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Research Paper

Heights of posets associated with Green's relations on semigroups

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ABSTRACT

Given a semigroup S, for each Green's relation $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}\}$ on S, the $\mathcal{K}$-height of S, denoted by $H_{\mathcal{K}}(S)$, is the height of the poset of $\mathcal{K}$-classes of S. More precisely, if there is a finite bound on the sizes of chains of $\mathcal{K}$-classes of S, then $H_{\mathcal{K}}(S)$ is defined as the maximum size of such a chain; otherwise, we say that S has infinite $\mathcal{K}$-height. We discuss the relationships between these four $\mathcal{K}$-heights. The main results concern the class of stable semigroups, which includes all finite semigroups. In particular, we prove that a stable semigroup has finite $\mathcal{L}$-height if and only if it has finite $\mathcal{R}$-height if and only if it has finite $\mathcal{J}$-height. In fact, for a stable semigroup S, if $H_{\mathcal{L}}(S) = n$ then $H_{\mathcal{R}}(S) \leq 2^n - 1$ and $H_{\mathcal{J}}(S) \leq 2^n - 1$, and we exhibit a family of examples to prove that these bounds are sharp. Furthermore, we prove that if $2 \leq H_{\mathcal{L}}(S) < \infty$ and $2 \leq H_{\mathcal{R}}(S) < \infty$, then $H_{\mathcal{J}}(S) \leq H_{\mathcal{L}}(S) + H_{\mathcal{R}}(S) - 2$. We also show that for each $n \in \mathbb{N}$ there exists a semigroup S such that $H_{\mathcal{L}}(S) = H_{\mathcal{R}}(S) = 2^n + n - 3$ and $H_{\mathcal{J}}(S) = 2^{n+1} - 4$. By way of contrast, we prove that for a regular semigroup the $\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{H}$-heights coincide with each other, and are greater or equal to the $\mathcal{J}$-height. Moreover, in a stable, regular semigroup the $\mathcal{L}$-, $\mathcal{R}$-, $\mathcal{H}$- and $\mathcal{J}$-heights are all equal.

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1. Introduction

Green's relations, five equivalence relations based on mutual divisibility, are arguably the most important tools for analysing the structure of semigroups. For four of these relations, namely $\mathcal{L}, \mathcal{R}, \mathcal{J}$ and $\mathcal{H}$, there is a natural associated poset and thus a height parameter. Specifically, Green's preorders $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$, $\leq_{\mathcal{J}}$ and $\leq_{\mathcal{H}}$ are defined as follows:

- $a \leq_{\mathcal{L}} b$ if and only if there exists $s \in S^1$ such that $a = sb$;
- $a \leq_{\mathcal{R}} b$ if and only if there exists $s \in S^1$ such that $a = bs$;
- $a \leq_{\mathcal{J}} b$ if and only if there exist $s, t \in S^1$ such that $a = sbt$;
- $a \leq_{\mathcal{H}} b$ if and only if there exist $s, t \in S^1$ such that $a = sb = bt$.

(Throughout this paper, S^1 stands for the monoid obtained from S by adjoining an identity $1 \notin S$.) Observe that $\leq_{\mathcal{H}} = \leq_{\mathcal{L}} \cap \leq_{\mathcal{R}}$, $\leq_{\mathcal{L}} \subseteq \leq_{\mathcal{J}}$ and $\leq_{\mathcal{R}} \subseteq \leq_{\mathcal{J}}$. Now, letting $\mathcal{K}$ stand for any of $\mathcal{L}, \mathcal{R}, \mathcal{J}$ and $\mathcal{H}$, Green's relation $\mathcal{K}$ is defined by

$$a \, \mathcal{K} \, b \Leftrightarrow a \leq_{\mathcal{K}} b \text{ and } b \leq_{\mathcal{K}} a.$$

The preorder $\leq_{\mathcal{K}}$ induces a partial order on the set of $\mathcal{K}$-classes:

$$K_a \leq K_b \Leftrightarrow a \leq_{\mathcal{K}} b \qquad \text{(where } K_s \text{ is the } \mathcal{K}\text{-class of } s \in S\text{)}.$$

Definition 1.1. The *$\mathcal{K}$-height* of S, denoted by $H_{\mathcal{K}}(S)$, is defined as follows. If there is a finite bound on the sizes of chains of $\mathcal{K}$-classes of S, then $H_{\mathcal{K}}(S)$ is defined as the minimum such bound; otherwise, we say that S has *infinite $\mathcal{K}$-height* and write $H_{\mathcal{K}}(S) = \infty$.

The $\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{J}$-heights were first explicitly defined and studied in [8], in the context of finite transformation semigroups. However, these parameters play an implicit role in the Rhodes expansion of a semigroup, a powerful tool in complexity theory, and in variants of this construction; see [2] [7, Chapter XII]. Moreover, for $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}\}$, having finite $\mathcal{K}$-height is clearly stronger than satisfying the minimal condition on $\mathcal{K}$-classes. The minimal conditions on $\mathcal{L}$-classes, $\mathcal{R}$-classes and $\mathcal{J}$-classes, denoted by M_L, M_R and M_J, as well as the related conditions M_L^* and M_R^*, have played an important role in the development of the structure theory of semigroups; see [5, Section 6.6] [9, p. 23-29] [11] [13].

The article [12] investigates the relationship between the $\mathcal{R}$-heights of semigroups and their bi-ideals (which include left-, right- and two-sided ideals). In particular, it is shown that the property of having finite $\mathcal{R}$-height is inherited by bi-ideals. Of course, the results of that paper have obvious duals in terms of $\mathcal{L}$-heights. In the final section of [12] there is a brief discussion about the relationships between the $\mathcal{L}$-, $\mathcal{R}$-, $\mathcal{J}$- and $\mathcal{H}$-heights. This is the subject of the present article.

In the following section we provide some preliminary definitions and results regarding minimality in the posets of $\mathcal{L}$-, $\mathcal{R}$-, $\mathcal{J}$- and $\mathcal{H}$-classes. In Section 3 we consider the notion of stability and its relationship to the $\mathcal{K}$-heights, and in Section 4 we discuss semigroups with $\mathcal{K}$-heights equal to 1 or 2. In order to prove the main results of the paper, in Section 5 we develop some machinery concerning quotients and ideal extensions. The main results, contained in Section 6, include the following, where S is a semigroup.

- If $H_{\mathcal{L}}(S) = n < \infty$, then S is stable if and only if $H_{\mathcal{R}}(S) < \infty$, in which case $\lceil \log_2(n+1) \rceil \leq H_{\mathcal{R}}(S) \leq 2^n - 1$.
- If S is stable, then $H_{\mathcal{L}}(S)$ is finite if only if $H_{\mathcal{J}}(S)$ is finite. Moreover, if $H_{\mathcal{L}}(S) = n < \infty$ then $n \leq H_{\mathcal{J}}(S) \leq 2^n - 1$.
- For every $n \in \mathbb{N}$ and $m \in \{n, \ldots, 2^n - 1\}$, there exists a semigroup T such that $H_{\mathcal{L}}(T) = n$ and $H_{\mathcal{R}}(T) = H_{\mathcal{J}}(T) = |T| = m$.
- If $2 \leq H_{\mathcal{L}}(S) < \infty$ and $2 \leq H_{\mathcal{R}}(S) < \infty$ then, with $\min\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big) = n$, we have

$$\max\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big) \leq H_{\mathcal{J}}(S) \leq \min(2^n - 1, H_{\mathcal{L}}(S) + H_{\mathcal{R}}(S) - 2).$$

Finally, in Section 7 we consider the relationships between the $\mathcal{K}$-heights for semisimple semigroups and, in particular, for regular semigroups.

2. Preliminaries

Throughout, S denotes an arbitrary semigroup, and $\mathcal{K}$ stands for any of Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{J}$ and $\mathcal{H}$. Recall that each of Green's relations is an equivalence relation. Furthermore, $\mathcal{L}$ is a right congruence, and $\mathcal{R}$ is a left congruence (a *left/right congruence* on S is an equivalence relation on S that is preserved under left/right multiplication).

For two elements $a, b \in S$, we write $a <_{\mathcal{K}} b$ if $a \leq_{\mathcal{K}} b$ but a and b are not $\mathcal{K}$-related. We define a *$\mathcal{K}$-chain* in S to be a sequence of elements of S that is strictly decreasing under the preorder $\leq_{\mathcal{K}}$. Note that S has finite $\mathcal{K}$-height if and only if there is a finite bound on the lengths of $\mathcal{K}$-chains in S. Thus, S may have infinite $\mathcal{K}$-height even if all its chains of $\mathcal{K}$-classes are finite.

Notice that a semigroup with finite $\mathcal{K}$-height has minimal and maximal $\mathcal{K}$-classes. A semigroup can have at most one minimal $\mathcal{J}$-class; if this $\mathcal{J}$-class exists, it is called the *minimal ideal*. On the other hand, a semigroup may possess multiple minimal $\mathcal{L}/\mathcal{R}$-classes (also known as minimal left/right ideals). If a semigroup has minimal $\mathcal{L}/\mathcal{R}$-classes, then it has a minimal ideal, which is equal to the union of all the minimal $\mathcal{L}/\mathcal{R}$-classes [3, Theorem 2.1].

A semigroup is *left/right simple* if it has a single $\mathcal{L}/\mathcal{R}$-class, and *simple* if it has a single $\mathcal{J}$-class. Note that a semigroup is simple if and only if it has $\mathcal{J}$-height 1. Certainly left/right simple semigroups are simple. It turns out that minimal $\mathcal{L}/\mathcal{R}$-classes are left/right simple subsemigroups [3, Theorem 2.4], and minimal ideals are simple subsemigroups [3, Theorem 1.1]. A *completely simple* semigroup is a simple semigroup that possesses both minimal $\mathcal{L}$-classes and minimal $\mathcal{R}$-classes.

From the preceding discussion, we immediately deduce the following lemma.

Lemma 2.1. *For $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$, if $H_{\mathcal{K}}(S)$ is finite then S has a minimal ideal, which is the union of the minimal $\mathcal{K}$-classes of S.*

In fact, the statement of Lemma 2.1 also holds for $\mathcal{K} = \mathcal{H}$, as a consequence of the following stronger result.

