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ON GOOD A_1 SUBGROUPS, SPRINGER MAPS, AND OVERGROUPS OF DISTINGUISHED UNIPOTENT ELEMENTS IN REDUCTIVE GROUPS

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Dedicated to the fond memory of Gary Seitz

Abstract. Suppose G is a simple algebraic group defined over an algebraically closed field of good characteristic p. In 2018 Korhonen showed that if H is a connected reductive subgroup of G which contains a distinguished unipotent element u of G of order p, then H is G-irreducible in the sense of Serre. We present a short and uniform proof of this result under an extra hypothesis using so-called *good* A_1 subgroups of G, introduced by Seitz. In the process we prove some new results about good A_1 subgroups of G and their properties. We also formulate a counterpart of Korhonen's theorem for overgroups of u which are finite groups of Lie type. Moreover, we generalize both results above by removing the restriction on the order of u under a mild condition on p depending on the rank of G, and we present an analogue of Korhonen's theorem for Lie algebras.

1. Introduction and main results

Throughout, G is a connected reductive linear algebraic group defined over an algebraically closed field k of characteristic p and H is a closed subgroup of G.

Following Serre [35], we say that H is *G-completely reducible* (G-cr for short) provided that whenever H is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup of P, and that H is *G-irreducible* (G-ir for short) provided H is not contained in any proper parabolic subgroup of G at all. Clearly, if H is G-irreducible, it is trivially G-completely reducible, and an overgroup of a G-irreducible subgroup is again G-irreducible; for an overview of this concept see [5], [34] and [35]. Note that in case $G = \mathrm{GL}(V)$ a subgroup H is G-completely reducible exactly when V is a semisimple H-module and it is G-irreducible precisely when V is an irreducible H-module. Recall that if H is G-completely reducible, then the identity component H° of H is reductive, [35, Prop. 4.1].

A unipotent element u of G is *distinguished* provided any torus in the centraliser $C_G(u)$ of u in G is central in G. Likewise, a nilpotent element X of the Lie algebra $\mathfrak{g}$ of G is *distinguished* provided any torus in the centraliser $C_G(X)$ of X in G is central in G, see [10, §5.9] and [16, §4.1]. For instance, regular unipotent elements in G are distinguished, and so are regular nilpotent elements in $\mathfrak{g}$ [39, III 1.14] (or [10, Prop. 5.1.5]). The converse is true in type A, since a distinguished unipotent or nilpotent element must clearly consist of a single Jordan block. Overgroups of regular unipotent elements have attracted much attention in the literature, e.g., see [42], [32], [44], [22], and [7].

In [17], Korhonen proves the following remarkable result.

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Theorem 1.1 ([17, Thm. 6.5]). *Suppose G is simple and p is good for G. Let H be a reductive subgroup of G. Suppose H° contains a distinguished unipotent element of G of order p. Then H is G-irreducible.*

One can easily extend this theorem to arbitrary connected reductive G by reducing to the simple case: see Remark 6.3.

Korhonen's proof of Theorem 1.1 depends on checks for the various possible Dynkin types for simple G. E.g., for G simple of exceptional type, Korhonen's argument relies on long exhaustive case-by-case investigations from [20], where all connected reductive non-G-cr subgroups are classified in the exceptional type groups in good characteristic. For classical G, Korhonen requires an intricate classification of all SL_2-representations on which a non-trivial unipotent element of SL_2 acts with at most one Jordan block of size p. Our main aim is to give a short uniform proof of Theorem 1.1 in §6 without resorting to further case-by-case checks, but imposing an extra hypothesis which allows us to use a landmark result by Seitz (see §5.1).

Theorem 1.2. *Suppose p is good for G. Let H be a connected reductive subgroup of G. Suppose H contains a distinguished unipotent element of G of order p. Suppose also that*

(†) *there exists a Springer map ϕ for H such that $\phi(u)$ is a distinguished element of $\mathfrak{g}$.*

Then H is G-irreducible.

For a discussion of Springer maps, see Section 4.1.

Remark 1.3. Suppose as in Theorem 1.1, that G is simple classical with natural module V, and $p \geq \dim V > 2$. Then, thanks to [15, Prop. 3.2], V is semisimple as an H°-module, and by [35, (3.2.2(b))], this is equivalent to H° being G-cr. Then H is G-ir, by Lemma 3.1. This gives a short uniform proof of the conclusion of Theorem 1.1 in this case, as the bound $p \geq \dim V > 2$ ensures that every distinguished unipotent element (including the regular ones) is of order p. The conclusion can fail if the bound is not satisfied: see Theorem 1.5.

We say that a subgroup of G is of *type* A_1 if it is isomorphic to SL_2 or PGL_2. Our proof of Theorem 1.2 involves the notion of a *good* A_1 subgroup, which was introduced by Seitz in [33]. We consider the interaction of good A_1 subgroups with associated cocharacters and Springer maps; we identify a useful class of Springer maps (Definition 5.16), which we call *logarithmic* Springer maps, and we prove some results that are of interest in their own right (see Corollary 5.20 and Lemma 5.30). Our main result on good A_1 subgroups is the following (see Section 5.2 for definitions).

Theorem 1.4. *Suppose p is good for G and let A be an A_1 subgroup of G. The following are equivalent.*

(i) *A is sub-principal.*
(ii) *A is optimal.*
(iii) *A is good.*

Theorem 1.1 covers the situation when p is good for G. There are only a few cases when G is simple, p is bad for G, and G admits a distinguished unipotent element of order p, by work of Proud-Saxl-Testerman [31, Lem. 4.1, Lem. 4.2] (see Lemmas 2.5 and 2.7). In this case the conclusion of Theorem 1.1 fails precisely in one instance, as observed in [17, Prop. 1.2]

(Example 2.6), else it is valid (Example 2.8). Combining the cases when p is bad for G with Theorem 1.2, we recover Korhonen's main theorem [17, Thm. 1.3] (assuming that (†) from Theorem 1.2 holds).

Theorem 1.5. *Suppose G is simple and let H be a reductive subgroup of G. Suppose H° contains a distinguished unipotent element of G of order p, and suppose that (†) holds. Then H is G-irreducible, unless $p = 2$, G is of type C_2, and H is a type A_1 subgroup of G.*

Our next goal is an extension of Theorem 1.2 to finite groups of Lie type in G. Let $\sigma : G \to G$ be a Steinberg endomorphism of G, so that the finite fixed point subgroup $G_\sigma = G(q)$ is a finite group of Lie type over the field $\mathbb{F}_q$ of q elements. For a Steinberg endomorphism σ of G and a connected reductive σ-stable subgroup H of G, σ is also a Steinberg endomorphism for H with finite fixed point subgroup $H_\sigma = H \cap G_\sigma$, [40, 7.1(b)]. Obviously, one cannot directly appeal to Theorem 1.2 to deduce anything about H_σ, because $(H_\sigma)^\circ$ is trivial. For the notion of a q-Frobenius endomorphism, see §2.3.

Theorem 1.6. *Let H be a connected reductive subgroup of G and suppose p is good for G. Let $\sigma : G \to G$ be a Steinberg endomorphism stabilizing H such that $\sigma|_H$ is a q-Frobenius endomorphism of H. If G admits components of exceptional type, then assume $q > 7$. Suppose H_σ contains a distinguished unipotent element of G of order p, and suppose that (†) holds. Then H_σ is G-irreducible.*

Combining Theorem 1.6 with the aforementioned results from [31], we are able to deduce the following analogue of Theorem 1.5 for finite subgroups of Lie type in G.

Theorem 1.7. *Let H be a connected reductive subgroup of G. Let $\sigma : G \to G$ be a Steinberg endomorphism stabilizing H such that $\sigma|_H$ is a q-Frobenius endomorphism of H. If G is of exceptional type, then assume $q > 7$. Suppose H_σ contains a distinguished unipotent element of G of order p, and suppose that (†) holds. Then H_σ is G-irreducible, unless $p = 2$, G is of type C_2, and H is a type A_1 subgroup of G.*

In the special instance in Theorems 1.6 and 1.7 when H_σ contains a regular unipotent element u from G, the conclusion of both theorems holds without any restriction on the order of u and without any restriction on q (and without any exceptions of the type seen in Theorem 1.7); see [7, Thm. 1.3].

In our final main result we show that we can remove condition (†) and the condition that u has order p from Theorem 1.2, at the cost of increasing our bound on p. We also obtain an analogue under the hypothesis that $\operatorname{Lie}(H)$ contains a distinguished nilpotent element of $\mathfrak{g}$. For a unipotent element $u \in G$ to be distinguished is a mere condition on the structure of the centralizer $C_G(u)$ of u in G. The extra condition for u to have order p is thus somewhat artificial. This restriction in Theorems 1.1 and 1.2 is due to the methods used in [17] and in our proofs in §6, which require the unipotent element to lie in a subgroup of type A_1; such an element must obviously have order p.

To state our theorem, we need to introduce an invariant $a(G)$ of G from [35, §5.2]: for G simple, set $a(G) = \operatorname{rk}(G) + 1$, where $\operatorname{rk}(G)$ is the rank of G. For reductive G, let $a(G) = \max\{1, a(G_1), \dots, a(G_r)\}$, where $G_1, \dots, G_r$ are the simple components of G.

Theorem 1.8. *Suppose $p \geq a(G)$. Let H be a reductive subgroup of G. Suppose H° contains a distinguished unipotent element of G or $\operatorname{Lie}(H)$ contains a distinguished nilpotent element of $\mathfrak{g}$. Then H is G-irreducible.*

Section 2 contains background material. In Section 3 we prove Theorem 1.8, along with some analogues for finite subgroups of Lie type. In Section 4 we discuss Springer maps and associated cocharacters. We recall Seitz's notion of good A_1 subgroups in Section 5 and we prove Theorem 1.4 in Section 5.2 (see Theorem 5.24). Theorems 1.2 and 1.5–1.7 are proved in Section 6.

2. Preliminaries

2.1. **Notation.** Throughout, we work over an algebraically closed field k of characteristic p. For convenience we assume that $p > 0$ unless otherwise stated; most of our results hold for $p = 0$ with obvious modifications and in many cases the proof is much easier (see Remark 3.3(vi), for example). All affine varieties are considered over k and are identified with their sets of k-points. A linear algebraic group H over k has identity component H°; if $H = H^\circ$, then we say that H is *connected*. We denote by $R_u(H)$ the *unipotent radical* of H; if $R_u(H)$ is trivial, then we say H is *reductive*.

Throughout, G denotes a connected reductive linear algebraic group over k. All subgroups of G considered are closed. By $\mathscr{D}G$ we denote the derived subgroup of G, and likewise for subgroups of G. We denote the Lie algebra of G by $\mathrm{Lie}(G)$ or by $\mathfrak{g}$. If $p > 0$ then we denote the p-power map on $\mathfrak{g}$ by $X \mapsto X^{[p]}$. By a Levi subgroup of G we mean a Levi subgroup of some parabolic subgroup of G. Recall that a homomorphism $f\colon G_1 \to G_2$ of connected algebraic groups is a *central isogeny* if f is surjective, $\ker(f)$ is finite and the kernel of the derivative df is central in $\mathrm{Lie}(G_1)$.

Let $Y(G) = \mathrm{Hom}(\mathbb{G}_m, G)$ denote the set of cocharacters of G. For $\mu \in Y(G)$ and $g \in G$ we define the *conjugate cocharacter* $g \cdot \mu \in Y(G)$ by $(g \cdot \mu)(t) = g\mu(t)g^{-1}$ for $t \in \mathbb{G}_m$; this gives a left action of G on $Y(G)$. For H a subgroup of G, let $Y(H) := Y(H^\circ) = \mathrm{Hom}(\mathbb{G}_m, H)$ denote the set of cocharacters of H. There is an obvious inclusion $Y(H) \subseteq Y(G)$.

Fix a Borel subgroup B of G containing a maximal torus T. Let $\Phi = \Phi(G,T)$ be the root system of G with respect to T, let $\Phi^+ = \Phi(B,T)$ be the set of positive roots of G, and let $\Sigma = \Sigma(G,T)$ be the set of simple roots of Φ^+. For each $\alpha \in \Phi$ we have a root subgroup U_α of G. For α in Φ, let $x_\alpha : \mathbb{G}_a \to U_\alpha$ be a parametrization of the root subgroup U_α of G.

