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O'Neill's theorem for PL approximations

Srihari Govindan and Lucas Pahl

Abstract. We present a version of O'Neill's theorem (Theorem 5.2 in O'Neill in Am J Math 75(3):497–509, 1953) for piecewise linear approximations.

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1. Introduction

Theorem 5.2 of Ref. [6] asserts that if f is a continuous function from a topological polyhedron to itself, C is a component of the set of fixed points of f, U is a Euclidean neighborhood of C containing no other fixed points of f, $r_1, \ldots, r_k$ are integers whose sum is the fixed-point index of C, and $x_1, \ldots, x_k$ are distinct points of C, then there is a map arbitrarily close to f whose fixed points in U are $x_1, \ldots, x_k$, with the fixed-point index of each x_i being r_i. This note establishes a version of this result in the PL category. Specifically: (i) we allow for the polyhedron to be a subset of a topological manifold, and not homeomorphic to an Euclidean neighborhood; (ii) we weaken the restriction that the component C be in the interior of the polyhedron and, consequently, have to allow for the x_i's to be arbitrarily close to it; (iii) we add the restriction that the manifold be the space of a simplicial complex and that the approximating function be piecewise linear; (iv) in order to obtain a regularity property for fixed points, we insist that they be interior points—barycenters, even—of full-dimensional simplices and that the displacement map of the approximating function be a homeomorphism locally around these fixed points, if the r_i's are ± 1.

Our interest in this problem was motivated by its intended use in game theory. Nash equilibria of games obtain as fixed points of self maps on strategy spaces. It is a frequent (and robust) feature of games that components of equilibria lie on the boundary of the strategy space, which prompts the weakening of O'Neill's condition sub (ii) above. Also, fixed-point problems arising

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from games have a special structure, since the payoff functions of games are multilinear. Hence, perturbations of a given fixed-point map associated with a game have to satisfy certain conditions if they are to be associated with fixed points of games, prompting us to investigate a multilinear version of O'Neill's theorem for games (see Ref. [2] for details). This paper presents a linear version of the problem, where a stronger result is possible, and is possibly of wider interest as well.

2. Statement of the theorem

We first set a few notational conventions and recall some definitions that will be required for the statement of the main theorem (Theorem 2.1) and its proof.

2.1. (Notational) conventions

For $\zeta > 0$, define $B_\zeta(x)$ to be the ball around x with radius ζ. The symbol id_X denotes the identity map on the set X. Given $A \subseteq \mathbb{R}^n$ and a map $f : A \to \mathbb{R}^n$, $d_f(x) \equiv x - f(x)$. Let $X \subset \mathbb{R}^n$ be compact, and $f, g : X \to \mathbb{R}^n$ two continuous maps, we denote $\|f - g\| \equiv \sup_{x \in X} \|f(x) - g(x)\|_p$, where $\| \cdot \|_p$ denotes the ℓ_p-norm in $\mathbb{R}^n$. Unless explicitly stated otherwise, we will assume that $p = 2$ and will omit the subscript p for notational convenience. If $C \subseteq \mathbb{R}^n, x \in \mathbb{R}^n$, let $d(x, C) \equiv \inf_{y \in C} \|x - y\|$.

2.2. Triangulations, polyhedra and pseudomanifolds

Our terminology and notation for polyhedral complexes is mostly standard. In particular, we follow the convention of piecewise linear topology according to which a map from $X \subset \mathbb{R}^m$ to $\mathbb{R}^n$ is *linear* if it is the restriction to X of a map that is affine in the sense of linear algebra, i.e., the composition of a linear transformation and a translation.

As always, a *polytope* $P \subset \mathbb{R}^m$ is the convex hull of a finite set of points; an equivalent definition is that a polytope is an intersection of finitely many closed half-spaces that happens to be bounded, hence compact. The *dimension* of P is the dimension of its affine hull. The *faces* of P are P, the empty face, and the intersections of P with the boundaries of closed half-spaces that contain P; faces other than P are *proper*. A (finite, bounded) *polyhedral complex* Z in $\mathbb{R}^m$ is a finite collection of polytopes that contains each face of each of its elements, such that the intersection of any two of its elements is a face of both. If Y is a subset of Z that contains each of the faces of each of its elements, then Y is a *subcomplex* of Z. For $n = 0, \ldots, m$, let Z^n be the set of n-dimensional elements of Z. Elements of Z^0 are *vertices* of Z. The *dimension* of Z is the largest n such that $Z^n \neq \emptyset$. The *mesh* of Z is the maximum of the diameters of the elements of Z. The *space* of Z is $|Z| = \bigcup_{P \in Z} P$. A set $P \subset \mathbb{R}^m$ is a *polyhedron* if it is the space of a polyhedral complex, and its *dimension* is the dimension of any such complex.