Lemma 2.2. *A semigroup has minimal $\mathcal{H}$-classes if and only if it has a completely simple minimal ideal (which is the union of all the minimal $\mathcal{H}$-classes). Consequently, a semigroup with finite $\mathcal{H}$-height has a completely simple minimal ideal.*

Proof. This lemma is probably well known, but we provide a proof for completeness.

Suppose that S is a semigroup with minimal $\mathcal{H}$-classes, and let J denote the union of all the minimal $\mathcal{H}$-classes. For any $x \in J$ and $s, t \in S^1$, we have $x \geq_{\mathcal{H}} xsxtx$, so that $x \,\mathcal{H}\, xsxtx$ by minimality, which implies that $x \,\mathcal{J}\, sxt$. Thus J is the minimal ideal of S, and hence J is simple. Moreover, for each $x \in J$ we have $x \,\mathcal{H}\, x^2$, so that, by Green's Theorem [4, Theorem 2.16], the $\mathcal{H}$-class of x is a group. Thus J is a union of groups. Hence, by [4, Theorem 4.5], J is completely simple.

Conversely, if S has a completely simple minimal ideal, say J, then J is the union both of the minimal $\mathcal{L}$-classes and of the minimal $\mathcal{R}$-classes [3, Theorem 3.2]. Consequently, the $\mathcal{H}$-classes of S contained in J are minimal. □

For semigroups with zero, the theory of minimal $\mathcal{K}$-classes becomes trivial, so we require the notion of 0-minimality. Suppose that S has a zero element 0. A $\mathcal{K}$-class of S is called *0-minimal* if $\{0\}$ is the only $\mathcal{K}$-class below it. The semigroup S is *left/right 0-simple* if $S^2 \neq \{0\}$ and the $\mathcal{L}/\mathcal{R}$-classes of S are $\{0\}$ and $S \setminus \{0\}$. Similarly, S is *0-simple* if $S^2 \neq 0$ and the $\mathcal{J}$-classes of S are $\{0\}$ and $S \setminus \{0\}$. A *completely 0-simple* semigroup is a 0-simple semigroup that possesses both 0-minimal $\mathcal{L}$-classes and 0-minimal $\mathcal{R}$-classes.

3. $\mathcal{K}$-heights and stability

We now introduce a crucial notion for this paper, namely stability.

Definition 3.1. A semigroup S is *left stable* if $\leq_{\mathcal{L}} \cap \mathcal{J} = \mathcal{L}$. Dually, S is *right stable* if $\leq_{\mathcal{R}} \cap \mathcal{J} = \mathcal{R}$. Finally, S is *stable* if it is both left stable and right stable.

The class of stable semigroups includes all group-bound semigroups (where *group-bound* means that every element has some power belonging to a subgroup) [6, Proposition 7], and hence all finite semigroups. The one Green's relation that we have hitherto not defined is $\mathcal{D} = \mathcal{L} \vee \mathcal{R}\,(= \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L})$, for which, in general, there is no natural associated preorder. However, we have $\mathcal{D} = \mathcal{J}$ in stable semigroups [6, Corollary 10].

For a simple semigroup, being completely simple is equivalent to being stable [6, Proposition 15]. Consequently, minimal ideals of stable semigroups are completely simple. This fact, together with Lemma 2.1, yields an analogue of Lemma 2.2 for stable semigroups.

Lemma 3.2. *Let $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$. A stable semigroup has minimal $\mathcal{K}$-classes if and only if it has a completely simple minimal ideal (which is the union of all the minimal $\mathcal{K}$-classes). Consequently, a stable semigroup with finite $\mathcal{K}$-height has a completely simple minimal ideal.*

It follows immediately from the definition that S is left stable if and only if every $\mathcal{J}$-class of S is a union of pairwise incomparable $\mathcal{L}$-classes. In fact, it turns out that S is left stable if and only if it satisfies the condition M_L^*, that for each $\mathcal{J}$-class J of S the set of $\mathcal{L}$-classes contained in J has a minimal element [5, Lemma 6.42]. Clearly, if S has finite $\mathcal{L}$-height then it satisfies M_L^*. Thus we have:

Lemma 3.3. *If $H_{\mathcal{L}}(S)$ (resp. $H_{\mathcal{R}}(S)$) is finite, then S is left (resp. right) stable. Consequently, if both $H_{\mathcal{L}}(S)$ and $H_{\mathcal{R}}(S)$ are finite, then S is stable.*

It turns out that semigroups with finite $\mathcal{H}$-height are also stable. In fact, we have the following stronger result.

Lemma 3.4. *If $H_{\mathcal{H}}(S) = n < \infty$, then for every $a \in S$ the element a^n belongs to a subgroup of S. In particular, S is group-bound (and hence stable).*

Proof. Let $a \in S$. Then

$$a \geq_{\mathcal{H}} a^2 \geq_{\mathcal{H}} \cdots \geq_{\mathcal{H}} a^{n+1} \geq_{\mathcal{H}} \cdots .$$

Since $H_{\mathcal{H}}(S) = n$, there exist $i, j \in \{1, \ldots, n+1\}$ with $i < j$ such that $a^i \,\mathcal{H}\, a^j$. Then, in particular $a^i \,\mathcal{L}\, a^j$, so there is $s \in S^1$ such that $a^i = sa^j$. It follows by an easy induction argument that $a^n = s^k a^{k(j-i)+n}$ for all $k \in \mathbb{N}$. Thus, we have $a^n = s^n a^{n(j-i+1)} = s^n a^{n(j-i-1)} a^{2n}$, and hence $a^n \,\mathcal{L}\, a^{2n}$. A dual argument proves that $a^n \,\mathcal{R}\, a^{2n}$, and hence $a^n \,\mathcal{H}\, a^{2n}$. Thus the $\mathcal{H}$-class of a^n is a subgroup of S, as required. □

The next result is the first on the main theme of the article: to explore the relationships between the four $\mathcal{K}$-heights.

Proposition 3.5. *Let S be a semigroup.*

(1) *Suppose that S is left stable. Then every $\mathcal{H}$-chain of S is an $\mathcal{R}$-chain, and every $\mathcal{L}$-chain of S is a $\mathcal{J}$-chain. Consequently, $H_{\mathcal{H}}(S) \leq H_{\mathcal{R}}(S)$ and $H_{\mathcal{L}}(S) \leq H_{\mathcal{J}}(S)$.*

(2) *Suppose that S is right stable. Then every $\mathcal{H}$-chain of S is an $\mathcal{L}$-chain, and every $\mathcal{R}$-chain of S is a $\mathcal{J}$-chain. Consequently, $H_{\mathcal{H}}(S) \leq H_{\mathcal{L}}(S)$ and $H_{\mathcal{R}}(S) \leq H_{\mathcal{J}}(S)$.*

(3) *Suppose that S is stable. Then every $\mathcal{H}$-chain of S is both an $\mathcal{L}$-chain and an $\mathcal{R}$-chain, and all $\mathcal{L}$-chains and $\mathcal{R}$-chains of S are $\mathcal{J}$-chains. Consequently,*

$$H_{\mathcal{H}}(S) \leq \min\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big) \quad \text{and} \quad \max\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big) \leq H_{\mathcal{J}}(S).$$

Proof. (1) Consider $a, b \in S$ with $a <_{\mathcal{H}} b$. Then $a \leq_{\mathcal{L}} b$ and $a \leq_{\mathcal{R}} b$. If $a \,\mathcal{R}\, b$, then certainly $a \,\mathcal{J}\, b$, which implies that $a \,\mathcal{L}\, b$, since S is left stable. Then $a \,\mathcal{H}\, b$, a contradiction. Thus, we must have $a <_{\mathcal{R}} b$. We conclude that every $\mathcal{H}$-chain is an $\mathcal{R}$-chain, and hence $H_{\mathcal{H}}(S) \leq H_{\mathcal{R}}(S)$.

It follows immediately from the definition of left stability that every $\mathcal{L}$-chain is a $\mathcal{J}$-chain, and hence $H_{\mathcal{L}}(S) \leq H_{\mathcal{J}}(S)$.

(2) holds by left-right duality, and (3) follows from (1) and (2). □

Corollary 3.6. *For any semigroup S, either $H_{\mathcal{H}}(S) \leq \min\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big)$ or $H_{\mathcal{H}}(S)$ is infinite.*

Proof. Suppose that $H_{\mathcal{H}}(S)$ is finite. Then S is stable by Lemma 3.3, and hence $H_{\mathcal{H}}(S) \leq \min\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big)$ by Proposition 3.5. □

4. Small $\mathcal{K}$-heights

In this section we consider the relationships between the different $\mathcal{K}$-heights where at least one of them is equal to 1 or 2.

Recall that $H_{\mathcal{J}}(S) = 1$ precisely when S is simple. The following result provides necessary and sufficient conditions for a semigroup to have $\mathcal{L}$-height 1 or $\mathcal{R}$-height 1 in terms of the $\mathcal{J}$-height and one-sided stability.

Proposition 4.1. *For a semigroup S, the following hold.*

(1) *$H_{\mathcal{L}}(S) = 1$ if and only if $H_{\mathcal{J}}(S) = 1$ and S is left stable.*
(2) *$H_{\mathcal{R}}(S) = 1$ if and only if $H_{\mathcal{J}}(S) = 1$ and S is right stable.*

Proof. Clearly it suffices to prove (1). If $H_{\mathcal{L}}(S) = 1$, then S, being the union of minimal $\mathcal{L}$-classes, is simple, so $H_{\mathcal{J}}(S) = 1$, and S is left stable by Lemma 3.3. The converse follows from Proposition 3.5(1). □

The assumptions of left stability and of right stability are necessary in the respective parts of Proposition 4.1. Indeed, there exist semigroups with $\mathcal{R}$-height 1 (and hence $\mathcal{J}$-height 1) but with infinite $\mathcal{L}$-height (or vice versa). For example, the Baer-Levi semigroup $\mathcal{BL}_X$ on an infinite set X, defined as the set of all injective maps $\alpha : X \to X$ with

$|X\backslash X\alpha| = |X|$, is right simple [5, Theorem 8.2], so has $\mathcal{R}$-height 1. On the other hand, we have $\alpha \leq_{\mathcal{L}} \beta$ in $\mathcal{BL}_X$ if and only if $X\alpha \subseteq X\beta$ and $|X\beta\backslash X\alpha| = |X|$ [10, Theorem 8]; consequently, $\mathcal{BL}_X$ has infinite $\mathcal{L}$-height.