We denote the unipotent variety of G by $\mathscr{U}_G$ and the nilpotent cone of $\mathfrak{g}$ by $\mathscr{N}_G$. We define

$$\mathscr{U}_G^{(1)} = \{u \in \mathscr{U}_G \mid u^p = 1\}$$

and

$$\mathscr{N}_G^{(1)} = \{X \in \mathscr{N}_G \mid X^{[p]} = 0\}.$$

If $u \in \mathscr{U}_G$ then we have a unique decomposition $u = u_1 \cdots u_r$, where $u_i \in G_i$ and the G_i are the simple factors of $\mathscr{D}G$; we call u_i the *projection of* u *onto* G_i. Clearly u is distinguished in G if and only if u_i is distinguished in G_i for each i.

2.2. **Good primes.** A prime p is said to be *good* for G if it does not divide any coefficient of any positive root when expressed as a linear combination of simple ones. Else p is called *bad* for G, [39, §4]. Explicitly, if G is simple, p is *good* for G provided $p > 2$ in case G is of Dynkin type B_n, C_n, or D_n; $p > 3$ in case G is of Dynkin type E_6, E_7, F_4 or G_2 and $p > 5$ in case G is of type E_8. If G is semisimple then we say that p is *separably good* for G if p is good for G and the canonical map from G_{sc} to G is separable, where G_{sc} is the simply connected cover of G. For arbitrary connected reductive G we say that p is *separably good* for G if it is separably good for $[G, G]$. We observe that if L is a Levi subgroup of G and p is good for G, then it is also good for L.

2.3. **Steinberg endomorphisms of G.** Recall that a *Steinberg endomorphism* of G is a surjective homomorphism $\sigma : G \to G$ such that the corresponding fixed point subgroup $G_\sigma := \{g \in G \mid \sigma(g) = g\}$ of G is finite. Frobenius endomorphisms σ_q of reductive groups over finite fields are familiar examples, giving rise to *finite groups of Lie type* $G(q)$. See Steinberg [40] for a detailed discussion (for this terminology, see [13, Def. 1.15.1b]). The set of all Steinberg endomorphisms of G is a subset of the set of all isogenies $G \to G$ (see [40, 7.1(a)]) which encompasses in particular all generalized Frobenius endomorphisms, i.e., endomorphisms of G some power of which are Frobenius endomorphisms corresponding to some $\mathbb{F}_q$-rational structure on G. In that case we also denote the finite group of Lie type G_σ by $G(q)$. If $\mathscr{S}$ is a σ-stable set of closed subgroups of G, then $\mathscr{S}_\sigma$ denotes the subset consisting of all σ-stable members of $\mathscr{S}$.

If $\sigma_q : G \to G$ is a standard q-power Frobenius endomorphism of G, then there exist a σ_q-stable maximal torus T and Borel subgroup $B \supseteq T$, and with respect to a chosen parametrisation of the root groups as above, we have $\sigma_q(x_\alpha(t)) = x_\alpha(t^q)$ for each $\alpha \in \Phi$ and $t \in \mathbb{G}_a$, see [13, Thm. 1.15.4(a)]. Following [31], we call a generalized Frobenius endomorphism σ a *q-Frobenius endomorphism* provided $\sigma = \tau\sigma_q$, where τ is an algebraic automorphism of G of finite order, σ_q is a standard q-power Frobenius endomorphism of G, and σ_q and τ commute. When p is bad for G, a q-Frobenius endomorphism does not involve a twisted Steinberg endomorphism, see [31, §3]. If G is simple and p is good for G, then any Steinberg endomorphism of G is a q-Frobenius endomorphism, see [40, §11]. If G is not simple and p is bad for G, then a generalized Frobenius map may fail to factor into a field and algebraic automorphism of G, e.g., see [14, Ex. 1.3].

2.4. **Bala-Carter Theory.** We recall some relevant results and concepts from Bala-Carter theory. Suppose p is good for G. A parabolic subgroup P of G admits a dense open orbit on its unipotent radical $R_u(P)$, the so-called *Richardson orbit*; see [10, Thm. 5.2.1]. A parabolic subgroup P of G is called *distinguished* provided $\dim(\mathscr{D}P/R_u(P)) = \dim(R_u(P)/\mathscr{D}R_u(P))$, see [30, §2.1]. For G simple, the distinguished parabolic subgroups of G (up to G-conjugacy) were worked out in [2] and [3]; see [10, pp. 174–177]. The notion of a distinguished parabolic subgroup of G also makes sense in case p is bad for G, cf. [16, §4.10].

The following is the celebrated Bala-Carter Theorem, see [10, Thm. 5.9.5, Thm. 5.9.6], which is valid in good characteristic, thanks to work of Pommerening [28], [29]. For the Lie algebra versions see also [16, Prop. 4.7, Thm. 4.13].

Theorem 2.1. *Suppose p is good for G.*

(i) *There is a bijective map between the G-conjugacy classes of distinguished unipotent elements of G and conjugacy classes of distinguished parabolic subgroups of G. The unipotent class corresponding to a given parabolic subgroup P contains the dense P-orbit on $R_u(P)$.*

(ii) *There is a bijective map between the G-conjugacy classes of unipotent elements of G and conjugacy classes of pairs (L, P), where L is a Levi subgroup of G and P is a distinguished parabolic subgroup of $\mathscr{D}L$. The unipotent class corresponding to the pair (L, P) contains the dense P-orbit on $R_u(P)$.*

Remark 2.2. (i). Let $1 \neq u \in \mathscr{U}_G$. Let S be a maximal torus of $C_G(u)$. Then u is distinguished in the Levi subgroup $C_G(S)$ of G, since S is the unique maximal torus of $C_{C_G(S)}(u)$. Conversely, if L is a Levi subgroup of G with u distinguished in L, then the connected center of L is a maximal torus of $C_G(u)^\circ$, see [16, Rem. 4.7].

(ii). Let $\sigma : G \to G$ be a Steinberg endomorphism of G and let $1 \neq u \in G_\sigma$ be unipotent. Then $C_G(u)^\circ$ is σ-stable. The set of all maximal tori of $C_G(u)^\circ$ is σ-stable and $C_G(u)^\circ$ is transitive on that set, [38, Thm. 6.4.1]. Thus the Lang-Steinberg Theorem, see [39, I 2.7], provides a σ-stable maximal torus, say S, of $C_G(u)^\circ$. Then, by part (i), $L = C_G(S)$ is a σ-stable Levi subgroup of G and u is distinguished in L.

2.5. **Cocharacters and parabolic subgroups of G.** Let $\lambda \in Y(G)$. Recall that λ affords a $\mathbb{Z}$-grading on $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}(j,\lambda)$, where $\mathfrak{g}(j,\lambda) := \{X \in \mathfrak{g} \mid \mathrm{Ad}(\lambda(t))X = t^j X \text{ for every } t \in \mathbb{G}_m\}$ is the j-weight space of $\mathrm{Ad}(\lambda(\mathbb{G}_m))$ on $\mathfrak{g}$, see [10, §5.5] or [16, §5.1]. Let $\mathfrak{p}_\lambda := \bigoplus_{j\geq 0} \mathfrak{g}(j,\lambda)$. Then there is a unique parabolic subgroup P_λ with $\mathrm{Lie}(P_\lambda) = \mathfrak{p}_\lambda$ and $C_G(\lambda) := C_G(\lambda(\mathbb{G}_m))$ is a Levi subgroup of P_λ. Since all maximal tori in G are conjugate, it suffices to describe these subgroups and subalgebras when $\lambda \in Y(T)$ for our fixed maximal torus T. In this case, letting $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ denote the character group of T, we have $U_\alpha \subseteq P_\lambda$ if and only if $\langle \lambda, \alpha \rangle \geq 0$, where $\langle\, , \,\rangle : Y(T) \times X(T) \to \mathbb{Z}$ is the usual pairing between cocharacters and characters. We have $U_\alpha \subseteq C_G(\lambda)$ if and only if $\langle \lambda, \alpha\rangle = 0$, and $R_u(P_\lambda)$ is generated by the U_α with $\langle \lambda, \alpha\rangle > 0$; see the proof of [38, Prop. 8.4.5].

Set $J := \{\alpha \in \Sigma \mid \langle \alpha, \lambda\rangle = 0\}$. Then $P_\lambda = P_J = \langle T, U_\alpha \mid \langle\alpha,\lambda\rangle \geq 0\rangle$ is the *standard parabolic subgroup* of G associated with $J \subseteq \Sigma$.

Let $\rho = \sum_{\alpha\in\Sigma} c_{\alpha\rho}\alpha$ be the highest root in Φ^+. Define $\mathrm{ht}_J(\rho) := \sum_{\alpha\in\Sigma\setminus J} c_{\alpha\rho}$. In view of Theorem 2.1, the following gives the order of a distinguished unipotent element in good characteristic.

Lemma 2.3 ([43, Order Formula 0.4])**.** *Suppose p is good for G. Let $P = P_J$ be a distinguished parabolic subgroup of G and let u be in the Richardson orbit of P on $R_u(P)$. Then the order of u is $\min\{p^a \mid p^a > \mathrm{ht}_J(\rho)\}$.*

2.6. **Overgroups of type A_1.** It has been understood for some time now that if p is good for G then one can study unipotent elements of G having order p by embedding them in A_1 subgroups of G. The existence of A_1 overgroups for unipotent elements of order p is guaranteed by the following fundamental results of Testerman [43, Thm. 0.1] if p is good for G and else by Proud-Saxl-Testerman [31]; these results were originally proved for semisimple G but the extension to arbitrary connected reductive G is immediate.

Theorem 2.4 ([43, Thm. 0.1, Thm. 0.2])**.** *Suppose p is good for G. Let σ be id_G or a Steinberg endomorphism of G. Let $u \in G_\sigma$ be unipotent of order p. Then there exists a σ-stable subgroup of G of type A_1 containing u.*

The proof of Theorem 2.4 is based on case-by-case checks and depends in part on computer calculations involving explicit unipotent class representatives. For a uniform proof of the theorem, we refer the reader to McNinch [23]. Conditions to ensure G-complete reducibility of such a subgroup were given in [26].

We now consider A_1 overgroups of distinguished unipotent elements in arbitrary characteristic. There are only a few instances when G is simple, p is bad for G, and G admits a distinguished unipotent element of order p. We recall the relevant results concerning the existence of A_1 overgroups of such elements from [31].

Lemma 2.5 ([31, Lem. 4.1])**.** *Let G be simple classical of type B_l, C_l, or D_l and suppose $p = 2$. Then G admits a distinguished involution u if and only if G is of type C_2 and u belongs to the subregular class $\mathscr{C}$ of G. If σ is id_G or a q-Frobenius endomorphism of G and $u \in \mathscr{C} \cap G_\sigma$, then there exists a σ-stable subgroup A of G of type A_1 containing u.*

Example 2.6. Let G be simple of type C_2 and let $p = 2$. Let σ be id_G or a q-Frobenius endomorphism of G, and suppose $u \in G_\sigma$ is a distinguished unipotent element of order 2. Then Lemma 2.5 provides a σ-stable subgroup A of type A_1 containing u. Thanks to [17, Prop. 1.2], there are such subgroups A which are not G-ir. In fact, according to *loc. cit.*, there are two G-conjugacy classes of such A_1 subgroups in G; see Example 5.15 below. Since A is contained in a proper parabolic subgroup of G, so is A_σ. So the latter is also not G-ir. By Lemma 3.1 below, A and A_σ are not G-cr, either.

Lemma 2.7 ([31, Lem. 3.3, Lem. 4.2])**.** *Let G be simple of exceptional type and suppose p is bad for G. Then G admits a distinguished unipotent element u of order p if and only if G is of type G_2, $p = 3$, and u belongs either to the subregular class $G_2(a_1)$*[1] *or to the class $A_1^{(3)}$ of G. Moreover, if σ is id_G or a q-Frobenius endomorphism of G and $u \in G_2(a_1) \cap G_\sigma$, then there exists a σ-stable subgroup A of G of type A_1 containing u. In case $u \in A_1^{(3)}$, there is no overgroup of u in G of type A_1.*

Example 2.8. Let G be simple of type G_2 and $p = 3$. Let H be a reductive subgroup of G containing a distinguished unipotent element u from G. Then, as $p = 3 = a(G_2)$, it follows from Theorem 1.8 that H° is G-ir, and so is H. This applies in particular to the subgroup A of G of type A_1 containing u when $u \in G_2(a_1)$. Since 3 is not a good prime for G, Theorem 1.1 does not apply in this case. See also [41, Cor. 2].

In case of the presence of a q-Frobenius endomorphism of G stabilizing H, we show in our proof of Theorem 1.7 that H_σ is also G-ir.

