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# SINGLE-SET CUBICAL CATEGORIES AND THEIR FORMALISATION WITH A PROOF ASSISTANT (EXTENDED VERSION)

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PHILIPPE MALBOS - TANGUY MASSACRIER - GEORG STRUTH

**Abstract** – We introduce a single-set axiomatisation of cubical  $\omega$ -categories, including connections and inverses. We justify these axioms by establishing a series of equivalences between the category of single-set cubical  $\omega$ -categories, and their variants with connections and inverses, and the corresponding cubical  $\omega$ -categories. We also report on the formalisation of cubical  $\omega$ -categories with the Isabelle/HOL proof assistant, which has been instrumental in developing the single-set axiomatisation.

**Keywords** – Cubical  $\omega$ -categories, formalised mathematics, Isabelle/HOL, higher-dimensional rewriting.

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<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Single-set cubical categories</b>	<b>4</b>
<b>3</b>	<b>Equivalence with classical cubical categories</b>	<b>15</b>
<b>4</b>	<b>Formalisation with Isabelle/HOL</b>	<b>26</b>
<b>5</b>	<b>Conclusion</b>	<b>35</b>
<b>A</b>	<b>Appendices</b>	<b>39</b>

## 1. INTRODUCTION

Cubical sets and categories are fundamental structures widely used in mathematics and theoretical computer science. Several lines of research have shaped their axioms. Cubical sets provide abstract descriptions of higher-dimensional cubes and their faces. They were first introduced in mathematics for modelling homotopy types [42, 55]. Their algebraic and categorical descriptions were subsequently obtained via topological cubical complexes and similar structures [16, 44]. Cubical categories, which equip cubical sets with compositions along faces of higher-dimensional cubes, were introduced by Brown and Higgins for their generalisation of van Kampen’s theorem to higher dimensions [12, 14]. These articles also introduce a notion of connection on cubical sets, essentially an operation of rotation of neighbouring faces. More recently, Lucas [45] has added a notion of inversion for cubes that imposes a groupoid structure on parts of the cubical structure. See [37] for a discussion of additional structure on cubical sets.

Formally, a cubical set is a family of sets  $(K_n)_{n \in \mathbb{N}}$  equipped with face maps  $\partial_{n,i}^\alpha : K_n \rightarrow K_{n-1}$  and degeneracy maps  $\varepsilon_{n,i} : K_{n-1} \rightarrow K_n$ , for  $1 \leq i \leq n$  and  $\alpha \in \{+, -\}$ . The former attach faces to higher

## 1. Introduction

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dimensional cubes; the latter represent lower dimensional cubes as degenerate higher dimensional ones. The cubical structure is imposed by the cubical relations

$$\begin{aligned} \partial_{n-1,i}^\alpha \partial_{n,j}^\beta &= \partial_{n-1,j-1}^\beta \partial_{n,i}^\alpha \quad (i < j), & \varepsilon_{n+1,i} \varepsilon_{n,j} &= \varepsilon_{n+1,j+1} \varepsilon_{n,i} \quad (i \leq j), \\ \partial_{n,i}^\alpha \varepsilon_{n,j} &= \varepsilon_{n-1,j-1} \partial_{n-1,i}^\alpha \quad (i < j), & \partial_{n,i}^\alpha \varepsilon_{n,j} &= id \quad (i = j), & \partial_{n,i}^\alpha \varepsilon_{n,j} &= \varepsilon_{n-1,j} \partial_{n-1,i-1}^\alpha \quad (i > j). \end{aligned}$$

In a cubical category, compositions of cubes along their faces are defined for each direction  $i$  in a way compatible with face and degeneracy maps. Adding connections and inverses to cubical sets and categories imposes further axioms, as expected.

The category of cubical sets with connections and structure-preserving maps between them forms a strict test category *à la* Grothendieck [51], which makes it suitable for studying homotopy [15, 62]. Compared to simplicial models, they facilitate the handling of products. Further, Al-Agl, Brown and Steiner have shown that categories of cubical categories with connections and those of globular categories (strict  $\omega$ -categories), another kind of higher categories, are equivalent [2].

In computer science, some fundamental models of homotopy type theory are based on cubical sets [9, 21]; see [3] for an overview. They support a constructive approach to Kan fibrations in the simplicial set model of homotopy type theory [43], several properties of which are undecidable [8]. This prevents a computational interpretation of Voevodsky's univalence axiom, which is possible in the cubical model [10].

A second application of cubical sets in computer science lies in geometrical and topological models of concurrency [26]. A prominent example are higher-dimensional-automata [54, 63]. Here,  $n$ -cells represent transitions of a concurrent system where  $n$  concurrent events are active, and the cubical cell structure comes from the fact that each concurrent event in an  $n$ -dimensional cell can either be active or inactive in each of its  $2n$   $(n-1)$ -dimensional faces. Higher-dimensional automata subsume many other models of concurrency [63]. They have been studied from homological [33, 34, 41], homotopical [28, 32, 36], language theoretic [24] and algorithmic [27] points of views.

Finally, at the interface of mathematics and computing, cubical categories have recently been proposed as a tool for higher-dimensional rewriting [4, 46], a categorification of term rewriting [61] with applications in categorical algebra. Diagrammatic statements and proofs of abstract rewriting results, such as Newman's lemma or the Church-Rosser theorem, use indeed cubical shapes; confluence diagrams associated with critical pairs, triples and  $n$ -tuples form squares, cubes and  $n$ -cubes, respectively. A categorical description leads to the notion of polygraphic resolution, which allows the study of homotopical properties of rewriting systems [4, 39, 40]. Explicit constructions of such resolutions lend themselves naturally to a formalisation in cubical categories [45, 46].

The ubiquity of cubical sets and cubical categories alone merits a formalisation with a proof assistant to support reasoning with these highly combinatorial structures in applications (the axioms for cubical  $\omega$ -categories with connections and inverses in Subsection 3.1 below, for instance, cover about two pages). Yet instead of merely typing an extant axiomatisation into a prover and checking some well known properties by machine, we use the Isabelle/HOL proof assistant [53] to develop an alternative axiomatisation for cubical categories. It is based on single-set categories [47], where only arrows are modelled explicitly, while objects remain implicit via their one-to-one correspondence with identity arrows. Single-set approaches have a long history in category theory [48]; they feature in well-known textbooks [29, 30, 47] and form the basis of three encyclopaedic formalisations of category theory with Isabelle [56–58]. Formally, a single-set category is a set  $\mathcal{S}$  with source and target maps  $\delta^-, \delta^+ : \mathcal{S} \rightarrow \mathcal{S}$

and a composition  $\circ$ , a partial operation such that  $x \circ y$  is defined if and only if  $\delta^+x = \delta^-y$  for  $x, y \in \mathcal{S}$ , which satisfy

$$\begin{aligned} \delta^-(x \circ y) &= \delta^-x, & \delta^+(x \circ y) &= \delta^+y, & x \circ \delta^+x &= x, & \delta^-x \circ x &= x, \\ (x \circ y) \circ z &= x \circ (y \circ z), & \delta^- &= \delta^+\delta^-, & \delta^+ &= \delta^-\delta^+. \end{aligned}$$

Identity arrows arise as fixed points of  $\delta^-$  or equivalently those of  $\delta^+$ . Single-set categories are therefore algebraically simpler than their classical siblings defined via objects and arrows. Functors and natural transformations are simply functions [29]. Single-set higher categories may thus be more suitable for symbolic reasoning and automated proof search than their classical counterparts. Indeed, single-set globular categories are used widely [2, 13, 47, 59] and have been formalised with Isabelle [19, 20]. Yet single-set cubical categories remain to be defined.

This is not entirely straightforward. We had to introduce symmetry maps that relate the sets of fixed points modelling higher identities in different directions as replacements of the traditional degeneracy maps. Initially, this led to an unwieldy number of axioms, which would have been tedious to use and would have inflated the categorical equivalence proof, which justifies them relative to their classical counterparts.

Isabelle has been instrumental in taming these axioms due to its powerful support for proof automation and counterexample search, which sets it apart from other proof assistants. Its proof automation comes from internal simplification and proof procedures and external proof search tools – so-called hammers – for first-order logic. Counterexample search uses SAT solvers and decision procedures, for instance for linear arithmetic. This combination supports not only a natural mathematical workflow with proofs and refutations, it also allows checking axiom systems for redundancy (via deduction) and irredundancy (via counterexamples) rapidly and effectively. It has already proved its worth for developing other algebraic axiomatisations [19, 23, 31]. Here, Isabelle has helped us to bring the single-set axiomatisation for cubical categories to a manageable size without compromising its structural coherence, to simplify candidate axioms and to analyse candidate axioms that emerged during our development rapidly. Starting from an around 40 initial candidate axioms, we have used Isabelle in an iterative process, simplifying candidate axioms and removing redundant ones, then attempting an equivalence proof, and adding new axioms if that failed. Without our confidence in Isabelle’s automated proof support, we might not have attempted this research.

The single-set axiomatisation for cubical categories thus forms the main conceptual contribution in this article. Our main technical contribution consists in the proofs of categorical equivalence mentioned, and our main engineering contribution is the formalisation of a mathematical component for cubical categories with Isabelle. In combination, these results constitute a case study in innovative, not merely reconstructive formalised mathematics.

The overall structure of this article is simple: our axioms for single-set cubical categories are introduced in Section 2, the proofs that the resulting categories are essentially the same as their classical counterparts are given in Section 3, our Isabelle formalisation and the workflow leading to our axioms are discussed in Section 4. Finally, we summarise our results and present some avenues for future work in Section 5.

More specifically, we recall the variant of single-set categories [22, 25, 60], on which our axioms for single-set cubical categories are based, in Subsection 2.1. Subsection 2.2 introduces single-set cubical  $\omega$ -categories, axiomatised as a set  $\mathcal{S}$  equipped with families of maps indexed by directions  $i \in \mathbb{N}_+$ : face

## 2. Single-set cubical categories

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maps  $\delta_i^-$  and  $\delta_i^+$ , composition maps  $\circ_i$ , symmetry maps  $s_i$  and reverse symmetry maps  $\tilde{s}_i$ . We show how single-set cubical  $n$ -categories appear as truncations. We define the category  $\text{SCub}_\omega$  with single-set cubical  $\omega$ -categories as objects and functions corresponding to functors of classical  $\omega$ -categories as morphisms. We also list some structural properties of these categories. In Subsection 2.3 we add connections, in Subsection 2.4 we further add inverses in each dimension greater than  $p$ . This yields the categories  $\text{SCub}_\omega^Y$  and  $\text{SCub}_{(\omega,p)}^Y$  as well as truncated variants for each dimension  $n$ . Inverses are relevant to constructive proofs in homotopy type theory and higher-dimensional rewriting.

In Subsection 3.1 we recall the classical axioms for cubical  $\omega$ - and  $n$ -categories, including connections and inverses. This leads to the categories  $\text{Cub}_\omega$ ,  $\text{Cub}_\omega^\Gamma$  and  $\text{Cub}_{(\omega,p)}^\Gamma$  with classical cubical  $\omega$ -categories as objects and functors as morphisms. We then present proofs of the equivalences  $\text{SCub}_\omega \simeq \text{Cub}_\omega$  in Theorem 3.2.1,  $\text{SCub}_\omega^Y \simeq \text{Cub}_\omega^\Gamma$  in Theorem 3.3.1 and  $\text{SCub}_{(\omega,p)}^Y \simeq \text{Cub}_{(\omega,p)}^\Gamma$  in Theorem 3.4.1 in Subsections 3.2, 3.3 and 3.4 respectively. Straightforward modifications yield similar equivalences between  $n$ -categories, which we do not list explicitly.

Subsection 4.1 contains a brief overview of Isabelle, Subsection 4.2 recalls the formalisation of single-set categories with Isabelle [60], which underlies our formalisation of cubical  $\omega$ -categories with and without connections in Subsection 4.3. For technical reasons, we do not formalise cubical  $(\omega, p)$ -categories. But we show how a non-trivial proof about  $(\omega, 0)$ -categories (of Proposition 2.4.8) can be formalised with our axiomatisation at the same level of granularity.

While this article can be read as an exercise in formalised mathematics, we did not aim to formalise all our results, as it would distract from our main goal: to showcase the unique benefits of Isabelle's proof automation in the analysis of higher categories. Alternatively, disregarding Section 4, it can be read as a mathematical paper with contributions beyond Isabelle. While the calculational lemmas in Section 2 have been checked by machine, and proofs therefore been omitted, the categorical equivalences in Section 3 have not been formalised, though that would have been possible at least in parts. Once again: our main use case for Isabelle in this work has been the development of the axioms in  $\text{SCub}_\omega$ ,  $\text{SCub}_\omega^Y$  and  $\text{SCub}_{(\omega,p)}^Y$ . Further work with proof assistants on higher categories, higher rewriting and higher automata is left for future work. Our Isabelle components for cubical categories, including a PDF proof document, can be found in the Archive of Formal Proofs [52].

## 2. SINGLE-SET CUBICAL CATEGORIES

In this section we introduce our axiomatisation of single-set cubical categories. In Subsection 2.1 we recall a previous axiomatisation of single-set categories. In Subsection 2.2 we introduce single-set cubical  $\omega$ -categories and  $n$ -categories. Extensions of these categories with connections and inverses are presented in Subsections 2.3 and 2.4.

### 2.1. Single-set categories

We start with recalling the definition and basic properties of single-set categories. While any axiomatisation would work for our purposes, we have chosen one in which the partiality of arrow composition is captured by a multioperation that maps pairs of elements to sets of elements, including the empty set [22, 25]. It is already well-supported by Isabelle components [60] and has previously served as a basis for formalising globular single-set  $\omega$ -categories [19, 20].

2.1.1. A single-set category  $(\mathcal{S}, \delta^-, \delta^+, \odot)$  consists of the following data:

- a set  $\mathcal{S}$  of cells,
- face maps  $\delta^\alpha : \mathcal{S} \rightarrow \mathcal{S}$  for  $\alpha \in \{-, +\}$ , which are extended to  $\mathcal{P}(\mathcal{S}) \rightarrow \mathcal{P}(\mathcal{S})$  by taking images,
- a composition map  $\odot : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{P}(\mathcal{S})$ , which is extended to  $\mathcal{P}(\mathcal{S}) \times \mathcal{P}(\mathcal{S}) \rightarrow \mathcal{P}(\mathcal{S})$  as

$$X \odot Y = \bigcup_{x \in X, y \in Y} x \odot y, \quad \text{for all } X, Y \subseteq \mathcal{S}.$$

It satisfies, for all  $x, y, z \in \mathcal{S}$ ,

- (i) *associativity*:  $\{x\} \odot (y \odot z) = (x \odot y) \odot \{z\}$ ,
- (ii) *units*:  $x \odot \delta^+ x = \{x\}$  and  $\delta^- x \odot x = \{x\}$ ,
- (iii) *locality*:  $x \odot y \neq \emptyset \Leftrightarrow \delta^+ x = \delta^- y$ ,
- (iv) *functionality*:  $\forall z, z' \in x \odot y, z = z'$ .

The cells of single-set categories correspond to arrows of classical categories. The face maps  $\delta^-$  and  $\delta^+$  send each cell in  $\mathcal{S}$  to its *source cell* and *target cell*, respectively, which are *identity cells*. These are in bijective correspondence with objects of classical categories.

Henceforth we tacitly assume that upper indices such as  $\alpha$  in all face maps  $\delta^\alpha$  range over  $\{-, +\}$ . We also write  $\delta^{-\alpha}$  to indicate that values of  $\alpha$  are exchanged relative to an occurrence of  $\delta^\alpha$ . Further, in order to avoid lengthy technical terms, we henceforth refer to single-set categories simply as categories wherever possible.

**2.1.2. Remark.** Omitting the functionality axiom and the right-to-left direction of the locality axiom from the definition of single-set categories yields axioms for *catoids*. Removing locality, functionality and the unit axioms yields *multisemigroups*. See [25] for details.

**2.1.3. Composition as a partial operation.** Two cells  $x, y$  of a category  $\mathcal{S}$  are *composable* if  $x \odot y \neq \emptyset$ , in which case we write  $\Delta(x, y)$  for short. Functionality makes  $\odot$  a partial operation  $\circ : \Delta \hookrightarrow \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$ , which sends each  $(x, y) \in \mathcal{S} \times \mathcal{S}$  to the unique  $z \in x \odot y$  whenever  $\Delta(x, y)$ . As we can recover

$$x \odot y = \begin{cases} \{x \circ y\} & \text{if } \Delta(x, y), \\ \emptyset & \text{otherwise} \end{cases}$$

from  $\delta^-, \delta^+$  and  $\circ$ , we henceforth write  $(\mathcal{S}, \delta^-, \delta^+, \circ)$  instead of  $(\mathcal{S}, \delta^-, \delta^+, \odot)$  and work with  $\circ$  instead of  $\odot$ .

2.1.4. A morphism  $f : \mathcal{S} \rightarrow \mathcal{S}'$  of categories  $\mathcal{S}$  and  $\mathcal{S}'$  is a map satisfying, for all  $x, y \in \mathcal{S}$ ,

$$f \delta^\alpha = \delta'^\alpha f \quad \text{and} \quad \Delta(x, y) \Rightarrow f(x \circ y) = f(x) \circ' f(y).$$

Such morphisms correspond to functors between classical categories.

## 2. Single-set cubical categories

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**2.1.5. Example.** With  $\circ$ , the associativity axiom of categories becomes

$$\begin{aligned}\Delta(x, y \circ z) \wedge \Delta(y, z) &\Leftrightarrow \Delta(x, y) \wedge \Delta(x \circ y, z), \\ \Delta(x, y \circ z) \wedge \Delta(y, z) &\Rightarrow x \circ (y \circ z) = (x \circ y) \circ z.\end{aligned}$$

By the first law, the left-hand side of the associativity law is defined if and only if its right-hand side is. By the second law, the two sides of this law are equal if either side is defined. Likewise, the unit axioms simplify to  $\Delta(x, \delta^+x)$ ,  $\Delta(\delta^-x, x)$ ,  $x \circ \delta^+x = x$  and  $\delta^-x \circ x = x$ .

**2.1.6. Example.** In preparation for the cubical categories below, suppose that the cells of a category are formed by squares that can be composed horizontally, for instance the commuting diagrams in an arrow category. The right unit axiom  $x \circ \delta^+x = x$  above can then be illustrated as

$$\begin{array}{c} \xrightarrow{\quad} \\ \downarrow \quad \downarrow \\ \boxed{x} \\ \downarrow \quad \downarrow \\ \xrightarrow{\quad} \end{array} \circ \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \quad \downarrow \\ \boxed{\delta^+x} \\ \downarrow \quad \downarrow \\ \xrightarrow{\quad} \end{array} = \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \quad \downarrow \\ \boxed{x} \\ \downarrow \quad \downarrow \\ \xrightarrow{\quad} \end{array}$$

The upper and lower faces of  $\delta^+x$  are drawn as equality arrows to indicate that the left and right faces of this cell, shown as dotted arrows, are equal. We assign a more precise semantics to such cubes below.

We frequently need the following laws in calculations. They have been verified with Isabelle [60].

**2.1.7. Lemma.** *Let  $\mathcal{S}$  be a category. Then*

- (i)  $\delta^\alpha \delta^\beta = \delta^\beta$ ,
- (ii)  $\delta^-x = x \Leftrightarrow \delta^+x = x$  for all  $x \in \mathcal{S}$ ,
- (iii)  $\delta^-(x \circ y) = \delta^-x$  and  $\delta^+(x \circ y) = \delta^+y$  for all  $x, y \in \mathcal{S}$  such that  $\Delta(x, y)$ .

**2.1.8. Example.** We can illustrate Lemma 2.1.7(i) as

$$\begin{array}{c} \xrightarrow{\quad} \\ \downarrow \quad \downarrow \\ \boxed{\quad} \\ \downarrow \quad \downarrow \\ \xrightarrow{\quad} \end{array} \xrightarrow{\delta^\beta} \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \quad \downarrow \\ \boxed{\quad} \\ \downarrow \quad \downarrow \\ \xrightarrow{\quad} \end{array} \xrightarrow{\delta^\alpha} \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \quad \downarrow \\ \boxed{\quad} \\ \downarrow \quad \downarrow \\ \xrightarrow{\quad} \end{array}$$

**2.1.9. Fixed points.** Lemma 2.1.7(i) and (ii) imply that the sets of fixed points of  $\delta^-$  and  $\delta^+$  in a category  $\mathcal{S}$  are equal and also equal to the sets of all left and right identities in  $\mathcal{S}$ . We write  $\mathcal{S}^\delta$  for the resulting set of all identities. It corresponds to the set of objects in (small) classical categories. In our examples, identities are illustrated by degenerated cubes, in which some opposite arrows are equality arrows.

## 2.2. Single-set cubical categories

Our single-set axiomatisation of cubical  $\omega$ -categories is based on a family of categories  $(\mathcal{S}, \delta_i^-, \delta_i^+, \circ_i)$  for each  $i \in \mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ . We thus equip our previous notation with indices. In particular we write  $\mathcal{S}^i$  for the set of fixed points of  $\delta_i^\alpha$ .