We now provide several equivalent characterisations for a semigroup to have both $\mathcal{L}$-height 1 and $\mathcal{R}$-height 1.

Proposition 4.2. *For a semigroup S, the following are equivalent:*

(1) $H_{\mathcal{L}}(S) = 1$ *and* $H_{\mathcal{R}}(S) = 1$*;*
(2) $H_{\mathcal{L}}(S) = 1$ *and* $H_{\mathcal{R}}(S)$ *is finite;*
(3) $H_{\mathcal{L}}(S) = 1$ *and S is stable;*
(4) $H_{\mathcal{R}}(S) = 1$ *and* $H_{\mathcal{L}}(S)$ *is finite;*
(5) $H_{\mathcal{R}}(S) = 1$ *and S is stable;*
(6) $H_{\mathcal{J}}(S) = 1$ *and S is stable;*
(7) $H_{\mathcal{H}}(S) = 1$*;*
(8) *S is completely simple.*

Proof. Let $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}\}$. If $H_{\mathcal{K}}(S) = 1$ then S, being the union of minimal $\mathcal{K}$-classes, is simple. Also, by Lemmas 3.3 and 3.4, each of (1), (2), (4) and (7) implies that S is stable. Thus, if any of (1)-(7) holds, then S is simple and stable, and hence S is completely simple by [6, Proposition 15].

Conversely, if S is completely simple, then S is stable and $H_{\mathcal{K}}(S) = 1$ for each $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}\}$. $\square$

An immediate consequence of Proposition 4.2 is that for a stable semigroup S,

$$H_{\mathcal{L}}(S) = 1 \Leftrightarrow H_{\mathcal{R}}(S) = 1 \Leftrightarrow H_{\mathcal{J}}(S) = 1 \Leftrightarrow H_{\mathcal{H}}(S) = 1 \Leftrightarrow S \text{ is completely simple.}$$

We now turn to the situation where one of the $\mathcal{K}$-heights is 2.

Proposition 4.3. *If S is a semigroup such that $H_{\mathcal{L}}(S) = 2$ or $H_{\mathcal{R}}(S) = 2$, then $H_{\mathcal{J}}(S) \in \{2, 3\}$.*

Proof. By symmetry, it suffices to consider the case that $H_{\mathcal{L}}(S) = 2$. By Lemma 2.1, S has a minimal ideal, say J, which is the union of all the minimal $\mathcal{L}$-classes of S. Since $H_{\mathcal{L}}(S) = 2$, the elements in $S\backslash J$ are all maximal under the $\mathcal{L}$-order, and in particular the $\mathcal{L}$-classes contained in $S\backslash J$ are incomparable.

As $H_{\mathcal{L}}(S)$ is finite, S is left stable by Lemma 3.3, and hence $H_{\mathcal{J}}(S) \geq 2$ by Proposition 3.5(1). Now suppose for a contradiction that $H_{\mathcal{J}}(S) \geq 4$. Then there exists a $\mathcal{J}$-chain (a, b, c, d) in S where $d \in J$ (and hence $a, b, c \in S\backslash J$). Thus, there are $s, t, u, v \in S^1$ such that $b = sat$ and $c = ubv$. We claim that $v \in S$ (i.e. that $v \neq 1$). Indeed, if $v = 1$, then $b \geq_{\mathcal{L}} c$, which implies that $b \,\mathcal{L}\, c$ (since the $\mathcal{L}$-classes in $S\backslash J$ are incomparable), but then

$b\,\mathcal{J}\,c$, a contradiction. Now, we have $v, c \in S\backslash J$ with $v \geq_{\mathcal{L}} c$, so $v\,\mathcal{L}\,c$ (as $H_{\mathcal{L}}(S) = 2$). Thus, there exists $x \in S^1$ such that $v = cx$. Therefore, we have

$$c = ubv = ubcx = ububvx = ubusatvx.$$

Thus $busa \geq_{\mathcal{J}} c$, so that $busa \in S\backslash J$ and hence $a\,\mathcal{L}\,busa$. Then $a\,\mathcal{J}\,busa \leq_{\mathcal{J}} b$, so $a \leq_{\mathcal{J}} b$. But this contradicts the assumption that $a >_{\mathcal{J}} b$. Thus $H_{\mathcal{J}}(S) \leq 3$, as required. □

For stable semigroups with $\mathcal{L}$-height 2, we obtain a far stronger statement than that of Proposition 4.3.

Proposition 4.4. *Let S be a semigroup. If S is stable and $H_{\mathcal{L}}(S) = 2$, then $H_{\mathcal{H}}(S) = 2$ and $H_{\mathcal{R}}(S) = H_{\mathcal{J}}(S) \in \{2, 3\}$. Moreover, the following are equivalent:*

(1) $H_{\mathcal{L}}(S) = H_{\mathcal{R}}(S) = 2$;
(2) $H_{\mathcal{J}}(S) = H_{\mathcal{H}}(S) = 2$;
(3) $H_{\mathcal{J}}(S) = 2$ *and S is stable.*

Proof. Suppose that S is stable with $H_{\mathcal{L}}(S) = 2$. Then, by Propositions 3.5(3), 4.2 and 4.3, we have $H_{\mathcal{H}}(S) = 2$ and $2 \leq H_{\mathcal{R}}(S) \leq H_{\mathcal{J}}(S) \leq 3$. Suppose for a contradiction that $H_{\mathcal{R}}(S) = 2$ and $H_{\mathcal{J}}(S) = 3$. By Lemma 3.2, S has a completely simple minimal ideal, say J. Thus, there exists a $\mathcal{J}$-chain (a, b, c) in S where $c \in J$. Now, there exist $s, t \in S^1$ such that $b = sat$. Then, as $b \notin J$, we have $sa, at \notin J$, and then

$$a \geq_{\mathcal{L}} sa >_{\mathcal{L}} csa \quad \text{and} \quad a \geq_{\mathcal{R}} at >_{\mathcal{R}} atc.$$

Since $H_{\mathcal{L}}(S) = H_{\mathcal{R}}(S) = 2$, it follows that $a\,\mathcal{L}\,sa$ and $a\,\mathcal{R}\,at$, so that there exist $u, v \in S^1$ such that $a = usa = atv$. But then we have $a = usatv = ubv$, contradicting the fact that $a >_{\mathcal{J}} b$. Thus $H_{\mathcal{R}}(S) = H_{\mathcal{J}}(S)$.

We now prove the equivalence of (1), (2) and (3). If (1) holds, then S is stable by Lemma 3.3, and then (2) follows from the first part of this proposition (just proved). That (2) implies (3) follows immediately from Lemma 3.4, and we have (3) implies (1) by Propositions 3.5(3) and 4.2. □

We conclude this section by exhibiting a pair of examples of semigroups to demonstrate that the conditions in Propositions 4.3 and 4.4 are necessary.

Our first example is a semigroup with $\mathcal{R}$-height 3 and infinite $\mathcal{J}$-height. This provides a negative answer to [12, Open Problem 5.1], which asks whether there is a general upper bound for the $\mathcal{J}$-height of a semigroup in terms of its $\mathcal{R}$-height.

Example 4.5. Let S be any right simple semigroup that is not completely simple (such as a Baer-Levi semigroup). Then $H_{\mathcal{R}}(S) = 1$ and S has infinite $\mathcal{L}$-height. Let $U = S \cup (S \times S) \cup \{0\}$, and define a multiplication on U, extending that of S, by

$$(a,b)c = (a,bc), \quad c(a,b) = (ca,b) \quad \text{and} \quad (a,b)(c,d) = (a,b)0 = 0(a,b) = 0^2 = 0$$

for all $a, b, c, d \in S$. It is straightforward to show that U is a semigroup under this multiplication.

It is easy to show that the poset the $\mathcal{R}$-classes of U is given as follows: $\{0\}$ is the minimum $\mathcal{R}$-class; S is the maximum $\mathcal{R}$-class; the remaining $\mathcal{R}$-classes are the sets $\{a\}\times S$ ($a \in S$), and these all lie between $\{0\}$ and S, and are pairwise incomparable. It follows that $H_{\mathcal{R}}(U) = 3$.

It is also straightforward to show that the poset the $\mathcal{J}$-classes of U is given as follows: $\{0\}$ is the minimum $\mathcal{J}$-class; S is the maximum $\mathcal{J}$-class; the remaining $\mathcal{J}$-classes are the sets $L_a \times S$ (where $a \in S$, and L_a denotes the $\mathcal{L}$-class of a in S), and $L_a \times S \leq L_b \times S$ if and only if $L_a \leq L_b$. Thus, the poset of $\mathcal{J}$-classes of U is isomorphic to the poset of $\mathcal{L}$-classes of S with minimum and maximum elements adjoined. Since $H_{\mathcal{L}}(S)$ is infinite, we conclude that $H_{\mathcal{J}}(U)$ is infinite.

Next, we show that it is possible for a (necessarily stable) semigroup to have $\mathcal{H}$-height 2 but infinite $\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{J}$-heights. In order to construct an example of such a semigroup, we first recall the notion of a Rees quotient. For an ideal I of S, the *Rees quotient of S by I*, denoted by S/I, is the semigroup $(S\backslash I) \cup \{0\}$, where $0 \notin S\backslash I$, with multiplication given by

$$a \cdot b = \begin{cases} ab & \text{if } a, b, ab \in S\backslash I, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.6. Let X be any infinite set, and let F denote the free semigroup on X, i.e. the set of all non-empty words over X. Let I be the set of all words over X in which some letter appears at least twice; that is,

$$I = \{w \in F : w = uxvxz \text{ for some } x \in X \text{ and } u, v, z \in F^1\}.$$

Then I is an ideal of F. Let S be the Rees quotient F/I. We note that S is $\mathcal{J}$-trivial (i.e. $\mathcal{J}$ is the equality relation on S). Choose distinct elements $x_i \in X$ ($i \in \mathbb{N}$). Then $(x_1, x_1x_2, x_1x_2x_3, \dots)$ is both an $\mathcal{R}$-chain and a $\mathcal{J}$-chain of S, so that $H_{\mathcal{R}}(S)$ and $H_{\mathcal{J}}(S)$ are infinite. Similarly, $H_{\mathcal{L}}(S)$ is infinite. We claim that $H_{\mathcal{H}}(S) = 2$. Indeed, let $u, v \in S$ with $u >_{\mathcal{H}} v$. We need to show that $v = 0$. Now, we have $v = wu = uw'$ for some $w, w' \in S$. If $w = 0$ or $w' = 0$, we are done, so assume that $w, w' \in F\backslash I$. If $|w| \leq |u|$, then, since $wu = uw'$, we have $u = wz$ for some $z \in F^1$, and hence $v = wwz = 0$. If $|w| > |u|$, then $w = uz'$ for some $z' \in F$, and hence $v = uz'u = 0$. Thus, in either case, we have $v = 0$, as required.

