The Structure of Classical Extensions of Quantum Probability Theory

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Abstract

On the basis of a suggestive definition of a classical extension of quantum mechanics in terms of statistical models, we prove that every such classical extension is essentially given by the so-called Misra-Bugajski reduction map. We consider how this map enables one to understand quantum mechanics as a reduced classical statistical theory on the projective Hilbert space as phase space and discuss features of the induced hidden-variables model. Moreover, some relevant technical results on the topology and Borel structure of the projective Hilbert space are reviewed.

Key words: Statistical model, classical extension of quantum mechanics, Misra-Bugajski map, projective Hilbert space.

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I.   INTRODUCTION

Every statistical (probabilistic) physical theory can be based on a set $\mathcal{S}$ of *states*, a set $\mathcal{E}$ of *effects*, and a *probability functional* associating each state $s \in \mathcal{S}$ and each effect $a \in \mathcal{E}$ with a real number $\langle s, a \rangle \in [0, 1]$, the latter being the *probability for the outcome 'yes' of the effect a in the state s* [14, 15, 17, 24, 25]. We summarize these basic concepts of a statistical theory by the pair $\langle \mathcal{S}, \mathcal{E} \rangle$; we call $\langle \mathcal{S}, \mathcal{E} \rangle$ a *statistical model* if the following properties are satisfied [1, 2, 21, 22]. Since states can be mixed, $\mathcal{S}$ has to be closed under such mixtures, and the probability functional must be affine in the states (mixture-preserving); moreover, we assume that the states and the effects separate each other (i.e., $\langle s_1, a \rangle = \langle s_2, a \rangle$ for all $a \in \mathcal{E}$ implies $s_1 = s_2$, and $\langle s, a_1 \rangle = \langle s, a_2 \rangle$ for all $s \in \mathcal{S}$ implies $a_1 = a_2$).

Given a statistical model $\langle \mathcal{S}_1, \mathcal{E}_1 \rangle$, assume only a subset $\mathcal{E}_2 \subseteq \mathcal{E}_1$ is accessible. In general, $\mathcal{E}_2$ no longer separates $\mathcal{S}_1$; call two states $s, \tilde{s} \in \mathcal{S}_1$ equivalent if $\langle s, a \rangle = \langle \tilde{s}, a \rangle$ for all $a \in \mathcal{E}_2$. Let $\mathcal{S}_2$ be the set of the equivalence classes and define

$$\langle [s], a \rangle := \langle s, a \rangle \tag{1}$$

where $[s] \in \mathcal{S}_2$ and $a \in \mathcal{E}_2$. Then $\mathcal{S}_2$ is a new set of states and $\langle \mathcal{S}_2, \mathcal{E}_2 \rangle$ a new statistical model; $\langle \mathcal{S}_2, \mathcal{E}_2 \rangle$ is a *reduction of* $\langle \mathcal{S}_1, \mathcal{E}_1 \rangle$, and $\langle \mathcal{S}_1, \mathcal{E}_1 \rangle$ is an *extension of* $\langle \mathcal{S}_2, \mathcal{E}_2 \rangle$. Let $R \colon \mathcal{S}_1 \to \mathcal{S}_2$ be the canonical projection, i.e., $R(s) := [s]$, and define the embedding map $R' \colon \mathcal{E}_2 \to \mathcal{E}_1$, i.e., $R'(a) := a$. Then Eq. (1) can be written as

$$\langle R(s), a \rangle = \langle s, R'(a) \rangle.$$

Note that R is affine and surjective, whereas R' is injective. We call R a *reduction map*.

Next let $\langle \mathcal{S}_1, \mathcal{E}_1 \rangle$ and $\langle \mathcal{S}_2, \mathcal{E}_2 \rangle$ be two arbitrary statistical models and $R \colon \mathcal{S}_1 \to \mathcal{S}_2$ a surjective affine mapping. Observe that $s_1 \mapsto \langle R(s_1), a_2 \rangle$ is an affine functional on $\mathcal{S}_1$ with values in the interval $[0, 1]$; assume that, for each effect $a_2 \in \mathcal{E}_2$, there exists an effect $a_1 \in \mathcal{E}_1$ such that

$$\langle R(s_1), a_2 \rangle = \langle s_1, a_1 \rangle \tag{2}$$

holds for all $s_1 \in \mathcal{S}_1$. Clearly, a_1 is uniquely determined, and we can define a map $R' \colon \mathcal{E}_2 \to \mathcal{E}_1$ according to $R'(a_2) := a_1$. Then Eq. (2) reads

$$\langle R(s_1), a_2 \rangle = \langle s_1, R'(a_2) \rangle, \tag{3}$$

and one easily shows that R' is injective. Moreover, we can call two states $s_1, \tilde{s}_1 \in \mathcal{S}_1$ equivalent if $R(s_1) = R(\tilde{s}_1)$; for effects of the form $R'(a_2)$, such equivalent states s_1 and $\tilde{s}_1$ give rise to the same probabilities. Because R is surjective, the states $s_2 \in S_2$ can be identified with the equivalence classes $[s_1] = R^{-1}(\{s_2\})$ where $s_2 = Rs_1$. Because R' is injective, we can further identify the effects $a_2 \in \mathcal{E}_2$ with the effects $R'(a_2)$, i.e., $\mathcal{E}_2$ can be considered as a subset of $\mathcal{E}_1$. By means of these identifications, Eq. (3) coincides with Eq. (1), and R takes the role of the canonical projection. Hence, the relation between the two statistical models of this paragraph is the same as that between the two statistical models of the preceding paragraph.

If $\langle \mathcal{S}_1, \mathcal{E}_1 \rangle$ and $\langle \mathcal{S}_2, \mathcal{E}_2 \rangle$ are two statistical models and R is a surjective affine mapping from $\mathcal{S}_1$ onto $\mathcal{S}_2$ for which, in the sense just described, a mapping R' exists, then we call $\langle \mathcal{S}_2, \mathcal{E}_2 \rangle$ a *reduction of* $\langle \mathcal{S}_1, \mathcal{E}_1 \rangle$, $\langle \mathcal{S}_1, \mathcal{E}_1 \rangle$ an *extension of* $\langle \mathcal{S}_2, \mathcal{E}_2 \rangle$, and R a *reduction map*. Since statistical models can be embedded into dual pairs of vector spaces (one vector space being a base-norm space and the other one an order-unit norm space, the pair forming a so-called *statistical duality* [21, 24, 25, 31]), the reduction-extension concept for statistical models can be reformulated in this general context. The reduction map R is then a surjective bounded linear map, and R' is the adjoint map of R which is linear, bounded, and injective. We do not consider this reformulation in complete generality, instead we shall study a reduction-extension concept specific to the subject of this paper which concerns the relation between classical and quantum probability.

It is the aim of this paper to revisit a particular classical extension of quantum mechanics defined by what we call the *Misra-Bugajski reduction map* [1, 6, 10, 16, 21, 26, 30], and to show that this map is essentially the only possible reduction map from a classical statistical model to the quantum statistical model, i.e., essentially the only possible way to obtain a classical extension of quantum probability theory. To this end, we first define the notions of quantum and classical statistical model. In doing so we also introduce most of the notations used in the paper.

Let a complex separable Hilbert space $\mathcal{H} \neq \{0\}$ be given. We denote the real vector space of the self-adjoint trace-class operators by $\mathcal{T}_s(\mathcal{H})$ and the convex set of the positive trace-class operators of trace 1 by $\mathcal{S}(\mathcal{H})$; the operators of $\mathcal{S}(\mathcal{H})$ are the density operators and describe the quantum states. The pair $(\mathcal{T}_s(\mathcal{H}), \mathcal{S}(\mathcal{H}))$ is a base-normed Banach space with closed positive cone, the base norm being the trace norm. We denote the real vector

space of all bounded self-adjoint operators by $\mathcal{B}_s(\mathcal{H})$ and the unit operator by I. The pair $(\mathcal{B}_s(\mathcal{H}), I)$ where $\mathcal{B}_s(\mathcal{H})$ is equipped with its order relation, is an order-unit normed Banach space with closed positive cone, the norm being the usual operator norm. The elements of the order-unit interval $\mathcal{E}(\mathcal{H}) := [0, I]$ describe the quantum mechanical effects. As is well known, $\mathcal{B}_s(\mathcal{H})$ can be considered as the dual space $(\mathcal{T}_s(\mathcal{H}))'$ where the duality is given by the trace functional

$$(V, A) \mapsto \langle V, A \rangle := \operatorname{tr} VA,$$

$V \in \mathcal{T}_s(\mathcal{H})$, $A \in \mathcal{B}_s(\mathcal{H})$. The restriction of this bilinear functional to $\mathcal{S}(\mathcal{H}) \times \mathcal{E}(\mathcal{H})$ is the quantum probability functional; $\operatorname{tr} WA$ is the probability for the outcome 'yes' of the effect $A \in \mathcal{E}(\mathcal{H})$ in the state $W \in \mathcal{S}(\mathcal{H})$. Thus, $\langle \mathcal{T}_s(\mathcal{H}), \mathcal{B}_s(\mathcal{H}) \rangle$ is a dual pair of vector spaces (in fact a statistical duality) and $\langle \mathcal{S}(\mathcal{H}), \mathcal{E}(\mathcal{H}) \rangle$ the *quantum statistical model* [11, 14, 22, 25].

Further we recall that the extreme points of the convex set $\mathcal{S}(\mathcal{H})$, i.e., the pure quantum states, are the one-dimensional orthogonal projections $P = P_\varphi := |\varphi\rangle\langle\varphi|$, $\|\varphi\| = 1$. We denote the set of these extreme points, i.e., the extreme boundary, by $\partial_e \mathcal{S}(\mathcal{H})$. The extreme points of the convex set $\mathcal{E}(\mathcal{H})$ are all orthogonal projections, these are sometimes called *sharp* effects whereas the other ones are called *unsharp* effects.—We also recall that $\sigma(\mathcal{T}_s(\mathcal{H}), \mathcal{B}_s(\mathcal{H}))$ is the weak Banach-space topology of $\mathcal{T}_s(\mathcal{H})$, i.e., the coarsest topology on $\mathcal{T}_s(\mathcal{H})$ in which the elements of $\mathcal{B}_s(\mathcal{H})$, considered as linear functionals on $\mathcal{T}_s(\mathcal{H})$, are continuous.

For a general measurable space (Ω, Σ) where Ω is a nonempty set and Σ an arbitrary σ-algebra of subsets of Ω, let $\mathcal{M}_\mathbb{R}(\Omega, \Sigma)$ be the real vector space of the real-valued measures on (Ω, Σ) (i.e., of the σ-additive real-valued set functions on Σ). We denote the convex subset of the positive normalized measures by $\mathcal{S}(\Omega, \Sigma)$; the elements of $\mathcal{S}(\Omega, \Sigma)$ are probability measures and describe classical states. The pair $(\mathcal{M}_\mathbb{R}(\Omega, \Sigma), \mathcal{S}(\Omega, \Sigma))$ is a base-normed Banach space with closed positive cone, the base norm being the total-variation norm. By $\mathcal{F}_\mathbb{R}(\Omega, \Sigma)$ we denote the real vector space of the bounded Σ-measurable functions on Ω and by χ_E the characteristic function of a set $E \in \Sigma$. The pair $(\mathcal{F}_\mathbb{R}(\Omega, \Sigma), \chi_\Omega)$ together with the order relation of $\mathcal{F}_\mathbb{R}(\Omega, \Sigma)$ is an order-unit normed Banach space with closed positive cone, the order-unit norm being the supremum norm. The elements of the order-unit interval $\mathcal{E}(\Omega, \Sigma) := [0, \chi_\Omega]$ describe the classical effects. By the bilinear functional given by the

integral

$$(\nu, f) \mapsto \langle \nu, f \rangle := \int_\Omega f d\nu,$$

$\nu \in \mathcal{M}_\mathbb{R}(\Omega, \Sigma)$, $f \in \mathcal{F}_\mathbb{R}(\Omega, \Sigma)$, the spaces $\mathcal{M}_\mathbb{R}(\Omega, \Sigma)$ and $\mathcal{F}_\mathbb{R}(\Omega, \Sigma)$ are placed in duality to each other; in particular, $\mathcal{F}_\mathbb{R}(\Omega, \Sigma)$ can be considered as a norm-closed subspace of the dual space $(\mathcal{M}_\mathbb{R}(\Omega, \Sigma))'$ where in general the dual space is larger than $\mathcal{F}_\mathbb{R}(\Omega, \Sigma)$. The restriction of $(\nu, f) \mapsto \langle \nu, f \rangle$ to $\mathcal{S}(\Omega, \Sigma) \times \mathcal{E}(\Omega, \Sigma)$ is the classical probability functional; $\int f d\nu$ is the probability for the outcome 'yes' of the effect $f \in \mathcal{E}(\Omega, \Sigma)$ in the state $\mu \in \mathcal{S}(\Omega, \Sigma)$. Again, $\langle \mathcal{M}_\mathbb{R}(\Omega, \Sigma), \mathcal{F}_\mathbb{R}(\Omega, \Sigma) \rangle$ is a dual pair of vector spaces (a statistical duality), whereas $\langle \mathcal{S}(\Omega, \Sigma), \mathcal{E}(\Omega, \Sigma) \rangle$ is the *classical statistical model* [8, 9, 15, 17, 18, 27, 29].

We remark that the Dirac measures δ_ω, $\omega \in \Omega$, are extreme points of the convex set $\mathcal{S}(\Omega, \Sigma)$, but in general there are also other extreme points. The extreme points of the convex set $\mathcal{E}(\Omega, \Sigma)$ are the characteristic functions χ_E, $E \in \Sigma$, these are the *sharp* classical effects (in the terminology of classical probability theory, the *events*), the other effects are *unsharp* or *fuzzy*.—Finally, we recall that $\sigma(\mathcal{M}_\mathbb{R}(\Omega, \Sigma), \mathcal{F}_\mathbb{R}(\Omega, \Sigma))$ is the coarsest topology on $\mathcal{M}_\mathbb{R}(\Omega, \Sigma)$ in which the elements of $\mathcal{F}_\mathbb{R}(\Omega, \Sigma)$, considered as linear functionals on $\mathcal{M}_\mathbb{R}(\Omega, \Sigma)$, are continuous.

