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The elastic stored energy of initially strained, or stressed, materials: restrictions and third-order expansions

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A large variety of engineering and biological materials have a non-zero internal stress distribution, even in the absence of applied forces. These stresses can arise from thermal expansion or volumetric growth, for example, in the production of the material. There are two approaches to modelling such materials that appear similar but are, in fact, distinct. The first defines a function, $\tilde{W}(\mathbf{F}, \boldsymbol{\tau})$, associated with a *fixed reference configuration*, $\mathcal{B}$, say, where each value of $\boldsymbol{\tau}$ corresponds to the initial stress in a *different elastic material* that occupies $\mathcal{B}$ (each with a different elastic constitutive equation, effectively). The second defines a function, $\overline{W}(\mathbf{F}, \boldsymbol{\tau})$, associated with a *single, fixed, initially stressed, elastic material* (with a single constitutive equation), where each value of $\boldsymbol{\tau}$ represents the stress in a *different configuration* of that material. Here, we discuss why stored energy functions of the latter type, and similar functions that are written in terms of an initial *strain*, need to satisfy some restrictions to avoid unphysical behaviours. To illustrate their need, we perform an asymptotic expansion to prove that these restrictions are required for consistency with strain energy functions of classical third-order weakly nonlinear elasticity.

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1. Introduction

The classical approach to modelling nonlinear elastic materials in the field theory of continuum mechanics is to define a strain energy density function, $\tilde{W} = \tilde{W}(\mathbf{F})$, where $\mathbf{F}$ is an elastic deformation gradient from the reference configuration to the current configuration. For such functions, the elastic material *and* the reference configuration are *both fixed*. Note the convention we introduce here that a tilde indicates a function that is only valid in a single reference configuration. The functional form of $\tilde{W}(\mathbf{F})$ depends upon the symmetry group corresponding to the underlying material microstructure [1,2]. In many materials, however, some level of stress is *always* present. This is especially the case in living materials, where mechano-transduction is known to regulate both biochemical activities and gene expression from cell to tissue level [3,4], and in manufactured solids, following thermal changes, fatigue, plasticity, etc. For such materials, allowing the stored energy function (SEF) to depend *only* on $\mathbf{F}$ is no longer appropriate.

One constitutive choice to model how an initial stress $\boldsymbol{\tau}$ (the Cauchy stress tensor in the reference configuration) affects the elastic response of a *fixed* material is to consider a strain energy function of the form $\tilde{W}(\mathbf{F}\mathbf{F}_1)$, where both $\mathbf{F}$ and $\mathbf{F}_1$ are elastic deformation gradients: $\mathbf{F}_1$ represents a deformation from a stress-free (possibly virtual) configuration to the configuration with stress $\boldsymbol{\tau}$, and $\mathbf{F}$ represents a further deformation from the stressed reference configuration to the current configuration. The product $\mathbf{F}\mathbf{F}_1$ is the deformation gradient associated with the combined deformation directly from the stress-free configuration to the current configuration. If both of these deformations are small, so that $\mathbf{F} = \mathbf{I} + \boldsymbol{\epsilon}$ and $\mathbf{F}_1 = \mathbf{I} + \boldsymbol{\epsilon}_1$, where $\mathbf{I}$ is the identity tensor and $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}_1$ are the infinitesimal displacement gradients, then $\mathbf{F}\mathbf{F}_1 \approx \mathbf{I} + \boldsymbol{\epsilon} + \boldsymbol{\epsilon}_1$. This method, adapted by Rodriguez *et al*. [5] from plasticity theory [6,7], is called the *multiplicative decomposition method* (see [8] for a historical account). It is possible to define a function of two variables such that $W(\mathbf{F}, \mathbf{F}_1) = \tilde{W}(\mathbf{F}\mathbf{F}_1)$ for all values of $\mathbf{F}$ and $\mathbf{F}_1$. For such a function, the value of $\mathbf{F}_1$ *defines* the reference configuration of W for the subsequent deformation $\mathbf{F}$. As we show in the paper, such functions must satisfy a restriction.

An alternative way to account for the presence of stress in the reference configuration is to consider an SEF that explicitly depends on the Cauchy stress tensor $\boldsymbol{\tau}$ in the reference configuration (to our knowledge, Johnson & Hoger [9] were the first to introduce this double dependence for the stress response). In this paper, we use the term 'stored energy function' instead of strain energy function to emphasize that such functions do not depend solely on the strain. There are two superficially similar, but distinct approaches to doing this. The first approach is to define a function $\tilde{W}(\mathbf{F}, \boldsymbol{\tau})$ that is associated with a *single, fixed reference configuration*, $\mathcal{B}$, say (we call this a type 1 function). In this case, each value of $\boldsymbol{\tau}$ corresponds to the initial stress in a *different elastic material* that occupies $\mathcal{B}$. By different elastic materials, we mean distinct physical bodies, which have different effective constitutive equations, but which are considered to occupy the fixed set of points that make up $\mathcal{B}$, regardless of the value of $\boldsymbol{\tau}$. The second approach is to define a function $\overline{W}(\mathbf{F}, \boldsymbol{\tau})$ that is associated with a *single, fixed, initially stressed, elastic material* (we call this a type 2 function). By a single elastic material, we mean a single physical body with the same constitutive response for different values of $\boldsymbol{\tau}$. Changing the value of $\boldsymbol{\tau}$ corresponds to changing the reference configuration of that material.

Type 2 functions are useful when solving inverse boundary value problems (BVPs) for the initial stress. For example, because growth and remodelling processes have much larger characteristic times than elastic processes, they are classically treated as elastic distortions, whose geometrical incompatibility may create stress in the reference configuration [5]. This approach relies on assuming the existence of a stress-free virtual state that may not necessarily be a realizable configuration of the material. The existence of a stress-free state is not only difficult to prove theoretically [10] but also practically impossible to detect experimentally [11,12]. Using type 2 functions greatly simplifies this challenge. In the classical approach, the initial stress can be determined by solving a nonlinear BVP as a function of the initial strain

distribution, which is not a trivial task. With type 2 functions, the variational problem is a function of the geometric deformation tensor, which is measurable in experiments, and the initial stress tensor $\boldsymbol{\tau}$, which is subjected to bounded variations and is in the functional space of symmetric, divergence-free, second-order tensors, thus making the inverse problem much easier to handle. Hence, we used this approach [4] to derive the residual stress distribution of an isotropic elastic tube subjected to a given internal pressure, such that the resulting Cauchy stress distribution is optimal in the sense that it minimizes the stress gradient functional, as empirically assumed in [13]. Furthermore, the incremental (small-on-large) theory can be employed in this class of problems to detect the initial strain threshold at the onset of structural instability. The critical elastic strain is large in many problems (e.g. biological materials), thus sticking with a fixed initial configuration may end up causing excessive mesh distortion in finite-element solvers, e.g. unavoidable locking phenomena [14]. With type 2 functions, we can instead use the finitely deformed configuration as the new reference configuration of the material keeping the same variational formulation, thus avoiding these numerical issues [15]. Further advances have recently been made for computational applications in the incompressible limit [16]. In the field of acousto-elasticity, we used type 2 functions to show how residual stresses could be experimentally determined from wave propagation measurements [4] and deduced new identities relating the wave speed and stress in initially stressed materials without [17–19] and with [20] fibres.

In [21], we established that type 2 functions of the form $\overline{W}(\mathbf{F}, \boldsymbol{\tau})$ need to satisfy an additional restriction that is not required of classical elastic strain energy functions. We called this restriction *Initial Stress Reference Independence* (ISRI), and specific cases of this restriction for SEFs that depend *only* on $\mathbf{F}$ and $\boldsymbol{\tau}$ were derived [21,22]. Such restrictions are *not* necessary for type 1 functions, but they *are* necessary for type 2 functions.

In this paper, we first prove that SEFs of the form $W(\mathbf{F}, \mathbf{F}_1)$ must satisfy a restriction that is analogous to the ISRI restriction for $\overline{W}(\mathbf{F}, \boldsymbol{\tau})$, relying only on the classical assumptions of nonlinear elasticity, as laid out by Marsden & Hughes [23], for example. In essence, the restriction on $W(\mathbf{F}, \mathbf{F}_1)$ states that the SEF can only depend on $\mathbf{F}$ and $\mathbf{F}_1$ through the term $\mathbf{F}\mathbf{F}_1$, which makes these SEFs equivalent to the multiplicative decomposition method. From this restriction, we later prove corresponding restrictions on $\overline{W}(\mathbf{F}, \boldsymbol{\tau})$ by inverting the relation between $\boldsymbol{\tau}$ and $\mathbf{F}_1$.

The article is structured as follows. In §2, we provide a rigorous derivation for a restriction on strain energy functions that depend on both an initial deformation gradient and a subsequent deformation gradient, starting from the classical assumptions of the field theory of nonlinear elasticity. This provides a direct way to state that these restrictions are natural and necessary. In §3, we deduce similar restrictions on type 2 SEFs that depend on an initial *stress* and a subsequent deformation gradient. Finally, in §4, we prove that restrictions on SEFs that depend on the strain and initial stress up to third-order terms are required to make them consistent with classical third-order elasticity. In appendixes B and D, we repeat the derivations of §2 and §3 in the simpler case of a one-dimensional uniaxial deformation of an incompressible material and consider examples of SEFs that do and do not satisfy the restrictions.

2. SEFs for initially strained materials

We first investigate SEFs of the form $W := W(\mathbf{F}_2, \mathbf{F}_1, \boldsymbol{X}_2)$, where $\mathbf{F}_2$ is the elastic deformation gradient from the (variable) reference configuration $\mathcal{B}_2$, $\mathbf{F}_1$ is the initial deformation gradient from the fixed configuration $\mathcal{B}_1$ to the variable configuration $\mathcal{B}_2$ and $\boldsymbol{X}_2 \in \mathcal{B}_2$ is the position vector; see figure 1 and its caption, where we define our notation.

Apart from an arbitrary translation, the value of $\mathbf{F}_1$ effectively *defines* the set of points that make up the configuration $\mathcal{B}_2$. We note that when the material is in the configuration $\mathcal{B}_1$, it may not be stress-free, and the source of this stress may be a non-elastic process such as growth or

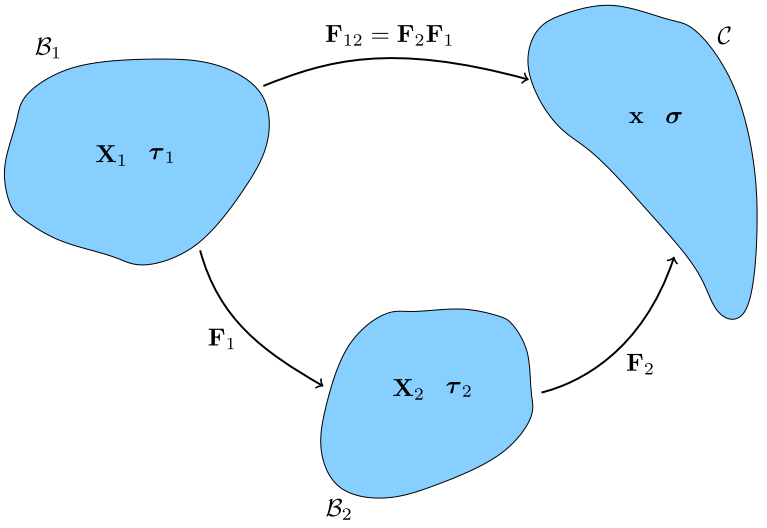

Figure 1. One material can be elastically deformed between different configurations, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{C}$. We relate the configurations through elastic maps ϕ_1, ϕ_2 and ϕ_{12}, where $\boldsymbol{X}_2 = \phi_1(\boldsymbol{X}_1)$, and $\boldsymbol{x} = \phi_2(\boldsymbol{X}_2) = \phi_{12}(\boldsymbol{X}_1)$. The deformation gradients are defined as $\mathbf{F}_1 = \partial\phi_1(\boldsymbol{X}_1)/\partial\boldsymbol{X}_1$, $\mathbf{F}_2 = \partial\phi_2(\boldsymbol{X}_2)/\partial\boldsymbol{X}_2$ and $\mathbf{F}_{12} = \partial\phi_{12}(\boldsymbol{X}_1)/\partial\boldsymbol{X}_1$, and their determinants are $J_1 = \det \mathbf{F}_1$, $J_2 = \det \mathbf{F}_2$ and $J_{12} = \det \mathbf{F}_{12} = \det(\mathbf{F}_1\mathbf{F}_2) = J_1J_2$, respectively. The Cauchy stress tensors in $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{C}$ are $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ and $\boldsymbol{\sigma}$, respectively. The set of points that makes up the configuration $\mathcal{B}_1$ is fixed, and the material can have an inhomogeneous anisotropic constitutive behaviour that depends on properties, such as the microstructure of the material, through the position vector $\boldsymbol{X}_1$. For a *given elastic material*, the stress $\boldsymbol{\tau}_1$ in $\mathcal{B}_1$ is also assumed to be fixed. In other words, changing the value of $\boldsymbol{\tau}_1$ in $\mathcal{B}_1$ without changing the set of points that make up $\mathcal{B}_1$ corresponds to changing the elastic material considered (this is the approach taken when using type 1 SEFs). In contrast, the set of points, $\boldsymbol{X}_2$, that makes up $\mathcal{B}_2$ is *defined* by the value of $\mathbf{F}_1$ (or, equivalently, the value of $\boldsymbol{\tau}_2$, which is the approach taken when using type 2 SEFs).

thermal expansion during the formation of the material. In general, this may lead to inhomogeneous, anisotropic constitutive behaviour from this configuration, and, indeed, there may be other causes of inhomogeneity and anisotropy, such as a complex material microstructure. All of this dependence is tracked via the third argument of the function W. When the third argument is evaluated at $\boldsymbol{X}_1$, it describes these properties of the material in $\mathcal{B}_1$. When it is evaluated at $\boldsymbol{X}_2$, it gives the push-forward of these properties to the configuration $\mathcal{B}_2$.