5. Quotients and ideal extensions

In this section we investigate the relationship between the $\mathcal{K}$-heights of a semigroup S and those of certain quotients of S. Our main purpose here is to establish some critical machinery for proving the main results of the paper, appearing in Section 6.

Proposition 5.1. *Let S be a semigroup and let ρ be a congruence on S.*

(1) *For each $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$ we have $H_{\mathcal{K}}(S) \geq H_{\mathcal{K}}(S/\rho)$.*
(2) *For each $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$, if $\leq_{\mathcal{K}} \cap \rho = \mathcal{K} \cap \rho$ then $H_{\mathcal{K}}(S) = H_{\mathcal{K}}(S/\rho)$.*
(3) *If S is left stable and $\leq_{\mathcal{L}} \cap \rho \subseteq \mathcal{J}$, then $H_{\mathcal{L}}(S) = H_{\mathcal{L}}(S/\rho)$.*
(4) *If S is right stable and $\leq_{\mathcal{R}} \cap \rho \subseteq \mathcal{J}$, then $H_{\mathcal{R}}(S) = H_{\mathcal{R}}(S/\rho)$.*

Proof. (1) We just consider the case $\mathcal{K} = \mathcal{J}$. The proofs for $\mathcal{L}$ and $\mathcal{R}$ are similar but slightly more straightforward.

Let $T = S/\rho$, and consider a $\mathcal{J}$-chain $(b_0, \ldots, b_n)$ in T. For $i \in \{1, \ldots, n\}$, there exist $u_i, v_i \in T^1$ such that $b_i = u_i b_{i-1} v_i$. If $u_i \in T$, choose $s_i \in S$ such that $u_i = [s_i]_\rho$; otherwise, let $s_i = 1$. Likewise, if $v_i \in T$, choose $t_i \in S$ such that $v_i = [t_i]_\rho$; otherwise, let $t_i = 1$. Now let $a_0 \in S$ be such that $[a_0]_\rho = b_0$, and for $i \in \{1, \ldots, n\}$ set $a_i = s_i \ldots s_1 a_0 t_1 \ldots t_i$. We then have $[a_i]_\rho = b_i$, and $a_0 \geq_{\mathcal{J}} a_1 \geq_{\mathcal{J}} \cdots \geq_{\mathcal{J}} a_n$ in S. We cannot have $a_{i-1} \mathrel{\mathcal{J}} a_i$ (in S), for that would imply that $b_{i-1} \mathrel{\mathcal{J}} b_i$ in T. Hence, we have a $\mathcal{J}$-chain $(a_0, \ldots, a_n)$ in S. Thus $H_{\mathcal{J}}(S) \geq H_{\mathcal{J}}(T)$.

(2) Again, we just consider the case $\mathcal{K} = \mathcal{J}$. By (1) we have $H_{\mathcal{J}}(S) \geq H_{\mathcal{J}}(S/\rho)$, so it remains to prove the reverse inequality. So, consider $a, b \in S$ with $a >_{\mathcal{J}} b$. Then $[a]_\rho \geq_{\mathcal{J}} [b]_\rho$ in S/ρ. Suppose that $[a]_\rho \mathrel{\mathcal{J}} [b]_\rho$. It then follows that $[a]_\rho = [sbt]_\rho$ for some $s, t \in S^1$. But then, since $a >_{\mathcal{J}} b \geq_{\mathcal{J}} sbt$, we have $(a, sbt) \in \leq_{\mathcal{J}} \cap \rho$ and $(a, sbt) \notin \mathcal{J}$, contradicting the assumption. Thus $[a]_\rho >_{\mathcal{J}} [b]_\rho$. We conclude that $H_{\mathcal{J}}(S) \leq H_{\mathcal{J}}(S/\rho)$, and hence $H_{\mathcal{J}}(S) = H_{\mathcal{J}}(S/\rho)$, as required.

We may now quickly prove (3); the proof of (4) is dual. Since $\leq_{\mathcal{L}} \cap \rho \subseteq \mathcal{J}$, we have

$$\leq_{\mathcal{L}} \cap \rho = (\leq_{\mathcal{L}} \cap \rho) \cap \mathcal{J} = (\leq_{\mathcal{L}} \cap \mathcal{J}) \cap \rho = \mathcal{L} \cap \rho,$$

where for the final equality we use the assumption that S is left stable. Thus, by (2), we have $H_{\mathcal{L}}(S) = H_{\mathcal{L}}(S/\rho)$. $\square$

It is possible for the $\mathcal{H}$-height of a semigroup S to be less than the $\mathcal{H}$-height of a quotient S/ρ of S, even when $\leq_{\mathcal{H}} \cap \rho = \mathcal{H} \cap \rho$, as demonstrated by the following example.

Example 5.2. Consider the semigroup S from Example 4.6. Recall that S is the Rees quotient of the free semigroup F on an arbitrary infinite set X by the ideal I of all words with multiple appearances of the same letter, and that $H_{\mathcal{H}}(S) = 2$.

For $w \in S\backslash\{0\}$, let $C(w)$ denote the set of elements of X appearing in w. We define a relation ρ on S by $u\,\rho\,v$ if and only if either $u = v = 0$ or $u, v \in F\backslash I$ with $C(u) = C(v)$. It is straightforward to prove that ρ is a congruence. In fact, ρ is the smallest congruence on S such that the resulting quotient is commutative. Since $u <_{\mathcal{H}} v$ if and only if $u = 0$, and $[0]_\rho = \{0\}$, it follows that $\leq_{\mathcal{H}} \cap\, \rho = \mathcal{H} \cap \rho$. It is elementary that, for $u, v \in F\backslash I$, we have $[u]_\rho \geq_{\mathcal{H}} [v]_\rho$ if and only if $C(u) \subseteq C(v)$. Thus, choosing distinct $x_i \in X$ ($i \in \mathbb{N}$), we have an $\mathcal{H}$-chain $([x_1]_\rho, [x_1x_2]_\rho, [x_1x_2x_3]_\rho, \dots)$, and hence S/ρ has infinite $\mathcal{H}$-height.

In the remainder of this section we focus on Rees quotients, which were introduced in Section 4. Although we defined Rees quotients without reference to congruences, they do in fact arise from congruences. Specifically, for an ideal I of S, the Rees quotient S/I is isomorphic to the quotient of S by the *Rees congruence* $\rho_I = \{(a, a) : a \in S\backslash I\} \cup (I \times I)$.

Proposition 5.3. *Let S be a semigroup and let $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$.*

(1) *For any ideal I of S, we have $H_{\mathcal{K}}(S) \geq H_{\mathcal{K}}(S/I)$.*
(2) *If S has minimal $\mathcal{K}$-classes, and J is the minimal ideal of S, then $H_{\mathcal{K}}(S) = H_{\mathcal{K}}(S/J)$.*
(3) *If S has a completely simple minimal ideal J, then $H_{\mathcal{K}}(S) = H_{\mathcal{K}}(S/J)$.*

Proof. (1) This is an immediate corollary of Proposition 5.1(1).

(2) Recalling that J is the union of the minimal $\mathcal{K}$-classes, it is straightforward to show that $\leq_{\mathcal{K}} \cap\, \rho_J = \mathcal{K} \cap \rho_J$. Hence, using Proposition 5.1(2) and the fact that $S/\rho_J \cong S/J$, we have $H_{\mathcal{K}}(S) = H_{\mathcal{K}}(S/J)$.

(3) This follows from (2), upon recalling that a completely simple minimal ideal is the union of all the minimal $\mathcal{K}$-classes. □

For $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}\}$, and for an ideal I of S, we define $H^S_{\mathcal{K}}(I)$ as follows. If there is a finite bound on the sizes of chains of $\mathcal{K}$-classes of S contained in I, then $H^S_{\mathcal{K}}(I)$ is the minimum such bound; otherwise $H^S_{\mathcal{K}}(I) = \infty$. Observing that $[b \in I, a \leq_{\mathcal{K}} b \Rightarrow a \in I]$, it is clear that any $\mathcal{K}$-chain of S splits into two subchains by restricting to I and to $S\backslash I$, and the latter subchain is a $\mathcal{K}$-chain of S/I to which 0 can be appended. It follows that

$$H_{\mathcal{K}}(S) \leq H^S_{\mathcal{K}}(I) + H_{\mathcal{K}}(S/I) - 1. \qquad (*)$$

For a semigroup S with zero 0, the *left socle* of S is the union of $\{0\}$ and all the 0-minimal $\mathcal{L}$-classes of S. It turns out that the left socle of S is a two-sided ideal of S; for a proof of this fact and for more information about the left socle, see [5, Section 6.3].

Theorem 5.4. *Let S be a semigroup with zero, and let I be the left socle of S.*

(1) *$H_{\mathcal{L}}(S)$ is finite if and only if $H_{\mathcal{L}}(S/I)$ is finite, in which case $H_{\mathcal{L}}(S) = H_{\mathcal{L}}(S/I) + 1$.*
(2) *If S is right stable, then $H_{\mathcal{R}}(S)$ is finite if and only if $H_{\mathcal{R}}(S/I)$ is finite, in which case $H_{\mathcal{R}}(S) \leq 2H_{\mathcal{R}}(S/I) + 1$.*

(3) *If S is right stable, then $H_{\mathcal{J}}(S)$ is finite if and only if $H_{\mathcal{J}}(S/I)$ is finite, in which case*

$$H_{\mathcal{J}}(S) \leq H_{\mathcal{R}}(S/I) + H_{\mathcal{J}}(S/I) + 1 \leq 2H_{\mathcal{J}}(S/I) + 1.$$

Proof. We may unambiguously let 0 denote both the zero of S and of S/I.

