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An Efficient Asymptotic Extraction Approach for the Green's Functions of Conformal Antennas in Multilayered Cylindrical Media

Jun Wu, Salam K. Khamas, and G. G. Cook

***Abstract*—Asymptotic expressions are derived for the dyadic Green's functions of antennas radiating in the presence of a multilayered cylinder, where analytic representation of the asymptotic expansion coefficients eliminates the computational cost of numerical evaluation. As a result, the asymptotic extraction technique has been applied only once for a large summation order n. In addition, the Hankel function singularity encountered for source and evaluation points at the same radius has been eliminated using analytical integration.**

***Index Terms*—Cylindrical antennas, dyadic Green's function, method of moments, multilayered media.**

I. Introduction

Efficient computation of dyadic Green's functions (DGF) for antennas in the vicinity of a layered dielectric cylinder has been the focus of several studies in recent years [1]–[7]. The infinite series involved converges slowly however, or even diverge, when the source and observation points are located at the same dielectric interface, that is when $\rho' = \rho = a_i$. A procedure to accelerate the convergence has been introduced in [4], which has been enhanced further in a subsequent study [5]. The expressions developed in [5] can be used to model source and field points that are at the same dielectric interface with a small azimuth separation $\Delta\phi$. However, they are singular when $\phi = \phi'$, hence further investigations have been reported to eliminate the singularity, involving either a hybrid approach [6], or a small argument Hankel function approximation [7]. In the aforementioned studies asymptotic extraction has been considered twice; once for the proper truncation of the infinite series as $n \to \infty$ where an n independent asymptotic expansion coefficient has been introduced, and secondly as $k_z \to \infty$ for the efficient computation of the Sommerfeld integral. Owing to the rather complex nature of reflections at the dielectric interfaces, the expansion coefficients have been determined using numerical approaches such as extrapolation [5]. Asymptotic DGF expressions have then been achieved using the Sommerfeld identity and the Hankel functions addition theorem [8].

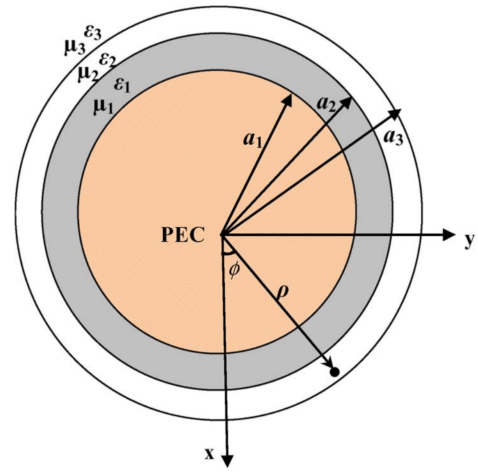

Fig. 1. Cross section of a multilayered dielectric cylinder.

In this article, alternative asymptotic DGF representations are derived using the large order principal asymptotic forms of the Bessel and Hankel functions [9], where simpler non-singular expressions have been obtained together with analytic formulations of the asymptotic expansion coefficients. Use of these expressions saves computation time and provides considerable algebraic simplifications include analytical integration of the Hankel function singularity using the Sommerfeld identity, so that no further measures are needed to handle the singularity. Furthermore, it has been shown that a single asymptotic extraction, for larger n, is sufficient to formulate closed form DGF expressions, which results in a computationally more efficient model. The algorithm has been evaluated using a method of moments (MoM) technique where cylindrically conformal wire antennas have been analyzed. Enhanced convergences of computed input impedances validate the accuracy and efficiency of the proposed formulation.

II. Formulation

General representations of cylindrical DGF are given briefly in this section based on those reported in [1], [2]. This is followed by the proposed asymptotic DGF formulation, which is fundamentally different from that presented in earlier studies [4]–[7].

A. General DGF Expansions

A multilayered cylindrical structure is illustrated in Fig. 1, where each dielectric layer has a permittivity and a permeability of ε_i and μ_i respectively. When the source and observation points are in the i^{th} layer, the spatial domain dyadic Green's function for a conformal current element is given by [1]

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = -\frac{1}{8\omega\varepsilon_i}\int_{-\infty}^{\infty} e^{-jk_z(z-z')}\sum_{n=-\infty}^{n=\infty} e^{jn(\phi-\phi')} \times\left\{\widetilde{G}_{zz}\hat{\mathbf{z}}\hat{\mathbf{z}}+\widetilde{G}_{\phi z}\hat{\boldsymbol{\phi}}\hat{\mathbf{z}}+\widetilde{G}_{z\phi}\hat{\mathbf{z}}\hat{\boldsymbol{\phi}}+\widetilde{G}_{\phi\phi}\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}\right\}dk_z \quad (1)$$

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where

$$\widetilde{G}_{zz} = k_{i\rho}^2 f_{11} \tag{2}$$

$$\widetilde{G}_{z\phi} = \frac{nk_z}{\rho}\left(f_{11} + \frac{j\omega\mu_i\rho}{nk_z}\frac{\partial f_{21}}{\partial \rho}\right) \tag{3}$$

$$\widetilde{G}_{\phi z} = \frac{nk_z}{\rho'}\left(f_{11} - \frac{j\omega\varepsilon_i\rho'}{nk_z}\frac{\partial f_{12}}{\partial \rho'}\right) \tag{4}$$

$$\begin{aligned}\widetilde{G}_{\phi\phi} = &\frac{n^2k_i^2}{\rho'\rho k_{i\rho}^2}\left(f_{11} - \frac{k_z j\omega\varepsilon_i\rho'}{k_i^2 n}\frac{\partial f_{12}}{\partial \rho'}\right)\\ &+\frac{k_i^2}{k_{i\rho}^2}\left(\frac{\partial^2 f_{22}}{\partial\rho\partial\rho'} + \frac{jn\omega\mu_i}{k_z\rho'}\frac{\partial f_{21}}{\partial\rho}\right)\\ &-\frac{n^2}{\rho'\rho}\left(f_{11} + \frac{j\omega\mu_i\rho}{k_z n}\frac{\partial f_{21}}{\partial\rho}\right)\end{aligned} \tag{5}$$