(a) Classical nonlinear elasticity

To begin our discussion, we summarize one classical way [23] to deduce the functional forms of elastic SEFs. We restrict our attention to isothermal deformations and thus neglect the influence of temperature on the potential energy for simplicity, as it does not change any of the results presented here. Let $\psi(\boldsymbol{x})$ be the stored energy density in the current configuration, $\mathcal{C}$, per unit volume of $\mathcal{C}$. Let $\phi: \boldsymbol{X} \mapsto \boldsymbol{x}$ be a map describing the elastic deformation from the reference configuration $\mathcal{R}$ to $\mathcal{C}$. We remark that the reference configuration $\mathcal{R}$ does not need to be stress-free (if it is stressed, the constitutive behaviour of the material is, in general, anisotropic and inhomogeneous, and thus varies with the initial position $\boldsymbol{X}$).

A set of classical assumptions [23, chapter 3—Constitutive Theory] implies that there exists a function $\tilde{W}$ in the form

$$J^{-1}\tilde{W}(\mathbf{F}, \boldsymbol{X}) = \psi(\boldsymbol{x}), \tag{2.1}$$

for every $\mathbf{F}$, $\boldsymbol{X}$, $\boldsymbol{x}$ and ϕ such that $\mathbf{F} = \partial\phi(\boldsymbol{X})/\partial\boldsymbol{X} = \partial\boldsymbol{x}/\partial\boldsymbol{X}$, with $J = \det \mathbf{F}$. Here, we reiterate that throughout this paper, we used the convention that tildes are always associated with a single reference configuration (and therefore cannot be used from any other configuration). Some of

the basic assumptions we use include that the elastic map is at least C^2 differentiable and that the deformation conserves energy.

The classical assumptions in [23] also imply that the Cauchy stress tensor in $\mathcal{C}$ is given by

$$\tilde{\boldsymbol{\sigma}}(\mathbf{F}, \boldsymbol{X}) = J^{-1}\frac{\partial \tilde{W}(\mathbf{F}, \boldsymbol{X})}{\partial \mathbf{F}}\mathbf{F}^{\mathrm{T}}, \tag{2.2}$$

where differentiation with respect to a tensor is defined component-wise as $(\partial/\partial\mathbf{A})_{ij} = \partial/\partial A_{ij}$, for any tensor $\mathbf{A}$ that has components A_{ij} with respect to a given basis.

Locality assumption: One assumption we want to draw attention to is *locality* (see [23, 2.2 Axiom of Locality]). It states that the energy density $\psi(\boldsymbol{x})$ depends on the map $\phi: \boldsymbol{X} \mapsto \boldsymbol{x}$ only through quantities that are defined in the neighbourhoods of $\boldsymbol{X}$ and $\boldsymbol{x}$. This assumption is essential to deduce that $\tilde{W}$ depends on the map ϕ through the deformation gradient $\mathbf{F}$.

We note that relaxing the assumptions in [23, chapter 3—Constitutive Theory] leads to other formulations such as either implicit constitutive theories [24], where the stress and strain can be related through an implicit equation, or first and second gradient theories [25], where the strain energy can depend on the gradient of the strain. Here we consider only the classical assumptions described in [23].

(b) A restriction on SEFs for initially strained materials

We start by considering SEFs of the form $W(\mathbf{F}_2, \mathbf{F}_1, \boldsymbol{X}_2)$, where $\mathbf{F}_2$ is the elastic deformation gradient associated with deformations *away from* the reference configuration $\mathcal{B}_2$, $\mathbf{F}_1$ is an initial deformation gradient associated with an elastic deformation *into* $\mathcal{B}_2$ from $\mathcal{B}_1$ and $\boldsymbol{X}_2$ is a position vector in $\mathcal{B}_2$. See figure 1 for an illustration of these configurations.

Note that, mathematically and conceptually, these SEFs are very different from classical strain energy functions, which depend on one deformation gradient only and are associated with a given reference configuration. Hence, $\tilde{W}(\mathbf{F}_1, \boldsymbol{X}_1)$ is associated with $\mathcal{B}_1$. If we say that $\tilde{W}_2(\mathbf{F}_2, \boldsymbol{X}_2)$ is the classical strain energy function associated with $\mathcal{B}_2$, then

$$\tilde{W}(\mathbf{F}_2\mathbf{F}_1, \boldsymbol{X}_1) = J_1\tilde{W}_2(\mathbf{F}_2, \phi_1(\boldsymbol{X}_1)), \tag{2.3}$$

a connection that is well established; see, for example, [26–28].

Here, we ask whether there exists a function, W, which gives the stored energy density in the current configuration, $\mathcal{C}$, per unit volume of $\mathcal{B}_2$, such that

$$J_2^{-1}W(\mathbf{F}_2, \mathbf{F}_1, \boldsymbol{X}_2) = \psi(\boldsymbol{x}), \tag{2.4}$$

for *every* deformation $\mathbf{F}_1$ from $\mathcal{B}_1$ to $\mathcal{B}_2$, and *every* deformation $\mathbf{F}_2$ from $\mathcal{B}_2$ to $\mathcal{C}$, where ψ is the potential energy stored in the current configuration, $\mathcal{C}$, per unit volume of $\mathcal{C}$.

Taking $\mathcal{B}_1$ as the reference configuration, for any elastic map ϕ_{12} and associated deformation gradient $\mathbf{F}_{12}$, the assumptions of classical elasticity imply that there exists a function $\tilde{W}$ such that

$$J_{12}^{-1}\tilde{W}(\mathbf{F}_{12}, \boldsymbol{X}_1) = J_1^{-1}J_2^{-1}\tilde{W}(\mathbf{F}_2\mathbf{F}_1, \boldsymbol{X}_1) = \psi(\boldsymbol{x}), \tag{2.5}$$

where $\tilde{W}$ is the stored energy per unit volume of $\mathcal{B}_1$ (note that $\mathcal{B}_1$ could be under stress). We emphasize that the above-given equation holds true for every $\mathbf{F}_1$ and $\mathbf{F}_2$ and that the function $\tilde{W}$ is strictly associated with the reference configuration $\mathcal{B}_1$. By comparing (2.4) and (2.5), we conclude that, as (2.5) holds for every $\mathbf{F}_1$ and $\mathbf{F}_2$, we can define

$$W(\mathbf{F}_2, \mathbf{F}_1, \boldsymbol{X}_2) := J_1^{-1}\tilde{W}(\mathbf{F}_2\mathbf{F}_1, \phi_1^{-1}(\boldsymbol{X}_2)), \tag{2.6}$$

for every $\mathbf{F}_2$, $\mathbf{F}_1$ and $\boldsymbol{X}_2$. The above uniquely defines W, given $\tilde{W}$, because ϕ_1 is invertible and (2.4) holds for every value of $\mathbf{F}_2$, $\mathbf{F}_1$ and $\boldsymbol{X}_2$.

Equation (2.6) naturally restricts the form of the function W, as follows. First note that equation (2.6) holds for *any* values of the arguments of W, so we can set the first argument to be $\mathbf{F}_{12}$ (i.e. $\mathbf{F}_2 \to \mathbf{F}_{12}$) and the second argument to be the identity tensor $\mathbf{I}$ (i.e. $\mathbf{F}_1 \to \mathbf{I}$). Hence, we are considering the initial map to be the trivial case of no initial deformation, which implies that $\boldsymbol{X}_2 \to \boldsymbol{X}_1$, and, on the right side $J_1^{-1} \to 1$, $\mathbf{F}_2\mathbf{F}_1 \to \mathbf{F}_{12}\mathbf{I}$ and $\phi_1^{-1}(\boldsymbol{X}_2) \to \boldsymbol{X}_1$. Using these substitutions in (2.6) leads to

$$W(\mathbf{F}_{12}, \mathbf{I}, \boldsymbol{X}_1) = W(\mathbf{F}_2\mathbf{F}_1, \mathbf{I}, \boldsymbol{X}_1) = \tilde{W}(\mathbf{F}_{12}\mathbf{I}, \boldsymbol{X}_1) = \tilde{W}(\mathbf{F}_2\mathbf{F}_1, \boldsymbol{X}_1)\,. \tag{2.7}$$

From (2.6), we have that $\tilde{W}(\mathbf{F}_2\mathbf{F}_1, \boldsymbol{X}_1) = J_1 W(\mathbf{F}_2, \mathbf{F}_1, \boldsymbol{X}_2)$, which used in the above leads to the restriction

$$W(\mathbf{F_2F_1}, \mathbf{I}, \boldsymbol{X_1}) = J_1 W(\mathbf{F_2}, \mathbf{F_1}, \boldsymbol{X_2}), \tag{2.8}$$

which must hold for every $\mathbf{F}_1$, $\mathbf{F}_2$, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$. Let us call this restriction *Initial Strain Reference Independence*. See appendix A for an alternative derivation of this restriction.

Next, we use equation (2.8) to derive a restriction on the Cauchy stress. From (2.6), we can rewrite the Cauchy stress functional (2.2), now as a function of three variables, in the form

$$\boldsymbol{\sigma}(\mathbf{F}_2, \mathbf{F}_1, \boldsymbol{X}_2) := J_2^{-1} \frac{\partial W(\mathbf{F}_2, \mathbf{F}_1, \boldsymbol{X}_2)}{\partial \mathbf{F}_2} \mathbf{F}_2^{\mathrm{T}} = \tilde{\boldsymbol{\sigma}}(\mathbf{F}_{12}, \boldsymbol{X}_1)\,. \tag{2.9}$$

To prove the latter equality, we use the chain rule as follows:

$$J_2^{-1} \frac{\partial W(\mathbf{F}_2, \mathbf{F}_1, \mathbf{X}_2)}{\partial \mathbf{F}_2} \mathbf{F}_2^{\mathrm{T}} = J_2^{-1} \left[J_1^{-1} \frac{\partial \tilde{W}(\mathbf{F}_{12}, \mathbf{X}_1)}{\partial \mathbf{F}_{12}} : \frac{\partial \mathbf{F}_{12}}{\partial \mathbf{F}_2} \right] \mathbf{F}_2^{\mathrm{T}} = J_{12}^{-1} \frac{\partial \tilde{W}(\mathbf{F}_{12}, \boldsymbol{X}_1)}{\partial \mathbf{F}_{12}} \mathbf{F}_{12}^{\mathrm{T}} = \tilde{\boldsymbol{\sigma}}(\mathbf{F}_{12}, \boldsymbol{X}_1), \tag{2.10}$$

where the double-contraction operator is defined such that $(\mathbf{A}:\mathbf{B})_{kl} = A_{ij}B_{ijkl}$, for second- and fourth-order tensors $\mathbf{A}$ and $\mathbf{B}$, with components A_{ij} and B_{ijkl}, respectively.

