]. \tag{5.28}$$

By combining (5.23) and (5.28) we conclude that

$$\sum_{j=-j_a(k)}^{j_b(k)-1} I_{k,j} \ll \max\left\{2, \frac{2^{\widetilde{r}/\widetilde{p}}}{2^{\widetilde{r}/\widetilde{p}}-1}\right\} \left[\sum_{j=0}^{j_b(k-1)-1} \gamma_{k-1,j} + \sum_{j=-j_a(k)}^{j_b(k)-1} \gamma_{k,j}\right]$$

$$\leqslant \max\left\{2, \frac{2^{\widetilde{r}/\widetilde{p}}}{2^{\widetilde{r}/\widetilde{p}}-1}\right\} \left[\sum_{j=-j_a(k-1)}^{j_b(k-1)-1} \gamma_{k-1,j} + \sum_{j=-j_a(k)}^{j_b(k)-1} \gamma_{k,j}\right]. \tag{5.29}$$

It follows from (5.17) and (5.29) that

$$\|f\|_{\widetilde{p}}^{\widetilde{p}} \ll \sum_{k\in\mathbb{Z}} \left[\sum_{j=-j_a(k-1)}^{j_b(k-1)-1} \gamma_{k-1,j} + \sum_{j=-j_a(k)}^{j_b(k)-1} \gamma_{k,j}\right] \leqslant 2\sum_{k\in\mathbb{Z}} \sum_{j=-j_a(k)}^{j_b(k)-1} \gamma_{k,j}. \tag{5.30}$$

Finally, we have by combining (5.14) and (5.30) that

$$\|\mathscr{G}\|_{L_{\widetilde{p},1}\to L_{\widetilde{q},\widetilde{u}}} \gg \left(\sum_{k\in\mathbb{Z}} \sum_{j=-j_a(k)}^{j_b(k)-1} \gamma_{k,j}\right)^{\frac{1}{\widetilde{r}}}$$

$$\geqslant \left(\sum_{k\in\mathbb{Z}} \sum_{j=-j_a(k)}^{j_b(k)-1} \int_{m_k^j}^{m_k^{j+1}} \left[\int_{\theta^-(x)} \widetilde{u}^{\widetilde{q}}(s)ds\right]^{\frac{\widetilde{r}}{\widetilde{p}}} [b(x)-a(x)]^{-\frac{\widetilde{r}}{\widetilde{p}}} \widetilde{u}^{\widetilde{q}}(x)dx\right)^{\frac{1}{\widetilde{r}}}$$

$$= (\widetilde{\mathscr{B}}_{\mathscr{G}})^-.$$

Analogously we can prove the inequality

$$\|\mathscr{G}\|_{L_{\widetilde{p},1}\to L_{\widetilde{q},\widetilde{u}}} \gg (\widetilde{\mathscr{B}}_{\mathscr{G}})^+.$$

Hence, in view of the equivalence $\mathscr{B}_{\mathscr{G}} \approx (\mathscr{B}_{\mathscr{G}})^- + (\mathscr{B}_{\mathscr{G}})^+$ and since $\widetilde{r}/\widetilde{p} = r/p$ and $\widetilde{u^q} = u^q$ the lower bound for $\|\mathscr{G}\|_{L_{p,v}\to L_{q,w}}$ is proved.

6. Applications

6.1. Embeddings

Let $u,$ v and w be non-negative measurable weight functions such that for $0 < p,q,s < \infty,$ $p > 1$ the powers $u^s,$ $w^q,$ v^{-p} and $v^{p'}$ are locally integrable on $(0,\infty)$ and $0 < u(x), v(x), w(x) < \infty$ for a.e. $x \in \mathbb{R}^+$. The Sobolev space $W^1_{p,s}$ consists of all absolutely continuous functions on $(0,\infty)$ such that

$$\|F\|_{W^1_{p,s}} := \|Fu\|_s + \|F'/v\|_p < \infty. \tag{6.1}$$

We assume for simplicity that $W^1_{p,s}$ is a closure of the set of all finitely supported differentiable functions with respect to the norm (6.1), i.e.