Now assume that, for the two statistical models $\langle \mathcal{S}_1, \mathcal{E}_1 \rangle = \langle \mathcal{S}(\Omega, \Sigma), \mathcal{E}(\Omega, \Sigma) \rangle$ and $\langle \mathcal{S}_2, \mathcal{E}_2 \rangle = \langle \mathcal{S}(\mathcal{H}), \mathcal{E}(\mathcal{H}) \rangle$, a reduction map $R : \mathcal{S}(\Omega, \Sigma) \to \mathcal{S}(\mathcal{H})$ is given. It is not hard to show that the surjective affine mapping R can uniquely be extended to a surjective linear map from $\mathcal{M}_\mathbb{R}(\Omega, \Sigma)$ onto $\mathcal{T}_s(\mathcal{H})$ which we also call R; the linear map R is automatically positive and bounded. According to Eq. (3) the injective mapping $R' : \mathcal{E}(\mathcal{H}) \to \mathcal{E}(\Omega, \Sigma)$ satisfies

$$\mathrm{tr}\,(R\mu)A = \langle R\mu, A \rangle = \langle \mu, R'A \rangle = \int_\Omega R'A \, d\mu \tag{4}$$

for all $\mu \in \mathcal{S}(\Omega, \Sigma)$ and all $A \in \mathcal{E}(\mathcal{H})$; R' is also affine. Moreover, from (4) it follows that the adjoint map of R w.r.t. the dual pairs $\langle \mathcal{M}_\mathbb{R}(\Omega, \Sigma), \mathcal{F}_\mathbb{R}(\Omega, \Sigma) \rangle$ and $\langle \mathcal{T}_s(\mathcal{H}), \mathcal{B}_s(\mathcal{H}) \rangle$ exists, this adjoint map $R' : \mathcal{B}_s(\mathcal{H}) \to \mathcal{F}_\mathbb{R}(\Omega, \Sigma)$ is a unique linear extension of the affine mapping $R' : \mathcal{E}(\mathcal{H}) \to \mathcal{E}(\Omega, \Sigma)$ and is also injective.

The existence of the adjoint map R' w.r.t. the considered dual pairs is equivalent to $R^* \mathcal{B}_s(\mathcal{H}) \subseteq \mathcal{F}_\mathbb{R}(\Omega, \Sigma)$ where $R^* : \mathcal{B}_s(\mathcal{H}) \to (\mathcal{M}_\mathbb{R}(\Omega, \Sigma))'$ is the Banach-space adjoint map of R. According to general results in duality theory, the existence of the linear map R' is also equivalent to the $\sigma(\mathcal{M}_\mathbb{R}(\Omega, \Sigma), \mathcal{F}_\mathbb{R}(\Omega, \Sigma))$-$\sigma(\mathcal{T}_s(\mathcal{H}), \mathcal{B}_s(\mathcal{H}))$ continuity of R.—The crucial

properties of the linear map R are summarized in the following definition.

Definition We call a linear map $R \colon \mathcal{M}_{\mathbb{R}}(\Omega, \Sigma) \to \mathcal{T}_s(\mathcal{H})$ a *reduction map* if

   (i) $R\mathcal{S}(\Omega, \Sigma) = \mathcal{S}(\mathcal{H})$;

   (ii) R is $\sigma(\mathcal{M}_{\mathbb{R}}(\Omega, \Sigma), \mathcal{F}_{\mathbb{R}}(\Omega, \Sigma))$-$\sigma(\mathcal{T}_s(\mathcal{H}), \mathcal{B}_s(\mathcal{H}))$-continuous.

We will say that the linear map R (or its affine restriction) together with the dual map R' constitutes a *reduction* of the classical statistical model $\langle \mathcal{S}(\Omega, \Sigma), \mathcal{E}(\Omega, \Sigma) \rangle$ to the quantum statistical model $\langle \mathcal{S}(\mathcal{H}), \mathcal{E}(\mathcal{H}) \rangle$. In particular, we will say that R and R' constitute a *classical extension of quantum mechanics*.

The properties of R stated in this definition imply again that R is bounded, positive, and surjective and that R' exists and is injective. Furthermore, one easily shows that R' is positive and that $R'I = \chi_\Omega$ and $R'\mathcal{E}(\mathcal{H}) \subseteq \mathcal{E}(\Omega, \Sigma)$. The restrictions of R and R' to $\mathcal{S}(\Omega, \Sigma)$ and $\mathcal{E}(\Omega, \Sigma)$, respectively, are affine; clearly, the restriction of R to $\mathcal{S}(\Omega, \Sigma)$ is a reduction map as defined previously in the context of two general statistical models $\langle \mathcal{S}_1, \mathcal{E}_1 \rangle$ and $\langle \mathcal{S}_2, \mathcal{E}_2 \rangle$.

It is not clear that classical extensions of quantum mechanics do exist, in fact, this may be considered surprising. The typical example of a reduction map is the so-called *Misra-Bugajski map* which we present in Section IV. In Section V we prove our result that every reduction map giving a classical extension of quantum mechanics is essentially equivalent to the Misra-Bugajski map. Thus, the Misra-Bugajski map is essentially unique and yields a canonical classical extension of quantum mechanics.

Sections II and III provide prerequisite results on the topology and the Borel structure of the projective Hilbert space which will be identified with the extreme boundary $\partial_e \mathcal{S}(\mathcal{H})$ of $\mathcal{S}(\mathcal{H})$. In Section VI some examples of reduction maps different from the Misra-Bugajski map are presented. Finally, in Section VII the physical interpretation of the results of Sections IV and V is discussed.

II. THE TOPOLOGY OF THE PROJECTIVE HILBERT SPACE

In this section we undertake a systematic review and comparison, sketched out in this context previously by Bugajski [7], of the various topologies on the set of the pure quantum

states or, alternatively, on the projective Hilbert space associated with a nontrivial separable complex Hilbert space $\mathcal{H} \neq \{0\}$.

Call two vectors of $\mathcal{H}^* := \mathcal{H} \setminus \{0\}$ equivalent if they differ by a complex factor, and define the *projective Hilbert space* $\mathcal{P}(\mathcal{H})$ to be the set of the corresponding equivalence classes which are often called *rays*. Instead of $\mathcal{H}^*$ one can consider only the unit sphere of $\mathcal{H}$, $S := \{\varphi \in \mathcal{H} \,|\, \|\varphi\| = 1\}$. Then two unit vectors are called equivalent if they differ by a phase factor, and the set of the corresponding equivalence classes, i.e., the set of the *unit rays*, is denoted by S/S^1 (in this context, S^1 is understood as the set of all phase factors, i.e., as the set of all complex numbers of modulus 1). Clearly, S/S^1 can be identified with the projective Hilbert space $\mathcal{P}(\mathcal{H})$. Furthermore, we can consider the elements of $\mathcal{P}(\mathcal{H})$ also as the one-dimensional subspaces of $\mathcal{H}$ or, equivalently, as the one-dimensional orthogonal projections $P = P_\varphi = |\varphi\rangle\langle\varphi|$, $\|\varphi\| = 1$.

The set $\mathcal{H}^*$ and the unit sphere S carry the topologies induced by the metric topology of $\mathcal{H}$. Using the canonical projections $\mu : \mathcal{H}^* \to \mathcal{P}(\mathcal{H})$, $\mu(\varphi) := [\varphi]$, and $\nu : S \to S/S^1$, $\nu(\chi) := [\chi]_S$, where $[\varphi]$ is a ray and $[\chi]_S$ a unit ray, we can equip the quotient sets $\mathcal{P}(\mathcal{H})$ and S/S^1 with their quotient topologies $\mathcal{T}_\mu$ and $\mathcal{T}_\nu$. Considering $\mathcal{T}_\nu$, a set $O \subseteq S/S^1$ is called open if $\nu^{-1}(O)$ is open.

Theorem 1 *The set S/S^1, equipped with the quotient topology $\mathcal{T}_\nu$, is a second-countable Hausdorff space, and ν is an open continuous mapping.*

Proof. By definition of $\mathcal{T}_\nu$, ν is continuous. To show that ν is open, let U be an open set of S. From

$$\nu^{-1}(\nu(U)) = \nu^{-1}(\{[\chi]_S \,|\, \chi \in U\}) = \bigcup_{\lambda \in S^1} \lambda U,$$

$S^1 = \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$, it follows that $\nu^{-1}(\nu(U)) \subseteq S$ is open. So $\nu(U) \subseteq S/S^1$ is open; hence, ν is open.

Next consider two different unit rays $[\varphi]_S$ and $[\psi]_S$ where $\varphi, \psi \in S$ and $|\langle\varphi|\psi\rangle| = 1 - \varepsilon$, $0 < \varepsilon \leq 1$. Since the mapping $\chi \mapsto |\langle\varphi|\chi\rangle|$, $\chi \in S$, is continuous, the sets

$$U_1 := \left\{\chi \in S \,\middle|\, |\langle\varphi|\chi\rangle| > 1 - \tfrac{\varepsilon}{2}\right\} \tag{5}$$

and

$$U_2 := \left\{\chi \in S \,\middle|\, |\langle\varphi|\chi\rangle| < 1 - \tfrac{\varepsilon}{2}\right\} \tag{6}$$

are open neighborhoods of φ and ψ, respectively. Consequently, the sets $O_1 := \nu(U_1)$ and $O_2 := \nu(U_2)$ are open neighborhoods of $[\varphi]_S$ and $[\psi]_S$, respectively. Assume $O_1 \cap O_2 \neq \emptyset$. Let $[\xi]_S \in O_1 \cap O_2$, then $[\xi]_S = \nu(\chi_1) = \nu(\chi_2)$ where $\chi_1 \in U_1$ and $\chi_2 \in U_2$. It follows that χ_1 and χ_2 are equivalent, so $|\langle \varphi | \chi_1 \rangle| = |\langle \varphi | \chi_2 \rangle|$, in contradiction to $\chi_1 \in U_1$ and $\chi_2 \in U_2$. Hence, O_1 and O_2 are disjoint, and $\mathcal{T}_\nu$ is separating.

Finally, let $\mathcal{B} = \{U_n \,|\, n \in \mathbb{N}\}$ be a countable base of the topology of S and define the open sets $O_n := \nu(U_n)$. We show that $\{O_n \,|\, n \in \mathbb{N}\}$ is a base of $\mathcal{T}_\nu$. For $O \in \mathcal{T}_\nu$, we have that $\nu^{-1}(O)$ is an open set of S and consequently $\nu^{-1}(O) = \bigcup_{n \in M} U_n$ where $U_n \in \mathcal{B}$ and $M \subseteq \mathbb{N}$. Since ν is surjective, it follows that

$$O = \nu(\nu^{-1}(O)) = \nu\left(\bigcup_{n \in M} U_n\right) = \bigcup_{n \in M} \nu(U_n) = \bigcup_{n \in M} O_n.$$

Hence, $\{O_n \,|\, n \in \mathbb{N}\}$ is a countable base of $\mathcal{T}_\nu$. $\square$

Analogously, it can be proved that the topology $\mathcal{T}_\mu$ on $\mathcal{P}(\mathcal{H})$ is separating and second-countable and that the canonical projection μ is open (and continuous by the definition of $\mathcal{T}_\mu$). Moreover, one can show that the natural bijection $\beta \colon \mathcal{P}(\mathcal{H}) \to S/S^1$, $\beta([\varphi]) := \left[\frac{\varphi}{\|\varphi\|}\right]_S$, $\beta^{-1}([\chi]_S) = [\chi]$, is a homeomorphism. Thus, identifying $\mathcal{P}(\mathcal{H})$ and S/S^1 by β, the topologies $\mathcal{T}_\mu$ and $\mathcal{T}_\nu$ are the same.

The above definition of $\mathcal{P}(\mathcal{H})$ and S/S^1 as well as of their quotient topologies is related to a geometrical point of view. From an operator-theoretical point of view, it is more obvious to identify $\mathcal{P}(\mathcal{H})$ with $\partial_e \mathcal{S}(\mathcal{H})$, the extreme boundary of $\mathcal{S}(\mathcal{H})$, and to restrict one of the various operator topologies to $\partial_e \mathcal{S}(\mathcal{H})$. A further definition of a topology on $\partial_e \mathcal{S}(\mathcal{H})$ is suggested by the interpretation of the one-dimensional projections $P \in \partial_e \mathcal{S}(\mathcal{H})$ as the pure quantum states and by the requirement that the transition probabilities between two pure states are continuous functions. Next we consider, taking account of $\partial_e \mathcal{S}(\mathcal{H}) \subseteq \mathcal{S}(\mathcal{H}) \subset \mathcal{T}_s(\mathcal{H}) \subseteq \mathcal{B}_s(\mathcal{H})$, the metric topologies on $\partial_e \mathcal{S}(\mathcal{H})$ induced by the trace-norm topology of $\mathcal{T}_s(\mathcal{H})$, resp., by the norm toplogy of $\mathcal{B}_s(\mathcal{H})$. After that we introduce the weak topology on $\partial_e \mathcal{S}(\mathcal{H})$ defined by the transition-probability functions as well as the restrictions of several weak operator topologies to $\partial_e \mathcal{S}(\mathcal{H})$. Finally, we shall prove the surprising result that all the many toplogies on $\mathcal{P}(\mathcal{H}) \cong S/S^1 \cong \partial_e \mathcal{S}(\mathcal{H})$ are equivalent.

Theorem 2 *Let $P_\varphi = |\varphi\rangle\langle\varphi| \in \partial_e \mathcal{S}(\mathcal{H})$ and $P_\psi = |\psi\rangle\langle\psi| \in \partial_e \mathcal{S}(\mathcal{H})$ where $\|\varphi\| = \|\psi\| = 1$. Then*

(a)

$$\rho_n(P_\varphi, P_\psi) := \|P_\varphi - P_\psi\| = \sqrt{1 - |\langle\varphi|\psi\rangle|^2} = \sqrt{1 - \operatorname{tr} P_\varphi P_\psi}$$

where the norm $\|\cdot\|$ is the usual operator norm;

(b)

$$\rho_{\mathrm{tr}}(P_\varphi, P_\psi) := \|P_\varphi - P_\psi\|_{\mathrm{tr}} = 2\|P_\varphi - P_\psi\|,$$

in particular, the metrics ρ_n and ρ_{tr} on $\partial_e S(\mathcal{H})$ induced by the operator norm $\|\cdot\|$ and the trace norm $\|\cdot\|_{\mathrm{tr}}$ are equivalent;

(c)

$$\|P_\varphi - P_\psi\| \leq \|\varphi - \psi\|,$$

in particular, the mapping $\varphi \mapsto P_\varphi$ from S into $\partial_e S(\mathcal{H})$ is continuous, $\partial_e S(\mathcal{H})$ being equipped with ρ_n or ρ_{tr}.