$k_i = \omega\sqrt{\mu_i\varepsilon_i}$, $k_{i\rho} = \sqrt{k_i^2 - k_z^2}$. Also [1]

$$\begin{aligned}\begin{bmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix} = &\Big\{J_n(k_{i\rho}\rho')H_n^{(2)}(k_{i\rho}\rho)\\ &\times(\bar{\mathbf{I}} + \widetilde{\overline{\mathbf{M}}}_{i+}\widetilde{\overline{\mathbf{R}}}_{i,i-1}\widetilde{\overline{\mathbf{R}}}_{i,i+1})\\ &+H_n^{(2)}(k_{i\rho}\rho')H_n^{(2)}(k_{i\rho}\rho)\widetilde{\overline{\mathbf{R}}}_{i,i-1}\\ &+J_n(k_{i\rho}\rho')J_n(k_{i\rho}\rho)\widetilde{\overline{\mathbf{R}}}_{i,i+1}\\ &+J_n(k_{i\rho}\rho)H_n^{(2)}(k_{i\rho}\rho')\\ &\times\widetilde{\overline{\mathbf{M}}}_{i-}\widetilde{\overline{\mathbf{R}}}_{i,i+1}\widetilde{\overline{\mathbf{R}}}_{i,i-1}\Big\}\end{aligned} \tag{6}$$

with $J_n(k_{i\rho}\rho)$, $H_n^{(2)}(k_{i\rho})$ represent the cylindrical Bessel and second type Hankel functions, respectively, $\widetilde{\overline{\mathbf{M}}}_{i\pm} = (\bar{\mathbf{I}} - \widetilde{\overline{\mathbf{R}}}_{i,i\pm1}\cdot\widetilde{\overline{\mathbf{R}}}_{i,i\pm1})^{-1}$,and $\widetilde{\overline{\mathbf{R}}}_{i,j}$ is the 2×2 multiple reflection matrix at the interface of layers i and j, given by [1]

$$\begin{aligned}\widetilde{\overline{\mathbf{R}}}_{i,i-1} = &\overline{\mathbf{R}}_{i,i-1} + \overline{\mathbf{T}}_{i-1,i}\widetilde{\overline{\mathbf{R}}}_{i-1,i-2}\\ &\times(\bar{\mathbf{I}} - \overline{\mathbf{R}}_{i-1,i}\widetilde{\overline{\mathbf{R}}}_{i-1,i-2})^{-1}\overline{\mathbf{T}}_{i,i-1}.\end{aligned} \tag{7}$$

Explicit expressions for the local reflection and transmission matrices,$\overline{\mathbf{R}}_{i,j}$ and $\overline{\mathbf{T}}_{i,j}$, are given in the Appendix. In order to simplify the present formulation considerably, (3)–(6) are presented in a different format compared to those reported in earlier studies.

B. Asymptotic DGF Expansions

When the summation index n is sufficiently large, the large order asymptotic expansion of the Bessel and Hankel functions can be employed. The following principal asymptotic forms [9] have been adopted as they lead to significant simplification of the final DGF expressions;

$$J_n(kr) \approx \sqrt{\frac{1}{2\pi n}}\left(\frac{ekr}{2n}\right)^n \tag{8a}$$

$$Y_n(kr) \approx -\sqrt{\frac{2}{\pi n}}\left(\frac{ekr}{2n}\right)^{-n} \tag{8b}$$

where e is the Euler's number. The large order Hankel function may be derived using

$$H_n^{(2)}(kr) = J_n(kr) - jY_n(kr) \tag{8c}$$

which can be expressed as

$$H_n^{(2)}(kr) \approx j\sqrt{\frac{2}{\pi n}}\left(\frac{ekr}{2n}\right)^{-n}\left\{1 - j\frac{1}{2}\left(\frac{ekr}{2n}\right)^{2n}\right\} \tag{8d}$$

Since $(ekr/2n)^{2n} \ll 1$ for larger n, then (8d) reduces to

$$H_n^{(2)}(kr) \approx j\sqrt{\frac{2}{\pi n}}\left(\frac{ekr}{2n}\right)^{-n}. \tag{8e}$$

Useful identities can be then attained as

$$\frac{\partial J_n(k_{i\rho}r)}{J_n(k_{i\rho}r)\partial r} = -\frac{\partial H_n^{(2)}(k_{i\rho}r)}{H_n^{(2)}(k_{i\rho}r)\partial r} \approx \frac{n}{r} \tag{9a}$$

$$\frac{J_n(k_\rho r_1)}{J_n(k_\rho r_2)} = \frac{H_n^{(2)}(k_\rho r_2)}{H_n^{(2)}(k_\rho r_1)} \approx \left(\frac{r_1}{r_2}\right)^n. \tag{9b}$$

