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PARTIAL ACTIONS OF MONOIDS

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ABSTRACT. We investigate *partial monoid actions*, in the sense of Megrelishvili and Schröder [12]. These are equivalent to a class of premorphisms, which we call *strong* premorphisms. We describe two distinct methods for constructing a monoid action from a partial monoid action: the *expansion* method provides a generalisation of a result of Kellendonk and Lawson [10] in the group case, whilst the approach via *globalisation* extends results of both [12] and [10].

INTRODUCTION

Partial *group* actions have been studied as a natural generalisation of group actions, in particular, by Exel [3] and by Kellendonk and Lawson [10]. The latter authors list a number of applications of partial group actions, for example, to model theory, partial symmetries, tilings, $\mathbb{R}$-trees, and to constructing C^*-algebras, which was Exel's original motivation. One question addressed by Kellendonk and Lawson is the following: given a partial action, is it possible to construct an action? They describe two distinct methods for doing this. The first of these, the 'expansion method', involves enlarging the group in question, and completes a construction of Exel. The second method, the 'globalisation' method, which involves enlarging the set upon which the group acts partially, appears in a range of other contexts—see [10, p. 89].

By comparison with partial group actions, partial *monoid* actions have seen little study. For example, Megrelishvili and Schröder [12] consider so-called *confluent* partial monoid actions on topological and metric spaces and show that these can be 'globalised' in a manner analogous to that of Kellendonk and Lawson. Megrelishvili and Schröder then go on to study the connection between their starting space (with partial action) and their larger space (with action), proving embedding theorems and describing certain preservation properties.

In the present paper, we will approach partial monoid actions in much the same way as Kellendonk and Lawson, obtaining analogues of some of their results. A monoid analogue of their 'globalisation' theorem for partial group actions has

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already been obtained by Megrelishvili and Schröder, but we will extend this result.

In the 'expansion' method for constructing an action from a partial action, Kellendonk and Lawson take a partial group action and obtain the action of an inverse monoid. We might therefore expect that when we apply the same construction to a partial monoid action, we obtain the action of some generalisation of an inverse monoid. This is indeed the case. The generalisation in question is a *weakly left E-ample monoid*; in Section 1, we give a brief overview of such monoids.

In Section 2, we define partial monoid actions and investigate their connection with a particular type of function called a *premorphism*.

Section 3 introduces the *Szendrei expansion* of a monoid. This is a vital ingredient for the 'expansion' method of constructing an action from a partial action, which we describe in Section 4. Section 5 details the 'globalisation' method, including some results on categories of globalisations.

Finally, Section 6 contains a brief consideration of the injectivity of partial monoid actions.

1. An overview of weakly left E-ample monoids

A summary of weakly left E-ample monoids can be found in [9]. We give here only a brief overview. Most of the following details apply equally well to semigroups, but we will only consider monoids, since it is the actions and partial actions of these which we will deal with in this paper.

Weakly left E-ample monoids arise very naturally from partial transformation monoids in the same way that inverse monoids arise from symmetric inverse monoids. We therefore begin by defining partial transformation monoids.

A *partial transformation* on a set X is a function $A \to B$, where $A, B \subseteq X$. The collection of all partial transformations on X is denoted $\mathcal{PT}_X$. We can compose $\alpha, \beta \in \mathcal{PT}_X$ (from left to right) according to the rule

$$\operatorname{dom} \alpha\beta = [\operatorname{im} \alpha \cap \operatorname{dom} \beta]\, \alpha^{-1},$$

where α^{-1} denotes the preimage under α, and $x(\alpha\beta) = (x\alpha)\beta$, for all $x \in \operatorname{dom} \alpha\beta$. Under this composition, $\mathcal{PT}_X$ forms a monoid—the *partial transformation monoid on X*.

A partial transformation monoid possesses an obvious partial order:

$$\alpha \leq \beta \iff \alpha = \beta|_{\operatorname{dom} \alpha}. \tag{1.1}$$

Note that $\mathcal{T}_X$, the full transformation monoid on X, is a submonoid of $\mathcal{PT}_X$. Since the elements of $\mathcal{T}_X$ are defined on the whole of X, the ordering becomes trivial.

In order to define a weakly left E-ample monoid, we must consider the idempotents of $\mathcal{PT}_X$. The only idempotents in which we will be interested are those of the form I_Z, for $Z \subseteq X$, i.e., those idempotents which are identities on their domains. We will refer to such idempotents as *partial identities.* Let $E_X \subseteq E(\mathcal{PT}_X)$ be the set of partial identities of $\mathcal{PT}_X$.

We now define a unary operation $^+$ on $\mathcal{PT}_X$ by $\alpha^+ = I_{\operatorname{dom}\alpha}$, for each $\alpha \in \mathcal{PT}_X$. Let S be a submonoid of $\mathcal{PT}_X$ and let E be the set $E = \{I_Z \in E_X : Z = \operatorname{dom}\alpha, \text{ for some } \alpha \in S\}$. If S is closed under $^+$, i.e., if $E \subseteq S$, then we call S a *weakly left E-ample monoid.* Such a submonoid can be regarded as a (2,1,0)-subalgebra of $\mathcal{PT}_X$. Since $\mathcal{PT}_X$ itself is closed under $^+$, $\mathcal{PT}_X$ is weakly left E_X-ample [9, Proposition 5.1].

Weakly left E-ample monoids also have a useful abstract characterisation of which we will make extensive use. Let S be a monoid and suppose that $E \subseteq E(S)$ is a commutative submonoid of idempotents of S. We define the (equivalence) relation $\widetilde{\mathcal{R}}_E$ on S by the rule that

$$a \,\widetilde{\mathcal{R}}_E\, b \Longleftrightarrow \forall e \in E[ea = a \Leftrightarrow eb = b],$$

for $a, b \in S$. Thus, two elements a, b are $\widetilde{\mathcal{R}}_E$-related if, and only if, they have the same left identities in E.

Definition 1.1. A monoid S with commutative submonoid $E \subseteq E(S)$ of idempotents is *weakly left E-ample* if

(1) every element a is $\widetilde{\mathcal{R}}_E$-related to a (unique) idempotent in E, denoted a^+;
(2) $\widetilde{\mathcal{R}}_E$ is a left congruence;
(3) for all $a \in S$ and all $e \in E$, $ae = (ae)^+a$.

Thus $a \,\widetilde{\mathcal{R}}_E\, b$ if, and only if, $a^+ = b^+$. The idempotent a^+ is a left identity for a. It is also clear that if $e \in E$, then $e^+ = e$.

The following theorem connects Definition 1.1 with the original characterisation of a weakly left E-ample monoid as a (2,1,0)-subalgebra of a partial transformation monoid:

Theorem 1.2. [9, Theorem 5.2] *Let S be a weakly left E-ample monoid, regarded as an algebra of type (2,1,0), for some $E \subseteq E(S)$. Then the mapping $\phi : S \to \mathcal{PT}_S$ given by $s\phi = \rho_s$, where*

$$\operatorname{dom}\rho_s = Ss^+ \qquad \textit{and} \qquad x\rho_s = xs, \ \forall x \in \operatorname{dom}\rho_s,$$

is a representation of S as a (2,1,0)-subalgebra of $\mathcal{PT}_S$.

Thus weakly left E-ample monoids are *precisely* the (2,1,0)-subalgebras of the class of partial transformation monoids.

For *any* monoid S, if $E = E(S)$, then we denote $\widetilde{\mathcal{R}}_E$ by $\widetilde{\mathcal{R}}$. Note that $\widetilde{\mathcal{R}} \subseteq \widetilde{\mathcal{R}}_E$, for any E. If S is weakly left E-ample with $E = E(S)$, then we call S simply *weakly left ample.*

Weakly left ample monoids generalise inverse monoids, since every inverse monoid is weakly left ample with $a^+ = aa^{-1}$ (in an inverse monoid, $\widetilde{\mathcal{R}}$ coincides with Green's relation $\mathcal{R}$). Note that every monoid is weakly left $\{1\}$-ample, with $a^+ = 1$, for all elements a. A unipotent monoid is therefore weakly left ample. Weakly left E-ample monoids also generalise left E-ample monoids (see Section 6).

We note a useful identity involving $^+$ which follows easily from the fact that $\widetilde{\mathcal{R}}_E$ is a left congruence:

Lemma 1.3. *Let S be a weakly left E-ample monoid, for some $E \subseteq E(S)$, and let $s, t \in S$. Then $(st)^+ = (st^+)^+$.*

In the abstract charcterisation of a weakly left E-ample monoid S, the ordering of (1.1) becomes the following natural partial order (i.e., a partial order which is compatible with multiplication and which restricts to the usual partial order on idempotents):

$$a \leq b \iff a = eb, \tag{1.2}$$

for some idempotent $e \in E$. Equivalently,

$$a \leq b \iff a = a^+b.$$

To see this equivalence, we start with $a = eb$ and use Lemma 1.3 to obtain

$$a^+ = (eb)^+ = (eb^+)^+ = eb^+,$$

so that $a = eb = eb^+b = a^+b$, as required. The converse is clear.