Proof. To prove (a) and (b), assume $P_\varphi \neq P_\psi$, otherwise the statements are trivial. Then the range of $P_\varphi - P_\psi$ is a two-dimensional subspace of $\mathcal{H}$ and is spanned by the two linearly independent unit vectors φ and ψ. Since eigenvectors of $P_\varphi - P_\psi$ belonging to eigenvalues $\lambda \neq 0$ must lie in the range of $P_\varphi - P_\psi$, they can be written as $\chi = \alpha\varphi + \beta\psi$. Therefore, the eigenvalue problem $(P_\varphi - P_\psi)\chi = \lambda\chi$, $\chi \neq 0$, is equivalent to the two linear equations

$$(1 - \lambda)\alpha + \langle\varphi|\psi\rangle\beta = 0$$
$$-\langle\psi|\varphi\rangle\alpha - (1 + \lambda)\beta = 0$$

where $\alpha \neq 0$ or $\beta \neq 0$. It follows that $\lambda = \pm\sqrt{1 - |\langle\varphi|\psi\rangle|^2} =: \lambda_{1,2}$. Hence, $P_\varphi - P_\psi$ has the eigenvalues λ_1, 0, and λ_2. Now, from $\|P_\varphi - P_\psi\| = \max\{|\lambda_1|, |\lambda_2|\}$ and $\|P_\varphi - P_\psi\|_{\mathrm{tr}} = |\lambda_1| + |\lambda_2|$, we obtain the statements (a) and (b).—From

$$\begin{aligned}
\|P_\varphi - P_\psi\|^2 &= 1 - |\langle\varphi|\psi\rangle|^2 = \|\varphi - \langle\psi|\varphi\rangle\psi\|^2 = \|(I - P_\psi)\varphi\|^2 \\
&\leq \|(I - P_\psi)\varphi\|^2 + \|\psi - P_\psi\varphi\|^2 \\
&= \|(I - P_\psi)\varphi - (\psi - P_\psi\varphi)\|^2 \\
&= \|\varphi - \psi\|^2
\end{aligned}$$

we conclude statement (c). $\square$

According to statement (b) of Theorem 2, the metrics ρ_n and ρ_{tr} give rise to the same topology $\mathcal{T}_n = \mathcal{T}_{\mathrm{tr}}$ as well as to the same uniform structures.

Theorem 3 *Equipped with either of the two metrics ρ_n and ρ_{tr}, $\partial_e \mathcal{S}(\mathcal{H})$ is separable and complete.*

Proof. As a metric subspace of the separable Hilbert space $\mathcal{H}$, the unit sphere S is separable. Therefore, by statement (c) of Theorem 2, the metric space $(\partial_e \mathcal{S}(\mathcal{H}), \rho_n)$ is separable and so is $(\partial_e \mathcal{S}(\mathcal{H}), \rho_{\mathrm{tr}})$ (the latter, moreover, implies the trace-norm separability of $\mathcal{T}_s(\mathcal{H})$). Now let $\{P_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $(\partial_e \mathcal{S}(\mathcal{H}), \rho_{\mathrm{tr}})$. Then there exists an operator $A \in \mathcal{T}_s(\mathcal{H})$ such that $\|P_n - A\|_{\mathrm{tr}} \to 0$ as well as $\|P_n - A\| \to 0$ as $n \to \infty$ (remember that, on $\mathcal{T}_s(\mathcal{H})$, $\|\cdot\|_{\mathrm{tr}}$ is stronger than $\|\cdot\|$). From

$$\left\|P_n - A^2\right\| = \left\|A^2 - P_n^2\right\| \;\leq\; \left\|A^2 - AP_n\right\| + \left\|AP_n - P_n^2\right\|$$
$$\leq\; \|A\| \, \|A - P_n\| + \|A - P_n\|$$
$$\to\; 0$$

as $n \to \infty$ we obtain $A = \lim_{n \to \infty} P_n = A^2$; moreover,

$$\operatorname{tr} A = \operatorname{tr} AI = \lim_{n \to \infty} \operatorname{tr} P_n I = 1.$$

Hence, A is a one-dimensional orthogonal projection, i.e., $A \in \partial_e \mathcal{S}(\mathcal{H})$. $\square$

Next we equip $\partial_e \mathcal{S}(\mathcal{H})$ with the topology $\mathcal{T}_0$ generated by the functions

$$P \mapsto h_Q(P) := \operatorname{tr} PQ = |\langle \varphi | \psi \rangle|^2 \tag{7}$$

where $P = |\psi\rangle\langle\psi| \in \partial_e \mathcal{S}(\mathcal{H})$, $Q = |\varphi\rangle\langle\varphi| \in \partial_e \mathcal{S}(\mathcal{H})$, and $\|\psi\| = \|\varphi\| = 1$. That is, $\mathcal{T}_0$ is the coarsest topology on $\partial_e \mathcal{S}(\mathcal{H})$ such that all the real-valued functions h_Q are continuous. Note that $\operatorname{tr} PQ = |\langle \varphi | \psi \rangle|^2$ can be interpreted as the transition probability between the two pure states P and Q.

Lemma 1 *The set $\partial_e \mathcal{S}(\mathcal{H})$, equipped with the topology $\mathcal{T}_0$, is a second-countable Hausdorff space. A countable base of $\mathcal{T}_0$ is given by the finite intersections of the open sets*

$$\begin{aligned}
U_{klm} &:= h_{Q_k}^{-1}\left(\,]q_l - \tfrac{1}{m}, q_l + \tfrac{1}{m}[\,\right) \\
&= \left\{ P \in \partial_e \mathcal{S}(\mathcal{H}) \,\middle|\, |\operatorname{tr} PQ_k - q_l| < \tfrac{1}{m} \right\}
\end{aligned} \tag{8}$$

where $\{Q_k\}_{k \in \mathbb{N}}$ is a sequence of one-dimensional orthogonal projections being ρ_n-dense in $\partial_e \mathcal{S}(\mathcal{H})$, $\{q_l\}_{l \in \mathbb{N}}$ is a sequence of numbers being dense in $[0,1] \subseteq \mathbb{R}$, and $m \in \mathbb{N}$.

Proof. Let P_1 and P_2 be any two different one-dimensional projections. Choosing $Q = P_1$ in (7), we obtain $h_{P_1}(P_1) = 1 \neq h_{P_1}(P_2) = 1 - \varepsilon$, $0 < \varepsilon \leq 1$. The sets

$$U_1 := \left\{ P \in \partial_e \mathcal{S}(\mathcal{H}) \,\big|\, h_{P_1}(P) > 1 - \tfrac{\varepsilon}{2} \right\}$$

and

$$U_2 := \left\{ P \in \partial_e \mathcal{S}(\mathcal{H}) \,\big|\, h_{P_1}(P) < 1 - \tfrac{\varepsilon}{2} \right\}$$

(cf. Eqs. (5) and (6)) are disjoint open neighborhoods of P_1 and P_2, respectively. So $\mathcal{T}_0$ is separating.

For an open set $O \subseteq \mathbb{R}$, $h_Q^{-1}(O)$ is $\mathcal{T}_0$-open. We next prove that

$$U := h_Q^{-1}(O) = \bigcup_{U_{klm} \subseteq U} U_{klm} \tag{9}$$

with U_{klm} according to (8). Let $P \in U$. Then there exists an $\varepsilon > 0$ such that the interval $]h_Q(P) - \varepsilon, h_Q(P) + \varepsilon[$ is contained in O. Choose $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0} < \frac{\varepsilon}{2}$, and choose a member q_{l_0} of the sequence $\{q_l\}_{l \in \mathbb{N}}$ and a member Q_{k_0} of $\{Q_k\}_{k \in \mathbb{N}}$ such that $|\operatorname{tr} PQ - q_{l_0}| < \frac{1}{2m_0}$ and $\|Q_{k_0} - Q\| < \frac{1}{2m_0}$. It follows that

$$
\begin{aligned}
|\operatorname{tr} PQ_{k_0} - q_{l_0}| &\leq |\operatorname{tr} PQ_{k_0} - \operatorname{tr} PQ| + |\operatorname{tr} PQ - q_{l_0}| \\
&\leq \|Q_{k_0} - Q\| + |\operatorname{tr} PQ - q_{l_0}| \\
&< \tfrac{1}{m_0}
\end{aligned}
$$

which, by (8), means that $P \in U_{k_0 l_0 m_0}$. We further have to show that $U_{k_0 l_0 m_0} \subseteq U$. To that end, let $\widetilde{P} \in U_{k_0 l_0 m_0}$. Then, from

$$\left| \operatorname{tr} \widetilde{P}Q - \operatorname{tr} PQ \right| \leq \left| \operatorname{tr} \widetilde{P}Q - \operatorname{tr} \widetilde{P}Q_{k_0} \right| + \left| \operatorname{tr} \widetilde{P}Q_{k_0} - q_{l_0} \right| + \left| q_{l_0} - \operatorname{tr} PQ \right|$$

where the first term on the right-hand side is again smaller than $\|Q - Q_{k_0}\|$ and, by (8), the second term is smaller than $\frac{1}{m_0}$, it follows that

$$\left| h_Q(\widetilde{P}) - h_Q(P) \right| = \left| \operatorname{tr} \widetilde{P}Q - \operatorname{tr} PQ \right| \leq \tfrac{1}{2m_0} + \tfrac{1}{m_0} + \tfrac{1}{2m_0} = \tfrac{2}{m_0} < \varepsilon.$$

This implies that $h_Q(\widetilde{P}) \in]h_Q(P) - \varepsilon, h_Q(P) + \varepsilon[\subseteq O$, i.e., $\widetilde{P} \in h_Q^{-1}(O) = U$. Hence, $U_{k_0 l_0 m_0} \subseteq U$.

Summarizing, we have shown that, for $P \in U$, $P \in U_{k_0 l_0 m_0} \subseteq U$. Hence, $U \subseteq \bigcup_{U_{klm} \subseteq U} U_{klm} \subseteq U$, and assertion (9) has been proved. The finite intersections of sets

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of the form $U = h_Q^{-1}(O)$ constitute a basis of the topology $\mathcal{T}_0$. Since every set $U = h_Q^{-1}(O)$ is the union of sets U_{klm}, the intersections of finitely many sets $U = h_Q^{-1}(O)$ is the union of finite intersections of sets U_{klm}. Thus, the finite intersections of the sets U_{klm} constitute a countable base of $\mathcal{T}_0$. $\square$

Later we shall see that the topological space $(\partial_e \mathcal{S}(\mathcal{H}), \mathcal{T}_0)$ is homeomorphic to $(\partial_e \mathcal{S}(\mathcal{H}), \mathcal{T}_n)$ as well as to $(S/S^1, \mathcal{T}_\nu)$. So it is also clear by Theorem 3 or Theorem 1 that $(\partial_e \mathcal{S}(\mathcal{H}), \mathcal{T}_0)$ is a second-countable Hausdorff space. The reason for stating Lemma 1 is that later we shall make explicit use of the particular countable base given there.

The weak operator topology on the space $\mathcal{B}_s(\mathcal{H})$ of the bounded self-adjoint operators on $\mathcal{H}$ is the coarsest topology such that the linear functionals

$$A \mapsto \langle \varphi | A\psi \rangle$$

where $A \in \mathcal{B}_s(\mathcal{H})$ and $\varphi, \psi \in \mathcal{H}$, are continuous. It is sufficient to consider only the functionals

$$A \mapsto \langle \varphi | A\varphi \rangle \tag{10}$$

where $\varphi \in \mathcal{H}$ and $\|\varphi\| = 1$. The topology $\mathcal{T}_w$ induced on $\partial_e \mathcal{S}(\mathcal{H}) \subset \mathcal{B}_s(\mathcal{H})$ by the weak operator topology is the coarsest topology on $\partial_e \mathcal{S}(\mathcal{H})$ such that the restrictions of the linear functionals (10) to $\partial_e \mathcal{S}(\mathcal{H})$ are continuous. Since these restrictions are given by

$$P \mapsto \langle \varphi | P\varphi \rangle = \operatorname{tr} PQ = h_Q(P)$$

where $P \in \partial_e \mathcal{S}(\mathcal{H})$ and $Q := |\varphi\rangle\langle\varphi| \in \partial_e \mathcal{S}(\mathcal{H})$, the topology $\mathcal{T}_w$ on $\partial_e \mathcal{S}(\mathcal{H})$ is, according to (7), just our topology $\mathcal{T}_0$.

Now we compare the weak topology $\mathcal{T}_0$ with the metric topology $\mathcal{T}_n$.

Theorem 4 *The weak topology $\mathcal{T}_0$ on $\partial_e \mathcal{S}(\mathcal{H})$ and the metric topology $\mathcal{T}_n$ on $\partial_e \mathcal{S}(\mathcal{H})$ are equal.*

Proof. According to (7), a neighborhood base of $P \in \partial_e \mathcal{S}(\mathcal{H})$ w.r.t. $\mathcal{T}_0$ is given by the

open sets

$$U(P; Q_1, \ldots, Q_n; \varepsilon)$$
$$:= \bigcap_{i=1}^{n} h_{Q_i}^{-1}(\,]h_{Q_i}(P) - \varepsilon, h_{Q_i}(P) + \varepsilon[\,) \tag{11}$$
$$= \left\{ \widetilde{P} \in \partial_e \mathcal{S}(\mathcal{H}) \,\middle|\, \left| h_{Q_i}(\widetilde{P}) - h_{Q_i}(P) \right| < \varepsilon \text{ for } i = 1, \ldots, n \right\}$$
$$= \left\{ \widetilde{P} \in \partial_e \mathcal{S}(\mathcal{H}) \,\middle|\, \left| \operatorname{tr} \widetilde{P} Q_i - \operatorname{tr} P Q_i \right| < \varepsilon \text{ for } i = 1, \ldots, n \right\}$$

where $Q_1, \ldots, Q_n \in \partial_e \mathcal{S}(\mathcal{H})$ and $\varepsilon > 0$; a neighborhood base of P w.r.t. $\mathcal{T}_n$ is given by the open balls

$$K_\varepsilon(P) := \left\{ \widetilde{P} \in \partial_e \mathcal{S}(\mathcal{H}) \,\middle|\, \left\| \widetilde{P} - P \right\| < \varepsilon \right\}. \tag{12}$$

If $\left\| \widetilde{P} - P \right\| < \varepsilon$, then

$$\left| \operatorname{tr} \widetilde{P} Q_i - \operatorname{tr} P Q_i \right| = \left| \operatorname{tr} Q_i (\widetilde{P} - P) \right| \le \| Q_i \|_{\operatorname{tr}} \left\| \widetilde{P} - P \right\| = \left\| \widetilde{P} - P \right\| < \varepsilon;$$

hence, $K_\varepsilon(P) \subseteq U(P; Q_1, \ldots, Q_n; \varepsilon)$. To show some converse inclusion, take account of Theorem 2, part (a), and note that

$$\left\| \widetilde{P} - P \right\|^2 = 1 - \operatorname{tr} \widetilde{P} P = \left| \operatorname{tr} \widetilde{P} P - \operatorname{tr} P P \right|.$$

In consequence, by (11) and (12), $U(P; P; \varepsilon^2) = K_\varepsilon(P)$. Hence, $\mathcal{T}_0 = \mathcal{T}_n$. $\square$

It looks surprising that the topolgies $\mathcal{T}_0$ and $\mathcal{T}_n$ coincide. In fact, consider the sequence $\{P_{\varphi_n}\}_{n \in \mathbb{N}}$ where the vectors $\varphi_n \in \mathcal{H}$ constitute an orthonormal system. Then, w.r.t. the weak operator topology, $P_{\varphi_n} \to 0$ as $n \to \infty$ whereas $\left\| P_{\varphi_n} - P_{\varphi_{n+1}} \right\| = 1$ for all $n \in \mathbb{N}$. However, $0 \notin \partial_e \mathcal{S}(\mathcal{H})$; so $\{P_{\varphi_n}\}_{n \in \mathbb{N}}$ is convergent neither w.r.t. $\mathcal{T}_w = \mathcal{T}_0$ nor w.r.t. $\mathcal{T}_n$. Finally, like in the case of the weak operator topology, there is a natural uniform structure inducing $\mathcal{T}_0$. The uniform structures that are canonically related to $\mathcal{T}_0$ and $\mathcal{T}_n$ are different: $\{P_{\varphi_n}\}_{n \in \mathbb{N}}$ is a Cauchy sequence w.r.t. the uniform structure belonging to $\mathcal{T}_0$ but not w.r.t. that belonging to $\mathcal{T}_n$, i.e., w.r.t. the metric ρ_n.