With the aid of (9), asymptotic expressions for the normalized reflection matrix, which is given by (38), can be obtained by the matrix equation (10), shown at the bottom of page. The asymptotic transmission matrix can then be expressed as

$$\bar{\mathbf{t}}_{i,i\pm1}^a = \bar{\mathbf{r}}_{i,i\pm1}^a + \bar{\mathbf{I}} \tag{11}$$

where the superscript a denotes an asymptotic form. As n increases, local reflections at the boundaries of the i^{th} layer persist while multiple reflections from other layers' boundaries decline significantly. For instance, in a three layers cylinder it can be shown that

$$\begin{aligned}\widetilde{\overline{\mathbf{R}}}_{3,2}^a = &\overline{\mathbf{R}}_{3,2}^a + \frac{J_n(k_{3\rho}a_2)}{H_n^{(2)}(k_{3\rho}a_2)}\left(\frac{a_1}{a_2}\right)^{2n}\bar{\mathbf{t}}_{2,3}^a\bar{\mathbf{r}}_{2,1}^a\\ &\times\left(\bar{\mathbf{I}} - \left(\frac{a_1}{a_2}\right)^{2n}\bar{\mathbf{r}}_{2,3}^a\bar{\mathbf{r}}_{2,1}^a\right)^{-1}\bar{\mathbf{t}}_{3,2}^a.\end{aligned} \tag{12}$$

$$\begin{aligned}\bar{\mathbf{r}}_{i,i\pm1}^a = &\frac{1}{(\varepsilon_i+\varepsilon_{i\pm1})(\mu_i+\mu_{i\pm1})}\\ &\times\begin{bmatrix}(\varepsilon_i-\varepsilon_{i\pm1})(\mu_i+\mu_{i\pm1}) + 2\varepsilon_i\mu_i\left(\frac{k_{(i\pm1)\rho}^2}{k_{i\rho}^2}-1\right) & \mp2\varepsilon_i\mu_i\left(\frac{k_{(i\pm1)\rho}^2}{k_{i\rho}^2}-1\right)\left(\frac{k_z}{-j\varepsilon_i\omega}\right)\\ \mp2\varepsilon_i\mu_i\left(\frac{k_{(i\pm1)\rho}^2}{k_{i\rho}^2}-1\right)\left(\frac{k_z}{j\mu_i\omega}\right) & (\varepsilon_i+\varepsilon_{i\pm1})(\mu_i-\mu_{i\pm1}) + 2\varepsilon_i\mu_i\left(\frac{k_{(i\pm1)\rho}^2}{k_{i\rho}^2}-1\right)\end{bmatrix}\end{aligned} \tag{10a}$$

or

$$\begin{aligned}\bar{\mathbf{r}}_{i,i\pm1}^a = &\frac{1}{(\varepsilon_i+\varepsilon_{i\pm1})(\mu_i+\mu_{i\pm1})}\\ &\times\begin{bmatrix}(\varepsilon_i+\varepsilon_{i\pm1})(\mu_{i\pm1}-\mu_i) + 2\varepsilon_i\mu_i\frac{k_{(i\pm1)\rho}^2}{k_{i\rho}^2} - 2\varepsilon_{i\pm1}\mu_{i\pm1} & \frac{jk_i^2}{\omega\varepsilon_i k_z^2}\left(\mp2\varepsilon_i\mu_i\frac{k_{(i\pm1)\rho}^2}{k_{i\rho}^2} + 2\varepsilon_{i\pm1}\mu_{i\pm1}\right)\\ \frac{k_i^2}{j\omega\mu_i k_z^2}\left(\mp2\varepsilon_i\mu_i\frac{k_{(i\pm1)\rho}^2}{k_{i\rho}^2} + 2\varepsilon_{i\pm1}\mu_{i\pm1}\right) & (\varepsilon_{i\pm1}-\varepsilon_i)(\mu_i+\mu_{i\pm1}) + 2\varepsilon_i\mu_i\frac{k_{(i\pm1)\rho}^2}{k_{i\rho}^2} - 2\varepsilon_{i\pm1}\mu_{i\pm1}\end{bmatrix}.\end{aligned} \tag{10b}$$

For simplicity moderately thick cylindrical layers are considered first, in which the factor $(a_1/a_2)^{2n}$ decays rapidly as $n \rightarrow \infty$. Therefore, a simplified expression for the multiple reflection matrix can be written as

$$\tilde{\overline{\mathbf{R}}}^a_{3,2} \approx \overline{\mathbf{R}}^a_{3,2} \quad (13)$$

i.e. the multiple reflections matrix asymptotes to the local counterpart for larger n. With no loss of generality, a recursive formula can be expressed as

$$\tilde{\overline{\mathbf{R}}}^a_{i,i\pm1} \approx \overline{\mathbf{R}}^a_{i,i\pm1}. \quad (14)$$

Similarly

$$\tilde{\overline{\mathbf{R}}}^a_{i,i\pm1}\tilde{\overline{\mathbf{R}}}^a_{i,i\mp1} \approx \left(\frac{a_{i-1}}{a_i}\right)^{2n} \overline{\mathbf{r}}^a_{i,i\pm1}\overline{\mathbf{r}}^a_{i,i\mp1}. \quad (15)$$

Numerical computations of (14) may result in $\tilde{\overline{\mathbf{R}}}^a_{i,i\pm1} \approx \overline{\mathbf{R}}^a_{i,i\pm1} = 0$ owing to under-flows in the computations of Bessel functions when $n \rightarrow \infty$. Therefore, normalized expressions have been employed in which n independent asymptotic matrices are developed as shown in (10), which must then be multiplied by the Bessel and Hankel functions de-normalization factors given in (34), (35) to obtain $\overline{\mathbf{R}}^a_{i,i\pm1}$. Then substituting (14) in (6) provides