By substituting equation (2.8) into equation (2.9) and making use of some of the results from (2.10), we obtain the following restriction on the Cauchy stress functional:

$$\boldsymbol{\sigma}(\mathbf{F_2F_1}, \mathbf{I}, \boldsymbol{X_1}) = \boldsymbol{\sigma}(\mathbf{F_2}, \mathbf{F_1}, \boldsymbol{X_2}), \tag{2.11}$$

for every $\mathbf{F}_1$, $\mathbf{F}_2$, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$.

In this section, we deduced that, for all materials that satisfy the classical assumptions of elasticity, as described in §2a, there exists a function W that satisfies (2.8) for every initial deformation gradient $\mathbf{F}_1$ and subsequent elastic deformation gradient $\mathbf{F}_2$.

3. SEFs for initially stressed materials

In this section, we move from SEFs of the form $W(\mathbf{F}_2, \mathbf{F}_1, \boldsymbol{X}_2)$ to SEFs of the form $\overline{W}(\mathbf{F}_2, \boldsymbol{\tau}_2, \boldsymbol{X}_2)$, where $\boldsymbol{\tau}_2$ is the initial stress in $\mathcal{B}_2$ (see figure 1). We show that the Initial Strain Reference Independence restriction on W can be used to derive a restriction on $\overline{W}$ by connecting $\boldsymbol{\tau}_2$ to an initial stretch tensor $\mathbf{U}_1$, corresponding to a deformation from a (potentially virtual) configuration $\mathcal{B}_1$ to $\mathcal{B}_2$. We find that, for an isotropic material, this restriction is equivalent to that introduced in [21], which we called ISRI.

(a) Initial stress from initial strain

Using the polar decomposition theorem [29], we can write $\mathbf{F}_1$ uniquely as $\mathbf{F}_1 = \mathbf{R}_1\mathbf{U}_1$, where $\mathbf{U}_1 = \mathbf{U}_1^{\mathrm{T}}$ is the right stretch tensor and $\mathbf{R}_1$ is a proper rotation tensor. Combining the polar decomposition with (2.2), it can be shown [2] that

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$$\tilde{\boldsymbol{\sigma}}(\mathbf{F}_1, \boldsymbol{X}_1) = \mathbf{R}_1\tilde{\boldsymbol{\sigma}}(\mathbf{U}_1, \boldsymbol{X}_1)\mathbf{R}_1^{\mathrm{T}}, \quad \text{where } \tilde{\boldsymbol{\sigma}}(\mathbf{U}_1, \boldsymbol{X}_1) := J_1^{-1}\frac{\partial\tilde{W}(\mathbf{U}_1, \boldsymbol{X}_1)}{\partial\mathbf{U}_1}\mathbf{U}_1, \tag{3.1}$$

by using the fact that $\tilde{W}(\mathbf{R}_1\mathbf{U}_1, \boldsymbol{X}_1) = \tilde{W}(\mathbf{U}_1, \boldsymbol{X}_1)$ due to objectivity.

To define an appropriate form for the SEF, $\overline{W}(\mathbf{F}_2, \boldsymbol{\tau}_2, \boldsymbol{X}_2)$, we need, for any stress tensor $\boldsymbol{\tau}_2$ and position vector $\boldsymbol{X}_2$ in $\mathcal{B}_2$, a function $\tilde{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\tau}_2, \boldsymbol{X}_2) = \mathbf{U}_1$ that gives the stretch tensor $\mathbf{U}_1$ that led to $\boldsymbol{\tau}_2$ from $\mathcal{B}_1$. In appendix C, we explain why an inverse function $\tilde{\boldsymbol{\sigma}}^{-1}$ should exist and can be uniquely defined for a wide range of constitutive choices (those that are stable under traction). However, to simplify the discussion below, we simply *assume* that such an inverse function exists. An example of an inverse function $\tilde{\boldsymbol{\sigma}}^{-1}$ is given for third-order energies in §4b by equation (4.18).

As another example for which an inverse function does exist, consider the stress–stretch relation for an incompressible neo-Hookean material: $\boldsymbol{\tau}_2 = -p\boldsymbol{I} + \mu\boldsymbol{F}_1\boldsymbol{F}_1^{\mathrm{T}} = \boldsymbol{R}_1\tilde{\boldsymbol{\sigma}}(\boldsymbol{U}_1)\boldsymbol{R}_1^{\mathrm{T}}$, where $\tilde{\boldsymbol{\sigma}} = -p\boldsymbol{I} + \mu\boldsymbol{U}_1^2$ (p is a Lagrange multiplier, and μ is the shear modulus). This equation can be inverted to find $\boldsymbol{U}_1 = \tilde{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\tau}_2, \boldsymbol{X}_2) = [(\boldsymbol{\tau}_2 + p_0\boldsymbol{I})/\mu]^{1/2}$, where p_0 is expressed in terms of $\boldsymbol{\tau}_2$ (see [22, §3] for details). A similar inversion to find $\boldsymbol{U}_1^2$ in terms of $\boldsymbol{\tau}_2$ can be achieved for the Mooney–Rivlin model [30,31] and, indeed, for any hyperelastic, isotropic model [31], so that $\boldsymbol{U}_1$, the square root of $\boldsymbol{U}_1^2$, can be expressed explicitly [32].

Using the inverse function, $\tilde{\boldsymbol{\sigma}}^{-1}$, we can define an SEF, $\overline{W}$, that depends on strain and initial stress, as

$$\overline{W}(\mathbf{F}_2, \boldsymbol{\tau}_2, \boldsymbol{X}_2) := W(\mathbf{F}_2, \tilde{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\tau}_2, \boldsymbol{X}_2), \boldsymbol{X}_2). \tag{3.2}$$

We then define a new functional for the Cauchy stress, which now depends on the initial *stress* $\boldsymbol{\tau}_2$ instead of the initial *strain* $\mathbf{F}_1$, as follows:

$$\boldsymbol{\sigma}_{\tau}(\mathbf{F}_2, \boldsymbol{\tau}_2, \boldsymbol{X}_2) := J_2^{-1}\frac{\partial\overline{W}(\mathbf{F}_2, \boldsymbol{\tau}_2, \boldsymbol{X}_2)}{\partial\mathbf{F}_2}\mathbf{F}_2^{\mathrm{T}} = \boldsymbol{\sigma}(\mathbf{F}_2, \tilde{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\tau}_2, \boldsymbol{X}_2), \boldsymbol{X}_2), \tag{3.3}$$

where $\boldsymbol{\sigma}$ is defined in equation (2.9); the second equality follows from equation (3.2).

We are now ready to prove a restriction on $\overline{W}$. In equation (3.3), which is valid for any values of its arguments, we let $\mathbf{F}_2 \to \mathbf{U}_1$, $\boldsymbol{\tau}_2 \to \boldsymbol{\tau}_1$ and $\boldsymbol{X}_2 \to \boldsymbol{X}_1$ to obtain

$$\boldsymbol{\sigma}_{\tau}(\mathbf{U}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1) = \boldsymbol{\sigma}(\mathbf{U}_1, \tilde{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\tau}_1, \boldsymbol{X}_1), \boldsymbol{X}_1) = \boldsymbol{\sigma}(\mathbf{U}_1, \mathbf{I}, \boldsymbol{X}_1) = \tilde{\boldsymbol{\sigma}}(\mathbf{U}_1, \boldsymbol{X}_1), \tag{3.4}$$

where the latter equality follows from equation (2.9). We substitute this equation into equation (3.2) to obtain

$$J_1\overline{W}(\mathbf{F}_2, \boldsymbol{\sigma}_{\tau}(\mathbf{U}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1), \boldsymbol{X}_2) = J_1\overline{W}(\mathbf{F}_2, \tilde{\boldsymbol{\sigma}}(\mathbf{U}_1, \boldsymbol{X}_1), \boldsymbol{X}_2) = J_1W(\mathbf{F}_2, \mathbf{U}_1, \boldsymbol{X}_2). \tag{3.5}$$

We can rewrite the last term above by using (2.8) followed by (3.2) to obtain

$$J_1W(\mathbf{F}_2, \mathbf{U}_1, \boldsymbol{X}_2) = W(\mathbf{F}_2\mathbf{U}_1, \mathbf{I}, \boldsymbol{X}_1) = \overline{W}(\mathbf{F}_2\mathbf{U}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1). \tag{3.6}$$

Finally, equating the last term of (3.6) with the first term in (3.5) leads to

$$\overline{W}(\mathbf{F}_2\mathbf{U}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1) = J_1\overline{W}(\mathbf{F}_2, \boldsymbol{\sigma}_{\tau}(\mathbf{U}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1), \boldsymbol{X}_2), \tag{3.7}$$

which holds for every $\mathbf{F}_2$ and $\mathbf{U}_1$.

Differentiating both sides of equation (3.7) with respect to $\mathbf{F}_2$ and using (3.3) and (2.9), we obtain

$$\boldsymbol{\sigma}_{\tau}(\mathbf{F}_2\mathbf{U}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1) = \boldsymbol{\sigma}_{\tau}(\mathbf{F}_2, \boldsymbol{\sigma}_{\tau}(\mathbf{U}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1), \boldsymbol{X}_2), \tag{3.8}$$

for every $\mathbf{F}_2$ and $\mathbf{U}_1$.

(b) Material symmetry

We now specialize the results given above to the case where there is no underlying *structural* anisotropy in the material (so that the material would be isotropic in the absence of initial stress). In this case, the stored energy does not change when the reference configuration of $\mathbf{F}_1$ is rotated, which leads to

$$\overline{W}(\mathbf{F}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1) = \overline{W}(\mathbf{F}_1\mathbf{R}_1^{\mathrm{T}}, \mathbf{R}_1\boldsymbol{\tau}_1\mathbf{R}_1^{\mathrm{T}}, \boldsymbol{X}_1), \tag{3.9}$$

for every $\mathbf{F}_1$, $\boldsymbol{\tau}_1$ and $\boldsymbol{X}_1$, where $\mathbf{R}_1$ is an orthogonal tensor that satisfies $\mathbf{F}_1 = \mathbf{U}_1\mathbf{R}_1^{\mathrm{T}}$. Using this result in equation (3.3), we also have that

$$\boldsymbol{\sigma}_\tau(\mathbf{F}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1) = \boldsymbol{\sigma}_\tau(\mathbf{F}_1\mathbf{R}_1^{\mathrm{T}}, \mathbf{R}_1\boldsymbol{\tau}_1\mathbf{R}_1^{\mathrm{T}}, \boldsymbol{X}_1), \tag{3.10}$$

for every $\mathbf{F}_1$, $\boldsymbol{\tau}_1$, $\boldsymbol{X}_1$ and orthogonal $\mathbf{R}_1$. By defining $\boldsymbol{\tau}_R = \mathbf{R}_1^{\mathrm{T}}\boldsymbol{\tau}_1\mathbf{R}_1$ and using equations (3.9) and (3.10), we obtain

$$\overline{W}(\mathbf{F}_2\mathbf{U}_1, \boldsymbol{\tau}_R, \boldsymbol{X}_1) = \overline{W}(\mathbf{F}_2\mathbf{U}_1\mathbf{R}_1^{\mathrm{T}}, \mathbf{R}_1\boldsymbol{\tau}_R\mathbf{R}_1^{\mathrm{T}}, \boldsymbol{X}_1), \tag{3.11}$$

$$J_1\overline{W}(\mathbf{F}_2, \boldsymbol{\sigma}_\tau(\mathbf{U}_1, \boldsymbol{\tau}_R, \boldsymbol{X}_1), \boldsymbol{X}_2) = J_1\overline{W}(\mathbf{F}_2, \boldsymbol{\sigma}_\tau(\mathbf{U}_1\mathbf{R}_1^{\mathrm{T}}, \mathbf{R}_1\boldsymbol{\tau}_R\mathbf{R}_1^{\mathrm{T}}, \boldsymbol{X}_1), \boldsymbol{X}_2). \tag{3.12}$$

The terms on the left side of both equations are equal due to equation (3.7); therefore, the terms on the right side are also equal. Equating the terms on the right and using $\mathbf{F}_1 = \mathbf{U}_1\mathbf{R}_1^{\mathrm{T}}$ and $\mathbf{R}_1\boldsymbol{\tau}_R\mathbf{R}_1^{\mathrm{T}} = \boldsymbol{\tau}_1$ leads to

$$\overline{W}(\mathbf{F}_2\mathbf{F}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1) = J_1\overline{W}(\mathbf{F}_2, \boldsymbol{\sigma}_\tau(\mathbf{F}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1), \boldsymbol{X}_2), \tag{3.13}$$

for every $\mathbf{F}_2$ and $\mathbf{F}_1$. Differentiating both sides of equation (3.13) with respect to $\mathbf{F}_2$ and using (3.3) and (2.9), we obtain

$$\boldsymbol{\sigma}_\tau(\mathbf{F}_2\mathbf{F}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1) = \boldsymbol{\sigma}_\tau(\mathbf{F}_2, \boldsymbol{\sigma}_\tau(\mathbf{F}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1), \boldsymbol{X}_2). \tag{3.14}$$

