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Modelling Network Travel Time Reliability Under Stochastic Demand

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Abstract – A technique is proposed for estimating the probability distribution of total network travel time, in the light of normal day-to-day variations in the travel demand matrix over a road traffic network. A solution method is proposed, based on a single run of a standard traffic assignment model, which operates in two stages. In stage one, moments of the total travel time distribution are computed by an analytic method, based on the multivariate moments of the link flow vector. In stage two, a flexible family of density functions is fitted to these moments. It is discussed how the resulting distribution may in practice be used to characterise unreliability. Illustrative numerical tests are reported on a simple network, where the method is seen to provide a means for identifying sensitive or vulnerable links, and for examining the impact on network reliability of changes to link capacities. Computational considerations for large networks, and directions for further research, are discussed.

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1. INTRODUCTION

Transport planning has been historically concerned with travel behaviour and the transport system in some nominally 'typical' conditions. The emerging topic of transport network reliability has begun to challenge this ideology. While the initial impetus appears to have derived from the study of major natural events – such as earthquakes (Bell & Iida, 1997) – affecting the 'connectivity' of a road network, it has had a wider impact on the thinking of the way in which less severe, but more frequently-occurring, events may affect the operation of a network. These events include minor accidents, on-street parking violations, snow, flooding, road maintenance and traffic signal failures, all of which would lead to variations in link capacities or free-run speeds. In addition, daily variations in activity patterns, manifested in the Origin-Destination (O-D) trip matrix, mean that the flows on the roads also have a major part to play in explaining variations in network performance.

If planners were able to quantify the impact on variable network performance of such elements, then it would open the possibility of directing both the design (Asakura *et al*, 2001) and economic appraisal (Du & Nicholson, 1997) of transport policy measures toward an improved treatment of such uncertainty. A practical need therefore arises for the development of *modelling techniques* that are able to quantify such impacts. In response to this need, there has been considerable activity in developing a diverse range of techniques, with five broad classes that may be identified.

The first class comprises *connectivity reliability* methods (Bell & Iida, 1997; Asakura *et al*, 2001), whereby each link of a network is assumed to have an independent, probabilistic, binary mode of operation. This binary mode may be open/closed, or may more generally reflect some subjective definition of the successful function of a link, such as the flow to capacity ratio being less than some given value. The objective is to compute the probability that a particular path or O-D movement will be 'connected', or more generally will 'function' as desired.

The second class consists of *travel time reliability* methods (Asakura & Kashiwadani, 1991; Asakura, 1996; Du & Nicholson, 1997; Bell *et al*, 1999; Yang *et al*, 2000), whereby a continuous probabilistic treatment is made of link, and hence path, travel times. For example, Asakura & Kashiwadani propose a simulation-based method for examining the impact of variability in O-D demand levels, whereby an O-D demand matrix is sampled and an equilibrium assignment performed for each sampled demand. Bell *et al* (1999) used a similar philosophical approach, but used equilibrium sensitivity analysis to overcome some of the computational overheads. The philosophy underlying the methods of Du & Nicholson (1997) is again broadly similar, but with a specific focus on network degradations in a multi-modal context. Like Bell *et al*, Du & Nicholson employ differential sensitivity analysis to their (multi-modal) equilibrium model, in this instance to examine the sensitivity of equilibrium 'system surplus' (a measure of performance of a multi-modal system) to various unreliable events, such as capacity degradation.

The third class encompasses methods to study *capacity reliability* (Chen *et al*, 2000, 2002; see Yang *et al*, 2000, for a comparison with travel time reliability methods). For example, in Chen *et al* (2000) the problem is to determine the maximum global O-D matrix multiplier such that the resulting link flows when assigned are within their respective link capacities. They also discuss how in the lower level (route choice) problem, an allowance may also be made for the risk-taking approach of drivers in the assignment model. In Chen *et al* (2002), alternative notions of reliability are examined in the context of variations in link capacities, using sensitivity analysis to estimate the impact of a perturbation on equilibrium flows. They also extend this approach by mixing it with Monte Carlo simulation, in order to estimate sensitivities under more complex model assumptions such as correlated link capacities.

The fourth class consists of *behavioural reliability* methods, whereby an effect on mean network performance is presumed to arise from the modified, mean behaviour of drivers in their attitude to the unpredictable variation and/or the 'risks' perceived. The issue is then how to represent, in an equilibrium framework, the impact on the 'typical' route choice pattern (Mirchandani & Soroush, 1987; Lo & Tung, 2000; Yin & Ieda 2001; Gordon *et al*, 2001; Liu *et al*, 2002; Watling, 2002a), or on other responses such as departure time choice (Uchida & Iida, 1993; Noland *et al*, 1998; Noland & Polak, 2002).

The fifth and final class consists of methods to examine the *potential reliability* of a network; rather than aiming to model performance based on some defined probabilistic model, these are ‘pessimistic’ methods that aim more to identify potential weak points/problems and their effect. In this context, Berdica (2001, 2002) proposed various simple tests of network *vulnerability*, to examine the impact on various output measures (in equilibrium) to changes in the input variables to a network model. D’Este & Taylor (2001) likewise considered notions of vulnerability, with a network node considered vulnerable if the loss of a small number of links significantly diminishes the ‘accessibility’ of the node. Bell (2000) and Bell & Cassir (2002) avoided the difficult issue of defining performance probabilities by supposing that they arose from a ‘game’ between the drivers and an evil entity, suggesting they could be used as a cautious basis for network design when users are pessimistic about the performance.

The technique to be proposed in the present paper falls within the class of travel time reliability methods, specifically examining the impact of variable O-D demand flows on network performance. As we shall see, however, the approach differs in philosophical foundation to previous studies of reliability—specifically in its use of the equilibrium paradigm—as well as in its solution technique, relying neither on sensitivity analysis nor Monte Carlo simulation, and in aiming to reconstruct a full probability distribution for the network performance measure.

2. FRAMEWORK FOR NETWORK RELIABILITY ASSESSMENT

The proposed method is based on an original modelling approach for representing variable network performance under stochastic O-D demands (to be described in §3), placed within a framework for reliability assessment. The purpose of the present section is to describe this latter framework, which is supposed to have a number of elements:

1. *Planning state*. The planning state is a representative set of assumptions concerning the state of the road network and demand data that is chosen subjectively by the planner, for the purpose of devising transport policy and traffic control measures. For example, the planning state may involve assuming O-D flow levels for a ‘typical’

weekday peak-hour when there are no public holidays or special events, and assuming a network where all links have the potential to operate at their full capacity.

2. *Performance measure.* This is a scalar measure used to describe the operation of the complete network or of prescribed elements of the network. Without loss of generality, to simplify later discussions, we suppose the measure is defined so that larger values of the measure are generally undesirable. For example, on a network-wide level, we might use proxies for congestion, such as total network travel time of all drivers or the negative of average network speed, or measures of total fuel consumption or pollution.

3. *Critical value.* Recall that we assume the performance measure to be defined such that large values are undesirable. The critical value is a pre-specified value of the performance level, above which the network would be considered to be performing "unreliably", relative to the planning state. A special case is where the critical value is exactly equal to the value of the performance measure in the planning state: any performance poorer than the planned situation is then considered subjectively unreliable. More generally we may define the critical value as some percentage excess of the value of the performance measure in the planning state.

4. *State distribution.* The state distribution is a joint density / probability distribution, describing the possible O-D demand and road network states that may actually prevail. In particular, this distribution can be used to infer the probability distribution for link flows and travel times across the network, and thereby the probability distribution of the performance measure.

Combining elements 1 and 2, we then suppose that we have a network model that is able to estimate the value of the performance measure in the planning state[2]. From this value, we define the critical value in 3 as an absolute or percentage excess of the value in the planning state. In parallel, combining elements 2 and 4 with the network model yields a probability distribution for the actual values of the performance measure. This distribution may then be compared with the critical value, and

[2]One key reason for defining a performance measure is typically to examine how it changes when applying certain hypothetical policy measures. That is to say, the performance measure to which we refer above is implicitly conditional on the values of the policy variables.

summary measures relating to the critical value produced, e.g. probability of exceeding critical value, mean performance value when critical value exceeded.

[FIGURE 1 HERE]

In Figure 1 we illustrate such a case, for an example where the performance measure is a network-wide, continuous attribute. The probability distribution of the actual values of the performance measure is illustrated. The planning state occurs when the performance measure equals the mode of around 1; the critical value is defined as a tolerance of 400% above the performance measure value in the planning state, yielding a critical value of 5. Then we could define *unreliability*, for example, in terms of the probability of exceeding the critical value $\Pr(M > 5)$, i.e. the area under the curve in the range labelled 'degraded performance'. So in percentage terms we might say the *reliability* is $\rho = 100\left(1 - \Pr(M > 5)\right)\%$.