$$\begin{bmatrix} f^a_{11} & f^a_{12} \\ f^a_{21} & f^a_{22} \end{bmatrix} \approx J_n(k_{i\rho}\rho')H^{(2)}_n(k_{i\rho}\rho) \times \left\{ \overline{\mathbf{I}} + \frac{J_n(k_{i\rho}\rho)}{H^{(2)}_n(k_{i\rho}\rho)}\frac{H^{(2)}_n(k_{i\rho}a_i)}{J_n(k_{i\rho}a_i)}\overline{\mathbf{r}}^a_{i,i+1} + \frac{H^{(2)}_n(k_{i\rho}\rho')}{J_n(k_{i\rho}\rho')}\frac{J_n(k_{i\rho}a_{i-1})}{H^{(2)}_n(k_{i\rho}a_{i-1})}\overline{\mathbf{r}}^a_{i,i-1} \right\} \quad (16)$$

where two terms of (6) have been eliminated owing to the aforementioned factor of $(a_{i-1}/a_i)^{2n}$. Asymptotic expressions for the DGF can then be obtained by substituting the elements of (16) into the (2)–(5). Hence computationally efficient expansions can be achieved by subtracting the spectral asymptotic expansion elements from, and subsequently adding their Fourier transforms to, the overall DGF components. For example, G_{zz} can be expressed as

$$G_{zz} = \frac{-1}{8\omega\varepsilon_i}\left\{ \int_{-\infty}^{\infty} e^{-jk_z(z-z')} \sum_{n=-\infty}^{n=\infty} e^{jn(\phi-\phi')}k^2_{i\rho} \times (f_{11} - f^a_{11})\, dk_z + G^a_{zz} \right\} \quad (17)$$

where

$$G^a_{zz} = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{n=\infty} C(k_z)e^{jn(\phi-\phi')}J_n(k_{i\rho}\rho') \times H^{(2)}_n(k_{i\rho}\rho)e^{-jk_z(z-z')}dk_z. \quad (18)$$

In earlier studies, no analytical expression has been formulated for the asymptotic expansion coefficient $C(k_z)$, hence it has been determined numerically [5], [7]. As a result, it wasn't possible to make a direct use of the Sommerfeld identity for integrals such as (18). Instead, the addition theorem was used to replace the infinite summation by the Hankel function $H^{(2)}_0(k_{i\rho}|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$, with the singularity at $\boldsymbol{\rho} = \boldsymbol{\rho}'$ handled using the small argument approximation [7]. Furthermore another level of asymptotic extraction has been used as $k_z \rightarrow \infty$ for the efficient computation of the Sommerfeld integral in (18).

In the present study, an analytic expression for $C(k_z)$ can be deduced from (16) for each of the spatial asymptotic DGF components. In the case of G^a_{zz}, the expansion coefficient can be derived as

$$C(k_z) = \left(k^2_i - \frac{\partial^2}{\partial z\partial z'}\right) + \left(k^2_i\upsilon_{i-1} - w_{i-1}\frac{\partial^2}{\partial z\partial z'}\right) \times \frac{H_n(k_{i\rho}\rho')}{J_n(k_{i\rho}\rho')}\frac{J_n(k_{i\rho}a_{i-1})}{H^{(2)}_n(k_{i\rho}a_{i-1})} + \left(k^2_i\upsilon_i - w_i\frac{\partial^2}{\partial z\partial z'}\right) \times \frac{J_n(k_{i\rho}\rho)}{H^{(2)}_n(k_{i\rho}\rho)}\frac{H^{(2)}_n(k_{i\rho}a_i)}{J_n(k_{i\rho}a_i)} \quad (19)$$

where $\upsilon_i = (\mu_{i+1} - \mu_i)/(\mu_{i+1} + \mu_i)$, $\upsilon_{i-1} = (\mu_{i-1} - \mu_i)/(\mu_{i-1} + \mu_i)$, $w_{i-1} = (\varepsilon_i - \varepsilon_{i-1})/(\varepsilon_i + \varepsilon_{i-1})$, and $w_i = (\varepsilon_i - \varepsilon_{i+1})/(\varepsilon_i + \varepsilon_{i+1})$.

With the aid of (8), the following asymptotic identities can be defined

$$J_n(k_{i\rho}\rho)\frac{H^{(2)}_n(k_{i\rho}a_i)}{J_n(k_{i\rho}a_i)} \approx H^{(2)}_n(k_{i\rho}d_i) \quad \text{(20a)}$$

$$H_n(k_{i\rho}\rho)\frac{J_n(k_{i\rho}a_{i-1})}{H^{(2)}_n(k_{i\rho}a_{i-1})} \approx J_n(k_{i\rho}d_{i-1}) \quad \text{(20b)}$$

where $d_i = a^2_i/\rho$ and $d_{i-1} = (a^2_{i-1})/\rho$. As a result, (19) can be written as

$$C(k_z) = \left(k^2_i - \frac{\partial^2}{\partial z\partial z'}\right) + \left(k^2_i\upsilon_{i-1} - w_{i-1}\frac{\partial^2}{\partial z\partial z'}\right) \times \frac{J_n(k_{i\rho}d_{i-1})}{J_n(k_{i\rho}\rho')} + \left(k^2_i\upsilon_i - w_i\frac{\partial^2}{\partial z\partial z'}\right)\frac{H^{(2)}_n(k_{i\rho}d_i)}{H^{(2)}_n(k_{i\rho}\rho)}. \quad (21)$$

In the same way it can be shown that

$$\begin{bmatrix} f^a_{11} & f^a_{12} \\ f^a_{21} & f^a_{22} \end{bmatrix} \approx \left\{ J_n(k_{i\rho}\rho')H^{(2)}_n(k_{i\rho}\rho)\overline{\mathbf{I}} + J_n(k_{i\rho}\rho')H^{(2)}_n \times (k_{i\rho}d_i)\overline{\mathbf{r}}^a_{i,i+1} + H^{(2)}_n(k_{i\rho}\rho')J_n(k_{i\rho}d_{i-1})\overline{\mathbf{r}}^a_{i,i-1} \right\}. \quad (22)$$