We observed earlier that a^+ is a left identity for a. We can now say a little more: a^+ is the *least* left identity for a, with respect to $\leq$. To see this, suppose that $a = fa$, for some $f \in E$. Then, by reasoning identical to that in the previous paragraph, we have $a^+ = fa^+$, or $a^+ = a^+f$, since idempotents in E commute. Hence $a^+ \leq f$. The next result now follows:

Lemma 1.4. [5, Proposition 1.6] *Let S be a weakly left E-ample monoid with partial order $\leq$, and let $s, t \in S$. Then $(st)^+ \leq s^+$.*

2. Partial monoid actions and premorphisms

Recall the definition of the action of a monoid M on a set X:

Definition 2.1. A monoid M *acts* on a set X (on the right) if there is a mapping $X \times M \to X$, given by $(x, s) \mapsto x \cdot s$, and such that

(1) $x \cdot 1 = x$, for all $x \in X$;
(2) $(x \cdot s) \cdot t = x \cdot st$, for all $x \in X$, for all $s, t \in M$.

The concept of a monoid action generalises to that of a *partial* monoid action, in which the action $x \cdot s$ is not necessarily defined for all pairs $(x, s) \in X \times M$. We will write "$\exists x \cdot s$" to mean "the action of $s \in M$ on $x \in X$ is defined". We adopt a definition of partial action which is modelled on that of Kellendonk and Lawson [10, p. 87]:

Definition 2.2. A monoid M *acts partially* on a set X (on the right) if there is a partial mapping $X \times M \to X$, given by $(x, s) \mapsto x \cdot s$, and such that[1]

(PA1) $\exists x \cdot 1$ and $x \cdot 1 = x$, for all $x \in X$;
(PA2′) $\exists x \cdot s$ and $\exists (x \cdot s) \cdot t \Rightarrow \exists x \cdot st$ and $(x \cdot s) \cdot t = x \cdot st$.

Whenever we need to emphasise the distinction between the actions of Definitions 2.1 and 2.2, we will refer to the former as a *global action*.

Note that there is an alternative notion of partial action (as found, for example, in [13, §3] for inverse semigroups) in which condition (PA2′) is an "if, and only if" statement. Given this type of 'partial action', we can easily build a global action by first adjoining an extra element, say 0, to X, and then demanding that all previously undefined actions be equal to 0. Notice that a 'partial action' according to this definition is also a partial action in our sense. By adopting the above definition of partial action, rather than the alternative one, we are getting something new, since it is no longer so trivial to construct a global action from a partial action.

Example 2.3. Let $\langle a \rangle^1$ be the monogenic semigroup on the element a, with adjoined identity. Then $\langle a \rangle^1$ acts partially on $\mathbb{N}$, with the action given by $n \cdot 1 = n$ and

$$n \cdot a^i = \begin{cases} n + i & \text{if } n \in \{1, \ldots, i\}; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In order to obtain the desired generalisations of results from the group case, we will require a slightly stronger concept of partial monoid action:

Definition 2.4. The partial action of a monoid M on a set X will be called *strong* if the following extra condition holds:

(PA3) $\exists x \cdot s$ and $\exists x \cdot st \Rightarrow \exists (x \cdot s) \cdot t$, in which case, $x \cdot st = (x \cdot s) \cdot t$.

Observe that (PA3) is a partial converse for (PA2′). In fact, we can combine these two axioms into a single new axiom:

(PA2) $\exists x \cdot s \Rightarrow [\exists (x \cdot s) \cdot t \Leftrightarrow \exists x \cdot st$, in which case, $x \cdot st = (x \cdot s) \cdot t]$.

The notion of partial monoid action previously adopted by Megrelishvili and Schröder [12] is precisely that of Definition 2.4. Whenever we need to emphasise the distinction between the partial actions of Definitions 2.2 and 2.4, we will refer to the former as a *weak partial action*.

Definition 2.4 now begs the question: is it, in fact, the case that *every* partial monoid action is strong? The answer to this question is 'no', indeed, Example 2.3 is an example of a partial action which is not strong, however, we will defer the proof until the end of this section.

[1]We place a ′ on the second condition, since we will very shortly replace (PA2′) by a stronger condition and we wish to reserve the label "(PA2)" for this.

Example 2.5. Let $H = \{e, a^3\}$ be a group, with $(a^3)^2 = e$ (the reasons for naming the nonidentity element "a^3" will be become clear in a later example). We define a partial action of $\mathbb{N}^0$ on H by

$$\exists h \cdot n \iff n \equiv 0 \,(\mathrm{mod}\, 3),$$

in which case, $h \cdot n = ha^n$, i.e., $e \cdot n = a^n$ and $a^3 \cdot n = a^{3+n}$. Using the simple properties of congruences, it is easy to verify that this is a strong partial action. Note also that $\exists h \cdot (m + n)$ does not imply that $\exists (h \cdot m) \cdot n$: we can have $m + n \equiv 0 \,(\mathrm{mod}\, 3)$ with neither $m \equiv 0 \,(\mathrm{mod}\, 3)$ nor $n \equiv 0 \,(\mathrm{mod}\, 3)$. In order to make this deduction, we need the extra condition that $\exists h \cdot m$, so this is *not* a partial action in the alternative sense.

Example 2.6. Let M be a monoid and let ρ be a right congruence on M. We define a strong partial action of M on M/ρ by

$$\exists (x\rho) \cdot s \iff xs \,\rho\, x, \text{ in which case, } (x\rho) \cdot s = (xs)\rho.$$

(So this is the 'identity action' whenever it is defined, since $x\rho = (xs)\rho$.) It is easy to confirm that this is indeed a strong partial action:

(PA1) Clearly, $x \,\rho\, x1$, for all $x \in M$, so $\exists (x\rho) \cdot 1$, for all x.

(PA2) If $\exists (x\rho) \cdot s$ and $\exists \,[(x\rho) \cdot s] \cdot t$, then $x \,\rho\, xs$ and $xs \,\rho\, xst$, whence $x \,\rho\, xst$, by transitivity. Therefore $\exists (x\rho) \cdot st$. Conversely, suppose that $\exists (x\rho) \cdot s$ and $\exists (x\rho) \cdot st$, i.e., $x \,\rho\, xs$ and $x \,\rho\, xst$. Then $xs \,\rho\, xst$, as required. Note that, in general, $\exists (x\rho) \cdot st$ does not imply that $\exists \,[(x\rho) \cdot s] \cdot t$; once again, in order to make this deduction, we need the extra condition that $\exists (x\rho) \cdot s$, so this is not a partial action in the alternative sense. This can be seen more clearly in the following concrete example.

Consider the bicyclic monoid $B = \mathbb{N}^0 \times \mathbb{N}^0$ with multiplication

$$(a, b)(c, d) = (a - b + \max\{b, c\}, d - c + \max\{b, c\}).$$

The relation σ, given by

$$(a, b) \,\sigma\, (c, d) \iff a - b = c - d,$$

is a (two-sided) congruence: the minimum group congruence on B. Let $[a, b]$ denote the congruence class of (a, b). Then B acts strongly and partially on B/σ, with

$$\begin{aligned} \exists [a, b] \cdot (c, d) &\iff (a, b)(c, d) \,\sigma\, (a, b) \\ &\iff (a - b + \max\{b, c\}, d - c + \max\{b, c\}) \,\sigma\, (a, b) \\ &\iff a - b + \max\{b, c\} - d + c - \max\{b, c\} = a - b \\ &\iff c = d \\ &\iff (c, d) \text{ is idempotent,} \end{aligned}$$

and $[a, b] \cdot (c, d) = [a, b]$. Note that if $(e, e) = (m, n)(p, q)$ is an idempotent in B (so that $\exists [a, b] \cdot (e, e)$, for any $[a, b] \in B/\sigma$), we cannot conclude that (m, n) and

(p, q) are idempotent; for example: $(3, 3) = (2, 1)(2, 3)$. So this is not a partial action in the alternative sense. However, if we know that (m, n) is idempotent, i.e., that $m = n$, then we can easily conclude that (p, q) must be also.