Clearly, the critical value has an important role to play in this measure, yet it will typically be difficult to justify objectively testing against a *single* such value. More usefully, then, the reliability could be assessed by reporting such a probability ρ corresponding to a number of critical values, or by reporting standard upper quantiles of the distribution, or ultimately by reference to the complete upper tail of the performance measure distribution. Thus, the motivation in the present paper will be to reconstruct the full distribution, to provide the maximum information for such an assessment. This may be contrasted with methods in which the objective is to compute a single reliability value, in which case more efficient computational techniques may be available.

For any specific reliability analysis, a first step is therefore to define the performance measure to be used. Looking to the literature, Bell & Iida (1997) define it from the road user's perspective as 'the probability that a trip can reach its destination within a given period'. Such a definition may be applied at the path or O-D level. Asakura & Kashiwadani (1991) suggest an alternative definition to be 'the upper limit of travel time by which one can travel … for given probability'. The focus of Nicholson & Du

(1997), on the other hand, was the complete socio-economic impact of unreliability, gained by examining the effect on 'system surplus', an economic benefit measure appropriate for multi-modal networks. Chen *et al* (2002) distinguish between unreliability due to normal variations in daily demand, and that due to capacity variations arising from network degradation. Focusing on the latter case, they consider for each O-D movement the ratio of travel time in a degraded state to the travel time in a non-degraded state. Travel time reliability is then defined as the probability that this ratio will be less than some pre-defined acceptable level. For a whole network, they note the difficulty in rigorously extending this definition, while allowing for the inter-dependencies between O-D travel times. Thus, at the network level various pragmatic measures are defined, based on either the weighted average or worst reliability across all O-D movements.

The measure to be adopted in the present paper follows a similar philosophy to that of Nicholson & Du (1997), in the sense that we aim to examine reliability at the network level. In the case of a single mode, fixed demand traffic assignment model, Nicholson & Du's 'system surplus' simplifies to be total travel cost. In fact, in this paper we shall treat cost purely as time (it is straightforward to include other flow-independent attributes in the definition of cost, but as this is not a central issue the possibility is not explicitly considered here). Therefore, the measure considered is total travel time, a measure commonly used as an indicator of network performance/congestion.

3. ESTIMATING THE TOTAL TRAVEL TIME DENSITY FUNCTION

Following the framework of §2, the key to estimating reliability is the computation of a probability density function for the performance measure in question; here, we focus on total travel time as the performance measure. This will be approached in three steps, by: proposing a statistical model for the underlying variability (§3.1), whereby moments of the total travel time distribution may be computed (§3.2), which are in turn used to 'fit' an approximating distribution (§3.3). Two key elements of the proposed approach are that: i) maximum use is made of the information that exists in a conventional traffic assignment application, ii) extensive Monte Carlo simulation is avoided by use of an 'analytic' approach.

3.1 Notation and assumptions

Define:

v_a = flow on link a $(a = 1,2,...,A)$, $\mathbf{v}$ the vector of flows across all links

q_w = mean demand on O-D movement w $(w = 1,2,...,W)$

$\mathbf{q}$ = W-vector of mean demands

R_w = index set of acyclic paths serving O-D movement w

δ_{ar} = indicator variable, equal to 1 if path r contains link a, 0 otherwise

$t_a(v_a)$ = travel time on link a as a function of v_a $(a = 1,2,...,A)$

$\mathbf{t}(\mathbf{v})$ = vector of functions $t_a(v_a)$ $(a = 1,2,...,A)$.

The key statistical model assumptions are then:

1. The actual O-D demand on any day is independently distributed across inter-zonal movements, and for each movement w is distributed as a stationary Poisson random variable with constant mean $q_w > 0$.

2. Conditional on the O-D movement w demand realised on any one day, drivers are assumed to choose independently between the alternative routes $r \in R_w$ with constant probabilities p_r $(r \in R_w)$ for each $w = 1,2,...,W$.

Assumptions 1 and 2 together imply that for each $w = 1,2,...,W$, the *route* flows F_r $(r \in R_w)$ are random samples of a Poisson process with mean q_w and sampling rate p_r. It follows that the route flows F_r $(r \in R_w)$ are independent Poisson random variables with means $p_r q_w$ $(r \in R_w)$, for each $w = 1,2,....,W$ (a proof of this result can be found in many standard texts, for example: Karlin & Taylor, 1981; Stuart & Ord, 1987, p 207 (5.20)).

Before proceeding, it is worth clarifying two potential misunderstandings:

- These assumptions are *not* equivalent to $F_r = p_r Q_w$, where p_r is a constant and Q_w is the stochastic O-D demand for movement *w*. If such an assumption had been adopted, then we would have:

$$\frac{\mathrm{var}[F_r]}{\mathrm{E}[F_r]} = \frac{{p_r}^2\,\mathrm{var}[Q_w]}{p_r \mathrm{E}[Q_w]} = p_r \quad (\text{assuming } Q_w \text{ Poisson})$$

and since $p_r \leq 1$ we would generally have less-than-Poisson variation for the route flows (ratio of variance to mean less than one). On the contrary, above we assume there are two sources of random variation: the Poisson O-D demands *and* the route choice fractions: p_r are constant probabilities, *not* constant proportions.

- The *conditional* route flow random variables $F_r | Q_w \ (r \in R_w)$ are multinomially distributed and therefore definitely not independent, since they must satisfy conservation-of-flow conditions: given the realised value of Q_w, one of the route flows is entirely determined once values are selected for all other route flows in R_w. However our interest is in the *unconditional* route flow random variables F_r $(r \in R_w)$: still they must sum to Q_w, but this itself is a random variable. Hence, it does not violate conservation-of-flow to claim that the *unconditional* route flows are independent.

Now, since the link flow random variables are related to the route flow random variables via the identities:

$$V_a = \sum_{w=1}^{W} \sum_{r \in R_w} \delta_{ar} F_r \qquad (a = 1,2,\ldots, A) \qquad (1)$$

then assumptions 1 and 2 imply that the means of the link flows (1) are:

$$\mathrm{E}[V_a] = \sum_{w=1}^{W} \sum_{r \in R_w} \delta_{ar} p_r q_w \qquad (a = 1,2,\ldots, A) \qquad (2)$$

and the covariances:

$$\mathrm{cov}[V_a, V_b] = \sum_{w=1}^{W} \sum_{r \in R_w} \delta_{ar} \delta_{br} p_r q_w \qquad (a = 1,2,\ldots, A;\ b = 1,2,\ldots, A) \qquad .$$

(3)

We then make the additional assumption:

3. The variation in link flows across the network may be approximated by a multivariate Normal distribution (with means and covariances as given above).

The assumption of approximate multivariate Normal link flows is partially supported by the assumption of Poisson demands for movements with 'large' mean q_w, since the path flows F_r $(r \in R_w)$ are (as noted above) also independent Poisson random variables with means $p_r q_w$ $(r \in R_w)$. Then, for the (dominant) paths with large mean $p_r q_w$ (say, greater than 10), independent Normal approximations are supported for their flows, which clearly mix into multivariate Normal link flows. See Hazelton (2001) for a more detailed discussion of the validity of this assumption.

The assumptions above require knowledge of the route choice probabilities p_r $(r \in R_w;\, w = 1,2,...,W)$. It is important to note that the specification of these probabilities is external to the present paper, in the sense that the methods to be subsequently described make no assumptions as to how these probabilities are derived. However, we propose that one sensible approach would be to estimate them by applying a standard network equilibrium model to the mean demands **q**. The output of the equilibrium model may be viewed as a set of equilibrium route choice fractions—route flows divided by corresponding mean O-D demand—and it is these fractions that may then be used to estimate the required route choice probabilities.

Any kind of network equilibrium model will serve the purpose above (including the various 'behavioural reliability' methods in §1), but in the later example we favour use of a stochastic user equilibrium (SUE) model (Sheffi, 1985). There are a number of reasons in support of this choice of model. Firstly, since we require outputs at the level of route flows, rather than link flows, it seems sensible to select an equilibrium model that is able to provide unique outputs at this level. It is well known that generally the deterministic user equilibrium model is non-unique at the route flow level, but that relatively mild conditions exist to ensure unique SUE route flows (see, for example, Cantarella & Cascetta, 1995). Secondly, there are theoretical results in support of SUE as a large-demand approximation to the *mean* of more general stochastic models that explicitly represent drivers' information acquisition in a stochastic environment (Davis & Nihan, 1993; Cantarella & Cascetta, 1995; Hazelton, 1998; Watling, 2002b; Hazelton & Watling, 2003), also supporting the interpretation of SUE route flow fractions as choice probabilities. Finally, it is noted that this approximation may be improved by a further simple refinement (see Appendix A).