$$W^1_{p,s} = \overset{\circ}{W}{}^1_{p,s}. \tag{6.2}$$

In [15] necessary and sufficient conditions for u and v are pointed out under which (6.2) holds. In this case by arguing as in [15] it is possible to construct the boundary functions $a(x)$ and $b(x)$ satisfying (1.2) such that $a(x) < x < b(x)$ and for all $x \in (0,\infty)$

$$\int_{\Delta^-(x)} v^{p'}(y)dy = \int_{\Delta^+(x)} v^{p'}(y)dy, \tag{6.3}$$

$$\left(\int_{\Delta(x)} u^s(z)dz\right)^{\frac{1}{s}} \left(\int_{\Delta(x)} v^{p'}(y)dy\right)^{\frac{1}{p'}} = 1, \tag{6.4}$$

where $\Delta(x) = [a(x), b(x)],$ $\Delta^-(x) = [a(x), x],$ $\Delta^+(x) = [x, b(x)]$. The equality (6.3) says that the fairway $\sigma(x) = x,$ and (6.4) guarantees inequalities

$$\left(\int_{\Delta(x)} |F(z)|^s u^s(z)dz\right)^{\frac{1}{s}} \leqslant 2\left(\int_{\Delta(x)} |F'(y)|^p v^{-p}(y)dy\right)^{\frac{1}{p}} \tag{6.5}$$

for every absolutely continuous function F such that $F(t) = 0$ for some $t \in \Delta(x)$ and

$$\sup_{t\in\Delta(x)} |F(t)| \ll \left(\int_{\Delta(x)} v^{p'}(y)dy\right)^{\frac{1}{p'}} \left[\left(\int_{\Delta(x)} |F(z)|^s u^s(z)dz\right)^{\frac{1}{s}} + \left(\int_{\Delta(x)} |F'(y)|^p v^{-p}(y)dy\right)^{\frac{1}{p}}\right],$$

which follow by applying Hölder's inequality.

Let $\mathscr{L} \subset \overset{\circ}{W}{}^1_{p,s}$ be the set of functions satisfying Definition 4.1. Using (6.5) it is easy to see that for $1 < p \leqslant s < \infty$

$$\|F\|_{W^1_{p,s}} \approx \|F'/v\|_p, \qquad F \in \mathscr{L}. \tag{6.6}$$

Applying Theorem 4.1 we obtain the following assertion, where the constants $\mathscr{A}_M$ and $\mathscr{B}_{MR}$ are given by (4.2) and (4.4), respectively.

THEOREM 6.1. *Let* $1 < p \leqslant q < \infty,\ s > 0$. *Then the inequality*

$$\|Fw\|_q \leqslant C\|F\|_{W^1_{p,s}}, \qquad F \in \mathscr{L} \tag{6.7}$$

is valid if and only if $\mathscr{A}_M < \infty$. *Moreover, for the least constant* C *in* (6.7) *the equivalence* $C \approx \mathscr{A}_M$ *holds. If* $0 < q < p < \infty,\ s \geqslant p > 1$, *then* (6.7) *is valid if and only if* $\mathscr{B}_{MR} < \infty$, *moreover* $C \approx \mathscr{B}_{MR}$.