Recall now that the (global) action of a monoid M on a set X, as given in Definition 2.1, is equivalent to a monoid morphism $\varphi : M \to \mathcal{T}_X$ with $s\varphi : x \mapsto x \cdot s$. Similarly, the alternative notion of a partial monoid action (with an "if, and only if" in (PA2′)) is equivalent to a monoid morphism $\psi : M \to \mathcal{PT}_X$. We now seek a function to which a partial monoid action (in our sense) is equivalent. To this end, we generalise a concept first introduced by McAlister and Reilly [11] and used also by Kellendonk and Lawson [10] in their treatment of partial group actions: that of a *premorphism*. According to the original definition [11, Definition 4.1][2], this is a mapping $\theta : S \to T$, where S and T are inverse semigroups, such that $(s\theta)^{-1} = s^{-1}\theta$ and $(s\theta)(t\theta) \leq (st)\theta$, where $\leq$ is the natural partial order in T. The mapping to which a partial monoid action is equivalent will be a mapping into a partial transformation monoid, which, as we have observed, is weakly left E_X-ample. We therefore adapt the concept of a premorphism to the following:

Definition 2.7. Let S and T be monoids, where T is weakly left E-ample, for some $E \subseteq E(T)$. Then $\theta : S \to T$ is a *premorphism* if

(PM1) $1\theta = 1$;

(PM2′) $(s\theta)(t\theta) \leq (st)\theta$,

where $\leq$ is the natural partial order in T, as defined by (1.2).

The following result is easy to see:

Proposition 2.8. *The (weak) partial action of a monoid M on a set X is equivalent to a premorphism $\theta : M \to \mathcal{PT}_X$.*

Just as we introduced a stronger concept of partial monoid action, we now introduce a stronger concept of premorphism. Observe that if we multiply (PM2′) on the left by $(s\theta)^+$, then we obtain $(s\theta)(t\theta) \leq (s\theta)^+(st)\theta$, since $(s\theta)^+ = I_{\mathrm{dom}\, s\theta}$ is a left identity for $s\theta$. We augment our definition of premorphism as follows:

Definition 2.9. A premorphism $\theta : S \to T$ will be called *strong* if the following condition holds:

(PM2) $(s\theta)(t\theta) = (s\theta)^+(st)\theta$.

Note that (PM2′) follows from (PM2), so we can drop this condition and define a strong premorphism via the conditions (PM1) and (PM2).

Whenever we wish to emphasise the distinction between the functions of Definitions 2.7 and 2.9, we will refer to the former as a *weak premorphism*.

Proposition 2.10. *A strong partial action is equivalent to a strong premorphism.*

[2]in which it was called a $\wedge$-*prehomomorphism*

Proof. Let $\theta : M \to \mathcal{PT}_X$ be a strong premorphism, for some monoid M and some set X. Then θ is equivalent to a (weak) partial action, by Proposition 2.8. Suppose now that $\exists x \cdot s$ and $\exists x \cdot st$. Then $x \in \operatorname{dom} s\theta = \operatorname{dom}(s\theta)^+$, so $x = x(s\theta)^+ \in \operatorname{dom}(st)\theta$, since $\exists x \cdot st$. Thus $x \in \operatorname{dom}(s\theta)^+(st)\theta = \operatorname{dom}(s\theta)(t\theta)$, so $\exists (x \cdot s) \cdot t$.

Now suppose that a monoid M acts strongly and partially on a set X. By Proposition 2.8, this is equivalent to a (weak) premorphism $\theta : M \to \mathcal{PT}_X$. Suppose that $x \in \operatorname{dom}(s\theta)^+(st)\theta$. Then $x \in \operatorname{dom}(s\theta)^+ = \operatorname{dom} s\theta$ and $x = x(s\theta)^+ \in \operatorname{dom}(st)\theta$. We have $\exists x \cdot s$ and $\exists x \cdot st$ so we can use (PA2) to deduce that $\exists (x \cdot s) \cdot t$. Thus $x \in \operatorname{dom}(s\theta)(t\theta)$ and $(s\theta)^+(st)\theta \leq (s\theta)(t\theta)$, as required. □

Our replacement of (PM2′) by (PM2) has a precedent in [8, p. 398], in which (PM2) is a necessary condition for a related function to be an FA-morphism. Note also that (PM2) holds for Kellendonk and Lawson's 'unital (group) premorphism': we simply take the left-right dual of condition (ii) of their Proposition 2.1 and multiply on the left by $s\theta$ to obtain (PM2), remembering, of course, that $aa^{-1} = a^+$ in an inverse semigroup. This, together with Proposition 2.10, tells us that every partial group action is strong in our sense.

We return, at last, to the question of whether every partial monoid action is strong and rephrase it in terms of premorphisms: is every premorphism is strong? The following counterexample demonstrates that the answer is 'no':

Example 2.11. Let $M = \langle a \rangle^1$, and let $X = \mathbb{N}$. We denote by $\mathcal{I}_\mathbb{N}$ the symmetric inverse monoid on $\mathbb{N}$. Define $\theta : M \to \mathcal{I}_\mathbb{N}$ by $1\theta = 1$ and $a^i\theta = \alpha_i$, for $i \in \mathbb{N}$, where $\operatorname{dom} \alpha_i = \{1, 2, \ldots, i\}$ and $n\alpha_i = n + i$, for all $n \in \operatorname{dom} \alpha_i$. Then α_{i+j} is the map with domain $\{1, 2, \ldots, i + j\}$, given by $n\alpha_{i+j} = n + i + j$. Also, $n\alpha_i\alpha_j = n + i + j$, so $n\alpha_i\alpha_j = n\alpha_{i+j}$, for those $n \in \mathbb{N}$ for which both maps are defined. We determine the domain of $\alpha_i\alpha_j$:

$$\begin{aligned}
\operatorname{dom} \alpha_i\alpha_j &= [\{1 + i, \ldots, 2i\} \cap \{1, \ldots, j\}]\, \alpha_i^{-1} \\
&= \begin{cases} \{1, \ldots, i\} & \text{if } 2i \leq j \\ \{1, \ldots, j - i\} & \text{if } 1 + i \leq j < 2i \\ \emptyset & \text{if } j < 1 + i \end{cases}
\end{aligned}$$

In each case, $\operatorname{dom} \alpha_i\alpha_j \subsetneq \operatorname{dom} \alpha_{i+j}$, so $(a^i\theta)(a^j\theta) \leq a^{i+j}\theta$. Note also that $I_\mathbb{N}\alpha_i = \alpha_i = \alpha_i I_\mathbb{N}$, so $(1\theta)(a^i\theta) \leq (1a^i)\theta$ and $(a^i\theta)(1\theta) \leq (a^i1)\theta$. Therefore θ is a premorphism; in fact, it is the premorphism equivalent to the partial action of Example 2.3.

We now demonstrate that θ is not strong. In $\mathcal{I}_\mathbb{N}$, β^+ is simply the identity on the domain of β. From this, together with our earlier observation that $n\alpha_i\alpha_j = n\alpha_{i+j}$, it is easy to see that $n\alpha_i^+\alpha_{i+j} = n\alpha_i\alpha_j$, for those $n \in \mathbb{N}$ for which both maps are defined. We consider the domain of $\alpha_i^+\alpha_{i+j}$:

$$\begin{aligned}\operatorname{dom}\alpha_i^+\alpha_{i+j} &= [\{1,\ldots,i\}\cap\{1,\ldots,i+j\}]\,(\alpha_i^+)^{-1}\\ &= \{1,\ldots,i\}\\ &\neq \operatorname{dom}\alpha_i\alpha_j,\ \text{ if } j<2i\end{aligned}$$

So in general, we do not have $(a^i\theta)(a^j\theta) = (a^i\theta)^+(a^{i+j}\theta)$. Then, since θ is an example of a premorphism which is not strong, Example 2.3 is an example of a partial monoid action which is not strong.

3. The Szendrei expansion of a monoid

Given the partial action of a monoid M on a set X, our aim is to construct the global action of some new monoid M' on X. We do this in much the same way as in [10]: by taking an *expansion* of M.

The concept of an expansion was first introduced by Birget and Rhodes [1]. Formally, an expansion is defined as follows:

Definition 3.1. [1, p. 241] An *expansion* is a functor F from one category of semigroups to a larger one such that there exists a natural transformation η from F to the identity functor, with each arrow[3] η_S surjective.

Since an expansion is a functor, we should not only specify its effect on the objects of a category but also its effect on the arrows. However, the 'objects' part is the only aspect of an expansion which we will need in connection with actions and partial actions. We will therefore omit all reference to an expansion's effect on arrows.