In addition to the statistical and model assumptions above, we shall focus specifically on link travel time functions of a polynomial form:

$$t_a(v_a) = \sum_{j=0}^{m} b_{ja} v_a{}^{j} \quad . \tag{4}$$

The power-law form of the commonly used Bureau of Public Roads functions are a special case of (4); for other functional forms, a polynomial Taylor series approximation may be used to obtain a form (4).

3.2 Computing moments for total travel time

Based on (4), we introduce the following random variables, a transformation of the link flow random variables:

$$W_a = V_a t_a(V_a) = \sum_{j=0}^{m} b_{ja} V_a{}^{j+1} \tag{5}$$

where V_a is a random variable representing the flow on link *a*, and W_a is the total travel time on link *a* (throughout the paper the convention is used that a random variable is denoted by a capital letter). Our interest will be in the total travel time random variable *T* given by

$$T = \sum_{a=1}^{A} V_a t_a(V_a) = \sum_{a=1}^{A} W_a \quad . \tag{6}$$

In particular, we shall aim to deduce moments of *T*, namely the mean $\mu_T = \mathrm{E}[T]$ and the expectations of the form $\mathrm{E}[(T-\mu_T)^n]$ $(n = 2,3,...)$, the *order n moments of T about the mean*. Now, by a Binomial expansion, it follows that

$$\mathrm{E}[(T-\mu_T)^n] = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (-\mu_T)^{n-k}\, \mathrm{E}[T^k] \qquad (n = 2,3,...) \tag{7}$$

and so the problem is equivalently to determine the moments of *T about the origin*, namely $\mathrm{E}[T^n]$ $(n = 1,2,...)$.

Now, for positive integers m and n, define the subset of m-dimensional integers:

$$I(m,n) = \left\{ (i_1, i_2, ..., i_m) : i_j \text{ a non negative integer and } \sum_{j=1}^{m} i_j = n \right\} . \tag{8}$$

Then by (6), and a second (multinomial) expansion:

$$\mathrm{E}[T^n] = \mathrm{E}\left[\left(\sum_{a=1}^{A} W_a \right)^n \right] = \sum_{(i_1, i_2, ..., i_A) \in I(A,n)} \frac{n!}{\prod_{a=1}^{A} i_a !} \, \mathrm{E}\left[\prod_{a=1}^{A} W_a^{\,i_a} \right] . \tag{9}$$

Let us now turn attention to W_a, and from its definition (5) write it in the form:

$$W_a = \sum_{j=0}^{m} b_{ja} \left((V_a - \mu_a) + \mu_a \right)^{j+1} . \tag{10}$$

Performing a further Binomial expansion yields:

$$\begin{aligned} W_a &= \sum_{j=0}^{m} b_{ja} \sum_{i=0}^{j+1} \frac{(j+1)!}{i!(j+1-i)!} (V_a - \mu_a)^i \mu_a^{\,j+1-i} \\ &= \sum_{j=0}^{m} b_{ja} \mu_a^{\,j+1} + \sum_{j=0}^{m} \sum_{i=1}^{j+1} b_{ja} \frac{(j+1)!}{i!(j+1-i)!} (V_a - \mu_a)^i \mu_a^{\,j+1-i} \end{aligned} \tag{11}$$

where on the second line the (constant) terms relating to each i = 0 have been separated. The order of summation in the second term of (11) may then be reversed:

$$W_a = \sum_{j=0}^{m} b_{ja} \mu_a^{\,j+1} + \sum_{i=1}^{m+1} \sum_{j=i-1}^{m} b_{ja} \frac{(j+1)!}{i!(j+1-i)!} (V_a - \mu_a)^i \mu_a^{\,j+1-i} \tag{12}$$

which may then be written in the form

$$W_a = \tilde{b}_{0a} + \sum_{i=1}^{m+1} \tilde{b}_{ia} (V_a - \mu_a)^i = \sum_{i=0}^{m+1} \tilde{b}_{ia} (V_a - \mu_a)^i \tag{13}$$

where the coefficients $\tilde{b}_{ia}$ $(i = 0,1,..., m+1)$ are given by:

$$\tilde{b}_{0a} = \sum_{j=0}^{m} b_{ja} \mu_a^{\,j+1} \; ; \; \tilde{b}_{ia} = \sum_{j=i-1}^{m} b_{ja} \frac{(j+1)!}{i!(j+1-i)!} \mu_a^{\,j+1-i} \;\; (i = 1,2,..., m+1) . \tag{14}$$

When (13) is substituted into (9), the latter becomes a sum of multivariate moments about the mean of the (assumed multivariate Normal) vector link flow random variable **V**. Therefore, combining (7), (9), (13) and (14), we have shown how moments of the total travel time random variable T may be written as a sum of multivariate moments of **V**. In order to compute the moments of **V**, results due to Isserlis (1918) are applied, which allow the computation of appropriate multivariate

Normal moments for any powers of any number of variables. See Appendix B for a description of the key elements of this work, and the computational methods adopted.

In this paper, we shall only aim to compute moments of T up to order $n=4$, and so for such cases we present the explicit formulae for the expressions deduced above. Now, from (7), after simplification:

$$\mathrm{E}[(T-\mu_T)^2]=\mathrm{E}[T^2]-\mu_T{}^2 \;; \quad \mathrm{E}[(T-\mu_T)^3]=\mathrm{E}[T^3]-3\mu_T\mathrm{E}[T^2]+2\mu_T{}^3$$

$$\mathrm{E}[(T-\mu_T)^4]=\mathrm{E}[T^4]-4\mu_T\mathrm{E}[T^3]+6\mu_T{}^2\mathrm{E}[T^2]-3\mu_T{}^4$$

and so we focus then on $\mu_T, \mathrm{E}[T^2], \mathrm{E}[T^3], \mathrm{E}[T^4]$. In particular, (9) then yields:

$$\mathrm{E}[T]=\sum_{a=1}^{A}\mathrm{E}[W_a]; \qquad \mathrm{E}[T^2]=\sum_{a=1}^{A}\mathrm{E}[W_a{}^2]+2\sum_{a=1}^{A}\sum_{b=a+1}^{A}\mathrm{E}[W_aW_b]$$

$$\mathrm{E}[T^3]=\sum_{a=1}^{A}\mathrm{E}[W_a{}^3]+3\sum_{a=1}^{A}\sum_{\substack{b=1\\(b\neq a)}}^{A}\mathrm{E}[W_a{}^2W_b]+6\sum_{a=1}^{A}\sum_{b=a+1}^{A}\sum_{c=b+1}^{A}\mathrm{E}[W_aW_bW_c]$$

$$\mathrm{E}[T^4]=\sum_{a=1}^{A}\mathrm{E}[W_a{}^4]+4\sum_{a=1}^{A}\sum_{\substack{b=1\\(b\neq a)}}^{A}\mathrm{E}[W_a{}^3W_b]+6\sum_{a=1}^{A}\sum_{b=a+1}^{A}\mathrm{E}[W_a{}^2W_b{}^2]$$

$$+12\sum_{a=1}^{A}\sum_{\substack{b=1\\b\neq a}}^{A}\sum_{\substack{c=b+1\\c\neq a}}^{A}\mathrm{E}[W_aW_bW_c]+24\sum_{a=1}^{A}\sum_{b=a+1}^{A}\sum_{c=b+1}^{A}\sum_{d=c+1}^{A}\mathrm{E}[W_aW_bW_cW_d]$$

and all the right-hand side expectations may be computed (using (13)) from:

$$\mathrm{E}[W_a]=\sum_{i=0}^{m+1}\tilde{b}_{ia}\mathrm{E}[(V_a-\mu_a)^i] \tag{15a}$$

$$\mathrm{E}[W_aW_b]=\sum_{i=0}^{m+1}\sum_{j=0}^{m+1}\tilde{b}_{ia}\tilde{b}_{jb}\mathrm{E}[(V_a-\mu_a)^i(V_b-\mu_b)^j] \tag{15b}$$

$$\mathrm{E}[W_aW_bW_c]=\sum_{i=0}^{m+1}\sum_{j=0}^{m+1}\sum_{k=0}^{m+1}\tilde{b}_{ia}\tilde{b}_{jb}\tilde{b}_{kc}\mathrm{E}[(V_a-\mu_a)^i(V_b-\mu_b)^j(V_c-\mu_c)^k] \tag{15c}$$

$$\mathrm{E}[W_aW_bW_cW_d]=\sum_{i=0}^{m+1}\sum_{j=0}^{m+1}\sum_{k=0}^{m+1}\sum_{l=0}^{m+1}\tilde{b}_{ia}\tilde{b}_{jb}\tilde{b}_{kc}\tilde{b}_{ld}\mathrm{E}[(V_a-\mu_a)^i(V_b-\mu_b)^j(V_c-\mu_c)^k(V_d-\mu_d)^l] \tag{15d}$$

The only remaining task in applying the expressions derived is then to compute the coefficients in (14); for example, in the case $m=2$ in (4) (travel time functions of quadratic form), they simplify to

$$\widetilde{b}_{0a} = b_{0a}\mu_a + b_{1a}\mu_a{}^2 + b_{2a}\mu_a{}^3$$

$$\widetilde{b}_{1a} = b_{0a} + 2b_{1a}\mu_a + 3b_{2a}\mu_a{}^2; \quad \widetilde{b}_{2a} = b_{1a} + 3b_{2a}\mu_a; \quad \widetilde{b}_{3a} = b_{2a} \ .$$

3.3 Curve fitting

Having computed from §3.2 the first four moments about the mean of the total travel time T, the customary moment-based summary measures may be defined:

Mean: $\mu = \mathrm{E}[T]$

Variance: $\sigma^2 = \mathrm{E}[(T-\mu)^2]$

Skewness: $\sqrt{\beta_1} = \dfrac{\mathrm{E}[(T-\mu)^3]}{\sigma^3}$

Kurtosis: $\beta_2 = \dfrac{\mathrm{E}[(T-\mu)^4]}{\sigma^4}$.