Therefore, G^a_{zz} may be obtained by substituting (21) in (18), that is,

$$G^a_{zz} = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{n=\infty} e^{jn(\phi-\phi')} \times \left\{ \left(k^2_i - \frac{\partial^2}{\partial z\partial z'}\right)J_n(k_{i\rho}\rho')H^{(2)}_n(k_{i\rho}\rho) + \left(k^2_i\upsilon_{i-1} - w_{i-1}\frac{\partial^2}{\partial z\partial z'}\right) \times H^{(2)}_n(k_{i\rho}\rho')J_n(k_{i\rho}d_{i-1}) + \left(k^2_i\upsilon_i - w_i\frac{\partial^2}{\partial z\partial z'}\right)J_n(k_{i\rho}\rho')H^{(2)}_n(k_{i\rho}d_i) \right\} \times e^{-jk_z(z-z')}dk_z. \quad (23)$$

It can be seen from (23) that G^a_{zz} has been decomposed into three terms; the first represents radiation from a source in an infinite homogenous medium while the remaining two correspond to reflections at the boundaries of the i^{th} layer. Further, each term consists of a k_z independent coefficient that is multiplied by a Bessel and a Hankel function. Hence, these terms can be integrated analytically using the Sommerfeld identity in conjunction with the Hankel function addition theorem, that is, [8]

$$\frac{e^{-jk_i|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{-j}{2}\int_{-\infty}^{+\infty} dk_z e^{-jk_z(z-z')} \times \sum_{n=-\infty}^{\infty} J_n(k_{i\rho}\rho')H^{(2)}_n(k_{i\rho}\rho)e^{jn(\phi-\phi')}. \quad (24)$$

The closed form representation of G^a_{zz} can then be written as

$$G^a_{zz}=k_i^2\left(\frac{e^{-jk_iR}}{R}+\upsilon_i\frac{e^{-jk_iR_i}}{R_i}+\upsilon_{i-1}\frac{e^{-jk_iR_{i-1}}}{R_{i-1}}\right)-\frac{\partial^2}{\partial z\partial z'}\left(\frac{e^{-jk_iR}}{R}+w_i\frac{e^{-jk_iR_i}}{R_i}+w_{i-1}\frac{e^{-jk_iR_{i-1}}}{R_{i-1}}\right)\quad(25)$$

where $R_m=\sqrt{(z-z')^2+\rho'^2+d_m^2-2\rho' d_m\cos(\phi-\phi')}, m=i, i-1$, and $R=\sqrt{(z-z')^2+|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}$. Thus numerical evaluation of the improper integral involving $H_0^{(2)}(k_{i\rho}|\boldsymbol{\rho}-\boldsymbol{\rho}'|)$ has been avoided. As a consequence of this, simplified asymptotic DGF expansions have been developed by employing asymptotic extraction once only. It should be noted that numerical computation of (17) has been implemented using (22) for the subtracted asymptotic component. This is because the closed form representation of G^a_{zz} has been obtained by employing (21), and a valid asymptotic extraction requires the subtraction and addition of the same quantities.

The asymptotic expressions developed so far involve two quasi-static images that correspond to local reflections at the boundaries of a moderately thick cylindrical layer. For thinner layers, local reflections at the adjacent boundaries need to be incorporated into the model. This can be achieved by extracting two more quasi static images from the spectral domain Green's function, which are then added back in the closed form DGF representation. Again taking G^a_{zz} as an example therefore, from (22)

$$G^a_{zz}=\int_{-\infty}^{\infty}\sum_{n=-\infty}^{n=\infty}e^{jn(\phi-\phi')}\times\Bigg\{\left(k_i^2-\frac{\partial^2}{\partial z\partial z'}\right)J_n(k_{i\rho}\rho')H_n^{(2)}(k_{i\rho}\rho)+\sum_{\ell=i-2}^{i-1}\left(k_i^2\upsilon_\ell-w_\ell\frac{\partial^2}{\partial z\partial z'}\right)J_n(k_{i\rho}d_\ell)\times H_n^{(2)}(k_{i\rho}\rho')+\sum_{\ell=i}^{i+1}\left(k_i^2\upsilon_\ell-w_\ell\frac{\partial^2}{\partial z\partial z'}\right)\times J_n(k_{i\rho}\rho')H_n^{(2)}(k_{i\rho}d_\ell)\Bigg\}e^{-jk_z(z-z')}dk_z\quad(26)$$

and hence from (25)

$$G^a_{zz}=k_i^2\left(\frac{e^{-jk_iR}}{R}+\sum_{\ell=i-2}^{i+1}\upsilon_\ell\frac{e^{-jk_iR_\ell}}{R_\ell}\right)-\frac{\partial^2}{\partial z\partial z'}\left(\frac{e^{-jk_iR}}{R}+\sum_{\ell=i-2}^{i+1}w_\ell\frac{e^{-jk_iR_\ell}}{R_\ell}\right)\quad(27)$$

where

$$\upsilon_{i-2}=\frac{\mu_{i-2}-\mu_{i-1}}{\mu_{i-2}+\mu_{i-1}},\quad \upsilon_{i+1}=\frac{\mu_{i+2}-\mu_{i+1}}{\mu_{i+2}+\mu_{i+1}}$$
$$w_{i-2}=\frac{\varepsilon_{i-1}-\varepsilon_{i-2}}{\varepsilon_{i-1}+\varepsilon_{i-2}},\quad w_{i+1}=\frac{\varepsilon_{i+1}-\varepsilon_{i+2}}{\varepsilon_{i+1}+\varepsilon_{i+2}}.$$