(1) For any $\mathcal{L}$-chain $(a_1, \dots, a_n, 0)$ in S, we have $a_i \in S \backslash I$ for each $i \in \{1, \dots, n-1\}$ (since $a_i >_{\mathcal{L}} a_n \neq 0$), and hence there is an $\mathcal{L}$-chain $(a_1, \dots, a_{n-1}, 0)$ in S/I. Thus $H_{\mathcal{L}}(S) \leq H_{\mathcal{L}}(S/I) + 1$.

Now consider an $\mathcal{L}$-chain $(b_1, \dots, b_n, 0)$ in S/I. Then $b_n \notin I$, so, by definition, there exists some $c \in S\backslash\{0\}$ with $b_n >_{\mathcal{L}} c$. Hence, we have an $\mathcal{L}$-chain $(b_1, \dots, b_n, c, 0)$ in S. Thus $H_{\mathcal{L}}(S) \geq H_{\mathcal{L}}(S/I) + 1$, and hence $H_{\mathcal{L}}(S) = H_{\mathcal{L}}(S/I) + 1$.

(2) By Proposition 5.3(1), if $H_{\mathcal{R}}(S)$ is finite then so is $H_{\mathcal{R}}(S/I)$. Suppose then that $H_{\mathcal{R}}(S/I)$ is finite. We shall prove that $H^S_{\mathcal{R}}(I) \leq H_{\mathcal{R}}(S/I) + 2$. Then, using (∗), we have

$$H_{\mathcal{R}}(S) \leq (H_{\mathcal{R}}(S/I) + 2) + H_{\mathcal{R}}(S/I) - 1 = 2H_{\mathcal{R}}(S/I) + 1.$$

Assume for a contradiction that $H^S_{\mathcal{R}}(I) > H_{\mathcal{R}}(S/I) + 2$. Then, with $n = H_{\mathcal{R}}(S/I) + 1$ (≥ 2), there is an $\mathcal{R}$-chain $(a_0, a_1, \dots, a_n, 0)$ of S where $a_i \in I$ for all $i \in \{0, \dots, n\}$. Let $s_i \in S$ be such that $a_i = a_{i-1}s_i$ ($1 \leq i \leq n$). For $i \in \{2, \dots, n\}$, let $t_i = s_2 \dots s_i$, so that $a_i = a_1 t_i$. Note that $t_n \neq 0$, as $a_n \neq 0$.

Suppose first that $t_n \in I$, i.e. t_n belongs to a 0-minimal $\mathcal{L}$-class of S. Since $a_n = a_1 t_n \leq_{\mathcal{L}} t_n$, it follows that $a_n \,\mathcal{L}\, t_n$. Thus, there exists $x \in S^1$ such that $t_n = x a_n$, and hence

$$t_n = x a_1 t_n = x a_1 x a_n = x a_1 x a_0 s_1 t_n.$$

It follows that $a_0 \geq_{\mathcal{L}} a_1 x a_0 \neq 0$. Since $a_0 \in I$, we have $a_0 \,\mathcal{L}\, a_1 x a_0$, which, together with $a_0 >_{\mathcal{R}} a_1$, implies that $a_0 \,\mathcal{J}\, a_1$. But this contradicts the fact that S is right stable.

Now suppose that $t_n \in S \backslash I$. Then we have a chain

$$t_2 \geq_{\mathcal{R}} t_3 \geq_{\mathcal{R}} \dots \geq_{\mathcal{R}} t_n >_{\mathcal{R}} 0$$

in S/I. It cannot be the case that all the inequalities in this chain are strict, for then there would be an $\mathcal{R}$-chain of length n in S/I, contradicting the fact that $H_{\mathcal{R}}(S/I) = n - 1$. Thus, there exists $i \in \{2, \dots, n-1\}$ such that $t_i \,\mathcal{R}\, t_{i+1}$. But then, using the fact that $\mathcal{R}$ is a left congruence on S, we have $a_i = a_1 t_i \,\mathcal{R}\, a_1 t_{i+1} = a_{i+1}$, contradicting the fact that $(a_0, \dots, a_n, 0)$ is an $\mathcal{R}$-chain. This completes the proof.

(3) By Proposition 5.3(1), if $H_{\mathcal{J}}(S)$ is finite then so is $H_{\mathcal{J}}(S/I)$. Suppose then that $H_{\mathcal{J}}(S/I)$ is finite. By assumption S is right stable, which implies that S/I is right stable. Therefore, by Proposition 3.5(2), we have $H_{\mathcal{R}}(S/I) \leq H_{\mathcal{J}}(S/I)$. The second inequality in the statement immediately follows. To prove the first inequality, we show that $H^S_{\mathcal{J}}(I) \leq H_{\mathcal{R}}(S/I) + 2$, and the inequality then follows from (∗).

Consider a $\mathcal{J}$-chain $(a_1, \ldots, a_n, 0)$ of S where $a_i \in I$ for all $i \in \{1, \ldots, n\}$. We need to show that $n \leq H_{\mathcal{R}}(S/I) + 1$. For each $i \in \{1, \ldots, n-1\}$, there exist $s_i, t_i \in S^1$ such that $a_{i+1} = s_i a_i t_i$, so $a_{i+1} = s_i \ldots s_1 a_1 t_1 \ldots t_i$. For $i \in \{0, \ldots, n-1\}$, let $b_{i+1} = s_i \ldots s_1 a_1$ and $c_{i+1} = a_1 t_1 \ldots t_i$, interpreting $b_1 = c_1 = a_1$. Then $b_i, c_i \in I \backslash \{0\}$, and we have

$$b_1 \geq_{\mathcal{L}} \cdots \geq_{\mathcal{L}} b_n >_{\mathcal{L}} 0 \qquad \text{and} \qquad c_1 \geq_{\mathcal{R}} \cdots \geq_{\mathcal{R}} c_n >_{\mathcal{R}} 0.$$

Since the elements of $I \backslash \{0\}$ belong to 0-minimal $\mathcal{L}$-classes, it follows that all the b_i are $\mathcal{L}$-related. Now, aiming for a contradiction, suppose that $c_i \,\mathcal{R}\, c_{i+1}$ for some $i \in \{1, \ldots, n-1\}$. Then there exist $x, y \in S^1$ such that $b_i = x b_{i+1}$ and $c_i = c_{i+1} y$. But then

$$\begin{aligned} a_i &= s_{i-1} \ldots s_1 a_1 t_1 \ldots t_{i-1} = b_i t_1 \ldots t_{i-1} = x b_{i+1} t_1 \ldots t_{i-1} = x s_i \ldots s_1 a_1 t_1 \ldots t_{i-1} \\ &= x s_i \ldots s_1 c_i = x s_i \ldots s_1 c_{i+1} y = x s_i \ldots s_1 a_1 t_1 \ldots t_i y = x a_{i+1} y, \end{aligned}$$

contradicting the fact that $a_i >_{\mathcal{J}} a_{i+1}$. We conclude that $(c_1, \ldots, c_n, 0)$ is an $\mathcal{R}$-chain of S contained in I. We have already established, in the proof of (2), that $H^S_{\mathcal{R}}(I) \leq H_{\mathcal{R}}(S/I) + 2$, so we conclude that $n \leq H_{\mathcal{R}}(S/I) + 1$, as required. □

We note that one can of course dually define the *right socle* of a semigroup with zero, and Theorem 5.4 has an obvious counterpart in terms of the right socle.

To conclude this section we shall prove that the upper bounds established in parts (2) and (3) of Theorem 5.4 are sharp. To this end, we introduce a construction in the form of a specific ideal extension of a null semigroup. (A semigroup N is *null* if it contains an element 0 such that $N^2 = \{0\}$.)

Construction 5.5. Let S be a semigroup with zero z. Let $\{x_s : s \in S^1\}$ be a set disjoint from S in one-to-one correspondence with S^1, and let $\mathcal{U}(S) = S \cup \{x_s : s \in S^1\}$. Define a multiplication on $\mathcal{U}(S)$, extending that on S, by

$$a x_s = x_s, \quad x_s a = x_{sa} \quad \text{and} \quad x_s x_t = x_z$$

for all $a \in S$ and $s, t \in S^1$. It is straightforward to show that $\mathcal{U}(S)$ is a semigroup with zero x_z.

We focus on this construction for the rest of the section. We first prove some elementary facts about the structure of $\mathcal{U}(S)$, and then describe its $\mathcal{L}$- and $\mathcal{R}$-heights in terms of those of S.

Lemma 5.6. *Let S be a semigroup with zero z, and let $U = \mathcal{U}(S)$.*

(1) *S is a subsemigroup of U, and $\{x_s : s \in S^1\}$ is a null semigroup and an ideal of U.*
(2) *The left socle of U is $I = \{x_s : s \in S^1\} \cup \{z\}$, and $U/I \cong S$.*
(3) *If S is right stable then so is U.*

Proof. (1) This follows immediately from the definition of the multiplication in U.

(2) For each $u \in I$ we have $U^1u = \{u, x_z\}$, and hence I is contained in the left socle of U. For $v \notin I$, we have $v \in S\backslash\{z\}$, and hence $v >_{\mathcal{L}} z >_{\mathcal{L}} x_z$ in U, so that v is not in the left socle of U. Thus I is the left socle of U.

Letting 0 denote the zero of U/I, it is straightforward to see that there is an isomorphism $U/I \to S$ given by $s \mapsto s$ ($s \in S\backslash\{z\}$) and $0 \mapsto z$.

(3) It is easy to show that the relations $\mathcal{R}$ and $\mathcal{J}$ on U restricted to S coincide with the corresponding relations on S. Moreover, $\{x_1\}$ is a singleton $\mathcal{J}$-class of U, and for $a, b \in S$ we have $x_a \leq_{\mathcal{J}} x_b$ if and only if $x_a \leq_{\mathcal{R}} x_b$ if and only if $a \leq_{\mathcal{R}} b$. Consequently, if S is right stable then so is U. □

Proposition 5.7. *Let S be a semigroup with zero z, and let $U = \mathcal{U}(S)$. Then:*

(1) *$H_{\mathcal{L}}(U)$ is finite if and only if $H_{\mathcal{L}}(S)$ is finite, in which case $H_{\mathcal{L}}(U) = H_{\mathcal{L}}(S) + 1$.*
(2) *$H_{\mathcal{R}}(U)$ is finite if and only if $H_{\mathcal{R}}(S)$ is finite, in which case $H_{\mathcal{R}}(U) = 2H_{\mathcal{R}}(S) + 1$.*

Proof. (1) This follows from Theorem 5.4(1) and Lemma 5.6(2).