Among the specific expansions introduced by Birget and Rhodes, was the *prefix expansion* [1, p. 266]: $\mathrm{Pr}(M)$ for a monoid M. This expansion has a somewhat restrictive definition, but it was observed by Szendrei [14] that if M is a group, then $\mathrm{Pr}(M)$ has a particularly simple form. Szendrei's observations can be used to define a new expansion, for any monoid:

Definition 3.2. [6, p. 252] Let M be a monoid, and let $\mathcal{P}_1^f(M)$ denote the collection of all finite subsets of M which contain 1. The *Szendrei expansion* of M is the set

$$\mathrm{Sz}(M) = \{(A,a)\in\mathcal{P}_1^f(M)\times M : a\in A\},$$

together with the multiplication given by

$$(A,a)(B,b) = (A\cup aB, ab).$$

Note that in general, for an arbitrary monoid M, $\mathrm{Pr}(M)$ and $\mathrm{Sz}(M)$ will be different. In [6, p. 252], it is shown that if M is a monoid, then $\mathrm{Sz}(M)$ is also a monoid, with identity $(\{1\},1)$. We extend this result:

[3] Since we will be using the word 'morphism' to mean a homomorphism, we will use the word 'arrow' in the category context.

Proposition 3.3. *If M is a monoid, then* $\mathrm{Sz}(M)$ *is a weakly left* $\mathcal{E}$*-ample monoid, where* $\mathcal{E} = \{(A, 1) : A \in \mathcal{P}_1^f(M)\}$.

Proof. First of all, $\mathcal{E}$ certainly forms a semilattice:

$$(A, 1)(B, 1) = (A \cup B, 1) = (B \cup A, 1) = (B, 1)(A, 1).$$

We now show that every element (A, a) is $\widetilde{\mathcal{R}}_{\mathcal{E}}$-related to an idempotent, namely $(A, 1)$. This is certainly a left identity for (A, a):

$$(A, 1)(A, a) = (A \cup 1A, 1a) = (A, a).$$

Now let $(E, 1)$ be any idempotent in $\mathcal{E}$ and suppose that $(E, 1)(A, a) = (A, a)$:

$$\begin{aligned}(E, 1)(A, a) = (A, a) &\Rightarrow (E \cup A, a) = (A, a)\\ &\Rightarrow E \subseteq A\\ &\Rightarrow (E \cup A, 1) = (A, 1)\\ &\Rightarrow (E, 1)(A, 1) = (A, 1).\end{aligned}$$

Thus $(A, a)^+ = (A, 1)$.

The left ample identity holds:

$$\begin{aligned}\left[(A, a)(E, 1)\right]^+ (A, a) &= (A \cup aE, 1)(A, a)\\ &= (A \cup aE, a)\\ &= (A, a)(E, 1).\end{aligned}$$

It only remains to show that $\widetilde{\mathcal{R}}_{\mathcal{E}}$ is a left congruence. Suppose that $(A, a)\, \widetilde{\mathcal{R}}_{\mathcal{E}}\, (B, b)$ and that $(C, c) \in \mathrm{Sz}(M)$:

$$\begin{aligned}(A, a)\, \widetilde{\mathcal{R}}_{\mathcal{E}}\, (B, b) &\Rightarrow (A, a)^+ = (B, b)^+\\ &\Rightarrow A = B\\ &\Rightarrow (C \cup cA, 1) = (C \cup cB, 1)\\ &\Rightarrow [(C, c)(A, a)]^+ = [(C, c)(B, b)]^+\\ &\Rightarrow (C, c)(A, a)\, \widetilde{\mathcal{R}}_{\mathcal{E}}\, (C, c)(B, b),\end{aligned}$$

hence $\widetilde{\mathcal{R}}_{\mathcal{E}}$ is a left congruence. □

Corollary 3.4. [7, Corollary 5.8.] *If M is a unipotent monoid, then* $\mathrm{Sz}(M)$ *is weakly left ample.*

Proof. A general idempotent in $\mathrm{Sz}(M)$ has the form (E, e), where e is idempotent in M and $eE \subseteq E$. If M is unipotent, then the only idempotents of $\mathrm{Sz}(M)$ are those of the form $(E, 1)$, i.e., $E(\mathrm{Sz}(M)) = \mathcal{E}$. Therefore $\mathrm{Sz}(M)$ is weakly left ample. □

We define the mapping $\iota : M \to \mathrm{Sz}(M)$ by $s\iota = (\{1, s\}, s)$. This is an injection but fails to be an embedding as it is not a morphism.

Proposition 3.5. *The Szendrei expansion* $\mathrm{Sz}(M)$ *of a monoid* M *is generated by elements of the form* $s\iota$*, using both multiplication and* $^+$*.*

Proof. Take a typical element $(\{1, s_1, \ldots, s_n\}, s_n) \in \mathrm{Sz}(M)$ and notice that

$$\begin{aligned}(\{1, s_1, \ldots, s_n\}, s_n) &= (\{1, s_1\}, 1)(\{1, s_2\}, 1) \cdots (\{1, s_n\}, 1)(\{1, s_n\}, s_n) \\ &= (\{1, s_1\}, s_1)^+(\{1, s_2\}, s_2)^+ \cdots (\{1, s_n\}, s_n)^+(\{1, s_n\}, s_n) \\ &= (s_1\iota)^+(s_2\iota)^+ \cdots (s_n\iota)^+(s_n\iota)\end{aligned}$$

Thus $\mathrm{Sz}(M)$ is generated by elements of the form $s\iota$. (Note that, strictly speaking, we don't need to include the factor of $(s_n\iota)^+$, as this is a left identity for $s_n\iota$.) $\square$

4. 'Expansion' of partial monoid actions

At the beginning of the previous section, we stated our desire to find a monoid M' such that if a monoid M acts partially on a set X, then M' is a larger monoid which acts globally on that same set. It will perhaps come as no surprise that we can take M' to be $\mathrm{Sz}(M)$. The connection between the partial action of M and the global action of $\mathrm{Sz}(M)$ is given by the following theorem, a generalisation of [10, Theorem 2.4].

Theorem 4.1. *Let* M *and* S *be monoids, with* S *weakly left* E*-ample, for some* $E \subseteq E(S)$*. Then for every strong premorphism* $\theta : M \to S$ *there is a unique (2,1,0)-morphism* $\overline{\theta} : \mathrm{Sz}(M) \to S$ *such that* $\iota\overline{\theta} = \theta$*, i.e., such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "\theta"] \arrow[d, "\iota"'] & S \\ \mathrm{Sz}(M) \arrow[ur, "\overline{\theta}"'] & \end{tikzcd}$$

Conversely, if $\overline{\theta} : \mathrm{Sz}(M) \to S$ *is a (2,1,0)-morphism, for some monoid* M*, then* $\theta = \iota\overline{\theta}$ *is a strong premorphism.*

Proof. Suppose that we have a strong premorphism $\theta : M \to S$. Following the pattern of Kellendonk and Lawson [10], we define $\overline{\theta}$ by

$$(A, s)\overline{\theta} = (a_1\theta)^+ \cdots (a_k\theta)^+(s\theta),$$

where $A = \{a_1, \ldots, a_k\}$. Then $(\{1\}, 1)\overline{\theta} = (1\theta)^+(1\theta) = 1\theta = 1$. If we put $A = \{1, s_1, \ldots, s_n\}$, then it is easy to see that we can omit the factor of $(1\theta)^+$ and write

$$(A, s)\overline{\theta} = (s_1\theta)^+ \cdots (s_n\theta)^+(s\theta).$$

We suppose, without loss of generality, that $s = s_n$. Note that the order in which we write elements of A is immaterial, since all the factors $(s_i\theta)^+$ are idempotent, and therefore commute. Similarly, it does not matter if there are any repetitions in A.

Recall that $(A,s)^+ = (A,1)$. We have $(A,s)^+\overline{\theta} = (A,1)\overline{\theta} = (s_1\theta)^+\cdots(s_n\theta)^+$, and

$$\begin{aligned}\left[(A,s)\overline{\theta}\right]^+ &= \left[(s_1\theta)^+\cdots(s_n\theta)^+(s\theta)\right]^+,\\ &= \left[(s_1\theta)^+\cdots(s_n\theta)^+(s\theta)^+\right]^+, \text{ by Lemma 1.3,}\\ &= \left[(s_1\theta)^+\cdots(s_n\theta)^+\right]^+, \text{ since } s=s_n,\\ &= (s_1\theta)^+\cdots(s_n\theta)^+, \text{ since } E \text{ is a semilattice,}\\ &= (A,s)^+\overline{\theta}.\end{aligned}$$

So $\overline{\theta}$ preserves $^+$.