The approach then is to fit the computed values of these four measures to a flexible family of probability densities known as *Johnson curves* (Johnson, 1949), according to the techniques described in Hill *et al* (1976) and Hill (1976). This family consists of distributions obtained by monotonic transformations of a Normal variate, with additional parameters incorporated to permit a flexible fit to observed data. Hill *et al* (1976) focus on three special cases studied at some length by Johnson, for the random variable X:

i) the lognormal system S_L, where $\gamma + \delta \ln(X - \xi) \sim \mathrm{Nor}(0,1)$ (for $X > \xi$);

ii) the unbounded system S_U, where $\gamma + \delta \sinh^{-1}\left(\dfrac{X-\xi}{\lambda}\right) \sim \mathrm{Nor}(0,1)$;

iii) the bounded system S_B, where $\gamma + \delta \ln\left(\dfrac{X-\xi}{\xi+\lambda-X}\right) \sim \mathrm{Nor}(0,1)$ (for $\xi < X < \xi + \lambda$)

where Nor (0,1) denotes a Normal distribution with mean 0 and variance 1. Thus, S_L is a three-parameter system, whereas S_U and S_B each depend on four parameters. Hill *et al* supplement these three systems with a special case family (S_T) on the boundary of validity of the S_B system. The approach adopted by Hill *et al* (1976) is to use the third and fourth moments β_1 and β_2 to select an appropriate system from the Johnson family, and then combine this information with the remaining moments to estimate the parameters of the chosen system.

In outline, the approach is to first (by the method of moments) estimate the parameter δ as if the data were explained by an S_L system. In fact, rather than δ, some simplification is possible if instead we estimate $\omega = \exp(\delta^{-2})$, by solving for ω:

$$(\omega - 1)(\omega + 2)^2 = \beta_1 \ .$$

Denoting the solution as $\hat{\omega}$ $(\hat{\omega} > 1)$, the implied S_L fourth moment is compared with the desired β_2: if $\beta_2 < \hat{\omega}^4 + 2\hat{\omega}^3 + 3\hat{\omega}^2 - 3$ then S_B (or S_T) is appropriate, but if $\beta_2 > \hat{\omega}^4 + 2\hat{\omega}^3 + 3\hat{\omega}^2 - 3$ then S_U is appropriate. The S_L distribution itself lies on the boundary of this inequality; in practice, if the equality is satisfied within some tolerance, then S_L is selected.

In our tests, S_U and S_T were never selected, and so are not described further here. The details of the estimation procedure for an S_B curve are somewhat lengthy but are eloquently described by Hill *et al* (1976), with corresponding program code, and so are not repeated here. It should be noted, however, that fitting an S_B curve is potentially problematic, especially in view of the limited range of validity of the parameters, and it is possible that the algorithm may fail to converge. In such cases, Hill *et al*'s algorithm resorts to fitting S_L or S_T as appropriate. It is noted in passing that in our tests reported in §4, S_L was selected on occasion as the most appropriate curve (difference between desired and implied S_L kurtosis within tolerance of 0.01), and on occasion due to failure to converge of the S_B algorithm.

As will be discussed in §5, the S_L system is particularly attractive for larger networks. For this system, the model fit is the most straightforward; having evaluated $\hat{\omega}$ according to the method above, the parameters are then estimated from:

$$\hat{\delta} = (\ln \hat{\omega})^{-\frac{1}{2}} \qquad \hat{\gamma} = \tfrac{1}{2}\hat{\delta} \ln\{\hat{\omega}(\hat{\omega} - 1)/\sigma^2\} \qquad \hat{\xi} = \mu - \exp\{(1/(2\hat{\delta}) - \hat{\gamma})/\hat{\delta}\} \ . \quad (16)$$

4. ILLUSTRATIVE EXAMPLE

[FIGURE 2 HERE]

A five-link test network will be considered, previously studied in the literature (e.g. Suwansirikul *et al*, 1987; Cho & Lo, 1999), and illustrated in Figure 2. It will later prove useful to label the routes available, with route A consisting of links 1 and 4, route B comprising links 2 and 5, and route C comprising links 1, 3 and 5. To define the base route choice probabilities, a probit-based SUE model is adopted, based on the original (rather than Poisson-corrected) travel time functions, and using independent link perception errors, distributed for link *a* as $\mathrm{Nor}\left(0,(\phi t_a(0))^2\right)$, with $\phi = 0.3$ used in the tests below. This SUE was estimated using 32,000 iterations of a route-based Method of Successive Averages (MSA) algorithm, with one stochastic network load sampled per MSA iteration. The resulting SUE link flows are provided in Table 1, inferring route choice probabilities of $p_A = 0.4310$, $p_B = 0.4452$ and $p_C = 0.1239$.

[TABLE 1 HERE]

Mindful that direct use of the quartic travel time functions in the subsequent reliability analysis could lead to a high computational cost in larger networks, a local quadratic (second order Taylor series) approximation about the SUE solution was therefore adopted, and the resulting error later investigated. The Taylor series coefficients (b_{ia}) and the transformed coefficients ($\tilde{b}_{ia}$) are also provided in Table 1.

The link flow covariance matrix (3), and thence the link-based total travel time moments (15a)–(15d), may then be estimated, whereby the total network travel time moments about the origin can be computed as:

$$\mathrm{E}[T] = 1298.39 \qquad \mathrm{E}[T^2] = 1{,}761{,}951.13$$

$$\mathrm{E}[T^3] = 2{,}501{,}598{,}503 \qquad \mathrm{E}[T^4] = 3{,}719{,}186{,}185{,}961.$$

It is then straightforward to obtain the summary measures:

$$\mu = 1298.39 \qquad \sigma = 275.95 \qquad \sqrt{\beta_1} = 0.7696 \qquad \beta_2 = 3.9755.$$

For comparison, these summary measures were also estimated using Monte Carlo simulation, with each of 1000 pseudo-random draws of the Poisson O-D matrix assigned probabilistically to the routes according to the base SUE route probabilities.

For each simulation draw, link travel times—and hence total network travel time—are computed, using either the exact quartic functions or quadratic approximations.

[FIGURE 3 HERE]

[TABLE 2 HERE]

The empirical frequency distributions obtained from four replications of this Monte Carlo procedure are illustrated in Figure 3. Even with as many as 1000 Monte Carlo draws per replication (a large number for a realistic scale network), considerable between-replication variability in the shape of the distribution may be observed. Table 2 compares the resulting summary statistics between the (exact) analytical, empirical BPR (Monte Carlo, exact quartic functions) and empirical Taylor approximation (Monte Carlo, approximate quadratic functions) methods. For both empirical methods 25 replications were performed, with both the mean and standard deviation across the replications presented. Although the Monte Carlo estimates have a bias no greater than 5%, their standard errors are as much as 15%–20% of their mean value for the third and fourth moments. While this error may be reduced (at a computational cost) by increasing the number of simulated draws, the potential unreliability of Monte Carlo methods for estimating shape parameters is evident, an especially significant factor in the study of an asymmetric/tail feature such as travel time unreliability.

[FIGURE 4 HERE]

Based on the analytical moments, the algorithm of §3.3 selected a Johnson S_L curve with parameters

$$\gamma = -28.1754 \qquad \delta = 4.04184 \qquad \xi = 200.067$$

which is labelled "Fixed" in Figure 4. The second curve represents a comparison with the equilibrium philosophy behind most existing reliability analyses reviewed in §1, whereby equilibrium is reached for each realisation of the system state (in this case, the realised O-D flows). This is achieved by approximating the equilibrium response by sensitivity analysis (Clark & Watling, 2002), obtaining equilibrium link flows $\mathbf{v}^*(\mathbf{q}) \approx \mathbf{d} + \mathbf{B}\mathbf{q}$, with $\mathbf{q}$ the vector of O-D demands, $\mathbf{d}$ a constant A-vector and $\mathbf{B}$ a constant $A \times A$ matrix. As a linear model for the random O-D demand vector $\mathbf{Q}$, the link flow covariance matrix of the induced random link flows is then:

$$\mathrm{var}[\mathbf{v}^*(\mathbf{Q})] \approx \mathrm{var}[\mathbf{d} + \mathbf{B}\mathbf{Q}] = \mathbf{B}\,\mathrm{var}[\mathbf{Q}]\mathbf{B}^{\mathrm{T}} \qquad (17)$$

with (17) then used in place of (3) before applying §3.2 and §3.3.