Proof. The upper bounds $C \ll \mathscr{A}_M$ and $C \ll \mathscr{B}_{MR}$ follow from (4.11) and the upper bounds for the norm $\|\mathscr{H}\|$ established in Theorem 4.1. The estimate $C \gg \mathscr{B}_{MR}$ follows from the proof of Theorem 4.1 and (6.6). To justify $C \gg \mathscr{A}_M$ we fix an arbitrary number $t > 0$ and consider a test function $F_t := F(x)$ determined by

$$F(x) = \begin{cases} 0, & x \notin [(t^-)^-, t^+], \\ \displaystyle\int_{(t^-)^-}^{x} v^{p'}(y)dy, & x \in [(t^-)^-, t^-], \\ \displaystyle\int_{(t^-)^-}^{t^-} v^{p'}(y)dy, & x \in [t^-, t], \\ \displaystyle\int_x^{t^+} v^{p'}(y)dy \frac{\displaystyle\int_{(t^-)^-}^{t^-} v^{p'}(y)dy}{\displaystyle\int_t^{t^+} v^{p'}(y)dy}, & x \in [t, t^+]. \end{cases}$$

Obviously, F is absolutely continuous, $\operatorname{supp} F \subseteq [(t^-)^-, t^+]$ and

$$\left(\int_{\Delta^-(t)} w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right) \leqslant \|Fw\|_q. \tag{6.8}$$

Let us show that

$$\|F\|_{W^1_{p,s}} \ll \left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^{\frac{1}{p}}. \tag{6.9}$$

We have

$$\|Fu\|_s^s = \int_{\Delta^-(t^-)} \left(\int_{(t^-)^-}^{z} v^{p'}(y)dy\right)^s u^s(z)dz + \left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^s \int_{\Delta^-(t)} u^s(z)dz$$
$$+ \left(\frac{\displaystyle\int_{\Delta^-(t^-)} v^{p'}(y)dy}{\displaystyle\int_{\Delta^+(t)} v^{p'}(y)dy}\right)^s \int_{\Delta^+(t)} \left(\int_z^{t^+} v^{p'}(y)dy\right)^s u^s(z)dz =: I_1 + I_2 + I_3,$$

$$\|F'/v\|_p^p = \int_{\Delta^-(t^-)} v^{p'}(y)dy + \left(\frac{\displaystyle\int_{\Delta^-(t^-)} v^{p'}(y)dy}{\displaystyle\int_{\Delta^+(t)} v^{p'}(y)dy}\right)^p \int_{\Delta^+(t)} v^{p'}(y)dy =: I_4 + I_5.$$

Using (6.3) and (6.4) we find

$$I_1 \leqslant \left(\int_{\Delta^-(t^-)} u^s(z)dz\right)\left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^s \leqslant \left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^{\frac{s}{p}},$$

$$I_2 \leqslant \left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^{\frac{s}{p}}\left(\int_{\Delta(t^-)} v^{p'}(y)dy\right)^{\frac{s}{p'}}\int_{\Delta(t)} u^s(z)dz$$

$$= 2^{s/p'}\left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^{\frac{s}{p}}\left(\int_{t^-}^{(t^-)^+} v^{p'}(y)dy\right)^{\frac{s}{p'}}\int_{\Delta(t)} u^s(z)dz$$

$$\leqslant 2^{s/p'}\left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^{\frac{s}{p}},$$

$$I_3 \leqslant \left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^{\frac{s}{p}}\left(\int_{t^-}^{(t^-)^+} v^{p'}(y)dy\right)^{\frac{s}{p'}}\left(\int_{\Delta^+(t)} u^s(z)dz\right)$$

$$\leqslant \left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^{\frac{s}{p}},$$

$$I_5 \leqslant 2^{p-1}\int_{\Delta^-(t^-)} v^{p'}(y)dy.$$

Putting these estimates together we get (6.9). Now we represent the function F in the form

$$F = F_1 + F_2,$$

where

$$F_1(x) = \begin{cases} 0, & x \notin \Delta(t^-), \\ \int_{(t^-)^-}^{x} v^{p'}(y)dy, & x \in \Delta^-(t^-), \\ \int_{x}^{(t^-)^+} v^{p'}(y)dy, & x \in \Delta^+(t^-). \end{cases}$$