It now remains to show that $\overline{\theta}$ preserves the semigroup multiplication. Let

$$\begin{aligned}(A,s) &= (\{1,s_1,\ldots,s_n\},s_n),\\ (B,t) &= (\{1,t_1,\ldots,t_m\},t_m),\end{aligned}$$

so that $(A,s)(B,t) = (\{1,s_1,\ldots,s_n,st_1,\ldots,st_m\},st)$ and

$$[(A,s)(B,t)]\,\overline{\theta} = (s_1\theta)^+\cdots(s_n\theta)^+\,[(st_1)\theta]^+\ldots[(st_m)\theta]^+\,[(st)\theta]\,. \tag{4.1}$$

We will now show that

$$(A,s)\overline{\theta}(B,t)\overline{\theta} = (s_1\theta)^+\cdots(s_n\theta)^+(s\theta)(t_1\theta)^+\cdots(t_m\theta)^+(t\theta)$$

is equal to (4.1).

We begin by considering the product $(s\theta)(t_1\theta)^+\cdots(t_m\theta)^+$. Since S is weakly left E-ample, we have the rule $ae = (ae)^+a$, for all $a\in S$ and all $e\in E$. By using this rule with $a = s\theta$ and $e = (t_1\theta)^+$, we see that this product can be rewritten as $[(s\theta)(t_1\theta)^+]^+\,(s\theta)(t_2\theta)^+\cdots(t_m\theta)^+$. We continue in this way with each factor $(s\theta)(t_i\theta)^+$ $(i=1,\ldots,m)$ to obtain

$$(s\theta)(t_1\theta)^+\cdots(t_m\theta)^+ = \left[(s\theta)(t_1\theta)^+\right]^+\cdots\left[(s\theta)(t_m\theta)^+\right]^+(s\theta).$$

Thus

$$\begin{aligned}(A,s)\overline{\theta}(B,t)\overline{\theta} &= (s_1\theta)^+\cdots(s_n\theta)^+\left[(s\theta)(t_1\theta)^+\right]^+\cdots\left[(s\theta)(t_m\theta)^+\right]^+(s\theta)(t\theta),\\ &= (s_1\theta)^+\cdots(s_n\theta)^+\left[(s\theta)(t_1\theta)^+\right]^+\cdots\left[(s\theta)(t_m\theta)^+\right]^+(s\theta)^+\,[(st)\theta]\,,\end{aligned}$$

by (PM2). We now repeatedly commute idempotents to bring the factor of $(s\theta)^+ = (s_n\theta)^+$ next to its earlier occurrence in the product and reduce it to a single power to obtain:

$$(A,s)\overline{\theta}(B,t)\overline{\theta} = (s_1\theta)^+\cdots(s_n\theta)^+\left[(s\theta)(t_1\theta)^+\right]^+\cdots\left[(s\theta)(t_m\theta)^+\right]^+[(st)\theta]\,.$$

We now consider the factors $[(s\theta)(t_i\theta)^+]^+$, for $i = 1, \ldots, m$:

$$\begin{aligned}\left[(s\theta)(t_i\theta)^+\right]^+ &= [(s\theta)(t_i\theta)]^+ \\ &= \left[(s\theta)^+(st_i)\theta\right]^+ \\ &= \left[(s\theta)^+\left[(st_i)\theta\right]^+\right]^+ \\ &= (s\theta)^+\left[(st_i)\theta\right]^+.\end{aligned}$$

Thus

$$(A,s)\overline{\theta}(B,t)\overline{\theta} = (s_1\theta)^+ \cdots (s_n\theta)^+(s\theta)^+\left[(st_1)\theta\right]^+ \cdots (s\theta)^+\left[(st_m)\theta\right]^+\left[(st)\theta\right].$$

Again, we can repeatedly commute idempotents to bring each occurrence of $(s\theta)^+$ next to the earliest occurrence and reduce it to a single power:

$$\begin{aligned}(A,s)\overline{\theta}(B,t)\overline{\theta} &= (s_1\theta)^+ \cdots (s_n\theta)^+\left[(st_1)\theta\right]^+ \cdots \left[(st_m)\theta\right]^+\left[(st)\theta\right] \\ &= \left[(A,s)(B,t)\right]\overline{\theta}.\end{aligned}$$

Therefore, $\overline{\theta}$ is a $(2,1,0)$-morphism.

We now consider

$$s\iota\overline{\theta} = (\{1,s\},s)\overline{\theta} = (s\theta)^+(s\theta) = s\theta.$$

So $\iota\overline{\theta} = \theta$.

Finally, we show that $\overline{\theta}$ is unique. Suppose that there is another (2,1,0)-morphism $\varphi : \mathrm{Sz}(M) \to S$ with $\iota\varphi = \theta$. Then $\overline{\theta}$ and φ have the same effect on each element of the form $s\iota$. By Proposition 3.5, these are generators for $\mathrm{Sz}(M)$. Hence $\overline{\theta} = \varphi$.

We turn now to the converse of the theorem. Suppose that we have a (2,1,0)-morphism $\overline{\theta} : \mathrm{Sz}(M) \to S$. We define $\theta : M \to S$ by $\theta = \iota\overline{\theta}$ and claim that θ is a strong premorphism.

We certainly have $1\theta = 1\iota\overline{\theta} = (\{1\},1)\overline{\theta} = 1$, since $\overline{\theta}$ is a (2,1,0)-morphism.

For (PM2), note that $(st)\theta = (\{1,st\},st)\overline{\theta}$ and

$$\begin{aligned}(s\theta)(t\theta) &= (\{1,s\},s)\overline{\theta}(\{1,t\},t)\overline{\theta} \\ &= \left[(\{1,s\},s)(\{1,t\},t)\right]\overline{\theta} \\ &= (\{1,s,st\},st)\overline{\theta}.\end{aligned}$$

Notice that $(\{1,s,st\},st) = (\{1,s\},1)(\{1,st\},st) = (\{1,s\},s)^+(\{1,st\},st)$. So

$$\begin{aligned}(s\theta)(t\theta) &= \left[(\{1,s\},s)^+(\{1,st\},st)\right]\overline{\theta} \\ &= (\{1,s\},s)^+\overline{\theta}(\{1,st\},st)\overline{\theta} \\ &= \left[(\{1,s\},s)\overline{\theta}\right]^+(\{1,st\},st)\overline{\theta} \\ &= (s\theta)^+\left[(st)\theta\right].\end{aligned}$$

Therefore, θ is a strong premorphism. ∎

Suppose now that we are given the strong partial action of a monoid M on a set X, as represented by a strong premorphism $\theta : M \to \mathcal{PT}_X$. By putting $S = \mathcal{PT}_X$ in the above theorem, we can construct a (2,1,0)-morphism $\overline{\theta} : \mathrm{Sz}(M) \to \mathcal{PT}_X$, i.e., a global action of $\mathrm{Sz}(M)$ on X. Conversely, if we are given just such a global action, we can construct the underlying strong partial action. Note that a partial action is strong if, and only if, it can be 'expanded' in this way.

We will hijack the terminology already applied to monoids and refer to this procedure for replacing a partial action by a global action as the *expansion* of the partial action.

If we take Theorem 4.1 and make M a group and S an inverse monoid, then $\mathrm{Sz}(M)$ is an inverse monoid ([2, (II.6) Proposition] and [14, Proposition 1]), and we obtain Kellendonk and Lawson's Theorem 2.4.

5. Globalisation of partial monoid actions

Given the partial action of a monoid M on a set X, we have seen that we can construct a global action of $\mathrm{Sz}(M)$ on X, via a generalisation of Kellendonk and Lawson's Theorem 2.4 [10]. We now take a different approach, in which we leave the monoid M fixed and modify the set which is being acted upon. This was one of the approaches taken for partial group actions by Kellendonk and Lawson and was subsequently generalised to partial monoid actions by Megrelishvili and Schröder [12]. An important difference between the methods of [10] and [12] is the fact that Kellendonk and Lawson define their 'globalisation' in terms of generators, whilst Megrelishvili and Schröder do not. We will compromise between these two methods by describing the 'globalisation' of a partial monoid action in terms of generators.

Before we proceed to obtain the results of this section, we will first introduce (the left-right dual of) some of the machinery employed by previous authors. Let G be a group with symmetric generating set Z, and let G act partially on a set X, in the sense of [10]. Kellendonk and Lawson [10, p. 97] defined the relation[4] $\approx_Z$ on $X \times G$ by the rule that $(x, g) \approx_Z (x', g')$ if, and only if, there exist $g_1, \ldots, g_n \in Z$ and $x = x_1, x_2, \ldots, x_{n+1} = x' \in X$ such that $g = g_1 \cdots g_n g'$ and

$$x_1 \cdot g_1 = x_2,\ x_2 \cdot g_2 = x_3,\ \ldots,\ x_n \cdot g_n = x_{n+1}.$$

In their Lemma 3.1, Kellendonk and Lawson showed that $\approx_Z$ is an equivalence relation on $X \times G$.