**2.2.1.** A single-set cubical  $\omega$ -category consists of a family of single-set categories  $(\mathcal{S}, \delta_i^-, \delta_i^+, \circ_i)_{i \in \mathbb{N}_+}$  with symmetry maps  $s_i : \mathcal{S} \rightarrow \mathcal{S}$  and reverse symmetry maps  $\tilde{s}_i : \mathcal{S} \rightarrow \mathcal{S}$  for each  $i \in \mathbb{N}_+$ . These satisfy, for all  $w, x, y, z \in \mathcal{S}$  and  $i, j \in \mathbb{N}_+$ ,

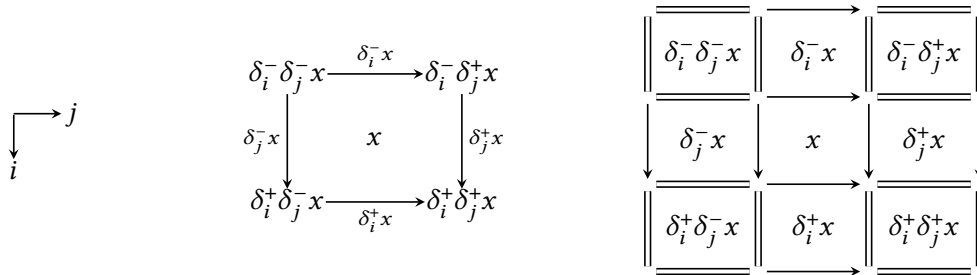
- (i)  $\delta_i^\alpha \delta_j^\beta = \delta_j^\beta \delta_i^\alpha$  if  $i \neq j$ ,
- (ii)  $\delta_i^\alpha(x \circ_j y) = \delta_i^\alpha x \circ_j \delta_i^\alpha y$  if  $i \neq j$  and  $\Delta_j(x, y)$ ,
- (iii)  $(w \circ_i x) \circ_j (y \circ_i z) = (w \circ_j y) \circ_i (x \circ_j z)$  if  $i \neq j$ ,  $\Delta_i(w, x)$ ,  $\Delta_i(y, z)$ ,  $\Delta_j(w, y)$  and  $\Delta_j(x, z)$ ,
- (iv)  $s_i(\mathcal{S}^i) \subseteq \mathcal{S}^{i+1}$  and  $\tilde{s}_i(\mathcal{S}^{i+1}) \subseteq \mathcal{S}^i$ ,
- (v)  $\tilde{s}_i s_i x = x$  and  $s_i \tilde{s}_i y = y$  if  $x \in \mathcal{S}^i$  and  $y \in \mathcal{S}^{i+1}$ ,
- (vi)  $\delta_j^\alpha s_j x = s_j \delta_{j+1}^\alpha x$  and  $\delta_i^\alpha s_j x = s_j \delta_i^\alpha x$  if  $i \neq j, j+1$  and  $x \in \mathcal{S}^j$ ,
- (vii)  $s_i(x \circ_{i+1} y) = s_i x \circ_i s_i y$  and  $s_i(x \circ_j y) = s_i x \circ_j s_i y$  if  $j \neq i, i+1$ ,  $x, y \in \mathcal{S}^i$  and  $\Delta_j(x, y)$ ,
- (viii)  $s_i x = x$  if  $x \in \mathcal{S}^i \cap \mathcal{S}^{i+1}$ ,
- (ix)  $s_i s_j x = s_j s_i x$  if  $|i - j| \geq 2$  and  $x \in \mathcal{S}^i \cap \mathcal{S}^j$ ,
- (x)  $\exists k \in \mathbb{N} \forall i \geq k+1, x \in \mathcal{S}^i$ .

A single-set cubical  $n$ -category, for  $n \in \mathbb{N}$ , is defined by the same data, but the index  $i \in \mathbb{N}_+$  is restricted to  $1 \leq i \leq n$ , and the  $s_i$  and  $\tilde{s}_i$  to  $1 \leq i \leq n-1$ . Likewise, all  $\omega$ -axioms are restricted to these ranges, and Axiom (x) is omitted, as it is now entailed. All results for  $\omega$ -categories in this article restrict to  $n$ -categories. As the  $\omega$ -categories or  $n$ -categories considered in this article are usually cubical, we drop this adjective whenever possible. Hence we often refer to single-set cubical  $\omega$ -categories simply as  $\omega$ -categories and likewise for  $n$ -categories.

As previously, we call the elements of  $\omega$ -categories *cells*. The face maps  $\delta_i^-$  and  $\delta_i^+$  attach *lower* and *upper faces in direction  $i$*  to them. The symmetry maps  $s_i$  and reverse symmetry maps  $\tilde{s}_i$  rotate identities for  $\circ_i$  to identities for  $\circ_{i+1}$  and vice versa.

**2.2.2. Explanation of axioms.** The cells of categories model higher-dimensional cubes, possibly with degenerate faces, which can be composed by gluing them together along their faces. While the examples in Subsection 2.1 provide some intuition for squares in low dimensions, we can illustrate higher dimensional cells and their compositions only through projections.

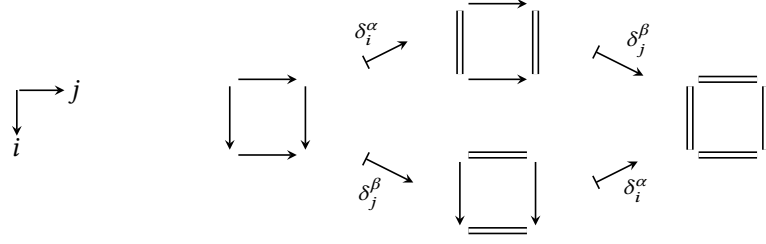
A cell  $x$  and its faces in the directions  $i$  and  $j$  can be illustrated as



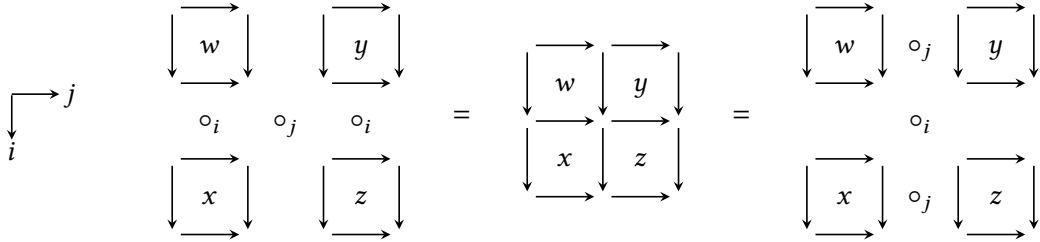
## 2. Single-set cubical categories

The arrows on the left indicate the directions  $i$  and  $j$ . The diagram on the right shows the faces of  $x$  as degenerate cells. They are identities of the composition in the same direction. Many of the axioms  $\omega$ -categories can be illustrated by such diagrams.

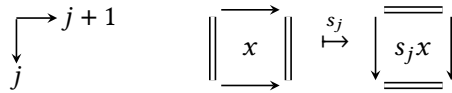
- Axiom (i) determines the cubical cell shape. It can be depicted as



- Axioms (i) and (ii) make face maps morphisms and hence functors with respect to the underlying categories in any other direction.
- The *interchange law* in Axiom (iii) makes composition in any direction bifunctorial with respect to the compositions in any other direction. It can be depicted as



- The *typing axioms* in (iv) restrict  $s_i$  to  $\mathcal{S}^i \rightarrow \mathcal{S}^{i+1}$  and  $\tilde{s}_i$  to  $\mathcal{S}^{i+1} \rightarrow \mathcal{S}^i$ . Though the  $s_i$  and  $\tilde{s}_i$  are defined as total maps on  $\mathcal{S}$ , only their typed versions matter.
- Axiom (v) states that each  $s_i : \mathcal{S}^i \rightarrow \mathcal{S}^{i+1}$  and  $\tilde{s}_i : \mathcal{S}^{i+1} \rightarrow \mathcal{S}^i$  forms a bijective pair.
- Axiom (vi) captures the action of a symmetry map  $s_j$  on a cell  $x \in \mathcal{S}^j$ : it rotates it together with its faces into a cell in  $\mathcal{S}^{j+1}$ :



- Axioms (vi) and (vii) state that the symmetry maps  $s_i$  are morphisms, hence functors, with respect to the categories in any direction  $j \neq i, i + 1$ .
- The *dimensionality* axiom (viii) imposes that  $s_i$  is an identity map on  $\mathcal{S}^i \cap \mathcal{S}^{i+1}$ .
- Axiom (ix) is a *braiding* axiom for symmetries. An illustration would require higher-dimensional cubes.

- Axiom (x) imposes that every cell has finite dimension, as defined below. It is important for the proof of equivalence between single-set cubical categories and their classical counterparts as well as for the coherence of the entire approach.

A more structural explanation of symmetries and their reverses is given in the following subsections.

**2.2.3. Lattice of fixed points.** Let  $\mathcal{S}$  be an  $\omega$ -category. It is easy to see that the set of all  $\mathcal{S}^I = \bigcap_{i \in I} \mathcal{S}^i$ , for  $I \subseteq \mathbb{N}_+$ , forms a lattice with respect to set inclusion. In fact,  $\mathcal{S}^I \subseteq \mathcal{S}^J$  if and only if  $I \supseteq J$ . In this lattice, the  $\mathcal{S}^I$  with co-finite  $I$  model sets of cells of finite dimension.

Formally, a cell  $x \in \mathcal{S}$  has dimension at most  $k \in \mathbb{N}_+$  if  $x \in \mathcal{S}^I$  for some  $I \subseteq \mathbb{N}_+$  with  $|\mathbb{N}_+ \setminus I| = k$ ; it has dimension  $k$  if  $k$  is the least positive integer for which it has dimension at most  $k$ .

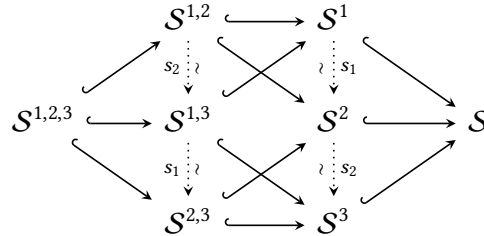
If the complements of the co-finite sets  $I, J \subseteq \mathbb{N}_+$  have the same (finite) cardinality, then the restrictions of symmetries, their reverses and their compositions induce bijections  $\mathcal{S}^I \simeq \mathcal{S}^J$ . These identify cells of the same dimension; they respect face maps and compositions.

For finite  $I = \{i_1, \dots, i_k\}$ , we write  $\mathcal{S}^{i_1, \dots, i_k}$ . In this case, Axiom (2.2.1) (i) implies that  $x \in \mathcal{S}^{i_1, \dots, i_k}$  if and only if  $\delta_{i_1}^{\alpha_1} \dots \delta_{i_k}^{\alpha_k} x = x$  for all  $\alpha_1, \dots, \alpha_k$ .

Finally, we write  $\mathcal{S}^{>n}$  for  $\mathcal{S}^I$  when  $I = \{i \in \mathbb{N} \mid i > n\}$ . All cells in  $\mathcal{S}^{>n}$  have dimension at most  $n$ . Axiom (x) implies that  $\mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}^{>n}$ . Hence every cell  $x \in \mathcal{S}$  has indeed some finite dimension, which is crucial for the constructions of the equivalence proof relating single-set cubical categories with their classical counterparts in Section 3. In particular, Axiom (x) thus imposes that the gradation of  $\mathcal{S}$  in terms of  $\mathcal{S}^{>n}$  corresponds to the gradation of the cubical sets  $K_n$  shown in the introduction.

**2.2.4. Remark.** The definition of symmetry in (2.2.1) differs from the notions of symmetric cubical monoid and category, introduced by Grandis and Mauri [37] and Grandis [35], respectively.

**2.2.5. Example.** The lattice of fixed points for the 3-category  $\mathcal{S}$  is given by the Hasse diagram



All cells in  $\mathcal{S}^{1,2,3}$  have dimension 0, those in  $\mathcal{S}^{1,2}$ ,  $\mathcal{S}^{1,3}$  and  $\mathcal{S}^{2,3}$  have dimension at most 1, those in  $\mathcal{S}^1$ ,  $\mathcal{S}^2$  and  $\mathcal{S}^3$  dimension at most 2, and those in  $\mathcal{S}$  dimension at most 3.

**2.2.6. Categories of cubical categories.** A morphism  $f : \mathcal{S} \rightarrow \mathcal{S}'$  of  $\omega$ -categories  $\mathcal{S}$  and  $\mathcal{S}'$  is a morphism of the underlying categories, for each  $i \in \mathbb{N}_+$ , which preserves symmetries restricted to their types. Hence, for  $i \geq 1$  and  $x \in \mathcal{S}^i$ ,  $f s_i x = s'_i f x$ . This defines the category  $\text{SCub}_\omega$  of single-set cubical  $\omega$ -categories.

Owing to the definition in (2.1.4), a morphism  $f : \mathcal{S} \rightarrow \mathcal{S}'$  of  $\omega$ -categories restricts to a map  $\mathcal{S}^I \rightarrow \mathcal{S}'^I$  for each  $I \subseteq \mathbb{N}_+$ . These restrictions also commute with the symmetries.

A morphism of  $n$ -categories is defined as above, but with symmetries ranging over  $1 \leq i \leq n - 1$ . This defines the category  $\text{SCub}_n$  of single-set cubical  $n$ -categories.

**2.2.7. Lemma.** Morphisms in  $\text{SCub}_\omega$  preserve typed reverse symmetry maps:  $f \tilde{s}_i x = \tilde{s}'_i f x$ , for all  $f : \mathcal{S} \rightarrow \mathcal{S}'$ ,  $i \in \mathbb{N}_+$  and  $x \in \mathcal{S}^{i+1}$ .

## 2. Single-set cubical categories

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**2.2.8. Truncations.** There is a truncation functor  $U_m^n : \text{SCub}_n \rightarrow \text{SCub}_m$ , for all  $1 \leq m \leq n$ , which forgets the  $k$ -dimensional structure for each  $k > m$ . It sends each  $n$ -category  $\mathcal{S}$  to the  $m$ -category on the set  $\mathcal{S}^{m+1, \dots, n}$  and restricts the ranges of face maps and compositions to  $1 \leq i \leq m$ , as well as that of the symmetry maps to  $1 \leq i \leq m-1$ .

There is also a truncation functor  $U_m : \text{SCub}_\omega \rightarrow \text{SCub}_m$ , which sends each  $\omega$ -category  $\mathcal{S}$  to the  $m$ -category with set  $\mathcal{S}^{>m}$  and restricts face and symmetry maps as well as compositions as above.

**2.2.9. Properties of symmetries.** The following facts provide further structural properties of symmetries and reverse symmetries. They have been proved with Isabelle.

**2.2.10. Lemma.** *Let  $\mathcal{S}$  be an  $\omega$ -category. For all  $i, j \in \mathbb{N}_+$ ,*

- (i)  $\delta_{j+1}^\alpha s_j x = s_j \delta_j^\alpha x$  if  $x \in \mathcal{S}^j$ ,
- (ii)  $s_i(x \circ_i y) = s_i x \circ_{i+1} s_i y$  if  $x, y \in \mathcal{S}^i$  and  $\Delta_j(x, y)$ ,
- (iii) *Yang-Baxter:*  $s_i s_{i+1} s_i x = s_{i+1} s_i s_{i+1} x$  if  $x \in \mathcal{S}^{i, i+1}$ .

An automatic proof of Lemma 2.2.10(ii), called *sym-func1* in our Isabelle component, is shown in Section 4.3.

Using Axiom 2.2.1(v), we can further derive properties for  $\tilde{s}$  that are dual to the symmetry axioms.

**2.2.11. Lemma.** *Let  $\mathcal{S}$  be an  $\omega$ -category. For all  $i, j \in \mathbb{N}_+$ ,*

- (i) if  $x \in \mathcal{S}^{j+1}$ , then

$$\delta_i^\alpha \tilde{s}_j x = \begin{cases} \tilde{s}_j \delta_{j+1}^\alpha x & \text{if } i = j, \\ \tilde{s}_j \delta_j^\alpha x & \text{if } i = j + 1, \\ \tilde{s}_j \delta_i^\alpha x & \text{otherwise.} \end{cases}$$

- (ii) if  $x, y \in \mathcal{S}^{i+1}$  and  $\Delta_j(x, y)$ , then

$$\tilde{s}_i(x \circ_j y) = \begin{cases} \tilde{s}_i x \circ_{i+1} \tilde{s}_i y & \text{if } j = i, \\ \tilde{s}_i x \circ_i \tilde{s}_i y & \text{if } j = i + 1, \\ \tilde{s}_i x \circ_j \tilde{s}_i y & \text{otherwise,} \end{cases}$$

- (iii)  $\tilde{s}_i x = x$  if  $x \in \mathcal{S}^{i, i+1}$ ,

- (iv) if  $|i - j| \geq 2$ ,  $x \in \mathcal{S}^{i, j+1}$ ,  $y \in \mathcal{S}^{i+1, j}$  and  $z \in \mathcal{S}^{i+1, j+1}$ , then

$$s_i \tilde{s}_j x = \tilde{s}_j s_i x, \quad \tilde{s}_i s_j y = s_j \tilde{s}_i y, \quad \tilde{s}_i \tilde{s}_j z = \tilde{s}_j \tilde{s}_i z,$$

- (v)  $\tilde{s}_i \tilde{s}_{i+1} \tilde{s}_i x = \tilde{s}_{i+1} \tilde{s}_i \tilde{s}_{i+1} x$  if  $x \in \mathcal{S}^{i+1, i+2}$ .

Lemma 2.2.11(i) can be depicted, for  $x \in \mathcal{S}^{j+1}$ , as

$$\begin{array}{ccc} \begin{array}{c} \xrightarrow{j+1} \\ \downarrow \\ \xrightarrow{j} \end{array} & \begin{array}{c} \overline{\overline{x}} \\ \downarrow \\ \overline{\overline{x}} \end{array} & \xrightarrow{\tilde{s}_j} \begin{array}{c} \overline{\overline{\tilde{s}_j x}} \\ \downarrow \\ \overline{\overline{\tilde{s}_j x}} \end{array} \end{array}$$

An interactive proof of the third case in this part of Lemma 2.2.11, called *inv-sym-face* in our Isabelle component, is also shown in Subsection 4.3.



## 2. Single-set cubical categories

- The first two equations of the *corner axiom* (ii) can be shown as

$$\begin{array}{c} \rightarrow i+1 \\ \downarrow i \end{array} \quad \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \circ_{i+1} \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} y = \begin{array}{|c|c|} \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \\ \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} x & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} y \\ \hline \end{array}$$

- Axioms (i) and (ii) impose that connection maps are morphisms and hence functors with respect to the underlying categories in any direction  $j \neq i, i+1$ .
- By Axiom (iii),  $\gamma_i^\alpha$  is an identity map on  $\mathcal{S}^{i,i+1}$ .
- The *zigzag axiom*, (iv) is depicted as

$$\begin{array}{c} \rightarrow i+1 \\ \downarrow i \end{array} \quad \begin{array}{|c|c|} \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \\ \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} x & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} x \\ \hline \end{array} = \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} x$$

- The *braiding axiom* (v) is hard to illustrate: four dimensions would be required for drawing it.
- Finally, Axiom (vi) relates connections in different directions:

$$\begin{array}{c} \rightarrow i+2 \\ \rightarrow i+1 \\ \downarrow i \end{array} \quad \gamma_{i+1}^+ x = \begin{array}{|c|c|c|} \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \\ \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \\ \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \\ \hline \end{array} \xrightarrow{s_i} \begin{array}{|c|c|c|} \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \\ \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \\ \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \\ \hline \end{array} \xrightarrow{s_{i+1}} \begin{array}{|c|c|c|} \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \\ \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \\ \hline \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \\ \hline \end{array} = \gamma_i^+ s_{i+1} x$$

**2.3.3. Category of  $\omega$ -categories with connections.** A morphism  $f : \mathcal{S} \rightarrow \mathcal{S}'$  of  $\omega$ -categories with connections is a morphism of  $\omega$ -categories that preserves connections:  $f\gamma_i^\alpha x = \gamma_i'^\alpha f x$ , for all  $i \in \mathbb{N}_+$  and  $x \in \mathcal{S}^i$ . This defines the category  $\text{SCub}_\omega^Y$  of single-set cubical  $\omega$ -categories with connections. In a same way, we define the category  $\text{SCub}_n^Y$  of  $n$ -categories with connections.

For  $1 \leq m \leq n$ , there is a truncation functor  $U_m^n : \text{SCub}_n^Y \rightarrow \text{SCub}_m^Y$  that forgets the  $k$ -dimensional structure for  $k > m$ . It sends any  $n$ -category with connections  $\mathcal{S}$  to the  $m$ -category with connections with set  $\mathcal{S}^{m+1, \dots, n}$ , keeping only the face maps and compositions up to  $m$  and the symmetries and connections up to  $m-1$ .

There is also a truncation functor  $U_m : \text{SCub}_\omega^Y \rightarrow \text{SCub}_m^Y$  that forgets the  $k$ -dimensional structure for  $k > m$ . It sends any  $\omega$ -category with connections  $\mathcal{S}$  to the  $m$ -category with connections with set  $\mathcal{S}^{>m}$ , keeping only the face maps and compositions indexed up to  $m$  and the symmetries and connections indexed up to  $m-1$ .

The following properties of connections have been proved using Isabelle.

**2.3.4. Lemma.** Let  $\mathcal{S}$  be an  $\omega$ -category with connections. For all  $i, j \in \mathbb{N}_+$ ,

- (i)  $\delta_j^\alpha \gamma_j^{-\alpha} x = \delta_{j+1}^\alpha x$  and  $\delta_{j+1}^\alpha \gamma_j^{-\alpha} x = \delta_{j+1}^\alpha x$  if  $x \in \mathcal{S}^i$ ,
- (ii) if  $j \neq i, i+1, x, y \in \mathcal{S}^i$  and  $\Delta_{i+1}(x, y)$ , then
- $$\gamma_i^+(x \circ_{i+1} y) = (\gamma_i^+ x \circ_i x) \circ_{i+1} (s_i x \circ_i \gamma_i^+ y), \quad \text{and} \quad \gamma_i^-(x \circ_{i+1} y) = (\gamma_i^- x \circ_i s_i y) \circ_{i+1} (y \circ_i \gamma_i^- y),$$
- (iii)  $\gamma_i^\alpha s_j x = s_j \gamma_i^\alpha x$  and  $\gamma_i^\alpha \widetilde{s}_j y = \widetilde{s}_j \gamma_i^\alpha y$  if  $|i - j| \geq 2, x \in \mathcal{S}^{i,j}$  and  $y \in \mathcal{S}^{i,j+1}$ ,
- (iv)  $\widetilde{s}_i \widetilde{s}_{i+1} \gamma_i^\alpha x = \gamma_{i+1}^\alpha \widetilde{s}_{i+1} x$  if  $x \in \mathcal{S}^{i,i+2}$ .

## 2.4. Inverses

Applications in higher rewriting and homotopy type theory require cubical  $(\omega, p)$ -categories, where cells of dimension strictly greater than  $p$  are invertible. More specifically,  $p = 0$  is needed for homotopy type theory [9] and the categorification of abstract rewriting, while the categorification of string rewriting requires  $p = 1$ , and that of term and diagram rewriting  $p = 2$  [4]. Here, we translate an approach introduced by Lucas [45] to single-sets, following his notation closely.