$F_1 \in \mathscr{L}$ because of (6.3), therefore $F_2 := F - F_1$ belongs $\mathscr{L}$ too. It is easy to see, that

$$\|F_1\|_{W^1_{p,s}} \leqslant \int_{\Delta^-(t^-)} v^{p'}(y)dy\left(\int_{\Delta^-(t^-)} u^s(z)dz\right)^{\frac{1}{s}} + \int_{\Delta^-(t^-)} v^{p'}(y)dy\left(\int_{\Delta^+(t^-)} u^s(z)dz\right)^{\frac{1}{s}}$$

$$+2\left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^{\frac{1}{p}} \ll \left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^{\frac{1}{p}} \tag{6.10}$$

and by the inequality (6.9)

$$\|F_2\|_{W^1_{p,s}} \leqslant \|F\|_{W^1_{p,s}} + \|F_1\|_{W^1_{p,s}} \ll \left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^{\frac{1}{p}}. \tag{6.11}$$

From (6.8), (6.10) and (6.11) applying (6.7) to F_1 and F_2 we find

$$\left(\int_{\delta^+(t^-)} w^q(x)dx\right)^{\frac{1}{q}}\left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right) \leqslant \|F_1w\|_q + \|F_2w\|_q$$

$$\leqslant C\left[\|F_1\|_{W^1_{p,s}} + \|F_2\|_{W^1_{p,s}}\right] \ll C\left(\int_{\Delta^-(t^-)} v^{p'}(y)dy\right)^{\frac{1}{p}}.$$

It implies

$$\left(\int_{\delta^+(t^-)} w^q(x)dx\right)^{\frac{1}{q}}\left(\int_{\Delta(t^-)} v^{p'}(y)dy\right)^{\frac{1}{p'}} \ll C,$$

where $\delta^+(t) = [t, a^{-1}(t)]$, $\delta^-(t) = [b^{-1}(t), t]$. Similar inequality with $\delta^-(t^+)$ and $\Delta(t^+)$ instead of $\delta^+(t^-)$ and $\Delta(t^-)$ follows from analogous consideration with test function $G_t := G$ of the form

$$G(x) = \begin{cases} 0, & x \notin [t^-, (t^+)^+], \\ \displaystyle\int_{t^-}^x v^{p'}(y)dy \frac{\int_{t^+}^{(t^+)^+} v^{p'}(y)dy}{\int_{t^-}^t v^{p'}(y)dy}, & x \in [t^-, t], \\ \displaystyle\int_{t^+}^{(t^+)^+} v^{p'}(y)dy, & x \in [t, t^+], \\ \displaystyle\int_{x}^{(t^+)^+} v^{p'}(y)dy, & x \in [t^+, (t^+)^+]. \end{cases}$$

Since $t > 0$ was arbitrary we obtain $\mathscr{A}_M \ll C$.

6.2. Inequalities on monotone functions

In this section we study the operator (1.1) from weighted L_p to L_q on subclasses of non-increasing ($f \downarrow$) or non-decreasing ($f \uparrow$) non-negative functions. The border functions $a(x)$ and $b(x)$ as before satisfy the conditions (1.2).

Applying the Sawyer criterion [21] we reduce the problem to the $L_p - L_q$ characterization of integral operators with Oinarov's kernels considered in Section 3. Using the results of Theorem 3.1 and 3.2 we characterize the operator $\mathscr{H}$ on the cones of monotone functions.

Let $U(x) := \int_0^x u^p(y)dy$. Put

$$\mathscr{A}_0^{\downarrow} = \left(\int_0^{\infty}\left[\int_{a(x)}^{b(x)} v(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}} U^{-1/p}(\infty),$$

$$\mathscr{A}_{1,0}^{\downarrow} := \sup_{t>0}\left(\int_0^t\left[\int_{a(x)}^{b(x)} v(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}}\left(\int_{b(t)}^{\infty} U^{-p'}(z)u^p(z)dz\right)^{\frac{1}{p'}},$$

$$\mathscr{A}_{1,1}^{\downarrow} = \sup_{s>0}\sup_{s\leqslant t\leqslant a^{-1}(b(s))}\left[\mathbb{A}_0^{\downarrow}(s,t) + \mathbb{A}_1^{\downarrow}(s,t)\right],$$