We remark that besides $\mathcal{T}_0$ and $\mathcal{T}_w$ several further weak topologies can be defined on $\partial_e \mathcal{S}(\mathcal{H})$. Let $\mathcal{C}_s(\mathcal{H})$ be the Banach space of the compact self-adjoint operators and remember that $(\mathcal{C}_s(\mathcal{H}))' = \mathcal{T}_s(\mathcal{H})$. So the weak Banach-space topologies of $\mathcal{C}_s(\mathcal{H})$, $\mathcal{T}_s(\mathcal{H})$, and $\mathcal{B}_s(\mathcal{H})$ as well as the weak-* Banach-space topologies of $\mathcal{T}_s(\mathcal{H})$ and $\mathcal{B}_s(\mathcal{H})$ can be restricted to $\partial_e \mathcal{S}(\mathcal{H})$, thus giving the topologies $\mathcal{T}_1 := \sigma(\mathcal{C}_s(\mathcal{H}), \mathcal{T}_s(\mathcal{H})) \cap \partial_e \mathcal{S}(\mathcal{H})$, $\mathcal{T}_2 := \sigma(\mathcal{T}_s(\mathcal{H}), \mathcal{C}_s(\mathcal{H})) \cap \partial_e \mathcal{S}(\mathcal{H})$, $\mathcal{T}_3 := \sigma(\mathcal{T}_s(\mathcal{H}), \mathcal{B}_s(\mathcal{H})) \cap \partial_e \mathcal{S}(\mathcal{H})$, $\mathcal{T}_4 := \sigma(\mathcal{B}_s(\mathcal{H}), \mathcal{T}_s(\mathcal{H})) \cap \partial_e \mathcal{S}(\mathcal{H})$, and $\mathcal{T}_5 :=$

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$\sigma(\mathcal{B}_s(\mathcal{H}), (\mathcal{B}_s(\mathcal{H}))') \cap \partial_e \mathcal{S}(\mathcal{H})$. Moreover, the strong operator topology induces a topology $\mathcal{T}_s$ on $\partial_e \mathcal{S}(\mathcal{H})$. From the obvious inclusions

$$\mathcal{T}_w \subseteq \mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3 \subseteq \mathcal{T}_{\mathrm{tr}},$$

$$\mathcal{T}_1 = \mathcal{T}_4 \subseteq \mathcal{T}_5 = \mathcal{T}_1,$$

and

$$\mathcal{T}_w \subseteq \mathcal{T}_s \subseteq \mathcal{T}_n$$

as well as from the shown equality

$$\mathcal{T}_0 = \mathcal{T}_w = \mathcal{T}_n = \mathcal{T}_{\mathrm{tr}}$$

it follows that the topologies $\mathcal{T}_1, \ldots, \mathcal{T}_5$ and $\mathcal{T}_s$ also coincide with $\mathcal{T}_0$.

Finally, we show that all the topologies on $\partial_e \mathcal{S}(\mathcal{H})$ are equivalent to the quotient topologies $\mathcal{T}_\mu$ and $\mathcal{T}_\nu$ on $\mathcal{P}(\mathcal{H})$, resp., S/S^1.

Theorem 5 *The mapping $F \colon S/S^1 \to \partial_e \mathcal{S}(\mathcal{H})$, $F([\varphi]_S) := P_\varphi$ where $\varphi \in S$, is a homeomorphism between the topological spaces $(S/S^1, \mathcal{T}_\nu)$ and $(\partial_e \mathcal{S}(\mathcal{H}), \mathcal{T}_0)$.*

Proof. The mapping F is bijective. The map $h_Q \circ F \circ \nu \colon S \to \mathbb{R}$ where h_Q is any of the functions given by Eq. (7) and ν is the canonical projection from S onto S/S^1, reads explicitly

$$(h_Q \circ F \circ \nu)(\varphi) = h_Q(F([\varphi]_S)) = h_Q(P_\varphi) = \operatorname{tr} P_\varphi Q = \langle \varphi | Q \varphi \rangle;$$

therefore, $h_Q \circ F \circ \nu$ is continuous. Consequently, for an open set $O \subseteq \mathbb{R}$,

$$(h_Q \circ F \circ \nu)^{-1}(O) = \nu^{-1}(F^{-1}(h_Q^{-1}(O)))$$

is an open set of S. By the definition of the quotient topology $\mathcal{T}_\nu$, it follows that $F^{-1}(h_Q^{-1}(O))$ is an open set of S/S^1. Since the sets $h_Q^{-1}(O)$, $Q \in \partial_e \mathcal{S}(\mathcal{H})$, $O \subseteq \mathbb{R}$ open, generate the weak topology $\mathcal{T}_0$, $F^{-1}(U)$ is open for any open set $U \in \mathcal{T}_0$. Hence, F is continuous.

To show that F is an open mapping, let $V \in \mathcal{T}_\nu$ be an open subset of S/S^1 and let $[\varphi_0]_S \in V$. Since the canonical projection ν is continuous, there exists an $\varepsilon > 0$ such that

$$\nu(K_\varepsilon(\varphi_0) \cap S) \subseteq V \tag{13}$$

where $K_\varepsilon(\varphi_0) := \{\varphi \in \mathcal{H} \mid \|\varphi - \varphi_0\| < \varepsilon\}$. Without loss of generality we assume that $\varepsilon < 1$.

The topology $\mathcal{T}_0$ is generated by the functions h_Q according to (7); $\mathcal{T}_0$ is also generated by the functions $P \mapsto g_Q(P) := \sqrt{h_Q(P)} = \sqrt{\operatorname{tr} PQ}$. In consequence, the set

$$U_\varepsilon := g_Q^{-1}\left(\,]1 - \tfrac{\varepsilon}{2}, 1 + \tfrac{\varepsilon}{2}[\,\right) \cap h_Q^{-1}\left(\,]1 - \tfrac{\varepsilon^2}{4}, 1 + \tfrac{\varepsilon^2}{4}[\,\right)$$

where $Q := P_{\varphi_0}$ and φ_0 and ε are specified in the preceding paragraph, is $\mathcal{T}_0$-open. Using the identity

$$1 - |\langle\varphi_0|\varphi\rangle|^2 = \|\varphi - \langle\varphi_0|\varphi\rangle\varphi_0\|^2$$

where $\varphi \in \mathcal{H}$ is also a unit vector, we obtain

$$\begin{aligned}
U_\varepsilon &= \left\{ P_\varphi \in \partial_e \mathcal{S}(\mathcal{H}) \,\middle|\, |g_Q(P_\varphi) - 1| < \tfrac{\varepsilon}{2} \text{ and } |h_Q(P_\varphi) - 1| < \tfrac{\varepsilon^2}{4} \right\} \\
&= \left\{ P_\varphi \in \partial_e \mathcal{S}(\mathcal{H}) \,\middle|\, \big||\langle\varphi_0|\varphi\rangle| - 1\big| < \tfrac{\varepsilon}{2} \text{ and } \big||\langle\varphi_0|\varphi\rangle|^2 - 1\big| < \tfrac{\varepsilon^2}{4} \right\} \\
&= \left\{ P_\varphi \in \partial_e \mathcal{S}(\mathcal{H}) \,\middle|\, \big||\langle\varphi_0|\varphi\rangle| - 1\big| < \tfrac{\varepsilon}{2} \text{ and } \|\varphi - \langle\varphi_0|\varphi\rangle\varphi_0\| < \tfrac{\varepsilon}{2} \right\}.
\end{aligned}$$

Now let $P_\varphi \in U_\varepsilon$. Since $\varepsilon < 1$, we have that $\langle\varphi|\varphi_0\rangle \neq 0$. Defining the phase factor $\lambda := \frac{\langle\varphi|\varphi_0\rangle}{|\langle\varphi|\varphi_0\rangle|}$, it follows that

$$\begin{aligned}
\|\lambda\varphi - \varphi_0\| &= \|\lambda\varphi - \lambda\langle\varphi_0|\varphi\rangle\varphi_0\| + \|\lambda\langle\varphi_0|\varphi\rangle\varphi_0 - \varphi_0\| \\
&= \|\varphi - \langle\varphi_0|\varphi\rangle\varphi_0\| + \big\||\langle\varphi_0|\varphi\rangle|\varphi_0 - \varphi_0\big\| \\
&< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

That is, $P_\varphi \in U_\varepsilon$ implies that $\lambda\varphi \in K_\varepsilon(\varphi_0)$; moreover, $\lambda\varphi \in K_\varepsilon(\varphi_0) \cap S$.

Taking the result (13) into account, we conclude that, for $P_\varphi \in U_\varepsilon$, $[\varphi]_S = [\lambda\varphi]_S = \nu(\lambda\varphi) \in V$. Consequently, $P_\varphi = F([\varphi]_S) \in F(V)$. Hence, $U_\varepsilon \subseteq F(V)$. Since U_ε is an open neighborhood of P_{φ_0}, P_{φ_0} is an interior point of $F(V)$. So, for every $[\varphi_0]_S \in V$, $F([\varphi_0]_S) = P_{\varphi_0}$ is an interior point of $F(V)$, and $F(V)$ is a $\mathcal{T}_0$-open set. Hence, the continuous bijective map F is open and thus a homeomorphism. $\square$

In the following, we identify the sets $\mathcal{P}(\mathcal{H})$, S/S^1, and $\partial_e \mathcal{S}(\mathcal{H})$ and call the identified set the *projective Hilbert space* $\mathcal{P}(\mathcal{H})$. However, we preferably think about the elements of $\mathcal{P}(\mathcal{H})$ as the one-dimensional orthogonal projections $P = P_\varphi$. On $\mathcal{P}(\mathcal{H})$ then the quotient topologies $\mathcal{T}_\mu$, $\mathcal{T}_\nu$, the weak topologies $\mathcal{T}_0$, $\mathcal{T}_w$, $\mathcal{T}_1, \ldots, \mathcal{T}_5$, $\mathcal{T}_s$, and the metric topologies $\mathcal{T}_n$, $\mathcal{T}_{\mathrm{tr}}$ coincide. So we can say that $\mathcal{P}(\mathcal{H})$ carries a natural topology $\mathcal{T}$; $(\mathcal{P}(\mathcal{H}), \mathcal{T})$ is a second-countable Hausdorff space.

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For our purposes, it is suitable to represent this topology $\mathcal{T}$ as $\mathcal{T}_0$, $\mathcal{T}_n$, or $\mathcal{T}_{\text{tr}}$. As already discussed, the topologies $\mathcal{T}_0$, $\mathcal{T}_n$, and $\mathcal{T}_{\text{tr}}$ are canonically related to uniform structures. With respect to the uniform structure inducing $\mathcal{T}_0$, $\mathcal{P}(\mathcal{H})$ is not complete. The uniform structures related to $\mathcal{T}_n$ and $\mathcal{T}_{\text{tr}}$ are the same since they are induced by the equivalent metrics ρ_n and ρ_{tr}; $(\mathcal{P}(\mathcal{H}), \rho_n)$ and $(\mathcal{P}(\mathcal{H}), \rho_{\text{tr}})$ are separable complete metric spaces. So $\mathcal{T}$ can be defined by a complete separable metric, i.e., $(\mathcal{P}(\mathcal{H}), \mathcal{T})$ is a polish space.

III. THE MEASURABLE STRUCTURE OF $\mathcal{P}(\mathcal{H})$

It is almost natural to define a measurable structure on the projective Hilbert space $\mathcal{P}(\mathcal{H})$ by the σ-algebra $\Xi = \Xi(\mathcal{T})$ generated by the $\mathcal{T}$-open sets, i.e., Ξ is the smallest σ-algebra containing the open sets of the natural topology $\mathcal{T}$. In this way $(\mathcal{P}(\mathcal{H}), \Xi)$ becomes a measurable space where the elements $B \in \Xi$ are the Borel sets of $\mathcal{P}(\mathcal{H})$. However, since the topology $\mathcal{T}$ is generated by the transition-probability functions h_Q according to Eq. (7), it is also obvious to define the measurable structure of $\mathcal{P}(\mathcal{H})$ by the σ-algebra Σ generated by the functions h_Q, i.e., Σ is the smallest σ-algebra such that all the functions h_Q are measurable. A result due to Misra (1974) [26, Lemma 3] clarifies the relation between Ξ and Σ. Before stating that result, we recall the following simple lemma which we shall also use later.

Lemma 2 *Let $(M, \mathcal{T})$ be any second-countable topological space, $\mathcal{B} \subseteq \mathcal{T}$ a countable base, and $\Xi = \Xi(\mathcal{T})$ the σ-algebra of the Borel sets of M. Then $\Xi = \Xi(\mathcal{T}) = \Xi(\mathcal{B})$ where $\Xi(\mathcal{B})$ is the σ-algebra generated by $\mathcal{B}$; $\mathcal{B}$ is a countable generator of Ξ.*

Proof. Clearly, $\Xi(\mathcal{B}) \subseteq \Xi(\mathcal{T})$. Since every open set $U \in \mathcal{T}$ is the countable union of sets of $\mathcal{B}$, it follows that $U \in \Xi(\mathcal{B})$. Therefore, $\mathcal{T} \subseteq \Xi(\mathcal{B})$ and consequently $\Xi(\mathcal{T}) = \Xi(\mathcal{B})$. $\square$

Theorem 6 (Misra) *The σ-algebra $\Xi = \Xi(\mathcal{T})$ of the Borel sets of the projective Hilbert space $\mathcal{P}(\mathcal{H})$ and the σ-algbra Σ generated by the transition-probability functions h_Q, $Q \in \mathcal{P}(\mathcal{H})$, are equal.*

Proof. Since $\mathcal{T}$ is generated by the functions h_Q, the latter are continuous and consequently Ξ-measurable. Since Σ is the smallest σ-algebra such that the functions h_Q are measurable, it follows that $\Sigma \subseteq \Xi$.

Now, by Lemma 1, $\mathcal{T}$ is second-countable, and a countable base $\mathcal{B}$ of $\mathcal{T}$ is given by the finite intersections of the sets U_{klm} according to Eq. (8). Since $U_{klm} \in \Sigma$, it follows that $\mathcal{B} \subseteq \Sigma$. By Lemma 2, we conclude that $\Xi = \Xi(\mathcal{B}) \subseteq \Sigma$. Hence, $\Xi = \Sigma$. $\square$

We remark that our proof of Misra's theorem is much easier than Misra's proof from 1974. The reason is that we explicitly used the countable base $\mathcal{B}$ of $\mathcal{T}$ consisting of Σ-measurable sets.