Theorem 2.9 ([31, Thm. 5.1])**.** *Let G be semisimple and suppose p is bad for G. Let σ be id_G or a q-Frobenius endomorphism of G. Let $u \in G_\sigma$ be unipotent of order p. If $p = 3$, and G has a simple component of type G_2, assume that the projection of u into this component does not lie in the class $A_1^{(3)}$. Then there exists a σ-stable subgroup of G of type A_1 containing u.*

Corollary 2.10. *Let G be simple of type G_2, $p = 3$ and let σ be id_G or a q-Frobenius endomorphism of G. Let $u \in A_1^{(3)} \cap G_\sigma$. Then there is no proper semisimple subgroup H of G containing u. In particular, any such u is* semiregular, *that is, $C_G(u)$ does not contain a non-central semisimple element of G.*

Proof. By way of contradiction, suppose H is a proper semisimple subgroup of G containing u. Since $p = 3$ is good for H (e.g., see [41, Cor. 3]), there is a σ-stable A_1 subgroup A in H containing u, by Theorem 2.4. It follows from Lemma 2.7 that $u \in G_2(a_1)$ which contradicts the hypothesis that $u \in A_1^{(3)}$. □

The following result is needed in the proof of Theorem 1.2 below.

Lemma 2.11. *Suppose G is semisimple and $p = 3$ is good for G. Let H be a connected reductive subgroup of G. Let $u \in H$ be a unipotent element of order 3 which is distinguished in G. Then H does not admit a simple component of type G_2.*

[1] Throughout, we use the Bala-Carter notation for distinguished classes in the exceptional groups, see [10, §5.9].

Proof. (cf. [17, p. 387]) Since p is good for G, every simple component of G is of classical type. Let V' be the natural module of the simple component G' of G, and let H' be the projection of H into G'. Since the projection u' of u into G' has order 3, the largest Jordan block size of u' on V' is at most 3. Since u' is distinguished in G', the Jordan block sizes of u' are distinct and of the same parity. Hence $\dim V' \leq 4$. Since a non-trivial representation of a simple algebraic group of type G_2 has dimension at least 5, H' does not have a simple component of type G_2. Hence H has no simple component of type G_2. □

In summary, we see that if $1 \neq u \in \mathscr{U}_G^{(1)}$ then u is contained in an A_1 subgroup of G unless $p = 3$ and G has a simple G_2 factor such that the projection of u onto this factor lies in the class $A_1^{(3)}$.

3. Variations on Theorems 1.2 and 1.6

In this section we prove Theorem 1.8. We also state and prove some related results for finite subgroups of Lie type. We need the following analogue of [7, Cor. 4.6], which shows that in order to derive the G-irreducibility of H in Theorem 1.8, it suffices to show that H is G-cr; see also [17, Lem. 6.1]. This also applies to Theorem 1.2 and Theorem 1.6.

Lemma 3.1. *Let H be a G-completely reducible subgroup of G. Suppose that H contains a distinguished unipotent element u of G or* $\mathrm{Lie}(H)$ *contains a distinguished nilpotent element X of $\mathfrak{g}$. Then H is G-irreducible.*

Proof. Suppose H is contained in a parabolic subgroup P of G. Then, by hypothesis, H is contained in a Levi subgroup L of P. As the latter is the centraliser of a torus S in G, S centralises u (resp., X) and so S is central in G. Hence $L = G$, which implies $P = G$. □

Along with Lemma 3.1, the following theorem of Serre immediately yields Theorem 1.8.

Theorem 3.2 ([35, Thm. 4.4])**.** *Suppose $p \geq a(G)$ and $(H : H^\circ)$ is prime to p. Then H° is reductive if and only if H is G-completely reducible.*

Proof of Theorem 1.8. Since $p \geq a(G)$, Theorem 3.2 applied to H° shows the latter is G-cr. Thus H° is G-ir by Lemma 3.1, and so is H. □

Remarks 3.3. (i). The characteristic restriction in Theorem 1.8 (and Theorem 3.2) is needed; see Theorem 1.5.

(ii). The condition in Theorem 1.8 that the distinguished unipotent element of G belongs to H° (as opposed to H) is also necessary, as for instance the finite unipotent subgroup of G generated by a given distinguished unipotent element of G is not G-cr [35, Prop. 4.1].

(iii). Under the given hypotheses, Theorem 1.8 applies to an arbitrary distinguished unipotent element of G, irrespective of its order. For Theorem 1.1 to achieve the same uniform result, p has to be sufficiently large to guarantee that the chosen element has order p. For G simple classical with natural module V, this requires the bound $p \geq \dim V$; see Remark 1.3. For G simple of exceptional type, this requires the following bounds: $p > 11$ for E_6, $p > 17$ for E_7, $p > 29$ for E_8, $p > 11$ for F_4, and $p > 5$ for G_2; see [43, Prop. 2.2]. So in many cases the bound $p \geq a(G)$ from Theorem 1.8 is better.

(iv). For an instance when p is bad for G so that Theorem 1.1 does not apply, but Theorem 1.8 does, see Example 2.8.

(v). Theorem 1.8 generalizes [7, Thm. 3.2] which consists of the analogue in the special instance when the distinguished element is regular in G (or $\mathfrak{g}$). Note that in this case no restriction on p is needed, see [44, Thm. 1.2], [22, Thm. 1], [7, Thm. 3.2].

(vi). In characteristic 0, a subgroup H of G is G-cr if and only if it is reductive, [35, Prop. 4.1]. So in that case the conclusion of Theorem 1.8 follows directly from Lemma 3.1.

Once again, in the presence of a Steinberg endomorphism σ of G, one cannot appeal to Theorem 1.8 directly to deduce anything about H_σ, because $(H_\sigma)^\circ$ is trivial. In Corollary 3.5 we present an analogue of Theorem 1.8 for the finite groups of Lie type H_σ under an additional condition stemming from [4].

Note that for S a torus in G, we have $C_G(S) = C_G(s)$ for some $s \in S$, see [8, III Prop. 8.18].

Proposition 3.4 ([4, Prop. 3.2]). *Let $H \subseteq G$ be connected reductive groups. Let $\sigma\colon G \to G$ be a Steinberg endomorphism that stabilises H and a maximal torus T of H. Suppose*

(i) *$C_G(T) = C_G(t)$, for some $t \in T_\sigma$, and*

(ii) *H_σ meets every T-root subgroup of H non-trivially.*

Then H_σ and H belong to the same parabolic and the same Levi subgroups of G. In particular, H is G-completely reducible if and only if H_σ is G-completely reducible; similarly, H is G-irreducible if and only if H_σ is G-irreducible.

Without condition (i), the proposition is false in general, see [4, Ex. 3.2]. The following is an immediate consequence of Theorem 1.8 and Proposition 3.4.

Corollary 3.5. *Suppose G, H and σ satisfy the hypotheses of Proposition 3.4. Suppose in addition that $p \geq a(G)$. If H_σ contains a distinguished unipotent element of G, then H_σ is G-irreducible.*

Corollary 3.5 generalizes [7, Thm. 1.3] which consists of the analogue in the special instance when the distinguished element is regular in G. Note in this case no restriction on p is needed.

The following example shows that the conditions in Corollary 3.5 hold generically.

Example 3.6. Let $\sigma_q\colon \mathrm{GL}(V) \to \mathrm{GL}(V)$ be a standard Frobenius endomorphism which stabilises a connected reductive subgroup H of $\mathrm{GL}(V)$ and a maximal torus T of H. Pick $l \in \mathbb{N}$ such that firstly all the different T-weights of V are still distinct when restricted to $T_{\sigma_q^l}$ and secondly there is a $t \in T_{\sigma_q^l}$, such that $C_{\mathrm{GL}(V)}(T) = C_{\mathrm{GL}(V)}(t)$. Then for every $n \geq l$, both conditions in Corollary 3.5 are satisfied for $\sigma = \sigma_q^n$. Thus there are only finitely many powers of σ_q for which the conditions in Corollary 3.5 can fail. The argument here readily generalises to a Steinberg endomorphism of a connected reductive G which induces a generalised Frobenius morphism on H.

4. Springer maps and associated cocharacters

4.1. **Springer maps.** The notion of a Springer isomorphism was introduced in [37]. A *Springer isomorphism* is a G-equivariant isomorphism of varieties $\phi\colon \mathscr{U}_G \to \mathscr{N}_G$. It follows from work of Springer [37, Thm. 3.1] that a Springer isomorphism ϕ exists if p is good and G is simple and simply connected. We follow Springer and consider G-equivariant maps from $\mathscr{U}_G$ to $\mathscr{N}_G$, but note that several other authors consider G-equivariant maps from $\mathscr{N}_G$ to $\mathscr{U}_G$ instead (see, e.g., [36]).

We wish to consider versions of Springer maps for arbitrary connected reductive G. To prove existence, we need to weaken the definition slightly.

Definition 4.1. A *Springer map (for G)* is a G-equivariant homeomorphism of varieties $\phi\colon \mathscr{U}_G \to \mathscr{N}_G$.

Remark 4.2. It follows from G-equivariance that if ϕ is a Springer map then $\phi(1) = 0$ and for any $u \in \mathscr{U}_G$, u is distinguished if and only if $\phi(u)$ is distinguished.

Remark 4.3. If p is good for G then there exists a Springer map ϕ for G: see [25, Prop. 5]. Below we sketch the argument briefly, following *loc. cit.* and [36, §1.2]. Note first that a Springer map is uniquely determined by its value on a single regular unipotent element u of G: this follows from G-equivariance, and because the orbit $G\cdot u$ is dense in $\mathscr{U}_G$. If G is simple and p is separably good for G then we can prove existence of a Springer isomorphism by reversing this argument. Fix a regular unipotent element $u \in G$, and choose $X \in \mathscr{N}_G$ such that $C_G(u) = C_G(X)$. We have an obvious isomorphism from $G\cdot u$ to $G\cdot X$. Because $\mathscr{U}_G$ and $\mathscr{N}_G$ are normal (for references, see [34, Lecture 2]), one can show that this map extends to a unique G-equivariant isomorphism from $\mathscr{U}_G$ to $\mathscr{N}_G$. Let us say that G is *of separable type* if it is of the form $G = G_1 \times \cdots \times G_r$, where each G_i is simple and p is separably good for G. A similar argument to the above works for G of separable type: for $\mathscr{U}_G = \mathscr{U}_{G_1} \times \cdots \times \mathscr{U}_{G_r}$ is normal since each $\mathscr{U}_{G_i}$ is, and likewise $\mathscr{N}_G$ is normal.

Now let G be an arbitrary connected reductive group and assume p is good for G. Since $\mathscr{U}_G \subseteq \mathscr{D}G$ and $\mathscr{N}_G \subseteq \mathrm{Lie}(\mathscr{D}G)$, there is no harm in assuming that G is semisimple. Choose a central isogeny π from $\widetilde{G}$ to G, where $\widetilde{G} = \widetilde{G}_1 \times \cdots \times \widetilde{G}_r$ with each $\widetilde{G}_i$ simple and p separably good for $\widetilde{G}$. Then π (resp., $d\pi$) gives a homeomorphism from $\mathscr{U}_{\widetilde{G}}$ to $\mathscr{U}_G$ (resp., from $\mathscr{N}_{\widetilde{G}}$ to $\mathscr{N}_G$) [23, Lem. 27]. If $\widetilde{\phi}$ is a Springer map for $\widetilde{G}$ then the composition $\mathscr{U}_G \to \mathscr{U}_{\widetilde{G}} \stackrel{\widetilde{\phi}}{\to} \mathscr{N}_{\widetilde{G}} \to \mathscr{N}_G$ is a Springer map for G. This gives a bijection between the set of Springer maps for $\widetilde{G}$ and the set of Springer maps for G. Since $\widetilde{G}$ admits a Springer isomorphism, it follows that G admits a Springer map.

Note that if G is of separable type then any Springer map ϕ for G is an isomorphism. For fix a regular unipotent element $u \in G$ and let $X = \phi(u)$. By the above discussion, there is a unique Springer isomorphism ϕ' for G such that $\phi'(u) = X$; the uniqueness implies that $\phi' = \phi$. It also follows from the construction in the previous paragraph that if G is an arbitrary connected reductive group and p is good for G then the restriction of ϕ to any maximal unipotent subgroup U of G gives an isomorphism of varieties from U to $\mathrm{Lie}(U)$.