We recall that $\boldsymbol{\tau}_1$ is fixed in the equations above; however, the restrictions can be generalized to a wider context. To do so, we make the substitutions $\mathbf{F}_2 \to \mathbf{F}_3$, followed by $\mathbf{F}_1 \to \mathbf{F}_1\mathbf{F}_2$ (we now have three reference configurations, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$: $\mathbf{F}_1$ maps $\mathcal{B}_1 \mapsto \mathcal{B}_2$, $\mathbf{F}_2$ maps $\mathcal{B}_2 \mapsto \mathcal{B}_3$, and $\mathbf{F}_3$ maps $\mathcal{B}_3 \mapsto \mathcal{C}$) in equation (3.13), and use the equation above to obtain

$$\overline{W}(\mathbf{F}_3\mathbf{F}_2\mathbf{F}_1, \boldsymbol{\tau}_1, \boldsymbol{X}_1) = J_1J_2\,\overline{W}(\mathbf{F}_3, \boldsymbol{\sigma}_\tau(\mathbf{F}_2, \boldsymbol{\tau}_2, \boldsymbol{X}_2), \boldsymbol{X}_3), \tag{3.15}$$

which is valid for every choice of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$, and $\boldsymbol{\tau}_2$ (note $\boldsymbol{\tau}_2$ can be determined from $\mathbf{F}_1$). Finally, again using equation (3.13) on the left side of equation (3.15) and dividing through by J_1 leads to

$$\overline{W}(\mathbf{F}_3\mathbf{F}_2, \boldsymbol{\tau}_2, \boldsymbol{X}_2) = J_2\overline{W}(\mathbf{F}_3, \boldsymbol{\sigma}_\tau(\mathbf{F}_2, \boldsymbol{\tau}_2, \boldsymbol{X}_2), \boldsymbol{X}_3), \quad \text{(ISRI)}, \tag{3.16}$$

which is valid for every $\mathbf{F}_2$, $\mathbf{F}_3$ and $\boldsymbol{\tau}_2$. Now, even though stress $\boldsymbol{\tau}_1$ in $\mathcal{B}_1$ is fixed, the above-given equation holds for every initial stress $\boldsymbol{\tau}_2$.

This form of ISRI is the same as in [21] (in a slightly different form). Similarly, in this case, equation (3.8) can be rewritten as

$$\boldsymbol{\sigma}_\tau(\mathbf{F}_3\mathbf{F}_2, \boldsymbol{\tau}_2, \boldsymbol{X}_2) = \boldsymbol{\sigma}_\tau(\mathbf{F}_3, \boldsymbol{\sigma}_\tau(\mathbf{F}_2, \boldsymbol{\tau}_2, \boldsymbol{X}_2), \boldsymbol{X}_3), \quad \text{(Initial Stress Symmetry)}, \tag{3.17}$$

which was first introduced (again in a slightly different form) in [22].

We have thus demonstrated that for type 2 SEFs of the form $\overline{W}(\mathbf{F}, \boldsymbol{\tau}, \boldsymbol{X})$, the ISRI and initial stress symmetry restrictions are natural consequences of the classical assumptions of nonlinear elasticity.

4. Third-order SEFs

In this section, we discuss SEFs that depend on strain and stress up to the third order in an expansion for small values of these arguments. We show that these SEFs are only consistent with classical third-order strain energy functions when imposing the restriction (2.8) or the restriction (3.13).

In classical third-order elasticity theory, the strain energy function $\tilde{W}$ is expanded in powers of the strain. For example, if we consider the deformation from $\mathcal{B}_1$ to $\mathcal{C}$ in figure 1, this time, assuming the deformation is infinitesimal, then the strain energy function can be written as

$$\tilde{W}_{\mathbf{E}}(\mathbf{E}_{12}) = \frac{\lambda}{2}(\operatorname{tr}\mathbf{E}_{12})^2 + \mu \operatorname{tr}\mathbf{E}_{12}^2 + \frac{A}{3}\operatorname{tr}\mathbf{E}_{12}^3 + B\operatorname{tr}\mathbf{E}_{12}\operatorname{tr}\mathbf{E}_{12}^2 + \frac{C}{3}(\operatorname{tr}\mathbf{E}_{12})^3, \tag{4.1}$$

where λ, μ are the Lamé parameters, A, B, C are the Landau parameters and $\mathbf{E}_{12} = \frac{1}{2}\left(\mathbf{F}_{12}^{\mathrm{T}}\mathbf{F}_{12} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{F}_1^{\mathrm{T}}\mathbf{F}_2^{\mathrm{T}}\mathbf{F}_2\mathbf{F}_1 - \mathbf{I}\right)$ is the Green–Lagrange strain tensor. Note that there are other, equivalent, expansions [33], all involving five independent elastic constants. In this section, we assume that $\mathbf{E}_{12}$ is small, so that the expansion (4.1) is valid up to $\mathcal{O}(\mathbf{E}_{12}^3)$. We also assume that the material is homogeneous in $\mathcal{B}_1$, so we can omit the dependence on the position vectors, and that $\mathcal{B}_1$ is stress-free, so that $\boldsymbol{\tau}_1 = \mathbf{0}$. The strain energy function (4.1) is deduced by assuming that $\mathcal{B}_1$ is stress-free, so we make this assumption throughout this section for consistency; however, we reiterate that the restrictions (2.8) and (3.7) do not assume that $\mathcal{B}_1$ is stress-free.

For third-order elasticity [34,35], the most common way to understand how an initial stress $\boldsymbol{\tau}$ *affects* an elastic deformation is to consider an initial stress that is *due* to an elastic deformation. That is, to assume $\boldsymbol{\tau} = \tilde{\boldsymbol{\sigma}}(\mathbf{F}_1)$, and then take a further elastic deformation $\mathbf{F}_2$ on top of this stressed state. Following this route, we can define the initially stressed SEF $\overline{W}$ such that, for all values of $\mathbf{F}_2$ and $\boldsymbol{\tau} = \tilde{\boldsymbol{\sigma}}(\mathbf{F}_1)$, it satisfies the following equation:

$$J_1\overline{W}(\mathbf{F}_2, \boldsymbol{\tau}) = \tilde{W}_{\mathbf{E}}(\mathbf{E}_{12}), \tag{4.2}$$

where $\tilde{W}_{\mathbf{E}}$ is given by equation (4.1). The previous sections of this paper have demonstrated that SEFs of the form $\overline{W}(\mathbf{F}_2, \boldsymbol{\tau})$ must satisfy the ISRI restriction (3.16) when the material is isotropic in $\mathcal{B}_1$. For clarity, we repeat those steps here in the case where $\mathbf{F}_1 = \mathbf{I}$, so that $\boldsymbol{\tau} = \mathbf{0}$ and (4.2) becomes $\overline{W}(\mathbf{F}_2, \mathbf{0}) = \tilde{W}(\mathbf{E}_2)$ for every $\mathbf{F}_2$, where $\mathbf{E}_2 = \frac{1}{2}(\mathbf{F}_2^{\mathrm{T}}\mathbf{F}_2 - \mathbf{I})$. In this case, $\mathbf{F}_2 = \mathbf{F}_2\mathbf{F}_1 = \mathbf{F}_{12}$, which implies that $\overline{W}(\mathbf{F}_2\mathbf{F}_1, \mathbf{0}) = \overline{W}(\mathbf{F}_{12}, \mathbf{0}) = \tilde{W}(\mathbf{E}_{12})$, and, therefore, $J_1\overline{W}(\mathbf{F}_2, \boldsymbol{\tau}) = \overline{W}(\mathbf{F}_{12}, \mathbf{0})$, which is equivalent to the restriction (3.13) in the case where $\mathbf{F}_1 = \mathbf{I}$. That is, an initially stressed SEF defined via equation (4.2) automatically satisfies the ISRI restriction.

Below, we show that, without the ISRI restrictions, if we take a naive expansion of the SEF in terms of its invariants, we can obtain third-order SEFs that are not equivalent to (4.1) and have too many free parameters. The ISRI restrictions are necessary to resolve this inconsistency.

In the following, we undertake the asymptotic expansions in order of growing complexity, starting first from an initially *strained* third-order energy function and later dealing with an initially *stressed* third-order energy function.

(a) An initially strained, third-order SEF

Here, we deduce a third-order SEF of the form $W(\mathbf{F}_2, \mathbf{F}_1) = W_{\mathbf{e}}(\mathbf{E}, \mathbf{e})$, a function of the small initial strain $\mathbf{e} = \frac{1}{2}(\mathbf{F}_1\mathbf{F}_1^{\mathrm{T}} - \mathbf{I})$ and the small subsequent strain $\mathbf{E} = \frac{1}{2}(\mathbf{F}_2^{\mathrm{T}}\mathbf{F}_2 - \mathbf{I})$.

Let us expand $W_{\mathbf{e}}(\mathbf{F}_2, \mathbf{F}_1)$ up to third order. Since this stored energy density depends only on $\mathbf{E}$ and $\mathbf{e}$, we can write it in terms of the nine independent mixed invariants of $\mathbf{E}$ and $\mathbf{e}$ [36,37]

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$$I_1 = \mathrm{tr}(\mathbf{E}), \qquad I_2 = \mathrm{tr}(\mathbf{E}^2), \qquad I_3 = \mathrm{tr}(\mathbf{E}^3),$$
$$i_4 = \mathrm{tr}(\mathbf{e}), \qquad i_5 = \mathrm{tr}(\mathbf{e}^2), \qquad i_6 = \mathrm{tr}(\mathbf{e}^3),$$
$$i_7 = \mathrm{tr}(\mathbf{eE}), \qquad i_8 = \mathrm{tr}(\mathbf{e}^2\mathbf{E}), \qquad i_9 = \mathrm{tr}(\mathbf{eE}^2). \tag{4.3}$$

The Cauchy stress tensor arising from $W_{\mathbf{e}}(\mathbf{E}, \mathbf{e})$ can be written as

$$\boldsymbol{\sigma}(\mathbf{F}_2, \mathbf{F}_1) = J_2^{-1}\mathbf{F}_2\left(\frac{\partial W_{\mathbf{e}}}{\partial I_1}\mathbf{I} + 2\frac{\partial W_{\mathbf{e}}}{\partial I_2}\mathbf{E} + 3\frac{\partial W_{\mathbf{e}}}{\partial I_3}\mathbf{E}^2 + \frac{\partial W_{\mathbf{e}}}{\partial i_7}\mathbf{e} + \frac{\partial W_{\mathbf{e}}}{\partial i_8}\mathbf{e}^2 + \frac{\partial W_{\mathbf{e}}}{\partial i_9}(\mathbf{Ee} + \mathbf{eE})\right)\mathbf{F}_2^{\mathrm{T}}. \tag{4.4}$$

The next step is to assume that both $\mathbf{e}$ and $\mathbf{E}$ are small and to expand $W_{\mathbf{e}}(\mathbf{E}, \mathbf{e})$ asymptotically up to $\mathcal{O}(\mathbf{E}^3)$. Here, we choose $\mathbf{e} = \mathcal{O}(\mathbf{E})$, but we note that if we assumed, for example, that $\mathbf{e}$ is asymptotically smaller, e.g. $\mathbf{e} = \mathcal{O}(\mathbf{E}^2)$, or larger, e.g. $\mathbf{e} = \mathcal{O}(\mathbf{E}^{\frac{1}{2}})$, than $\mathbf{E}$, then the calculations below would be different, but the main result would be unchanged. Namely, without imposing ISRI, $W_{\mathbf{e}}(\mathbf{E}, \mathbf{e})$ has too many free constants, whereas only five independent constants remain after imposing ISRI, consistently with classical third-order elasticity.