Finally, consider the σ-algebra Ξ_0 in $\mathcal{P}(\mathcal{H})$ that is generated by all $\mathcal{T}$-continuous real-valued functions on $\mathcal{P}(\mathcal{H})$, i.e., Ξ_0 is the σ-algebra of the Baire sets of $\mathcal{P}(\mathcal{H})$. Obviously, $\Sigma \subseteq \Xi_0 \subseteq \Xi$; so Theorem 6 implies that $\Xi_0 = \Xi$. This result is, according to a general theorem, also a consequence of the fact that the topology $\mathcal{T}$ of $\mathcal{P}(\mathcal{H})$ is metrizable.

Summarizing, our result $\Sigma = \Xi_0 = \Xi$ manifests that the projective Hilbert space carries, besides its natural topology $\mathcal{T}$, also a very natural measurable structure Ξ.

IV.   THE MISRA-BUGAJSKI REDUCTION MAP

The expression $\operatorname{tr} WA$ where $W \in \mathcal{S}(\mathcal{H})$ is a density operator and A a self-adjoint operator, plays a central role in quantum mechanics. We are going to show how, for bounded self-adjoint operators $A \in \mathcal{B}_s(\mathcal{H})$, this expression can be represented as an integral over the projective Hilbert space $\mathcal{P}(\mathcal{H})$. This result was first obtained by Misra (1974) [26] and independently by Ghirardi, Rimini and Weber (1976) [16], and an elementary construction for the case of a two-dimensional Hilbert space was discussed by Holevo (1982) [21]. The significance of the representation of quantum expectations on $\mathcal{P}(\mathcal{H})$ was elucidated in seminal papers of Bugajski and Beltrametti [1, 6]. Further discussion can be found in [10, 30].

Theorem 7 *For every probability measure μ on $(\mathcal{P}(\mathcal{H}), \Xi)$, there exists a uniquely determined density operator $W_\mu \in \mathcal{S}(\mathcal{H})$ such that, for all $A \in \mathcal{B}_s(\mathcal{H})$,*

$$\operatorname{tr} W_\mu A = \int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} PA \ \mu(dP).$$

Proof.   Because of $|\operatorname{tr}(P - P_0)A| \leq \|P - P_0\|_{\operatorname{tr}} \|A\|$ where $P, P_0 \in \mathcal{P}(\mathcal{H})$, the function $P \mapsto \operatorname{tr} PA$ on $\mathcal{P}(\mathcal{H})$ is continuous w.r.t. the metric ρ_{tr} and in consequence $\mathcal{T}$-continuous and Ξ-measurable; in addition, because of $|\operatorname{tr}(PA| \leq \|A\|$, the function is bounded. Hence, the integral $\int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} PA \ \mu(dP)$ exists for every probability measure μ on $\mathcal{P}(\mathcal{H})$. Moreover,

the functional

$$A \mapsto \phi(A) := \int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} PA \; \mu(dP)$$

is linear, bounded, and positive. Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of bounded self-adjoint operators satisfying $0 \leq A_n \leq A_{n+1} \leq I$; $\{A_n\}_{n \in \mathbb{N}}$ converges to some $A \in \mathcal{B}_s(\mathcal{H})$, $A \leq I$, with respect to the weak operator topology, for instance. It follows that, for all $P \in \mathcal{P}(\mathcal{H})$,

$$0 \leq \operatorname{tr} PA_n \leq \operatorname{tr} PA_{n+1} \leq 1$$

and, writing $P = P_\psi$,

$$\operatorname{tr} PA_n = \langle \psi | A_n \psi \rangle \to \langle \psi | A \psi \rangle = \operatorname{tr} PA$$

as $n \to \infty$. By the monotone-convergence theorem we obtain

$$\phi(A_n) = \int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} PA_n \; \mu(dP) \to \int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} PA \; \mu(dP) = \phi(A),$$

i.e., the functional ϕ is normal. Since the normal functionals on $\mathcal{B}_s(\mathcal{H})$ can be represented by trace-class operators, there exists an operator $W_\mu \in \mathcal{T}_s(\mathcal{H})$ such that

$$\phi(A) = \operatorname{tr} W_\mu A = \int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} PA \; \mu(dP).$$

The operator W_μ is uniquely determined, self-adjoint, positive, and, because of $\operatorname{tr} W_\mu = \phi(I) = 1$, of trace 1, i.e., $W_\mu \in \mathcal{S}(\mathcal{H})$. $\square$

The next theorem summarizes the properties of the mapping $\mu \mapsto W_\mu$. Remember that the elements of $\mathcal{P}(\mathcal{H})$ are the extreme points of the convex set $\mathcal{S}(\mathcal{H})$.

Theorem 8 *The mapping $R : \mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi) \to \mathcal{S}(\mathcal{H})$, $R(\mu) = W_\mu$, where $\mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi)$ denotes the convex set of all probability measures on $(\mathcal{P}(\mathcal{H}), \Xi)$, has the following properties:*

(a) R is affine, i.e., for every convex linear combination $\mu = \alpha \mu_1 + (1 - \alpha)\mu_2$ of $\mu_1, \mu_2 \in \mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi)$, $0 \leq \alpha \leq 1$, we have $W_\mu = \alpha W_{\mu_1} + (1 - \alpha)W_{\mu_2}$;

(b) R is surjective, but not injective (provided that $\dim \mathcal{H} \geq 2$);

(c) $R(\mu) = P$, $P \in \mathcal{P}(\mathcal{H})$, holds if and only if μ is equal to the Dirac measure δ_P;

(d) R maps the Dirac measures on $(\mathcal{P}(\mathcal{H}), \Xi)$ bijectively onto the pure quantum states $P \in \mathcal{P}(\mathcal{H})$ and all other probability measures on $(\mathcal{P}(\mathcal{H}), \Xi)$ "many-to-one" onto the mixed quantum states $W \in \mathcal{S}(\mathcal{H})$.

Proof. The first statement is trivial. To prove statement (b), consider any $W \in \mathcal{S}(\mathcal{H})$ and a representation $W = \sum_{i=1}^{\infty} \alpha_i P_i$ where $\alpha_i \geq 0$, $\sum_{i=0}^{\infty} \alpha_i = 1$, $P_i \in \mathcal{P}(\mathcal{H})$, and the infinite sum converges in the trace norm. Define a probability measure $\mu \in \mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi)$ by $\mu := \sum_{i=1}^{\infty} \alpha_i \delta_{P_i}$ and note that the sum converges in the total-variation norm. Writing $\operatorname{tr} PA =: f_A(P)$ where $A \in \mathcal{B}_s(\mathcal{H})$ and $f_A \in \mathcal{F}_{\mathbb{R}}(\mathcal{P}(\mathcal{H}), \Xi)$, it follows that

$$
\begin{aligned}
\int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} PA \, \mu(dP) \; = \; \langle \mu, f_A \rangle &= \left\langle \sum_{i=1}^{\infty} \alpha_i \delta_{P_i}, f_A \right\rangle \\
&= \sum_{i=1}^{\infty} \alpha_i \langle \delta_{P_i}, f_A \rangle \\
&= \sum_{i=1}^{\infty} \alpha_i \int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} PA \, \delta_{P_i}(dP) \\
&= \sum_{i=1}^{\infty} \alpha_i \operatorname{tr} P_i A \\
&= \operatorname{tr} WA,
\end{aligned}
$$

which implies $W = W_\mu = R(\mu)$. Hence, R is surjective. Since every mixed quantum state can be represented in many ways as an infinite convex linear combination of one-dimensional orthogonal projections, not necessarily being mutually orthogonal (cf. [3, 25]), let

$$
W = \sum_{i=1}^{\infty} \alpha_i P_i = \sum_{i=1}^{\infty} \beta_i Q_i, \qquad \mu_1 := \sum_{i=1}^{\infty} \alpha_i \delta_{P_i}, \qquad \mu_2 := \sum_{i=1}^{\infty} \beta_i \delta_{Q_i},
$$

where two different representations of any $W \in \mathcal{S}(\mathcal{H}) \setminus \mathcal{P}(\mathcal{H})$ have been chosen. Then $W = R(\mu_1) = R(\mu_2)$ holds, but $\mu_1 \neq \mu_2$; that is, R is not injective.

Since $R(\delta_P) = P$ is a trivial fact, we have, in order to prove (c), only to show that $R(\mu) = P$ implies $\mu = \delta_P$. From $R(\mu) = P$, resp., $\operatorname{tr} PA = \int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} QA \, \mu(dQ)$ we obtain, setting $A = P$,

$$
1 = \int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} QP \, \mu(dQ)
$$

which can be rewritten as

$$
\int_{\mathcal{P}(\mathcal{H})} (1 - \operatorname{tr} QP) \, \mu(dQ) = 0.
$$

Because the integrand is nonnegative, it must vanish almost everywhere. It follows that

$$
\mu(\{Q \in \mathcal{P}(\mathcal{H}) \,|\, \operatorname{tr} QP = 1\}) = 1
$$

or, equivalently, $\mu(\{P\}) = 1$. That is, the probability measure μ is concentrated at the point $P \in \mathcal{P}(\mathcal{H})$ and consequently equal to the Dirac measure δ_P.

Statement (d) is a consequence of (c), (b), and the proof of the fact that R is not injective.
□

Consider now the unique linear extension $R \colon \mathcal{M}_{\mathbb{R}}(\mathcal{P}(\mathcal{H}), \Xi) \to \mathcal{T}_s(\mathcal{H})$ of the affine mapping $R \colon \mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi) \to \mathcal{S}(\mathcal{H})$. The extended map R is determined by

$$\operatorname{tr}(R\nu)A = \int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} PA \; \nu(dP) \tag{14}$$

where $\nu \in \mathcal{M}_{\mathbb{R}}(\mathcal{P}(\mathcal{H}), \Xi)$ and $A \in \mathcal{B}_s(\mathcal{H})$. From

$$\langle R\nu, A \rangle = \int_{\mathcal{P}(\mathcal{H})} \operatorname{tr} PA \; \nu(dP) = \langle \nu, f_A \rangle$$

where $f_A(P) = \operatorname{tr} PA$ it follows that the dual map R' of R w.r.t. the considered dualities $\langle \mathcal{M}_{\mathbb{R}}(\mathcal{P}(\mathcal{H}), \Xi), \mathcal{F}_{\mathbb{R}}(\mathcal{P}(\mathcal{H}), \Xi) \rangle$ and $\langle \mathcal{T}_s(\mathcal{H}), \mathcal{B}_s(\mathcal{H}) \rangle$ exists and is given by $R'A = f_A$. The existence of R' in this sense means that the range of the usual adjoint map $R^* \colon \mathcal{B}_s(\mathcal{H}) \to (\mathcal{M}_{\mathbb{R}}(\mathcal{P}(\mathcal{H}), \Xi))'$ is under $\mathcal{F}_{\mathbb{R}}(\mathcal{P}(\mathcal{H}), \Xi)$. According to the discussion in the introduction and the definition there, R is a reduction map and $\langle \mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi), \mathcal{E}(\mathcal{P}(\mathcal{H}), \Xi) \rangle$ a classical extension of the quantum statistical model $\langle \mathcal{S}(\mathcal{H}), \mathcal{E}(\mathcal{H}) \rangle$. We call the reduction map R given by (14) the *Misra-Bugajski map*. The affine mapping R was introduced by Misra in 1974 [26] who considered it as a new way of defining the notion of quantum state; it was the late S. Bugajski who realized that this map determines a classical extension of the quantum statistical duality and who initiated a research program to elucidate the physical significance of this extension—see, e.g., [1, 6].

The adjoint R' of the Misra-Bugajski map R associates the quantum mechanical effects $A \in \mathcal{E}(\mathcal{H})$ with the classical effects $R'A = f_A \in \mathcal{E}(\mathcal{P}(\mathcal{H}), \Xi)$. However, except for the trivial cases $A = 0$ or $A = I$, such a function f_A, $f_A(P) = \operatorname{tr} PA$, is never the characteristic function χ_B of some set $B \in \Xi$; that is, the functions f_A describe unsharp (fuzzy) effects.

V. THE REPRESENTATION OF CLASSICAL EXTENSIONS OF QUANTUM MECHANICS

Now we are going to show that every classical extension of quantum mechanics is essentially given by the Misra-Bugajski reduction map. This result was conjectured in [10], and the proof given here takes up elements of a very rough sketch given there.

Assume a classical extension on a measurable space (Ω, Σ) is given by the linear maps $R\colon \mathcal{M}_{\mathbb{R}}(\Omega, \Sigma) \to \mathcal{T}_s(\mathcal{H})$ and $R'\colon \mathcal{B}_s(\mathcal{H}) \to \mathcal{F}_{\mathbb{R}}(\Omega, \Sigma)$. Then, for $\mu \in \mathcal{S}(\Omega, \Sigma)$ and $A \in \mathcal{B}_s(\mathcal{H})$, we have

$$\operatorname{tr}(R\mu)A = \langle R\mu, A \rangle = \langle \mu, R'A \rangle = \int_{\Omega} R'A \, d\mu; \tag{15}$$

setting $\mu = \delta_{\omega}$ where δ_{ω} denotes the Dirac measure of a point $\omega \in \Omega$, we obtain

$$(R'A)(\omega) = \operatorname{tr}(R\delta_{\omega})A. \tag{16}$$

Hence,

$$\operatorname{tr}(R\mu)A = \int_{\Omega} \operatorname{tr}(R\delta_{\omega})A \, \mu(d\omega). \tag{17}$$

To prove our main result, Theorem 10 below, we need several lemmata.

Lemma 3 *For $P \in \mathcal{P}(\mathcal{H})$, the set $\{\omega \in \Omega \mid R\delta_{\omega} = P\}$ is measurable. If $P = R\mu$, then*

$$\mu(\{\omega \in \Omega \mid R\delta_{\omega} = P\}) = 1.$$

In particular, for every $P \in \mathcal{P}(\mathcal{H})$ there exists an $\omega \in \Omega$ such that $R\delta_{\omega} = P$.