A *simplicial complex* S in $\mathbb{R}^m$ is a polyhedral complex whose elements are all simplices. We say that S is a *triangulation* of $|S|$. The *carrier* $\Delta(x)$ of $x \in |S|$ in S is the smallest element of S that contains x, so it is the unique element of S whose interior contains x. If Z is a simplicial complex, we say that Y is a *subdivision* of Z if Y is a simplicial complex with $|Y| = |Z|$, and every simplex of Z is the union of simplices of Y.

When X is the space of a subcomplex of Z, we write $Z(X)$ to denote the subcomplex of Z composed by the simplices of Z which are contained in X.

If S, T are simplicial complexes, a function $f\colon |S| \to |T|$ is *simplicial* (relative to the triangulations S and T) if, for each $\sigma \in S$, there is a $\tau \in T$ such that f maps each vertex of σ to a vertex of τ and the restriction of f to σ is linear. If $P \subset \mathbb{R}^m$ and $Q \subset \mathbb{R}^\ell$ are polyhedra, a function $f\colon P \to Q$ is *piecewise linear* (PL) if there are simplicial subdivisions S of P and T of Q with respect to which f is simplicial. A sufficient condition for this (Theorem 2.14 of Ref. [7]) is that there is a simplicial subdivision S of P such that the restriction of f to each $\sigma \in S$ is linear.

A *polyhedron of homogeneous dimension* n is a polyhedron P that is the union of finitely many n-dimensional simplices, provided that the intersection of any two of the n-dimensional simplices is a (possibly empty) common face of both. The collection of the n-dimensional simplices together with all their faces then constitute a triangulation of P. If T is a triangulation of P, then ∂P is the union of those $\tau \in T^{n-1}$ that are a face of exactly one $\sigma \in T^n$; evidently ∂P is a polyhedron of homogeneous dimension $n-1$.

A polyhedron P of homogeneous dimension n is an n-*pseudomanifold*, provided the following hold for some triangulation T of P:

(1) Every element of T^{n-1} is a face of at most two elements of T^n;
(2) For any two n-simplices $\sigma, \sigma' \in T$ there is a finite chain $\sigma = \sigma_1, \ldots, \sigma_k = \sigma'$ of simplices in T^n such that $\sigma_i \cap \sigma_{i+1} \in T^{n-1}$.

2.3. Statement of the result

Let $(Y, \partial Y)$ be a topological n-manifold with ∂Y denoting its boundary and assume $Y \subseteq \mathbb{R}^m$ for some finite $m > 0$. Let $(X, \partial X)$ be an n-pseudomanifold with boundary ∂X with $X \subseteq Y$. Suppose Y is a polyhedron of homogenous dimension n with triangulation T, and X is the space of a subcomplex of Y of homogenous dimension n, as well. We can assume without loss of generality that $m \geq n+1$, by embedding Y in a Euclidean space of dimension larger than n, when $m = n$. Let $S \equiv T(X)$. Let $f : X \to Y$ be a continuous function satisfying the following assumptions: (A) either f has no fixed points on the boundary of X in Y, or $f(X) \subseteq X$; (B) the map f has a unique connected component of fixed points (cf. Remark 2.3 for a generalization). Thanks to assumption (A) about f, C has a well-defined index, call it c. Let U be a neighborhood of C in X with closure denoted $\bar{U}$.

Theorem 2.1. *For every $\varepsilon_0 > 0$, there exists $\delta_0 > 0$ such that for each $0 < \delta \leq \delta_0$ and each finite collection of points $x_1, \ldots, x_k$ and integers $r_1, \ldots, r_k$ such that*: (a) *for each $1 \leq i \leq k$, x_i belongs to the interior of an n-simplex of S, and $d(x_i, C) < \delta$, and* (b) *$\sum_i r_i = c$, there exist subdivisions S^* and T^* of S and T, resp., and a simplicial map $h^* : |S^*| \to |T^*|$ such that*:

(1) $\|f - h^*\| < \varepsilon_0$;

(2) $h^*(X) \subseteq X$, *if* $f(X) \subseteq X$;
(3) *the only fixed points of* h^* *in* $\bar{U}$ *are the* x_i*'s, and the index of each* x_i *is* r_i;
(4) *for each* i *such that* $r_i \in \{-1, +1\}$, *there exist simplices* $\sigma_i \in S^*$ *and* $\tau_i \in T^*$ *such that*:
   (a) $\sigma_i \subset \tau_i$ *and* x_i *is the barycenter of both* σ_i *and* τ_i;
   (b) h^* *maps* σ_i *homeomorphically onto* τ_i.