[TABLE 3 HERE]

It is clear from Figure 4, coupled with the corresponding 'reliability probabilities' (as defined in §2) in Table 3, that the impact of the equilibrium response is a more optimistic evaluation of reliability than the 'fixed' response (curve shifted to left), whereby drivers have sufficient knowledge of prevailing conditions to mitigate the impacts of system variation by adjusting their choice of route. Thus, we are able to contrast the impacts of variability that is not predictable by the drivers ('fixed', the primary approach of the present paper) and of predictable variation ('equilibrium' response), the latter perhaps more closely achievable with some kind of intelligent driver information system.

Focusing again on the primary method proposed in §3.1–§3.3, the impact is considered of adjusting the capacity of each link in turn (base values are given in the quartic term denominators of the travel time functions in Figure 2). In each case, a new probit SUE is first computed to obtain the route choice probabilities, and a new quadratic Taylor series approximation subsequently estimated. Figure 5 illustrates the resultant Johnson curves, with $S_L(0)$ denoting the base case, and $S_L(a)$/$S_B(a)$ denoting the case in which the capacity of link a alone was reduced by 10 units $(a = 1,2,...,5)$. Links 1 and 5 may be identified as the most 'sensitive' links, with the capacity reductions here having the most pronounced effect.

[FIGURE 5 HERE]

Figure 6 illustrates a further experiment, as the capacity of link 1 was gradually reduced. In addition to the anticipated location shift, which could be predicted by an existing equilibrium model, subtle dispersion and shape impacts are also evident. Specifically, an increase in dispersion and skewness (longer right hand tail) may be seen, while the left-hand tail is apparently anchored. These results are plausible, in the sense that 'spare capacity' allows a network to deal better with unexpected variation.

[FIGURE 6 HERE]

5. IMPLEMENTATION CONSIDERATIONS

The computational load of the method in §3 is dominated by the difficulty in computing the highest moment, $\mathrm{E}[T^4]$. For quadratic travel time functions (using a Taylor series approximation where necessary), $\mathrm{E}[T^4]$ will involve in the order of $(A(m+2))^4$ terms requiring Isserlis moments as high as $q_{12}(1,1,...,1)$ (see Appendix B). In §4, with $A=5$ and $m=2$, around 160,000 $q_{12}(1,1,...,1)$ evaluations were required, which was achieved in a matter of seconds, yet clearly this will increase rapidly with the size of the network (i.e. with A).

In large networks, an attractive simplification is to restrict attention to only the lognormal S_L system, whereby only moments up to $\mathrm{E}[T^3]$ are required, the difficulty then dominated by the computation of around $(A(m+2))^3$ Isserlis moments of the $q_8(1,1,...,1)$ kind. Hence both the number of terms, and the difficulty of computing each term, dramatically reduces. We have verified that practical run-times in the order of minutes could then be achieved on current fast personal computers, for a problem with $A=100$ links and $m=2$.

Still, there are cases where further computational savings may be required, such as in networks much larger than 100 links, or as part of a method to *optimise* network reliability where multiple applications of this method are required. By assuming structural relationships between the parameters, or fixing externally estimates of some parameters, it is then possible that only moments up to $\mathrm{E}[T^2]$ are required, the computational load reducing dramatically to requiring only in the order of $(A(m+2))^2$ Isserlis moments of the $q_6(1,1,...,1)$ kind. For example, in the case of S_B type curves, a method is provided in Bacon-Stone (1985) to estimate all four parameters with the first two moments and two boundary values. Prior estimates of such boundary values might be fixed from some reasonable assumptions about the minimum and maximum demand, or in an optimisation context from a previous full estimation with three or four moments.

In a similar spirit, it is straightforward to adapt the estimation of S_L curves to use a pre-specified value ξ_0 of the shift parameter ξ, which effectively represents the minimum total travel time. By eliminating $\hat{\gamma}$ and $\hat{\omega}$ from the expressions in (16) through substitution, and setting $\hat{\xi} = \xi_0$ we obtain:

$$\hat{\delta} = \left(\ln\left(1 + \left(\frac{\sigma}{\mu - \xi_0} \right)^2 \right) \right)^{-\frac{1}{2}} \qquad \hat{\gamma} = \frac{1}{2\hat{\delta}} - \hat{\delta}\ln(\mu - \xi_0) \tag{18}$$

whereby an S_L curve may be estimated with only knowledge of μ and σ. As an illustration, considering the moments given in Table 2 and fitted curve denoted $S_L(0)$ in Figure 5, then with the full estimation of all parameters we obtain an S_L curve with $\hat{\xi} = 200.07$, $\hat{\delta} = 4.04$ and $\hat{\gamma} = -28.18$. If, on the other hand, we make a very crude estimate of the minimum by setting $\xi_0 = 0$ and applying (18), we obtain rather similar parameter estimates $\hat{\delta} = 4.77$ and $\hat{\gamma} = -34.10$. In the last two columns of Table 3, the resulting favourable comparison is given of the reliability probabilities obtained from the full estimation ('Fixed' column), with those from the two parameter estimation with $\xi_0 = 0$ ('Approximate Fixed' column).

6. CONCLUSION

An approach has been proposed which departs in philosophy from previous analyses of this topic, a key element being an assumption of disequilibrium, with drivers assumed to face unpredictable variation to which they are not able to re-equilibrate. Although as illustrated in §4, it is possible to implement this approach with Monte Carlo methods, it has been shown that it is also theoretically possible to estimate *analytically* moments of the network travel time probability distribution under an assumption of stochastic demand (§3.1, §3.2), from which an estimate of the full distribution may be readily constructed (§3.3), and from which system reliability probabilities may be computed (§2). The analytical method presented is flexible, in that it may be tailored to the demands of the particular application: in large networks, one can restrict the computation of moments by selecting a restricted family of density functions for fitting or by fixing certain parameters (§5). The numerical tests

(§4) have demonstrated the application of the approach in understanding the impacts of capacity changes on the shape of the network travel time density, beyond the effect on mean and variance, as well as its use in identifying vulnerable or sensitive links, in terms of their impact on overall system performance.

There are many potential avenues for further research with this technique:

1. On a practical level, a large network case-study would clearly be valuable, including comparisons with any empirical data on variability.
2. In order to widen the opportunities for statistical fitting, there may be advantages in widening the range of underlying statistical model assumptions that are admissible, including those in which the O-D demand variability may itself be parameterised. One extension to the current assumptions would be to reflect correlations in O-D demand levels due to common underlying factors.
3. The reliability measures themselves may be extended beyond overall network travel time, such as O-D specific total travel time, and the day-to-day distribution of user-average O-D travel times.
4. There are opportunities to generalise the model itself in many ways, including extensions to reflect randomly varying link capacities, and the estimation of reliability for economic benefit measures under elastic (and stochastic) demand.
5. Finally, there is potential for embedding the proposed method of reliability evaluation within a bi-level (possibly multi-objective) optimisation framework, whereby network capacities, tolls or information sensors may be set with reliability considerations in mind. The use of an analytical approach would be expected to have particular advantages (over Monte Carlo methods), for devising efficient gradient-based or sensitivity-analysis based algorithms in such a context.

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APPENDIX A: Modified travel time functions under Poisson demand

It is reasonable to assume drivers base their decisions on long-run expected travel times, $\mathrm{E}[t_a(V_a)] = \sum_{j=0}^{m} b_{ja}\mathrm{E}[V_a{}^j]$ (under (4)). Consider the case $m = 2$. Since under the assumptions in §3.1, V_a is marginally Poisson with mean (and variance) μ_a,

$$\mathrm{E}[t_a(V_a)] = b_{0a} + b_{1a}\mathrm{E}[V_a] + b_{2a}\mathrm{E}[V_a{}^2] = t_a(\mu_a) + b_{2a}\mu_a \quad \left(= \hat{t}_a(\mu_a), \text{say}\right)$$

Use of the *Poisson-corrected travel time function* $\hat{t}_a(.)$ would therefore give greater model consistency than $t_a(.)$ when applied in an equilibrium framework approximating a stochastic flow environment. The same applies for $m > 2$, with higher order Poisson moments utilised (e.g.: $\mathrm{E}[(V_a - \mu_a)^3] = \mu_a$; $\mathrm{E}[(V_a - \mu_a)^4] = \mu_a(1 + 3\mu_a)$: Stuart & Ord, 1987, p 112). This refinement was first suggested for two-link networks in an unpublished note of Bell (1991), and represents a statistical approximation of the multinomial path flow model in Watling (2002c).