We now summarise some definitions made by Megrelishvili and Schröder. Let M be a monoid which acts strongly and partially on a set X. The relation $\sim$ is defined on $X \times M$ by

$$(x, st) \sim (x \cdot s, t),$$

[4]Kellendonk and Lawson denoted this relation by $\sim$, but we wish to reserve the symbol $\sim$ for a different use.

whenever $\exists x \cdot s$ [12, p. 125]. Unlike Kellendonk and Lawson's $\approx_Z$, this relation is not an equivalence relation, merely a preorder. Megrelishvili and Schröder denote by $\simeq$ the equivalence relation generated by $\sim$ and observe that $\simeq \subseteq \rho$, where ρ is defined on $X \times M$ by

$$(x, s)\, \rho\, (x', s') \iff x \cdot s \equiv x' \cdot s';$$

the symbol $\equiv$ denotes "strong equality", i.e., $\exists x \cdot s$ if, and only if, $\exists x' \cdot s'$, in which case they are equal.

With these relations established, we now turn to the question of 'globalisation'. Following Kellendonk and Lawson, we make the following definition:

Definition 5.1. Let M be a monoid with generating set N and let $\theta : M \to \mathcal{PT}_X$ be a strong premorphism. A *globalisation of θ with respect to N* is a pair (ι, φ) consisting of an injection $\iota : X \to Y$, for some $Y \supseteq X$, and a monoid morphism $\varphi : M \to \mathcal{T}_Y$ such that $s\theta = \iota(s\varphi)\iota^{-1}$, for each $s \in N$. The strong partial action of M on X, as defined by θ, is said to be a *restriction* of the global action of M on Y, as defined by φ.

As already commented, our construction of such a globalisation will be analogous to those of [10] and [12]. We define the relation $\sim_N$ on $X \times M$ by:

$$(x, nm) \sim_N (x \cdot n, m) \text{ whenever } x \cdot n \text{ is defined, for } n \in N.$$

This relation is not an equivalence relation, nor is it the direct monoid analogue of Kellendonk and Lawson's $\approx_Z$; we have only a single generator where they have a sequence of generators. Let $\simeq_N$ be the equivalence relation generated by $\sim_N$, i.e.,

$$\simeq_N = \left[1_{X\times M} \cup \sim_N \cup \sim_N^{-1}\right]^\infty,$$

where $1_{X\times M}$ is the identity relation on $X \times M$ and $^\infty$ denotes transitive closure. We show how $\simeq_N$ can be related to $\approx_Z$ under certain conditions.

Lemma 5.2. *Suppose that all elements of N are invertible and that Kellendonk and Lawson's 'inverse axiom' holds:*

(I) $\exists z \cdot m \Rightarrow \exists (z \cdot m) \cdot m^{-1}$ *and* $(z \cdot m) \cdot m^{-1} = z$.

Then $\simeq_N = \approx_Z$, for $Z = N \cup N^{-1}$, where $N^{-1} = \{n^{-1} : n \in N\}$.

Proof. We show that $\sim_N^{-1} = \sim_{N^{-1}}$. Suppose first of all that $(x, s) \sim_N (y, t)$, for some $(x, s), (y, t) \in X \times M$. There therefore exists an $a \in N$ with $x \cdot a = y$ and $s = at$. Then, by (I), $\exists (x \cdot a) \cdot a^{-1}$, i.e., $\exists y \cdot a^{-1}$ and $y \cdot a^{-1} = x$, with $t = a^{-1}s$. Hence $(y, t) \sim_{N^{-1}} (x, s)$, so $\sim_N^{-1} \subseteq \sim_{N^{-1}}$. Similarly, the reverse inclusion. We can now write:

$$\begin{aligned}
\simeq_N = \left[1_{X\times M} \cup \sim_N \cup \sim_N^{-1}\right]^\infty &= \left[1_{X\times M} \cup \sim_N \cup \sim_{N^{-1}}\right]^\infty \\
&= \left[1_{X\times M} \cup \sim_{N\cup N^{-1}}\right]^\infty \\
&= 1_{X\times M} \cup \left[\sim_{N\cup N^{-1}}\right]^\infty,
\end{aligned}$$

which is precisely $\approx_Z$ with $Z = N \cup N^{-1}$, since $\approx_Z$ contains $1_{X\times M}$, as $\approx_Z$ is an equivalence relation. $\square$

Returning now to the general situation, it is easy to see that our $\sim_N$ is contained in Megrelishvili and Schröder's $\sim$, hence $\simeq_N \subseteq \simeq \subseteq \rho$. Let us denote the $\simeq_N$-class of $(x,s) \in X \times M$ by $[x,s]_N$, and denote the collection of all $\simeq_N$-classes by $X_{(M,N)}$.

Lemma 5.3. *The function $X_{(M,N)} \times M \to X_{(M,N)}$, given by*

$$([x,s]_N, t) \mapsto [x, st]_N := [x,s]_N * t,$$

defines a (global) action $*$ of M on $X_{(M,N)}$.

Before we prove Lemma 5.3, we first make the easy observation that

$$(x, nm) \sim_N (x \cdot n, m) \Longrightarrow (x, nmp) \sim_N (x \cdot n, mp), \text{ for any } p \in M.$$

Proof of Lemma 5.3. We must first prove that this action is well-defined. Suppose that $[x,m]_N = [x',m']_N$, or $(x,m) \simeq_N (x',m')$. We therefore have a sequence of transitions

$$(x,m) = (x_1,m_1) \to (x_2,m_2) \to \cdots \to (x_k,m_k) = (x',m'),$$

in which either $(x_i,m_i) \sim_N (x_{i+1},m_{i+1})$ or $(x_{i+1},m_{i+1}) \sim_N (x_i,m_i)$, for each $i \in \{1,2,\ldots,k-1\}$. Let $p \in M$. By the observation preceding the proof, we have the sequence

$$(x,mp) = (x_1,m_1p) \to (x_2,m_2p) \to \cdots \to (x_k,m_kp) = (x',m'p),$$

in which either $(x_i,m_ip) \sim_N (x_{i+1},m_{i+1}p)$ or $(x_{i+1},m_{i+1}p) \sim_N (x_i,m_ip)$, for each $i \in \{1,2,\ldots,k-1\}$. Hence $(x,mp) \simeq_N (x',m'p)$. The action is therefore well-defined.

To complete the proof, we need the following properties: $[x,m]_N * 1 = [x,m]_N$ and $([x,m]_N * s) * t = [x,m]_N * st$. These properties follow immediately from the definition of the action. $\square$

Our aim is to find a globalisation of the strong premorphism θ. To that end, we define a map $\iota_N : X \to X_{(M,N)}$ by $x\iota_N = [x,1]_N$, and prove the following:

Lemma 5.4. *The map ι_N is injective.*

Proof. For ease of notation, we will drop the subscript from ι_N and write simply ι. Suppose that $x\iota = y\iota$, i.e., $[x,1]_N = [y,1]_N$, or $(x,1) \simeq_N (y,1)$. Then, since $\simeq_N \subseteq \simeq \subseteq \rho$, we have

$$(x,1) \simeq_N (y,1) \Rightarrow (x,1)\,\rho\,(y,1) \Rightarrow x \cdot 1 \equiv y \cdot 1$$

But, of course, $\exists z \cdot 1$ and $z \cdot 1 = z$, for all $z \in X$, so $x = y$, as required. $\square$

Our next step in finding a globalisation of θ involves the morphism $\varphi_N : M \to \mathcal{T}_{X_{(M,N)}}$, where $t\varphi_N$ is the map given by $[x,s]_N \mapsto [x,s]_N * t$. As with $\iota = \iota_N$, we will drop the subscript from φ_N.

Lemma 5.5. *For each $s \in N$, $s\theta = \iota(s\varphi)\iota^{-1}$.*

Proof. We begin by showing the inclusion $s\theta \subseteq \iota(s\varphi)\iota^{-1}$ (i.e., $s\theta$ is a restriction of $\iota(s\varphi)\iota^{-1}$). We do this in the same way as in [10].

Suppose that $\exists x \cdot s$. Then $x \in \operatorname{dom} s\theta$ and $x(s\theta) = x \cdot s$. Further, $x(s\theta)\iota = (x \cdot s)\iota = [x \cdot s, 1]_N$. We also have $x\iota(s\varphi) = [x, 1]_N * s = [x, s]_N$.

Clearly, $(x \cdot s, 1) \sim_N (x, s)$, so $(x \cdot s, 1) \simeq_N (x, s)$, i.e., $[x \cdot s, 1]_N = [x, s]_N$. Therefore, $x(s\theta)\iota = x\iota(s\varphi)$, for all $x \in \operatorname{dom} s\theta$. Hence $s\theta \subseteq \iota(s\varphi)\iota^{-1}$.