Remark 2.2. The triangulation T^* can be chosen such that X is the space of a subcomplex of T^*. Also, S^* can be chosen such that outside of a neighborhood of the x_i's, it subdivides the triangulation $T^*(X)$. Apparently, we are unable to get the stronger condition that S^* subdivides the triangulation induced by T^*.

Remark 2.3. If the map f has finitely many connected components of fixed points $C_1, \ldots, C_k$ (for example, if f is semialgebraic), the proof of Theorem 2.1 applies with insignificant modifications in order to obtain a simplicial approximation g of f where the result stated in Theorem 2.1 holds for each C_i.

Remark 2.4. When comparing Theorem 2.1 with Theorem 5.2 in Ref. [6], our statement, ignoring the PL structure and applying it to triangulable manifolds, provides a couple of generalizations. First, Theorem 2.1 allows for fixed-point components to intersect the boundary of X in Y, whereas in O'Neill, a fixed point component is located in the interior of the pseudomanifold X. Second, we allow for a pseudomanifold X that is the subset of a topological manifold of the same dimension as X and contained in a Euclidean space, while O'Neill requires X to be homeomorphic to a Euclidean neighborhood. When the first case occurs, then $f(X) \subseteq X$, by our assumption on f, and the index is well defined (explicitly, by the trace formula of O'Neill).

3. Auxiliary results

Lemma 3.1. *Let* τ *be a* n*-simplex in* $\mathbb{R}^m$ *with barycenter* x *and let* c *be an integer. There exists an* n*-simplex* $\sigma \subset \tau$ *with* x *as a barycenter and a PL map* $h : \sigma \to \tau$ *such that* x *is the unique fixed point of* h *and its index is* c. *Furthermore, if* $c \in \{-1, +1\}$, h *can be chosen to be an affine homeomorphism.*

Proof. Consider first the case where $|c| \neq 1$. We can assume without loss of generality that $m = n$ and $x = 0$. Take $\delta > 0$ such that ℓ_1-distance between 0 and $\partial\tau$ is greater than 2δ. Letting $B \subset \tau$ be the ℓ_1-ball of radius δ around 0, it is sufficient to construct a PL function $h : B \to \tau$ such that 0 is the unique fixed point of h and its index is c. We can further reduce the problem to the case $n = 2$: intersect τ (and B) with the linear subspace H of $\mathbb{R}^n$ consisting of points where the last $n-2$ coordinates are zero. If we have a PL function $h : H \cap B \to H \cap \tau$ where the index of 0 is c, we can extend it to B by composing it with the projection from B to $H \cap B$. The point 0 still has index c under the extension.

By the choice of δ, the problem is solved if we can find a PL function $d : B \to B$—to serve as the displacement of h—such that 0 is the only zero of d and has degree c. The case $c = 0$ is obvious: map 0 to 0, the boundary of B to some constant on ∂B and all other points by linear interpolation. Fix now c such that $|c| > 1$. The ℓ_1-ball B can be triangulated as the union of four triangles (one in each orthant). Subdivide each of the triangles into $|c|$ triangles all of which having 0 as a vertex. There now exists a PL map from B to itself that sends each of the $4c$ triangles of the subdivision to one of the triangles of B and that has degree c.

For the case $|c| = 1$, the lemma requires h to be an affine homeomorphism, so we approach the problem slightly differently. Let $w_0, \ldots, w_n$ be the vertices of τ. Take a simplex $\sigma \subset \tau$ of diameter less than δ, that has x as the barycenter, and is such that, letting $v_0, \ldots, v_n$ be the vertex set of σ, there is $\lambda > 1$ for which $w_i = x + \lambda(v_i - x)$ for all i.

For any permutation $\pi : \{0, \ldots, n\} \to \{0, \ldots, n\}$, we can define an affine homeomorphism $f^\pi : \sigma \to \tau$ that sends v_i to $w_{\pi(i)}$. Obviously x is the only fixed point of f^π. By virtue of the assumptions on σ, there is a retraction $r : \tau \to \sigma$ that sends w_i to v_i for each i, and that is affine on each face of τ. For a permutation π, x is also an isolated fixed point under $f^\pi \circ r$ and its index is the same under f^π and $f^\pi \circ r$.