APPENDIX B: Computation of multivariate normal moments

Suppose $\mathbf{X} = (X_1, X_2,..., X_k)$ is multivariate Normal with mean vector $(\mu_1, \mu_2,..., \mu_k)$ and covariances $\sigma_{ij}\ (i = 1,2,...,k;\ j = 1,2,...,k)$. For non-negative integers $(n_1, n_2,..., n_k)$, denote:

$$p_k(n_1, n_2,..., n_k) = \mathrm{E}\left[\prod_{i=1}^{k}(X_i - \mu_i)^{n_i}\right]$$

and the corresponding *reduced moment* by

$$q_k(n_1, n_2,..., n_k) = \frac{p_k(n_1, n_2,..., n_k)}{\sqrt{\prod_{i=1}^{k}\sigma_{ii}^{\ n_i}}}.$$

Now for any *even* positive integer *m*, define a *pairing of order m* to be a division of the set {1,2,…,*m*} into $m/2$ subsets, each consisting of a distinct pair of elements. For example, if *m* = 6 then one possible pairing of order 6 is $\{\{1,3\},\{2,4\},\{5,6\}\}$. In general, denote an arbitrary pairing of order *m* by:

$$(\mathbf{a}, \mathbf{b}) = \left\{\{a_i, b_i\} : i = 1,2,..., m/2 \text{ such that} \bigcup_{i=1}^{m/2}\{a_i, b_i\} = \{1,2,3,..., m\}\right\}.$$

and denote the collection of all possible such pairings of order *m* by $\Omega(m)$.

Then (Isserlis, 1918):

$$q_m(\underbrace{1,1,...,1}_{m\ \text{times}}) = \begin{cases} 0 & \text{if } m \text{ odd} \\ \sum_{(\mathbf{a},\mathbf{b})\in\Omega(m)} \prod_{i=1}^{m/2}\tilde{r}_{a_i b_i} & \text{if } m \text{ even} \end{cases}$$

where $\tilde{r}_{ij}\ (i = 1,2,...,m;\ j = 1,2,...,m)$ are correlation coefficients. This result can be used to compute an arbitrary reduced moment $q_k(n_1, n_2,..., n_k)$, simply by creating an *m*-vector multivariate Normal where $m = \sum_{i=1}^{k} n_i$, consisting of n_i duplicate occurrences of each X_i, such that $q_k(n_1, n_2,..., n_k) = q_m(\underbrace{1,1,...,1}_{m\ \text{times}})$.

For example, if $(X_1, X_2) \sim \mathrm{Nor}\left((\mu_1, \mu_2), \begin{pmatrix}\sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22}\end{pmatrix}\right)$, we would re-write:

$$\mathrm{E}[(X_1-\mu_1)^3(X_2-\mu_2)] = \mathrm{E}[(X_1-\mu_1)(X_1-\mu_1)(X_1-\mu_1)(X_2-\mu_2)]$$

with the matrix of correlation coefficients of (X_1, X_1, X_1, X_2) then:

$$\begin{pmatrix} \tilde{r}_{11} & \tilde{r}_{12} & \tilde{r}_{13} & \tilde{r}_{14} \\ & \tilde{r}_{22} & \tilde{r}_{23} & \tilde{r}_{24} \\ & & \tilde{r}_{33} & \tilde{r}_{34} \\ & & & \tilde{r}_{44} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & r_{12} \\ & 1 & 1 & r_{12} \\ & & 1 & r_{12} \\ & & & 1 \end{pmatrix} \quad \text{where } r_{12} = \sigma_{12} \Big/ \sqrt{\sigma_{11}\sigma_{22}} \, .$$

Then by Isserlis's result, $q_4(1,1,1,1) = \tilde{r}_{12}\tilde{r}_{34} + \tilde{r}_{13}\tilde{r}_{24} + \tilde{r}_{14}\tilde{r}_{23} = 3r_{12}$, the required reduced moment $q_2(3,1)$ in the original system. Finally, $p_2(3,1) = q_2(3,1)\sqrt{\sigma_{11}{}^3\,\sigma_{22}} = 3\sigma_{12}\sigma_{11}$. On the other hand, we would immediately know from the Isserlis result that $\mathrm{E}[(X_1-\mu_1)^4(X_2-\mu_2)] = 0$, since the sum of powers is 5, an odd number.

The utility of the Isserlis result is particularly evident for moments involving a larger number of variables of higher powers. In the present application, the highest order Isserlis moment required for an n^{th} order total travel time moment based on a k^{th} order polynomial travel time function is $n(k+1)$. So, for example, for a 4^{th} order moment based on quadratic travel time functions, we would require up to $q_{12}(1,1,...,1)$. In order to generate such higher order Isserlis expressions, we have found a simple recursive method to be efficient to code:

To determine $q_m(1,1,...,1)$ for $\mathbf{Y} = (Y_1, Y_2, Y_3, ..., Y_m)$, with m even, first denote Isserlis reduced moments for even subsets (marginal distributions) of $(Y_1, Y_2, Y_3, ..., Y_m)$ as:

$$Q_t(\{i_1, i_2, ..., i_t\}) = q_t(\underbrace{1,1,...,1}_{t \text{ times}}) \text{ for } (Y_{i_1}, Y_{i_2}, ..., Y_{i_t}) \qquad (2 \le t \le m,\, t \text{ even}).$$

Denoting the correlation coefficients of $\mathbf{Y}$ by $\tilde{r}_{ij}$ $(i = 1,2,...,m;\ j = 1,2,...,m)$, then for any given m the required reduced moment (which in this new notation is $Q_m(\{1,2,3,...,m\})$) can be generated through reduced moments of lower order, according to the recursion:

$$Q_2(\{i_1, i_2\}) = \tilde{r}_{i_1, i_2}$$

$$Q_t(\{i_1, i_2, ..., i_t\}) = \sum_{j=2}^{t} \tilde{r}_{i_1, i_j}\ Q_{t-2}(\{i_2, i_3, ..., i_t\} \cap \{i_j\}^{\mathrm{c}}) \qquad (t = 4,6,8,...)$$

where A^c denotes the complement of the set A. As a check, Isserlis (1918) provides the explicit formula for $q_6(1,1,1,1,1,1)$ as well as special cases of higher order moments.

a	v_a	b_{0a}	b_{1a}	b_{2a}	$\tilde{b}_{0a}$	$\tilde{b}_{1a}$	$\tilde{b}_{2a}$	$\tilde{b}_{3a}$
1	55.48	10.6637	-0.3203	0.00433	345.32	15.114	0.4005	0.00433
2	44.52	10.1417	-0.2481	0.00418	328.55	12.903	0.3101	0.00418
3	12.39	2.0016	-0.0004	0.00002	24.78	2.002	0.0004	0.00002
4	43.10	8.0320	-0.1876	0.00326	258.65	10.026	0.2339	0.00326
5	56.90	8.5289	-0.2591	0.00342	276.50	12.263	0.3247	0.00342

Table 1: Base SUE link flows and travel time function coefficients

	Analytical	Empirical BPR	Empirical Taylor
μ	1298.39	1265.16 (8.97)	1264.75 (7.75)
σ	275.95	271.24 (8.14)	267.14 (7.54)
$\sqrt{\beta_1}$	0.7696	0.8303 (0.1300)	0.7882 (0.1322)
β_2	3.9755	4.3275 (0.7485)	4.1720 (0.8330)

Table 2: Base solution total network travel time moments

Critical Value for Total Network Travel Time	Equilibrium	Fixed	Approximate Fixed
1250	0.4757	0.5233	0.5286
1500	0.1714	0.2108	0.2128
1750	0.0465	0.0649	0.0628
2000	0.0107	0.0169	0.0150

Table 3: Reliability probabilities for a number of notional critical values under 'equilibrium' and 'fixed' responses (see Figure 4)