**2.4.1. Single-set cubical categories with inverses.** A cell  $x$  of an  $\omega$ -category with connections  $\mathcal{S}$  is  $r_i$ -invertible, for  $i \in \mathbb{N}_+$ , if there exists a cell  $y \in \mathcal{S}$  such that

$$\Delta_i(x, y), \quad x \circ_i y = \delta_i^- x, \quad \Delta_i(y, x), \quad y \circ_i x = \delta_i^+ x.$$

A cell  $x \in \mathcal{S}^{>n}$  has an  $r_i$ -invertible  $n - 1$ -shell, for  $k, i \in \mathbb{N}_+$ , if  $\delta_j^\alpha x$  is  $r_i$ -invertible for each  $1 \leq j \leq n$  with  $j \neq i$ .

A single-set cubical  $(\omega, p)$ -category (with connections) is a single-set cubical  $(\omega, p)$ -category (with connections)  $\mathcal{S}$  such that every cell in  $\mathcal{S}^{>n}$  with an  $r_i$ -invertible  $n - 1$ -shell is  $r_i$ -invertible for all  $n \geq p + 1$  and  $1 \leq i \leq n$ .

We have shown with Isabelle that the  $r_i$ -inverse of any  $x$  is uniquely defined. We therefore write  $r_i x$  for the  $r_i$ -inverse of  $x$ .

We restrict these definitions to  $n$ -categories with connections as usual, removing the indices outside the range  $1 \leq i \leq n$ . In particular, a single-set cubical  $(n, p)$ -category is a single-set cubical  $n$ -category  $\mathcal{S}$  such that every cell in  $\mathcal{S}^{k+1, \dots, n}$  with an  $r_i$ -invertible  $k - 1$ -shell is  $r_i$ -invertible for all  $p + 1 \leq k \leq n$  and  $1 \leq i \leq k$ ,

**2.4.2. Category of  $(\omega, p)$ -categories.** A morphism of  $(\omega, p)$ -categories is simply a morphism in  $\text{SCub}_{\omega}^Y$ . Inverses are preserved because  $\Delta_i(x, r_i x)$  implies  $\Delta_i(f(x), f(r_i x))$  and  $f(x) \circ_i f(r_i x) = \delta_i^- f(x)$ , and likewise for the opposite order of composition. Thus  $f(x)$  is  $r_i$ -invertible with inverse  $f(r_i x)$ . The situation for  $(n, p)$ -categories is similar. This defines the categories  $\text{SCub}_{(\omega, p)}^Y$  and  $\text{SCub}_{(n, p)}^Y$ .

**2.4.3. Explanation of inverse maps.** A  $(1, 0)$ -category  $\mathcal{S}$  is a category with a map  $r_1 : \mathcal{S} \rightarrow \mathcal{S}$  such that  $\Delta_1(x, r_1 x)$ ,  $\Delta_1(r_1 x, x)$ ,  $x \circ_1 r_1 x = \delta_1^- x$  and  $r_1 x \circ_1 x = \delta_1^+ x$ , for every  $x \in \mathcal{S}$ . This defines a groupoid. The cell  $x$  is sent by  $r_1$  to the backward arrow in the following diagram

$$r_1 \left( \delta_1^- x \xrightarrow{x} \delta_1^+ x \right) = \delta_1^+ x \leftarrow \delta_1^- x.$$

## 2. Single-set cubical categories

A  $(2, 0)$ -category  $\mathcal{C}$  is a 2-category with maps  $r_1, r_2 : \mathcal{S} \rightarrow \mathcal{S}$  such that  $\Delta_i(x, r_i x), \Delta_i(r_i x, x), x \circ_i r_i x = \delta_i^- x, r_i x \circ_i x = \delta_i^+ x$ , for  $i \in \{1, 2\}$  and  $x \in \mathcal{S}$ . The inverses of  $x$  are depicted as

$$r_1 \left( \begin{array}{ccc} & \xrightarrow{\delta_1^- x} & \\ \delta_2^- x \downarrow & x & \downarrow \delta_2^+ x \\ & \xrightarrow{\delta_1^+ x} & \end{array} \right) = r_1 \delta_2^- x \begin{array}{ccc} & \xrightarrow{\delta_1^+ x} & \\ \uparrow r_1 x & & \uparrow r_1 \delta_2^+ x \\ & \xrightarrow{\delta_1^- x} & \end{array}, \quad r_2 \left( \begin{array}{ccc} & \xrightarrow{\delta_1^- x} & \\ \delta_2^- x \downarrow & x & \downarrow \delta_2^+ x \\ & \xrightarrow{\delta_1^+ x} & \end{array} \right) = \delta_2^+ x \begin{array}{ccc} & \xrightarrow{r_2 \delta_1^- x} & \\ \downarrow r_2 x & & \downarrow \delta_2^- x \\ & \xrightarrow{r_2 \delta_1^+ x} & \end{array}.$$

**2.4.4. Truncation.** The truncation functors  $U_m^n : \text{SCub}_n^Y \rightarrow \text{SCub}_m^Y$  and  $U_m : \text{SCub}_\omega^Y \rightarrow \text{SCub}_m^Y$  induce functors  $U_m^n : \text{SCub}_{(n,p)}^Y \rightarrow \text{SCub}_{(m,p)}^Y$  and  $U_m : \text{SCub}_{(\omega,p)}^Y \rightarrow \text{SCub}_{(m,p)}^Y$  for  $p \leq m \leq n$ .

**2.4.5. Remark.** For  $i \in \mathbb{N}_+$ , every cell in  $\mathcal{S}^i$  is its own  $r_i$ -inverse.

**2.4.6. Lemma.** Let  $\mathcal{S}$  be an  $(\omega, p)$ -category with connections. For every  $i, j \in \mathbb{N}_+$  and every  $r_i$ -invertible  $x, y \in \mathcal{S}$ ,

- (i)  $\delta_i^\alpha r_i x = \delta_i^{-\alpha} x$  and  $\delta_j^\alpha r_i x = r_i \delta_j^\alpha x$  if  $j \neq i$ ,
- (ii) if  $j \neq i$ , then  $r_i(x \circ_i y) = r_i y \circ_i r_i x$  if  $\Delta_i(x, y)$ , and  $r_i(x \circ_j y) = r_i x \circ_j r_i y$  if  $\Delta_j(x, y)$ ,
- (iii)  $r_i s_{i-1} x = s_{i-1} x$  and  $r_i s_j y = s_j r_i y$  if  $j \neq i-1, x \in \mathcal{S}^{i-1}$  and  $y \in \mathcal{S}^j$ ,
- (iv)  $r_i \tilde{s}_i x = \tilde{s}_i x$  and  $r_i \tilde{s}_j y = \tilde{s}_j r_i y$  if  $j \neq i, x \in \mathcal{S}^{i+1}$  and  $y \in \mathcal{S}^{j+1}$ ,
- (v)  $r_i \gamma_j^\alpha x = \gamma_j^\alpha r_i x$  if  $i \neq j, j+1$  and  $x \in \mathcal{S}^j$ .

*Proof.* We prove the first item of (i) as an example. By  $r_i$ -invertibility of  $x$ ,  $\Delta_i(x, r_i x)$  and  $\Delta_i(r_i x, x)$ . Hence  $\delta_i^+ x = \delta_i^- r_i x$  and  $\delta_i^+ r_i x = \delta_i^- x$ . Suppose  $j \neq i$ . Then  $z = \delta_j^\alpha r_i x$  satisfies

$$\Delta_i(\delta_j^\alpha x, z), \quad \delta_j^\alpha x \circ_i z = \delta_i^- \delta_j^\alpha x, \quad \Delta_i(z, \delta_j^\alpha x), \quad z \circ_i \delta_j^\alpha x = \delta_i^+ \delta_j^\alpha x.$$

Thus  $z = r_i \delta_j^\alpha x$ , because  $r_i$ -inverses are unique. The remaining proofs are similar.  $\square$

**2.4.7. Lemma.** Let  $\mathcal{S}$  be an  $(\omega, p)$ -category. If  $x \in \mathcal{S}^{>n}$  is  $r_i$ -invertible, then  $r_i x \in \mathcal{S}^{>n}$ .

*Proof.* Suppose  $x$  is  $r_i$ -invertible. For any  $m \geq n+1$ , apply  $\delta_m^\alpha$  to  $x \circ_i r_i x = \delta_i^- x$  and  $x \circ_i r_i x = \delta_i^- x$ . This yields  $\delta_m^\alpha r_i x \circ_i x = \delta_i^+ x$ . Uniqueness of  $r_i x$  then implies that  $\delta_m^\alpha r_i x = r_i x$ .  $\square$

**2.4.8. Proposition.** Every cell in an  $(\omega, 0)$ -category is  $r_i$ -invertible for each  $i \in \mathbb{N}_+$ .

*Proof.* By Axiom (x),  $\mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}^{>n}$ . We show by induction on the dimension  $n$  of cells that every cell  $x \in \mathcal{S}^{>n}$  is  $r_i$ -invertible for all  $i \in \mathbb{N}_+$ .

For  $n = 0$ , suppose  $x \in \mathcal{S}^{>0}$ . Then  $x$  is its own  $r_i$ -inverse for all  $i \in \mathbb{N}_+$ .

Suppose the property holds for  $n-1$ . Let  $i \in \mathbb{N}_+$  and  $x \in \mathcal{S}^{>n}$ . If  $i \geq n+1$ , then  $x$  is its own  $r_i$ -inverse, so suppose  $i \leq n$ . Let  $1 \leq j \leq n$  with  $i \neq j$ . We have  $\delta_j^\alpha x \in \mathcal{S}^{j, n+1, n+2, \dots}$  and thus  $s_{n-1} \dots s_j \delta_j^\alpha x \in \mathcal{S}^{>n}$  using Axioms (iv) and (vi) of Definition 2.2.1. There are two cases depending on the value of  $i$ .

- If  $j < i \leq n$ , then  $s_{n-1} \dots s_j \delta_j^\alpha x$  has an  $r_{i-1}$ -inverse  $y$  by the induction hypothesis. Writing  $z = \tilde{s}_j \dots \tilde{s}_{n-1} y$  and using property (ii) of Lemma 2.2.11,

$$\begin{aligned}
 z \circ_i \delta_j^\alpha x &= \tilde{s}_j \dots \tilde{s}_{n-1} y \circ_i \tilde{s}_j \dots \tilde{s}_{n-1} s_{n-1} \dots s_j \delta_j^\alpha x \\
 &= \tilde{s}_j \dots \tilde{s}_{n-1} (y \circ_{i-1} s_{n-1} \dots s_j \delta_j^\alpha x) \\
 &= \tilde{s}_j \dots \tilde{s}_{n-1} \delta_{i-1}^+ s_{n-1} \dots s_j \delta_j^\alpha x \\
 &= \delta_i^+ \delta_j^\alpha x.
 \end{aligned}$$

Similarly  $\delta_j^\alpha x \circ_i z = \delta_i^- \delta_j^\alpha x$ ,  $\Delta_i(z, \delta_j^\alpha x)$  and  $\Delta_i(\delta_j^\alpha x, z)$ . So  $z$  is the  $r_i$ -inverse of  $\delta_j^\alpha x$ .

- If  $i < j$  then  $s_{n-1} \dots s_j \delta_j^\alpha x$  has an  $r_i$ -inverse  $y$  by the induction hypothesis. Using the same abbreviation and lemma as before,

$$\begin{aligned}
 z \circ_i \delta_j^\alpha x &= \tilde{s}_j \dots \tilde{s}_{n-1} y \circ_i \tilde{s}_j \dots \tilde{s}_{n-1} s_{n-1} \dots s_j \delta_j^\alpha x \\
 &= \tilde{s}_j \dots \tilde{s}_{n-1} (y \circ_i s_{n-1} \dots s_j \delta_j^\alpha x) \\
 &= \tilde{s}_j \dots \tilde{s}_{n-1} \delta_i^+ s_{n-1} \dots s_j \delta_j^\alpha x \\
 &= \delta_i^+ \delta_j^\alpha x,
 \end{aligned}$$

using property (ii) of Lemma 2.2.11. Likewise,  $\delta_j^\alpha x \circ_i z = \delta_i^- \delta_j^\alpha x$ ,  $\Delta_i(z, \delta_j^\alpha x)$  and  $\Delta_i(\delta_j^\alpha x, z)$ , and it follows again that  $z$  is the  $r_i$ -inverse of  $\delta_j^\alpha x$ .

This shows that  $x$  has an  $r_i$ -invertible  $n - 1$ -shell and is therefore  $r_i$ -invertible. □

A formalisation of this proof with Isabelle is shown in Subsection 4.6.

## 3. EQUIVALENCE WITH CLASSICAL CUBICAL CATEGORIES

We now present our main theorems: a series of equivalences which justify our single-set axioms relative to the classical ones. We begin in Subsection 3.1 by recalling the classical axioms for cubical categories. For cubical categories with connection, we use the axioms of Al-Agl, Brown and Steiner [2]; for inverses, we follow Lucas [45]. These categories, with appropriate functors between them, form the categories  $\text{Cub}_\omega$  without connections,  $\text{Cub}_\omega^\Gamma$  with connections and  $\text{Cub}_{(\omega,p)}^\Gamma$  with inverses. In Subsection 3.2, we prove an equivalence of categories  $\text{SCub}_\omega \simeq \text{Cub}_\omega$ , in Subsection 3.3 we extend it to  $\text{SCub}_\omega^Y \simeq \text{Cub}_\omega^\Gamma$  and in Subsection 3.4 further to  $\text{SCub}_{(\omega,p)}^Y \simeq \text{Cub}_{(\omega,p)}^\Gamma$ .

Straightforward modifications of these proofs lead to equivalences between the corresponding  $n$ -categories. We do not list them explicitly.

### 3.1. Cubical categories

First we recall the classical definitions of cubical categories as cubical sets with cell compositions.

### 3. Equivalence with classical cubical categories

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**3.1.1.** A (cubical)  $\omega$ -category is a family  $C = (C_n)_{n \in \mathbb{N}}$  of sets of  $n$ -cells with face maps  $\partial_{n,i}^\alpha : C_n \rightarrow C_{n-1}$ , degeneracy maps  $\varepsilon_{n,i} : C_{n-1} \rightarrow C_n$  and compositions  $\star_{n,i} : C_n \times_{n,i} C_n \rightarrow C_n$ , for  $1 \leq i \leq n$ , where  $C_n \times_{n,i} C_n$  denotes the pullback of the cospan  $\partial_{n,i}^+ : C_n \rightarrow C_{n-1} \leftarrow C_n : \partial_{n,i}^-$ . These satisfy, for all  $1 \leq i, j \leq n$ ,

(i)  $a \star_{n,i} (b \star_{n,i} c) = (a \star_{n,i} b) \star_{n,i} c$  if either side is defined,

(ii)  $a \star_{n,i} \varepsilon_{n,i} \partial_{n,i}^+ a = \varepsilon_{n,i} \partial_{n,i}^- a \star_{n,i} a = a$ ,

(iii)  $\partial_{n-1,i}^\alpha \partial_{n,j}^\beta = \partial_{n-1,j-1}^\beta \partial_{n,i}^\alpha$  if  $i < j$ ,

(iv) if  $a, b$  are  $\star_{n,j}$ -composable then

$$\partial_{n,i}^\alpha (a \star_{n,j} b) = \begin{cases} \partial_{n,i}^\alpha a \star_{n,j-1} \partial_{n,i}^\alpha b & \text{if } i < j, \\ \partial_{n,i}^- a & \text{if } i = j \text{ and } \alpha = -, \\ \partial_{n,i}^+ b & \text{if } i = j \text{ and } \alpha = +, \\ \partial_{n,i}^\alpha a \star_{n,j} \partial_{n,i}^\alpha b & \text{if } i > j, \end{cases}$$

(v) if  $a, b$  are  $\star_{n,i}$ -composable,  $c, d$  are  $\star_{n,i}$ -composable,  $a, c$  are  $\star_{n,j}$ -composable and  $b, d$  are  $\star_{n,j}$ -composable, then  $(a \star_{n,i} b) \star_{n,j} (c \star_{n,i} d) = (a \star_{n,j} c) \star_{n,i} (b \star_{n,j} d)$ ,

(vi)

$$\partial_{n,i}^\alpha \varepsilon_{n,j} = \begin{cases} \varepsilon_{n-1,j-1} \partial_{n-1,i}^\alpha & \text{if } i < j, \\ id_{C_{n-1}} & \text{if } i = j, \\ \varepsilon_{n-1,j} \partial_{n-1,i-1}^\alpha & \text{if } i > j, \end{cases}$$

(vii) if  $a, b$  are  $\star_{n,j}$ -composable then

$$\varepsilon_{n+1,i} (a \star_{n,j} b) = \begin{cases} \varepsilon_{n+1,i} a \star_{n+1,j+1} \varepsilon_{n+1,i} b & \text{if } i \leq j, \\ \varepsilon_{n+1,i} a \star_{n+1,j} \varepsilon_{n+1,i} b & \text{if } i > j, \end{cases}$$

(viii)  $\varepsilon_{n+1,i} \varepsilon_{n,j} = \varepsilon_{n+1,j+1} \varepsilon_{n,i}$  if  $i \leq j$ .

A (cubical)  $n$ -category is a family  $C = (C_k)_{k \leq n}$  of sets of  $k$ -cells with face maps  $\partial_{k,i}^\alpha : C_k \rightarrow C_{k-1}$ , degeneracy maps  $\varepsilon_{k,i} : C_{k-1} \rightarrow C_k$  and composition maps  $\star_{k,i} : C_k \times_{k,i} C_k \rightarrow C_k$ , for  $1 \leq i \leq k \leq n$ . These satisfy the above axioms, removing those involving face maps, compositions and degeneracies outside the appropriate ranges.

Each degeneracy map  $\varepsilon_{n,i}$  yields identities for  $\star_{n,i}$ .

**3.1.2.** A (cubical)  $\omega$ -category with connections  $C$  is an  $\omega$ -category with connection maps  $\Gamma_{n,i}^\alpha : C_{n-1} \rightarrow C_n$  for  $1 \leq i < n$ , such that

(i)

$$\partial_{n,i}^\alpha \Gamma_{n,j}^\beta = \begin{cases} \Gamma_{n-1,j-1}^\beta \partial_{n-1,i}^\alpha & \text{if } i < j, \\ id_{C_{n-1}} & \text{if } i = j, j+1 \text{ and } \alpha = \beta, \\ \varepsilon_{n-1,j} \partial_{n-1,i}^\alpha & \text{if } i = j, j+1 \text{ and } \alpha = -\beta, \\ \Gamma_{n-1,j}^\beta \partial_{n-1,i-1}^\alpha & \text{if } i > j+1, \end{cases}$$

(ii) if  $a, b$  are  $\star_{n,j}$ -composable then

$$\Gamma_{n+1,i}^\alpha(a \star_{n,j} b) = \begin{cases} \Gamma_{n+1,i}^\alpha a \star_{n+1,j+1} \Gamma_{n+1,i}^\alpha b & \text{if } i < j, \\ (\Gamma_{n+1,i}^- a \star_{n+1,i} \varepsilon_{n+1,i+1} b) \star_{n+1,i+1} (\varepsilon_{n+1,i} b \star_{n+1,i} \Gamma_{n+1,i}^- b) & \text{if } i = j \text{ and } \alpha = -, \\ (\Gamma_{n+1,i}^+ a \star_{n+1,i} \varepsilon_{n+1,i} a) \star_{n+1,i+1} (\varepsilon_{n+1,i+1} a \star_{n+1,i} \Gamma_{n+1,i}^+ b) & \text{if } i = j \text{ and } \alpha = +, \\ \Gamma_{n+1,i}^\alpha a \star_{n+1,j} \Gamma_{n+1,i}^\alpha b & \text{if } i > j, \end{cases}$$

(iii)  $\Gamma_{n,i}^+ a \star_{n,i} \Gamma_{n,i}^- a = \varepsilon_{n,i+1} a$  and  $\Gamma_{n,i}^+ a \star_{n,i+1} \Gamma_{n,i}^- a = \varepsilon_{n,i} a$ ,

(iv)

$$\Gamma_{n+1,i}^\alpha \varepsilon_{n,j} = \begin{cases} \varepsilon_{n+1,j+1} \Gamma_{n,i}^\alpha & \text{if } i < j, \\ \varepsilon_{n+1,i} \varepsilon_{n,i} & \text{if } i = j, \\ \varepsilon_{n+1,j} \Gamma_{n,i-1}^\alpha & \text{if } i > j, \end{cases}$$

(v)

$$\Gamma_{n+1,i}^\alpha \Gamma_{n,j}^\beta = \begin{cases} \Gamma_{n+1,j+1}^\beta \Gamma_{n,i}^\alpha & \text{if } i < j, \\ \Gamma_{n+1,i+1}^\alpha \Gamma_{n,i}^\alpha & \text{if } i = j \text{ and } \alpha = \beta. \end{cases}$$

A (cubical)  $n$ -category with connections is an  $n$ -category with connection maps  $\Gamma_{k,i}^\alpha : C_{k-1} \rightarrow C_k$  for  $1 \leq i < k \leq n$ , satisfying axioms with the appropriate index restrictions.

There is some index shifting in the above axioms. For instance in the axiom describing the faces of compositions 3.1.1(iv) and the one describing the faces of connections 3.1.2(i), there is a  $j - 1$  index appearing in the case where  $j > i$ , which does not appear in the other cases. Furthermore, there are two indices, for the dimension  $k$  and the direction  $i$ . Their single-set counterparts 2.2.1(ii) and 2.3.1(i) do not have such explicit index shifting and the dimension index is missing.

**3.1.3. Cubical categories with inverses.** Let  $1 \leq i \leq n$ . An  $n$ -cell  $a$  of an  $\omega$ -category with connections  $C$  is  $R_{n,i}$ -invertible if there is a  $n$ -cell  $b$  such that

$$a \star_{n,i} b = \varepsilon_{n,i} \partial_{n,i}^- a \quad \text{and} \quad b \star_{n,i} a = \varepsilon_{n,i} \partial_{n,i}^+ a.$$

Each  $R_{n,i}$ -inverse of an  $n$ -cell  $a$  is unique and denoted  $R_{n,i} a$ . A  $n$ -cell  $a$  has an  $R_{n-1,i}$ -invertible shell if the cells  $\partial_{n,j}^\alpha a$  are  $R_{n-1,i-1}$ -invertible for all  $1 \leq j < i$ , and the cells  $\partial_{n,j}^\alpha a$  are  $R_{n-1,i}$ -invertible for all  $i < j \leq n$ .

A (cubical)  $(\omega, p)$ -category (with connections)  $C$ , for  $p \in \mathbb{N}$ , is an  $\omega$ -category with connections in which, for all  $n > p$  and  $1 \leq i \leq n$ , every  $n$ -cell with an  $R_{n-1,i}$ -invertible shell is  $R_{n,i}$ -invertible.