Suppose π is a cyclic permutation where the only cycle involves all $n+1$ elements. Then, the index of x under f^π is $+1$ as under $f^\pi \circ r$ it is the unique fixed point. To obtain a fixed point of index -1, consider a permutation π that leaves, say, 0 fixed, and is cyclic on the others. Under the map $f^\pi \circ r$, there are three fixed points, w_0, x, and the barycenter of the face opposite w_0. The index of the first and the last fixed points is $+1$, assigning x an index of -1. □

Lemma 3.2. *Let $\hat{T}$ be a triangulation of Y. Let $\{x_i\}_{i=1}^k$ be a subset of Y, with each x_i contained in the interior of a simplex $\tau_i \in \hat{T}^n$. For each $\delta > 0$, there exists a triangulation $\tilde{T}$ of Y that subdivides $\hat{T}$ and satisfies the following*:

(1) *The mesh of $\tilde{T}$ is less than δ*;
(2) *For each $i = 1, \ldots, k$, there exist n-simplices $\sigma_i \in \tilde{T}$ and $\tau_i \in \hat{T}$ with $\sigma_i \subset \tau_i$, x_i the barycenter of σ_i.*

Proof. For each $i = 1, \ldots, k$, consider an n-simplex $\sigma_i \subset \mathrm{int}(\tau_i)$ with diameter less than δ that has x_i as a barycenter. For each i, take a polyhedral subdivision P_i of τ_i that has σ_i as an n-dimensional polyhedron, without introducing new vertices in τ_i beyond those of σ_i and τ_i. There exists a triangulation $\hat{T}_i'$ of τ_i which subdivides P_i, without introducing new vertices (cf. Proposition 2.9 in Ref. [7]). The simplices of the triangulation $\hat{T}_i'$, for each i, together with the other simplices of the triangulation $\hat{T}$, form a triangulation $\hat{T}'$ of Y. Now iterating sufficiently many times the barycentric subdivision of $\hat{T}'$ modulo $\cup_i \sigma_i$, (cf. [10]), we obtain a triangulation $\tilde{T}$ that subdivides $\hat{T}'$ and has mesh less than δ as well. The triangulation $\tilde{T}$ satisfies both requirements of the lemma. □

Lemma 3.3. *Let $\hat{T}$ be a triangulation of Y and let $\sigma \in \hat{T}^n$. Let $\hat{S}$ be any triangulation of σ. There exists a triangulation $\tilde{T}$ of Y that subdivides $\hat{T}$ such that $\tilde{T}(\sigma) = \hat{S}$ and the simplices of $\hat{T}$ that are disjoint from σ are simplices of $\tilde{T}$.*

Proof. Let $\hat{\mathcal{S}}$ be the collection of simplices in $\hat{S}$ that are contained in maximal proper faces of σ. Let $\hat{\mathcal{T}}$ be the collection of simplices in $\hat{T}$ that intersect σ but are not contained in σ. Let $\hat{\mathcal{T}}_0 = \{ \tau \in \hat{T} \mid \tau \cap \sigma = \emptyset, \tau \subset \varrho \in \hat{\mathcal{T}} \}$. Let $f \in \hat{\mathcal{S}}$ and assume $\varrho \in \hat{\mathcal{T}}$ contains f. The convex closure of f with any simplex in $\mathcal{T}_0 \cap \varrho$ is a simplex. Taking the convex closure of simplices in $\varrho \cap \hat{S}$ and in $\hat{\mathcal{T}}_0 \cap \varrho$ produces a triangulation of ϱ, which adds no vertices to the faces ϱ that are not contained in σ. The simplices of $\hat{S}$, the simplices obtained by the triangulation just defined in the simplices of $\hat{\mathcal{T}}$ and simplices of the triangulation $\hat{T}$ which do not intersect σ, define the triangulation $\tilde{T}$ of the statement. $\square$

We say the triangulation $\tilde{T}$ from Lemma 3.3 *extends* the triangulation $\hat{S}$ from σ to Y.

Definition 3.4. A *fiber bundle (with fiber F)* is a tuple (E, B, F, p) where:

(1) $p : E \to B$ is a continuous surjective map from the *total space* E to the *base space* B;
(2) For each $x \in B$, there exists a neighborhood $U \subseteq B$ of x such that $h_x : p^{-1}(U) \to U \times F$ is a homeomorphism that satisfies $p = p_1 \circ h_x$, where p_1 is the projection over the first coordinate.

Two fiber bundles $(\bar{E}, \bar{B}, \bar{F}, \bar{p})$ and (E, B, F, p) are *isomorphic* if there exist homeomorphisms $\bar{h} : \bar{E} \to E$ and $h : \bar{B} \to B$ such that $h \circ \bar{p} = p \circ \bar{h}$. The fiber bundle (E, B, F, p) is *trivial* if $E = B \times F$ and p is the projection over the first coordinate. For notational convenience, we will say that a fiber bundle is *trivial* if it is isomorphic to a trivial bundle.