Following a similar procedure, asymptotic expressions for the other DGF components can be accomplished as

$$G^a_{z\phi}=-\frac{\partial^2}{\rho'\partial\phi'\partial z}\times\int_{-\infty}^{\infty}\sum_{n=-\infty}^{n=\infty}\widetilde{b}_n(k_z)e^{jn(\phi-\phi')}e^{-jk_z(z-z')}dk_z\quad(28)$$

$$G^a_{\phi z}=-\frac{\partial^2}{\rho\partial\phi\partial z'}\times\int_{-\infty}^{\infty}\sum_{n=-\infty}^{n=\infty}\widetilde{b}_n(k_z)e^{jn(\phi-\phi')}e^{-jk_z(z-z')}dk_z\quad(29)$$

$$G^a_{\phi\phi}=\int_{-\infty}^{\infty}\sum_{n=-\infty}^{n=\infty}e^{jn(\phi-\phi')}\frac{k_i^2}{k_{i\rho}^2}\times\Bigg\{\left(\frac{\partial^2}{\rho\rho'\partial\phi\partial\phi'}+\frac{\partial^2}{\partial\rho\partial\rho'}\right)J_n(k_{i\rho}\rho')H_n^{(2)}(k_{i\rho}\rho)+\sum_{\ell=i-2}^{i-1}\frac{d_\ell}{\rho}\upsilon_\ell\left(\frac{\partial^2}{\rho' d_\ell\partial\phi\partial\phi'}+\frac{\partial^2}{\partial\rho'\partial d_\ell}\right)\times J_n(k_{i\rho}d_\ell)H_n^{(2)}(k_{i\rho}\rho')+\sum_{\ell=i}^{i+1}\frac{d_\ell}{\rho}\upsilon_\ell\left(\frac{\partial^2}{\rho' d_\ell\partial\phi\partial\phi'}+\frac{\partial^2}{\partial\rho'\partial d_\ell}\right)\times J_n(k_{i\rho}\rho')H_n^{(2)}(k_{i\rho}d_\ell)-\frac{k_{i\rho}^2}{k_i^2}\frac{\partial^2\widetilde{b}_n(k_z)}{\rho\rho'\partial\phi\partial\phi'}\Bigg\}\times e^{-jk_z(z-z')}dk_z\quad(30)$$

where

$$\widetilde{b}_n(k_z)=J_n(k_{i\rho}\rho')H_n^{(2)}(k_{i\rho}\rho)+\sum_{\ell=i-2}^{i-1}w_\ell J_n(k_{i\rho}d_\ell)\times H_n^{(2)}(k_{i\rho}\rho')+\sum_{\ell=i}^{i+1}w_\ell J_n(k_{i\rho}\rho')H_n^{(2)}(k_{i\rho}d_\ell).\quad(31)$$

Closed form representations of (28)–(30) can then be obtained using (24) in conjunction with the following identity

$$\cos(\phi-\phi')\frac{e^{-jk_i|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}=-\frac{j}{2}\int_{-\infty}^{+\infty}dk_ze^{-jk_z(z-z')}\frac{1}{k_{i\rho}^2}\left(\frac{\partial^2}{\rho\rho'\partial\phi\partial\phi'}+\frac{\partial^2}{\partial\rho\partial\rho'}\right)\times\sum_{n=-\infty}^{+\infty}J_n(k_{i\rho}\rho')H_n^{(2)}(k_{i\rho}\rho)e^{jn(\phi-\phi')}\quad(32)$$

which gives a unified asymptotic DGF expansion as

$$\overline{\mathbf{G}}^a(\mathbf{r},\mathbf{r}')=-\frac{j}{4\omega\varepsilon_i}\times\Bigg\{k_i^2\left(\frac{e^{-jk_iR}}{R}+\sum_{\ell=i-2}^{i+1}\upsilon_\ell\frac{e^{-jk_iR_\ell}}{R_\ell}\right)\hat{\mathbf{z}}\hat{\mathbf{z}}+k_i^2\cos(\phi-\phi')\times\left(\frac{e^{-jk_iR}}{R}+\sum_{\ell=i-2}^{i+1}\upsilon_\ell\frac{d_\ell}{\rho}\frac{e^{-jk_iR_\ell}}{R_\ell}\right)\hat{\phi}\hat{\phi}-\nabla\nabla'\left(\frac{e^{-jk_iR}}{R}+\sum_{\ell=i-2}^{i+1}w_\ell\frac{e^{-jk_iR_\ell}}{R_\ell}\right)\Bigg\}.\quad(33)$$

It should be noted that employing (32) provides further simplifications in the $G^a_{\phi\phi}$ expression as it eliminates a few highly singular closed-form terms that appear in previous investigations [5]. The spectral domain components have been computed using a deformed path in the complex k_z plane of the Sommerfeld integral, where the first term of (17) has been sampled uniformly along that path and approximated into discrete complex images form using the generalized pencil of functions (GPOF) method [10], [11] and then transformed into the spatial domain, as previously reported in [1], [2].

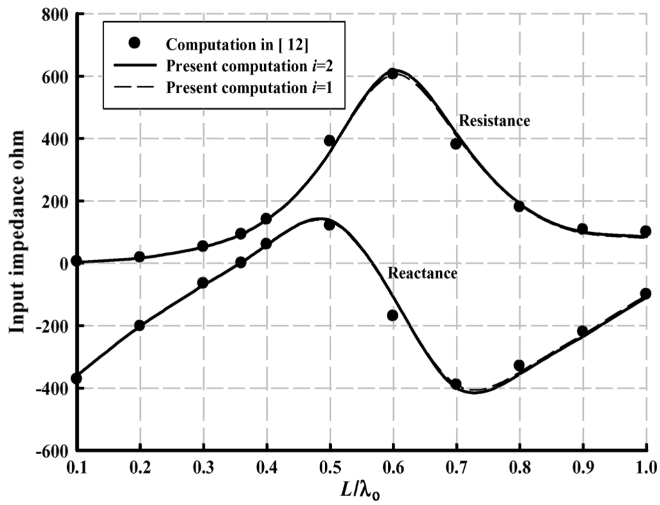

Fig. 2. Input impedance of a ϕ directed dipole of length L when the antenna is placed at dielectric interface, approaching it from either layer $i = 1$ or $i = 2$.