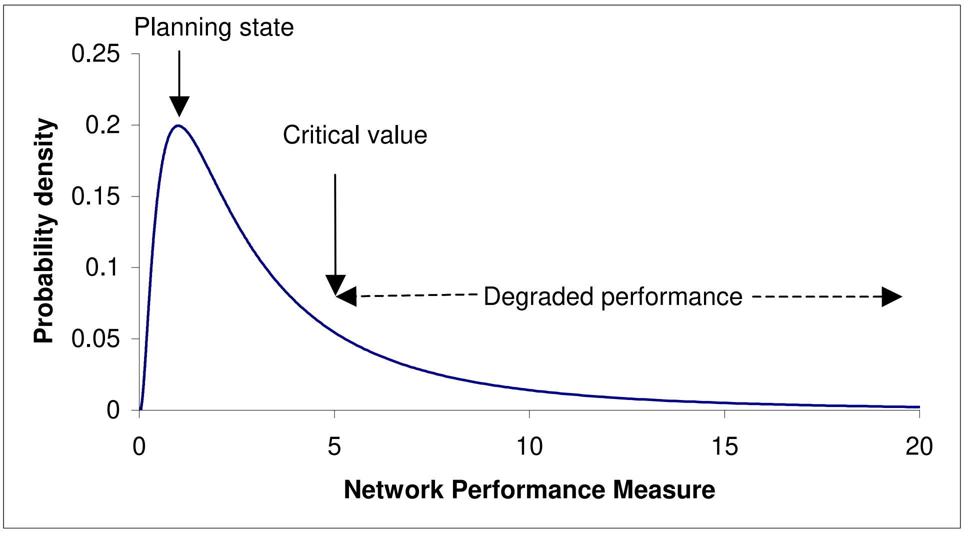

Figure 1: Illustrative example of performance measure distribution

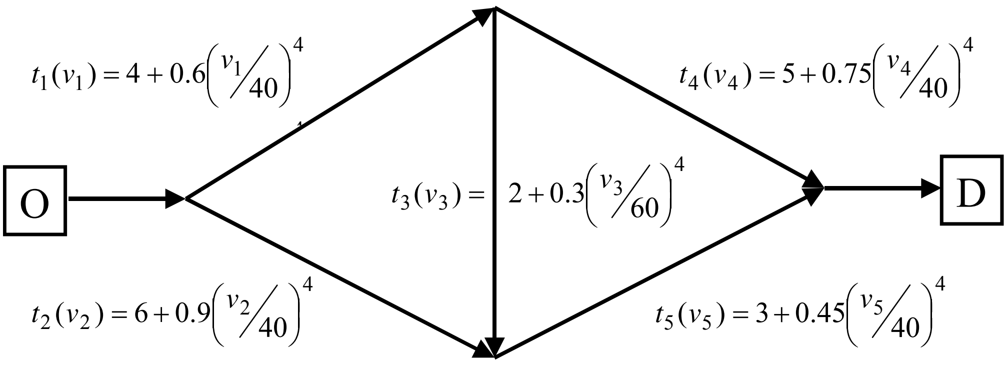

Figure 2: Example network (Suwansirikul *et al*, 1987), O-D demand $q = 100$

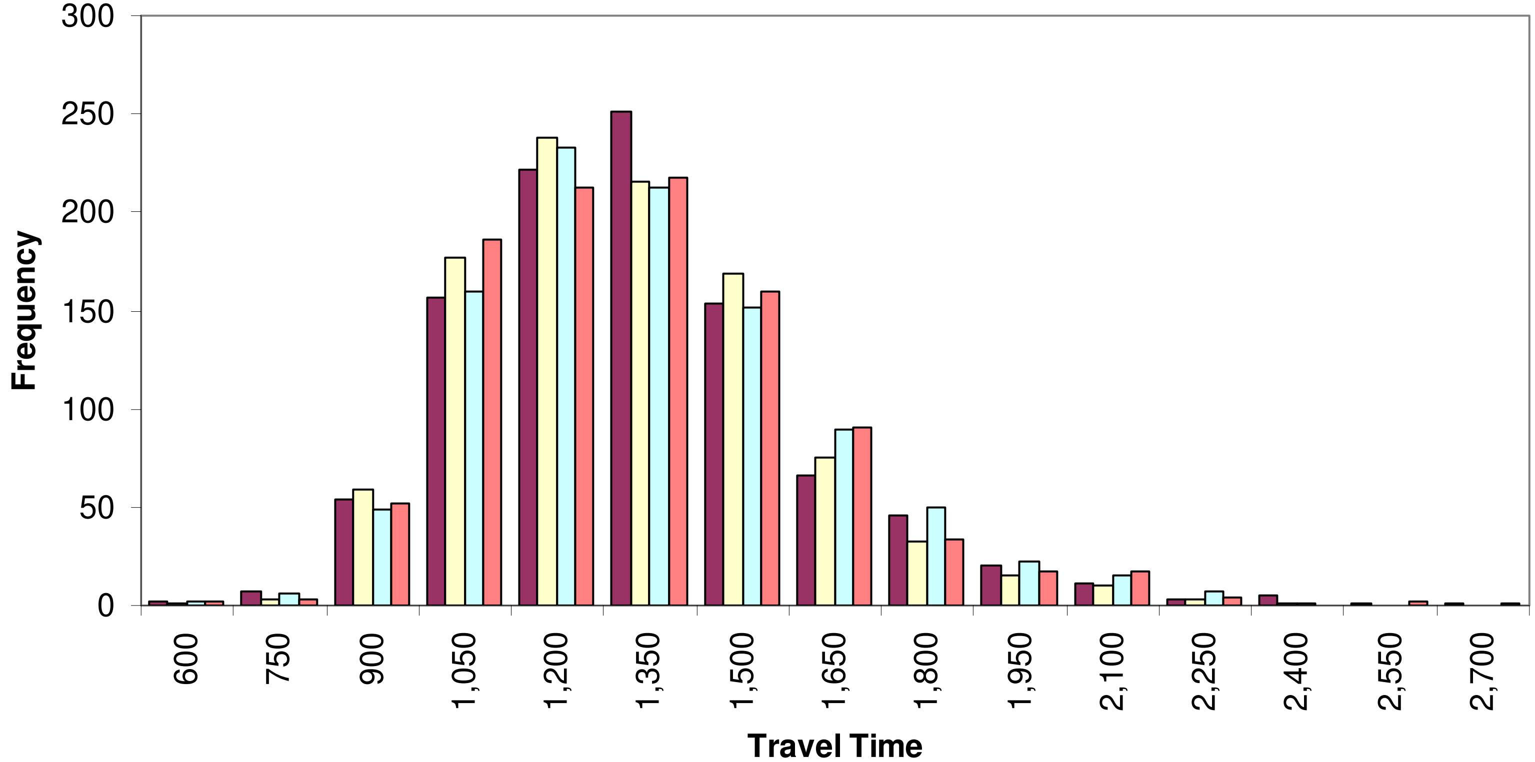

Figure 3: Empirical distribution of total network travel time for four replications of Monte Carlo method

(1000 simulation draws per replication; exact quartic travel time functions)

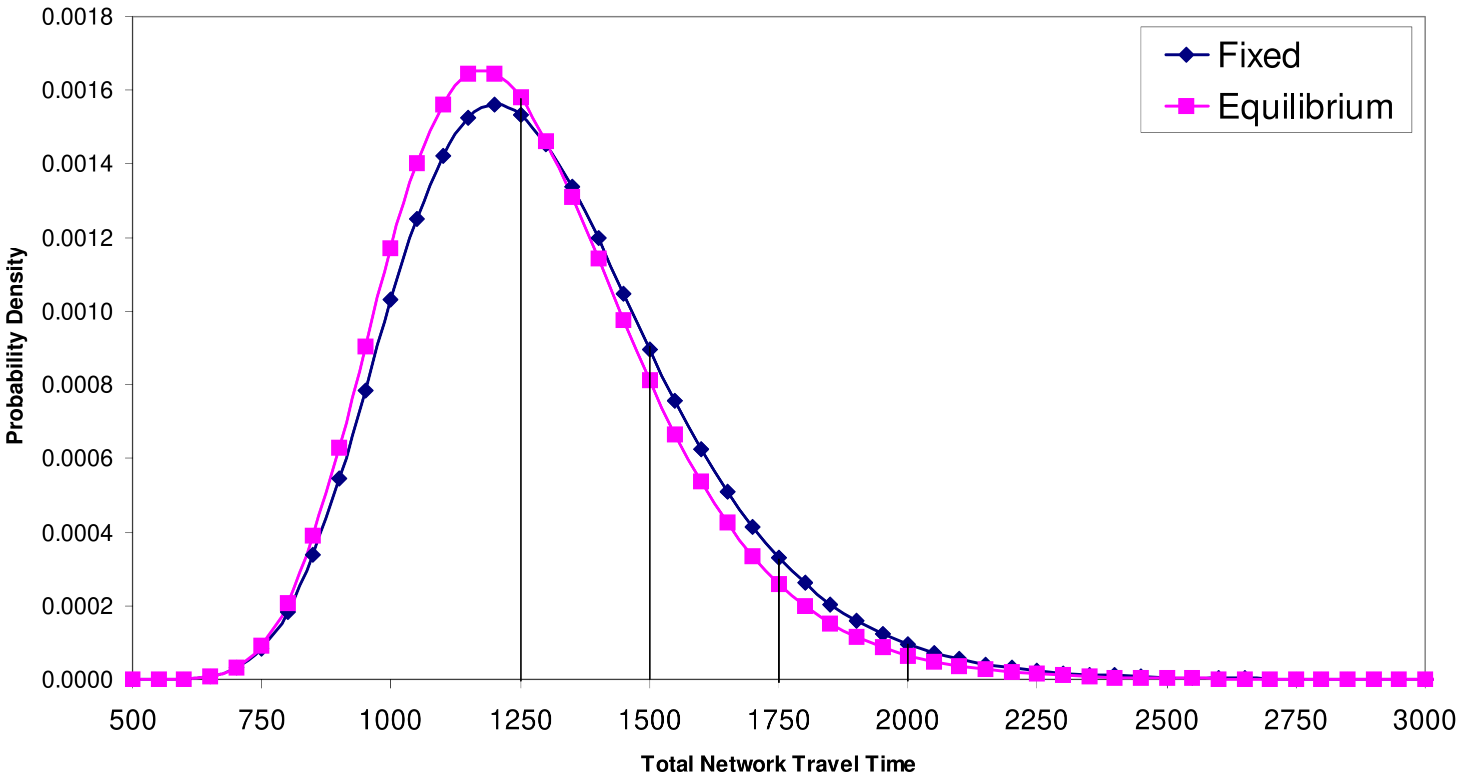

Figure 4: Distribution of total network travel time under “Fixed” and “Equilibrium” model assumptions