Definition 3.5. A *n-microbundle* over the base space B is a tuple (E, B, e, p) where $e : B \to E$ and $p : E \to B$ are continuous maps such that:

(1) $p \circ e = id_B$;
(2) For every $b \in B$, there are a neighborhood $U \subseteq B$ of b and a neighborhood $V \subseteq E$ of $e(b)$ such that $e(U) \subseteq V$, $p(V) \subseteq U$ and $h_V : V \to U \times B_1^n(0)$ a homeomorphism satisfying: (i) $p_1 \circ h_V = p|_V$, and (ii) $h \circ e|_U = i$, where $i : B \to B \times B_1^n(0)$, $i(b) \equiv (b, 0)$ and p_1 is the projection over the first coordinate.

Let $Y^* = Y \sqcup_{\partial Y} Y$ be the compact, connected, n-dimensional, boundaryless topological manifold containing Y, obtained by attaching Y with itself along its boundary. Let p_1 be the natural projection from $Y^* \times Y^*$ to its first factor. Let $\Delta = \{ (y, y) \in Y^* \times Y^* \}$. Let $D : Y^* \to Y^* \times Y^*$ be the diagonal map, which sends $x \in Y^*$ to $(x, x) \in \Delta$. For each $\delta > 0$, let $B_\delta(\Delta)$ be the set of $(x, y) \in Y^* \times Y^*$ such that $\|x - y\| \le \delta$. Let $B_1^n(0)$ be the unit ball of $\mathbb{R}^n$. Given open sets V in $Y^* \times Y^*$ and U in Y, we say that a homeomorphism $h : V \to U \times B_1^n(0)$ is *trivializing* for D if $h \circ D(x) = (x, 0)$.

We say h is *trivializing* for p_1 if $p_1 = q_1 \circ h$, where $q_1 : Y^* \times B_1^n(0) \to Y^*$ is the projection over the first coordinate.

The n-microbundle $(Y^* \times Y^*, Y^*, D, p_1)$ is called the *tangent microbundle of* Y^* (see Example (iii) in Chapter 14 of Ref. [9] or Ref. [5]).

Lemma 3.6. *For each $\delta > 0$, there exists a neighborhood Z_δ of Δ in $B_\delta(\Delta)$ such that the restriction of p_1 to Z_δ is a fiber bundle $(Z_\delta, Y^*, B_1^n(0), p_1|_{Z_\delta})$.*

Proof. We start by constructing a microbundle (O_δ, Y^*, D, p_1) where $O_\delta \subset B_\delta(\Delta)$. Consider the tangent microbundle of Y^*. For each $x \in Y^*$, there exist an open neighborhood $U_x \subset Y^*$ of x, an open neighborhood $V_x \subset Y^* \times Y^*$ of (x, x) and a trivializing homeomorphism $h_x : V_x \to U_x \times B_1^n(0)$ for both the diagonal map D and the projection p_1. By compactness of Δ, there exist finitely many $x_1, \ldots, x_k$ such that $\bigcup_{i=1}^k V_{x_i}$ is a neighborhood of the diagonal Δ. For each x_i, there exists $\lambda_i > 0$, such that $h_{x_i}^{-1}(U_{x_i} \times B_{\lambda_i}^n(0)) \subset B_\delta(\Delta)$. Take $\lambda = \min_i\{\lambda_i\}$ and let $W_i \equiv h_{x_i}^{-1}(U_{x_i} \times B_\lambda^n(0)) \subset B_\delta(\Delta)$. The union $O_\delta \equiv \bigcup_i W_i$ is, therefore, a microbundle such that $O_\delta \subset B_\delta(\Delta)$. Applying the Kister–Mazur Theorem (Theorem 2 in Ref. [4]), we obtain a neighborhood $Z_\delta \subset O_\delta$ of the diagonal Δ such that $(Z_\delta, Y^*, B_1^n(0), p_1|_{Z_\delta})$ is a fiber bundle. $\square$

We now present the final auxiliary result which will be used in the proof of Theorem 2.1. The result is known (see, for example, Corollary 2.6 in Ref. [1]).

Lemma 3.7. *Let (E, B, F, p) be a fiber bundle over a paracompact and contractible space B. Then, (E, B, F, p) is trivial.*

4. Proof of Theorem 2.1

With preparations complete, we proceed to the proof of Theorem 2.1 per se. Let $W \subset Y^*$ be a neighborhood of Y for which there exists a retraction $r_Y : W \to Y$. There exists $\tilde{\delta} > 0$ such that the $\tilde{\delta}$-neighborhood $Y(\tilde{\delta})$ around Y in Y^* is contained in W and the $\tilde{\delta}$-neighborhood $X(\tilde{\delta})$ around X in Y^* retracts to X. We denote this retraction by r_X for notational convenience. Define $\ell_X : [0, \tilde{\delta}] \to \mathbb{R}_+$ by the maximum of $\|x - r_X(x)\|$ over all $x \in Y^*$ such that $d(x, X) \leq \delta$. If else, define $\ell_Y : [0, \tilde{\delta}] \to \mathbb{R}_+$ by the maximum of $\|x - r_Y(x)\|$ over all $x \in W$ such that $d(x, Y) \leq \delta$. Observe that for $* \in \{X, Y\}$, ℓ_* is continuous and $\ell_*(0) = 0$. For $\delta > 0$, denote by $B_\delta(C)$ the δ-neighborhood around C in $\mathbb{R}^m$.