Letting $\mathbf{e} = O(\mathbf{E})$, we now follow a systematic expansion method [38] to expand $W_{\mathbf{e}}(\mathbf{E}, \mathbf{e})$ up to $O(\mathbf{E}^3)$ to obtain

$$\begin{aligned} W_{\mathbf{e}}(\mathbf{E}, \mathbf{e}) = {} & \alpha_0 I_1 + \alpha_1 I_1^2 + \alpha_2 I_2 + \alpha_3 I_3 + \alpha_4 I_1 I_2 + \alpha_5 I_1^3 + \alpha_6 i_4 + \alpha_7 i_4^2 + \alpha_8 i_5 + \alpha_9 i_6 + \alpha_{10} i_4 i_5 + \alpha_{11} i_4^3 \\ & + \alpha_{12} i_9 + \alpha_{13} I_1^2 i_4 + \alpha_{14} I_1 i_7 + \alpha_{15} I_2 i_4 + \alpha_{16} i_7 + \alpha_{17} I_1 i_4 + \alpha_{18} I_1 i_4^2 + \alpha_{19} I_1 i_5 + \alpha_{20} i_4 i_7 + \alpha_{21} i_8, \end{aligned} \tag{4.5}$$

where α_j, for $j = 0, \ldots, 21$), are constants. Note that the requirement $\boldsymbol{\sigma}(\mathbf{I}, \mathbf{I}) = \mathbf{0}$ implies that $\alpha_0 = 0$. This leaves a total of 21 free constants, whereas the strain energy function of classical third-order elasticity (4.1) has only five independent constants. These two SEFs are expanded to the same asymptotic order and account for the same quantities, yet have a difference of 16 degrees of freedom, which is clearly inconsistent. This simple observation highlights the *need for a restriction* to be imposed on SEFs of the form $W_{\mathbf{e}}(\mathbf{E}, \mathbf{e})$. This restriction is provided by (2.8).

The restriction (2.8) can be written in terms of the quantities introduced in this section as

$$W_{\mathbf{e}}(\mathbf{E}_{12}, \mathbf{0}) = J_1 W_{\mathbf{e}}(\mathbf{E}, \mathbf{e}), \tag{4.6}$$

for every $\mathbf{E}$ and $\mathbf{e}$. The left side is

$$W_{\mathbf{e}}(\mathbf{E}_{12}, \mathbf{0}) = \alpha_0 \mathrm{tr}\mathbf{E}_{12} + \alpha_1(\mathrm{tr}\mathbf{E}_{12})^2 + \alpha_2 \mathrm{tr}\mathbf{E}_{12}^2 + \alpha_3 \mathrm{tr}\mathbf{E}_{12}^3 + \alpha_4 \mathrm{tr}\mathbf{E}_{12}\mathrm{tr}\mathbf{E}_{12}^2 + \alpha_5(\mathrm{tr}\mathbf{E}_{12})^3, \tag{4.7}$$

which we can rewrite in terms of the mixed invariants of $\mathbf{E}$ and $\mathbf{e}$. Assuming that $\mathbf{E}$ is small and that $\mathbf{e} = \mathcal{O}(\mathbf{E})$, we find

$$\begin{aligned} & \mathrm{tr}\,\mathbf{E}_{12} = \mathrm{tr}\,\mathbf{E} + \mathrm{tr}\,\mathbf{e} + 2\,\mathrm{tr}\,(\mathbf{eE}) = I_1 + i_4 + 2i_7, \\ & \mathrm{tr}\,(\mathbf{E}_{12}^2) \approx \mathrm{tr}\,(\mathbf{E}^2) + \mathrm{tr}\,(\mathbf{e}^2) + 2\,\mathrm{tr}\,(\mathbf{eE}) + 4\,\mathrm{tr}\,(\mathbf{e}^2\mathbf{E}) + 4\,\mathrm{tr}\,(\mathbf{eE}^2) = I_2 + i_5 + 2i_7 + 4i_8 + 4i_9, \\ & \mathrm{tr}\,\mathbf{E}_{12}^3 \approx \mathrm{tr}\,(\mathbf{E}^3) + \mathrm{tr}\,(\mathbf{e}^3) + 3\,\mathrm{tr}\,(\mathbf{e}^2\mathbf{E}) + 3\,\mathrm{tr}\,(\mathbf{eE}^2) = I_3 + i_6 + 3i_8 + 3i_9, \end{aligned} \tag{4.8}$$

where the first equality is exact and the other two are correct up to the third order in the strain measures. We then expand J_1 as

$$J_1 = \det\mathbf{F}_1 = \sqrt{\det(\mathbf{I} + 2\mathbf{e})} = 1 + \mathrm{tr}\,\mathbf{e} + \tfrac{1}{2}(\mathrm{tr}\mathbf{e})^2 - \mathrm{tr}\mathbf{e}^2 + O(\mathbf{e}^3) = 1 + i_4 + \tfrac{1}{2}i_4^2 - i_5 + O(\mathbf{e}^3), \tag{4.9}$$

substitute this expression into the right side of equation (4.6), neglect $O(\mathbf{E}^4)$ terms and compare the coefficients multiplying each linearly independent combination of the invariants. This results in the following set of equations:

$$\begin{aligned} & \alpha_6 = 0, \quad \alpha_7 = \alpha_1, \quad \alpha_8 = \alpha_2, \quad \alpha_9 = \alpha_3, \quad \alpha_{10} = \alpha_4 - \alpha_2, \quad \alpha_{11} = \alpha_5 - \alpha_1, \quad \alpha_{12} = 4\alpha_2 + 3\alpha_3, \\ & \alpha_{13} = 3\alpha_5 - \alpha_1, \quad \alpha_{14} = 2(2\alpha_1 + \alpha_4), \quad \alpha_{15} = \alpha_4 - \alpha_2, \quad \alpha_{16} = 2\alpha_2, \quad \alpha_{17} = 2\alpha_1, \quad \alpha_{18} = 3\alpha_5 - 2\alpha_1, \\ & \alpha_{19} = \alpha_4, \quad \alpha_{20} = 2(2\alpha_1 - \alpha_2 + \alpha_4), \quad \alpha_{21} = 4\alpha_2 + 3\alpha_3. \end{aligned} \tag{4.10}$$

A Mathematica file is provided as electronic supplementary material to verify these calculations. These 16 non-trivial equations reduce the number of free parameters down to five, which we can write in terms of classical third-order constants. To rewrite them in terms of the Lamé and Landau parameters, we simply set $\tilde{W}(\mathbf{E}_{12}) = W_{\mathbf{e}}(\mathbf{E}_{12}, \mathbf{0})$ to find

$$\alpha_1 = \frac{\lambda}{2}, \quad \alpha_2 = \mu, \quad \alpha_3 = \frac{A}{3}, \quad \alpha_4 = B, \quad \alpha_5 = \frac{C}{3}. \tag{4.11}$$

For completeness, with the parameters relabelled as above, the SEF is given by

$$W_{\mathbf{e}}(\mathbf{E}, \mathbf{e}) = \frac{\lambda}{2}I_1^2 + \mu I_2 + \frac{A}{3}I_3 + BI_1I_2 + \frac{C}{3}I_1^3 + \frac{\lambda}{2}(1 - i_4)i_4^2 + \frac{C}{3}i_4^3 + \mu i_5 + \frac{A}{3}i_6 + (B - \mu)i_4i_5 + (4\mu + A)i_9 +$$
$$\left(C - \frac{\lambda}{2}\right)I_1^2i_4 + 2(\lambda + B)I_1i_7 + (B - \mu)I_2i_4 + 2\mu i_7 + \lambda I_1i_4 + (C - \lambda)I_1i_4^2 + BI_1i_5 + 2(\lambda - \mu + B)i_4i_7 + (4\mu + A)i_8\,. \tag{4.12}$$

This section proves that using an expansion of the SEF in terms of the invariants of $\mathbf{E}$ and $\mathbf{e}$, as shown in equation (4.5), leads to too many degrees of freedom. This is expected because, as shown in §2b, SEFs of the form $W_{\mathbf{e}}(\mathbf{E}, \mathbf{e})$ need to satisfy the restriction (2.8). Thus, we have shown that the restriction (2.8) naturally implies that all third-order expansions of $W_{\mathbf{e}}(\mathbf{E}, \mathbf{e})$ are equivalent to the classical third-order elastic strain energy function given in equation (4.1).

Next, we consider third-order expansions of SEFs of the form $\overline{W}(\mathbf{F}_2, \boldsymbol{\tau}) = \widehat{W}_{\boldsymbol{\tau}}(\mathbf{E}, \boldsymbol{\tau})$. Again, we show that a restriction is needed in the form of (3.13) to reduce the number of free parameters and make this initially stressed SEF equivalent to a classical third-order strain energy function.

(b) An initially stressed, third-order SEF

Let us consider a third-order expansion of an SEF of the form $\widehat{W}_{\boldsymbol{\tau}}(\mathbf{E}, \boldsymbol{\tau})$ by assuming that both the stress $\boldsymbol{\tau}$ and the strain $\mathbf{E} = \frac{1}{2}(\mathbf{F}_2^{\mathrm{T}}\mathbf{F}_2 - \mathbf{I})$ are small. To make this assumption rigorous, the SEF and stress need to be made dimensionless, for example, by dividing them with respect to μ, but we omit the details of this process here for brevity. We assume that the stored energy density depends on only $\mathbf{E}$ and $\boldsymbol{\tau}$, which implies that we can write $\widehat{W}_{\boldsymbol{\tau}}(\mathbf{E}, \boldsymbol{\tau})$ in terms of the nine independent invariants of $\mathbf{E}$ and $\boldsymbol{\tau}$ [36,37]

$$\begin{aligned} &I_1 = \mathrm{tr}(\mathbf{E}), && I_2 = \mathrm{tr}(\mathbf{E}^2), && I_3 = \mathrm{tr}(\mathbf{E}^3),\\ &I_4 = \mathrm{tr}(\boldsymbol{\tau}), && I_5 = \mathrm{tr}(\boldsymbol{\tau}^2), && I_6 = \mathrm{tr}(\boldsymbol{\tau}^3),\\ &I_7 = \mathrm{tr}(\boldsymbol{\tau}\mathbf{E}), && I_8 = \mathrm{tr}(\boldsymbol{\tau}^2\mathbf{E}), && I_9 = \mathrm{tr}(\boldsymbol{\tau}\mathbf{E}^2)\,. \end{aligned} \tag{4.13}$$

The Cauchy stress arising from $\widehat{W}_{\boldsymbol{\tau}}(\mathbf{E}, \boldsymbol{\tau})$ can be expressed in terms of these invariants, as

$$\widehat{\boldsymbol{\sigma}}_{\boldsymbol{\tau}}(\mathbf{F}_2, \boldsymbol{\tau}) = J_2^{-1}\mathbf{F}_2\left(\frac{\partial \widehat{W}_{\boldsymbol{\tau}}}{\partial I_1}\mathbf{I} + 2\frac{\partial \widehat{W}_{\boldsymbol{\tau}}}{\partial I_2}\mathbf{E} + 3\frac{\partial \widehat{W}_{\boldsymbol{\tau}}}{\partial I_3}\mathbf{E}^2 + \frac{\partial \widehat{W}_{\boldsymbol{\tau}}}{\partial I_7}\boldsymbol{\tau} + \frac{\partial \widehat{W}_{\boldsymbol{\tau}}}{\partial I_8}\boldsymbol{\tau}^2 + \frac{\partial \widehat{W}_{\boldsymbol{\tau}}}{\partial I_9}(\mathbf{E}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{E})\right)\mathbf{F}_2^{\mathrm{T}}. \tag{4.14}$$

We now assume that $\boldsymbol{\tau} = O(\mathbf{E})$ and expand $\widehat{W}_{\boldsymbol{\tau}}(\mathbf{E}, \boldsymbol{\tau})$ up to $O(\mathbf{E}^3)$. As discussed in the previous section, there are other choices than $\boldsymbol{\tau} = O(\mathbf{E})$, but the main message would be the same. Again, we follow a systematic expansion method similar to that given in [38] to obtain

$$\begin{aligned} \widehat{W}_{\boldsymbol{\tau}}(\mathbf{E}, \boldsymbol{\tau}) = {} & \beta_0 I_1 + \beta_1 I_1^2 + \beta_2 I_2 + \beta_3 I_3 + \beta_4 I_1 I_2 + \beta_5 I_1^3 + \beta_6 I_4 + \beta_7 I_4^2 + \beta_8 I_5 + \beta_9 I_6 + \beta_{10} I_4 I_5 + \beta_{11} I_4^3 \\ & + \beta_{12} I_9 + \beta_{13} I_1^2 I_4 + \beta_{14} I_1 I_7 + \beta_{15} I_2 I_4 + \beta_{16} I_7 + \beta_{17} I_1 I_4 + \beta_{18} I_1 I_4^2 + \beta_{19} I_1 I_5 + \beta_{20} I_4 I_7 + \beta_{21} I_8, \end{aligned} \tag{4.15}$$

where β_j, for $j = 0, \ldots, 21$, are constants. A useful feature of SEFs of the above form is that they lead to simple expressions for elastic wave speeds in terms of the initial stress, which facilitates measuring the initial stress [17].