(2) Assume that $H_{\mathcal{R}}(S) = n \in \mathbb{N}$. Then S is right stable by Lemma 3.3, and hence U is right stable by Lemma 5.6(3). Hence, by Theorem 5.4(2) and Lemma 5.6(2), we have $H_{\mathcal{R}}(U) \leq 2n + 1$.

Now, there exists an $\mathcal{R}$-chain $(a_1, \dots, a_{n-1}, a_n = z)$ in S. Since $U\backslash S$ is an ideal, it follows that $(a_1, \dots, a_{n-1}, z)$ is an $\mathcal{R}$-chain of U, and that $z >_{\mathcal{R}} zx_1 = x_1$ in U. Write $x_{a_i} = x_{i+1}$ for each $i \in \{1, \dots, n\}$, then $x_1a_1 = x_2$, and clearly $x_1 \notin a_1U^1$, so $x_1 >_{\mathcal{R}} x_2$. For each $i \in \{2, \dots, n\}$ there exists $t_i \in S$ such that $a_{i-1}t_i = a_i$, so $x_it_i = x_{i+1}$ and hence $x_i \geq_{\mathcal{R}} x_{i+1}$. Suppose that $x_i \,\mathcal{R}\, x_{i+1}$. Then $x_i = x_{i+1}u$ for some $u \in U$. Since $x_{i+1}x_s = x_z$ for each $s \in S^1$, we have $u \in S$. But then $a_{i-1} = a_iu$, contradicting the fact that $a_{i-1} >_{\mathcal{R}} a_i$. Thus $x_i >_{\mathcal{R}} x_{i+1}$. We conclude that there is an $\mathcal{R}$-chain

$$(a_1, \dots, a_n, x_1, x_2 \dots, x_{n+1})$$

in U, so that $H_{\mathcal{R}}(U) \geq 2n + 1$. Thus $H_{\mathcal{R}}(U) = 2n + 1$. □

We may now quickly deduce that the bounds given in parts (2) and (3) of Theorem 5.4 can be attained. Indeed, take any semigroup S with zero such that $H_{\mathcal{R}}(S) = H_{\mathcal{J}}(S) < \infty$, let $U = \mathcal{U}(S)$, and let I be the left socle of U. Then $S \cong U/I$ by Lemma 5.6. Thus, using Propositions 5.7(2), 3.5(2) and Theorem 5.4(3), we have

$$2H_{\mathcal{J}}(U/I) + 1 = 2H_{\mathcal{R}}(U/I) + 1 = H_{\mathcal{R}}(U) \leq H_{\mathcal{J}}(U) \leq 2H_{\mathcal{J}}(U/I) + 1,$$

implying that $H_{\mathcal{R}}(U) = H_{\mathcal{J}}(U) = 2H_{\mathcal{R}}(U/I) + 1 = 2H_{\mathcal{J}}(U/I) + 1$. See Fig. 1 for an illustration.

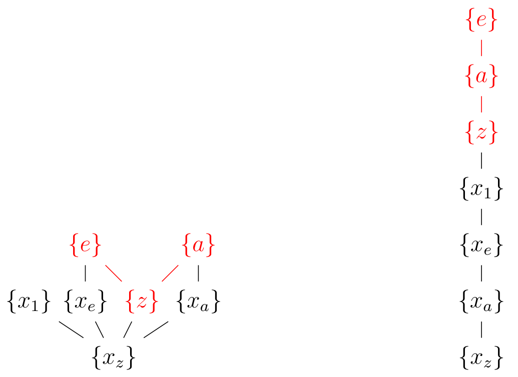

Fig. 1. Let $S = \{e, a, z\}$ be the semigroup with multiplication given by $e^2 = e$, $ea = a$ and $ae = sz = zs = z$ ($s \in S$). (Then $S \cong \mathcal{U}(\{e\})$.) The poset of $\mathcal{L}$-classes of $\mathcal{U}(S)$ is displayed on the left. The poset of $\mathcal{R}$-classes of $\mathcal{U}(S)$ coincides with the poset of $\mathcal{J}$-classes, and is displayed on the right. These contain, respectively, the posets of $\mathcal{L}$-classes of S and of $\mathcal{R}(=\mathcal{J})$-classes of S, which are displayed in red. (For interpretation of the colours in the figure, the reader is referred to the web version of this article.)

6. General bounds on $\mathcal{K}$-heights

This section contains the main results of the article. The first main result establishes upper and lower bounds on the $\mathcal{R}$-height of a stable semigroup with finite $\mathcal{L}$-height.

Theorem 6.1. *Let S be a semigroup. If $H_{\mathcal{L}}(S) = n < \infty$, then S is stable if and only if $H_{\mathcal{R}}(S) < \infty$, in which case*

$$\lceil \log_2(n+1) \rceil \leq H_{\mathcal{R}}(S) \leq 2^n - 1.$$

Proof. That finite $H_{\mathcal{R}}(S)$ (together with finite $H_{\mathcal{L}}(S)$) implies stability is precisely the statement of Lemma 3.3. Therefore, it suffices to prove that for a stable semigroup the bounds stated for $H_{\mathcal{R}}(S)$ hold. We prove that $H_{\mathcal{R}}(S) \leq 2^n - 1$ by induction. A dual argument then proves that $n = H_{\mathcal{L}}(S) \leq 2^m - 1$, where $m = H_{\mathcal{R}}(S)$, which yields the lower bound in the statement.

Suppose then that S is stable. If $n = 1$, then $H_{\mathcal{R}}(S) = 1 \,(= 2^1 - 1)$ by Proposition 4.2. Now take $n > 1$. By Lemma 3.2, S has a completely simple minimal ideal, say J. By Proposition 5.3(3), we may assume that $S = S/J$; that is, S has a zero. Let I denote the left socle of S. Then $H_{\mathcal{L}}(S/I) = n - 1$ by Theorem 5.4(1). By the inductive hypothesis we have $H_{\mathcal{R}}(S/I) \leq 2^{n-1} - 1$. Then, using Theorem 5.4(2), we have

$$H_{\mathcal{R}}(S) \leq 2H_{\mathcal{R}}(S/I) + 1 \leq 2(2^{n-1} - 1) + 1 = 2^n - 1,$$

completing the proof of the inductive step and hence of the theorem. □

Recall that it is possible for a semigroup to have finite $\mathcal{L}$-height but infinite $\mathcal{J}$-height (e.g. the dual of Example 4.5). However, a stable semigroup with finite $\mathcal{L}$-height *does* have finite $\mathcal{J}$-height. In fact, in this case the $\mathcal{J}$-height has the same upper bound as that for the $\mathcal{R}$-height (given in Theorem 6.1).

Theorem 6.2. *Let S be a stable semigroup. Then $H_{\mathcal{L}}(S)$ is finite if only if $H_{\mathcal{J}}(S)$ is finite. Moreover, if $H_{\mathcal{L}}(S) = n < \infty$ then*

$$n \leq H_{\mathcal{J}}(S) \leq 2^n - 1.$$

Proof. By Proposition 3.5, we have $H_{\mathcal{L}}(S) \leq H_{\mathcal{J}}(S)$. So, it suffices to prove the upper bound in the statement. The proof of this is essentially the same as that of Theorem 6.1: simply replace $\mathcal{R}$ with $\mathcal{J}$, and invoke Theorem 5.4(3) rather than Theorem 5.4(2). □

Using the construction from Section 5, we now show that all the possible $\mathcal{R}$- and $\mathcal{J}$-heights according to Theorems 6.1 and 6.2 are in fact attainable. See Table 1 for such values when $H_{\mathcal{L}}(S) = 1, \dots, 8$.

Theorem 6.3. *Let $n \in \mathbb{N}$. For every $m \in \{n, \dots, 2^n - 1\}$, there exists a (necessarily $\mathcal{J}$-trivial) semigroup S such that $H_{\mathcal{L}}(S) = n$ and $H_{\mathcal{R}}(S) = H_{\mathcal{J}}(S) = |S| = m$.*

Proof. We prove the result by induction. Note that a finite semigroup S is $\mathcal{J}$-trivial if and only if $H_{\mathcal{J}}(S) = |S|$. For $n = 1$ (in which case $n = 2^n - 1$), we take S to be the trivial semigroup, which clearly has the desired properties.

Now let $n \geq 2$. First consider $m \in \{n, \dots, 2^{n-1}\}$. By the inductive hypothesis, there exists a semigroup S such that $H_{\mathcal{L}}(S) = n - 1$ and $H_{\mathcal{R}}(S) = H_{\mathcal{J}}(S) = |S| = m - 1$. Then $H_{\mathcal{L}}(S^1) = n$ and $H_{\mathcal{R}}(S^1) = H_{\mathcal{J}}(S^1) = |S^1| = m$, as required.

Now consider $m \in \{2^{n-1} + 1, \dots, 2^n - 1\}$. By the inductive hypothesis, there exists a semigroup S such that $H_{\mathcal{L}}(S) = n-1$ and $H_{\mathcal{R}}(S) = H_{\mathcal{J}}(S) = |S| = 2^{n-1}-1$. Being finite and $\mathcal{J}$-trivial, the semigroup S has a zero, say z. Let $U = \mathcal{U}(S)$. Clearly $|U| = 2|S|+1 = 2^n - 1$. By Proposition 5.7, we have $H_{\mathcal{L}}(U) = n$ and $H_{\mathcal{R}}(U) = H_{\mathcal{J}}(U) = 2^n - 1$. In fact, following the proof of Proposition 5.7, and using the fact that $H_{\mathcal{R}}(U) = |U|$, we may write $U = \{a_1, \dots, a_{2^n-1}\}$, where $S = \{a_1, \dots, a_{2^{n-1}-1}(= z)\}$, such that $(a_1, \dots, a_{2^n-1})$ is an $\mathcal{R}(= \mathcal{J})$-chain of U. By Lemma 5.6, the left socle of U is

$$(U \backslash S) \cup \{z\} = \{a_{2^{n-1}-1}, \dots, a_{2^n-1}\}.$$

Let $I = \{a_m, \dots, a_{2^n-1}\}$. Then I is an ideal of U (since it consists of all elements less than or equal to a_m under the $\mathcal{J}$-order). Let T denote the Rees quotient

$$U/I = \{a_1, \dots, a_{m-1}, 0\},$$

and observe that $S \subseteq T$. It is immediate that for $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$ and $x, y \in T\backslash\{0\}$, we have $x >_{\mathcal{K}} y$ in T if and only if $x >_{\mathcal{K}} y$ in U. It follows that $H_{\mathcal{L}}(T) = n$ and $H_{\mathcal{R}}(T) = H_{\mathcal{J}}(T) = |T| = m$. This completes the proof. □

Table 1
For some small natural numbers n, the range of possible values of $H_{\mathcal{R}}(S)$ and of $H_{\mathcal{J}}(S)$ for a stable semigroup S with $H_{\mathcal{L}}(S) = n$.