Let $\varepsilon_0 > 0$. By continuity of $\ell_*(\cdot), * \in \{X, Y\}$, choose $\bar{\delta} > 0$ sufficiently small such that $\ell_*(\bar{\delta}) + \bar{\delta} < \varepsilon_0$. Fix $\delta_0 > 0$ such that

$$\mathrm{Graph}(f) \cap (B_{\delta_0}(C) \times Y) \subset Z_{\bar{\delta}}. \tag{0}$$

Fix any $\delta \in (0, \delta_0)$ and choose points $x_1, \ldots, x_k$ in the interior of n-simplices of S with $d(x_i, C) < \delta$. Let $r_1, \ldots, r_k$ be integers such that $\sum r_i = c$.

Apply now the Hopf Approximation Theorem (Theorem 2.5, Appendix C in Ref. [3]) to obtain two subdivisions T_0 and S_0 of T, with S_0 a subdivision of T_0, and a simplicial map $g : |S_0(X)| \to |T_0|$ such that:

(1) $\forall x \in X, d(f(x), g(x)) < \bar{\delta}$;
(2) $g(X) \subseteq X$ if $f(X) \subseteq X$;
(3) $\mathrm{Graph}(g) \cap (B_\delta(C) \times Y) \subset Z_{\bar{\delta}}$;
(4) g has finitely many fixed points, each of which is contained in the interior of an n-simplex in $S_0(X)$;
(5) The boundary of $B_\delta(C) \cap X$ in X has no fixed points of g and the index of g over $B_\delta(C)$ is c;
(6) All fixed points of g are contained in $B_\delta(C)$.

Let $F(g)$ be the set of fixed points of g in $B_\delta(C)$. Consider an open neighborhood $V \subset X \setminus \partial X$ of $F(g) \cup \bigcup_{i=1}^{k}\{x_i\}$ that is contractible and contained in $B_\delta(C) \cap (X \setminus \partial X)$. Using the fact that V is contractible, Lemmas 3.6 and 3.7 imply that the restriction of $p_1|_{Z_{\bar{\delta}}}$ to $Z_{\bar{\delta}}|_V \equiv (p_1|_{Z_{\bar{\delta}}})^{-1}(V)$ defines the trivial bundle $(Z_{\bar{\delta}}|_V, V, B_1^n(0), p_1)$. Therefore, letting $q_1 : V \times B_1^n(0) \to V$ be the natural projection on the first factor, there exists a homeomorphism $\varphi : Z_{\bar{\delta}}|_V \to V \times B_1^n(0)$ such that $p_1|_{Z_{\bar{\delta}}|_V} = q_1 \circ \varphi$. We note that $\mathrm{Graph}(g|_V) \subset Z_{\bar{\delta}}|_V$ (from (3) above). The restriction of φ to the x-section $(Z_{\bar{\delta}}|_V)_x = \{(x, y) \in Z_{\bar{\delta}}|_V\}$ is a homeomorphism with $\{x\} \times B_1^n(0)$. Let now $(h_x)_{x \in B_1^n(0)}$ be a continuous family of homeomorphisms from $B_1^n(0)$ to itself, such that h_x sends x to 0; let φ_2 be the coordinate map of φ mapping to $B_1^n(0)$. We can now define $\psi : Z_{\bar{\delta}}|_V \to V \times B_1^n(0)$ by $(x, y) \mapsto (x, h_{\varphi_2(x,y)} \circ \varphi_2(x, y))$; this is a homeomorphism that sends (y, y) to $y \times \{0\}$. Letting $Z^*_{\bar{\delta}}|_V \equiv Z_{\bar{\delta}}|_V - \{\Delta\}$, it follows that $\psi|_{Z^*_{\bar{\delta}}|_V}$ is a homeomorphism $Z^*_{\bar{\delta}}|_V \to V \times (B_1^n(0) - \{0\})$.