Before applying the ISRI restriction (3.7), we can first reduce the number of free coefficients by enforcing stress compatibility: $\widehat{\boldsymbol{\sigma}}_{\boldsymbol{\tau}}(\mathbf{I}, \boldsymbol{\tau}) = \boldsymbol{\tau}$ for every $\boldsymbol{\tau}$, which implies that $\beta_{16} = 1$ and $\beta_0 = \beta_{17} = \beta_{18} = \beta_{19} = \beta_{20} = \beta_{21} = 0$. See Shams *et al.* [36] for further details on initial stress compatibility.

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We see that there are still 15 constants left, whereas there are only five constants for classical third-order elasticity (4.1). Again, this is unexpected and illustrates that a restriction is needed to reduce the number of constants in the SEFs (4.15). Since we have assumed that $\widehat{W}_\tau$ depends on only $\mathbf{E}$ and $\boldsymbol{\tau}$, we can use the restriction (3.13). As we have further assumed that the material supports a stress-free state, we can take $\boldsymbol{\tau}_1 = \mathbf{0}$ and write (3.13) in terms of the quantities introduced in this section to obtain

$$\widehat{W}_\tau(\mathbf{E}_{12}, \mathbf{0}) = J_1 \widehat{W}_\tau(\mathbf{E}, \hat{\boldsymbol{\sigma}}_\tau(\mathbf{F}_1, \mathbf{0})), \tag{4.16}$$

for every $\mathbf{F}_1$ and $\mathbf{F}_2$, where J_1 is given by (4.9). We can use the restriction (4.16) to find the relationships between the constants. To do so, we first calculate the Cauchy stress in $\mathcal{B}_2$ to be

$$\hat{\boldsymbol{\sigma}}_\tau(\mathbf{F}_1, \mathbf{0}) = [2\beta_1 \operatorname{tr}\mathbf{e} + \beta_4 \operatorname{tr}(\mathbf{e}^2) - (2\beta_1 - 3\beta_5)(\operatorname{tr}\mathbf{e})^2]\mathbf{I} + 2[\beta_2 + (2\beta_1 - \beta_2 + \beta_4)\operatorname{tr}\mathbf{e}]\mathbf{e} + (4\beta_2 + 3\beta_3)\mathbf{e}^2 + O(\mathbf{e}^3). \tag{4.17}$$

We then note that $\boldsymbol{\tau} = \hat{\boldsymbol{\sigma}}_\tau(\mathbf{F}_1, \mathbf{0})$ and invert the above to obtain

$$\mathbf{e} = [a_1 \operatorname{tr}\boldsymbol{\tau} + a_2 \operatorname{tr}(\boldsymbol{\tau}^2) + a_3 \operatorname{tr}(\boldsymbol{\tau})^2]\mathbf{I} + (a_4 + a_5 \operatorname{tr}\boldsymbol{\tau})\boldsymbol{\tau} + a_6\boldsymbol{\tau}^2 + \mathcal{O}(\boldsymbol{\tau}^3), \tag{4.18}$$

where a_j, for $j = 1, \ldots, 6$, are constants that can be written in terms of β_i, $i = 1, .., 5$ (see the electronic supplementary material Mathematica file for further details). Substituting this expression into equation (4.8) gives the invariants of $\mathbf{E}_{12}$ in terms of the invariants of $\mathbf{E}$ and $\boldsymbol{\tau}$, which we then use to write the left side of (4.16) in terms of the mixed invariants of $\mathbf{E}$ and $\boldsymbol{\tau}$. For the right side of (4.16), we note that $\boldsymbol{\tau} = \hat{\boldsymbol{\sigma}}_\tau(\mathbf{F}_1, \mathbf{0})$ so we can use the form (4.15). Comparing the coefficients of each independent combination of invariants on both sides of (4.16) leads to

$$\begin{aligned}
&\beta_6 = 0, \quad \beta_7 = -\frac{\beta_1}{4\beta_2(3\beta_1 + \beta_2)}, \quad \beta_8 = \frac{1}{4\beta_2}, \quad \beta_9 = -\frac{2\beta_2 + \beta_3}{4\beta_2^3}, \quad \beta_{10} = \frac{2\beta_1(4\beta_2 + 3\beta_3) + \beta_2(\beta_2 - 2\beta_4)}{8\beta_2^3(3\beta_1 + \beta_2)},\\
&\beta_{11} = -\frac{12\beta_1^3(\beta_2 + \beta_3) + \beta_1^2\beta_2(7\beta_2 + 6\beta_3 - 6\beta_4) + \beta_1\beta_2^2(\beta_2 - 4\beta_4) + 2\beta_2^3\beta_5}{8\beta_2^3(3\beta_1 + \beta_2)^3}, \quad \beta_{12} = 2 + \frac{3\beta_3}{2\beta_2},\\
&\beta_{13} = -\frac{2\beta_1(2\beta_1 + \beta_4) + \beta_2(\beta_1 - 3\beta_5)}{2\beta_2(3\beta_1 + \beta_2)}, \quad \beta_{14} = \frac{2\beta_1 + \beta_4}{\beta_2}, \quad \beta_{15} = -\frac{\beta_1(4\beta_2 + 3\beta_3) + \beta_2(\beta_2 - \beta_4)}{2\beta_2(3\beta_1 + \beta_2)}.
\end{aligned} \tag{4.19}$$

This time, as required, we have 10 non-trivial equations, which, again, reduces the number of free parameters down to five. Again, if wished, we can set $\widetilde{W}_{\mathbf{E}}(\mathbf{E}_{12}) = \widehat{W}_\tau(\mathbf{E}_{12}, \mathbf{0})$ to find

$$\beta_1 = \frac{\lambda}{2}, \quad \beta_2 = \mu, \quad \beta_3 = \frac{A}{3}, \quad \beta_4 = B, \quad \beta_5 = \frac{C}{3}. \tag{4.20}$$

For completeness, substituting (4.19) into (4.15) and relabelling the parameters as per equation (4.20) leads to

$$\begin{aligned}
\widehat{W}_\tau(\mathbf{E}, \boldsymbol{\tau}) = \;& \frac{\lambda}{2}I_1^2 + \frac{C}{3}I_1^3 + \mu I_2 + I_2 I_1 B + \frac{A}{3}I_3 - \frac{\lambda}{12K\mu}I_4^2 + \frac{2BK - \lambda(2\lambda + \mu)}{36K^2\mu^2}I_4^3\\
&- \frac{3A\lambda^2(\lambda + \mu) + 4\mu^3(B + C)}{162K^3\mu^3}I_4^3 - \frac{2\lambda(B + \lambda) + \mu(\lambda - 2C)}{6K\mu}I_4 I_1^2 - \frac{A\lambda + 2\mu(2\lambda + \mu - B)}{6K\mu}I_2 I_4\\
&+ \frac{1}{4\mu}I_5 + \frac{A\lambda + \mu(-2B + 4\lambda + \mu)}{12K\mu^3}I_4 I_5 - \frac{A + 6\mu}{12\mu^3}I_6 + I_7 + \frac{B + \lambda}{\mu}I_7 I_1 + \frac{A}{2\mu}I_9 + 2I_9,
\end{aligned} \tag{4.21}$$

where $K = \lambda + 2\mu/3$, which is the bulk modulus.

The conclusion from this section is that expanding $\widehat{W}_\tau$ asymptotically in terms of its invariants, as shown in (4.15), leads to too many elastic constants. Using the ISRI restriction (3.13) makes $\overline{W}$ equivalent to a classical third-order strain energy function.

5. Discussion

This paper explores the development of a robust nonlinear elastic constitutive theory in scenarios where the reference configuration is initially stressed. In particular, we examined two distinct types of SEFs: type 1 and type 2, each suited to different modelling contexts in nonlinear elasticity. Type 1 functions, as proposed by Merodio *et al.* [39], for example, and denoted here as $\widetilde{W}(\mathbf{F}, \boldsymbol{\tau})$, model various elastic materials occupying a single, fixed reference configuration, with each value of the initial stress $\boldsymbol{\tau}$ corresponding to a distinct material. Thus, for type 1 functions, there is no need for ISRI, as they focus on capturing the range of responses across different materials in a single configuration. This clarification addresses potential misunderstandings of [21], in which we highlighted the physical inconsistency arising from using a type 1 function within a type 2 approach to model an initially stressed material subjected to further elastic deformations. In [21], we aimed to discuss the unphysical implications of using a type 1 function, such as that derived in [39] to model a single material under several superposed elastic deformations. Here, we clarify that this type 1 function does not need to satisfy ISRI when used solely within a type 1 framework. Type 2 functions, represented as $\overline{W}(\mathbf{F}, \boldsymbol{\tau}, \mathbf{X})$, are used to model the behaviour of a single elastic material as it undergoes further elastic deformations, resulting in distinct reference configurations with different stress values $\boldsymbol{\tau}$. Since type 2 functions describe a single material across various configurations, ISRI becomes essential to ensure consistency of the SEF's dependence on initial stress across these evolving configurations. In this latter case, we demonstrated that the need for ISRI emerges naturally from classical elasticity principles, providing a clear basis for understanding its mechanical significance. We emphasize that our previous work in the area of initially stressed materials [15,17–22,30] solely utilizes the type 2 approach and the restrictions mentioned therein thus only apply to this approach.

In §2, we deduced restrictions for stored energies of the form $W(\mathbf{F}, \mathbf{F}_1, \boldsymbol{X})$ from the assumptions of classical elasticity. In §3, we derived restrictions on stored energy of the form $\overline{W}(\mathbf{F}, \boldsymbol{\tau}, \boldsymbol{X})$ from those derived in §2. To do so, we further assumed that the stress could be inverted for the strain, which holds for a wide range of constitutive models in classical elasticity. An example of this stress inversion for third-order materials was given in (4.18). A similar inversion is possible for an asymptotic expansion of any order.

In §4, we gave some examples to illustrate the theory when the strain and stress are small by considering third-order materials. We showed that, when they are unrestricted, SEFs of the form $W_{\mathbf{e}}(\mathbf{E}, \mathbf{e})$ and $\widehat{W}_{\boldsymbol{\tau}}(\mathbf{E}, \boldsymbol{\tau})$, asymptotically expanded up to the third order, have too many free parameters compared with classical third-order elastic materials (4.1), which have only five constants. We expect the number of constants to be the same, as $W_{\mathbf{e}}(\mathbf{E}, \mathbf{e})$, $\widehat{W}_{\boldsymbol{\tau}}(\mathbf{E}, \boldsymbol{\tau})$ and the classical third-order strain energy functions all account for the same quantities, at the same asymptotic order. We showed that this inconsistency is resolved by enforcing the restrictions given in equations (4.19) and (4.16).

To conclude, we re-emphasize that the restrictions derived in §3 of this paper only apply to type 2 SEFs of the form $\overline{W}(\mathbf{F}, \boldsymbol{\tau}, \boldsymbol{X})$, which are used to model a *single material*, with different values of $\boldsymbol{\tau}$ corresponding to different reference configurations of the material. Type 1 SEFs of the form $\widetilde{W}(\mathbf{F}, \boldsymbol{\tau})$, where the *reference configuration is fixed* and different values of $\boldsymbol{\tau}$ correspond to *different elastic materials* that occupy the same reference configuration, do not need to satisfy these restrictions.

Data accessibility. Supplementary material available at [40].