where

$$\mathbb{A}_0^{\downarrow}(s,t) = \left(\int_s^t\left[\int_{a(x)}^{a(t)} v(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}}\left(\int_{a(t)}^{b(s)} U^{-p'}(z)u^p(z)dz\right)^{\frac{1}{p'}},$$

$$\mathbb{A}_1^{\downarrow}(s,t) = \left(\int_s^t w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} \left[\int_{a(t)}^{z} v(y)dy\right]^{p'} U^{-p'}(z)u^p(z)dz\right)^{\frac{1}{p'}}.$$

Also we notate

$$\mathscr{B}_{1,0}^{\downarrow} := \left(\int_0^\infty \left[\int_0^t \left\{\int_{a(x)}^{b(x)} v(y)dy\right\}^q w^q(x)dx\right]^{\frac{r}{p}} \right.$$
$$\left. \times \left[\int_{b(t)}^\infty U^{-p'}(z)u^p(z)dz\right]^{\frac{r}{p'}} \left[\int_{a(t)}^{b(t)} v(y)dy\right]^q w^q(t)dt\right)^{\frac{1}{r}},$$

$$\mathscr{B}_{1,1}^{\downarrow} = \left[\sum_{k\in\mathbb{Z}} \left(\mathbb{B}_{k,1}^{\downarrow}\right)^r + \left(\mathbb{B}_{k,2}^{\downarrow}\right)^r + \left(\mathbb{B}_{k,3}^{\downarrow}\right)^r + \left(\mathbb{B}_{k,4}^{\downarrow}\right)^r\right]^{\frac{1}{r}},$$

where

$$\mathbb{B}_{k,1}^{\downarrow} = \left(\int_{a(\xi_k)}^{a(\xi_{k+1})} \left[\int_{\xi_k}^{a^{-1}(t)} \left\{\int_{a(x)}^{t} v(y)dy\right\}^q w^q(x)dx\right]^{\frac{r}{q}} \right.$$
$$\left. \times \left[\int_t^{a(\xi_{k+1})} U^{-p'}(z)u^p(z)dz\right]^{\frac{r}{q'}} U^{-p'}(t)u^p(t)dt\right)^{\frac{1}{r}},$$

$$\mathbb{B}_{k,2}^{\downarrow} = \left(\int_{\xi_k}^{\xi_{k+1}} \left[\int_{\xi_k}^{t} w^q(x)dx\right]^{\frac{r}{p}} \right.$$
$$\left. \times \left[\int_{a(t)}^{a(\xi_{k+1})} \left\{\int_{a(t)}^{z} v(y)dy\right\}^{p'} U^{-p'}(z)u^p(z)dz\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}},$$

$$\mathbb{B}_{k,3}^{\downarrow} = \left(\int_{b(\xi_k)}^{b(\xi_{k+1})} \left[\int_{b^{-1}(t)}^{\xi_{k+1}} \left\{\int_{a(x)}^{a(\xi_{k+1})} v(y)dy\right\}^q w^q(x)dx\right]^{\frac{r}{q}} \right.$$
$$\left. \times \left[\int_{b(\xi_k)}^{t} U^{-p'}(z)u^p(z)dz\right]^{\frac{r}{q'}} U^{-p'}(t)u^p(t)dt\right)^{\frac{1}{r}},$$

$$\mathbb{B}_{k,4}^{\downarrow} = \left(\int_{\xi_k}^{\xi_{k+1}} \left[\int_{t}^{\xi_{k+1}} w^q(x)dx\right]^{\frac{r}{p}} \right.$$
$$\left. \times \left[\int_{b(\xi_k)}^{b(t)} \left\{\int_{a(\xi_{k+1})}^{z} v(y)dy\right\}^{p'} U^{-p'}(z)u^p(z)dz\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}},$$