Let $\delta_1 > 0$ be such that the set-distance $d(V, \partial X) > \delta_1$ and $\min_{i \neq j} d(x_i, x_j) \geq 3\delta_1$. Let $\delta_2 > 0$ be such that for each $i = 1, \ldots, k$, any n-simplex τ_i with barycenter at x_i and diameter less than δ_2 is contained in V and is such that $\tau_i \times \tau_i \subset Z_{\bar{\delta}}$. Consider now a closed connected neighborhood B of $F(g) \cup \bigcup_i \{x_i\}$ that is contained in the interior of V. Let $d(\partial V, B) \geq \delta_3 > 0$. Fix $\eta \equiv \min\{\delta_1, \delta_2, \delta_3\}$. Lemma 3.2 now gives a simplicial subdivision T_1 of T_0 with mesh less than η such that each x_i is the barycenter of an n-simplex $\tau_i \in T_1$. Our choice of η implies that the collection of simplices τ_i is pairwise disjoint. Let P be the closed star of B with respect to T_1. The set P is a orientable, connected n-pseudomanifold with boundary ∂P (with associated triangulation $T_1(P)$) contained in V.

For each i, using Lemma 3.1 in each τ_i, we obtain a n-simplex $\sigma_i \subseteq \tau_i$ and a PL map $h_i : \sigma_i \to \tau_i$ such that x_i is the barycenter of both σ_i and τ_i, and the only fixed point of h_i, with index r_i. Take now a subdivision T_2 of T_1 that has each σ_i as an n-simplex of T_2 if $|r_i| = 1$. Using Theorem 2.14 in Ref. [7], there exist for each i for which $|r_i| \neq 1$, simplicial subdivisions $\hat{S}(\sigma_i)$ and $\hat{T}(\tau_i)$ of σ_i and τ_i, such that $h_i : |\hat{S}(\sigma_i)| \to |\hat{T}(\tau_i)|$ is simplicial. Using Lemma 3.3, there exist subdivisions $\hat{S}$ of T_2 and $\hat{T}$ of T_1 that extend $\hat{S}(\sigma_1)$ and $\hat{T}(\tau_1)$. Since σ_2 and τ_2 are disjoint from σ_1 and τ_1, respectively, the same lemma guarantees that σ_2 is an n-simplex of $\hat{S}$, and τ_2 an n-simplex of $\hat{T}$. This observation applied iteratively together with Lemma 3.3 implies there exists a subdivision $\hat{S}_2$ of T_2 and $\hat{T}_2$ of T_1 such that for each $i = 1, \ldots, k$, $\hat{S}_2$ extends the triangulation $\hat{S}(\sigma_i)$ and $\hat{T}_2$ extends the triangulation $\hat{T}(\sigma_i)$. For notational convenience we drop the subscripts of $\hat{T}_2$ and $\hat{S}_2$ and refer to these triangulations only as $\hat{T}$ and $\hat{S}$. Note that if $|r_i| = 1$, then we can assume that $\sigma_i \in \hat{S}^n$ and $\tau_i \in \hat{T}^n$.

Define $q : \partial P \cup \bigcup_i \sigma_i \to Y^*$ by $q|_{\partial P} \equiv g|_{\partial P}$ and for each $i = 1, \ldots, k$, $q|_{\sigma_i} \equiv h_i$. Let $Q = P \setminus \bigcup_{x_i \in V}(\sigma_i \setminus \partial\sigma_i)$. The set Q is a connected, orientable, n-pseudomanifold with boundary $\partial Q = \partial P \cup \bigcup_{x_i \in V} \partial\sigma_i$. Define now a map $d_q : \partial Q \to B_1^n(0) - \{0\}$ by $d(x) = q_2(\psi(x, q(x)))$, where q_2 is the projection on the second factor. Clearly, the degree of d is zero. By the Hopf Extension Theorem (Corollary 18, Chapter 8 in Ref. [8]), d_q extends to a map over Q, still denoted d_q. This defines a map $h : Q \to Y^*$ by letting $h(x) = p_2(\psi^{-1}(x, d_q(x)))$, where p_2 is the projection on the second factor.

The graph of h is guaranteed to be in $B_{\bar{\delta}}(\Delta)$ but not in $Q \times Y$, so, from h we now construct another map whose graph is in $Q \times Y$. Since Graph$(h) \subset Z^*_{\bar{\delta}}|_V \subset B_{\bar{\delta}}(\Delta)$, if $h(x) \in Y^* \backslash Y$, then it follows that $h(x) \in Y(\bar{\delta})$; if $f(X) \subset X$, then we have that $h(x) \in X(\bar{\delta})$. In the latter case, define $\hat{h}_X : Q \to Y$ by $\hat{h} = r_X \circ h$; in the former case, let $\hat{h}_Y = r_Y \circ h$. Therefore, we have that for each $x \in Q \subset V \subset X \setminus \partial X$, if $f(X) \subseteq X$, then $\hat{h}_X(Q) \subseteq X$ and $\|x - \hat{h}_X(x)\| \leq \ell_X(\bar{\delta}) + \bar{\delta}$; if else, $\|x - \hat{h}_Y(x)\| \leq \ell_Y(\bar{\delta}) + \bar{\delta}$. In either case, we can extend the map $\hat{h}_*, * \in \{X, Y\}$ to a map over X by letting it be equal to g everywhere on $X \setminus P$, denoting the extension still by $\hat{h}_*$.