To prove the reverse inclusion, first suppose that $x\iota(s\varphi) \in X\iota$. Then $x\iota(s\varphi) = [x, s]_N = [x', 1]_N$, for some $x' \in X$. There must therefore be a sequence of transitions $(x, s) = (x_1, s_1) \to \cdots \to (x_n, s_n) = (x', 1)$, in which either $(x_i, s_i) \sim_N (x_{i+1}, s_{i+1})$ or $(x_{i+1}, s_{i+1}) \sim_N (x_i, s_i)$, for each $i \in \{1, \ldots, n-1\}$. Then, once again using the fact that $\sim_N \subseteq \rho$, we have

$$x_1 \cdot s_1 \equiv x_2 \cdot s_2 \equiv \cdots \equiv x_n \cdot s_n.$$

We know that $\exists x' \cdot 1 = x_n \cdot s_n$, so $\exists x_1 \cdot s_1 = x \cdot s$, hence $x \in \operatorname{dom} s\theta$. We conclude from this that $\iota(s\varphi)\iota^{-1} = s\theta$. ☐

The preceding sequence of lemmas can be combined to give the following generalisation of both [10, Proposition 3.3] and [12, Proposition 2.6]:

Theorem 5.6. *Let M, X, θ, $X_{(M,N)}$, $\iota = \iota_N$ and $\varphi = \varphi_N$ be as previously defined. Then the pair (ι, φ) is a globalisation of θ with respect to N.*

We have the following further theorem:

Theorem 5.7. *A premorphism $\theta : M \to \mathcal{PT}_X$ can be globalised if, and only if, it is strong.*

Proof. First suppose that θ is a strong premorphism. Then θ can be globalised, by Theorem 5.6.

Conversely, suppose that θ can be globalised. By the observation preceding Definition 2.9, we have $(s\theta)(t\theta) \leq (s\theta)^+(st)\theta$. To show that θ is strong, it is sufficient to prove that $\operatorname{dom}(s\theta)^+(st)\theta \subseteq \operatorname{dom}(s\theta)(t\theta)$.

Let $x \in \operatorname{dom}(s\theta)^+(st)\theta$. Then $x \in \operatorname{dom}(s\theta)^+ = \operatorname{dom} s\theta$ and $x \in \operatorname{dom}(st)\theta$. Now consider $x(s\theta)$. Since θ can be globalised, we have $x(s\theta) = x\iota(s\varphi)\iota^{-1}$. Note also that $x(s\theta) \in X$, since $x \in \operatorname{dom} s\theta$. We must therefore be able to compose $\iota(s\varphi)$ with ι^{-1} at x, i.e., $x\iota(s\varphi) \in X\iota$. Similarly

$$x \in \operatorname{dom}(st)\theta = \operatorname{dom} \iota(st)\varphi\iota^{-1} = \operatorname{dom} \iota(s\varphi)(t\varphi)\iota^{-1}$$

and

$$\begin{aligned} x(st)\theta = x\iota(s\varphi)(t\varphi)\iota^{-1} &= x\iota(s\varphi)I_{X\iota}(t\varphi)\iota^{-1}, \text{ since } x\iota(s\varphi) \in X\iota \\ &= x\iota(s\varphi)\iota^{-1}\iota(t\varphi)\iota^{-1} \\ &= x(s\theta)(t\theta) \end{aligned}$$

Hence $x(s\theta) \in \operatorname{dom} t\theta$, and $x \in \operatorname{dom}(s\theta)(t\theta)$, giving us the desired inclusion. Therefore θ is strong. ☐

The following example of a strong partial action from Section 2 was constructed by taking a global action and restricting it, in accordance with Theorem 5.6.

Example 5.8. Let $G = \{e, a, a^2, a^3, a^4, a^5\}$ be the cyclic group of order 6. We define a mapping $\theta : \mathbb{N}^0 \to G$ by $1\theta = a$, so that $n\theta = a^n$. We have a (global) action of $\mathbb{N}^0$ on G, given by: $a^i * n = a^{i+n}$. We now take the subgroup $H = \{e, a^3\}$, with the obvious inclusion mapping $\iota : H \to G$. We define a partial action of $\mathbb{N}^0$ on H by

$$\exists h \cdot n \Longleftrightarrow h, h * n \in H,$$

in which case, $h \cdot n = h * n = ha^n$. We know that $a^k \in H$ if, and only if, $k \equiv 0 \,(\mathrm{mod}\, 3)$. For $a^k * n$ to belong to H, we need $k + n \equiv 0 \,(\mathrm{mod}\, 3)$, hence $n \equiv 0 \,(\mathrm{mod}\, 3)$. This is, of course, the partial action of Example 2.5.

Since $\simeq_N \subseteq \simeq$, for any generating set N, $\simeq$ is the larger relation. Therefore, $X_M := (X \times M)/\simeq$ is a smaller set than $X_{(M,N)} = (X \times M)/\simeq_N$. In fact, the globalisation associated with X_M is the *smallest* globalisation of θ.

We can make this last comment more precise by proving a number of category-theoretic results, including an analogue of yet another result of [10]. Let $\theta : M \to \mathcal{PT}_X$ be a strong premorphism, where M is a monoid with generating set N. Following Kellendonk and Lawson, we define a category $\mathbf{G}_N$, whose objects are globalisations (κ, ψ) of θ with respect to N. Let $(\kappa, \psi), (\kappa', \psi') \in \mathrm{ob}\, \mathbf{G}_N$, where $\kappa : X \to Y$ and $\kappa' : X \to Y'$. Then an arrow in $\mathbf{G}_N$ is a function $\alpha : Y \to Y'$ such that

$$\kappa\alpha = \kappa' \quad \text{and} \quad y(s\psi)\alpha = y\alpha(s\psi'),$$

for $y \in Y$.

Before we prove an analogue of [10, Theorem 3.4], we first note the following useful result:

Lemma 5.9. *Suppose that (κ, ψ) is a globalisation of θ with respect to N. Let $(x, s), (y, t) \in X \times M$ and let $(x, s) \to (y, t)$ be a transition in which either $(x, s) \sim_N (y, t)$ or $(y, t) \sim_N (x, s)$. Then $x\kappa(s\psi) = y\kappa(t\psi)$.*

Proof. If $(x, s) \sim_N (y, t)$, then there exists an $a \in N$ with $x \cdot a = y$ and $s = at$. This means that $x(a\theta) = y$. Then, since (κ, ψ) is a globalisation of θ with respect to N, $x\kappa(a\psi)\kappa^{-1} = y$, or $x\kappa(a\psi) = y\kappa$. Thus

$$x\kappa(s\psi) = x\kappa(at)\psi = x\kappa(a\psi)(t\psi) = y\kappa(t\psi).$$

Similarly, if $(y, t) \sim_N (x, s)$. □

Theorem 5.10. *The globalisation $(\iota, \varphi) = (\iota_N, \varphi_N)$ is an initial object of the category $\mathbf{G}_N$.*

Proof. Let (κ, ψ) be a globalisation of θ with respect to N, with $\kappa : X \to Y$. We define a function $\alpha : X_{(M,N)} \to Y$ by $[x, s]_N\alpha = x\kappa(s\psi)$. We will show that α

is the unique arrow $X_{(M,N)} \to Y$ and begin by showing that α is well-defined. Suppose that $[x,s]_N = [x',s']_N$. This means that there is a sequence of transitions

$$(x,s) = (x_1,s_1) \to (x_2,s_2) \to \cdots \to (x_n,s_n) = (x',s')$$

where $x_1,\ldots,x_n \in X$, $s_1,\ldots,s_n \in N$ and either $(x_i,s_i) \sim_N (x_{i+1},s_{i+1})$ or $(x_{i+1},s_{i+1}) \sim_N (x_i,s_i)$, for $i \in \{1,\ldots,n-1\}$. Then, by Lemma 5.9,

$$x_1\kappa(s_1\psi) = x_2\kappa(s_2\psi) = \cdots = x_n\kappa(s_n\psi),$$

i.e., $x\kappa(s\psi) = x'\kappa(s'\psi)$, so α is well-defined.

We now show that α is an arrow in $\mathbf{G}_N$. From here on, the proof is much the same as that of [10]. First, recall that $x\iota = [x,1]_N$. Then $x\iota\alpha = [x,1]_N\alpha = x\kappa(1\psi) = x\kappa$, so $\iota\alpha = \kappa$. Now let $t \in M$. We have

$$[x,s]_N(t\varphi)\alpha = [x,st]_N\alpha = x\kappa(st)\psi = x\kappa(s\psi)(t\psi) = [x,s]_N\alpha(t\psi).$$

Therefore α is an arrow in $\mathbf{G}_N$.