Proof. Let $E_P := \{\omega \in \Omega \mid R\delta_{\omega} = P\}$. Since the statement $R\delta_{\omega} = P$ is equivalent to $\operatorname{tr}(R\delta_{\omega})P = 1$, it follows that

$$E_P = \{\omega \in \Omega \mid \operatorname{tr}(R\delta_{\omega})P = 1\}.$$

Setting $A = P$ in Eq. (16), we see that the function $P \mapsto \operatorname{tr}(R\delta_{\omega})P$ is measurable; therefore, the set E_P is measurable. Setting $P = R\mu$ and $A = P$ in Eq. (17), we obtain

$$\int_{\Omega} \operatorname{tr}(R\delta_{\omega})P \, \mu(d\omega) = 1$$

which can be rewritten as

$$\int_{\Omega} (1 - \operatorname{tr}(R\delta_{\omega})P) \, \mu(d\omega) = 0.$$

Since the integrand is nonnegative, it must vanish almost everywhere. Hence,

$$\mu(E_P) = \mu(\{\omega \in \Omega \mid 1 - \operatorname{tr}(R\delta_{\omega})P = 0\}) = 1.$$

Because R is surjective, every $P \in \mathcal{P}(\mathcal{H})$ is of the form $P = R\mu$. Then $\mu(E_P) = 1$ implies that E_P is not empty. $\square$

Lemma 4 *Let $P_n \in \mathcal{P}(\mathcal{H})$, $n \in \mathbb{N}$, and assume that, for some $W_0 \in \mathcal{S}(\mathcal{H})$,*

$$\lim_{n \to \infty} \mathrm{tr}\, W_0 P_n = 1. \tag{18}$$

Then there exists an element $P \in \mathcal{P}(\mathcal{H})$ such that $\lim_{n \to \infty} \|P_n - P\| = 0$; moreover, $W_0 = P$.

Proof. For each $n \in \mathbb{N}$, let φ_n be a unit vector in the range of P_n, and write $P_n = P_{\varphi_n}$. Since $\|\varphi_n\| = 1$, the weak compactness of the unit sphere of $\mathcal{H}$ entails that there is a subsequence $\{\varphi_{n_j}\}_{j \in \mathbb{N}}$ of $\{\varphi_n\}_{n \in \mathbb{N}}$ converging weakly to some $\psi \in \mathcal{H}$, $\|\psi\| \le 1$.

Let W be any element of $\mathcal{S}(\mathcal{H})$. We show that $\mathrm{tr}\, W P_{\varphi_{n_j}} \to \mathrm{tr}\,(W|\psi\rangle\langle\psi|)$ as $j \to \infty$. The density operator can be written as $W = \sum_{i=1}^{\infty} \alpha_i P_{\chi_i}$ where $\alpha_i \ge 0$, $\sum_{i=1}^{\infty} \alpha_i = 1$, $\chi_i \in \mathcal{H}$, and $\|\chi_i\| = 1$. Choose $\varepsilon > 0$ and a number $N_0 \in \mathbb{N}$ such that $\sum_{i=N_0+1}^{\infty} \alpha_i < \frac{\varepsilon}{4}$. Since the sequence $\{\varphi_{n_j}\}_{j \in \mathbb{N}}$ converges weakly to ψ, there is an integer $J(\varepsilon)$ such that for all $j \ge J(\varepsilon)$ and all $i = 1, \ldots, N_0$,

$$|\langle \chi_i | \varphi_{n_j} \rangle|^2 - |\langle \chi_i | \psi \rangle|^2 < \tfrac{\varepsilon}{2}.$$

It follows that, for all $j \ge J(\varepsilon)$,

$$
\begin{aligned}
\left| \mathrm{tr}\, W P_{\varphi_{n_j}} - \mathrm{tr}\,(W|\psi\rangle\langle\psi|) \right| &= \left| \sum_{i=1}^{\infty} \alpha_i |\langle\chi_i|\varphi_{n_j}\rangle|^2 - \sum_{i=1}^{\infty} \alpha_i |\langle\chi_i|\psi\rangle|^2 \right| \\
&\le \left| \sum_{i=1}^{N_0} \alpha_i \left(|\langle\chi_i|\varphi_{n_j}\rangle|^2 - |\langle\chi_i|\psi\rangle|^2 \right) \right| + 2 \sum_{i=N_0+1}^{\infty} \alpha_i \\
&< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Hence,

$$\lim_{j \to \infty} \mathrm{tr}\, W P_{\varphi_{n_j}} = \mathrm{tr}\,(W|\psi\rangle\langle\psi|). \tag{19}$$

For $W = W_0$, Eqs. (18) and (19) imply that

$$\mathrm{tr}\,(W_0|\psi\rangle\langle\psi|) = 1.$$

So $\psi \ne 0$; defining $\Psi := \frac{\psi}{\|\psi\|}$, we obtain $\|\psi\|^2 \,\mathrm{tr}\, W_0 P_\Psi = 1$. It follows immediately that $\|\psi\| = 1$ and $\mathrm{tr}\, W_0 P_\Psi = 1$. Hence, $\mathrm{tr}\, W_0 P_\psi = \langle\psi|W_0\psi\rangle = 1$, that is, W_0 has the eigenvalue 1 with multiples of ψ as eigenvectors, i.e., $W_0 = P_\psi =: P$.

It remains to show that $\|P_n - P\| \to 0$ as $n \to \infty$. From (18) and $W_0 = P$ it follows that $\mathrm{tr}\, P P_n \to 1$ as $n \to \infty$. But this is, according to Theorem 2, part (a), equivalent to

$$\|P_n - P\|^2 = 1 - \mathrm{tr}\, P P_n \to 0$$

as $n \to \infty$. $\square$

It can be shown that the norm convergence of a sequence $\{P_n\}_{n\in\mathbb{N}}$ in $\mathcal{P}(\mathcal{H})$, $P_n = P_{\varphi_n}$, to $P = P_\psi \in \mathcal{P}(\mathcal{H})$ entails the existence of a subsequence $\{\varphi_{n_j}\}_{j\in\mathbb{N}}$ of $\{\varphi_n\}_{n\in\mathbb{N}}$ such that $\lim_{j\to\infty} \|\varphi_{n_j} - e^{i\alpha}\psi\| = 0$ with some $\alpha \in \mathbb{R}$. The example

$$\varphi_n := e^{in\pi}\psi = (-1)^n\psi, \quad \|P_{\varphi_n} - P_\psi\| \to 0 \text{ as } n \to \infty$$

shows that convergence at the level of vectors can follow only for a subsequence. Concerning the sequences $\{\varphi_n\}_{n\in\mathbb{N}}$ and $\{\varphi_{n_j}\}_{j\in\mathbb{N}}$ introduced at the beginning of the preceding proof, it finally turns out that the subsequence $\{\varphi_{n_j}\}_{j\in\mathbb{N}}$ is even norm-convergent (which is not essential for the proof), however, the restriction of $\{\varphi_n\}_{n\in\mathbb{N}}$ to a subsequence is essential.

Lemma 5 *Let*

$$\widetilde{\Omega} := \{\omega \in \Omega \,|\, R\delta_\omega \in \mathcal{P}(\mathcal{H})\} = \{\omega \in \Omega \,|\, \mathrm{tr}\,(R\delta_\omega)P = 1 \text{ for some } P \in \mathcal{P}(\mathcal{H})\}.$$

Then $\widetilde{\Omega}$ is a measurable subset of Ω.

Proof. Let $\{P_m\}_{m\in\mathbb{N}}$ be a $\|\cdot\|$-dense sequence in $\mathcal{P}(\mathcal{H})$ and let

$$\Omega_{mn} := \left\{\omega \in \Omega \,\middle|\, \mathrm{tr}\,(R\delta_\omega)P_m > 1 - \tfrac{1}{n}\right\}$$

where $n \in \mathbb{N}$. We show that

$$\widetilde{\Omega} = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \Omega_{mn} \tag{20}$$

holds.

Let $\omega \in \widetilde{\Omega}$ and $R\delta_\omega = P$, i.e., $\mathrm{tr}\,(R\delta_\omega)P = 1$. For every $n \in \mathbb{N}$ there exists a member P_m of the dense sequence satisfying $\|P_m - P\| < \tfrac{1}{n}$, in consequence,

$$1 - \mathrm{tr}\,(R\delta_\omega)P_m = |\mathrm{tr}\,(R\delta_\omega)P_m - \mathrm{tr}\,(R\delta_\omega)P| \leq \|R\delta_\omega\|_{\mathrm{tr}}\,\|P_m - P\| < \tfrac{1}{n};$$

that is, $\mathrm{tr}\,(R\delta_\omega)P_m > 1 - \tfrac{1}{n}$. Hence, $\omega \in \bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\Omega_{mn}$.

Conversely, assume $\omega \in \bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\Omega_{mn}$. Then for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ with $\omega \in \Omega_{mn}$. In other words, for every $n \in \mathbb{N}$ there exists at least one P_m such that $\mathrm{tr}\,(R\delta_\omega)P_m > 1 - \tfrac{1}{n}$. Let P_{m_n} be such a P_m. Then it holds true that $1 - \tfrac{1}{n} < \mathrm{tr}\,(R\delta_\omega)P_{m_n} \leq 1$, which implies that

$$\mathrm{tr}\,(R\delta_\omega)P_{m_n} \to 1$$

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as $n \to \infty$. By virtue of Lemma 4, this entails $R\delta_\omega = P \in \mathcal{P}(\mathcal{H})$, that is, $\omega \in \widetilde{\Omega}$. Thus, Eq. (20) has been proved.

Due to the measurability of the functions $\omega \mapsto (R'A)(\omega) = \operatorname{tr}(R\delta_\omega)A$ for $A \in \mathcal{B}_s(\mathcal{H})$, the sets Ω_{mn} are measurable; from Eq. (20) one then concludes that $\widetilde{\Omega} \in \Sigma$. $\square$

Next we shall redefine our reduction map $R \colon \mathcal{M}_\mathbb{R}(\Omega, \Sigma) \to \mathcal{F}_\mathbb{R}(\Omega, \Sigma)$ w.r.t. the measurable space $(\widetilde{\Omega}, \widetilde{\Sigma})$ where $\widetilde{\Sigma} := \Sigma \cap \widetilde{\Omega}$ (since $\widetilde{\Omega}$ is measurable, we have that $\widetilde{\Sigma} = \{E \in \Sigma \,|\, E \subseteq \widetilde{\Omega}\} \subseteq \Sigma$). To that end, we introduce

$$\mathcal{N} := \big\{\nu \in \mathcal{M}_\mathbb{R}(\Omega, \Sigma) \,\big|\, \nu(E) = 0,\ E \in \Sigma,\ E \subseteq \Omega \setminus \widetilde{\Omega}\big\}$$

and

$$\begin{aligned}
\mathcal{S}_\mathcal{N} &:= \big\{\mu \in \mathcal{S}(\Omega, \Sigma) \,\big|\, \mu(\Omega \setminus \widetilde{\Omega}) = 0\big\} = \big\{\mu \in \mathcal{S}(\Omega, \Sigma) \,\big|\, \mu(\widetilde{\Omega}) = 1\big\} \\
&= \mathcal{N} \cap \mathcal{S}(\Omega, \Sigma).
\end{aligned}$$

The set $\mathcal{N}$ is a norm-closed subspace of $\mathcal{M}_\mathbb{R}(\Omega, \Sigma)$, and $\mathcal{S}_\mathcal{N}$ is a norm-closed face of $\mathcal{S}(\Omega, \Sigma)$. Moreover, $(\mathcal{N}, \mathcal{S}_\mathcal{N})$ is a base-normed Banach space with closed positive cone; we do not need these results here. The spaces $\mathcal{N}$ and $\mathcal{M}_\mathbb{R}(\widetilde{\Omega}, \widetilde{\Sigma})$ are canonically related by the linear map $J \colon \mathcal{N} \to \mathcal{M}_\mathbb{R}(\widetilde{\Omega}, \widetilde{\Sigma})$ defined by

$$\nu \mapsto \tilde{\nu} = J\nu := \nu|_{\widetilde{\Sigma}}$$

where $\nu|_{\widetilde{\Sigma}}$ denotes the restriction of ν to $\widetilde{\Sigma}$; J is a linear isomorphism preserving norm and order. The inverse J^{-1} is given by

$$\tilde{\nu} \mapsto \nu = J^{-1}\tilde{\nu}, \quad \nu(A) = \tilde{\nu}(A \cap \widetilde{\Omega})$$

where $A \in \Sigma$. We shall only use that J is a linear isomorphism.—In the context of the following theorem, $\tilde{\delta}_\omega$ denotes the restriction of the Dirac measure δ_ω, defined on Σ and concentrated at $\omega \in \widetilde{\Omega}$, to $\widetilde{\Sigma}$.

Theorem 9 *Let a linear map $\widetilde{R} \colon \mathcal{M}_\mathbb{R}(\widetilde{\Omega}, \widetilde{\Sigma}) \to \mathcal{T}_s(\mathcal{H})$ be defined according to $\widetilde{R}\tilde{\nu} := R\nu$ where $J\nu = \tilde{\nu}$, i.e., $\widetilde{R} = RJ^{-1}$. Then*

(i) $\widetilde{R}\mathcal{S}(\widetilde{\Omega}, \widetilde{\Sigma}) = \mathcal{S}(\mathcal{H})$;

(ii) $\widetilde{R}$ *is* $\sigma(\mathcal{M}_\mathbb{R}(\widetilde{\Omega}, \widetilde{\Sigma}), \mathcal{F}_\mathbb{R}(\widetilde{\Omega}, \widetilde{\Sigma}))$-$\sigma(\mathcal{T}_s(\mathcal{H}), \mathcal{B}_s(\mathcal{H}))$-*continuous;*

(iii) $\big\{\widetilde{R}\tilde{\delta}_\omega \,\big|\, \omega \in \widetilde{\Omega}\big\} = \mathcal{P}(\mathcal{H})$.

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That is, $\widetilde{R}$ is a reduction map with the additional property (iii).

Proof. We prove statement (iii) first. By the definition of $\widetilde{\Omega}$ in Lemma 5 it is clear that $\{R\delta_\omega \,|\, \omega \in \widetilde{\Omega}\} \subseteq \mathcal{P}(\mathcal{H})$. Let $P \in \mathcal{P}(\mathcal{H})$, then by virtue of Lemma 3 there exists an $\omega \in \Omega$ such that $R\delta_\omega = P$; again by the definition of $\widetilde{\Omega}$, $\omega \in \widetilde{\Omega}$. Hence, $\{R\delta_\omega \,|\, \omega \in \widetilde{\Omega}\} = \mathcal{P}(\mathcal{H})$; furthermore, $R\delta_\omega = \widetilde{R}\tilde{\delta}_\omega$ for $\omega \in \widetilde{\Omega}$.