Remark 4.4. Let G_1, G_2 be connected reductive groups and let ϕ_i be a Springer map for G_i for $i = 1, 2$. We claim that the map $\phi_1 \times \phi_2\colon \mathscr{U}_{G_1\times G_2} \to \mathscr{N}_{G_1\times G_2}$ given by $(\phi_1 \times \phi_2)((u_1, u_2)) = (\phi_1(u_1), \phi_2(u_2))$ is a Springer map for $G_1 \times G_2$. It is clear that $\phi_1 \times \phi_2$ is a $(G_1 \times G_2)$-equivariant bijection. The Zariski topology on the product of varieties is not the product topology, so it is not immediately clear that $\phi_1 \times \phi_2$ is a homeomorphism. To see this, we can pass to the case when G_1 and G_2 are of separable type, by Remark 4.3. Then ϕ_1 and ϕ_2 are isomorphisms, so $\phi_1 \times \phi_2$ is an isomorphism, and the claim follows. We show in Lemma 4.14 that every Springer map for $G_1 \times G_2$ arises in this way.

Remark 4.5. It follows from G-equivariance that a Springer map ϕ gives rise to a bijective map from the set of unipotent conjugacy classes of G to the set of nilpotent conjugacy classes of $\mathfrak{g}$. Serre shows [24, §10, Corollary] that this map does not depend on the choice of Springer map (the proof given in *loc. cit.* is for simple G, but the extension to arbitrary G follows easily from Remarks 4.3 and 4.4). In particular, the condition in (†) does not depend on the choice of Springer map for H.

Remark 4.6. Springer maps need not exist in bad characteristic. For instance, a simple group G of type F_4 with $p = 2$ does not admit a Springer map, because the numbers of unipotent classes in G and nilpotent G-orbits in $\mathrm{Lie}(G)$ are different (see [10, §5.11]).

The following result is [36, §1.2, Rem. 1].

Lemma 4.7. *Let ϕ be a Springer map for G. Then for any $u \in \mathscr{U}_G$, $\phi(u^p) = \phi(u)^{[p]}$.*

Remark 4.8. It follows from Lemma 4.7 that any Springer map for G induces a homeomorphism from $\mathscr{U}_G^{(1)}$ to $\mathscr{N}_G^{(1)}$.

In Section 4.2 we define the notion of an associated cocharacter for an element $u \in \mathscr{U}_G$, using a fixed Springer map to give a correspondence between $\mathscr{U}_G$ and $\mathscr{N}_G$. In many contexts one can fix a single Springer map once and for all. We need, however, to consider the interaction of Springer maps with subgroups of G. This motivates the following definition.

Definition 4.9. Let M be a connected subgroup of G. We say that a Springer map ϕ for G is *M-compatible* if $\phi(\mathscr{U}_M) \subseteq \mathscr{N}_M$, and we say that M is *Springer-compatible* if there exists an M-compatible Springer map for G.

If ϕ is M-compatible then in fact $\phi(\mathscr{U}_M) = \mathscr{N}_M$, since $\dim(\mathscr{U}_M) = \dim(\mathscr{N}_M)$; note that dimension can be defined in a purely topological way (via Krull dimension), so it is preserved by homeomorphisms. Note also that when M is reductive and ϕ is an M-compatible Springer map, the restriction of ϕ to $\mathscr{U}_M$ gives a Springer map for M, which we denote by ϕ_M.

Example 4.10. ([27, (3.3.1)(a)]) Let M be a connected reductive subgroup of the form $C_G(S)^\circ$, where $S \subseteq G$. It follows from G-equivariance that any Springer map for G is M-compatible, so M is Springer-compatible.

Example 4.11. The arguments in Remark 4.3 show that if G_i is a simple factor of G then any Springer map for G is G_i-compatible, so G_i is Springer-compatible.

Example 4.12. Assume $p > h(G)$, where $h(G)$ denotes the Coxeter number of G. The map $\log\colon \mathscr{U}_G \to \mathscr{N}_G$ from [34, Thm. 3] is a Springer map. Let H be a connected reductive subgroup of G. We see that log is H-compatible if and only if H is *saturated* in the sense of [34, Lecture 3]. For some properties of saturated subgroups, see [34] and [4].

Example 4.13. Let $G = \mathrm{SL}_2 \times \mathrm{SL}_2$. For q a positive power of p, let H_q be SL_2 diagonally embedded in G with a q-Frobenius twist in one of the factors: say, the second factor. Note that $\mathrm{Lie}(H_q) = \mathrm{Lie}(\mathrm{SL}_2) \oplus 0$, so $\mathrm{Lie}(H_q)$ contains no nilpotent elements that are distinguished in $\mathfrak{g}$. It follows from Remark 4.2 that no Springer map for G is H_q-compatible, so H_q is not Springer-compatible.

We can find a similar example for G simple. Let G be a simple group of type G_2 and assume $p > 2$. Define H_q to be SL_2 diagonally embedded in the $A_1\widetilde{A_1}$ regular subgroup of G with a q-Frobenius twist in one of the factors, and let $1 \neq u \in H_q$ be unipotent. Then u is a distinguished unipotent element of G by [19, Table 10, §4.1], but $\mathrm{Lie}(H_q)$ contains no nilpotent elements that are distinguished in $\mathfrak{g}$, so H_q is not Springer-compatible. We are grateful to Adam Thomas for this example.

Lemma 4.14. *Let G_1, G_2 be connected reductive groups and let ϕ be a Springer map for $G_1 \times G_2$. Then ϕ is G_1-compatible and G_2-compatible. Moreover, $\phi = \phi_1 \times \phi_2$, where ϕ_i is the restriction of ϕ to G_i.*

Proof. By Remark 4.4, we can reduce to the case when $G_1 \times G_2$ is of separable type. The G_i-compatibility of ϕ follows easily from the $(G_1 \times G_2)$-equivariance. Now fix regular $u_1 \in \mathscr{U}_{G_1}$ and $u_2 \in \mathscr{U}_{G_2}$, and set $X = (X_1, X_2)$, where $X_i = \phi_i(u_i)$ for $1 \leq i \leq 2$. Then X_i is a regular element of $\mathrm{Lie}(G_i)$ for $1 \leq i \leq 2$, $u = (u_1, u_2)$ is a regular element of G and X is a regular element of $\mathrm{Lie}(G)$. Clearly $C_{G_i}(u_i) = C_{G_i}(X_i)$ for $1 \leq i \leq 2$.

Let ϕ_i' be the unique Springer isomorphism for G_i such that $\phi_i'(u_i) = X_i$. We have $(\phi_1' \times \phi_2')((u_1, u_2)) = (\phi_1'(u_1), \phi_2'(u_2)) = (X_1, X_2) = \phi((u_1, u_2))$, so $\phi = \phi_1' \times \phi_2'$. Moreover, $(\phi_1(u_1), 0) = \phi((u_1, 0)) = (\phi_1' \times \phi_2')((u_1, 0)) = (\phi_1'(u_1), 0)$, so $\phi_1(u_1) = \phi_1'(u_1)$, so $\phi_1 = \phi_1'$. Likewise $\phi_2 = \phi_2'$, and the result follows. ☐

4.2. **Cocharacters associated to nilpotent and unipotent elements.** The Jacobson-Morozov Theorem allows one to associate an $\mathfrak{sl}(2)$-triple to any given non-zero element of $\mathscr{N}_G$ in characteristic zero or large positive characteristic. This is an indispensable tool in the Dynkin-Kostant classification of the nilpotent orbits in characteristic zero as well as in the Bala-Carter classification of unipotent conjugacy classes of G in large prime characteristic, see [10, §5.9]. In good characteristic there is a replacement for $\mathfrak{sl}(2)$-triples, so-called *associated cocharacters*; see Definition 4.15 below. These cocharacters are important tools in the classification theory of unipotent classes and nilpotent orbits of reductive algebraic groups in good characteristic, see for instance [16, §5] and [30]. We recall the relevant concept of cocharacters associated to a nilpotent element following [16, §5.3].

Definition 4.15. Let $X \in \mathscr{N}_G$. A cocharacter $\lambda \in Y(G)$ of G is *associated* to X (in G) provided $X \in \mathfrak{g}(2, \lambda)$ and there exists a Levi subgroup L of G such that X is distinguished nilpotent in $\mathrm{Lie}(L)$ and $\lambda(\mathbb{G}_m) \leq \mathscr{D}L$. Following [12, Def. 2.13], we write

$$\Omega_G^a(X) := \{\lambda \in Y(G) \mid \lambda \text{ is associated to } X\}$$

for the set of cocharacters of G associated to X. Likewise, for M a connected reductive subgroup of G such that $X \in \mathrm{Lie}(M)$, we write $\Omega_M^a(X)$ for the set of cocharacters of M that are associated to X. This notation stems from the fact that associated cocharacters are destabilising cocharacters of G for X in the sense of Kempf-Rousseau theory, see [30] and [24].

Let $u \in \mathscr{U}_G$. A cocharacter $\lambda \in Y(G)$ of G is *associated* to u (in G) provided it is associated to $\phi(u)$, where $\phi : \mathscr{U}_G \to \mathscr{N}_G$ is a fixed Springer map as in §4.1; see [25, § 3]. We write

$$\Omega_{G,\phi}^a(u) := \{\lambda \in Y(G) \mid \lambda \text{ is associated to } u\}$$

for the set of cocharacters of G associated to u. Likewise, for M a connected reductive subgroup of G containing u and ϕ' a Springer map for M, we write $\Omega_{M,\phi'}^a(u)$ for the set of cocharacters of M that are associated to u. If ϕ is understood then we sometimes write $\Omega_G^a(u)$ instead of $\Omega_{G,\phi}^a(u)$.

Remark 4.16. Let $u \in \mathscr{U}_G$, $\lambda \in \Omega_G^a(u)$, and $g \in C_G(u)$. Then $g \cdot \lambda$ is also associated to u, see [16, §5.3]. Proposition 4.19(ii) gives a converse to this property.

Remark 4.17. Let $G_1, \dots, G_r$ be connected reductive groups and set $G = G_1 \times \cdots \times G_r$. Let $u_i \in \mathscr{U}_{G_i}$ for each $1 \le i \le r$ and let L be a Levi subgroup of G. Then $L = L_1 \times \cdots \times L_r$ for some Levi subgroups L_i of G_i. Set $u = (u_1, \dots, u_r) \in L$. It is clear that u is distinguished in L if and only if u_i is distinguished in L_i for each i. Likewise, if $X = (X_1, \dots, X_r) \in \mathscr{N}_L$ then X is distinguished in $\mathrm{Lie}(L)$ if and only if X_i is distinguished in $\mathrm{Lie}(L_i)$ for each i.

Fix a Springer map for G. Let $\lambda \in Y(G)$. We can write $\lambda = \lambda_1 \times \cdots \times \lambda_r$ for some $\lambda_i \in Y(G_i)$. It follows from the previous paragraph that λ is associated to X in $\mathrm{Lie}(G)$ if and only if λ_i is associated to X_i in $\mathrm{Lie}(G_i)$ for each i, [16, §5.6]. We deduce the analogous statement for u from Remark 4.4: if $\phi = \phi_1 \times \cdots \times \phi_r$ is a Springer map for G then λ is associated to u in G if and only if λ_i is associated to u_i in G_i for each i.

Let $\psi\colon \widetilde{G} \to G$ be an epimorphism of connected reductive groups such that $\ker(d\psi)$ is central in $\mathrm{Lie}(\widetilde{G})$. Let $\widetilde{u} \in \mathscr{U}_{\widetilde{G}}$, let $\widetilde{X} \in \mathscr{N}_{\widetilde{G}}$, let $\widetilde{L}$ be a Levi subgroup of $\widetilde{G}$ and let $\widetilde{\lambda} \in Y(\widetilde{G})$. Set $u = \psi(\widetilde{u})$, $X = d\psi(\widetilde{X})$, $L = \psi(\widetilde{L})$ and $\lambda = \psi \circ \widetilde{\lambda}$. Let $\widetilde{\phi}$ be a Springer map for $\widetilde{G}$ and let ϕ be the corresponding Springer map for G as described in Remark 4.3. Using [16, §4.3] and Remark 4.4 we get analogues of the above statements: u is distinguished in L if and only if $\widetilde{u}$ is distinguished in $\widetilde{L}$, X is distinguished in $\mathrm{Lie}(L)$ if and only if $\widetilde{X}$ is distinguished in $\mathrm{Lie}(\widetilde{L})$ and λ is associated to X (resp., to u) if and only if $\widetilde{\lambda}$ is associated to $\widetilde{X}$ (resp., to $\widetilde{u}$).