and $\xi_k = (a^{-1}\circ b)^k(\xi_0), \quad \xi_0 = 1.$

THEOREM 6.2. *For the least possible constant C in the inequality*

$$\|\mathscr{H} f\|_q \leqslant C\|fu\|_p, \qquad f\downarrow, \tag{6.12}$$

the estimates

$$C \approx \begin{cases} \mathscr{A}_0^{\downarrow} + \mathscr{A}_{1,0}^{\downarrow} + \mathscr{A}_{1,1}^{\downarrow}, & 1 < p \leqslant q < \infty, \\ \mathscr{A}_0^{\downarrow} + \mathscr{B}_{1,0}^{\downarrow} + \mathscr{B}_{1,1}^{\downarrow}, & 1 < q < p < \infty, \end{cases}$$

hold.

Proof. We start with analysis of the Sawyer criterion for the three-weighted inequality

$$\|Tf\|_q \leqslant C\|fu\|_p, \qquad f\downarrow, \tag{6.13}$$

where

$$Tf(x) = w(x)\int_0^\infty t(x,y)f(y)v(y)dy$$

is an integral operator with a non-negative kernel $t(x,y)$. Let

$$T^*g(y) = v(y)\int_0^\infty t(x,y)g(x)w(x)dx$$

be a formally adjoint operator to T. By Sawyer criterion [21] the inequality (6.13) is equivalent, when $1 < p,q < \infty$, to the following two inequalities

$$\left(\int_0^\infty \left[U^{-1}(z)u^{p-1}(z)\int_0^z T^*g(y)dy\right]^{p'} dx\right)^{\frac{1}{p'}} \leqslant C_1\left(\int_0^\infty g^{q'}(x)dx\right)^{\frac{1}{q'}}, \qquad g \geqslant 0, \tag{6.14}$$

and

$$\int_0^\infty T^*g(y)dy \leqslant C_2\left(\int_0^\infty g^{q'}(x)dx\right)^{\frac{1}{q'}} U^{1/p}(\infty), \qquad g \geqslant 0. \tag{6.15}$$

We assume the constants C, C_1 and C_2 as the least possible. The second inequality (6.15) is easily characterized by the duality in Lebesgue's space and

$$C_2 = \left(\int_0^\infty \left[\int_0^\infty t(x,y)v(y)dy\right]^q w^q(x)dx\right)^{\frac{1}{q}} U^{-1/p}(\infty).$$

As for the first inequality (6.14) it is more convenient for our purpose to use its dual form

$$\left(\int_0^\infty \left[w(x)\int_0^\infty t(x,y)v(y)\left\{\int_y^\infty U^{-1}(z)u^{p-1}(z)g(z)dz\right\}dy\right]^q dx\right)^{\frac{1}{q}}$$
$$\leqslant C_1\left(\int_0^\infty g^p(z)dz\right)^{\frac{1}{p}}, \qquad g \geqslant 0. \tag{6.16}$$

Now, applying this scheme for the least possible constant C in (6.12) we have

$$C \approx \mathscr{A}_0^{\downarrow} + \mathscr{A}_1^{\downarrow},$$

where $\mathscr{A}_1^{\downarrow}$ is a characterization constant for the inequality

$$\left(\int_0^\infty \left[w(x)\int_{a(x)}^{b(x)} v(y)\left\{\int_y^\infty U^{-1}(z)u^{p-1}(z)g(z)dz\right\}dy\right]^q dx\right)^{\frac{1}{q}} \leqslant C_1\left(\int_0^\infty g^p(z)dz\right)^{\frac{1}{p}}, \qquad g \geqslant 0. \tag{6.17}$$