We finally show that α is unique. Suppose that β is another arrow $(\iota,\varphi) \to (\kappa,\psi)$, so that $\iota\beta = \kappa$ and $[x,s]_N(t\varphi)\beta = [x,s]_N\beta(t\psi)$. Notice that for any $[x,s]_N \in X_{(M,N)}$, we have $[x,s]_N = x\iota(s\varphi)$. Then

$$\begin{aligned}[x,s]_N\beta = x\iota(s\varphi)\beta &= x\iota\beta(s\psi), \text{ since } \beta \text{ is an arrow}\\ &= x\kappa(s\psi), \text{ since } \iota\beta = \kappa\\ &= [x,s]_N\alpha, \text{ by definition of } \alpha.\end{aligned}$$

Thus α is unique, and hence (ι,φ) is an initial object of the category $\mathbf{G}_N$. □

Unlike Kellendonk and Lawson, we are unable to show that our α is injective, so (ι,φ) is not *embedded* in every other globalisation.

It is, of course, possible that M could have more than one set of generators. The following proposition shows how, in certain instances, we can relate the categories arising from two different generating sets.

Proposition 5.11. *Let N and N' be generating sets for the monoid M, with $N \subseteq N'$. Then the category $\mathbf{G}_{N'}$ is a subcategory of the category $\mathbf{G}_N$.*

Proof. Let (κ,ψ) be an object in $\mathbf{G}_{N'}$, i.e., (κ,ψ) is a globalisation of θ with respect to N', in which case, the condition $s\theta = \kappa(s\psi)\kappa^{-1}$ holds for all $s \in N'$. It then clearly holds for all $s \in N$, since $N \subseteq N'$. Thus (κ,ψ) is also a globalisation of θ with respect to N, i.e., (κ,ψ) is an object of $\mathbf{G}_N$. It is clear that an arrow in $\mathbf{G}_{N'}$ is also an arrow in $\mathbf{G}_N$. □

Therefore, if $\mathbf{G}_M$ is the category of globalisations of θ with respect to the whole monoid M, then $\mathbf{G}_M$ is a subcategory of $\mathbf{G}_N$, for any generating set $N \subseteq M$. In particular, the (initial) object (ι_M,φ_M) of $\mathbf{G}_M$ is also an object of each $\mathbf{G}_N$, though, in general, it will not be initial in $\mathbf{G}_N$.

As a preliminary to our final category-theoretic result, we make the following definition: let $\mathbf{G}$ be the union of all the categories $\mathbf{G}_N$, where N ranges over all generating sets for M. Then $\mathbf{G}$ is itself a category. Let $\mathbf{I}$ be the collection of all initial objects of the original categories $\mathbf{G}_N$, together with all arrows between them in $\mathbf{G}$. Then $\mathbf{I}$ is a subcategory of $\mathbf{G}$. Note that, in general, the objects in $\mathbf{I}$ will not be initial in $\mathbf{G}$. We have the following result:

Proposition 5.12. *The object* (ι_M, φ_M) *is a terminal object in* $\mathbf{I}$.

Proof. We have already observed that (ι_M, φ_M) is an object in each $\mathbf{G}_N$. If (λ_N, ϕ_N) is an initial object in $\mathbf{G}_N$, then there is a unique arrow from (λ_N, ϕ_N) to (ι_M, φ_M), for each N. In other words, for each object in $\mathbf{I}$, there is a unique arrow from that object to (ι_M, φ_M), i.e., (ι_M, φ_M) is terminal in $\mathbf{I}$. □

6. One-one partial actions

In this final section, we consider the question of when a partial monoid action is one-one. We first set down the following definitions:

Definition 6.1. Suppose that a monoid M acts (globally) on a set X, with corresponding morphism $\varphi : M \to \mathcal{T}_X$. We say that the action is *one-one* if the image of φ contains only one-one maps.

Definition 6.2. Suppose that a monoid M acts partially on a set X, with corresponding premorphism $\theta : M \to \mathcal{PT}_X$. We say that the partial action is *one-one* if the image of θ contains only one-one maps.

It is clear that the partial group actions of Kellendonk and Lawson [10] are always one-one in the sense of Definition 6.2, since such a partial action is equivalent to a premorphism into $\mathcal{I}_X$, the symmetric inverse monoid on X.

Suppose now that a monoid M acts partially on a set X. If we demand that this partial action be one-one, then it is easy to see that the expanded action is also one-one: in Theorem 4.1, we put $S = \mathcal{I}_X$, so S is inverse and the global action is one-one. Conversely, if the expanded action is one-one, so that $\overline{\theta}$ is a (2,1,0)-morphism into $\mathcal{I}_X$, then $\iota\overline{\theta} = \theta$ also maps into $\mathcal{I}_X$, hence the underlying partial action is also one-one. The expansion of a partial group action is always one-one; this can be seen by applying similar reasoning to [10, Theorem 2.4].

We now consider the effect of globalisation on injectivity. If a globalised monoid action $\varphi : M \to \mathcal{O}_X$ is one-one, where $\mathcal{O}_X$ denotes the monoid of one-one maps on X, then it is clear that any restriction $\theta : M \to \mathcal{PT}_X$ must also be one-one: each $s\theta$ is the composition of the one-one maps $\iota : X \to Y$, $s\varphi : Y \to Y$ and $\iota^{-1} : X\iota \to X$, so is itself one-one. Conversely, however, if we start with a one-one partial action $\theta : M \to \mathcal{I}_X$, it is not clear that globalisation will lead to a morphism $\varphi : M \to \mathcal{O}_X$. In the group case, the globalisation is necessarily one-one, by the very nature of group actions.

One way in which we can make a partial monoid action one-one is to consider the partial action of a *right cancellative* monoid. It is natural in this case to augment Definition 2.4 with the following additional axiom to reflect the monoid under consideration:

(C) if $\exists x \cdot s$, $\exists y \cdot s$ and $x \cdot s = y \cdot s$, then $x = y$.

The addition of this extra condition clearly forces each mapping $s\theta$ to be one-one. Thus, with this extra requirement, the partial action of a right cancellative monoid is one-one. It is then clear that, with suitable modification of Theorem 4.1, the expansion of the partial action of a right cancellative monoid is also one-one. This modification requires a special case of a weakly left E-ample monoid which we now describe.

As commented in Section 1, weakly left E-ample monoids generalise left E-ample monoids. In particular, weakly left ample monoids generalise the left ample (formerly, left type-A) monoids of Fountain [4, 5]. To see this, let us define the (equivalence) relation $\mathcal{R}^*$ on a monoid[5] S:

$$a\,\mathcal{R}^*\,b \Longleftrightarrow \forall x, y \in S[xa = ya \Leftrightarrow xb = yb]. \tag{6.1}$$

Equivalently, $a\,\mathcal{R}^*\,b$ in S if, and only if, $a\,\mathcal{R}\,b$ in some oversemigroup T [5, Lemma 1.1]. Hence $\mathcal{R} \subseteq \mathcal{R}^*$. Notice also that if $a\,\mathcal{R}^*\,b$ in S, then we can set $y = 1$ and $x = e$ in (6.1), for any $e \in E \subseteq E(S)$, to obtain

$$ea = a \Leftrightarrow eb = b.$$

Hence $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$, for any $E \subseteq E(S)$.

We now define a left ample monoid, via the following:

Definition 6.3. A monoid S with commutative submonoid $E \subseteq E(S)$ of idempotents is *left E-ample* if

(1) every element a is $\mathcal{R}^*$-related to a (unique) idempotent in E, denoted a^+;
(2) for all $a \in S$ and all $e \in E$, $ae = (ae)^+a$.

If $E = E(S)$, then S is *left ample*.

Note that $\mathcal{R}^*$ is always a left congruence, so we do not need to demand this explicitly. By our observation that $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$, it is easy to see that every left E-ample monoid is weakly left E-ample, indeed, in a left E-ample monoid, $\mathcal{R}^* = \widetilde{\mathcal{R}}_E$, so there is no ambiguity in our use of $^+$ to denote the idempotent in an $\mathcal{R}^*$-class. In an inverse monoid, we have $\mathcal{R} = \mathcal{R}^* = \widetilde{\mathcal{R}}$.

We note the following extension of Corollary 3.4:

Proposition 6.4. [6, Proposition 4.4 (dual)] *If M is a right cancellative monoid, then* $\mathrm{Sz}(M)$ *is left ample.*

By modifying Theorem 4.1 to make S left ample, we see that the expansion of the partial action of a right cancellative monoid M on a set X (i.e., the global action of the left ample monoid $\mathrm{Sz}(M)$ on X) is one-one.

As in the case of an arbitrary monoid, however, the globalisation of the partial action of a right cancellative monoid is not (necessarily) one-one.

[5] If S were a semigroup but not a monoid, we would need $x, y \in S^1$ here.

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