The above definitions of invertibility and invertible shell extend to  $n$ -categories with connections, by removing the  $R_{k,i}$  with indices  $(k, i)$  whenever  $k > n$ . In particular, for  $0 \leq p \leq n$ , a (cubical)  $(n, p)$ -category (with connections)  $C$  is an  $n$ -category with connections in which, for all  $p + 1 \leq k \leq n$  and  $1 \leq i \leq k$ , every  $k$ -cell with an  $R_{k-1,i}$ -invertible shell is  $R_{k,i}$ -invertible.

### 3. Equivalence with classical cubical categories

**3.1.4. Categories of cubical categories.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  of  $\omega$ -categories is a family of maps  $(F_n : C_n \rightarrow \mathcal{D}_n)_{n \in \mathbb{N}}$  that preserve all face, degeneracy and composition maps:

$$F_{n-1} \partial_{n,i}^\alpha = \partial_{n,i}^\alpha F_n, \quad F_n(a \star_{n,i} b) = F_n a \star_{n,i} F_n b, \quad F_n \varepsilon_{n,i} = \varepsilon_{n,i} F_{n-1},$$

for all  $1 \leq i \leq n$  and  $\star_{n,i}$ -composable  $a, b \in C_n$ . Cubical  $\omega$ -categories and their functors form the category  $\text{Cub}_\omega$ .

Further, a functor of  $\omega$ -categories with connections is a functor between the underlying  $\omega$ -categories that preserves the connection maps:  $F_n \Gamma_{n,i}^\alpha = \Gamma_{n,i}^\alpha F_{n-1}$ , for all  $1 \leq i < n$ . Cubical  $\omega$ -categories with connections and their functors form the category  $\text{Cub}_\omega^\Gamma$ .

Finally, a functor of  $(\omega, p)$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor between the underlying  $\omega$ -categories with connections. As in the single-set case, inverses are preserved by such functors. Cubical  $(\omega, p)$ -categories and their functors form the category  $\text{Cub}_{(\omega,p)}^\Gamma$ .

The categories  $\text{Cub}_n$ ,  $\text{Cub}_n^\Gamma$  and  $\text{Cub}_{(n,p)}^\Gamma$  of cubical  $n$ -categories and their morphisms are defined by truncation as usual.

### 3.2. Equivalence for cubical $\omega$ -categories

We are now prepared for the following result.

**3.2.1. Theorem.** *There is an equivalence of categories*

$$(-)^c : \text{SCub}_\omega \xrightleftharpoons{\quad} \text{Cub}_\omega : (-)^s.$$

We develop the proof of  $\text{SCub}_\omega \simeq \text{Cub}_\omega$  in the remainder of this subsection. The functors  $(-)^c$  and  $(-)^s$  are defined in (3.2.2) and (3.2.4), the natural isomorphisms in (3.2.6) and (3.2.8) below. We henceforth refer to classical and single-set  $\omega$ -categories to distinguish between objects in  $\text{Cub}_\omega$  and  $\text{SCub}_\omega$ .

**3.2.2. The functor  $(-)^c$ .** For each category  $(\mathcal{S}, \delta, \circ, s)$  in  $\text{SCub}_\omega$ , we define the category  $(\mathcal{S}^c, \delta, \circ, s)^c = (\mathcal{S}^c, \partial, \varepsilon, \star)$  in  $\text{Cub}_\omega$  with

- (i) sets of  $n$ -cells  $\mathcal{S}_n^c = \mathcal{S}^{>n}$  for all  $n \in \mathbb{N}$ ,
- (ii)  $\partial_{n,i}^\alpha : \mathcal{S}_n^c \rightarrow \mathcal{S}_{n-1}^c$  such that  $\partial_{n,i}^\alpha = s_{n-1} \dots s_i \delta_i^\alpha$  for all  $1 \leq i \leq n$ ,
- (iii)  $\varepsilon_{n,i} : \mathcal{S}_{n-1}^c \rightarrow \mathcal{S}_n^c$  such that  $\varepsilon_{n,i} = \tilde{s}_i \dots \tilde{s}_{n-1}$  for all  $1 \leq i \leq n$ ,
- (iv) compositions  $\star_{n,i}$  such that  $x \star_{n,i} y = x \circ_i y$  if  $\Delta_i(x, y)$  and undefined otherwise, for all  $1 \leq i \leq n$  and  $x, y \in \mathcal{S}_n^c$ .

With each morphism  $f : \mathcal{S} \rightarrow \mathcal{S}'$  in  $\text{SCub}_\omega$ ,  $(-)^c$  associates the functor  $f^c : \mathcal{S}^c \rightarrow \mathcal{S}'^c$  on  $n$ -cells in  $\text{Cub}_\omega$  as the restriction of  $f$  to a map from  $\mathcal{S}_n^c = \mathcal{S}^{>n}$  to  $\mathcal{S}'_n^c = \mathcal{S}'^{>n}$ , for each  $n \in \mathbb{N}$ .

**3.2.3. Lemma.** *The functor  $(-)^c$  is well-defined.*

*Proof.* We need to show that  $(-)^c : \text{SCub}_\omega \rightarrow \text{Cub}_\omega$  sends each category in  $\text{SCub}_\omega$  to a category in  $\text{Cub}_\omega$  and each morphism in the former category to a functor in the latter, and that it is itself a functor.

First we show that  $(\mathcal{S}^c, \partial, \varepsilon, \star)$  is indeed a classical  $\omega$ -category, that is, we check the axioms in 3.1.1. Here we only consider some representative examples. Proofs for the remaining axioms shown in Appendix A.1. Suppose  $1 \leq i, j \leq n$  and  $a, b, c, d \in (\mathcal{S}^c)_n$ .

- (i) If  $a \star_{n,i} (b \star_{n,i} c)$  is defined, then  $a \star_{n,i} (b \star_{n,i} c) = a \circ_i (b \circ_i c) = (a \circ_i b) \circ_i c = (a \star_{n,i} b) \star_{n,i} c$  because  $\Delta_i(a, b \star_{n,i} c)$  and  $\Delta_i(b, c)$ , using Axioms 2.1.1(i), (iii), (iv) among others. The same holds if  $(a \star_{n,i} b) \star_{n,i} c$  is defined.
- (ii) The compositions  $a \star_{n,i} \varepsilon_{n,i} \partial_{n,i}^+ a$  and  $\varepsilon_{n,i} \partial_{n,i}^- a \star_{n,i} a$  are defined because  $\Delta_i(a, \varepsilon_{n,i} \partial_{n,i}^+ a)$  and  $\Delta_i(\varepsilon_{n,i} \partial_{n,i}^- a, a)$ , using Axioms 2.1.1(ii), (iii), (iv) among others. It follows that  $a \star_{n,i} \varepsilon_{n,i} \partial_{n,i}^+ a = a \circ_i \delta_i^+ a = a$  and  $\varepsilon_{n,i} \partial_{n,i}^- a \star_{n,i} a = \delta_i^- a \circ_i a = a$  because  $\varepsilon_{n,i} \partial_{n,i}^\alpha a = \delta_i^\alpha a$ .
- (iii) If  $i < j$ , then Axiom 2.2.1(i) and other facts imply that

$$\begin{aligned} \partial_{n-1, j-1}^\beta \partial_{n,i}^\alpha &= s_{n-2} \dots s_{j-1} \delta_{j-1}^\beta s_{n-1} \dots s_i \delta_i^\alpha \\ &= s_{n-2} \dots s_{j-1} s_{n-1} \dots s_j \delta_{j-1}^\beta s_{j-1} \dots s_i \delta_i^\alpha \\ &= s_{n-2} \dots s_{j-1} s_{n-1} \dots s_j \delta_j^\beta s_{j-2} \dots s_i \delta_i^\alpha \\ &= s_{n-2} \dots s_i \delta_i^\alpha s_{n-1} \dots s_j \delta_j^\beta \\ &= \partial_{n-1, i}^\alpha \partial_{n, j}^\beta. \end{aligned}$$

Next we show that  $f^c$  is a morphism in  $\text{Cub}_\omega$ . Suppose  $1 \leq i \leq n$  and  $a, b \in \mathcal{S}_n^c$ . Then

- (i)  $f_{n-1}^c \partial_{n,i}^\alpha = f s_{n-1} \dots s_i \delta_i^\alpha = s_{n-1} \dots s_i \delta_i^\alpha f = \partial_{n,i}^\alpha f_n^c$ ,
- (ii) if  $\partial_{n,i}^+ a = \partial_{n,i}^- b$  then  $\Delta_i(a, b)$  and  $f_n^c(a \star_{n,i} b) = f(a \circ_i b) = f(a) \circ_i f(b) = f_n^c a \star_{n,i} f_n^c b$ ,
- (iii)  $f_n^c \varepsilon_{n,i} = f \tilde{s}_i \dots \tilde{s}_{n-1} = \tilde{s}_i \dots \tilde{s}_{n-1} f = \varepsilon_{n,i} f_{n-1}^c$ .

Further, it is clear from the definition that  $f^c$  preserves sources, targets and compositions of morphisms in  $\text{Cub}_\omega$ .  $\square$

All calculations in this proof are performed within  $\text{SCub}_\omega$ . Their formalisation with our Isabelle component seems therefore routine.

**3.2.4. The functor  $(-)^s$ .** Next, we define the category  $(C, \partial, \varepsilon, \star)^s = (\mathcal{S}, \delta, \circ, s)$  in  $\text{SCub}_\omega$  for each cubical  $(C, \partial, \varepsilon, \star)$  in  $\text{Cub}_\omega$

- (i) The underlying set is the following colimit in the category  $\text{Set}$ :

$$\mathcal{S} = \text{colim}(C_0 \xrightarrow{\varepsilon_{1,1}} C_1 \xrightarrow{\varepsilon_{2,2}} C_2 \xrightarrow{\varepsilon_{3,3}} \dots) = \bigsqcup_{n \in \mathbb{N}} C_n / \sim,$$

where  $a \in C_m$  and  $b \in C_n$  with  $m \leq n$  are equivalent if and only if  $b = \varepsilon_{n,n} \dots \varepsilon_{m+1, m+1} a$  by injectivity of the degeneracy maps. We write  $\varphi_n : C_n \rightarrow \mathcal{S}$  for the maps forming a cocone to the colimit. They send each  $n$ -cell  $a$  to its equivalence class in the above quotient.

- (ii) The  $\delta_i^\alpha : \mathcal{S} \rightarrow \mathcal{S}$ , for  $i \in \mathbb{N}_+$ , are the unique morphisms in  $\text{Set}$  induced by the cocone  $(\varphi_n \varepsilon_{n,i} \partial_{n,i}^\alpha)_{n \geq i}$ . They send the equivalence class of  $a \in C_n$ ,  $n \geq i$ , to the set  $\delta_i^\alpha [a]_\sim = [\varepsilon_{n,i} \partial_{n,i}^\alpha a]_\sim$ .
- (iii) The  $s_i : \mathcal{S} \rightarrow \mathcal{S}$ , for  $i \in \mathbb{N}_+$ , are the unique morphisms in  $\text{Set}$  induced by the cocone  $(\varphi_n \varepsilon_{n, i+1} \partial_{n,i}^-)_{n \geq i+1}$ . They send the equivalence class of  $a \in C_n$  to  $s_i [a]_\sim = [\varepsilon_{n, i+1} \partial_{n,i}^- a]_\sim$ .

### 3. Equivalence with classical cubical categories

- (iv) The  $\tilde{s}_i : \mathcal{S} \rightarrow \mathcal{S}$ , for  $i \in \mathbb{N}_+$ , are the unique morphisms in  $\text{Set}$  induced by the cocone  $(\varphi_n \varepsilon_{n,i} \partial_{n,i+1}^-)_{n \geq i+1}$ . They send the equivalence class of  $a \in C_n$  to  $\tilde{s}_i[a]_{\sim} = [\varepsilon_{n,i} \partial_{n,i+1}^- a]_{\sim}$ .
- (v) The  $\circ_i : \mathcal{S} \times_{\Delta_i} \mathcal{S} \rightarrow \mathcal{S}$ , for  $i \in \mathbb{N}_+$ , send the equivalence classes of  $a, b \in C_n$ , to  $[a]_{\sim} \circ_i [b]_{\sim} = [a \star_{n,i} b]_{\sim}$ .

For each functor  $g : C \rightarrow C'$  in  $\text{Cub}_\omega$  we define the morphism  $g^s : C^s \rightarrow C'^s$  in  $\text{SCub}_\omega$  as the unique morphism in  $\text{Set}$  induced by the cocone  $(\varphi'_n \circ g_n : C_n \rightarrow C'^s)_{n \in \mathbb{N}}$ , where the  $\varphi'_n$  are the inclusion maps  $C'_n \rightarrow C'^s$ . It sends the equivalence class of  $a \in C_n$  to  $g^s([a]_{\sim}) = [g(a)]_{\sim}$  in  $C'_n$ .

In the following, we do not distinguish between equivalence classes and their representatives.

**3.2.5. Lemma.** *The functor  $(-)^s$  is well-defined.*

*Proof.* The proof is similar to that of Lemma 3.2.3. First we check that  $(\mathcal{S}, \delta, \circ, s)$  is a category in  $\text{SCub}_\omega$ , verifying the axioms in 2.2.1. Once again we only present some cases and refer to Appendix A.2 for the remaining ones. Suppose  $i, j \in \mathbb{N}_+$  and  $w, x, y, z \in \mathcal{S}$  with representatives in  $C_n$  for  $n \geq i, j$ .

- (i)  $\Delta_i(x, y) \Leftrightarrow \partial_{n,i}^+ x = \partial_{n,i}^- y$ , hence  $\Delta_i(x, y \circ_i z)$  and  $\Delta_i(y, z)$  if and only  $\Delta_i(x \circ_i y, z)$  and  $\Delta_i(x, y)$ . It follows that  $x \circ_i (y \circ_i z) = x \star_{n,i} (y \star_{n,i} z) = (x \star_{n,i} y) \star_{n,i} z = (x \circ_i y) \circ_i z$ .
- (ii)  $x \circ_i \delta_i^+ x = x \star_{n,i} \varepsilon_{n,i} \partial_{n,i}^+ x = x$  and  $\delta_i^- x \circ_i x = \varepsilon_{n,i} \partial_{n,i}^- x \star_{n,i} x = x$ .
- (iii) – If  $i < j$  then

$$\begin{aligned} \delta_i^\alpha \delta_j^\beta x &= \varepsilon_{n,i} \partial_{n,i}^\alpha \varepsilon_{n,j} \partial_{n,j}^\beta x \\ &= \varepsilon_{n,i} \varepsilon_{n-1,j-1} \partial_{n-1,i}^\alpha \partial_{n,j}^\beta x \\ &= \varepsilon_{n,j} \varepsilon_{n-1,i} \partial_{n-1,j-1}^\beta \partial_{n,i}^\alpha x \\ &= \varepsilon_{n,j} \partial_{n,j}^\beta \varepsilon_{n,i} \partial_{n,i}^\alpha x \\ &= \delta_j^\beta \delta_i^\alpha x, \end{aligned}$$

– if  $i > j$  then

$$\begin{aligned} \delta_i^\alpha \delta_j^\beta x &= \varepsilon_{n,i} \partial_{n,i}^\alpha \varepsilon_{n,j} \partial_{n,j}^\beta x \\ &= \varepsilon_{n,i} \varepsilon_{n-1,j} \partial_{n-1,i-1}^\alpha \partial_{n,j}^\beta x \\ &= \varepsilon_{n,j} \varepsilon_{n-1,i-1} \partial_{n-1,j}^\beta \partial_{n,i}^\alpha x \\ &= \varepsilon_{n,j} \partial_{n,j}^\beta \varepsilon_{n,i} \partial_{n,i}^\alpha x \\ &= \delta_j^\beta \delta_i^\alpha x. \end{aligned}$$

Next we show that  $g^s$  is a morphism in  $\text{SCub}_\omega$ . For all  $1 \leq i \leq n$  and  $x, y \in \mathcal{S}$  with representatives  $a, b$  in  $C_n$ ,

- (i)  $g^s \delta_i^\alpha x = [g \varepsilon_{n,i} \partial_{n,i}^\alpha x]_{\sim} = [\varepsilon'_{n,i} \partial'_{n,i}^\alpha g x]_{\sim} = \delta_i'^\alpha g^s x$ ,
- (ii) if  $\Delta_i(x, y)$ , then  $\partial_{n,i}^+ a = \partial_{n,i}^- b$ , so  $g^s(x \circ_i y) = [g(a \star_{n,i} b)]_{\sim} = [g(a) \star_{n,i} g(b)]_{\sim} = g^s(x) \circ_i g^s(y)$ ,
- (iii) if  $n \geq i + 1$ , then  $g^s s_i x = [g \varepsilon_{n,i+1} \partial_{n,i}^- x]_{\sim} = [\varepsilon'_{n,i+1} \partial'_{n,i}^- g x]_{\sim} = s_i' g^s x$ .

Finally,  $g^s$  is indeed a morphism: it is clear from its definition that it preserves sources and targets, while reservation of compositions of morphisms in  $\text{SCub}_\omega$  follows from properties of colimits.  $\square$

This time, the calculations in the proof above are performed in  $\text{Cub}_\omega$ . Their formalisation with Isabelle is beyond the scope of this work.

**3.2.6. Natural isomorphism  $(-)^s \circ (-)^c \Rightarrow id$ .** Let  $(\mathcal{S}, \delta, \circ, s)$  be a category of  $\text{SCub}_\omega$  and  $(C, \partial, \varepsilon, \star) = (\mathcal{S}, \delta, \circ, s)^c$  as previously. The category  $(\mathcal{S}', \delta', \circ', s') := ((\mathcal{S}, \delta, \circ, s)^c)^s$  in  $\text{SCub}_\omega$  is computed as follows. Let  $i \in \mathbb{N}_+$ .

(i)  $\mathcal{S}' = \text{colim}(C_0 \xrightarrow{\varepsilon_{1,1}} C_1 \xrightarrow{\varepsilon_{2,2}} \dots) = \text{colim}(\mathcal{S}^{>0} \xrightarrow{id} \mathcal{S}^{>1} \xrightarrow{id} \dots)$ . Axiom 2.2.1(x) then implies that

$$\mathcal{S}' = \bigsqcup_{n \in \mathbb{N}} \mathcal{S}^{>n} / \sim = \bigcup_{n \in \mathbb{N}} \mathcal{S}^{>n} = \mathcal{S}.$$

- (ii) The face maps are  $\delta'_i{}^\alpha = \delta_i^\alpha$ , because they send each  $x \in \mathcal{S}^{>n}$  with  $n \geq i$  to  $\delta'_i{}^\alpha x = \varepsilon_{n,i} \partial_{n,i}^\alpha x = \delta_i^\alpha x$ . Thus in particular  $\mathcal{S}'^i = \mathcal{S}^i$ .
- (iii) The symmetries and reverse symmetries are  $s'_i = s_i \delta_i^-$  and  $\tilde{s}'_i = \tilde{s}_i \delta_{i+1}^-$ , because they send  $x \in \mathcal{S}^{>n}$  with  $n \geq i+1$  to  $s'_i x = \varepsilon_{n,i+1} \partial_{n,i}^- x = s_i \delta_i^- x$  and  $\tilde{s}'_i x = \varepsilon_{n,i} \partial_{n,i+1}^- x = \tilde{s}_i \delta_{i+1}^-$ . Therefore,  $s'_i = s_i$  on  $\mathcal{S}^i$  and  $\tilde{s}'_i = \tilde{s}_i$  on  $\mathcal{S}^{i+1}$ .
- (iv) The compositions are  $\circ'_i = \circ_i$ , because for all  $x, y \in \mathcal{S}^{>n}$  with  $n \geq i$  such that  $\Delta'_i(x, y)$ , we have  $\Delta_i(x, y)$  and the compositions  $\circ'_i$  send  $x, y$  to  $x \circ'_i y = x \star_{n,i} y = x \circ_i y$ .

Further, any morphism  $f : \mathcal{S} \rightarrow \mathcal{T}$  in  $\text{SCub}_\omega$  satisfies  $(f^c)^s = f$ .

**3.2.7. Lemma.** *The maps  $id_{\mathcal{S}} : \mathcal{S}' \rightarrow \mathcal{S}$  induce a natural isomorphism  $\mu : (-)^s \circ (-)^c \Rightarrow id$ .*

*Proof.* First,  $id \delta'_i{}^\alpha = \delta_i^\alpha id$  and  $\Delta_i(x, y)$  implies  $id(x \circ'_i y) = id(x) \circ_i id(y)$ . Second,  $id(s'_i x) = s_i id(x)$  for all  $x \in \mathcal{S}'^i$ . So, as  $\mathcal{S}' = \mathcal{S}$ ,  $\mu_{\mathcal{S}} = id : \mathcal{S}' \rightarrow \mathcal{S}$  is a morphism in  $\text{SCub}_\omega$ .

Naturality of the family  $\mu$  and the invertibility of each component  $\mu_{\mathcal{S}}$  are clear.  $\square$

**3.2.8. Natural isomorphism  $id \Rightarrow (-)^c \circ (-)^s$ .** Let  $(C, \partial, \varepsilon, \star)$  be a category in  $\text{Cub}_\omega$  and let  $(\mathcal{S}, \delta, \circ, s) = (C, \partial, \varepsilon, \star)^s$  as before. The category  $(\mathcal{C}', \partial', \varepsilon', \star') = ((C, \partial, \varepsilon, \star)^s)^c$  in  $\text{Cub}_\omega$  is computed as follows. Let  $1 \leq i \leq n$ .

(i) The sets of  $n$ -cells are

$$C'_n = \mathcal{S}^{>n} = \left( \bigsqcup_{m \in \mathbb{N}} C_m / \sim \right)^{>n}.$$

(ii) The face maps  $\partial'_{n,i}{}^\alpha$  send each  $a \in C'_n$ , represented by some  $a_0 \in C_m$  with  $m \geq n$ , to

$$\partial'_{n,i}{}^\alpha a = s_{n-1} \dots s_i \delta_i^\alpha a = [\varepsilon_{m,n} \partial_{m,n-1}^- \dots \varepsilon_{m,i+1} \partial_{m,i}^- \varepsilon_{m,i} \partial_{m,i}^\alpha a_0] \sim = [\varepsilon_{m,n} \partial_{m,i}^\alpha a_0] \sim.$$

(iii) The degeneracies  $\varepsilon'_{n,i}$  send each  $a \in C'_n$ , represented by some  $a_0 \in C_m$  with  $m \geq n$ , to

$$\varepsilon'_{n,i} a = \tilde{s}_i \dots \tilde{s}_{n-1} a = [\varepsilon_{m,i} \partial_{m,i+1}^- \dots \varepsilon_{m,n-1} \partial_{m,n}^- a_0] \sim = [\varepsilon_{m,i} \partial_{m,n}^- a_0] \sim.$$

(iv) The compositions  $a \star'_{n,i} b$ , where  $a, b \in C'_n$  are represented by some  $a_0, b_0 \in C_m$  with  $m \geq n$ , satisfy  $a \star'_{n,i} b = [a_0 \star_{m,i} b_0] \sim$  whenever  $\Delta_i(a, b)$ , that is, when  $a_0 \star_{n,i} b_0$  is defined, and are undefined otherwise.