III. Results

The significance of the proposed solution is demonstrated using an MoM model of a conformal wire dipole, printed on an interface between two cylindrical dielectric layers surrounding a PEC core. Non magnetic dielectric layers have been considered, i.e. $\mu_i = \mu_0$. Each dipole is divided into eleven segments of equal length $\Delta\ell$. Piecewise sinusoidal expansion and testing functions, given by $\sin(k_i(\Delta\ell - |\ell - \ell_m|))/\sin(k_i\Delta\ell)$, are employed in a Galerkin's MoM procedure. Excitation is provided by a delta gap source at the center of the dipole, and the thin wire approximation has been adopted in which the source and observation points are located at $\rho' = \rho = a_2$, $z' = a$, and $z = 0$, where a is the wire radius. Following an example reported in [12], the geometry of Fig. 1 is considered assuming the PEC cylindrical core has a radius of $a_1 = 11.7$ cm and is covered by a dielectric substrate with a relative permittivity of $\varepsilon_{r1} = 2.45$ and a thickness of 1 cm with $\varepsilon_{r2} = \varepsilon_{r3} = 1$. A ϕ directed printed dipole driven at 6 GHz is modelled at the interface between layers $i = 1$ and $i = 2$ using a wire radius of 0.0125 cm. The computed input impedance of this antenna is illustrated in Fig. 2, where it can be observed that good agreement has been achieved compared with the results reported in [12]. The dipole has been modelled approaching the interface from each side, and as expected the input impedance is the same in both cases. This example demonstrates the validity of the suggested approach in the analyses of printed antennas when $\boldsymbol{\rho} = \boldsymbol{\rho}'$. Fig. 3 illustrates the convergence of the input impedance as a function of the number of terms in the infinite summation of spectral elements for a particular dipole length of $L/\lambda = 0.5$, where it can be seen that the impedance converges using approximately 30 terms when the proposed approach is employed, whereas non convergent results are obtained if the method reported in [1], [2] is implemented directly. This is to be expected when no asymptotic extraction is employed, as has been mentioned in earlier studies [4]–[7]. A ϕ directed dipole of length $L/\lambda = 0.4$ is next modelled at the interface between 1 mm thick dielectric layers $\varepsilon_{r1} = 4.0$, $\varepsilon_{r2} = 2.45$ and $\varepsilon_{r3} = 1$, having a PEC core radius of 12.6 cm at 6 GHz. The input impedance is illustrated in Fig. 4, where it is evident that convergence has been attained using 70 terms irrespective of the direction of approach to the interface. As another validation, the mutual impedance between two z directed current sources has been calculated and compared to that reported in [7] with good agreement as shown in Fig. 5 using the parameters given in the aforementioned article.

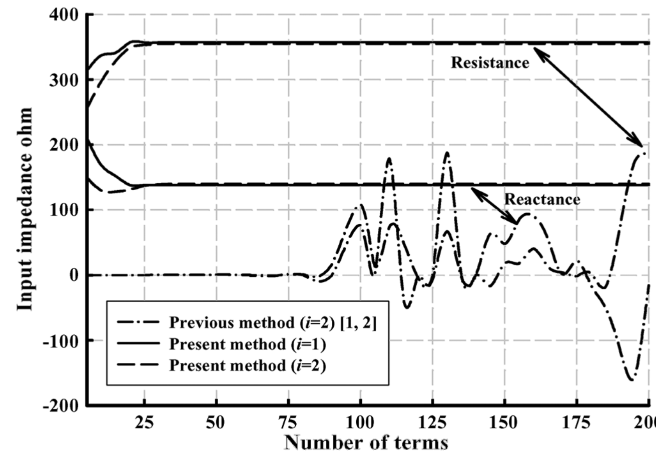

Fig. 3. Convergence of the input impedance of the ϕ directed printed cylindrical dipole for $L/\lambda = 0.5$, where i refers to the antenna's layer.

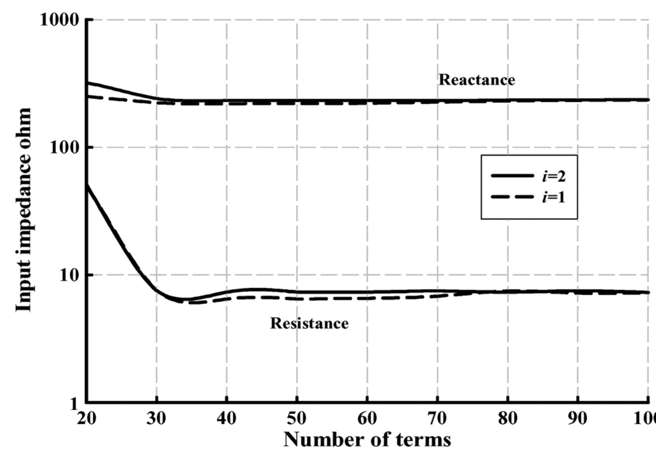

Fig. 4. Convergence of the input impedance of a ϕ directed cylindrical dipole printed at the interface of thin dielectric substrates when $L/\lambda = 0.4$.