Remark 4.18. The notion of an associated cocharacter for an element $u \in \mathscr{U}_G$ depends on the choice of the Springer map for G: see [24, Rem. 23]. We do, however, have the following. Let ϕ_1 and ϕ_2 be Springer maps for G. Let $1 \ne u_1 \in \mathscr{U}_G$ and let $\lambda \in \Omega^a_{G,\phi_1}(u_1)$. Then $\lambda \in \Omega^a_{G,\phi_2}(u_2)$, where $u_2 = \phi_2^{-1}(\phi_1(u))$. Note that u_2 is conjugate to u_1 by Remark 4.5.

We require some basic facts about cocharacters associated to unipotent elements. The following results are [16, Lem. 5.3; Prop. 5.9] for nilpotent elements (see also [30, Thm. 2.3, Prop. 2.5]); the versions for unipotent elements follow immediately.

Proposition 4.19. *Suppose p is good for G. Let $1 \ne u \in \mathscr{U}_G$.*

(i) *$\Omega^a_G(u) \ne \varnothing$, i.e., cocharacters of G associated to u exist.*
(ii) *$C_G(u)^\circ$ acts transitively on $\Omega^a_G(u)$.*
(iii) *Let $\lambda \in \Omega^a_G(u)$ and let P_λ be the parabolic subgroup of G defined by λ as in §2.5. Then P_λ depends only on u and not on the choice of λ.*
(iv) *Let $\lambda \in \Omega^a_G(u)$ and let $P(u) := P_\lambda$ be as in (iii). Then $C_G(u) \subseteq P(u)$.*

If u is distinguished in G, then the parabolic subgroup $P(u)$ of G from Proposition 4.19(iii) is a distinguished parabolic subgroup of G and u belongs to the Richardson orbit of $P(u)$ on its unipotent radical, see Theorem 2.1(i); see also [24, Prop. 22].

Remark 4.20. Let $p > 0$ and suppose $1 \ne u \in \mathscr{U}_G^{(1)}$ is contained in a subgroup A of G of type A_1. Such a subgroup A always exists when p is good, and when p is bad there is essentially only one exception, due to Testerman [43] and Proud-Saxl-Testerman [31] — see Theorems 2.4 and 2.9. Then, since p is good for A, by Proposition 4.19(i) there exists a cocharacter $\lambda \in \Omega^a_A(u)$. Note that $\lambda(\mathbb{G}_m)$ is a maximal torus in A.

It follows from the work of Pommerening [28], [29] that the description of the unipotent classes in characteristic 0 is identical to the one for G when p is good for G. In both instances these are described by so-called *weighted Dynkin diagrams*. As a result, a cocharacter associated to a unipotent element in good characteristic acts with the same weights on the Lie algebra of G as its counterpart does in characteristic 0. This fact is used in the proof of the following result by Lawther [18, Thm. 1]; see also the proof of [33, Prop. 4.2] and [24, Rem. 31]. The result is stated in *loc. cit.* for G simple, but the extension to arbitrary connected reductive G is immediate, using arguments like those in Remark 4.17; note that if $\psi\colon \widetilde{G} \to G$ is an epimorphism of connected reductive groups such that $\ker(d\psi)$ is central in $\mathrm{Lie}(G)$ then $d\psi$ gives an isomorphism from $\mathrm{Lie}(\widetilde{U})$ onto $\mathrm{Lie}(\psi(\widetilde{U}))$, where $\widetilde{U}$ is any maximal unipotent subgroup of $\widetilde{G}$, so the weights of $\widetilde{\lambda} \in Y(\widetilde{G})$ on $\mathrm{Lie}(\widetilde{G})$ are the same as the weights of $\psi \circ \widetilde{\lambda}$ on $\mathrm{Lie}(G)$.

Lemma 4.21. *Let $u \in \mathscr{U}_G$. Suppose p is good for G. Let $\lambda \in \Omega^a_G(u)$. Denote by ω_G the highest weight of $\lambda(\mathbb{G}_m)$ on $\mathfrak{g}$. Then u has order p if and only if $\omega_G \leq 2p - 2$.*

The concept of associated cocharacters is not only a convenient replacement for $\mathfrak{sl}(2)$-triples from the Jacobson-Morozov theory, it is a very powerful tool in the classification theory of unipotent conjugacy classes and nilpotent orbits. Specifically, in [30] Premet showcases a conceptual and uniform proof of Pommerening's extension of the Bala-Carter Theorem 2.1 to good characteristic. His proof uses the fact that associated cocharacters are *optimal* in the geometric invariant theory sense of Kempf-Rousseau-Hesselink.

4.3. **Cocharacters associated to distinguished elements.** The linchpin of our proofs of Theorems 1.2 and 1.6 is the following collection of facts.

Lemma 4.22 ([12, Lem. 3.1])**.** *Suppose p is good for G. Let M be a connected reductive subgroup of G. Let $X \in \mathrm{Lie}(M)$ be a distinguished nilpotent element of $\mathfrak{g}$. Then $\Omega^a_M(X) = \Omega^a_G(X) \cap Y(M)$.*

The assertion of the lemma fails in general if X is not distinguished in $\mathfrak{g}$, even when p is good for both M and G: e.g., see [16, Rem. 5.12]. However, we do have the following result for all nilpotent elements in good characteristic.

Lemma 4.23 ([12, Cor. 3.22])**.** *Suppose p is good for G. Let $L \subset G$ be a Levi subgroup of G. Let $X \in \mathscr{N}_L$. Then $\Omega^a_L(X) = \Omega^a_G(X) \cap Y(L)$.*

We need group-theoretic analogues of Lemmas 4.22 and 4.23. For the former we need an extra Springer compatibility assumption, otherwise the result can fail (see Remark 6.1).

Lemma 4.24. *Suppose p is good for G. Let M be a connected reductive subgroup of G. Suppose M is Springer-compatible and let ϕ be an M-compatible Springer map. Let $u \in M$ be a distinguished unipotent element of G. Then $\Omega^a_{M,\phi_M}(u) = \Omega^a_{G,\phi}(u) \cap Y(M)$.*

Proof. Let $X = \phi(u) = \phi_M(u)$. Then
$$\Omega^a_{M,\phi_M}(u) = \Omega^a_M(X) = \Omega^a_G(X) \cap Y(M) = \Omega^a_{G,\phi}(u) \cap Y(M),$$
where the middle equality is from Lemma 4.22. ☐

Lemma 4.25. *Suppose p is good for G. Let $L \subset G$ be a Levi subgroup of G and let ϕ be a Springer map for G. Let $u \in \mathscr{U}_L$. Then $\Omega^a_{L,\phi_L}(u) = \Omega^a_{G,\phi}(u) \cap Y(L)$.*

Proof. Since $L = C_G(S)$ for some torus S, ϕ is L-compatible by Example 4.10. The result now follows by the same argument as in Lemma 4.24. ☐

5. Good A_1 subgroups

5.1. **Good A_1 overgroups.** In his seminal work [33], Seitz defines an important class of A_1 overgroups of an element $1 \neq u \in \mathscr{U}_G^{(1)}$ for G simple (see [33, Sec. 1]). He establishes the existence and fundamental properties of these overgroups provided p is good for G. We recall some of these results and generalise them to arbitrary connected reductive G.

Definition 5.1. Following [23, §1], we say that a homomorphism $\beta\colon \mathrm{SL}_2 \to G$ is *good* if each weight of the corresponding representation of SL_2 on $\mathfrak{g}$ is at most $2p-2$. We say that a subgroup A of G of type A_1 is a *good* A_1 subgroup of G, or is *good for* G, if it is the image of a good homomorphism. Else we call A a *bad* A_1 subgroup of G. This is of course independent of the choice of a maximal torus of A. For $1 \neq u \in \mathscr{U}_G^{(1)}$, we define

$$\mathscr{A}(u) := \mathscr{A}_G(u) := \{A \subseteq G \mid A \text{ is a good } A_1 \text{ subgroup of } G \text{ containing } u\}$$

and analogously, for a connected reductive subgroup M of G we write $\mathscr{A}_M(u)$ for the set of all good A_1 subgroups of M containing u.

Clearly any conjugate of a good A_1 homomorphism (resp., subgroup) is good. If $A \subseteq H \subseteq G$ are connected reductive groups such that A is a good A_1 subgroup of G, then A is obviously also a good A_1 subgroup of H. We see in Lemma 5.30 that the converse holds under some extra hypotheses. The converse is false in general, however: e.g., just take $A = H$ to be a bad A_1 subgroup of G.

Example 5.2. Let V be an SL_2-module such that weights of a maximal torus T of SL_2 on V are less than p. Then the weights of T in the induced action on $\mathrm{Lie}(\mathrm{GL}(V)) \cong V \otimes V^*$ are at most $2p-2$. Thus the induced subgroup A in $\mathrm{GL}(V)$ is a good A_1. In this situation the highest weights of T on each composition factor of V are restricted, so V is a semisimple SL_2-module; see [1, Cor. 3.9]. Hence A is $\mathrm{GL}(V)$-cr; this is a special case of Theorem 5.4(iii) below.

We record parts of the main theorems from [33] for our purposes, using the notation above. These were formulated and proved in *loc. cit.* for simple G, but we need extensions to arbitrary connected reductive G. To obtain this, we need the following lemma.

Lemma 5.3. *Let G be a connected reductive group. Let $\beta_1, \beta_2 : \mathrm{SL}_2 \to G$ be good homomorphisms with the same image A. Then β_1 and β_2 are conjugate by an element of A.*

Proof. Assume first that $A \cong \mathrm{SL}_2$. Let $1 \leq i \leq 2$. Then we can regard β_i as an element of $\mathrm{End}(\mathrm{SL}_2)$, so it is an inner endomorphism followed by a Frobenius qth power map for $q = p^r$ for some $r \geq 0$. Let T be a maximal torus of SL_2. If $r \geq 1$ then the highest weight of T is at least $2q$, since SL_2 acts on $\mathrm{Lie}(A)$ with highest weight 2, which contradicts the goodness assumption. Therefore, $\beta_i \in \mathrm{Aut}(\mathrm{SL}_2)$. But all automorphisms of SL_2 are inner. The result follows.

For $A \cong \mathrm{PGL}_2$, we can factor β_i as $\mathrm{SL}_2 \to \mathrm{PGL}_2 \stackrel{\beta_i'}{\to} \mathrm{PGL}_2$, where the first map is the canonical projection. One can now apply an argument like the above one to the maps $\beta_i'\colon \mathrm{PGL}_2 \to A$. ∎

Theorem 5.4. *Let G be connected reductive. Suppose p is good for G and let $1 \neq u \in \mathscr{U}_G^{(1)}$. Then the following hold:*

(i) $\mathscr{A}(u) \neq \varnothing$.

(ii) *$R_u(C_G(u))$ acts transitively on $\mathscr{A}(u)$.*
(iii) *Let $A \in \mathscr{A}(u)$. Then A is G-completely reducible.*
(iv) *There is a unique 1-dimensional unipotent subgroup U of G such that $u \in U$ and U is contained in a good A_1 subgroup of G.*

Notation 5.5. We denote the subgroup U from Theorem 5.4(iv) by $\mathcal{U}(u)$.

Proof of Theorem 5.4. For G simple see [33, Thms. 1.1–1.3]. Now let G be connected reductive. Since SL_2 and PGL_2 are perfect, any A_1 subgroup of G is contained in $\mathscr{D}G$. Hence without loss we can assume that G is semisimple; note for (iii) that a subgroup of $\mathscr{D}G$ is $\mathscr{D}G$-cr if and only if it is G-cr [6, Prop. 2.8]. Moreover, let $\psi\colon \widetilde{G} \to G$ be a central isogeny of connected reductive groups. If $\widetilde{A}$ is an A_1 subgroup of $\widetilde{G}$ then $\widetilde{A}$ is good for $\widetilde{G}$ if and only if $\psi(\widetilde{A})$ is good for G: cf. the argument of the paragraph preceding Lemma 4.21. Note also for (iii) that if $\widetilde{H} \subseteq \widetilde{G}$ then $\widetilde{H}$ is $\widetilde{G}$-cr if and only if $\psi(\widetilde{H})$ is G-cr [5, Lem. 2.12]. Hence we can assume without loss that $G = G_1 \times \cdots \times G_r$, where each G_i is simple.