### 3. Equivalence with classical cubical categories

Further, for any morphism  $g : C \rightarrow \mathcal{D}$  in  $\text{Cub}_\omega$ ,  $(g^s)^c$  sends  $[a]_\sim$  in  $(C^s)_n^c$  to  $[g(a)]_\sim$  in  $(\mathcal{D}^s)_n^c$ .

Suppose  $\varphi_n : C_n \rightarrow \mathcal{S}$  is the morphism to the colimit, which sends any  $n$ -cell  $a$  to its equivalence class in  $\bigsqcup_{m \in \mathbb{N}} C_m / \sim$ . Its image is included in  $C'_n$  because  $a \sim \varepsilon_{i,i} \dots \varepsilon_{n+1,n+1} a = a'$  for each  $a \in C_n$  and  $i \geq n+1$ . Thus  $\delta_i^- \varphi_n a = [\varepsilon_{i,i} \delta_i^- a']_\sim = \varphi_n a$ . Let  $(\eta_C)_n : C_n \rightarrow C'_n$  be the induced map and  $\eta_C : C \rightarrow C'$  the family  $((\eta_C)_n)_{n \in \mathbb{N}}$ . Let further  $(\bar{\eta}_C)_n : C'_n \rightarrow C_n$  be the map that sends  $b$ , represented by some  $a \in C_m$  with  $m \geq n$ , to  $\partial_{n+1,n+1}^- \dots \partial_{m,m}^- a$ . It is well-defined because the image of  $b$  does not depend on the choice of the representative: indeed  $a \sim a'$  and  $a' \in C_l$  with  $l \geq m$  imply

$$\partial_{n+1,n+1}^- \dots \partial_{l,l}^- a' = \partial_{n+1,n+1}^- \dots \partial_{l,l}^- \varepsilon_{l,l} \dots \varepsilon_{m+1,m+1} a = \partial_{n+1,n+1}^- \dots \partial_{m,m}^- a.$$

Finally, we write  $\bar{\eta}_C : C' \rightarrow C$  for the family  $((\bar{\eta}_C)_n)_{n \in \mathbb{N}}$ .

**3.2.9. Lemma.** *The maps  $\eta_C : C \rightarrow C'$  induce a natural isomorphism  $\eta : id \Rightarrow (-)^c \circ (-)^s$ .*

*Proof.* We need to show that the maps  $\eta_C : C \rightleftarrows C' : \bar{\eta}_C$  are morphisms in  $\text{Cub}_\omega$ , which are natural and inverses of each other. Let  $a, b \in C_n$  and  $c \in C_{n-1}$ . Then

$$\begin{aligned} (\eta_C)_{n-1} \partial_{n,i}^\alpha a &= [\partial_{n,i}^\alpha a]_\sim = [\varepsilon_{n,n} \partial_{n,i}^\alpha a]_\sim = \partial_{n,i}^\alpha [a]_\sim = \partial_{n,i}^\alpha (\eta_C)_n a, \\ (\eta_C)_n (a \star_{n,i} b) &= [a \star_{n,i} b]_\sim = [a]_\sim \star'_{n,i} [b]_\sim = (\eta_C)_n a \star'_{n,i} (\eta_C)_n b, \\ (\eta_C)_n \varepsilon_{n,i} c &= [\varepsilon_{n,i} c]_\sim = [\varepsilon_{n,i} \partial_{n,n}^- \varepsilon_{n,n} c]_\sim = \varepsilon'_{n,i} [\varepsilon_{n,n} c]_\sim = \varepsilon'_{n,i} [c]_\sim = \varepsilon'_{n,i} (\eta_C)_{n-1} c. \end{aligned}$$

The  $\eta_C$  are therefore morphisms in  $\text{Cub}_\omega$ . The proof that the  $\bar{\eta}_C$  are morphisms in  $\text{Cub}_\omega$  is similar.

The maps  $\eta_C$  and  $\bar{\eta}_C$  are inverses. Indeed,  $\eta_C(\bar{\eta}_C(b)) = [\partial_{n+1,n+1}^- \dots \partial_{m,m}^- a]_\sim = b$  holds for every  $b \in C'_n$  represented by some  $a \in C_m$  with  $m \geq n$ , and likewise for the other composition.

Finally, the family  $\eta$  is natural because  $\eta_C g(a) = [g(a)]_\sim = (g^s)^c [a]_\sim = g(\bar{\eta}_C a)$  for every functor  $g : C \rightarrow \mathcal{D}$ .  $\square$

**3.2.10. Remark.** All axioms of single-set  $\omega$ -categories have been used in one direction of the proof of Theorem 3.2.1, and they have been derived in the other one. Single-set  $\omega$ -categories and their classical counterparts are therefore essentially the same.

Interestingly, in this proof, Axiom 2.2.1(x) has only been used to establish the natural isomorphism  $\mu$ , more precisely for showing that  $\mathcal{S} = \mathcal{S}'$  in the colimit construction. It is unnecessary for a proof of equivalence between  $\text{SCub}_n$  and  $\text{Cub}_n$ , where the colimit construction simplifies; see also 2.2. The proofs of all other properties work uniformly for  $n$  and  $\omega$ .

### 3.3. Equivalence for connections

In this subsection we extend the equivalence  $\text{SCub}_\omega \simeq \text{Cub}_\omega$  to connections.

**3.3.1. Theorem.** *There is an equivalence of categories*

$$(-)^{c\Gamma} : \text{SCub}_\omega^Y \rightleftarrows \text{Cub}_\omega^\Gamma : (-)^{sY}.$$

In the proof, we go through the same steps as before.

**3.3.2. The functor  $(-)^{c\Gamma}$ .** For every category  $(\mathcal{S}, \delta, \circ, \gamma)$  in  $\text{SCub}_\omega^Y$ , the category  $(\mathcal{C}, \partial, \varepsilon, \star, \Gamma) := (\mathcal{S}, \delta, \circ, \gamma)^{c\Gamma}$  in  $\text{Cub}_\omega^\Gamma$  is defined as follows:

- (i) the underlying  $\omega$ -category in  $\text{Cub}_\omega$  is  $(\mathcal{C}, \partial, \varepsilon, \star) = (\mathcal{S}, \delta, \circ, s)^c$ ,
- (ii) the connections are the restrictions  $\Gamma_{n,i}^\alpha : \mathcal{C}_{n-1} \rightarrow \mathcal{C}_n$  of  $\Gamma_{n,i}^\alpha = \gamma_i^\alpha \tilde{s}_i \dots \tilde{s}_{n-1}$  for  $1 \leq i < n$ .

For each morphism  $f : \mathcal{S} \rightarrow \mathcal{S}'$  in  $\text{SCub}_\omega^Y$  we define the morphisms  $f^{c\Gamma} : \mathcal{S}^{c\Gamma} \rightarrow \mathcal{S}'^{c\Gamma}$  as  $f^c$  on  $n$ -cells in  $\text{Cub}_\omega^\Gamma$  for each  $n \in \mathbb{N}$ .

**3.3.3. Lemma.** *The functor  $(-)^{c\Gamma}$  is well-defined.*

*Proof.* As before, we first check that  $(\mathcal{S}, \partial, \varepsilon, \star, \Gamma)^{c\Gamma}$  defines a category in  $\text{Cub}_\omega^\Gamma$ . We only need to consider the connection axioms. We show selected axioms only and refer to Appendix A.3 for the others. Let  $1 \leq i, j < n$  and  $a, b, c, d \in \mathcal{S}_n^{c\Gamma}$ .

- (i) – If  $i < j$ , then Axiom 2.3.1(i) and others imply that

$$\begin{aligned} \partial_{n,i}^\alpha \Gamma_{n,j}^\beta &= s_{n-1} \dots s_j s_{j-1} \gamma_j^\beta s_{j-2} \dots s_i \tilde{s}_j \dots \tilde{s}_{n-1} \delta_i^\alpha \\ &= s_{n-1} \dots s_{j+1} \gamma_{j-1}^\beta s_j s_{j-2} \dots s_i \tilde{s}_j \dots \tilde{s}_{n-1} \delta_i^\alpha \\ &= \gamma_{j-1}^\beta s_{j-2} \dots s_i \delta_i^\alpha \\ &= \Gamma_{n-1, j-1}^\beta \partial_{n-1, i}^\alpha \end{aligned}$$

$$- \partial_{n,i}^\alpha \Gamma_{n,i}^\alpha = s_{n-1} \dots s_i \delta_i^\alpha \gamma_i^\alpha \tilde{s}_i \dots \tilde{s}_{n-1} = id.$$

- (ii) If  $a, b$  are  $\star_{n,j}$ -composable, then Axiom 2.3.1(ii) and others imply that,

- if  $i < j$  then

$$\begin{aligned} \Gamma_{n+1, i}^\alpha (a \star_{n, j} b) &= \gamma_i^\alpha \tilde{s}_i \dots \tilde{s}_j (\tilde{s}_{j+1} \dots \tilde{s}_n a \circ_j \tilde{s}_{j+1} \dots \tilde{s}_n b) \\ &= \gamma_i^\alpha (\tilde{s}_i \dots \tilde{s}_n a \circ_{j+1} \tilde{s}_i \dots \tilde{s}_n b) \\ &= \Gamma_{n+1, i}^\alpha a \star_{n+1, j+1} \Gamma_{n+1, i}^\alpha b, \end{aligned}$$

- if  $i = j$  then

$$\begin{aligned} \Gamma_{n+1, i}^- (a \star_{n, i} b) &= \gamma_i^- (\tilde{s}_i \dots \tilde{s}_n a \circ_{i+1} \tilde{s}_i \dots \tilde{s}_n b) \\ &= (\gamma_i^- \tilde{s}_i \dots \tilde{s}_n a \circ_i s_i \tilde{s}_i \dots \tilde{s}_n b) \circ_{i+1} (\tilde{s}_i \dots \tilde{s}_n b \circ_i \gamma_i^- \tilde{s}_i \dots \tilde{s}_n b) \\ &= (\Gamma_{n+1, i}^- a \star_{n+1, i} \varepsilon_{n+1, i+1} b) \star_{n+1, i+1} (\varepsilon_{n+1, i} b \star_{n+1, i} \Gamma_{n+1, i}^- b). \end{aligned}$$

- (iv) If  $i < j$  then  $\Gamma_{n+1, i}^\alpha \varepsilon_{n, j} = \gamma_i^\alpha \tilde{s}_i \dots \tilde{s}_{j-1} \tilde{s}_{j+1} \dots \tilde{s}_n \tilde{s}_j \dots \tilde{s}_{n-1} = \tilde{s}_{j+1} \dots \tilde{s}_n \gamma_i^\alpha \tilde{s}_i \dots \tilde{s}_{n-1} = \varepsilon_{n+1, j+1} \Gamma_{n, i}^\alpha$ .

It remains to be shown that  $f^{c\Gamma}$  is a morphism in  $\text{Cub}_\omega^\Gamma$ . Indeed, for each  $1 \leq i \leq n$  and  $a, b \in \mathcal{S}_n^{c\Gamma}$ ,

$$f^{c\Gamma} \Gamma_{n, i}^\alpha a = f \gamma_i^\alpha \tilde{s}_i \dots \tilde{s}_{n-1} a = \gamma_i^\alpha \tilde{s}_i \dots \tilde{s}_{n-1} f a = \Gamma_{n, i}^\alpha f^{c\Gamma} a.$$

The claim then follows from Lemma 3.2.3. □

### 3. Equivalence with classical cubical categories

**3.3.4. The functor  $(-)^{s\gamma}$ .** For  $(C, \partial, \varepsilon, \star, \Gamma)$  in  $\text{Cub}_\omega^\Gamma$ , the category  $(C, \partial, \varepsilon, \star, \Gamma)^{s\gamma} := (\mathcal{S}, \delta, \odot, s, \gamma)$  in  $\text{SCub}_\omega^Y$  is defined as follows:

- (i) The underlying single-set cubical  $\omega$ -category in  $\text{SCub}_\omega$  is  $(C, \partial, \varepsilon, \star)^s = (\mathcal{S}, \delta, \odot, s)$ .
- (ii) The  $\gamma_i^\alpha : \mathcal{S} \rightarrow \mathcal{S}$ , for  $i \geq 1$ , are the unique morphisms in  $\text{Set}$  induced by the cocone  $(\varphi_n \Gamma_{n,i}^\alpha \partial_{n,i}^\alpha)_{n>i}$ . They send the equivalence class of each  $a \in C_n$  to  $\gamma_i^\alpha [a]_\sim = [\Gamma_{n,i}^\alpha \partial_{n,i}^\alpha a]_\sim$ .

For each morphism  $g : C \rightarrow C'$  in  $\text{Cub}_\omega$ , the morphism  $g^{s\gamma} : C^{s\gamma} \rightarrow C'^{s\gamma}$  in  $\text{SCub}_\omega^Y$  is defined as  $g^s$ .

**3.3.5. Lemma.** *The functor  $(-)^{s\gamma}$  is well-defined.*

*Proof.* As usual, we start with checking that  $(\mathcal{S}, \delta, \odot, s, \gamma)$  is a category in  $\text{SCub}_\omega^Y$ . It remains to consider the single-set axioms for connections. As usual, we only show some cases and refer to Appendix A.4 for the remaining ones. Let  $i, j \in \mathbb{N}_+$  and  $x, y \in \mathcal{S}$  with representatives  $a, b$  in  $C_n$  with  $n > i, j$ .

- (i) If  $i \neq j, j+1$  and  $x \in \mathcal{S}^j$ , then  $\partial_{n,j}^- x = \partial_{n,j}^+ x$  so  $\delta_j^\alpha \gamma_j^\alpha x = \varepsilon_{n,j} \partial_{n,j}^\alpha \Gamma_{n,j}^\alpha \partial_{n,j}^\alpha x = \delta_j^\alpha x = x$ .
- (ii) If  $j \neq i, i+1$ ,  $x, y \in \mathcal{S}^i$  and  $\Delta_{i+1}(x, y)$  then

$$\begin{aligned} \gamma_i^+(x \circ_{i+1} y) &= \Gamma_{n,i}^+(\partial_{n,i}^+ x \star_{n,i} \partial_{n,i}^+ y) \\ &= (\Gamma_{n,i}^+ \partial_{n,i}^+ x \star_{n+1,i+1} \varepsilon_{n,i+1} \partial_{n,i}^+ x) \star_{n+1,i} (\varepsilon_{n,i} \partial_{n,i}^+ x \star_{n+1,i+1} \Gamma_{n,i}^+ \partial_{n,i}^+ y) \\ &= (\gamma_i^+ x \circ_{i+1} s_i x) \circ_i (x \circ_{i+1} \gamma_i^+ y). \end{aligned}$$

- (v) If  $x \in \mathcal{S}^{i,j}$  and  $i < j-1$  then

$$\gamma_i^\alpha \gamma_j^\beta x = \Gamma_{n,i}^\alpha \Gamma_{n-1,j-1}^\beta \partial_{n-1,i}^\alpha \partial_{n,j}^\beta x = \Gamma_{n,j}^\beta \Gamma_{n-1,i}^\alpha \partial_{n-1,j-1}^\beta \partial_{n,i}^\alpha x = \gamma_j^\beta \gamma_i^\alpha x.$$

- (vi) If  $x \in \mathcal{S}^{i,i+1}$  then  $\partial_{n,i}^- a = \partial_{n,i}^+ a$  and  $\partial_{n,i+1}^- a = \partial_{n,i+1}^+ a$  so

$$s_{i+1} s_i \gamma_{i+1}^\alpha x = \varepsilon_{n,i+2} \Gamma_{n-1,i}^\alpha \partial_{n-1,i}^- \partial_{n,i+1}^\alpha x = \Gamma_{n,i}^\alpha \varepsilon_{n-1,i+1} \partial_{n-1,i}^\alpha \partial_{n,i+1}^- x = \gamma_i^\alpha s_{i+1} x.$$

It remains to show that  $g^{s\gamma}$  is a morphism in  $\text{SCub}_\omega^Y$ . For  $i \geq 1$  and  $x \in \mathcal{S}^i$ ,

$$g^{s\gamma} \gamma_i^\alpha x = [g \Gamma_{n,i}^\alpha \partial_{n,i}^\alpha x]_\sim = [\Gamma_{n,i}^\alpha \partial_{n,i}^\alpha g x]_\sim = \gamma_i^\alpha g^{s\gamma} x.$$

The claim then follows from Lemma 3.2.5. □

**3.3.6. Natural isomorphism  $(-)^{s\gamma} \circ (-)^{c\Gamma} \Rightarrow id$ .** Let  $(\mathcal{S}, \delta, \circ, s, \gamma)$  be a category in  $\text{SCub}_\omega^Y$  and let  $(C, \partial, \varepsilon, \star, \Gamma) = (\mathcal{S}, \delta, \circ, s, \gamma)^{c\Gamma}$ . The category  $(\mathcal{S}', \delta', \circ', s', \gamma') := ((\mathcal{S}, \delta, \circ, s, \gamma)^{c\Gamma})^{s\gamma}$  in  $\text{SCub}_\omega^Y$  is determined as follows. For  $i \in \mathbb{N}_+$ ,

- (i) the underlying  $\omega$ -category in  $\text{SCub}_\omega$  is  $(\mathcal{S}', \delta', \circ', s') = ((\mathcal{S}, \delta, \circ, s)^c)^s$
- (ii) the connections are  $\gamma_i'^\alpha = \gamma_i^\alpha \delta_i^\alpha$ ; they send  $x \in \mathcal{S}^{>n}$  with  $n > i$  to  $\gamma_i'^\alpha x = \Gamma_{n,i}^\alpha \partial_{n,i}^\alpha x = \gamma_i^\alpha \delta_i^\alpha x$ . Hence in particular  $\gamma_i'^\alpha = \gamma_i^\alpha$  on  $\mathcal{S}^i$ .

Also, for each morphism  $f : \mathcal{S} \rightarrow \mathcal{T}$  in  $\text{SCub}_\omega^Y$ ,  $(f^{c\Gamma})^{s\gamma} = (f^c)^s$ , so that  $(f^{c\Gamma})^{s\gamma} = f$ .

**3.3.7. Lemma.** *The maps  $\mu_{\mathcal{S}} : \mathcal{S}' \rightarrow \mathcal{S}$  induce a natural isomorphism  $\mu : (-)^{s\gamma} \circ (-)^{c\Gamma} \Rightarrow id$ .*

*Proof.* Relative to Lemma 3.2.7 it remains to show that  $\mu_{\mathcal{S}}$  is a morphism in  $\text{SCub}_\omega^Y$ . Indeed,  $id \gamma_i'^\alpha x = \gamma_i^\alpha id x$  for all  $x \in \mathcal{S}^i$ . □

**3.3.8. Natural isomorphism  $id \Rightarrow (-)^{c\Gamma} \circ (-)^{s\gamma}$ .** Let  $(C, \partial, \varepsilon, \star, \Gamma)$  be a category in  $\text{Cub}_\omega^\Gamma$  and let  $(\mathcal{S}, \delta, \circ, s, \gamma) := (C, \partial, \varepsilon, \star, \Gamma)^{s\gamma}$  as previously. In this case, the category  $(C', \partial', \varepsilon', \star', \Gamma') := ((C, \partial, \varepsilon, \star, \Gamma)^{s\gamma})^{c\Gamma}$  in  $\text{Cub}_\omega^\Gamma$  is determined as follows. For  $1 \leq i < n$ ,

(i) the underlying  $\omega$ -category in  $\text{Cub}_\omega$  is  $(C', \partial', \varepsilon', \star') = ((C, \partial, \varepsilon, \star)^s)^c$ .

(ii) the connections are  $\Gamma'_{n,i}{}^\alpha$ ; they send  $a \in C'_n$  represented by some  $a_0 \in C_m$  with  $m \geq n$  to

$$\Gamma'_{n,i}{}^\alpha a = \gamma_i^\alpha \widetilde{s}_i \dots \widetilde{s}_{n-1} a = [\Gamma_{m,i}^\alpha \partial_{m,i}^\alpha \varepsilon_{m,i} \partial_{m,i+1}^- \dots \varepsilon_{m,n-1} \partial_{m,n}^- a_0] = [\Gamma_{m,i}^\alpha \partial_{m,n}^- a_0].$$

Moreover, for each morphism  $g : C \rightarrow \mathcal{D}$  in  $\text{Cub}_\omega$ ,  $(g^{s\gamma})^{c\Gamma} = (g^s)^c$  sends  $[a]_\sim$  in  $(C^{s\gamma})_n^{c\Gamma}$  to  $[g(a)]$  in  $(\mathcal{D}^{s\gamma})_n^{c\Gamma}$ .

**3.3.9. Lemma.** *The maps  $\eta_C : C \rightarrow C'$  induce a natural isomorphism  $\eta : id \Rightarrow (-)^{c\Gamma} \circ (-)^{s\gamma}$ .*

*Proof.* We must show, relative to Lemma 3.2.9, that  $\eta_C$  and  $\overline{\eta}_C$  are morphisms in  $\text{Cub}_\omega^\Gamma$ . Indeed, for all  $1 \leq i < n$ ,  $a \in C_{n-1}$  and  $b \in C'_{n-1}$  represented by some  $b_0 \in C_{m-1}$ ,

$$\begin{aligned} \Gamma'_{n,i}{}^\alpha \eta_C a &= \Gamma'_{n,i}{}^\alpha [\varepsilon_{n,n} a] = [\Gamma_{n,i}^\alpha a] = \eta_C \Gamma_{n,i}^\alpha a, \\ \overline{\eta}_C \Gamma'_{n,i}{}^\alpha b &= \partial_{n+1,n+1}^- \dots \partial_{m,m}^- \Gamma_{m,i}^\alpha \partial_{m,n}^- b_0 = \Gamma_{n,i}^\alpha \partial_{n,n}^- \dots \partial_{m,m}^- b_0 = \Gamma_{n,i}^\alpha \overline{\eta}_C b. \end{aligned} \quad \square$$

### 3.4. Equivalence for inverses

Finally, we extend the equivalence  $\text{SCub}_\omega^Y \simeq \text{Cub}_\omega^\Gamma$  to the case  $(\omega, p)$ .