For notational convenience, because the proofs in the two cases ($f(X) \subseteq X$ and $f(X) \not\subset X$) are equal, we will omit the subscripts X and Y from ℓ_X and ℓ_Y, as well as from $\hat{h}_X$ and $\hat{h}_Y$, writing only ℓ and $\hat{h}$.

Recall that: (i) $P \subset V \subset B_\delta(C) \cap (X \setminus \partial X)$, so, from (0), Graph$(f|_P) \subset Z_{\bar{\delta}} \subset B_{\bar{\delta}}(\Delta)$, which implies that $\|f|_P - id_P\| \leq \bar{\delta}$; (ii) $\|id_Q - \hat{h}|_Q\| \leq \ell(\bar{\delta}) + \bar{\delta}$; (iii) for each i, since $\tau_i \times \tau_i \subset Z_{\bar{\delta}} \subset B_{\bar{\delta}}(\Delta)$, then $\|id_{\sigma_i} - \hat{h}|_{\sigma_i}\| \leq \bar{\delta}$. Since $P = Q \cup \bigcup_i \sigma_i$, (i)–(iii) imply $\|f|_P - \hat{h}|_P\| \leq \ell(\bar{\delta}) + 2\bar{\delta}$. In $X \setminus P$, the map $\hat{h}$ equals g, and therefore, from (1), $\|f|_{X \setminus P} - \hat{h}|_{X \setminus P}\| \leq \bar{\delta}$. Hence, we have $\|f - \hat{h}\| \leq \ell(\bar{\delta}) + 2\bar{\delta}$.

Note now that by construction $\hat{h}$ has no fixed points in $X \setminus \bigcup_i(\sigma_i \setminus \partial\sigma_i)$. Since this is a compact set, let $0 < \alpha < \bar{\delta}$ be such that $\|x - \hat{h}(x)\| > 3\alpha$ for all $x \in X \setminus \bigcup_i(\sigma_i \setminus \partial\sigma_i)$. By Lemma 3.2, we can take a subdivision T^* of $\hat{T}$ such that:

(1) The diameter of each simplex is less than α;
(2) for each i, τ_i is the space of a subcomplex $T^*(\tau_i)$ of T^*;
(3) For each i for which $|r_i| = 1$, there is a full-dimensional simplex τ_i^* of T^* that has x_i as its barycenter.

Recall that, for each i, the map $\hat{h}|_{\sigma_i} = h_i : \sigma_i \to \tau_i$ is simplicial by construction w.r.t. to triangulations $\hat{S}(\sigma_i)$ of σ_i and $\hat{T}(\tau_i)$ of τ_i. Since $T^*(\tau_i)$ is a subdivision of $\hat{T}(\tau_i)$, by Lemma 2.16 in Ref. [7], there exists, for each i, a subdivision $S^*(\sigma_i)$ of $\hat{S}(\sigma_i)$ such that $\hat{h}|_{\sigma_i} : |S^*(\sigma_i)| \to |T^*|$ is simplicial for each i. When $r_i = \pm 1$, as $\hat{h}|_{\sigma_i}$ is an affine homeomorphism to τ_i^*, the simplex σ_i^* of $S^*(\sigma_i)$ that maps to τ_i^* has x_i as the barycenter for each such i. As before, applying Lemma 3.3 recursively for $i = 1, \ldots, k$, we can extend the triangulation $S^*(\cup_i\sigma_i)$ to a triangulation $\hat{S}^*$ of X. Using now the Theorem and Addendum in Ref. [10], we can consider a sufficiently fine barycentric subdivision S^* of $\hat{S}^*$ modulo $S^*(\cup_i \sigma_i)$ and a simplicial map $h^* : |S^*| \to |T^*|$ such that the restriction of h^* to $\cup_i \sigma_i$ equals $\hat{h}$ and $\|h^* - \hat{h}\| < 2\alpha$. It is easily verified that h^* has all the stated properties of the theorem.

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Declarations

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Srihari Govindan
Department of Economics
University of Rochester
Rochester NY14627
USA
e-mail: s.govindan@rochester.edu

Lucas Pahl
Hausdorff Center for Mathematics and Institute for Microeconomics
University of Bonn
Adenauerallee 24-42
53113 Bonn
Germany
e-mail: pahl.lucas@gmail.com

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