We have $\widetilde{R}\mathcal{S}(\widetilde{\Omega},\widetilde{\Sigma}) = R\mathcal{S}_\mathcal{N} \subseteq R\mathcal{S}(\Omega,\Sigma) = \mathcal{S}(\mathcal{H})$, thus $\widetilde{R}\mathcal{S}(\widetilde{\Omega},\widetilde{\Sigma}) \subseteq \mathcal{S}(\mathcal{H})$. Let $W \in \mathcal{S}(\mathcal{H})$, and write $W = \sum_{i=1}^\infty \alpha_i P_i$ where $\alpha_i \geq 0$, $\sum_{i=1}^\infty \alpha_i = 1$, and $P_i \in \mathcal{P}(\mathcal{H})$. Defining $\tilde{\mu} := \sum_{i=1}^\infty \alpha_i \tilde{\delta}_{\omega_i}$ where $P_i = \widetilde{R}\tilde{\delta}_{\omega_i}$ and $\omega_i \in \widetilde{\Omega}$, we obtain a probability measure $\tilde{\mu} \in \mathcal{S}(\widetilde{\Omega},\widetilde{\Sigma})$. It follows that

$$\widetilde{R}\tilde{\mu} = \sum_{i=1}^\infty \alpha_i \widetilde{R}\tilde{\delta}_{\omega_i} = \sum_{i=1}^\infty \alpha_i P_i = W;$$

for this conclusion we have used that the sums converge in the respective norms and $\widetilde{R}$ is norm-continuous, the latter due to the linearity of $\widetilde{R}$ and the property $\widetilde{R}\mathcal{S}(\widetilde{\Omega},\widetilde{\Sigma}) \subseteq \mathcal{S}(\mathcal{H})$ already shown above. Hence, $\widetilde{R}\mathcal{S}(\widetilde{\Omega},\widetilde{\Sigma}) = \mathcal{S}(\mathcal{H})$.

Taking account of $\nu = J^{-1}\tilde{\nu} \in \mathcal{N}$ for $\tilde{\nu} \in \mathcal{M}_\mathbb{R}(\widetilde{\Omega},\widetilde{\Sigma})$ and using the abbreviation $f_A := R'A$ where $A \in \mathcal{B}_s(\mathcal{H})$, we obtain that

$$
\begin{aligned}
\langle \widetilde{R}\tilde{\nu}, A \rangle &= \mathrm{tr}\,(\widetilde{R}\tilde{\nu})A = \mathrm{tr}\,(R\nu)A \\
&= \int_\Omega R'A \, d\nu = \int_\Omega f_A \chi_{\widetilde{\Omega}} \, d\nu \\
&= \int_{\widetilde{\Omega}} f_A \, d\nu = \int_{\widetilde{\Omega}} f_A|_{\widetilde{\Omega}} \, d\tilde{\nu} \\
&= \langle \tilde{\nu}, f_A|_{\widetilde{\Omega}} \rangle \\
&= \langle \tilde{\nu}, \widetilde{R}'A \rangle;
\end{aligned}
$$

that is, the map $\widetilde{R}' : \mathcal{B}_s(\mathcal{H}) \to \mathcal{F}_\mathbb{R}(\widetilde{\Omega},\widetilde{\Sigma})$ being dual to $\widetilde{R}$ w.r.t. the dualities $\langle \mathcal{T}_s(\mathcal{H}), \mathcal{B}_s(\mathcal{H}) \rangle$ and $\langle \mathcal{M}_\mathbb{R}(\widetilde{\Omega},\widetilde{\Sigma}), \mathcal{F}_\mathbb{R}(\widetilde{\Omega},\widetilde{\Sigma}) \rangle$ exists. $\square$

In the sequel we omit the tilde notation and understand by $R : \mathcal{M}_\mathbb{R}(\Omega,\Sigma) \to \mathcal{T}_s(\mathcal{H})$ a linear map with the properties (i)-(iii) of Theorem 9. We have again that

$$\mathrm{tr}\,(R\mu)A = \int_\Omega R'A \, d\mu = \int_\Omega \mathrm{tr}\,(R\delta_\omega)A \; \mu(d\omega) \tag{21}$$

holds for all $\mu \in \mathcal{S}(\Omega,\Sigma)$ and $A \in \mathcal{B}_s(\mathcal{H})$ (cf. Eqs. (15)-(17)). Moreover, now the equality

$$\mathcal{P}(\mathcal{H}) = \{R\delta_\omega \,|\, \omega \in \Omega\} \tag{22}$$

is satisfied.

Lemma 6 *Let $\mathcal{T}$ be the natural topology of $\mathcal{P}(\mathcal{H})$ and $\Xi = \Xi(\mathcal{T})$ the σ-algebra of the Borel sets of $\mathcal{P}(\mathcal{H})$. The mapping $i \colon \Omega \to \mathcal{P}(\mathcal{H})$ defined by $i(\omega) := R\delta_\omega$ is Σ-Ξ-measurable.*

Proof. The topology $\mathcal{T}$ is generated by the functions h_Q defined by Eq. (7). According to

$$h_Q(i(\omega)) = \operatorname{tr} i(\omega)Q = \operatorname{tr}(R\delta_\omega)Q = (R'Q)(\omega)$$

where Eq. (16) has been taken into account, the functions $h_Q \circ i$ are Σ-measurable.

Let $O \subseteq \mathbb{R}$ be an open set. Then

$$U := h_Q^{-1}(O) \in \mathcal{T}. \tag{23}$$

From the measurability of the functions $h_Q \circ i$ it follows that

$$i^{-1}(U) = i^{-1}(h_Q^{-1}(O)) = (h_Q \circ i)^{-1}(O) \in \Sigma;$$

that is, for all U of the form (23) we have

$$i^{-1}(U) \in \Sigma. \tag{24}$$

According to Lemma 1, for a sequence $\{Q_k\}_{k \in \mathbb{N}}$ being dense in $\mathcal{P}(\mathcal{H})$, a sequence $\{q_l\}_{l \in \mathbb{N}}$ of numbers being dense in $[0, 1]$, and $m \in \mathbb{N}$, the finite intersections of the sets

$$U_{klm} = h_{Q_k}^{-1}\left(\left] q_l - \tfrac{1}{m}, q_l + \tfrac{1}{m} \right[\right)$$

form a countable basis $\mathcal{B}$ of the topology $\mathcal{T}$ of $\mathcal{P}(\mathcal{H})$. From this and from (24) we obtain that

$$i^{-1}(U) \in \Sigma$$

for all $U \in \mathcal{B}$.

In virtue of Lemma 2, the countable basis $\mathcal{B}$ of $\mathcal{T}$ is a (countable) generator of $\Xi(\mathcal{T})$. Since $i^{-1}(U) \in \Sigma$ for all sets U of a generator of $\Xi = \Xi(\mathcal{T})$, the mapping i is Σ-Ξ-measurable. $\square$

By virtue of Eq. (22), i is a surjective measurable mapping.

Theorem 10 *Any reduction map R with the property $\{R\delta_\omega \,|\, \omega \in \Omega\} = \mathcal{P}(\mathcal{H})$ can be represented according to*

$$\operatorname{tr}(R\mu)A = \int_\Omega \operatorname{tr} PA \,(\mu \circ i^{-1})(dP) \tag{25}$$

where $\mu \in \mathcal{S}(\Omega, \Sigma)$, $A \in \mathcal{B}_s(\mathcal{H})$, $i \colon \Omega \to \mathcal{P}(\mathcal{H})$ is the mapping $\omega \mapsto i(\omega) = R\delta_\omega$, and $\mu \circ i^{-1}$ the image measure.

Proof. The claim follows from (21), Lemma 6, and the transformation theorem for integrals:

$$\mathrm{tr}\,(R\mu)A \;=\; \int_\Omega \mathrm{tr}\,(R\delta_\omega)A\,\mu(d\omega) = \int_\Omega \mathrm{tr}\,i(\omega)A\,\mu(d\omega)$$

$$= \int_\Omega \mathrm{tr}\,PA\,(\mu \circ i^{-1})(dP).\;\;\square$$

Given any reduction map $R\colon \mathcal{M}_\mathbb{R}(\Omega,\Sigma) \to \mathcal{T}_s(\mathcal{H})$, every density operator $W \in \mathcal{S}(\mathcal{H})$ is the image of some probability measure $\mu \in \mathcal{S}(\Omega,\Sigma)$, i.e., $W = R\mu$. Theorem 10 now states that, after removing the redundant $\omega \in \Omega$ for which $R\delta_\omega \notin \mathcal{P}(\mathcal{H})$, W is the *weak integral*

$$R\mu = \int_{\mathcal{P}(\mathcal{H})} P\,(\mu \circ i^{-1})(dP) \tag{26}$$

of the elements $P \in \mathcal{P}(\mathcal{H})$ (i.e., of the identity map of $\mathcal{P}(\mathcal{H})$) w.r.t. the probability measure $\mu \in \mathcal{S}(\mathcal{P}(\mathcal{H}),\Xi)$. The classical *sample space* (Ω,Σ) can be replaced by the *phase space* $(\mathcal{P}(\mathcal{H}),\Xi)$ (for the interpretation of $\mathcal{P}(\mathcal{H})$ as a phase space, see Section VII), Eqs. (25) and (26) show the central role of $\mathcal{P}(\mathcal{H})$. Comparing Eq. (25) with Eq. (14), the latter specifying the Misra-Bugajski map R_{MB}, we obtain

$$R\mu = R_{MB}(\mu \circ i^{-1}). \tag{27}$$

If the surjective measurable map i also transforms the measurable sets of Σ into measurable sets of Ξ, then every probability measure $\mu' \in \mathcal{S}(\mathcal{P}(\mathcal{H}),\Xi)$ is of the form $\mu' = \mu \circ i^{-1}$. In this case R can be replaced by R_{MB}; in the case where not every μ' is of the form $\mu \circ i^{-1}$, R can be restated as some restriction of R_{MB}. Summarizing, every classical extension of quantum mechanics is essentially given by the Misra-Bugajski reduction map; therefore, R_{MB} is distinguished under all reduction maps.

However, the examples presented in the next section show that the mapping i is necessary for the statement of Theorem 10 even if $\Omega = \mathcal{P}(\mathcal{H})$.

VI.   EXAMPLES

The following examples of reduction maps are also of interest by themselves.

Example 1 Let $\mathcal{K}$ be an infinite-dimensional closed subspace of the Hilbert space $\mathcal{H}$, $V : \mathcal{H} \to \mathcal{H}$ a partial isometry satisfying $V\mathcal{K} = \mathcal{H}$ and $V\mathcal{K}^\perp = \{0\}$, and let $\mathcal{P}(\mathcal{K}) := \{P \in \mathcal{P}(\mathcal{H}) \,|\, P = P_\varphi, \|\varphi\| = 1, \varphi \in \mathcal{K}\}$ ($\mathcal{P}(\mathcal{K})$ can be identified with the projective Hilbert space associated with the Hilbert space $\mathcal{K}$). Using the general information given in the paragraph after the proof of Lemma 4, one easily proves that $\mathcal{P}(\mathcal{K})$ is a norm-closed subset of $\mathcal{P}(\mathcal{H})$; therefore, $\mathcal{P}(\mathcal{K})$ is Ξ-measurable, and the following integral in (28) makes sense. In fact, according to

$$\operatorname{tr} W_\mu A = \int_{\mathcal{P}(\mathcal{K})} \operatorname{tr} VPV^*A \; \mu(dP) \tag{28}$$

where $A \in \mathcal{B}_s(\mathcal{H})$, for each probability measure $\mu \in \mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi)$ concentrated on $\mathcal{P}(\mathcal{K})$, i.e., $\mu(\mathcal{P}(\mathcal{K})) = 1$, a density operator $W_\mu \in \mathcal{S}(\mathcal{H})$ is defined. We can identify the set of these probability measures with $\mathcal{S}(\mathcal{P}(\mathcal{K}), \Xi_\mathcal{K})$ where $\Xi_\mathcal{K} := \Xi \cap \mathcal{P}(\mathcal{K}) = \{B \in \Xi \,|\, B \subseteq \mathcal{P}(\mathcal{K})\} \subseteq \Xi$. Moreover, the affine mapping $\mu \mapsto W_\mu$ can be extended to a reduction map $R : \mathcal{M}_\mathbb{R}(\mathcal{P}(\mathcal{K}), \Xi_\mathcal{K}) \to \mathcal{T}_s(\mathcal{H})$; R maps the Dirac measures of $\mathcal{S}(\mathcal{P}(\mathcal{K}), \Xi_\mathcal{K})$ bijectively onto $\mathcal{P}(\mathcal{H})$, namely, $R\delta_P = VPV^*$, $P \in \mathcal{P}(\mathcal{K})$.

Setting $(\Omega, \Sigma) := (\mathcal{P}(\mathcal{K}), \Xi_\mathcal{K})$, it follows from Lemma 3 that, for $Q \in \mathcal{P}(\mathcal{H})$ and any $\mu \in \mathcal{S}(\mathcal{P}(\mathcal{K}), \Xi_\mathcal{K})$, $R\mu = Q$ if and only if $\mu = \delta_P$ with $P = V^*QV$. Furthermore, we have for the set $\widetilde{\Omega}$ introduced in Lemma 5 and for the mapping $i : \widetilde{\Omega} \to \mathcal{P}(\mathcal{H})$ of Lemma 6 that $\widetilde{\Omega} = \Omega$ and $i(P) = R\delta_P = VPV^*$. In particular, if $\mathcal{K} = \mathcal{H}$ (where $\mathcal{H}$ need not be infinite-dimensional) and V is a unitary operator, then $\Omega = \mathcal{P}(\mathcal{H}) = \widetilde{\Omega}$ and $i(P) = VPV^*$.

Example 2 Letting $\mathcal{K}$, V, and $\mathcal{P}(\mathcal{K})$ as in the preceding example, then for each probability measure $\mu \in \mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi)$ a density operator $W_\mu \in \mathcal{S}(\mathcal{H})$ is defined according to

$$\operatorname{tr} W_\mu A = \int_{\mathcal{P}(\mathcal{K})} \operatorname{tr} VPV^*A \; \mu(dP) + \int_{\mathcal{P}(\mathcal{H}) \setminus \mathcal{P}(\mathcal{K})} \operatorname{tr} PA \; \mu(dP) \tag{29}$$

where $A \in \mathcal{B}_s(\mathcal{H})$ and $\mathcal{P}(\mathcal{H}) \setminus \mathcal{P}(\mathcal{K})$ is the set-theoretical complement of $\mathcal{P}(\mathcal{K})$. Note that μ is a probability measure on $\mathcal{P}(\mathcal{H})$ whereas in the preceding example μ is a probability measure on $\mathcal{P}(\mathcal{K})$. The affine mapping $\mu \mapsto W_\mu$ given by (29) can be extended to a reduction map $R : \mathcal{M}_\mathbb{R}(\mathcal{P}(\mathcal{H}), \Xi) \to \mathcal{T}_s(\mathcal{H})$; R maps the Dirac measures of $\mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi)$ onto $\mathcal{P}(\mathcal{H})$, partially two-to-one:

$$R\delta_P = \begin{cases} VPV^* & \text{if } P \in \mathcal{P}(\mathcal{K}) \\ P & \text{if } P \in \mathcal{P}(\mathcal{H}) \setminus \mathcal{P}(\mathcal{K}). \end{cases}$$

In fact, from $R\delta_P = Q$ it follows that $P = V^*QV$ if $Q \in \mathcal{P}(\mathcal{K})$, and $P = V^*QV$ or $P = Q$ if $Q \in \mathcal{P}(\mathcal{H}) \setminus \mathcal{P}(\mathcal{K})$. By Lemma 3, $R\mu = Q$ for any $\mu \in \mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi)$ is equivalent to $\mu = \delta_{V^*QV}$ if $Q \in \mathcal{P}(\mathcal{K})$, resp., to $\mu = \alpha \delta_{V^*QV} + (1 - \alpha)\delta_Q$, $0 \leq \alpha \leq 1$, if $Q \in \mathcal{P}(\mathcal{H}) \setminus \mathcal{P}(\mathcal{K})$.