**3.4.1. Theorem.** *The categories  $\text{SCub}_{(\omega,p)}^Y$  and  $\text{Cub}_{(\omega,p)}^\Gamma$  are equivalent.*

*Proof.* More specifically, we show that the functors  $(-)^{c\Gamma} : \text{SCub}_\omega^Y \simeq \text{Cub}_\omega^\Gamma : (-)^{s\gamma}$  from Theorem 3.3.1 extend to  $\text{SCub}_{(\omega,p)}^Y \simeq \text{Cub}_{(\omega,p)}^\Gamma$ . First, suppose  $\mathcal{S}$  is a category in  $\text{SCub}_{(\omega,p)}^Y$ . Let  $C = \mathcal{S}^{c\Gamma}$  and, for all  $n > p$  and  $1 \leq i \leq n$ , pick an  $n$ -cell  $c$  in  $C_n$  with an  $R_{n-1,i}$ -invertible shell.

– Then  $\partial_{n,j}^\alpha c$  has an  $R_{n-1,i-1}$ -inverse,  $d$  say, for each  $1 \leq j < i$ , by the hypothesis. Therefore

$$\begin{aligned} \partial_{n,j}^\alpha c \star_{n-1,i-1} d &= \varepsilon_{n-1,i-1} \partial_{n-1,i-1}^- \partial_{n,j}^\alpha c, \\ s_{n-1} \dots s_j \delta_j^\alpha c \circ_{i-1} d &= \delta_{i-1}^- s_{n-1} \dots s_j \delta_j^\alpha c, \\ \delta_j^\alpha c \circ_i e &= \delta_i^- \delta_j^\alpha c, \end{aligned}$$

where we write  $e = \widetilde{s}_j \dots \widetilde{s}_{n-1} d$  for short. Similarly we can show that  $e \circ_i \delta_j^\alpha c = \delta_i^+ \delta_j^\alpha c$ ,  $\Delta_i(\delta_j^\alpha c, e)$  and  $\Delta_i(e, \delta_j^\alpha c)$ . Hence  $e$  is the  $r_i$ -inverse of  $\delta_j^\alpha c$ .

– Alternatively, the hypothesis implies that  $\partial_{n,j}^\alpha c$  has an  $R_{n-1,i}$ -inverse,  $d$  say, for each  $i < j \leq n$ . So

$$\begin{aligned} \partial_{n,j}^\alpha c \star_{n-1,i} d &= \varepsilon_{n-1,i} \partial_{n-1,i}^- \partial_{n,j}^\alpha c, \\ s_{n-1} \dots s_j \delta_j^\alpha c \circ_i d &= \delta_i^- s_{n-1} \dots s_j \delta_j^\alpha c, \\ \delta_j^\alpha c \circ_i e &= \delta_i^- \delta_j^\alpha c, \end{aligned}$$

where we abbreviate  $e = \widetilde{s}_j \dots \widetilde{s}_{n-1} d$ . Again we can prove that  $e \circ_i \delta_j^\alpha c = \delta_i^+ \delta_j^\alpha c$ ,  $\Delta_i(\delta_j^\alpha c, e)$  and  $\Delta_i(e, \delta_j^\alpha c)$ , and  $e$  is the  $r_i$ -inverse of  $\delta_j^\alpha c$ .

## 4. Formalisation with Isabelle/HOL

This shows that  $c$  has an  $r_i$ -invertible  $(n-1)$ -shell and hence the  $r_i$ -inverse  $r_i c \in \mathcal{S}^{>n}$  by Lemma 2.4.7. It satisfies  $\Delta_i(c, r_i c)$ ,  $\Delta_i(r_i c, c)$ ,  $c \circ_i r_i c = \delta_i^- c$  and  $r_i c \circ_i c = \delta_i^+ c$ . It thus follows that  $r_i c \in C_n$ , and that

$$\begin{aligned} c \star_{n,i} r_i c &= c \circ_i r_i c = \delta_i^- c = \tilde{s}_i \dots \tilde{s}_{n-1} s_{n-1} \dots s_i \delta_i^- c = \varepsilon_{n,i} \partial_{n,i}^- c, \\ r_i c \star_{n,i} c &= r_i c \circ_i c = \delta_i^+ c = \tilde{s}_i \dots \tilde{s}_{n-1} s_{n-1} \dots s_i \delta_i^+ c = \varepsilon_{n,i} \partial_{n,i}^+ c. \end{aligned}$$

Therefore,  $c$  is  $R_{n,i}$ -invertible in  $C$  and  $C$  is a cubical  $(\omega, p)$ -category.

Second, suppose  $C$  is a classical  $(\omega, p)$ -category. Let  $\mathcal{S} = C^{s\mathcal{Y}}$  and, for all  $n > p$  and  $i \geq 1$ , pick a cell  $s$  in  $\mathcal{S}^{>n}$  with an  $r_i$ -invertible  $(n-1)$ -shell. For each  $j \geq 1$  with  $i \neq j$ ,  $\delta_j^\alpha s$  then has an  $r_i$ -inverse  $t = r_i \delta_j^\alpha s$ . Pick representatives  $c$  and  $d$  in  $C_n$  of  $s$  and  $t$ , respectively, which is possible because  $s \in \mathcal{S}^{>n}$ . Then

$$\delta_j^\alpha s \circ_i t = \delta_i^- \delta_j^\alpha s = \delta_i^- \delta_j^\alpha s \quad \text{and} \quad \varepsilon_{n,j} \partial_{n,j}^\alpha c \star_{n,i} d = \varepsilon_{n,j} \partial_{n,j}^\alpha c \varepsilon_{n,i} \partial_{n,i}^- c.$$

- If  $i < j$ , then  $\partial_{n,j}^\alpha c \star_{n-1,i} e = \partial_{n,j}^\alpha \varepsilon_{n,i} \partial_{n,i}^- c = \varepsilon_{n-1,i} \partial_{n-1,i}^- \partial_{n,j}^\alpha c$ , where we have abbreviated  $e = \partial_{n,j}^\alpha d$ . Similarly we can show that  $e \star_{n-1,i} \partial_{n,j}^\alpha c = \varepsilon_{n-1,i} \partial_{n-1,i}^+ \partial_{n,j}^\alpha c$ . Thus  $e$  is the  $R_{n-1,i}$ -inverse of  $\partial_{n,j}^\alpha c$ .
- Alternatively, if  $i > j$ , then  $\partial_{n,j}^\alpha c \star_{n-1,i-1} e = \partial_{n,j}^\alpha \varepsilon_{n,i} \partial_{n,i}^- c = \varepsilon_{n-1,i-1} \partial_{n-1,i-1}^- \partial_{n,j}^\alpha c$ , where we abbreviate  $e = \partial_{n,j}^\alpha d$ . Similarly we can show that  $e \star_{n-1,i-1} \partial_{n,j}^\alpha c = \varepsilon_{n-1,i-1} \partial_{n-1,i-1}^+ \partial_{n,j}^\alpha c$ . Hence once again  $e$  is the  $R_{n-1,i-1}$ -inverse of  $\partial_{n,j}^\alpha c$ .

It follows from the definition that  $c$  has an  $R_{n-1,i}$ -invertible shell and hence an  $R_{n,i}$ -inverse  $R_{n,i} c \in C_n$ . It satisfies  $c \star_{n,i} R_{n,i} c = \varepsilon_{n,i} \partial_{n,i}^- c$  and  $R_{n,i} c \star_{n,i} c = \varepsilon_{n,i} \partial_{n,i}^+ c$ . Suppose  $u = [R_{n,i} c]_{\sim} \in \mathcal{S}$ . The previous equations then imply that  $s \circ_i u = \delta_i^- s$  and  $u \circ_i s = \delta_i^+ s$ . Hence  $u$  is the  $r_i$ -inverse of  $s$ . This shows that  $\mathcal{S}$  is a single-set  $(\omega, p)$ -category.  $\square$

## 4. FORMALISATION WITH ISABELLE/HOL

The Isabelle/HOL proof assistant [53] has been indispensable for developing our axiomatisation of single-set cubical  $\omega$ -categories in Section 2. In this section we describe our Isabelle components, report on our particular usage of its support for proof automation in taming these axioms and give an example of a non-trivial proof (of Proposition 2.4.8) using our formalisation.

### 4.1. Isabelle/HOL in a nutshell

Isabelle/HOL is based on a simply-typed classical higher-order logic, which in practice often gives the impression of working in typed set theory. Among similar proof assistants such as Coq [7] or Lean [1], it stands out due to its support for proof automation. On one hand, Isabelle employs internal simplification and proof procedures as well as external proof search tools for first-order logic, which can be invoked using the *Sledgehammer* tactic. On the other, it integrates SAT solvers for counterexample search using the *Nitpick* tactic.

Yet these strengths come at a price: Isabelle's type classes, one of its two main mechanisms for modelling and working with algebraic hierarchies, allow only one single type parameter, which essentially imposes a single-set approach to categories when using type classes. Consequently, the numbers  $n$  or  $\omega$

do not feature as type parameters in the formalisation of  $n$ - or  $\omega$ -categories and cannot be instantiated easily to fixed finite dimensions. Further, Isabelle’s type system does not support dependent types, which may sometimes be desirable in mathematical specifications.

To overcome the first limitation, Isabelle offers locales as a more set-based specification mechanism. This allows more than one type parameter and hence the standard approach to categories with objects and arrows. Yet a locale-based approach to formalising mathematics [5], which has been developed over many years, had recently to be revised [6] for modelling more advanced mathematical concepts such as Grothendieck schemes [11], at the expense of losing many benefits of types and type checking as well as a weaker coupling with Isabelle’s main libraries. It might therefore be desirable to use proof assistants with more expressive type systems, such as Coq or Lean, for formalising more advanced features of higher categories. However, more work is needed to evaluate the strengths and weaknesses of different proof assistants in this regard. The formalisation of higher categories would certainly provide excellent test cases.

### 4.2. Formalising single-set categories

As previously for globular  $\omega$ -categories [20], the basic Isabelle type class for our formalisation of cubical  $\omega$ -categories is that of a *catoid*, a structure mentioned en passant in Remark 2.1.2. We start with recalling the basic features of the formalisation of catoids and single-set categories with Isabelle from the Archive of Formal proofs [60].

```

class multimagma =
  fixes mcomp :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a set (infixl  $\odot$  70)

class multisetgroup = multimagma +
  assumes assoc:  $(\bigcup v \in y \odot z. x \odot v) = (\bigcup v \in x \odot y. v \odot z)$ 

class st-op =
  fixes src :: 'a  $\Rightarrow$  'a ( $\sigma$ )
  and tgt :: 'a  $\Rightarrow$  'a ( $\tau$ )

class st-multimagma = multimagma + st-op +
  assumes Dst:  $x \odot y \neq \{\}$   $\implies$   $\tau x = \sigma y$ 
  and s-absorb [simp]:  $\sigma x \odot x = \{x\}$ 
  and t-absorb [simp]:  $x \odot \tau x = \{x\}$ 

class catoid = st-multimagma + multisetgroup

```

The type classes *multimagma* and *st-op* introduce a multioperation and source and target maps, together with notation  $\odot$ ,  $\sigma$  and  $\tau$ . The classes *multisetgroup*, *st-op* and *catoid* add structure, extending the classes previously defined. The *catoid* class, for instance, extends the *st-multimagma* and *multisetgroup* classes, while the *multisetgroup* class extends the class defining its multiplication.

Catoids are extended to single-set categories by imposing the locality and functionality axioms from Section 2.1.1.

```

class single-set-category = functional-catoid + local-catoid

```

## 4. Formalisation with Isabelle/HOL

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Each of the above type class features one single type parameter  $'a$  (spelled  $\alpha$ ) and is polymorphic in this parameter. It can therefore be instantiated to more concrete types. In the class *multisemigroup*, for instance,  $\alpha$  could be instantiated to the type of strings and  $\odot$  to the shuffle operation on strings.

Note also that multisemigroups or catoids have been specified without carrier sets. While such sets can be added easily, it often suffices to regard the type  $\alpha$  roughly as a set.

The standard function type in proof assistants such as Coq, Isabelle or Lean models total functions. Partiality is usually modelled using a monad or by adjoining a zero. Here instead we take the multioperation  $\odot$  as a basis, mapping to the empty set when two elements cannot be composed. We define the resulting domain of definition as

**abbreviation**  $\Delta x y \equiv (x \odot y \neq \{\})$

and the partial operation  $\otimes$  (denoted  $\circ$  in previous sections), using the definite description operator *THE* as

**definition**  $pcomp :: 'a \Rightarrow 'a \Rightarrow 'a$  (**infixl**  $\otimes$  70) **where**  
 $x \otimes y = (THE z. z \in x \odot y)$

If  $x \odot y$  is empty, Isabelle maps  $x \otimes y$  to a value about which nothing particular can be proved. Using  $\otimes$  in place of  $\odot$  with single-set categories allows us to be precise about definedness conditions in higher categories, which may be subtle, while avoiding clumsy specifications with many set braces and a proliferation of cases in proofs due to undefinedness. In proof assistants with dependent types, the partiality of composition in categories can alternatively be formalised at type level, defining arrow composition on homsets. This approach might be harder to integrate with tools like Sledgehammer.

### 4.3. Formalising single-set cubical $\omega$ -categories

To formalise  $\omega$ -categories, we have first created indexed variants of the classes leading to *single-set-category* above, that is, classes based on  $\odot_i$ ,  $\otimes_i$  for the compositions and  $\partial_i \alpha$  for face maps. Unlike in previous sections, our indices start with 0 and we write  $\partial$  instead of  $\delta$  for single-set face maps. With Isabelle, we can formally link these indexed classes with the non-indexed ones (using so-called sublocale statements between classes) so that all theorems about single-set categories are in scope in the indexed variants. In particular, we have introduced the definedness conditions *DD i* and linked them formally with  $\Delta$ .

Using the class *icategory* for indexed single-set categories, we have first defined an auxiliary class for  $\omega$ -categories without symmetries.

**class** *semi-cubical-omega-category* = *icategory* +  
**assumes** *face-comm*:  $i \neq j \implies \partial_i \alpha \circ \partial_j \beta = \partial_j \beta \circ \partial_i \alpha$   
**and** *face-func*:  $i \neq j \implies DD j x y \implies \partial_i \alpha (x \otimes_j y) = \partial_i \alpha x \otimes_j \partial_i \alpha y$   
**and** *interchange*:  $i \neq j \implies DD i w x \implies DD i y z \implies DD j w y \implies DD j x z$   
 $\implies (w \otimes_i x) \otimes_j (y \otimes_i z) = (w \otimes_j y) \otimes_i (x \otimes_j z)$   
**and** *fin-fix*:  $\exists k. \forall i. k \leq i \implies ff x i x$

### 4.3. Formalising single-set cubical $\omega$ -categories

In the last axiom,  $fFx\ i\ x \equiv (\partial\ i\ ff\ x = x)$ , which we use in place of the predicate  $x \in \mathcal{S}^i$  from Section 2.2, has been defined in the context of a class on which *icategory* is based.

We have further extended this class to one for  $\omega$ -categories with the remaining axioms for symmetries and reverse symmetries from Definition 2.2.1, after introducing a separate class for these two maps.

```

class symmetry-ops =
  fixes symmetry :: nat  $\Rightarrow$  'a  $\Rightarrow$  'a ( $\sigma$ )
  and inv-symmetry :: nat  $\Rightarrow$  'a  $\Rightarrow$  'a ( $\vartheta$ )

class cubical-omega-category = semi-cubical-omega-category + symmetry-ops +
  assumes sym-type:  $\sigma\ \sigma\ i\ (\text{face-fix } i) \subseteq \text{face-fix } (i + 1)$ 
  and inv-sym-type:  $\vartheta\ \vartheta\ i\ (\text{face-fix } (i + 1)) \subseteq \text{face-fix } i$ 
  and sym-inv-sym:  $fFx\ (i + 1)\ x \Longrightarrow \sigma\ i\ (\vartheta\ i\ x) = x$ 
  and inv-sym-sym:  $fFx\ i\ x \Longrightarrow \vartheta\ i\ (\sigma\ i\ x) = x$ 
  and sym-face1:  $fFx\ i\ x \Longrightarrow \partial\ i\ \alpha\ (\sigma\ i\ x) = \sigma\ i\ (\partial\ (i + 1)\ \alpha\ x)$ 
  and sym-face2:  $i \neq j \Longrightarrow i \neq j + 1 \Longrightarrow fFx\ j\ x \Longrightarrow \partial\ i\ \alpha\ (\sigma\ j\ x) = \sigma\ j\ (\partial\ i\ \alpha\ x)$ 
  and sym-func:  $i \neq j \Longrightarrow fFx\ i\ x \Longrightarrow fFx\ i\ y \Longrightarrow DD\ j\ x\ y \Longrightarrow$ 
     $\sigma\ i\ (x \otimes_j y) = (\text{if } j = i + 1 \text{ then } \sigma\ i\ x \otimes_i \sigma\ i\ y \text{ else } \sigma\ i\ x \otimes_j \sigma\ i\ y)$ 
  and sym-fix:  $fFx\ i\ x \Longrightarrow fFx\ (i + 1)\ x \Longrightarrow \sigma\ i\ x = x$ 
  and sym-sym-braid:  $\text{diffSup } i\ j\ 2 \Longrightarrow fFx\ i\ x \Longrightarrow fFx\ j\ x \Longrightarrow \sigma\ i\ (\sigma\ j\ x) = \sigma\ j\ (\sigma\ i\ x)$ 

```

In Axioms *sym-type* and *inv-sym-type*, the functions  $\sigma\sigma$  and  $\vartheta\vartheta$  are the image maps corresponding to symmetries and reverse symmetries. Further *face-fix*  $i$  denotes the set  $\mathcal{S}^i$  of fixed points of the lower face map in direction  $i$ . It is defined as *face-fix*  $i \equiv \text{Fix } (\partial\ i\ ff)$  as in Section 2.2. Other axioms in the class use the predicate *diffSup*  $i\ j\ k \equiv (i - j \geq k \vee j - i \geq k)$ .

Though our formalisation is a so-called shallow embedding of categories in Isabelle, as it uses Isabelle's built-in types for functions, sets and numbers to axiomatise  $\omega$ -categories, it has nevertheless some deep features, as we do not define a (sub)type for each  $fFx\ i\ x \equiv (\partial\ i\ ff\ x = x)$  and we capture the partiality of cell composition in terms of the predicate *DD*, but not at type level. Such typing or composition conditions must therefore be declared explicitly in axioms and lemmas, and they need to be checked explicitly in proofs.

Finally, we have provided a class for connections and defined a class for  $\omega$ -categories with connections.

```

class cubical-omega-category-connections = cubical-omega-category + connection-ops +
  assumes conn-face1:  $fFx\ j\ x \Longrightarrow \partial\ j\ \alpha\ (\Gamma\ j\ \alpha\ x) = x$ 
  and conn-face2:  $fFx\ j\ x \Longrightarrow \partial\ (j + 1)\ \alpha\ (\Gamma\ j\ \alpha\ x) = \sigma\ j\ x$ 
  and conn-face3:  $i \neq j \Longrightarrow i \neq j + 1 \Longrightarrow fFx\ j\ x \Longrightarrow \partial\ i\ \alpha\ (\Gamma\ j\ \beta\ x) = \Gamma\ j\ \beta\ (\partial\ i\ \alpha\ x)$ 
  and conn-corner1:  $fFx\ i\ x \Longrightarrow fFx\ i\ y \Longrightarrow DD\ (i + 1)\ x\ y \Longrightarrow$ 
     $\Gamma\ i\ tt\ (x \otimes_{(i + 1)} y) = (\Gamma\ i\ tt\ x \otimes_{(i + 1)} \sigma\ i\ x) \otimes_i (x \otimes_{(i + 1)} \Gamma\ i\ tt\ y)$ 
  and conn-corner2:  $fFx\ i\ x \Longrightarrow fFx\ i\ y \Longrightarrow DD\ (i + 1)\ x\ y \Longrightarrow$ 
     $\Gamma\ i\ ff\ (x \otimes_{(i + 1)} y) = (\Gamma\ i\ ff\ x \otimes_{(i + 1)} y) \otimes_i (\sigma\ i\ y \otimes_{(i + 1)} \Gamma\ i\ ff\ y)$ 
  and conn-corner3:  $j \neq i \wedge j \neq i + 1 \Longrightarrow fFx\ i\ x \Longrightarrow fFx\ i\ y \Longrightarrow DD\ j\ x\ y \Longrightarrow \Gamma\ i\ \alpha\ (x \otimes_j y) = \Gamma\ i\ \alpha\ x \otimes_j \Gamma\ i\ \alpha\ y$ 
  and conn-fix:  $fFx\ i\ x \Longrightarrow fFx\ (i + 1)\ x \Longrightarrow \Gamma\ i\ \alpha\ x = x$ 
  and conn-zigzag1:  $fFx\ i\ x \Longrightarrow \Gamma\ i\ tt\ x \otimes_{(i + 1)} \Gamma\ i\ ff\ x = x$ 
  and conn-zigzag2:  $fFx\ i\ x \Longrightarrow \Gamma\ i\ tt\ x \otimes_i \Gamma\ i\ ff\ x = \sigma\ i\ x$ 

```

## 4. Formalisation with Isabelle/HOL

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**and conn-conn-braid:**  $\text{diffSup } i \ j \ 2 \implies \text{fF}x \ j \ x \implies \text{fF}x \ i \ x \implies \Gamma \ i \ \alpha \ (\Gamma \ j \ \beta \ x) = \Gamma \ j \ \beta \ (\Gamma \ i \ \alpha \ x)$   
**and conn-shift:**  $\text{fF}x \ i \ x \implies \text{fF}x \ (i + 1) \ x \implies \sigma \ (i + 1) \ (\sigma \ i \ (\Gamma \ (i + 1) \ \alpha \ x)) = \Gamma \ i \ \alpha \ (\sigma \ (i + 1) \ x)$

### 4.4. Example proofs

We present two Isabelle proofs as examples. The first one shows a proof of Lemma 2.2.10(ii) by automated proof search.