We write for the left hand side of (6.17)

$$\begin{aligned} F &:= \int_0^\infty \left[w(x)\int_{a(x)}^{b(x)} v(y)\left\{\int_y^\infty U^{-1}(z)u^{p-1}(z)g(z)dz\right\}dy\right]^q dx \\ &\approx \int_0^\infty \left[w(x)\int_{a(x)}^{b(x)} v(y)dy\left\{\int_{b(x)}^\infty U^{-1}(z)u^{p-1}(z)g(z)dz\right\}\right]^q dx \\ &\quad + \int_0^\infty \left[w(x)\int_{a(x)}^{b(x)} U^{-1}(z)u^{p-1}(z)g(z)\left\{\int_{a(x)}^z v(y)dy\right\}dz\right]^q dx. \end{aligned}$$

Hence,

$$\begin{aligned} F &\approx \int_0^\infty \left[w(x)\int_{a(x)}^{b(x)} v(y)dy\left\{\int_{b(x)}^\infty U^{-1}(z)u^{p-1}(z)g(z)dz\right\}\right]^q dx \\ &\quad + \int_0^\infty \left[w(x)\int_{a(x)}^{b(x)} k(z,x)g(z)U^{-1}(z)u^{p-1}(z)dz\right]^q dx, \end{aligned}$$

where $k(z,x) := \int_{a(x)}^z v(y)dy$. Clearly, the kernel $k(z,x) \geqslant 0$ satisfies the condition (3.4). Thus, the inequality (6.17) is equivalent to the following two inequalities

$$\left(\int_0^\infty \left[w(x)\int_{a(x)}^{b(x)} v(y)dy\left\{\int_{b(x)}^\infty U^{-1}(z)u^{p-1}(z)g(z)dz\right\}\right]^q dx\right)^{\frac{1}{q}} \leqslant C_{1,0}\left(\int_0^\infty g^p(z)dz\right)^{\frac{1}{p}}, \qquad g \geqslant 0, \tag{6.18}$$

and

$$\left(\int_0^\infty \left[w(x)\int_{a(x)}^{b(x)} k(z,x)g(z)U^{-1}(z)u^{p-1}(z)dz\right]^q dx\right)^{\frac{1}{q}} \leqslant C_{1,1}\left(\int_0^\infty g^p(z)dz\right)^{\frac{1}{p}}, \qquad g \geqslant 0, \tag{6.19}$$

with $C_1 \approx C_{1,0} + C_{1,1}$. The constant $C_{1,0}$ of (6.18) is characterized by Lemma 2.2 and 2.4: for $1 < p \leqslant q < \infty$ we have $C_{1,0} \approx \mathscr{A}_{1,0}^{\downarrow}$, and $C_{1,0} \approx \mathscr{B}_{1,0}^{\downarrow}$ for $1 < q < p < \infty$. Applying Theorem 3.1 for the inequality (6.19) we finish the proof.

REMARK 6.1. The similar result as Theorem 6.2 is also true for non-decreasing functions. We omit details.

Bibliographical Remarks

Section 1. Investigation of the operators (1.1) in a primitive form was started in [2], the case $a(x) = x,\ b(x) = 2x$ was later completely characterized in PhD Thesis by E.N. Batuev [1]. The regular study of the $L_p - L_q$ boundedness of (1.1) was initiated in [7] and for (1.4) in [6] in the Banach function spaces setting and continued in [28], [5]. In particular, the important conception of a fairway-function was introduced in [28].

Section 2. The standard references for the weighted Hardy inequalities are the monographs [14], [10], [9] and [12] (§1.3) with original papers [31], [13], [4] and [23]. The Hardy type operators were studied in [16], [3], [26] (see also [20]). The characterization constant (2.7) was discovered in [18].

Section 3. Lemma 3.1 was stated in [29]. Theorems 3.1, 3.2 improve and correct the related results of [28].

Section 4. Less general form of Theorems 4.1 and 4.2 can be found in [28], [30].

Section 5. Theorem 5.1 in case $0 < p \leqslant q < \infty$ is proved in [17].

Section 6. Theorem 6.1 is closely related to the results of [15] and Theorem 6.2 to the results of [24], [25], [7], [22], [19], [8].

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