We need a description of good A_1 subgroups of G in terms of good A_1 subgroups of the G_i. Let T be a maximal torus of SL_2. Denote by π_i the projection from G to G_i. Let $\beta : \mathrm{SL}_2 \to G$ be a homomorphism and define $\beta_i := \pi_i \circ \beta$. For notational convenience, we assume that each β_i is non-trivial. The weights of T on $\mathrm{Lie}(G_i)$ form a subset of the set of weights of T on $\mathrm{Lie}(G)$, since $\mathrm{Lie}(G) = \bigoplus \mathrm{Lie}(G_i)$. Therefore, if β is a good homomorphism for G, then β_i is a good homomorphism for G_i or trivial. Conversely, if $\beta_i : \mathrm{SL}_2 \to G_i$ is a non-trivial homomorphism for each i, define $\beta := \beta_1 \times \cdots \times \beta_r$ to be the diagonal embedding into G. Then the maximal weight ω_G of T on $\mathrm{Lie}(G)$ is given by $\max\{\omega_{G_i}\}$, where ω_{G_i} is the maximal weight of T on $\mathrm{Lie}(G_i)$. Thus, β is good if and only if the β_i are good. Now (i) and (iii) are immediate from the above observations, [5, Lem. 2.12] and the results for G simple.

For (ii), let A^1 and A^2 be good A_1 subgroups of $G = G_1 \times \cdots \times G_r$ containing $u = (u_1, \ldots, u_r)$ with $u_i \neq 1$ for each i. Choose two good homomorphisms $\beta^1, \beta^2 : \mathrm{SL}_2 \to G$ such that $\mathrm{Im}(\beta^i) = A^i$. By the observations above, there are good homomorphisms $\beta_i^1, \beta_i^2 : \mathrm{SL}_2 \to G_i$ with images A_i^1, A_i^2 containing u_i. Now [33, Thm. 1.1(ii)] implies that $A_i^2 = g_i A_i^1 g_i^{-1}$ for some $g_i \in R_u(C_{G_i}(u_i))$. Lemma 5.3 (applied to G_i) implies that $h_i g_i \cdot \beta_i^1 = \beta_i^2$ for some $h_i \in A_i^2$. Hence $\beta^2 = hg \cdot \beta^1$, where $g = (g_1, \ldots, g_r) \in R_u(C_G(u))$ and $h = (h_1, \ldots, h_r) \in A^2$. It follows that $A^2 = g \cdot A^1$.

For (iv), let $G = G_1 \times \cdots \times G_r$ and let $u = (u_1, \ldots, u_r) \in \mathscr{U}_G^{(1)}$ with $u_i \neq 1$ for each i. Choose an $A \in \mathscr{A}_G(u)$ which is the image of the good homomorphism β. As before, we get good homomorphisms β_i with images $A_i \in \mathscr{A}_{G_i}(u_i)$, and $\beta = \beta_1 \times \cdots \times \beta_r$. Without loss we can assume the β_i are non-trivial. Fix a 1-dimensional unipotent subgroup V of SL_2. After conjugating β by an element of A, we can assume that $\mathcal{U}(u_i) = \beta_i(V)$ for each i. Define $\mathcal{U}(u) = (\beta_1 \times \cdots \times \beta_r)(V)$. This is a 1-dimensional unipotent subgroup of G containing u and is contained in the good A_1 subgroup A. This proves the existence. For the uniqueness, let U' be another 1-dimensional unipotent subgroup of G such that $u \in U' \subseteq A'$ for some $A' \in \mathscr{A}_G(u)$. By (ii), $A = gA'g^{-1}$ for some $g \in C_G(u)$, and so $gU'g^{-1} = \mathcal{U}(u)$. Write $g = (g_1, \ldots, g_r)$ with $g_i \in C_{G_i}(u_i)$. By [33, Thm. 1.2(i)] g_i centralises $\mathcal{U}(u_i)$, hence g centralises $\mathcal{U}(u)$. Thus, $U' = \mathcal{U}(u)$. □

Example 5.6. Let H_q be the bad A_1 subgroup of $G = \mathrm{SL}_2 \times \mathrm{SL}_2$ from Example 4.13. Here $\beta_1 = \mathrm{id}_{\mathrm{SL}_2}$, while $\beta_2 : \mathrm{SL}_2 \to \mathrm{SL}_2$ is the qth power map, which is not a good homomorphism. On the other hand, the projection of H_q onto each factor is just SL_2, which is a good A_1

subgroup of SL_2, so we cannot detect the badness of H_q just by looking at its images in the simple factors of G.

Remark 5.7. Let $1 \neq u \in \mathscr{U}_G^{(1)}$. We claim that

$$C_G(\mathcal{U}(u)) = C_G(u) = C_G(\mathrm{Lie}(\mathcal{U}(u))). \tag{5.8}$$

To see this, suppose first that $\widetilde{G}$ is of the form $G_1 \times \cdots \times G_r$, where each G_i is simple, and let $\pi_i\colon \widetilde{G} \to G_i$ be the canonical projection. Let $\widetilde{u} = (u_1, \ldots, u_r) \in \mathscr{U}_{\widetilde{G}}^{(1)}$ with $u_i \neq 1$ for each i. Choose a good homomorphism $\widetilde{\beta}\colon \mathrm{SL}_2 \to \widetilde{G}$ such that $\mathcal{U}(\widetilde{u}) \subseteq \widetilde{A} := \mathrm{Im}(\mathrm{SL}_2)$, and set $\beta_i = \pi_i \circ \widetilde{\beta}$ and $A_i = \beta_i(\mathrm{SL}_2)$. It follows from [33, Thm. 1.2(i)] that $C_{G_i}(\mathcal{U}(u_i)) = C_{G_i}(u_i) = C_{G_i}(\mathrm{Lie}(\mathcal{U}(u_i)))$ for each i. We deduce from the arguments in the proof of Theorem 5.4 that $C_{\widetilde{G}}(\mathcal{U}(\widetilde{u})) = C_{\widetilde{G}}(\widetilde{u}) = C_{\widetilde{G}}(\mathrm{Lie}(\mathcal{U}(\widetilde{u})))$; note that $d\beta_i\colon \mathrm{Lie}(\mathrm{SL}_2) \to \mathrm{Lie}(A_i)$ is surjective for each i because β_i does not involve a Frobenius twist.

If $\psi\colon \widetilde{G} \to G$ is a central isogeny and $1 \neq \widetilde{u} \in \mathscr{U}_{\widetilde{G}}^{(1)}$, then it is clear that $\mathcal{U}(u) = \psi(\mathcal{U}(\widetilde{u}))$, where $u = \psi(\widetilde{u})$, and we deduce that

$$C_G(\mathcal{U}(u)) = C_G(u) = C_G(\mathrm{Lie}(\mathcal{U}(u))). \tag{5.9}$$

Now let G be an arbitrary connected reductive group and let $1 \neq u \in \mathscr{U}_G^{(1)}$. Then $\mathcal{U}(u) \subseteq \mathscr{D}G$. Now (5.8) follows easily from (5.9) applied to the semisimple group $\mathscr{D}G$.

We deduce from (5.8) and Theorem 5.4(ii) that $\mathcal{U}(u)$ is contained in **every** good A_1 overgroup of u.

Lemma 5.10. *Suppose p is good for G. Let $1 \neq u \in \mathscr{U}_G^{(1)}$ and let A be an A_1 subgroup of G containing $\mathcal{U}(u)$. Then A is good in G.*

Proof. Let A' be a good A_1 subgroup containing $\mathcal{U}(u)$. Then A and A' have a common maximal unipotent subgroup $\mathcal{U}(u)$. By [21, Thm. 1.1], A and A' are G-conjugate. Hence A is good, because A' is. ☐

Lemma 5.11. *Suppose p is good for G. Let A be an A_1 subgroup of G and let $\lambda \in Y(A)$. Suppose that*

(i) *$\lambda \in \Omega^a_G(X)$ for some $0 \neq X \in \mathscr{N}_G^{(1)}$, or*
(ii) *$\lambda \in \Omega^a_{G,\phi}(u)$ for some $1 \neq u \in \mathscr{U}_G^{(1)}$ and some Springer map ϕ for G.*

Then A is a good A_1 subgroup of G.

Proof. Let ϕ be a Springer map for G, and suppose $\lambda \in \Omega^a_{G,\phi}(u)$ for some $1 \neq u \in \mathscr{U}_G^{(1)}$. It follows from Lemma 4.21 that the weights of λ on $\mathfrak{g}$ are at most $2p-2$. Define $\beta\colon \mathrm{SL}_2 \to A$ to be an isomorphism if $A \cong \mathrm{SL}_2$, and the usual central isogeny $\mathrm{SL}_2 \to \mathrm{PGL}_2$ followed by an isomorphism from PGL_2 onto A if $A \cong \mathrm{PGL}_2$. Then there exists $\mu\colon \mathbb{G}_m \to \mathrm{SL}_2$ such that μ is an isomorphism onto a maximal torus of SL_2 and $\lambda = \beta \circ \mu$. The weights of μ on $\mathfrak{g}$ are at most $2p-2$ by construction, so A is good. Hence A is good if (ii) holds.

If (i) holds then $\lambda \in \Omega^a_{G,\phi}(u)$, where $u := \phi^{-1}(X)$. But $u \in \mathscr{U}_G^{(1)}$, by Lemma 4.7, so (ii) holds, so A is good by the argument above. ☐

In the next theorem we recall parts of the analogue of Theorem 5.4 for finite overgroups of type A_1.

Theorem 5.12. *Let G be connected reductive. Suppose p is good for G. Let $\sigma : G \to G$ be a Steinberg endomorphism of G. Suppose $u \in G_\sigma$ is unipotent of order p.*

(i) $\mathscr{A}(u)_\sigma \neq \varnothing$.
(ii) $R_u(C_G(u))_\sigma$ *acts transitively on* $\mathscr{A}(u)_\sigma$.
(iii) *Let* $A \in \mathscr{A}(u)_\sigma$. *Suppose that* $q > 7$ *if* G *is of exceptional type. Then* A_σ *is* σ*-completely reducible.*
(iv) *There is a unique* σ*-stable 1-dimensional unipotent subgroup* U *of* G *such that* $u \in U$ *and* U *is contained in a good* A_1 *subgroup of* G*.*

Proof. (i)–(iii): The simple case is proved by Seitz in [33, Thm. 1.4]. For connected reductive groups we use an argument similar to the one in the proof of Theorem 5.4.

(iv) By (i) we can choose some $A \in \mathscr{A}(u)_\sigma$. Now $\mathcal{U}(u) \subseteq A$ by Remark 5.7. Clearly $\mathcal{U}(u)$ is the unique 1-dimensional unipotent subgroup of A that contains u, so $\mathcal{U}(u)$ must be σ-stable. Hence $U := \mathcal{U}(u)$ has the desired properties. □

Remark 5.13. Parts (i) and (ii) of Theorem 5.12 follow from parts (i) and (ii) of Theorem 5.4 and the Lang-Steinberg Theorem, see [33, Prop. 9.1].

Remark 5.14. (i). Concerning the terminology in Theorem 5.12(iii), following [14], a subgroup H of G is said to be σ*-completely reducible*, provided that whenever H lies in a σ-stable parabolic subgroup P of G, it lies in a σ-stable Levi subgroup of P. This notion is motivated by certain rationality questions concerning G-complete reducibility; see [14] for details. For a σ-stable subgroup H of G, this property is equivalent to H being G-cr, thanks to [14, Thm. 1.4].

(ii). Apart from the special conjugacy class of good A_1 subgroups in G asserted in Theorem 5.4, there might be a plethora of conjugacy classes of bad A_1 subgroups in G even when p is good for G. Just take a non-semisimple representation $\beta : \mathrm{SL}_2 \to \mathrm{SL}(V) = G$ in characteristic $p > 0$. Then the A_1 subgroup $\beta(\mathrm{SL}_2)$ is bad in G, while p is good for G. For a concrete example, see [16, Rem. 5.12]. This can only happen if p is sufficiently small compared to the rank of G, thanks to Theorem 3.2.

The subgroups H_q of $\mathrm{SL}_2 \times \mathrm{SL}_2$ in Example 4.13 are also bad A_1 subgroups: see Remark 6.1.

(iii). The proofs of Theorems 5.4 and 5.12 for G simple by Seitz in [33] depend on separate considerations for each Dynkin type and involve in part intricate arguments for the component groups of centralizers of unipotent elements. In [24], McNinch presents uniform proofs of Seitz's theorems for G strongly standard reductive, which are almost entirely free of any case-by-case checks, utilizing methods from geometric invariant theory. However, McNinch's argument (see [24, Thm. 44]) of the conjugacy result in Theorem 5.4(ii) depends on the fact that for a good A_1 subgroup A of G, the A-module $\mathfrak{g}$ is a tilting module. The latter is established by Seitz in [33, Thm. 1.1].