**lemma** *sym-func1:*

**assumes**  $\text{fF}x \ i \ x$

**and**  $\text{fF}x \ i \ y$

**and**  $DD \ i \ x \ y$

**shows**  $\sigma \ i \ (x \otimes_i \ y) = \sigma \ i \ x \otimes_{(i+1)} \sigma \ i \ y$

**by** (*metis* *assms* *icid.ts-compat* *local.iDst* *local.icat.sscatml.l0-absorb* *sym-type-var1*)

Isabelle's Sledgehammer tactic has returned the proof shown. Sledgehammer invokes external proof-search tools for first-order logic, which are internally reconstructed by Isabelle's *metis* tool, which itself has been verified using Isabelle to increase trustworthiness. The proof statement lists the lemmas used. All of them are part of our Isabelle component for cubical  $\omega$ -categories and the components on which it builds.

The second proof shows how a proof by hand can be typed into Isabelle line by line, and each line then be verified automatically using Isabelle's proof tactics – here the third case in Lemma 2.2.11(i).

**lemma** *inv-sym-face:*

**assumes**  $i \neq j$

**and**  $i \neq j + 1$

**and**  $\text{fF}x \ (j + 1) \ x$

**shows**  $\partial \ i \ \alpha \ (\vartheta \ j \ x) = \vartheta \ j \ (\partial \ i \ \alpha \ x)$

**proof** –

**have**  $\sigma \ j \ (\partial \ i \ \alpha \ (\vartheta \ j \ x)) = \sigma \ j \ (\partial \ i \ \alpha \ (\partial \ j \ \text{ff}' \ (\vartheta \ j \ x)))$

**using** *assms(3)* *inv-sym-type-var* **by** *simp*

**also have**  $\dots = \partial \ i \ \alpha \ (\sigma \ j \ (\partial \ j \ \alpha \ (\vartheta \ j \ x)))$

**by** (*metis* *assms(1)* *assms(2)* *assms(3)* *inv-sym-type-var* *local.fF}x-prop* *sym-face-var1*)

**also have**  $\dots = \partial \ i \ \alpha \ (\sigma \ j \ (\vartheta \ j \ x))$

**using** *assms(1)* *assms(2)* *assms(3)* *calculation* *inv-sym-type-var* *local.sym-face* **by** *presburger*

**also have**  $\dots = \partial \ i \ \alpha \ (\partial \ (j + 1) \ \alpha \ x)$

**by** (*metis* *assms(3)* *local.face-compat-var* *sym-inv-sym-var1*)

**finally have**  $\sigma \ j \ (\partial \ i \ \alpha \ (\vartheta \ j \ x)) = \partial \ i \ \alpha \ (\partial \ (j + 1) \ \alpha \ x)$ .

**thus** *?thesis*

**by** (*smt* (*z3*) *assms(3)* *icid.st-eq1* *inv-sym-type-var* *local.face-comm-var* *local.inv-sym-sym*)

**qed**

Beyond fully automatic proofs found using Sledgehammer and interactive proofs using Isabelle's proof scripting language Isar, as here, Isabelle also offers so-called apply-style proofs, in which simplification steps, application of rules or substitutions of particular formulas can be combined step-wise with Sledgehammer proofs. Examples can be found in our Isabelle component.

### 4.5. Taming $\omega$ -categories with Isabelle

We have already outlined in the introduction how Isabelle has helped developing the single-set axioms for  $\omega$ -categories. Here we provide more details. Recall that we have justified these axioms via the equivalence proofs in Section 3. Their selection was driven by the construction of the functors  $(-)^c$  and  $(-)^s$  and their extensions, which relate classical and single-set concepts. We aimed for a small set of structurally meaningful axioms to make the equivalence proofs smooth and simple.

We started with translating the axioms for  $\partial_{n,i}^\alpha : C_n \rightarrow C_{n-1}$  into those for  $\delta_i^\alpha : \mathcal{S} \rightarrow \mathcal{S}$ . This was straightforward, for instance, for (i), (ii) or (iii) in (2.2.1), but others required encoding the index-shift in  $C_n \rightarrow C_{n-1}$  of face maps in the single-set axioms, where graded sets  $C_n$  are not immediately available. Instead we used the fixed point sets  $\mathcal{S}^i$  or the face maps  $\delta_i^\alpha$  as guards in the single-set axioms; Isabelle helped us to bring them into convenient form. While this obviated degeneracies, we had to introduce symmetries to relate fixed points at the same dimension but in different directions, and to model the rotations of degenerated cubes through the interactions of symmetries with face maps and compositions. Starting from lattices like that in Subsection 2.2.3, the translations between symmetries and degeneracies in  $(-)^c$  and  $(-)^s$  and geometric intuitions, we used Isabelle, in particular Sledgehammer in combination with other proof tactics, in an iterative process to adapt or simplify candidate axioms, to analyse their dependencies, and to add axioms in light of the equivalence proof. Beyond symmetries, the equivalence proof for  $\omega$ -categories led us to experiment with dimensionality axioms using Isabelle. This resulted in Axiom (x), and enabled the colimit and filtration constructions for  $(-)^s$ .

During this process, we compressed the single-set axiomatisation for  $\omega$ -categories by a factor  $>2$  to a size similar to the classical one. A significant part of the process was automatic. Most redundant candidate axioms now feature in Lemmas 2.2.10 and 2.2.11. Our work flow for  $\omega$ -categories with connections has been similar and resulted in a similar compression. Redundant laws are shown in Lemma 2.3.4. Interestingly, we found Axiom (vi) quite late through the equivalence proof.

Our insights in the strengths and weaknesses of Isabelle’s proof automation might be valuable for mathematicians working with higher categories, where proofs tend to be highly combinatorial, axiomatisations often fill pages and there can be a big formalisation gap with respect to geometric or (string) diagrammatical intuition. In our work, we were sometimes surprised when Sledgehammer managed to derive seemingly independent conjectures, such as the Yang-Baxter identity in Lemma 2.2.10 or the face identities in Lemma 2.2.11(1). But we also spent hours feeding paper-and-pencil proofs into Isabelle and hard-coding rule applications, including the proof in the following subsection. Overall, interactive proofs with higher categories at the granularity of paper and pencil proofs seem nowadays feasible – and highly beneficial for activities like the one described in this article. Yet a main source of disappointment was that, unlike in previous work, we could not use Isabelle’s Nitpick tool for verifying the irredundancy of our axiomatisation: it seems that the underlying SAT solver cannot cope with the arithmetic constraints in our axioms, though that should certainly be possible for SMT solvers.

### 4.6. A non-trivial proof

At the end of this section, we show our formalisation at work, presenting a proof of Proposition 2.4.8. This example shows that Isabelle’s proof automation smoothly supports interactive proofs in higher category theory. For this we have formalised  $(\omega, 0)$ -categories with Isabelle. A formalisation of  $(\omega, p)$ -categories based on type classes seems impossible as it would require more than one type parameter.

#### 4. Formalisation with Isabelle/HOL

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Defining a type class for  $(\omega, 0)$ -categories needs some preliminary definitions. First we have defined compositions of sequences of symmetries and reverse symmetries.

**primrec** *symcomp* ::  $\text{nat} \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'a (\Sigma)$  **where**  
 $\Sigma i 0 x = x$   
 $|\Sigma i (\text{Suc } j) x = \sigma (i + j) (\Sigma i j x)$

**primrec** *inv-symcomp* ::  $\text{nat} \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'a (\Theta)$  **where**  
 $\Theta i 0 x = x$   
 $|\Theta i (\text{Suc } j) x = \Theta i j (\vartheta (i + j) x)$

Then we have defined  $r_i$ -invertibility and shell  $r_i$ -invertibility, following Definition 2.4.1.

**abbreviation** (in *cubical-omega-category-connections*)  
 $\text{ri-inv } i x y \equiv (DD i x y \wedge DD i y x \wedge x \otimes_i y = \partial i \text{ff } x \wedge y \otimes_i x = \partial i \text{tt } x)$

**abbreviation** (in *cubical-omega-category-connections*)  
 $\text{ri-inv-shell } k i x \equiv (\forall j \alpha. j + 1 \leq k \wedge j \neq i \longrightarrow (\exists y. \text{ri-inv } i (\partial j \alpha x) y))$

This allowed us to specify a class for  $(\omega, 0)$ -categories following Definition 2.4.1.

**class** *cubical-omega-zero-category-connections* = *cubical-omega-category-connections* +  
**assumes** *ri-inv*:  $k \geq 1 \Longrightarrow i \leq k - 1 \Longrightarrow \text{dim-bound } k x \Longrightarrow \text{ri-inv-shell } k i x \Longrightarrow \exists y. \text{ri-inv } i x y$

In the axiom *ri-inv*, the predicate  $\text{dim-bound } k x \equiv (\forall i. k \leq i \longrightarrow \text{ff } x i x)$ , which we use in place of the set  $\mathcal{S}^{>k}$  from Section 2.2, has been defined in the context of the class *icategory*.

We have shown uniqueness of  $r_i$ -inverses and used this property, together with Isabelle's definite description operator *THE*, to define an inversion map.

**lemma** *ri-unique*:  $(\exists y. \text{ri-inv } i x y) = (\exists !y. \text{ri-inv } i x y)$   
 $\langle \text{proof} \rangle$

**definition**  $\text{ri } i x = (\text{THE } y. \text{ri-inv } i x y)$

Our proof of Proposition 2.4.8 with Isabelle follows that in Subsection 2.4 quite directly. We do not show the verifications of individual proof steps by Isabelle's proof tools. Details can be found in our Isabelle component [52]. The main part of the proof is captured by a technical lemma that proceeds by induction on the dimension  $k$  of the cell  $x$ .

**lemma** *every-dim-k-ri-inv*:  
**assumes** *dim-bound*  $k x$   
**shows**  $\forall i. \exists y. \text{ri-inv } i x y$  **using** *dim-bound*  $k x$   
**proof** (*induction*  $k$  *arbitrary*:  $x$ )  
**case** 0  
**thus** ?*case*  
 $\langle \text{proof} \rangle$   
**next**  
**case** (*Suc*  $k$ )

```
{fix i
  have  $\exists y. ri\text{-inv } i \ x \ y$ 
```

Here we start a proof by cases for  $i \geq k + 1$  as in Section 2.4. As in the proof by hand, the first case is trivial, and automatic with Isabelle.

```
proof (cases Suc k  $\geq$  i)
  case True
  thus ?thesis
  <proof>
next
  case False
  {fix j  $\alpha$ 
    assume  $h: j \leq k \wedge j \neq i$ 
```

While the proof of  $\delta_j^\alpha x \in \mathcal{S}^{j, k+1, k+2, \dots}$  with Isabelle is automatic, we need to check  $s_{k-1} \dots s_j \delta_j^\alpha x \in \mathcal{S}^{>k}$ .

```
hence  $a: \text{has-dim-bound } k \ (\Sigma j \ (k - j) \ (\partial j \ \alpha \ x))$ 
  <proof>
  have  $\exists y. ri\text{-inv } i \ (\partial j \ \alpha \ x) \ y$ 
```

To construct an  $r_i$ -inverse of  $\delta_j^\alpha x$ , we perform a proof by cases on  $j$ .

```
proof (cases  $j < i$ )
  case True
```

For  $j < i$ , we introduce  $y$  as the  $r_{i-1}$ -inverse of  $s_{k-1} \dots s_j \delta_j^\alpha x$  using the induction hypothesis.

```
obtain  $y$  where  $b: ri\text{-inv } (i - 1) \ (\Sigma j \ (k - j) \ (\partial j \ \alpha \ x)) \ y$ 
  <proof>
```

We check that  $\tilde{s}_j \dots \tilde{s}_{k-1} y$  and  $\delta_j^\alpha x$  are composable and then show that these expressions are inverses.

```
have  $c: \text{dim-bound } k \ y$ 
  <proof>
hence  $d: DD \ i \ (\partial j \ \alpha \ x) \ (\Theta j \ (k - j) \ y)$ 
  <proof>
hence  $e: DD \ i \ (\Theta j \ (k - j) \ y) \ (\partial j \ \alpha \ x)$ 
  <proof>
have  $f: (\partial j \ \alpha \ x) \otimes_i \ (\Theta j \ (k - j) \ y) = \Theta j \ (k - j) \ ((\Sigma j \ (k - j) \ (\partial j \ \alpha \ x)) \otimes_{(i-1)} \ y)$ 
  <proof>
have  $(\Theta j \ (k - j) \ y) \otimes_i \ (\partial j \ \alpha \ x) = \Theta j \ (k - j) \ (y \otimes_{(i-1)} \ (\Sigma j \ (k - j) \ (\partial j \ \alpha \ x)))$ 
  <proof>
thus ?thesis
  <proof>
```

We proceed similarly in the case of  $j > i$ .

#### 4. Formalisation with Isabelle/HOL

---

```

next
  case False
  obtain y where b: ri-inv i (Σ j (k - j) (∂ j α x)) y
    < proof >
  have c: dim-bound k y
    < proof >
  hence d: DD i (∂ j α x) (Θ j (k - j) y)
    < proof >
  hence e: DD i (Θ j (k - j) y) (∂ j α x)
    < proof >
  have f: (∂ j α x) ⊗i (Θ j (k - j) y) = Θ j (k - j) ((Σ j (k - j) (∂ j α x)) ⊗i y)
    < proof >
  have (Θ j (k - j) y) ⊗i (∂ j α x) = Θ j (k - j) (y ⊗i (Σ j (k - j) (∂ j α x)))
    < proof >
  thus ?thesis
    < proof >
qed

```

This shows that  $x$  is  $r_i$ -invertible. We can now conclude that  $x$  is  $r_i$ -invertible in each direction  $i$ .

```

  thus ?thesis
    < proof >
qed
thus ?case
  < proof >
qed

```

Every cell in a single-set cubical  $(\omega, 0)$ -category has finite dimension. Lemma *every-dim-k-ri-inv* therefore shows that every cell is  $r_i$ -invertible in every direction  $i$ , which is Proposition 2.4.8.

```

lemma every-ri-inv:  $\exists y. ri\text{-inv } i \ x \ y$ 
  using every-dim-k-ri-inv local.fin-fix by blast

```

Relative to the proof by hand, we had to prove several fixed points and definedness conditions for compositions due the deep features of our embedding. These are usually left implicit in proofs by hand or with proper shallow embeddings, where they are discharged by type inference. Except for such proof steps, the granularity of this formal proof is similar to that of a proof by hand owing to Isabelle's proof automation. Nevertheless, in some longer formulas we had to tell Isabelle in detail how assumptions had to be matched with proof goals. These details are visible in our proof document.

As already mentioned, formalising  $(\omega, p)$ -categories with Isabelle for a finite  $p$  is at least complicated with Isabelle's type classes. Particular instances of  $p$  can be given, but an arbitrary  $p$  would require more than one type parameter, and therefore locales.

## 5. CONCLUSION

We have introduced single-set axiomatisations of cubical categories with additional structure such as connections and inverses. We have justified their adequacy through equivalence proofs relative to their classical counterparts. The Isabelle/HOL proof assistant, with its powerful support for proof automation and counterexample search, has been instrumental in this development. We might not have undertaken this research without it. Cubical set and categories have a broad range of applications in mathematics and computer science: from homotopy theory and algebraic topology to homotopy type theory, concurrency theory and rewriting theory. Our formalisation might therefore support innovative applications of proof assistants in these fields. In this regard, our results allow us to outline several lines of research that we hope to explore in the future.

First, our formalisation work yields an initial step towards formal tools and methods that tame the combinatorial complexity of proofs in higher categories. These are meant to support users in reasoning formally with higher categories, based on geometric intuitions if available, and with a high degree of automation. For this, a single-set approach seems relevant because of its algebraic simplicity. As our axiomatisation is essentially single-sorted first-order, we expect it to work well with SMT solvers and similar tools. Yet further experiments, and in particular comparisons with formalisations using locales in Isabelle or proof assistants such as Coq [7] and Lean [1], are needed to identify the most suitable approach. Dependent types, as supported by Coq and Lean, might offer advantages in specifying and reasoning with higher categories orthogonal to the proof automation supplied by Isabelle.

Second, this work is part of a programme on proof support for higher-dimensional rewriting. Higher globular and cubical categories and higher globular and cubical polygraphs, which correspond essentially to higher path categories organised in globes or cubes, are particularly appropriate for this [4]. Cubical categories, for instance, allow natural explicit formulations of confluence results such as the Church-Rosser theorem, Newman's lemma and their higher-dimensional extensions. Higher algebras such as globular Kleene algebras and quantales have already been developed for reasoning about such properties [18, 19] and the initial steps of the globular approach have already been formalised with Isabelle [20]. The construction of similar Kleene algebras and quantales related to cubical categories and their formalisation along similar lines seem equally desirable.

Third, one can construct polygraphic resolutions related to rewriting properties in globular or cubical categories [45, 46]. In a companion article [49] we are using our single-set approach to formalise normalisation strategies for higher abstract rewriting systems, which provide a constructive approach to their polygraphic resolutions in the cubical setting. Based on this, we aim to formalise polygraphic constructions of higher-dimensional rewriting on categories [39, 40] and higher algebras [38, 50] using proof assistants and the mathematical components created for them.

A fourth line of work might concern the categorical constructions from Section 3. Theorem 3.2.1, in particular shows that the category  $\text{SCub}_\omega$  of single-set cubical  $\omega$ -categories is equivalent to its classical counterpart  $\text{Cub}_\omega$ . This equivalence of categories lives in the 2-category of categories, where the natural isomorphisms 3.2.6 and 3.2.8 are 2-cells. One may wonder how to formulate such an equivalence in a single-set globular 2-category. This would constitute a formalisation of the proof of justification of the single-set axiomatisation within a single-set approach. The question arises in the same way for the equivalences of Theorems 3.3.1 and 3.4.1.

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## A. APPENDICES

### A.1. End of the proof of Lemma 3.2.3

To show that  $\mathcal{S}^c$  is a cubical  $\omega$ -category, we derive the remaining axioms:

(iv) if  $x, y$  are  $\star_{k,j}$ -composable, by Axioms 2.2.1(ii), (vii) and other ones,

– if  $i < j$  then

$$\begin{aligned}
 \partial_{k,i}^\alpha(x \star_{k,j} y) &= s_{k-1} \dots s_i \delta_i^\alpha(x \circ_j y) \\
 &= s_{k-1} \dots s_{j-1} (s_{j-2} \dots s_i \delta_i^\alpha x \circ_j s_{j-2} \dots s_i \delta_i^\alpha y) \\
 &= s_{k-1} \dots s_j (s_{j-1} \dots s_i \delta_i^\alpha x \circ_{j-1} s_{j-1} \dots s_i \delta_i^\alpha y) \\
 &= s_{k-1} \dots s_i \delta_i^\alpha x \circ_{j-1} s_{k-1} \dots s_i \delta_i^\alpha y \\
 &= \partial_{k,i}^\alpha x \star_{k,j-1} \partial_{k,i}^\alpha y,
 \end{aligned}$$

## A. Appendices

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– if  $i > j$  then

$$\partial_{k,i}^\alpha(x \star_{k,j} y) = s_{k-1} \dots s_i \delta_i^\alpha(x \circ_j y) = s_{k-1} \dots s_i \delta_i^\alpha x \circ_j s_{k-1} \dots s_i \delta_i^\alpha y = \partial_{k,i}^\alpha x \star_{k,j} \partial_{k,i}^\alpha y,$$

– if  $i = j$  then

$$\partial_{k,i}^-(x \star_{k,i} y) = s_{k-1} \dots s_i \delta_i^-(x \circ_i y) = s_{k-1} \dots s_i \delta_i^- x = \partial_{k,i}^- x,$$

$$\partial_{k,i}^+(x \star_{k,i} y) = s_{k-1} \dots s_i \delta_i^+(x \circ_i y) = s_{k-1} \dots s_i \delta_i^+ y = \partial_{k,i}^+ y,$$

(v) if  $a, b$  are  $\star_{k,i}$ -composable,  $c, d$  are  $\star_{k,i}$ -composable,  $a, c$  are  $\star_{k,j}$ -composable,  $b, d$  are  $\star_{k,j}$ -composable, and if  $i \neq j$ , then  $\Delta_i(a, b)$ ,  $\Delta_i(c, d)$ ,  $\Delta_j(a, c)$  and  $\Delta_j(b, d)$ , so by Axiom 2.2.1(iii) and other ones

$$\begin{aligned} (a \star_{k,i} b) \star_{k,j} (c \star_{k,i} d) &= (a \circ_i b) \circ_j (c \circ_i d) \\ &= (a \circ_j c) \circ_i (b \circ_j d) \\ &= (a \star_{k,j} c) \star_{k,i} (b \star_{k,j} d), \end{aligned}$$

(vi) – if  $i < j$  then by Axioms 2.2.1(iv), (v), (vi), (viii) and other ones

$$\begin{aligned} \partial_{k,i}^\alpha \varepsilon_{k,j} &= s_{k-1} \dots s_i \delta_i^\alpha \widetilde{s}_j \dots \widetilde{s}_{k-1} \\ &= s_{k-1} \dots s_i \widetilde{s}_j \dots \widetilde{s}_{k-1} \delta_i^\alpha \\ &= s_{k-1} \dots s_j s_{j-2} \dots s_i \widetilde{s}_j \dots \widetilde{s}_{k-1} \delta_i^\alpha \\ &= s_{j-2} \dots s_i s_{k-1} \dots s_j \widetilde{s}_j \dots \widetilde{s}_{k-1} \delta_i^\alpha \\ &= s_{j-2} \dots s_i \delta_i^\alpha \\ &= \varepsilon_{k-1, j-1} \partial_{k-1, i}^\alpha, \end{aligned}$$

– if  $i > j$  then similarly  $\partial_{k,i}^\alpha \varepsilon_{k,j} = s_{k-1} \dots s_i \delta_i^\alpha \widetilde{s}_j \dots \widetilde{s}_{k-1} = s_{j-1} \dots s_{i-1} \delta_{i-1}^\alpha = \varepsilon_{k-1, j} \partial_{k-1, i-1}^\alpha$ ,

– if  $i = j$  then  $\partial_{k,i}^\alpha \varepsilon_{k,i} = s_{k-1} \dots s_i \delta_i^\alpha \widetilde{s}_i \dots \widetilde{s}_{k-1} = s_{k-1} \dots s_i \widetilde{s}_i \dots \widetilde{s}_{k-1} = id$ ,