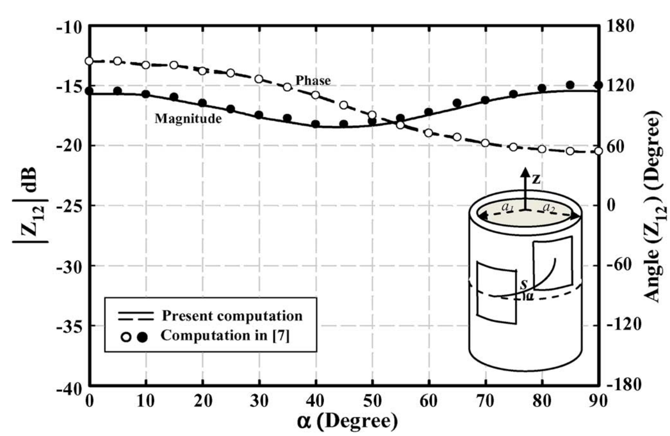

Fig. 5. Mutual impedance between two z directed current sources printed on a two-layer dielectric cylinder when $s = 0.5\lambda_o$.

IV. Conclusion

A new approach to compute the DGF for antennas in the vicinity of layered cylindrical media has been introduced, where simplified closed form expressions have been obtained and validated for the spatial asymptotic DGF components. These expressions have been achieved through a single application of asymptotic extraction for larger n. This results in a significant saving in the required computation time as it eliminates the need for another level of asymptotic extraction for larger k_z as well as the need to determine the expansion coefficients numerically. Furthermore, the Hankel function singularity when $\boldsymbol{\rho} = \boldsymbol{\rho}'$ has been handled through analytical integration. A simplification in the formulae for the spectral asymptotic expansion elements has also been achieved. The model's advantage reduces as the radial distance between field and source points increases as well as when the antenna is not located close to a dielectric boundary.

APPENDIX

The local reflection and transmission matrices can be defined as [1]

$$\overline{\mathbf{R}}_{i,i-1} = \frac{J_n(k_{i\rho}a_{i-1})}{H_n^{(2)}(k_{i\rho}a_{i-1})}\overline{\mathbf{r}}_{i,i-1} \tag{34}$$

$$\overline{\mathbf{R}}_{i,i+1} = \frac{H_n^{(2)}(k_{i\rho}a_i)}{J_n(k_{i\rho}a_i)}\overline{\mathbf{r}}_{i,i+1} \tag{35}$$

$$\overline{\mathbf{T}}_{i,i-1} = \frac{J_n(k_{i\rho}a_{i-1})}{J_n\left(k_{(i-1)\rho}a_{i-1}\right)}\overline{\mathbf{t}}_{i,i-1} \tag{36}$$

$$\overline{\mathbf{T}}_{i,i+1} = \frac{H_n^{(2)}(k_{i\rho}a_i)}{H_n^{(2)}\left(k_{(i+1)\rho}a_i\right)}\overline{\mathbf{t}}_{i,i+1} \tag{37}$$

where the matrix $\overline{\mathbf{r}}$ can be expressed in a convenient form as

$$\overline{\mathbf{r}}_{i+1,i} = \frac{1}{\xi}\begin{bmatrix} \varepsilon_{i+1}\Delta_{i+1}f_i(\mu) - \xi & \frac{\mu_{i+1}}{j\omega}\Delta_{i+1}Q_i \\ -\frac{\varepsilon_{i+1}}{j\omega}\Delta_{i+1}Q_i & \mu_{i+1}\Delta_{i+1}f_i(\varepsilon) - \xi \end{bmatrix} \tag{38a}$$

$$\overline{\mathbf{r}}_{i,i+1} = \frac{1}{\xi}\begin{bmatrix} \varepsilon_i\Delta_i f_i(\mu) - \xi & \frac{\mu_i}{j\omega}\Delta_i Q_i \\ -\frac{\varepsilon_i}{j\omega}\Delta_i Q_i & \mu_i\Delta_i f_i(\varepsilon) - \xi \end{bmatrix} \tag{38b}$$

where $\xi = f_i(\varepsilon)f_i(\mu) - (Q_i^2/\omega^2)$, and

$$f_i(x) = \frac{x_i \partial J_n(k_{i\rho}a_i)}{k_{i\rho}J_n(k_{i\rho}a_i)\partial(k_{i\rho}a_i)} - \frac{x_{i+1}\partial H_n^{(2)}\left(k_{(i+1)\rho}a_i\right)}{k_{(i+1)\rho}H_n^{(2)}\left(k_{(i+1)\rho}a_i\right)\partial\left(k_{(i+1)\rho}a_i\right)} \tag{39}$$

$$\Delta_i = \frac{\partial J_n(k_{i\rho}a_i)}{k_{i\rho}J_n(k_{i\rho}a_i)\partial(k_{i\rho}a_i)} - \frac{\partial H_n^{(2)}(k_{i\rho}a_i)}{k_{i\rho}H_n^{(2)}(k_{i\rho}a_i)\partial(k_{i\rho}a_i)} \tag{40}$$

$$Q_i^2 = (nk_z)^2\left(\frac{1}{k_{i\rho}^2} - \frac{1}{k_{(i+1)\rho}^2}\right)^2 = n^2\omega^2\left(\frac{\varepsilon_i\mu_i}{k_{i\rho}^4} + \frac{\varepsilon_{i+1}\mu_{i+1}}{k_{(i+1)\rho}^4} - \frac{\varepsilon_i\mu_i + \varepsilon_{i+1}\mu_{i+1}}{k_{i\rho}^2 k_{(i+1)\rho}^2}\right). \tag{41}$$

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