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

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editing. All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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Appendix A. Alternative derivation of *Initial Strain Reference Independence*

Similarly to the derivation in §2b, we could set the first argument of W in equation (2.6) to be the identity tensor $\mathbf{I}$ (i.e. $\mathbf{F}_2 \to \mathbf{I}$) and the *second* argument to be $\mathbf{F}_{12}$ (i.e. $\mathbf{F}_1 \to \mathbf{F}_{12}$), so that we are considering the case where *all* of the deformation is initial deformation and there is no *subsequent* deformation, which implies that $\boldsymbol{X}_2 \to \boldsymbol{x}$, and, on the right side $J_1^{-1} \to J_{12}^{-1}$, $\mathbf{F}_2\mathbf{F}_1 \to \mathbf{I}\mathbf{F}_{12}$ and $\phi_1^{-1}(\boldsymbol{X}_2) \to \phi_{12}^{-1}(\boldsymbol{x}) = \boldsymbol{X}_1$. Using these substitutions leads to

$$W(\mathbf{I}, \mathbf{F}_{12}, \boldsymbol{x}) = W(\mathbf{I}, \mathbf{F}_2\mathbf{F}_1, \boldsymbol{x}) = J_{12}^{-1}\tilde{W}(\mathbf{I}\mathbf{F}_{12}, \boldsymbol{X}_1) = J_{12}^{-1}\tilde{W}(\mathbf{F}_2\mathbf{F}_1, \boldsymbol{X}_1), \tag{A 1}$$

and, comparing this with equations (2.6) and (2.8), we obtain

$$W(\mathbf{F_2F_1}, \mathbf{I}, \boldsymbol{X_1}) = J_1 W(\mathbf{F_2}, \mathbf{F_1}, \boldsymbol{X_2}) = J_1 J_2 W(\mathbf{I}, \mathbf{F_2F_1}, \boldsymbol{x}), \tag{A 2}$$

which must hold for every $\mathbf{F}_1$, $\mathbf{F}_2$, $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ and $\boldsymbol{x}$. We remark that these three equalities are mathematically equivalent.

By substituting equation (A 1) into equation (2.9) and making use of some of the results from (2.10), we obtain the following restriction on the Cauchy stress functional:

$$\boldsymbol{\sigma}(\mathbf{F_2F_1}, \mathbf{I}, \boldsymbol{X_1}), \; = \boldsymbol{\sigma}(\mathbf{F_2}, \mathbf{F_1}, \boldsymbol{X_2}) = \boldsymbol{\sigma}(\mathbf{I}, \mathbf{F_2F_1}, \boldsymbol{X_1}), \tag{A 3}$$

for every $\mathbf{F}_1$, $\mathbf{F}_2$, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$.

Appendix B. SEFs for initially strained materials—uniaxial, incompressible case

Here, we derive the *Initial Strain Reference Independence* restriction (2.8) in the special case of a uniaxial deformation of a three-dimensional, incompressible solid. We assume that the boundary conditions are such that the stress tensor only has a single non-zero component so that the problem reduces to a scalar-valued problem.

We represent the uniaxial deformation in terms of a single stretch, λ, and we write the energy density per unit volume as $\tilde{W}(\lambda, \boldsymbol{X}_1)$ (noting that we do not need to specify the reference volume because of incompressibility) and the uniaxial stress in any deformed configuration as

$$\tilde{\sigma} = \lambda \frac{\partial \tilde{W}(\lambda, \boldsymbol{X}_1)}{\partial \lambda}. \tag{B 1}$$

We call $\tilde{W}$ a *classical strain energy function*: it depends only on the stretch for a deformation from the reference configuration $\mathcal{B}_1$ (and on the position in that configuration).
We now ask whether there exists a strain energy function, W, which gives the energy density in the current configuration, $\mathcal{C}$, such that

$$W(\lambda_2, \lambda_1, \boldsymbol{X}_2) = \psi(\boldsymbol{x}), \tag{B 2}$$

for *every* deformation λ_1 from $\mathcal{B}_1$ to $\mathcal{B}_2$, and *every* deformation λ_2 from $\mathcal{B}_2$ to $\mathcal{C}$, where $\psi(\boldsymbol{x})$ is the energy density in the configuration $\mathcal{C}$ expressed as a function of the deformed variable, $\boldsymbol{x}$. Note that this SEF W, which depends on two stretches, is distinct from the classical strain energy function $\tilde{W}$, which depends on one stretch only.
In terms of W, the stress in the configuration $\mathcal{C}$ is given by

$$\sigma(\lambda_2, \lambda_1, \boldsymbol{X}_2) := \lambda_2 \frac{\partial W(\lambda_2, \lambda_1, \boldsymbol{X}_2)}{\partial \lambda_2}. \tag{B 3}$$

The configurations in figure 1 are chosen to deduce the energy density in $\mathcal{C}$ when using $\mathcal{B}_2$ as the reference configuration. Using the classical assumptions of nonlinear elasticity (such as locality and history independence [23]), we know that there exists a function $\tilde{W}$ (which uses $\mathcal{B}_1$ as its fixed reference configuration) such that

$$\tilde{W}(\lambda_1\lambda_2, \boldsymbol{X}_1) = \psi(\boldsymbol{x}). \tag{B 4}$$

From (B 4) and (B 2), we deduce that

$$W(\lambda_2, \lambda_1, \boldsymbol{X}_2) := \tilde{W}(\lambda_2\lambda_1, \phi_1^{-1}(\boldsymbol{X}_2)). \tag{B 5}$$

Because this relation is to hold *for all values* of the arguments of W, we may write it when its first argument is $\lambda_2\lambda_1$ and the second is 1. Then the left-hand side of the equation above is $W(\lambda_2\lambda_1, 1, X_2)$ while the right-hand side is unchanged, which leads to the relation

$$W(\lambda_2\lambda_1, 1, \boldsymbol{X}_1) = W(\lambda_2, \lambda_1, \boldsymbol{X}_2), \tag{B 6}$$

for every λ_1, λ_2 and $X_1 = \phi_1^{-1}(\boldsymbol{X}_2)$. This is the incompressible version of the *Initial Strain Reference Independence* restriction (2.8) in the case of a uniaxial deformation. Differentiating both sides of this equation with respect to λ_2 gives

$$\sigma(\lambda_2\lambda_1, 1, \boldsymbol{X}_1) = \sigma(\lambda_2, \lambda_1, \boldsymbol{X}_2), \tag{B 7}$$

which is the stress version of the *Initial Strain Reference Independence* restriction (2.11) in the case of a uniaxial deformation. Changing λ_1 is equivalent to changing the reference configuration of W.
Now we can ask the question: *how do we find explicit examples of such functions, W?* This depends on what we assume about the material. We can either make an assumption about the form of $\tilde{W}$, or an assumption about the form of W.

B.1. Example SEFs for initially strained materials

Let us present some examples to illustrate the results from the previous section. Let us investigate the consequence of assuming, for example, that $\tilde{W}$ is of the form

$$\tilde{W}(\lambda, \boldsymbol{X}_1) = \mu(\lambda - \ln\lambda). \tag{B 8}$$

Using equation (B 5), we find the form of W to be

$$W(\lambda_2, \lambda_1, \boldsymbol{X}_2) = \tilde{W}(\lambda_2\lambda_1, \boldsymbol{X}_1) = \mu(\lambda_2\lambda_1 - \ln(\lambda_2\lambda_1)), \tag{B 9}$$

which clearly satisfies the restriction (B 6), as it should. In fact, following the same reasoning, we see that any assumed form for $\tilde{W}$ leads to a W that satisfies equation (B 6). Conversely, if we assume that W is in a form that satisfies equation (B 6), then we can find the corresponding classical strain energy function $\tilde{W}$ of the solid. However, what would be the consequence of choosing a function W that does not satisfy the restriction (B 6)?

Let us assume that W is of the form

$$W(\lambda_2, \lambda_1, \boldsymbol{X}_2) := \mu(\lambda_2 - \ln \lambda_2) + \mu(\lambda_1 - 1)\lambda_2, \tag{B 10}$$

which does not satisfy the restriction (2.8). We could attempt to repeat the steps of the previous subsection to conclude that

$$W(\lambda_2, \lambda_1, \boldsymbol{X}_2) = \tilde{W}(\lambda_2\lambda_1, \boldsymbol{X}_1) = \mu(\lambda_2 - \ln \lambda_2) + \mu(\lambda_1 - 1)\lambda_2\,. \tag{B 11}$$

However, because W does not satisfy the restriction (2.8), we cannot say that this equation should hold for all λ_2 and λ_1. The right side of the equation cannot be written as a function of $\lambda_2\lambda_1$, and we, therefore, cannot find a classical strain energy function, $\tilde{W}$, that corresponds to the proposed form for W. Writing equation (B 11) when $\lambda_2 = \lambda$ and $\lambda_1 = 1$ gives

$$\tilde{W}(\lambda, \boldsymbol{X}) = \mu(\lambda - \ln \lambda), \tag{B 12}$$

whereas writing it when $\lambda_2 = 1$ and $\lambda_1 = \lambda$ gives

$$\tilde{W}(\lambda, \boldsymbol{X}) = \mu\lambda, \tag{B 13}$$

a different classical strain energy function for the same solid—a clear contradiction.

In summary, because (B 10) does not satisfy the restriction (B 6), it cannot be used to model the strain energy density of the material for every choice of λ_2 and λ_1, and we cannot find the form of the associated classical strain energy function $\tilde{W}$ from (B 5) because it gives different expressions for different choices of λ_2 and λ_1.

Appendix C. An inverse function for the stress

To define a stored energy of the form $\overline{W}(\mathbf{F}_2, \boldsymbol{\tau}_2, \boldsymbol{X}_2)$ in terms of an SEF of the form $W(\mathbf{F}_2, \mathbf{F}_1, \boldsymbol{X}_2)$, we need to invert the stress function $\tilde{\boldsymbol{\sigma}}$. Below, we demonstrate that this is possible for a wide class of functions W. Defining $\overline{W}$ from a function W has been previously carried out by Hoger [41]. In §4b, we give an example of this stress inversion for third-order elasticity (see equation (4.18)). A similar inversion could be carried out for an asymptotic expansion of the SEF to any order.

The outline of our explanation is as follows:

(1) We show the function $\tilde{\boldsymbol{\sigma}}$ is locally invertible for a large class of materials. That is, given $\mathbf{U}_1$, then, locally, around the point $\boldsymbol{X}_2$, where $\boldsymbol{\tau}_2 = \tilde{\boldsymbol{\sigma}}(\mathbf{U}_1, \boldsymbol{X}_1)$, there exists a unique inverse function $\tilde{\boldsymbol{\sigma}}^{-1}$ that satisfies $\tilde{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\tau}_2, \boldsymbol{X}_2) = \mathbf{U}_1$.

(2) We let $\mathbf{U}_0$ be such that $\tilde{W}(\mathbf{U}_0, \boldsymbol{X}_1)$ is the global energy minimum. Then we show that there exists a unique global inverse $\tilde{\boldsymbol{\sigma}}^{-1}$ that passes through this energy minimum, that is, that satisfies $\tilde{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\tau}_0, \boldsymbol{X}_1) = \mathbf{U}_0$, where $\tilde{\boldsymbol{\sigma}}(\mathbf{U}_0, \boldsymbol{X}_1) = \boldsymbol{\tau}_0$.

1. Local inversion: For a wide class of materials, we expect $\tilde{\boldsymbol{\sigma}}$ to have a local inverse because local invertibility is a result of the material being stable under traction. We expect most materials to be stable for every $\mathbf{U}$ in some set $\mathcal{U}$, which we call the region of stability.

To explain the connection between stability and a local inverse, we note that being stable under traction means that small changes in stress lead to small changes in strain, which implies that the partial derivative $\partial\mathbf{U}/\partial\boldsymbol{\sigma}$ is unique, exists, and is continuously differentiable [42,43]. The aforementioned properties of $\partial\mathbf{U}/\partial\boldsymbol{\sigma}$ also guarantee that there exists a local inverse of the function $\tilde{\boldsymbol{\sigma}}: \mathbf{U} \mapsto \boldsymbol{\sigma}$ for $\mathbf{U} \in \mathcal{U}$.

The above explains why we expect there exists a local inverse for a large class of constitutive choices. However, to check if some given $\tilde{\boldsymbol{\sigma}}$ is locally invertible, it is simpler to check whether

$$\left\| \frac{\partial \tilde{\boldsymbol{\sigma}}(\mathbf{U}, \boldsymbol{X}_1)}{\partial \mathbf{U}} : \delta \mathbf{U} \right\| > c, \tag{C 1}$$

for some positive real constant $c > 0$, and any symmetric tensor $\delta\mathbf{U}$, where $\|\boldsymbol{v}\|$ is the Euclidean norm of any vector $\boldsymbol{v}$. Even for $c = 0$, the condition (C 1) is enough to guarantee that there exists a local inverse (see [44, theorem 1.1.7.] and [45]). However, having $c > 0$ allows us to extend this inverse, as we do below.