$H_{\mathcal{L}}(S)$	Possible values for $m = H_{\mathcal{R}}(S)$	Possible values for $p = H_{\mathcal{J}}(S)$
1	$m = 1$	$p = 1$
2	$m = 2, 3$	$p = 2, 3$
3	$2 \leq m \leq 7$	$3 \leq p \leq 7$
4	$3 \leq m \leq 15$	$4 \leq p \leq 15$
5	$3 \leq m \leq 31$	$5 \leq p \leq 31$
6	$3 \leq m \leq 63$	$6 \leq p \leq 63$
7	$3 \leq m \leq 127$	$7 \leq p \leq 127$
8	$4 \leq m \leq 255$	$8 \leq p \leq 255$

Corollary 6.4. *Let $n \in \mathbb{N}$. For every $m \in \{\lceil \log_2(n+1) \rceil, \ldots, 2^n - 1\}$, there exists a semigroup S such that $H_{\mathcal{L}}(S) = n$, $H_{\mathcal{R}}(S) = m$ and $H_{\mathcal{J}}(S) = |S| = \max(m, n)$.*

Proof. If $m \geq n$ then we apply Theorem 6.3. If $m < n \, (\leq 2^m - 1)$, then we apply the dual of Theorem 6.3 to obtain a semigroup S such that $H_{\mathcal{R}}(S) = m$ and $H_{\mathcal{L}}(S) = H_{\mathcal{J}}(S) = |S| = n$. □

Recall Proposition 4.2, which provides several equivalent conditions for a semigroup S to have $H_{\mathcal{L}}(S) = H_{\mathcal{R}}(S) = 1$, and Proposition 4.4, which gives equivalent conditions for S to have $H_{\mathcal{L}}(S) = H_{\mathcal{R}}(S) = 2$. Our next result is an analogue, describing when the $\mathcal{L}$- and $\mathcal{R}$-heights of a semigroup are both finite.

Theorem 6.5. *For a semigroup S, the following are equivalent:*

(1) *$H_{\mathcal{L}}(S)$ and $H_{\mathcal{R}}(S)$ are finite;*
(2) *$H_{\mathcal{L}}(S)$ and $H_{\mathcal{H}}(S)$ are finite;*
(3) *$H_{\mathcal{L}}(S)$ is finite and S is stable;*
(4) *$H_{\mathcal{J}}(S)$ and $H_{\mathcal{H}}(S)$ are finite;*
(5) *$H_{\mathcal{J}}(S)$ is finite and S is stable;*
(6) *$H_{\mathcal{R}}(S)$ and $H_{\mathcal{H}}(S)$ are finite;*
(7) *$H_{\mathcal{R}}(S)$ is finite and S is stable;*

Proof. If (1) holds, then S is stable by Lemma 3.3, and hence $H_{\mathcal{H}}(S)$ is finite by Proposition 3.5(3), so (2) holds. Each of (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (7) follows from Lemma 3.4. That (3) $\Rightarrow$ (4) follows from Theorem 6.2 and Proposition 3.5(3), and that (5) $\Rightarrow$ (6) follows from Proposition 3.5(3). Finally, (7) $\Rightarrow$ (1) follows from Proposition 3.5(3) and the dual of Theorem 6.1. □

An important aspect of Theorem 6.5 is:

Corollary 6.6. *If S is a stable semigroup, then*

$$H_{\mathcal{L}}(S) \text{ is finite} \Leftrightarrow H_{\mathcal{R}}(S) \text{ is finite} \Leftrightarrow H_{\mathcal{J}}(S) \text{ is finite},$$

in which case $H_{\mathcal{H}}(S)$ is finite.

Recall that for a stable semigroup S we have $H_{\mathcal{L}}(S) = 1$ if and only if $H_{\mathcal{R}}(S) = 1$ if and only if $H_{\mathcal{J}}(S) = 1$. The next result establishes lower and upper bounds on the $\mathcal{J}$-height in terms of finite (but greater than 1) $\mathcal{L}$- and $\mathcal{R}$-heights.

Theorem 6.7. *Let S be a semigroup such that $2 \le H_{\mathcal{L}}(S) < \infty$ and $2 \le H_{\mathcal{R}}(S) < \infty$. Then, letting* $\min\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big) = n$, *we have*

$$\max\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big) \le H_{\mathcal{J}}(S) \le \min(2^n - 1, H_{\mathcal{L}}(S) + H_{\mathcal{R}}(S) - 2).$$

Proof. By Lemma 3.3, the semigroup S is stable. Thus, by Proposition 3.5(3), the lower bound in the statement holds. The inequality $H_{\mathcal{J}}(S) \le 2^n - 1$ follows from Theorem 6.2 and its dual (which bounds the $\mathcal{J}$-height in terms of the $\mathcal{R}$-height). It remains to show that $H_{\mathcal{J}}(S) \le H_{\mathcal{L}}(S) + H_{\mathcal{R}}(S) - 2$.

By Lemma 3.2, S has a completely simple minimal ideal, which, by Proposition 5.3(3), we may assume is trivial, i.e. S has a zero 0. Consider a $\mathcal{J}$-chain $(a_1, \dots, a_n, 0)$ in S. We need to show that $n \le H_{\mathcal{L}}(S) + H_{\mathcal{R}}(S) - 3$. For each $i \in \{1, \dots, n-1\}$, there exist $s_i, t_i \in S^1$ such that $a_{i+1} = s_i a_i t_i$, so that $a_{i+1} = s_i \dots s_1 a_1 t_1 \dots t_i$. For $i \in \{0, \dots, n-1\}$, let $b_{i+1} = s_i \dots s_1 a_1$ and $c_{i+1} = a_1 t_1 \dots t_i$, interpreting $b_1 = c_1 = a_1$. We then have chains

$$b_1 \ge_{\mathcal{L}} \cdots \ge_{\mathcal{L}} b_n >_{\mathcal{L}} 0 \qquad \text{and} \qquad c_1 \ge_{\mathcal{R}} \cdots \ge_{\mathcal{R}} c_n >_{\mathcal{R}} 0.$$

Now, for each $i \in \{1, \dots, n-1\}$, either $b_i >_{\mathcal{L}} b_{i+1}$ or $c_i >_{\mathcal{R}} c_{i+1}$. Indeed, if we had $b_i \,\mathcal{L}\, b_{i+1}$ and $c_i \,\mathcal{R}\, c_{i+1}$ for some $i \in \{1, \dots, n-1\}$, then, as in the proof of Theorem 5.4(3), we would have $a_i \,\mathcal{J}\, a_{i+1}$, a contradiction. It follows that there exist $k, l \le n$ such that $n \le k + l - 1$, there is an $\mathcal{L}$-chain $(b_{i_1}, \dots, b_{i_k}, 0)$ where $1 = i_1 < \cdots < i_k \le n$, and there is an $\mathcal{R}$-chain $(c_{j_1}, \dots, c_{j_l}, 0)$ where $1 = j_1 < \cdots < j_l \le n$. We must then have $k \le H_{\mathcal{L}}(S) - 1$ and $l \le H_{\mathcal{R}}(S) - 1$, whence

$$n \le k + l - 1 \le H_{\mathcal{L}}(S) + H_{\mathcal{R}}(S) - 3,$$

as required. □

It is an open question as to whether the upper bound on $H_{\mathcal{J}}(S)$ given in Theorem 6.7 can be attained for every possible combination of $H_{\mathcal{L}}(S)$ and $H_{\mathcal{R}}(S)$ (as determined by Theorem 6.1 and its dual). However, it is asymptotically sharp:

Theorem 6.8. *For each $n \in \mathbb{N}$ there exists a semigroup U_n such that $H_{\mathcal{L}}(U_n) = H_{\mathcal{R}}(U_n) = 2^n + n - 3$ and $H_{\mathcal{J}}(U_n) = 2^{n+1} - 4$. Moreover, we have*

$$\lim_{n\to\infty} \frac{H_{\mathcal{J}}(U_n)}{H_{\mathcal{L}}(U_n) + H_{\mathcal{R}}(U_n) - 2} = 1.$$

Proof. Recall that a *left identity* of a semigroup S is an element $e \in S$ such that $es = s$ for all $s \in S$. Right identities are defined dually. Recalling Construction 5.5, observe that if e is a left identity of S then it is also a left identity of $\mathcal{U}(S)$.

Let $n \in \mathbb{N}$. By the above observation and the proof of Theorem 6.3, there exists a semigroup S with a left identity e such that $H_{\mathcal{L}}(S) = n$ and $H_{\mathcal{R}}(S) = H_{\mathcal{J}}(S) = |S| = 2^n - 1$. Dually, there exists a semigroup T with a right identity f such that $H_{\mathcal{R}}(T) = n$ and $H_{\mathcal{L}}(T) = H_{\mathcal{J}}(T) = |T| = 2^n - 1$. Let 0_S and 0_T denote the zeros of S and T, respectively, and let $I = (S \times \{0_T\}) \cup (\{0_S\} \times T)$. Clearly I is an ideal of the direct product $S \times T$. Now let $U (= U_n) = (S \times T)/I$.