Setting $(\Omega, \Sigma) := (\mathcal{P}(\mathcal{H}), \Xi)$, we obtain $\widetilde{\Omega} = \Omega$ and $i : \widetilde{\Omega} \to \mathcal{P}(\mathcal{H})$, $i(P) = R\delta_P = \chi_{\mathcal{P}(\mathcal{K})}(P) V P V^* + \chi_{\mathcal{P}(\mathcal{H}) \setminus \mathcal{P}(\mathcal{K})}(P) P$ where $\chi_{\mathcal{P}(\mathcal{K})}$, for instance, is the characteristic function of the set $\mathcal{P}(\mathcal{K})$.

Example 3 Now let $\mathcal{K}$ be an infinite-dimensional closed subspace of $\mathcal{H}$ with an infinite dimensional orthocomplement $\mathcal{K}^\perp$ and let V_1 and V_2 be partial isometries satisfying

$$V_1 \mathcal{K} = \mathcal{H}, \qquad V_1 \mathcal{K}^\perp = \{0\}$$
$$V_2 \mathcal{K}^\perp = \mathcal{H}, \qquad V_2 \mathcal{K} = \{0\}.$$

Then each probability measure $\mu \in \mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi)$ determines a density operator $W_\mu \in \mathcal{S}(\mathcal{H})$ according to

$$\operatorname{tr} W_\mu A = \int_{\mathcal{P}(\mathcal{H})} \operatorname{tr}(V_1 P V_1^* + V_2 P V_2^*) A \; \mu(dP) \tag{30}$$

where $A \in \mathcal{B}_s(\mathcal{H})$. The affine mapping $\mu \mapsto W_\mu$ given by (30) again extends to a reduction map $R : \mathcal{M}_\mathbb{R}(\mathcal{P}(\mathcal{H}), \Xi) \to \mathcal{T}_s(\mathcal{H})$; R maps the Dirac measures of $\mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi)$ onto the quantum states

$$R\delta_P = V_1 P V_1^* + V_2 P V_2^* = |V_1\varphi\rangle\langle V_1\varphi| + |V_2\varphi\rangle\langle V_2\varphi|$$
$$= \|\chi_1\|^2 P_{\frac{\chi_1}{\|\chi_1\|}} + \|\chi_2\|^2 P_{\frac{\chi_2}{\|\chi_2\|}}$$

where $P = P_\varphi$, $\chi_1 := V_1\varphi$, $\chi_2 := V_2\varphi$, and $\|\chi_1\|^2 + \|\chi_2\|^2 = 1$. In general, the states $R\delta_P$ are mixed; $R\delta_P \in \mathcal{P}(\mathcal{H})$ is equivalent to $P = P_\varphi$ with $\varphi = a\varphi_1 + b\varphi_2$, $\varphi_1 \in \mathcal{K}$, $\varphi_2 \in \mathcal{K}^\perp$, $\|\varphi_1\| = \|\varphi_2\| = 1$, $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$, and $V_1\varphi_1 = V_2\varphi_2$. In particular, for each $Q \in \mathcal{P}(\mathcal{H})$, there is one unit vector $\varphi_1 \in \mathcal{K}$ and one unit vector $\varphi_2 \in \mathcal{K}^\perp$ such that $R\delta_{P_{\varphi_1}} = R\delta_{P_{\varphi_2}} = Q$, φ_1 and φ_2 are uniquely determined up to phase factors. Let $\mathcal{K}_Q$ be the two-dimensional subspace of $\mathcal{H}$ that is spanned by φ_1 and φ_2 and let $\mathcal{P}(\mathcal{K}_Q) := \{P \in \mathcal{P}(\mathcal{H}) \mid P = P_\varphi, \|\varphi\| = 1, \varphi \in \mathcal{K}_Q\}$. Then $R\delta_P = Q$ if and only if $P \in \mathcal{P}(\mathcal{K}_Q)$, and by Lemma 3, $R\mu = Q$ for any $\mu \in \mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi)$ if and only if μ is concentrated on $\mathcal{P}(\mathcal{K}_Q)$, i.e., $\mu(\mathcal{P}(\mathcal{K}_Q)) = 1$.

It follows that $\mathcal{K}_{Q_1} \cap \mathcal{K}_{Q_2} = \{0\}$ as well as $\mathcal{P}(\mathcal{K}_{Q_1}) \cap \mathcal{P}(\mathcal{K}_{Q_2}) = \emptyset$ for $Q_1 \neq Q_2$ and that $\bigcup_{Q \in \mathcal{P}(\mathcal{H})} \mathcal{K}_Q \neq \mathcal{H}$ as well as $\bigcup_{Q \in \mathcal{P}(\mathcal{H})} \mathcal{P}(\mathcal{K}_Q) \neq \mathcal{P}(\mathcal{H})$. Writing $(\Omega, \Sigma) := (\mathcal{P}(\mathcal{H}), \Xi)$, we obtain $\widetilde{\Omega} = \{P \in \mathcal{P}(\mathcal{H}) \mid P \in \mathcal{P}(\mathcal{K}_Q) \text{ for some } Q \in \mathcal{P}(\mathcal{H})\} = \bigcup_{Q \in \mathcal{P}(\mathcal{H})} \mathcal{P}(\mathcal{K}_Q)$, $\widetilde{\Omega} \neq \Omega$, and $i : \widetilde{\Omega} \to \mathcal{P}(\mathcal{H})$, $i(P) = R\delta_P = V_1 P V_1^* + V_2 P V_2^*$.

VII.   PHYSICAL INTERPRETATION

Interpreting the bounded self-adjoint operators on $\mathcal{H}$ as quantum observables with real values, the expectation value of $A \in \mathcal{B}_s(\mathcal{H})$ in the state $W \in \mathcal{S}(\mathcal{H})$ is given by $\operatorname{tr} WA$. Analogously, if Ω is a classical phase space with the Borel structure Σ, the states are described by the probability measures on Ω and the observables by the (bounded) measurable functions on Ω; the expectation value of a classical observable $f \in \mathcal{F}_{\mathbb{R}}(\Omega, \Sigma)$ in the state $\mu \in \mathcal{S}(\Omega, \Sigma)$ is $\int f d\mu$. According to Theorems 7 and 8, each $W \in \mathcal{S}(\mathcal{H})$ is of the form $W = R\mu = W_\mu$, μ being some probablity measure on $\Omega = \mathcal{P}(\mathcal{H})$. That is, for every $W \in \mathcal{S}(\mathcal{H})$ there exists a probability measure $\mu \in \mathcal{S}(\mathcal{P}(\mathcal{H}), \Xi)$ such that for all $A \in \mathcal{B}_s(\mathcal{H})$, $A = A^*$,

$$\operatorname{tr} WA = \int_{\mathcal{P}(\mathcal{H})} f_A d\mu \tag{31}$$

holds where f_A is the function $P \mapsto f_A(P) = \operatorname{tr} PA$ on $\mathcal{P}(\mathcal{H})$. Viewing the projective Hilbert space as a classical phase space, this result means that the quantum states can be seen as classical states and the quantum observables as classical ones where the expectation values can be expressed in classical terms. However, the injective map $A \mapsto f_A$ is not surjective, as is easily seen. That is, not all classical observables on $\mathcal{P}(\mathcal{H})$ represent quantum ones, which is related to the fact that the quantum states W correspond to the equivalence classes $R^{-1}(\{W\})$ of classical states, each member of an equivalence class giving the same quantum mechanical expectation values.

Taking up the notion of quantum statistical model reviewed in the introduction, the result (31) can, much more fundamentally, be interpreted in terms of probabilities if the operators A are specified to be effects; in that case, $\operatorname{tr} WA$ is interpreted to be the probability for the occurrence of 'yes' of the effect A in the state W. Eq. (31) then states that the quantum mechanical effects $A \in \mathcal{E}(\mathcal{H})$ can classically be described by measurable functions taking values between the numbers 0 and 1, i.e., by the classical effects $f_A \in \mathcal{E}(\mathcal{P}(\mathcal{H}), \Xi)$. In the context of classical probability theory, such effects can be interpreted as "unsharp" measurements of events, these being the classical analogs of the quantum mechanical effects and extending probability theory to *operational* or *fuzzy probability theory* (cf. [8, 17, 18, 29]). Again, the map $A \mapsto f_A$, $0 \leq A \leq 1$, into the measurable functions f on $\mathcal{P}(\mathcal{H})$, $0 \leq f \leq 1$, is injective, but not surjective. In particular, the orthogonal projections, describing the ideal quantum mechanical yes–no measurements, are not mapped onto the characteristic

functions, except for the trivial cases; the "sharp" classical events do not correspond to any quantum mechanical effects.

In general, quantum observables with values in some space M, (M, Υ) being a measurable space, are operationally described by *positive operator-valued measures (POVMs) $F \colon \Upsilon \to \mathcal{B}_s(\mathcal{H})$*, $b \mapsto F(b)$, $0 \leq F(b) \leq 1$;

$$b \mapsto \operatorname{tr} W F(b)$$

is the probability distribution of the observable F in the state $W \in \mathcal{S}(\mathcal{H})$. The analogous classical concept is that of *fuzzy random variables* which generalizes the usual concept of random variables (cf. [9, 18, 27, 29]). Given a classical sample or phase space (Ω, Σ) and a space (M, Υ) of possible measurement results, a fuzzy random variable is a Markov kernel $K \colon \Omega \times \Upsilon \to [0, 1]$, i.e., for each $b \in \Upsilon$, $K(\,.\,, b)$ is a measurable function on Ω and, for each $\omega \in \Omega$, $K(\omega, \,.\,)$ is a probability measure on Υ;

$$b \mapsto \int_{\Omega} K(\omega, b) \, \mu(d\omega)$$

is the probability distribution of the observable, resp., fuzzy random variable K in the state $\mu \in \mathcal{M}(\Omega)$. Now, in the case of a POVM F on (M, Υ), Eq. (31) can be rewritten according to

$$\operatorname{tr} W F(b) = \int_{\mathcal{P}(\mathcal{H})} K(P, b) \, \mu(dP) \tag{32}$$

where the Markov kernel $K \colon \mathcal{P}(\mathcal{H}) \times \Upsilon \to [0, 1]$ is defined by $K(P, b) := \operatorname{tr} P F(b)$. That is, every quantum observable can be represented by a classical observable; however, there are many more fuzzy random variables $K \colon \mathcal{P}(\mathcal{H}) \times \Upsilon \to [0, 1]$ than POVMs $F \colon \Upsilon \to \mathcal{B}_s(\mathcal{H})$.

Summarizing, the statistical scheme of quantum mechanics can be reformulated in classical terms by virtue of the Misra-Bugajski map. This reformulation is complete in the sense that all quantum states and quantum effects are represented as probability measures and functions on the phase space $\mathcal{P}(\mathcal{H})$, respectively; however, not all classically possible observables are quantum ones. Quantum mechanics can thus be understood as a fuzzy probability theory on $\mathcal{P}(\mathcal{H})$ with a selection rule for the observables; briefly, quantum mechanics is a *reduced* fuzzy probability theory. Moreover, the projective Hilbert space is a differentiable manifold carrying a natural symplectic structure which allows one to reformulate quantum dynamics in terms of Hamiltonian mechanics (cf. [4, 5, 12, 13, 19, 23]). Hence, quantum mechanics can be interpreted to be a reduced classical statistical mechanics on the phase space $\mathcal{P}(\mathcal{H})$.

As already observed by Bugajski in 1991, the classical embedding of quantum mechanics induced by the Misra-Bugajski map contains all ingredients of a hidden-variables, or ontological, model of quantum mechanics. In fact, there is a phase space whose points may be taken to play the role of *ontic* states describing the hypothetical underlying reality of the quantum system. Next, there is the set of probability measures μ over the phase space, which can be interpreted as *epistemic* states describing the lack of information about the actual ontic state in a preparation of the system represented by μ. Finally, there is the correspondence (31) between quantum and classical expectation values which determines the correspondences $\mu \mapsto W_\mu$ and $A \mapsto f_A$ between the quantum states and observables on the one hand and the classical epistemic states and functions on phase space on the other hand.

This ontological model is noncontextual with respect to measurements since to every quantum effect probabilities are assigned that are independent of the observables to which this effect may belong. However, the model does display contextuality with respect to preparations, in the sense defined by Spekkens [28]: two preparations that are statistically indistinguishable and hence represented by one and the same density operator W are generally represented by different probability measures μ and μ' on the phase space $\mathcal{P}(\mathcal{H})$ such that $W = W_\mu = W_{\mu'}$. This was demonstrated in the proof of Theorem 8, part (b).

The function $P \mapsto K(P, b)$ appearing in (32) can be interpreted as the probability for the outcome of a measurement of the observable F to lie in the set b, given that the ontic state of the system is P. This is to say that the present ontological model constitutes a so-called stochastic or non-deterministic hidden-variables model.

An ontological model of quantum mechanics can be said to ascribe reality to the pure quantum states if any change in a pure state must be associated with a corresponding change in the ontic state of the system [28]. The Misra-Bugajski map satisfies this condition since the correspondence between pure quantum states and point measures is given by a map $\delta_P \mapsto R\delta_P = P$.

In [20], Hardy has given a proof of the fact that any ontological model that reproduces the quantum mechanical expectations must carry a large amount of "quantum ontological excess baggage"; more precisely, it is shown that even for a finite-dimensional quantum system, any ontological model that accounts for all quantum probabilities is based on a classical phase space with infinitely many points, so that the epistemic states form an infinite-dimensional

simplex.

The requirements Hardy stipulates of an ontological model of quantum mechanics are essentially those of our definition of a reduction map R. If one accepts, in addition, the seemingly innocent requirement that the adjoint map R^* associates bounded quantum observables with bounded measurable functions on phase space, then Theorem 10 asserts that, after removing redundant points from the phase space, R is related to the Misra-Bugajski map via the map i according to (25) and (27), so that essentially all ontological models arise from some classical reduction map as defined in the present paper. The uncountable infinity of point measures in the set of epistemic states is now an immediate consequence of Theorem 10.

It is evident that preparation contextuality is necessary for any classical reduction map. As Examples 2 and 3 show, the correspondence $\delta_P \mapsto R\delta_P$ may be many-to-one, and there may be point measures (hence ontic states) that are mapped to mixed quantum states. The ontological model induced by the Misra-Bugajski map is thus essentially distinguished (modulo similarity) by a minimality or nonredundancy property in the sense that a bijective correspondence is established between the pure quantum states and the points of the associated classical phase space. As Example 1 shows, this correspondence identifies Dirac measures with pure quantum states up to a similarity transformation.

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