(vii) if  $a, b$  are  $\star_{k,j}$ -composable,

– if  $i \leq j$  then

$$\begin{aligned} \varepsilon_{k+1, i}(a \star_{k, j} b) &= \widetilde{s}_i \dots \widetilde{s}_k (a \circ_j b) \\ &= \widetilde{s}_i \dots \widetilde{s}_j (\widetilde{s}_{j+1} \dots \widetilde{s}_k a \circ_j \widetilde{s}_{j+1} \dots \widetilde{s}_k b) \\ &= \widetilde{s}_i \dots \widetilde{s}_{j-1} (\widetilde{s}_j \dots \widetilde{s}_k a \circ_{j+1} \widetilde{s}_j \dots \widetilde{s}_k b) \\ &= \widetilde{s}_i \dots \widetilde{s}_k a \circ_{j+1} \widetilde{s}_i \dots \widetilde{s}_k b \\ &= \varepsilon_{k+1, i} a \star_{k+1, j+1} \varepsilon_{k+1, i} b, \end{aligned}$$

– if  $i > j$  then

$$\varepsilon_{k+1, i}(a \star_{k, j} b) = \widetilde{s}_i \dots \widetilde{s}_k (a \circ_j b) = \widetilde{s}_i \dots \widetilde{s}_k a \circ_j \widetilde{s}_i \dots \widetilde{s}_k b = \varepsilon_{k+1, i} a \star_{k+1, j} \varepsilon_{k+1, i} b,$$

(viii) if  $i \leq j$  then by Axioms 2.2.1(viii), (ix) and other ones

$$\begin{aligned} \varepsilon_{k+1, j+1} \varepsilon_{k, i} &= \widetilde{s}_{j+1} \dots \widetilde{s}_k \widetilde{s}_i \dots \widetilde{s}_{k-1} \\ &= \widetilde{s}_i \dots \widetilde{s}_{j-1} \widetilde{s}_{j+1} \dots \widetilde{s}_k \widetilde{s}_j \dots \widetilde{s}_{k-1} \\ &= \widetilde{s}_i \dots \widetilde{s}_k \widetilde{s}_j \dots \widetilde{s}_{k-1} \\ &= \varepsilon_{k+1, i} \varepsilon_{k, j}. \end{aligned}$$

A.2. End of the proof of Lemma 3.2.5

To show that  $C^s$  is a single-set cubical  $\omega$ -category, we derive the remaining axioms:

(iv) if  $\Delta_j(x, y)$ ,

– if  $i < j$  then

$$\begin{aligned}\delta_i^\alpha(x \circ_j y) &= \varepsilon_{n,i} \partial_{n,i}^\alpha(x \star_{n,j} y) \\ &= \varepsilon_{n,i} (\partial_{n,i}^\alpha x \star_{n-1,j-1} \partial_{n,i}^\alpha y) \\ &= \varepsilon_{n,i} \partial_{n,i}^\alpha x \star_{n,j} \varepsilon_{n,i} \partial_{n,i}^\alpha y \\ &= \delta_i^\alpha x \circ_j \delta_i^\alpha y,\end{aligned}$$

– if  $i > j$  then

$$\begin{aligned}\delta_i^\alpha(x \circ_j y) &= \varepsilon_{n,i} \partial_{n,i}^\alpha(x \star_{n,j} y) \\ &= \varepsilon_{n,i} (\partial_{n,i}^\alpha x \star_{n-1,j} \partial_{n,i}^\alpha y) \\ &= \varepsilon_{n,i} \partial_{n,i}^\alpha x \star_{n,j} \varepsilon_{n,i} \partial_{n,i}^\alpha y \\ &= \delta_i^\alpha x \circ_j \delta_i^\alpha y,\end{aligned}$$

(v) exchange law: if  $i \neq j$ ,  $\Delta_i(w, x)$ ,  $\Delta_i(y, z)$ ,  $\Delta_j(w, y)$  and  $\Delta_j(x, z)$  then

$$(w \circ_i x) \circ_j (y \circ_i z) = (w \star_{n,i} x) \star_{n,j} (y \star_{n,i} z) = (w \star_{n,j} y) \star_{n,i} (x \star_{n,j} z) = (w \circ_j y) \circ_i (x \circ_j z),$$

(vi) – if  $x \in \mathcal{S}^i$  then  $\delta_{i+1}^- s_i x = \varepsilon_{n,i+1} \partial_{n,i+1}^- \varepsilon_{n,i+1} \partial_{n,i}^- x = s_i x$ , so  $s_i x \in \mathcal{S}^{i+1}$ ,

– if  $y \in \mathcal{S}^{i+1}$  then  $\delta_i^- \tilde{s}_i y = \varepsilon_{n,i} \partial_{n,i}^- \varepsilon_{n,i} \partial_{n,i+1}^- y = \tilde{s}_i y$ , so  $\tilde{s}_i y \in \mathcal{S}^i$ ,

(vii) – if  $x \in \mathcal{S}^i$  then  $\tilde{s}_i s_i x = \varepsilon_{n,i} \partial_{n,i+1}^- \varepsilon_{n,i+1} \partial_{n,i}^- x = \delta_i^- x = x$ ,

– if  $y \in \mathcal{S}^{i+1}$  then  $s_i \tilde{s}_i y = \varepsilon_{n,i+1} \partial_{n,i}^- \varepsilon_{n,i} \partial_{n,i+1}^- y = \delta_{i+1}^- y = y$ ,

(viii) if  $x \in \mathcal{S}^j$ ,

– then

$$\delta_j^\alpha s_j x = \varepsilon_{n,j} \partial_{n,j}^\alpha \varepsilon_{n,j+1} \partial_{n,j}^- x = \varepsilon_{n,j+1} \varepsilon_{n-1,j} \partial_{n-1,j}^\alpha \partial_{n,j+1}^- x = \varepsilon_{n,j} \varepsilon_{n-1,j} \partial_{n-1,j}^\alpha \partial_{n,j}^- x = s_j \delta_{j+1}^\alpha x,$$

– if  $i < j$  then

$$\delta_i^\alpha s_j x = \varepsilon_{n,i} \partial_{n,i}^\alpha \varepsilon_{n,j+1} \partial_{n,j}^- x = \varepsilon_{n,i} \varepsilon_{n-1,j} \partial_{n-1,i}^\alpha \partial_{n,j}^\beta x = \varepsilon_{n,j+1} \varepsilon_{n-1,i} \partial_{n-1,j-1}^\alpha \partial_{n,i}^\beta x = s_j \delta_i^\alpha x,$$

–  $i > j + 1$  then

$$\delta_i^\alpha s_j x = \varepsilon_{n,i} \partial_{n,i}^\alpha \varepsilon_{n,j+1} \partial_{n,j}^- x = \varepsilon_{n,i} \varepsilon_{n-1,j+1} \partial_{n-1,i-1}^\alpha \partial_{n,j}^\beta x = \varepsilon_{n,j+1} \varepsilon_{n-1,i-1} \partial_{n-1,j}^\alpha \partial_{n,i}^\beta x = s_j \delta_i^\alpha x,$$

(ix) if  $x, y \in \mathcal{S}^i$  and  $\Delta_j(x, y)$ ,

## A. Appendices

---

– if  $j = i + 1$  then

$$\begin{aligned} s_i(x \circ_{i+1} y) &= \varepsilon_{n,i+1} \partial_{n,i}^- (x \star_{n,i+1} y) \\ &= \varepsilon_{n,i+1} (\partial_{n,i}^- x \star_{n,i} \partial_{n,i}^- y) \\ &= \varepsilon_{n,i+1} \partial_{n,i}^- x \star_{n,i} \varepsilon_{n,i+1} \partial_{n,i}^- y \\ &= s_i x \circ_i s_i y \end{aligned}$$

– if  $j < i$  then

$$\begin{aligned} s_i(x \circ_j y) &= \varepsilon_{n,i+1} \partial_{n,i}^- (x \star_{n,j} y) \\ &= \varepsilon_{n,i+1} (\partial_{n,i}^- x \star_{n,j} \partial_{n,i}^- y) \\ &= \varepsilon_{n,i+1} \partial_{n,i}^- x \star_{n,j} \varepsilon_{n,i+1} \partial_{n,i}^- y \\ &= s_i x \circ_j s_i y \end{aligned}$$

– if  $j > i + 1$  then

$$\begin{aligned} s_i(x \circ_j y) &= \varepsilon_{n,i+1} \partial_{n,i}^- (x \star_{n,j} y) \\ &= \varepsilon_{n,i+1} (\partial_{n,i}^- x \star_{n,j-1} \partial_{n,i}^- y) \\ &= \varepsilon_{n,i+1} \partial_{n,i}^- x \star_{n,j} \varepsilon_{n,i+1} \partial_{n,i}^- y \\ &= s_i x \circ_j s_i y \end{aligned}$$

(x) if  $x \in \mathcal{S}^{i,i+1}$  then

$$s_i x = s_i \delta_i^- \delta_{i+1}^- x = \varepsilon_{n,i+1} \partial_{n,i}^- \varepsilon_{n,i+1} \partial_{n,i+1}^- x = \varepsilon_{n,i+1} \partial_{n,i+1}^- \varepsilon_{n,i} \partial_{n,i}^- x = \delta_{i+1}^- \delta_i^- x = x,$$

(xi) if  $x \in \mathcal{S}^{i,j}$ ,

– if  $i < j - 1$  then

$$s_i s_j x = \varepsilon_{n,i+1} \partial_{n,i}^- \varepsilon_{n,j+1} \partial_{n,j}^- x = \varepsilon_{n,i+1} \varepsilon_{n,j} \partial_{n,i}^- \partial_{n,j}^- x = \varepsilon_{n,j+1} \varepsilon_{n-1,i+1} \partial_{n-1,j-1}^- \partial_{n,i}^- x = s_j s_i x,$$

– if  $i > j + 1$  then

$$s_i s_j x = \varepsilon_{n,i+1} \partial_{n,i}^- \varepsilon_{n,j+1} \partial_{n,j}^- x = \varepsilon_{n,i+1} \varepsilon_{n,j+1} \partial_{n,i-1}^- \partial_{n,j}^- x = \varepsilon_{n,j+1} \varepsilon_{n-1,i} \partial_{n-1,j}^- \partial_{n,i}^- x = s_j s_i x,$$

(xii) each  $x \in \mathcal{S}$  has a representative  $a \in C_n$  for some  $n \in \mathbb{N}$ , so let  $i \geq n + 1$ , then  $a' = \varepsilon_{i,i} \dots \varepsilon_{n+1,n+1} a \in C_i$  represents  $x$ , so by definition

$$\delta_i^- x = cl_{\sim}(\varepsilon_{i,i} \partial_{i,i}^- a') = cl_{\sim}(\varepsilon_{i,i} \partial_{i,i}^- \varepsilon_{i,i} \dots \varepsilon_{n+1,n+1} a) = cl_{\sim}(\varepsilon_{i,i} \dots \varepsilon_{n+1,n+1} a) = x,$$

so  $x \in \mathcal{S}^{>n}$ .

### A.3. End of the proof of Lemma 3.3.3

To show that  $\mathcal{S}^{\Gamma}$  is a cubical  $\omega$ -category with connections, we prove the remaining axioms:

- (i) –  $\partial_{k,i+1}^{\alpha} \Gamma_{k,i}^{\alpha} = s_{k-1} \dots s_{i+1} \delta_{i+1}^{\alpha} \gamma_i^{\alpha} \tilde{s}_i \dots \tilde{s}_{k-1} = id$ ,  
–  $\partial_{k,i}^{\alpha} \Gamma_{k,i}^{-\alpha} = s_{k-1} \dots s_i \delta_{i+1}^{\alpha} \tilde{s}_i \dots \tilde{s}_{k-1} = \tilde{s}_i \dots \tilde{s}_{k-2} s_{k-2} \dots s_i \delta_i^{\alpha} = \varepsilon_{k-1,i} \partial_{k-1,i}^{\alpha}$ ,  
–  $\partial_{k,i+1}^{\alpha} \Gamma_{k,i}^{-\alpha} = s_{k-1} \dots s_{i+1} \delta_{i+1}^{\alpha} \tilde{s}_i \dots \tilde{s}_{k-1} = \tilde{s}_i \dots \tilde{s}_{k-2} s_{k-2} \dots s_i \delta_i^{\alpha} = \varepsilon_{k-1,i} \partial_{k-1,i}^{\alpha}$ ,  
– if  $i > j + 1$  then

$$\partial_{k,i}^{\alpha} \Gamma_{k,j}^{\beta} = \gamma_j^{\beta} s_{k-1} \dots s_i \tilde{s}_j \dots \delta_i^{\alpha} \tilde{s}_{i-1} \tilde{s}_i \dots \tilde{s}_{k-1} = \gamma_j^{\beta} \tilde{s}_j \dots \tilde{s}_{i-2} \delta_{i-1}^{\alpha} = \Gamma_{k-1,j}^{\beta} \partial_{k-1,i-1}^{\alpha}$$

- (ii) if  $a, b$  are  $\star_{k,j}$ -composable,

- $i = j$  then

$$\begin{aligned} \Gamma_{k+1,i}^+(a \star_{k,i} b) &= \gamma_i^+(\tilde{s}_i \dots \tilde{s}_k a \circ_{i+1} \tilde{s}_i \dots \tilde{s}_k b) \\ &= (\gamma_i^+ \tilde{s}_i \dots \tilde{s}_k a \circ_i \tilde{s}_i \dots \tilde{s}_k a) \circ_{i+1} (s_i \tilde{s}_i \dots \tilde{s}_k a \circ_i \gamma_i^+ \tilde{s}_i \dots \tilde{s}_k b) \\ &= (\Gamma_{k+1,i}^+ a \star_{k+1,i} \varepsilon_{k+1,i} a) \star_{k+1,i+1} (\varepsilon_{k+1,i+1} a \star_{k+1,i} \Gamma_{k+1,i}^+ b), \end{aligned}$$

- if  $i > j$  then  $\Gamma_{k+1,i}^{\alpha}(a \star_{k,j} b) = \gamma_i^{\alpha}(\tilde{s}_i \dots \tilde{s}_k a \circ_j \tilde{s}_i \dots \tilde{s}_k b) = \Gamma_{k+1,i}^{\alpha} a \star_{k+1,j} \Gamma_{k+1,i}^{\alpha} b$ ,

- (iii) –  $\Gamma_{k,i}^+ a \star_{k,i} \Gamma_{k,i}^- a = \gamma_i^+ \tilde{s}_i \dots \tilde{s}_{k-1} a \circ_i \gamma_i^- \tilde{s}_i \dots \tilde{s}_{k-1} a = \varepsilon_{k,i+1} a$  by Axiom 2.3.1(iv) and other ones,  
–  $\Gamma_{k,i}^+ a \star_{k,i+1} \Gamma_{k,i}^- a = \gamma_i^+ \tilde{s}_i \dots \tilde{s}_{k-1} a \circ_{i+1} \gamma_i^- \tilde{s}_i \dots \tilde{s}_{k-1} a = \varepsilon_{k,i} a$  by Axiom 2.3.1(iv) and other ones,

- (iv) – by Axiom 2.3.1(iii) and other ones  $\Gamma_{k+1,i}^{\alpha} \varepsilon_{k,i} = \gamma_i^{\alpha} \tilde{s}_i \dots \tilde{s}_k \tilde{s}_i \dots \tilde{s}_{k-1} = \varepsilon_{k+1,i} \varepsilon_{k,i}$ ,  
– if  $i > j$  then

$$\Gamma_{k+1,i}^{\alpha} \varepsilon_{k,j} = \tilde{s}_j \dots \tilde{s}_{i-2} \gamma_i^{\alpha} \tilde{s}_i \dots \tilde{s}_k \tilde{s}_{i-1} \dots \tilde{s}_{k-1} = \tilde{s}_j \dots \tilde{s}_i \gamma_{i-1}^{\alpha} \tilde{s}_{i+1} \dots \tilde{s}_k \tilde{s}_{i-1} \dots \tilde{s}_{k-1} = \varepsilon_{k+1,j} \Gamma_{k,i-1}^{\alpha}$$

- (v) – if  $i < j$  then using Axioms 2.3.1(v), (vi) and other ones

$$\begin{aligned} \Gamma_{k+1,i}^{\alpha} \Gamma_{k,j}^{\beta} &= \gamma_i^{\alpha} \tilde{s}_i \dots \tilde{s}_j \tilde{s}_{j+1} \gamma_j^{\beta} \tilde{s}_{j+2} \dots \tilde{s}_k \tilde{s}_j \dots \tilde{s}_{k-1} \\ &= \gamma_i^{\alpha} \tilde{s}_i \dots \tilde{s}_{j-1} \gamma_{j+1}^{\beta} \tilde{s}_{j+1} \dots \tilde{s}_k \tilde{s}_j \dots \tilde{s}_{k-1} \\ &= \Gamma_{k+1,j+1}^{\beta} \Gamma_{k,i}^{\alpha} \end{aligned}$$

- $\Gamma_{k+1,i}^{\alpha} \Gamma_{k,i}^{\alpha} = \tilde{s}_i \tilde{s}_{i+1} \gamma_i^{\alpha} \tilde{s}_{i+2} \dots \tilde{s}_k \gamma_i^{\alpha} \tilde{s}_i \dots \tilde{s}_{k-1} = \gamma_{i+1}^{\alpha} \tilde{s}_{i+1} \dots \tilde{s}_k \gamma_i^{\alpha} \tilde{s}_i \dots \tilde{s}_{k-1} = \Gamma_{k+1,i+1}^{\alpha} \Gamma_{k,i}^{\alpha}$ .

### A.4. End of the proof of Lemma 3.3.5

To show that  $C^{SY}$  is a single-set cubical  $\omega$ -category with connections, we prove the remaining axioms:

- (i) if  $i \neq j, j + 1$  and  $x \in \mathcal{S}^j$ , then  $\partial_{n,j}^- x = \partial_{n,j}^+ x$  so

## A. Appendices

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- $\delta_{j+1}^\alpha \gamma_j^\alpha x = \varepsilon_{n,j+1} \partial_{n,j+1}^\alpha \Gamma_{n,j}^\alpha \partial_{n,j}^\alpha x = \varepsilon_{n,j+1} \partial_{n,j}^\alpha x = s_j x$ ,
- if  $i < j$  then  $\delta_i^\alpha \gamma_j^\beta x = \varepsilon_{n,i} \Gamma_{n-1,j-1}^\beta \partial_{n-1,i}^\alpha \partial_{n,j}^\beta x = \Gamma_{n,j}^\beta \varepsilon_{n-1,i} \partial_{n-1,j-1}^\beta \partial_{n,i}^\alpha x = \gamma_j^\beta \delta_i^\alpha x$ ,
- if  $i > j + 1$  then  $\delta_i^\alpha \gamma_j^\beta x = \varepsilon_{n,i} \Gamma_{n-1,j}^\beta \partial_{n-1,i-1}^\alpha \partial_{n,j}^\beta x = \Gamma_{n,j}^\beta \varepsilon_{n-1,i-1} \partial_{n-1,j}^\beta \partial_{n,i}^\alpha x = \gamma_j^\beta \delta_i^\alpha x$ ,

(ii) if  $j \neq i, i+1, x, y \in \mathcal{S}^i$ ,

- if  $\Delta_{i+1}(x, y)$  then

$$\begin{aligned} \gamma_i^-(x \circ_{i+1} y) &= \Gamma_{n,i}^-(\partial_{n,i}^- x \star_{n,i} \partial_{n,i}^- y) \\ &= (\Gamma_{n,i}^- \partial_{n,i}^- x \star_{n+1,i+1} \varepsilon_{n,i} \partial_{n,i}^- y) \star_{n+1,i} (\varepsilon_{n,i+1} \partial_{n,i}^- y \star_{n+1,i+1} \Gamma_{n,i}^- \partial_{n,i}^- y) \\ &= (\gamma_i^- x \circ_{i+1} y) \circ_i (s_i y \circ_{i+1} \gamma_i^- y), \end{aligned}$$

- if  $j < i$  and  $\Delta_j(x, y)$  then  $\gamma_i^\alpha(x \circ_j y) = \Gamma_{n,i}^\alpha(\partial_{n,i}^\alpha x \star_{n-1,j} \partial_{n,i}^\alpha y) = \gamma_i^\alpha x \circ_j \gamma_i^\alpha y$ ,
- if  $j > i + 1$  and  $\Delta_j(x, y)$  then  $\gamma_i^\alpha(x \circ_j y) = \Gamma_{n,i}^\alpha(\partial_{n,i}^\alpha x \star_{n-1,j-1} \partial_{n,i}^\alpha y) = \gamma_i^\alpha x \circ_j \gamma_i^\alpha y$ ,

(iii) if  $x \in \mathcal{S}^{i,i+1}$  then  $x = \varepsilon_{n,i} \varepsilon_{n-1,i} \partial_{n-1,i}^- \partial_{n,i+1}^- x$  so

$$\gamma_i^\alpha x = \Gamma_{n,i}^\alpha \varepsilon_{n-1,i} \partial_{n-1,i}^- \partial_{n,i+1}^- x = \varepsilon_{n,i} \varepsilon_{n-1,i} \partial_{n-1,i}^- \partial_{n,i+1}^- x = x,$$

(iv) if  $x \in \mathcal{S}^i$ , then  $\partial_{n,i}^+ x = \partial_{n,i}^- x$  so

- $\gamma_i^+ x \circ_{i+1} \gamma_i^- x = \Gamma_{n,i}^+ \partial_{n,i}^+ x \star_{n,i+1} \Gamma_{n,i}^- \partial_{n,i}^- x = \varepsilon_{n,i} \partial_{n,i}^+ x = x$ ,
- $\gamma_i^+ x \circ_i \gamma_i^- x = \Gamma_{n,i}^+ \partial_{n,i}^+ x \star_{n,i} \Gamma_{n,i}^- \partial_{n,i}^- x = \varepsilon_{n,i+1} \partial_{n,i}^+ x = s_i x$ ,

(v) if  $x \in \mathcal{S}^{i,j}$  and  $i > j + 1$  then

$$\gamma_i^\alpha \gamma_j^\beta x = \Gamma_{n,i}^\alpha \Gamma_{n-1,j}^\beta \partial_{n-1,i-1}^\alpha \partial_{n,j}^\beta x = \Gamma_{n,j}^\beta \Gamma_{n-1,i-1}^\alpha \partial_{n-1,j}^\beta \partial_{n,i}^\alpha x = \gamma_j^\beta \gamma_i^\alpha x.$$

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