In [33, §9], Seitz exhibits instances when there is no good A_1 overgroup of an element of order p when p is bad for G. As we explain next, Example 2.6 gives a counterexample to Theorem 5.4(iii) in case p is bad for G: that is, it gives a good A_1 subgroup A such that A is not G-cr. Specifically, we show that some of the A_1 subgroups in that example are good A_1 subgroups of G, but thanks to Example 2.6, they are not G-cr.

Example 5.15 (Example 2.6 continued)**.** Let G be simple of type C_2 and $p = 2$. Let σ be id_G or a q-Frobenius endomorphism of G. Let $\mathscr{C}$ denote the subregular unipotent class of G. Suppose $u \in \mathscr{C} \cap G_\sigma$. Then by Example 2.6 there are σ-stable subgroups A of G of type A_1 containing u that are not G-cr. Specifically, let E be the natural module for SL_2. Consider the two conjugacy classes of embeddings of SL_2 into $G = \mathrm{Sp}(V)$, where we take either $V \cong E \perp E$ or $V \cong E \otimes E$, as an SL_2-module. The images of both embeddings meet the class $\mathscr{C}$ non-trivially. One checks that the highest weight of a maximal torus of SL_2 on $\mathfrak{g}$ is 4 in the second instance. So in this case the image of SL_2 in G is not a good A_1. In contrast, in the first instance the highest weight of a maximal torus of SL_2 on $\mathfrak{g}$ is $2 = 2p-2$, by Example 5.2. So the image of SL_2 in G is a good A_1 in $\mathrm{SL}(V)$, and so it is a good A_1 in G as well.

5.2. **Characterisations of good A_1 subgroups.** In this section we investigate some other types of A_1 subgroup which were introduced by McNinch. We prove that these other notions are all equivalent to goodness (Theorem 5.24). The key ingredient we need is work of Sobaje, who proved the existence of a Springer map for G with especially nice properties. We assume throughout the section that p is good for G.

We recall a construction from [33, Prop. 5.2] (see also [36]). Let P be a parabolic subgroup of G, and set $U = R_u(P)$. It can be shown that any Springer map for G maps U to $\mathrm{Lie}(U)$. Suppose U has nilpotency class less than p; in this case we say that P is *restricted*. In particular, any distinguished parabolic subgroup of G corresponding to a distinguished unipotent element of order p is restricted [24, Prop. 24]. We endow $\mathrm{Lie}(U)$ with the structure of an algebraic group using the Baker-Campbell-Hausdorff formula. There is a unique P-equivariant isomorphism of algebraic groups $\exp_P\colon \mathrm{Lie}(U) \to U$ such that the derivative of $\exp_P$ is the identity on $\mathrm{Lie}(U)$ (this is established in [33, Prop. 5.2] for semisimple G, but the extension to connected reductive G is immediate). We denote the inverse of $\exp_P$ by $\log_P\colon \mathrm{Lie}(U) \to U$.

Definition 5.16. We say that a Springer map ϕ for G is *logarithmic* if the following holds: for any $1 \neq u \in \mathscr{U}_G^{(1)}$, the restriction of ϕ gives an isomorphism ϕ_u of algebraic groups from $\mathcal{U}(u)$ to $\mathrm{Lie}(\mathcal{U}(u))$, and $d\phi_u$ is the identity on $\mathrm{Lie}(\mathcal{U}(u))$.

Proposition 5.17. (i) *There exists a logarithmic Springer map for G.*
(ii) *Let ϕ be a logarithmic Springer map for G. Then for every restricted parabolic subgroup P, the restriction of ϕ to $R_u(P)$ is $\log_P$.*
(iii) *Any two logarithmic Springer maps induce the same map from $\mathscr{U}_G^{(1)}$ to $\mathscr{N}_G^{(1)}$.*

Proof. First assume that G is simple and p is separably good for G. Part (ii) follows from [36, Prop. 2.1]. For part (i), let $\varphi\colon \mathscr{N}_G \to \mathscr{U}_G$ be a G-equivariant isomorphism of varieties as in [36, Thm. 4.1]. Fix a maximal unipotent subgroup U of G. By [36, Thm. 1.1], $d\varphi\colon \mathrm{Lie}(U) \to \mathrm{Lie}(U)$ is a scalar multiple of the identity. Condition (1) of [36, Thm. 4.1] implies that this scalar is 1, so $d\varphi$ is the identity map. Let $1 \neq u \in \mathscr{U}_G^{(1)}$ and set $X = \varphi^{-1}(u)$. Then $X \in \mathscr{N}_G^{(1)}$ by Remark 4.8, so by [36, Cor. 4.3(1)], φ gives an isomorphism from kX onto a 1-dimensional unipotent subgroup U' of G which is contained in a good A_1 subgroup of G. By construction, $U' = \mathcal{U}(u)$. Since $d\varphi$ is the identity map, X belongs to $\mathrm{Lie}(\mathcal{U}(u))$, so φ gives an isomorphism of algebraic groups from $\mathrm{Lie}(\mathcal{U}(u))$ to $\mathcal{U}(u)$. It follows that φ^{-1} is a logarithmic Springer map for G, so (i) is proved.

Now let $1 \neq u \in \mathscr{U}_G^{(1)}$. Choose a good A_1 overgroup A of u in G. Choose a maximal torus T of A such that T normalises $\mathcal{U}(u)$. Definition 5.16 and the T-equivariance of φ^{-1} imply that the map from $\mathcal{U}(u)$ to $\mathrm{Lie}(\mathcal{U}(u))$ induced by φ^{-1} does not depend on the choice of φ^{-1}. This proves part (iii).

The result now follows for arbitrary connected reductive G using Remark 4.3. □

Remark 5.18. If $p > h(G)$ then the map log from Example 4.12 is a logarithmic Springer map (see [34, Thm. 3] and [24, Rem. 27]). In this case any Borel subgroup of G is a restricted parabolic, so the restriction of any logarithmic Springer map for G to $R_u(B)$ is $\log_B$ by Proposition 5.17(ii). Hence log is the unique logarithmic Springer map for G.

Remark 5.19. We saw above that the condition on ϕ in Definition 5.16 implies part (ii) of Proposition 5.17. Sobaje observes at the beginning of [36, §2] that the converse also holds. The reason is that every $1 \neq u \in \mathscr{U}_G^{(1)}$ belongs to $R_u(P)$ for some restricted parabolic subgroup P of G: this follows from [9, Thm. 2.4]. We also deduce that the restriction of $\log_P$ to $\mathcal{U}(u)$ is ϕ_u for every restricted parabolic subgroup P of G and every $u \in R_u(P)$ such that u has order p.

Corollary 5.20. *Let ϕ be a logarithmic Springer map for G. Then for any A_1 subgroup A of G, A is good for G if and only if ϕ is A-compatible.*

Proof. Suppose A is good. Let $1 \neq u \in \mathscr{U}_A$. Then $\mathcal{U}(u) \subseteq A$ and $\phi(\mathcal{U}(u)) = \mathrm{Lie}(\mathcal{U}(u)) \subseteq \mathrm{Lie}(A)$, so ϕ is A-compatible. Conversely, suppose ϕ is A-compatible. Let $1 \neq u \in \mathscr{U}_A$ and set $X = \phi(u) \in \mathrm{Lie}(\mathcal{U}(u))$. Now $X \in \mathrm{Lie}(A)$ by the A-compatibility, so $kX \subseteq \mathrm{Lie}(A)$. Hence $\mathcal{U}(u) = \phi^{-1}(kX) \subseteq A$ by the A-compatibility. We deduce from Lemma 5.10 that A is good for G. □

We now recall the other types of A_1 subgroup that we need, namely optimal and sub-principal A_1 subgroups. These were introduced by McNinch in [24] and [23].

Definition 5.21. We call a homomorphism $\beta\colon \mathrm{SL}_2 \to G$ *optimal* if there is a maximal torus T of SL_2 such that the restriction λ of β to $T \cong \mathbb{G}_m$ is a cocharacter associated in G to some nilpotent $0 \neq X \in \mathrm{Im}(d\beta)$. We call an A_1 subgroup of G *optimal* if it is the image of an optimal homomorphism.

Remark 5.22. This is equivalent to the definition in [24, §1]: for it is clear that if T is the standard maximal torus of SL_2 and λ is associated to some nilpotent $0 \neq X \in \mathrm{Lie}(\mathrm{SL}_2)$ then X is a scalar multiple of $d\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$.

Definition 5.23. Fix a Springer map ϕ for G. We call a homomorphism $\beta\colon \mathrm{SL}_2 \to G$ *sub-principal* if there is a maximal torus T of SL_2 such that the restriction λ of β to $T \cong \mathbb{G}_m$ is a cocharacter associated in G to some nilpotent $0 \neq X \in \mathrm{Im}(d\beta)$ and $\phi^{-1}(X)$ is G-conjugate to an element of $\mathrm{Im}(\beta)$. Note that the latter condition does not depend on the choice of ϕ, by Lemma 4.5. We call an A_1 subgroup of G *sub-principal* if it is the image of a sub-principal homomorphism.

The next result implies Theorem 1.4.

Theorem 5.24. *Let A be an A_1 subgroup of G. Let ϕ be a logarithmic Springer map for G. The following conditions are equivalent.*

(i) *A is sub-principal.*
(ii) *A is optimal.*

(iii) *There exist* $u \in \mathscr{U}_G^{(1)}$ *and* $\lambda \in Y(A)$ *such that* $\lambda \in \Omega^a_{G,\phi}(u)$.
(iv) A *is good.*

Proof. The implication (i) $\implies$ (ii) is immediate from the definitions, and (iii) $\implies$ (iv) follows from Lemma 5.11. If A is optimal then there exist $0 \neq X \in \mathscr{N}_A$ and $\lambda \in Y(A)$ such that $\lambda \in \Omega^a_G(X)$. Then $\lambda \in \Omega^a_{G,\phi}(u')$, where $u' = \phi^{-1}(X)$, and u' has order p by Lemma 4.7. Hence (ii) $\implies$ (iii).

By [23, Rem. 21], there exists at least one sub-principal A_1 subgroup A of G such that $u \in A$, and A is good by the arguments above. It is clear from the definition that any $C_G(u)$-conjugate of A is sub-principal. Since $C_G(u)$ acts transitively on $\mathscr{A}(u)$ (Theorem 5.4(ii)), it follows that any good A_1 subgroup of G that contains u is sub-principal. This shows that (iv) $\implies$ (i). Hence (i)–(iv) are all equivalent. □

Remark 5.25. If the equivalent conditions from Theorem 5.24 hold then there exist $\lambda \in Y(A)$ and $0 \neq X \in \mathscr{N}_A$ such that $\lambda \in \Omega^a_G(X)$. Then $\lambda \in \Omega^a_G(u)$, where $u = \phi^{-1}(X)$, which belongs to A by Corollary 5.20. Hence we can take the element u from Theorem 5.24(iii) to belong to A if we wish.

Remark 5.26. It is implicit in the discussion in [23, §1] that a sub-principal A_1 subgroup of G is good. McNinch also proved that goodness and optimality are equivalent for A_1 subgroups under the extra assumption that G is strongly standard (see [24, Prop. 53]).