2. Defining a global inverse: By using the condition (C 1) for every $\mathbf{U} \in \mathcal{U}$, we can define an inverse $\tilde{\boldsymbol{\sigma}}^{-1} : \mathcal{S} \mapsto \mathcal{U}$, where the open set $\mathcal{S} := \{\tilde{\boldsymbol{\sigma}}(\mathbf{U}, \boldsymbol{X}_1) : \mathbf{U} \in \mathcal{U}\}$. To define this inverse uniquely, we need to choose one point, or state, for this inverse to pass through. That is, we need to choose a deformation $\mathbf{U}_0$ and a stress $\boldsymbol{\tau}_0$ such that $\tilde{\boldsymbol{\sigma}}(\mathbf{U}_0, \boldsymbol{X}_1) = \boldsymbol{\tau}_0$ and $\boldsymbol{U}_0 \in \mathcal{U}$. Often, a natural choice is $\mathbf{U}_0 = \mathbf{I}$ and $\boldsymbol{\tau}_0 = \tilde{\boldsymbol{\sigma}}(\mathbf{I}, \boldsymbol{X}_1)$, which is usually a stress-free state. However, to be more general, we choose:

$$\mathbf{U}_0 = \arg\min_{\mathbf{U}} \tilde{W}(\mathbf{U}, \boldsymbol{X}_1). \tag{C 2}$$

Now, we assume that there is an open set $\mathcal{U}$ around the point $\mathbf{U}_0$ such that the material is stable under traction for every $\mathbf{U} \in \mathcal{U}$, and, therefore, (C 1) holds. As a consequence of (C 1), there is a unique local inverse function $\tilde{\boldsymbol{\sigma}}^{-1}$ such that $\tilde{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\tau}_0, \boldsymbol{X}_1) = \mathbf{U}_0$. We can uniquely extend this local inverse, by repeatedly using (C 1), to define an inverse function $\tilde{\boldsymbol{\sigma}}^{-1} : \mathcal{S} \to \mathcal{U}$.

In conclusion, $\tilde{\boldsymbol{\sigma}}^{-1}$ is an inverse of $\tilde{\boldsymbol{\sigma}}$ in the sense that

$$\tilde{\boldsymbol{\sigma}}^{-1}(\tilde{\boldsymbol{\sigma}}(\mathbf{U}, \boldsymbol{X}_1), \boldsymbol{X}_1) = \mathbf{U} \quad \text{and} \quad \tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\tau}, \boldsymbol{X}_1), \boldsymbol{X}_1) = \boldsymbol{\tau}, \tag{C 3}$$

for every $\mathbf{U} \in \mathcal{U}$ and $\boldsymbol{\tau} \in \mathcal{S}$, and $\tilde{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\tau}_0, \boldsymbol{X}_1) = \mathbf{U}_0$.

Appendix D. SEFs for initially stressed materials—uniaxial, incompressible case

Here, we derive the ISRI restriction (3.16) in the special case of a uniaxial deformation of a three-dimensional, incompressible solid. We begin by asking whether there exists an SEF of the form $\overline{W}(\lambda_2, \tau_2, \boldsymbol{X}_2)$ for *every* value of the stretch λ_2 and *every* value of the uniaxial stress τ_2. In appendix B, we answered the same question for a function $W = W(\lambda_2, \lambda_1, \boldsymbol{X}_2)$. Here, to make a link with that appendix, we simply express τ_2 in terms of λ_1 in $\overline{W}(\lambda_2, \tau_2, \boldsymbol{X}_2)$.

We again assume that there exists a classical strain energy function $\tilde{W}$, which can be used to calculate the stress via equation (B 1), so that the inverse stress function,

$$\tilde{\sigma}^{-1}(\tau, X) = \lambda, \tag{D 1}$$

is well-defined. The existence of this function implies that an SEF of the sought form, $\overline{W}(\lambda_2, \tau_2, \boldsymbol{X}_2)$, *does* exist and it can be defined in terms of W as

$$\overline{W}(\lambda_2, \tau_2, \boldsymbol{X}_2) := W(\lambda_2, \tilde{\sigma}^{-1}(\tau_2, \boldsymbol{X}_2), \boldsymbol{X}_2). \tag{D 2}$$

We can now show that the fact that $\overline{W}$ is the stored energy density in the configuration $\mathcal{C}$ for *any* deformation λ_2 and initial stress τ_2 leads to a restriction. The first step is to show that

$$\sigma_\tau(\lambda_2, \tau_2, \boldsymbol{X}_2) := \lambda_2 \frac{\partial \overline{W}(\lambda_2, \tau_2, \boldsymbol{X}_2)}{\partial \lambda_2} = \sigma(\lambda_2, \tilde{\sigma}^{-1}(\tau_2, \boldsymbol{X}_2), \boldsymbol{X}_2), \tag{D 3}$$

where we used the definitions (D 2) and (B 3). The above holds for every λ_2, τ_2 and $\boldsymbol{X}_2$. In particular, when $\lambda_2 = \lambda_1$, $\tau_2 = \tau_1$ and $\boldsymbol{X}_2 = \boldsymbol{X}_1$, we have

$$\tilde{\sigma}^{-1}(\tau_1, \boldsymbol{X}_1) = 1, \tag{D4}$$

where we used the fact that $\tilde{\sigma}^{-1}(\tau_1, \boldsymbol{X}_1) = 1$ and the definition (B 3). Substituting the above into equation (D 2) leads to

$$\overline{W}\,(\lambda_2, \sigma_\tau(\lambda_1, \tau_1, \boldsymbol{X}_1), \boldsymbol{X}_2) = \overline{W}\,(\lambda_2, \tilde{\sigma}(\lambda_1, \boldsymbol{X}_1), \boldsymbol{X}_2) = W\,(\lambda_2, \lambda_1, \boldsymbol{X}_2)\,. \tag{D 5}$$

We can rewrite the last term above by using (B 6) followed by (D 2) to obtain

$$W\,(\lambda_2, \lambda_1, \boldsymbol{X}_2) = W\,(\lambda_2\lambda_1, 1, \boldsymbol{X}_2) = \overline{W}\,(\lambda_2\lambda_1, \tau_1, \boldsymbol{X}_1), \tag{D 6}$$

which holds for every λ_1, λ_2 and $\boldsymbol{X}_1$. Finally, equating the first term of (D 5) with the last term of (D 6) leads to

$$\overline{W}\,(\lambda_2\lambda_1, \tau_1, \boldsymbol{X}_1) = \overline{W}\,(\lambda_2, \sigma_\tau(\lambda_1, \tau_1, \boldsymbol{X}_1), \boldsymbol{X}_2), \tag{D 7}$$

which holds for every λ_2, λ_1 and $\boldsymbol{X}_1 = \phi_1^{-1}(\boldsymbol{X}_2)$. Differentiating both sides of this equation with respect to λ_2 then leads to

$$\sigma_\tau\,(\lambda_2\lambda_1, \tau_1, \boldsymbol{X}_1) = \sigma_\tau(\lambda_2, \sigma_\tau\,(\lambda_1, \tau_1, \boldsymbol{X}_1), \boldsymbol{X}_2)\,. \tag{D 8}$$

We recall that $\tau_2 = \sigma_\tau(\lambda_1, \tau_1, \boldsymbol{X}_1)$, so that equation (D 8) is equivalent to

$$\sigma_\tau(\lambda_1\lambda_2, \tau_1, \boldsymbol{X}_1) = \sigma_\tau(\lambda_2, \tau_2, \boldsymbol{X}_2)\,. \tag{D 9}$$

Next, we make the substitution $\lambda_2 \longrightarrow \lambda_3$ followed by $\lambda_1 \longrightarrow \lambda_1\lambda_2$ in equation (D 7), and equation (D 9), to obtain

$$\overline{W}\,(\lambda_3\lambda_2\lambda_1, \tau_1, \boldsymbol{X}_1) = \overline{W}\,(\lambda_3, \sigma_\tau(\lambda_2, \tau_2, \boldsymbol{X}_2), \boldsymbol{X}_3), \tag{D 10}$$

which is valid for every choice of τ_2, λ_2 and λ_3 (note λ_1 can be determined from τ_2). If we wished to visualize the configurations, we would have three reference configurations, where λ_1 maps $\mathcal{B}_1 \mapsto \mathcal{B}_2$, λ_2 maps $\mathcal{B}_2 \mapsto \mathcal{B}_3$ and λ_3 maps $\mathcal{B}_3 \mapsto \mathcal{C}$. Finally, again using equation (D 7) on the left side of the above leads to

$$\overline{W}\,(\lambda_3\lambda_2, \tau_2, \boldsymbol{X}_2) = \overline{W}\,(\lambda_3, \sigma_\tau(\lambda_2, \tau_2, \boldsymbol{X}_2), \boldsymbol{X}_3), \tag{D 11}$$

which is valid for every λ_3, λ_2 and τ_2. Now, even if the stress τ_1 in $\mathcal{B}_1$ is fixed, the above-given equation holds for every initial stress τ_2. This is the incompressible version of ISRI (3.16) in the case of a uniaxial deformation. Differentiating both sides of this equation with respect to λ_3 gives, after some algebraic manipulation,

$$\sigma_\tau(\lambda_3\lambda_2, \tau_2, \boldsymbol{X}_2) = \sigma_\tau(\lambda_3, \sigma_\tau(\lambda_2, \tau_2, \boldsymbol{X}_2), \boldsymbol{X}_3), \tag{D 12}$$

which is the incompressible version of the ISS condition (3.17) in the case of a uniaxial deformation. In the next section, we discuss some examples of SEFs that do and do not satisfy these restrictions.

D.1 Example SEFs for initially stressed materials

Let us now write down explicit forms for $\overline{W}$. For the first example, we assume that there exists a classical strain energy function $\tilde{W}$, which depends only on the deformation λ and has the form

$$\tilde{W}(\lambda, \boldsymbol{X}_1) := \mu(\lambda - \ln\lambda)\,. \tag{D 13}$$

In this case, if the material is subjected to the stretch λ, the stress is $\tau = \mu(\lambda - 1)$ and, therefore, the inverse stress function $\tilde{\sigma}^{-1}$ is

$$\tilde{\sigma}^{-1}(\tau, \boldsymbol{X}) := \tau/\mu + 1. \tag{D 14}$$

We may then define $\overline{W}$, using (D 2) and (B 9), as $\overline{W}(\lambda_2, \tau_2, \boldsymbol{X}_2) := \mu\lambda_2(\tau_2/\mu + 1) - \mu\ln[\lambda_2(\tau_2/\mu + 1)]$ or, generically,

$$\overline{W}(\lambda, \tau, \boldsymbol{X}) := \mu\lambda(\tau/\mu + 1) - \mu\ln[\lambda(\tau/\mu + 1)]. \tag{D 15}$$

It is easily checked that this SEF satisfies ISRI (D 7) for all λ, τ and $\boldsymbol{X}$, as expected.

As another example, consider the case where, instead of prescribing the classical strain energy function $\tilde{W}(\lambda, \boldsymbol{X})$, we prescribe $\overline{W}(\lambda, \tau, \boldsymbol{X})$. Specifically, take

$$\overline{W}(\lambda, \tau, \boldsymbol{X}) := \mu(\lambda - \ln\lambda) + \tau\lambda, \quad \Rightarrow \quad \sigma_\tau(\lambda, \tau, \boldsymbol{X}) = \lambda(\mu + \tau) - \mu. \tag{D 16}$$

In this case, ISRI is not satisfied. Because it does not satisfy the restriction (3.7), one of the assumptions used to derive ISRI must not hold. In this case, it is the assumption that there exists a classical strain energy function $\tilde{W}$ used in (2.6). To demonstrate this, we use equations (D 4), (D 2) and (B 5) with the above choice of $\overline{W}$ to obtain

$$\tilde{W}(\lambda_2\lambda_1, \boldsymbol{X}_1) = \overline{W}(\lambda_2, \sigma_\tau(\lambda_1, \tau_1), \boldsymbol{X}_2) = -\mu\ln\lambda_2 + \lambda_2\lambda_1(\mu + \tau_1). \tag{D 17}$$

It is not possible to find a $\tilde{W}$ that satisfies the above-given equation as the right side cannot be written as a function of $\lambda_2\lambda_1$ only. As in the case of initial strain, because (D 16) does not satisfy the restriction (D 7), it cannot be used to model the strain energy density of the material for every choice of λ and τ, and we cannot find an associated classical strain energy function $\tilde{W}$.

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