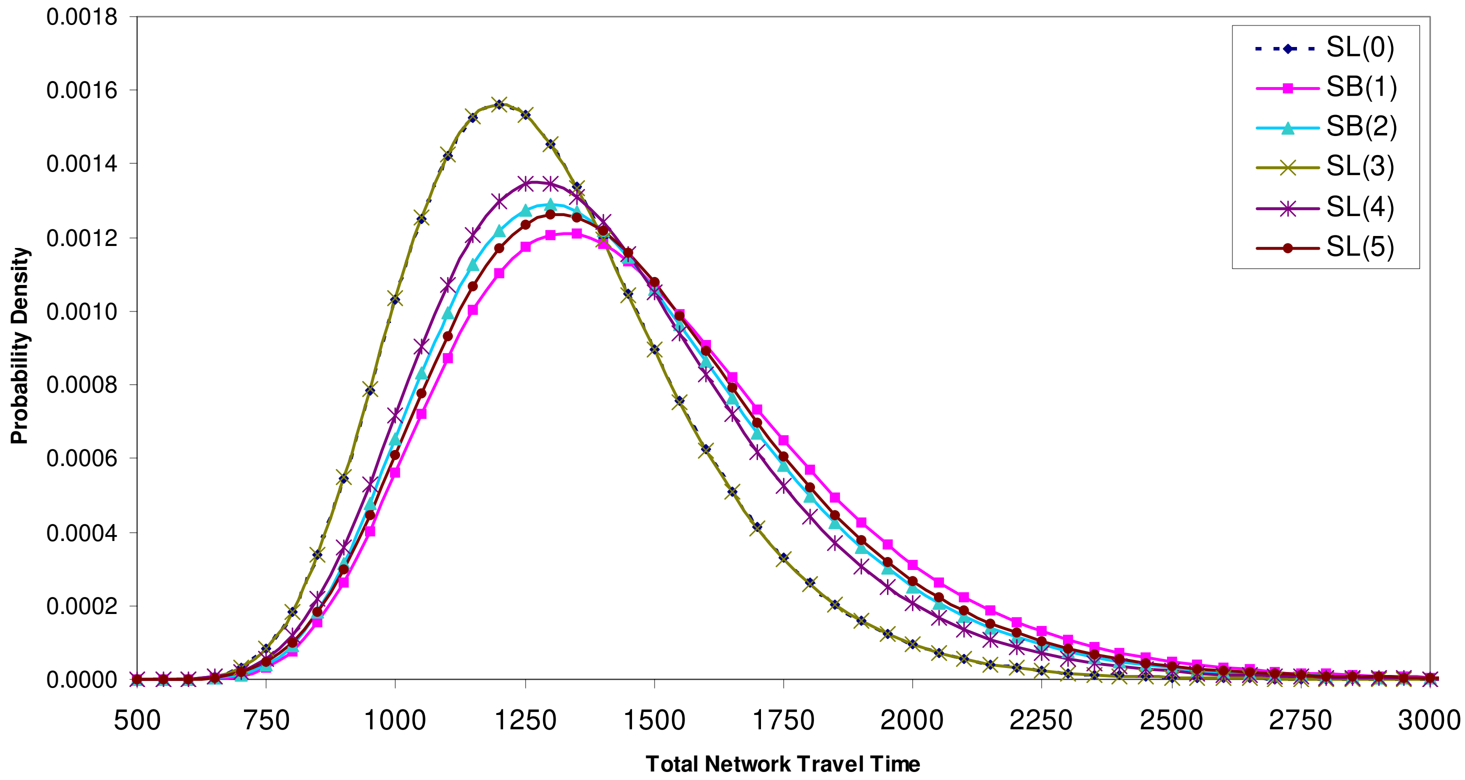

Figure 5: Distribution of total network travel time for a 10 unit reduction, in turn, in the capacity of each link

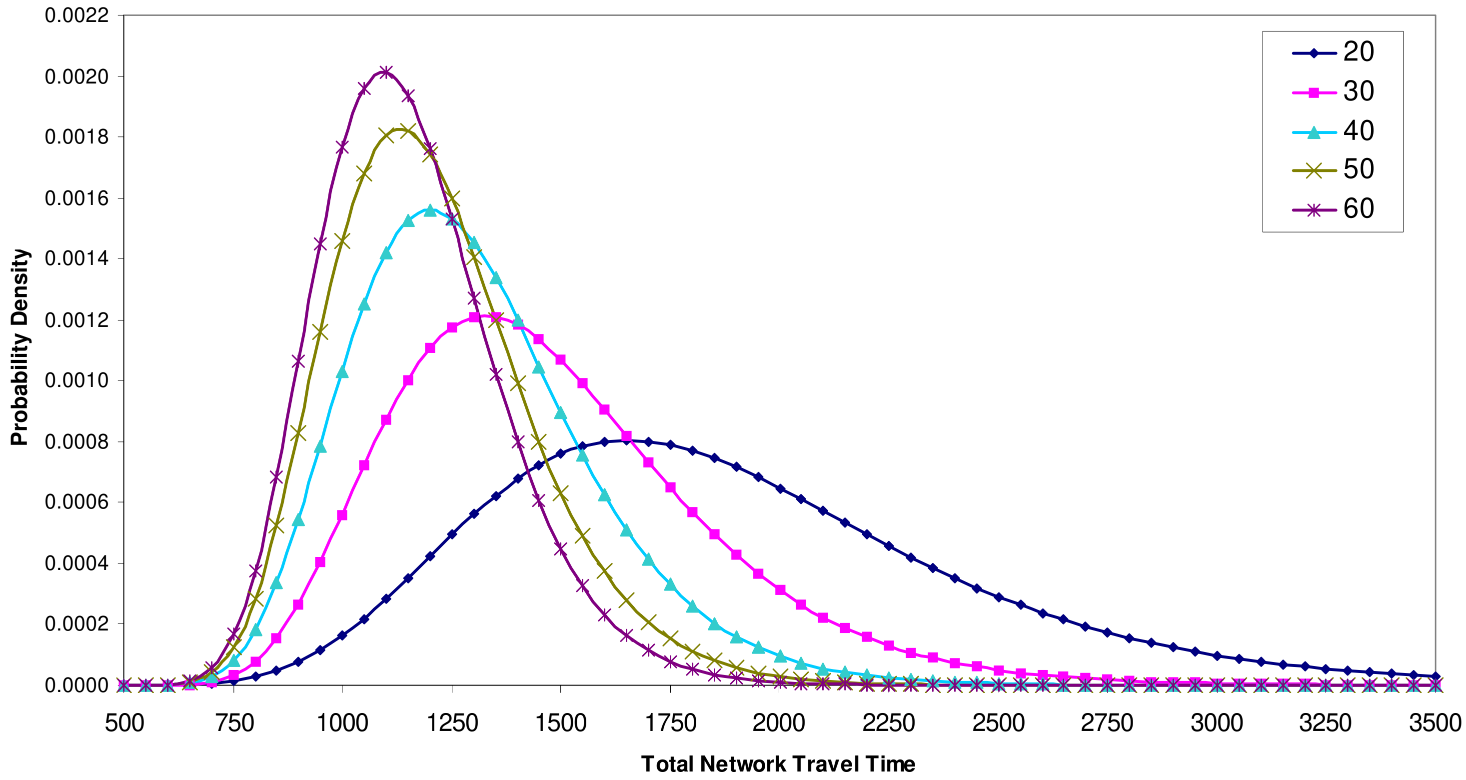

Figure 6: Empirical distribution of total network travel time for a range of capacities on link 1