Proposition 5.27. *Let L be a Levi subgroup of G and let A be a good A_1 subgroup of L. Then A is a good A_1 subgroup of G.*

Proof. Since p is good for G, p is good for L. By Theorem 5.24, A is optimal in L, so there exist $0 \neq X \in \mathscr{N}_A$ and $\lambda \in Y(A)$ such that $\lambda \in \Omega^a_L(X)$. Lemma 4.23 implies that $\lambda \in \Omega^a_G(X)$, so A is optimal in G. Hence A is a good A_1 subgroup of G by Theorem 5.24. □

Corollary 5.28. *Let A be a good A_1 subgroup of G and let $1 \neq u \in \mathscr{U}_A$. Then there is a Levi subgroup L of G such that $A \subseteq L$ and u is a distinguished unipotent element of L.*

Proof. Pick a Levi subgroup L' of G such that u is a distinguished unipotent element of L'. By Theorem 5.4(i) we can choose a good A_1 subgroup A' of L' such that $u \in A'$. Now A' is a good A_1 subgroup of G by Proposition 5.27, so there exists $g \in C_G(u)$ such that $gA'g^{-1} = A$ (Theorem 5.4(ii)). Then $A \subseteq L$, where $L := gL'g^{-1}$. Clearly, L is a Levi subgroup of G and u is a distinguished unipotent element of L. □

Corollary 5.29. *Let ϕ be a logarithmic Springer map for G and let L be a Levi subgroup of G. Then ϕ_L is a logarithmic Springer map for L.*

Proof. Let $1 \neq u \in \mathscr{U}_L^{(1)}$. Choose a good A_1 overgroup A of u in L. Then A is good for G by Proposition 5.27, so $\mathcal{U}(u) \subseteq A$. We see that $\mathcal{U}(u)$ is both the unique 1-dimensional overgroup of u that is contained in a good A_1 subgroup of L, and the unique 1-dimensional overgroup of u that is contained in a good A_1 subgroup of G. The result now follows from the definition of a logarithmic Springer map. □

Lemma 5.30. *Let H be a connected reductive subgroup of G, and assume p is good for H. Let $u \in \mathscr{U}_H^{(1)}$ such that u is distinguished in G. Let $A \in \mathscr{A}_H(u)$. Suppose there is a Springer map ϕ for H such that $\phi(u)$ is a distinguished element of $\mathfrak{g}$. Then A is good for G.*

Proof. By Theorem 5.24, A is a sub-principal A_1 subgroup of H, so there exist $\lambda \in Y(A)$ and $0 \neq X \in \mathscr{N}_A$ such that λ is associated to X in H and $\phi(u)$ is H-conjugate to X. Since by hypothesis $\phi(u)$ is a distinguished element of $\mathfrak{g}$, X is also a distinguished element of $\mathfrak{g}$. Lemma 4.22 implies that λ is associated to X in G. It follows that A is an optimal A_1 subgroup of G, so A is a good A_1 subgroup of G by Theorem 5.24. □

Remark 5.31. Let H be a connected reductive subgroup of G and assume p is good for H. Suppose H is Springer-compatible. Let A be an A_1 subgroup of H containing a distinguished unipotent element u of G. Let ϕ be the restriction to H of any H-compatible Springer map for G. Then $\phi(u)$ is a distinguished element of $\mathfrak{g}$ by Remark 4.2, so the hypotheses of Lemma 5.30 hold. Hence if A is good for H then A is good for G.

The following relates the set of cocharacters of G that are associated to some $1 \neq u \in \mathscr{U}_G^{(1)}$ to those stemming from good A_1 overgroups of u in G.

Corollary 5.32. *Let $1 \neq u \in \mathscr{U}_G^{(1)}$. Let ϕ be a logarithmic Springer map for G. We have a disjoint union*

$$\Omega^a_{G,\phi}(u) = \dot{\bigcup_{A \in \mathscr{A}(u)}} \Omega^a_{A,\phi_A}(u),$$

where ϕ_A denotes the restriction of ϕ to A.

Proof. Note that it makes sense to speak of the restriction of ϕ to a good A_1 subgroup A of G, by Corollary 5.20. We first prove that the union above is disjoint. Let $A, \tilde{A} \in \mathscr{A}(u)$ and suppose there exists some $\lambda \in \Omega^a_{A,\phi_A}(u) \cap \Omega^a_{\tilde{A},\phi_{\tilde{A}}}(u)$. Then A and $\tilde{A}$ share the common Borel subgroup $\lambda(\mathbb{G}_m)\mathcal{U}(u)$. It follows from [21, Lem. 2.4] that $A = \tilde{A}$.

Let $A \in \mathscr{A}(u)$ and let $\lambda \in \Omega^a_{A,\phi_A}(u)$. By Corollary 5.28 there is a Levi subgroup L of G such that $A \subseteq L$ and u is a distinguished unipotent element of L. It follows from Lemma 4.24 (applied to the inclusion $A \subseteq L$) and Lemma 4.25 that $\lambda \in \Omega^a_{G,\phi}(u)$. Hence $\Omega^a_{G,\phi}(u) \supseteq \dot{\bigcup}_{A \in \mathscr{A}(u)} \Omega^a_{A,\phi_A}(u)$. Since $\mathscr{A}(u) \neq \varnothing$ (Theorem 5.4(i)), $C_G(u)$ acts transitively on both $\mathscr{A}(u)$ (Theorem 5.4(ii)) and $\Omega^a_{G,\phi}(u)$ (Proposition 4.19(ii)), we see that the reverse inclusion follows. □

6. Proofs of Theorems 1.2 and 1.5–1.7

Armed with the results from above, we prove Theorems 1.2 and 1.6 simultaneously.

Proof of Theorems 1.2 and 1.6. We may assume that G is semisimple, since any unipotent element of G is contained in the derived subgroup $\mathscr{D}G^\circ$. Likewise, we may also assume that H is connected and semisimple, as any unipotent element of H° is contained in the derived subgroup $\mathscr{D}H^\circ$, and H is G-ir if $\mathscr{D}H^\circ$ is. Let $u \in \mathscr{U}_H^{(1)}$ be distinguished in G.

First suppose p is bad for H. If $p > 2$ then H admits a simple component H' of exceptional type. If $u \in H$ is a distinguished unipotent element of G then the projection u' of u onto H' is a distinguished unipotent element of H', so $p = 3$ and H' is of type G_2, by Lemma 2.7. But this is impossible by Lemma 2.11 since p is good for G. Hence $p = 2$. It follows that each simple component of G is of type A. Now distinguished unipotent elements are regular in type A, so u is a regular element of G. It follows from [7, Thm. 1.1] (resp., [7, Thm. 1.3]) that H (resp., H_σ) is G-ir.

Hence we can assume that p is good for H. By Theorem 5.4(i) (resp., Theorem 5.12(i)) there is a good A_1 subgroup (resp., good σ-stable A_1 subgroup) A of H such that $u \in A$. By Lemma 5.30 and hypothesis (†), A is a good A_1 subgroup of G. Hence A (resp., A_σ) is G-cr by Theorem 5.4(iii)) (resp., Theorem 5.12)(iii)), so A (resp., A_σ) is G-ir by Lemma 3.1. We conclude that H (resp., H_σ) is G-ir. □

Remark 6.1. If we remove hypothesis (†) from Lemma 5.30, Theorem 1.2, etc., then our arguments break down. For instance, let $G = \mathrm{SL}_2 \times \mathrm{SL}_2$, q and H_q be as in Example 4.13. Let u be any unipotent element of H_q such that the projection of u onto each SL_2-factor of H_q is non-trivial; then u is distinguished (in fact, regular) in G. It is easy to see that H_q is not a good A_1 subgroup of G and there does not exist $\lambda \in Y(H_q)$ such that λ is associated to u in G; in particular, the conclusion of Lemma 5.30 does not hold for H_q. Of course Theorem 1.1 still applies, alternately so does Theorem 1.8, so H_q is G-ir.

As a consequence of Theorems 1.2 and 1.6 we obtain the following.

Corollary 6.2. *Let G be a connected reductive group. Suppose p is good for G. Let σ be id_G or a q-Frobenius endomorphism of G. Let $u \in G_\sigma$ be unipotent of order p. Suppose u is distinguished in the σ-stable Levi subgroup L of G (see Remark 2.2(ii)). Let H be a σ-stable connected reductive subgroup of L containing u, and suppose there is a Springer map ϕ for L such that $\phi(u)$ is a distinguished element of $\mathrm{Lie}(L)$. Then H_σ is G-completely reducible.*

Proof. As p is also good for L (see §2.2), it follows from Theorem 1.2 (resp. 1.6) applied to L that H_σ is L-ir and so is L-cr. Thus, H_σ is G-cr, by [35, Prop. 3.2]. □

Remark 6.3. In the setting of Theorem 1.1 the following argument allows to reduce the case when G is connected reductive to the simple case. As in the proof of Theorem 1.2 above, we can assume that G is semisimple. Let $G_1, \dots, G_r$ be the simple factors of G. Multiplication gives an isogeny from $G_1 \times \cdots \times G_r$ to G. Thus, by [5, Lem. 2.12(ii)(b)] and [16, §4.3], we can replace G with $G_1 \times \cdots \times G_r$, so we can assume G is the product of its simple factors. Finally, thanks to [5, Lem. 2.12(i)], [16, §4.3], we can reduce to the case when G is simple.

Finally, we address Theorems 1.5 and 1.7.

Proof of Theorems 1.5 and 1.7. By Theorems 1.2 and 1.6, the only cases we need to consider are when p is bad for G. If G is classical, then we are in the situation of Lemma 2.5 and Example 2.6, so we are done.

We are left to consider the case when G is of exceptional type. Then owing to Lemma 2.7, G is of type G_2 and $p = 3$. There is no harm in assuming that H is semisimple. It follows from Example 2.8 that H is G-ir. Thus Theorem 1.5 follows. So consider the setting of Theorem 1.7 when $\sigma|_H$ is a q-Frobenius endomorphism of H in this case. By Corollary 2.10, u belongs to the subregular class of G_2. It follows from the proof of Lemma 2.7 in [31] that u is contained in a σ-stable maximal rank subgroup of G of type $A_1\widetilde{A}_1$ and this type is unique. Since H is proper and semisimple, $H \subseteq M$, where M is a σ-stable maximal rank subgroup of G of type $A_1\widetilde{A}_1$. Since p is good for H, there is a σ-stable subgroup A of H of type A_1 containing u, by Theorem 2.4. Thus $A \subseteq H \subseteq M$. Since u is also distinguished in M and $p = 3$ is good for M, Theorem 1.6 shows that A_σ is M-ir. Note that M is the centralizer of a semisimple element of G of order 2 (by Deriziotis' Criterion, see [11, 2.3]). Since A_σ is M-cr, it is G-cr, owing to [5, Cor. 3.21]. Once again, by Lemma 3.1, A_σ is G-ir and so is H_σ. Theorem 1.7 follows. □

Remark 6.4. In [17, §7], Korhonen gives counterexamples to Theorem 1.1 when the order of the distinguished unipotent element of G is greater than p (even when p is good for G [17, Prop. 7.1]). Theorem 1.8 implies that this can only happen when $p < a(G)$. For instances of overgroups of distinguished unipotent elements of G of order greater than p for $p \geq a(G)$ (and p good for G), so that Theorem 1.8 applies, see Examples 6.6 and 6.7.

Remark 6.5. In view of Remark 6.4, it is natural to ask for instances of G, u and H when the conclusion of Theorem 1.8 holds even when $p < a(G)$ but p is still good for G. If p is good for G and G is simple classical, non-regular distinguished unipotent elements always belong to a maximal rank semisimple subgroup H of G, by [43, Prop. 3.1, Prop. 3.2]. For G simple of exceptional type this is also the case in almost all instances of non-regular distinguished unipotent elements, see [43, Lem. 2.1]. Each such H is obviously G-irreducible. This is independent of p of course and thus applies in particular when $p < a(G)$. For instance, let G be of type E_7, $p = 5$, and suppose u belongs to the distinguished class $E_7(a_3)$ (resp. $E_7(a_4)$, $E_7(a_5)$). Then $\mathrm{ht}_J(\rho) = 9$ (resp. 7, 5), so u has order 5^2, by Lemma 2.3 in each case. Since u does not have order 5, Theorem 1.1 does not apply, and since $5 < 8 = a(G)$ neither does Theorem 1.8. Nevertheless, in each case u is contained in a maximal rank subgroup H of type A_1D_6, see [43, p. 52], and each such H is G-ir.

We close the section with several additional higher order examples in good characteristic when Theorem 1.1 does not apply but Theorem 1.8 does.

Example 6.6. Let G be of type E_6. Suppose p is good for G. In [43, Lem. 2.7], Testerman exhibits the existence of a simple subgroup H of G of type C_4 whose regular unipotent class belongs to the subregular class $E_6(a_1)$ of G. Let u be regular unipotent in H. For $p = 7$, the order of u is 7^2, by Lemma 2.3, so Theorem 1.1 can't be invoked to say anything about H. However, for $p = 7 = a(G)$, we infer from Theorem 1.8 that H is G-ir.

Example 6.7. Let G be of type E_8. Suppose $p = 11$. Let u be in the distinguished class $E_8(a_3)$ (resp. $E_8(a_4)$, $E_8(b_4)$, $E_8(a_5)$, or $E_8(b_5)$). From the corresponding weighted Dynkin diagram corresponding to u we get $\mathrm{ht}_J(\rho) = 17$ (resp. 14, 13, 11, or 11), see [10, p. 177]. It follows from Lemma 2.3 that in each of these instances u has order 11^2. So we can't appeal to Theorem 1.1 to deduce anything about reductive overgroups of u. But as $11 = p \geq a(G) = 9$, Theorem 1.8 applies and allows us to conclude that each such overgroup is G-ir. For example, in each instance above, u is contained in a maximal rank subgroup H of G of type A_1E_7 or D_8, see [43, p. 52].

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