Consider a $\mathcal{K}$-chain $(u_1, \ldots, u_m, 0)$ in U, where $u_i = (x_i, y_i)$. Then, for each $i \in \{1, \ldots, m-1\}$, either $x_i >_{\mathcal{K}} x_{i+1}$ in S or $y_i >_{\mathcal{K}} y_{i+1}$ in T. It follows that there exist $k, l \leq m$ such that $m \leq k+l-1$, there is an $\mathcal{K}$-chain $(x_{i_1}, \ldots, x_{i_k}, 0)$ where $1 = i_1 < \cdots < i_k \leq m$, and there is an $\mathcal{K}$-chain $(y_{j_1}, \ldots, y_{j_l}, 0)$ where $1 = j_1 < \cdots < j_l \leq m$. We must then have $k \leq H_{\mathcal{K}}(S)-1$ and $l \leq H_{\mathcal{K}}(T)-1$, whence $m \leq k+l-1 \leq H_{\mathcal{K}}(S)+H_{\mathcal{K}}(T)-3$. It follows that $H_{\mathcal{K}}(U) \leq H_{\mathcal{K}}(S) + H_{\mathcal{K}}(T) - 2$. Thus, we have $H_{\mathcal{L}}(U) \leq 2^n + n - 3$, $H_{\mathcal{R}}(U) \leq 2^n + n - 3$ and $H_{\mathcal{J}}(U) \leq 2^{n+1} - 4$.

Note that, for any $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$ and $(a, b), (c, d) \in U \backslash \{0\}$, if $(a, b) \,\mathcal{K}\, (c, d)$ then $a \,\mathcal{K}\, c$ in S and $b \,\mathcal{K}\, d$ in T.

Now, let $(a_1, \ldots, a_{n-1}, 0_S)$ be a maximal $\mathcal{L}$-chain in S, and let $(b_1, \ldots, b_{2^n-2}, 0_T)$ be *the* maximum $\mathcal{L}$-chain in T. Observe that $b_1 = f$. For $i \in \{1, \ldots, n-2\}$ let $s_i \in S$ be such that $a_{i+1} = s_i a_i$, and for $j \in \{1, \ldots, 2^n-3\}$ let $t_j \in T$ be such that $b_{j+1} = t_j b_j$. Then, for each such i and j, we have $(a_{i+1}, f) = (s_i, f)(a_i, f)$ and $(a_{n-1}, b_{j+1}) = (e, t_j)(a_{n-1}, b_j)$. It follows that

$$(a_1, f) >_{\mathcal{L}} \cdots >_{\mathcal{L}} (a_{n-1}, f) = (a_{n-1}, b_1) >_{\mathcal{L}} \cdots >_{\mathcal{L}} (a_{n-1}, b_{2^n-2}) >_{\mathcal{L}} 0,$$

so that $H_{\mathcal{L}}(U) \geq 2^n + n - 3$. Thus $H_{\mathcal{L}}(U) = 2^n + n - 3$. Similarly, we have $H_{\mathcal{R}}(U) = 2^n + n - 3$.

Now let $(c_1, \ldots, c_{2^n-2}, 0_S)$ be the maximum $\mathcal{R}$-chain in S. Letting $s_i' \in S$ $(1 \leq i \leq 2^n - 3)$ be such that $c_{i+1} = c_i s_i'$, we have $(c_{i+1}, f) = (c_i, f)(s_i', f)$. Also, with b_j and t_j as above, we have $(c_{2^n-2}, b_{j+1}) = (e, t_j)(c_{2^n-2}, b_j)$. It follows that

$$(c_1, f) >_{\mathcal{J}} \cdots >_{\mathcal{J}} (c_{2^n-2}, f) = (c_{2^n-2}, b_1) >_{\mathcal{J}} \cdots >_{\mathcal{J}} (c_{2^n-2}, b_{2^n-2}) >_{\mathcal{J}} 0,$$

so that $H_{\mathcal{J}}(U) \geq 2(2^n - 2) = 2^{n+1} - 4$. Thus $H_{\mathcal{J}}(U) = 2^{n+1} - 4$. This completes the proof of the first part of the statement. The second part now follows immediately. □

An immediate consequence of Theorem 6.8 is that the upper bound of Theorem 6.7 is sharp in the case that the $\mathcal{L}$- and $\mathcal{R}$-heights are both 3: $H_{\mathcal{L}}(U_2) = H_{\mathcal{R}}(U_2) = 3$ and $H_{\mathcal{J}}(U_2) = 4\,(= 3+3-2)$. In the proof of Theorem 6.8 for the case $n = 2$, the semigroup S must be isomorphic to $\mathcal{U}(\{e\})$, and T must be anti-isomorphic to $\mathcal{U}(\{e\})$. Writing $S = \{e, x, 0_S\}$ and $T = \{f, y, 0_T\}$, and letting $a = (e, f)$, $b = (e, y)$, $c = (x, f)$ and

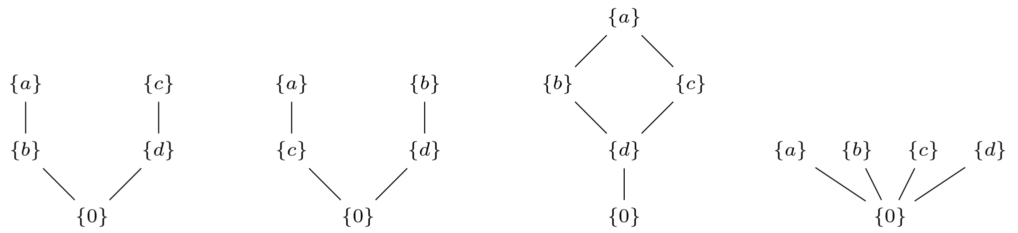

Fig. 2. For the semigroup U_2, the posets of $\mathcal{L}$-classes (left), $\mathcal{R}$-classes (middle left), $\mathcal{J}$-classes (middle right) and $\mathcal{H}$-classes (right).

$d = (x, y)$, we have $U_2 = \{a, b, c, d, 0\}$, and the posets of $\mathcal{L}$-, $\mathcal{R}$-, $\mathcal{J}$- and $\mathcal{H}$-classes of U_2 are as displayed in Fig. 2. We note that, since $H_{\mathcal{H}}(U_2) = 2$, both the inequalities in Proposition 3.5(3) can be strict.

7. Semisimple and regular semigroups

A crucial reason for the disparities between certain $\mathcal{K}$-heights of the semigroups of Example 4.6 and Theorems 6.3 and 6.8 is the presence of $\mathcal{J}$-classes J such that $J^2 \cap J = \emptyset$. In this section we shall see that for stable semigroups without such $\mathcal{J}$-classes all the four $\mathcal{K}$-heights coincide.

The *principal factors* of (the $\mathcal{J}$-classes of) a semigroup S are defined as follows. If S has a minimal $\mathcal{J}$-class (i.e. a minimal ideal), then this $\mathcal{J}$-class is defined to be its own principal factor. For any non-minimal $\mathcal{J}$-class J of S, the principal factor of J is the Rees quotient of the subsemigroup S^1JS^1 of S by its ideal $S^1JS^1 \backslash J$. Observe that this principal factor has universe $J \cup \{0\}$, where 0 denotes the zero element.

Recall that the minimal ideal of a semigroup is simple. All other principal factors are either 0-simple or null [4, Lemma 2.39]. A semigroup is *semisimple* if all its principal factors are either simple or 0-simple.

Proposition 7.1. *Let S be a semisimple semigroup. Then*

$$H_{\mathcal{J}}(S) \leq \min\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big).$$

Proof. Consider a $\mathcal{J}$-chain $(a_0, a_1, \dots, a_n)$ in S. Let J_i denote the $\mathcal{J}$-class of a_i ($i \in \{0, \dots, n\}$). Since $J_i = J_i^2 = J_i^3$, and $J_i \subseteq S^1 a_{i-1} S^1$, there exist $x_i, y_i \in J_i$ and $s_i, t_i \in S^1$ such that $a_i = (x_i s_i) a_{i-1} (t_i y_i)$. Put $u_i = x_i s_i$ and $v_i = t_i y_i$. We certainly have $u_i \dots u_1 a_0 \geq_{\mathcal{L}} u_{i+1} \dots u_1 a_0$ for each $i \in \{1, \dots, n-1\}$. Also, for each $i \in \{1, \dots, n\}$, we have

$$x_i \geq_{\mathcal{J}} u_i \dots u_1 a_0 \geq_{\mathcal{J}} u_i \dots u_1 a_0 v_1 \dots v_i = a_i,$$

implying that $u_i \dots u_1 a_0 \in J_i$ (since $x_i, a_i \in J_i$). It follows that

$$a_0 >_{\mathcal{L}} u_1 a_0 >_{\mathcal{L}} u_2 u_1 a_0 >_{\mathcal{L}} \cdots >_{\mathcal{L}} u_n \dots u_1 a_0,$$

which is an $\mathcal{L}$-chain of length $n+1$. We conclude that $H_{\mathcal{L}}(S) \geq H_{\mathcal{J}}(S)$. A similar argument using the v_i proves that $H_{\mathcal{R}}(S) \geq H_{\mathcal{J}}(S)$. □

Given a semigroup S, there is a natural partial order on the set $E = E(S)$ of idempotents of S given by $e \geq f$ if and only if $ef = fe = f$. We denote the height of the poset $(E, \leq)$ by $H_E(S)$.

It is straightforward to show that, for $e, f \in E$, we have

$$e \geq_{\mathcal{L}} f \Leftrightarrow fe = f \qquad \text{and} \qquad e \geq_{\mathcal{R}} f \Leftrightarrow ef = f,$$

from which it follows that $[e \geq f \Leftrightarrow e \geq_{\mathcal{H}} f]$ and $[e > f \Leftrightarrow e >_{\mathcal{L}} f$ and $e >_{\mathcal{R}} f]$. Moreover, we have $[e >_{\mathcal{L}} f \Leftrightarrow e > ef$ and $e \geq_{\mathcal{L}} f]$. Indeed, suppose that $e >_{\mathcal{L}} f$. Then certainly $e \geq_{\mathcal{L}} f$, and we have $(ef)e = e(fe) = ef = e(ef)$, so $e \geq ef$. Since $e \not\leq_{\mathcal{L}} f$, it follows that $e \neq ef$ and hence $e > ef$. Conversely, if $e > ef$ and $e \geq_{\mathcal{L}} f$, then we cannot have $e \,\mathcal{L}\, f$, for that would imply that $e = ef$, so we must have $e >_{\mathcal{L}} f$.

From the above discussion we deduce:

Lemma 7.2. *Let S be a semigroup, let E denote the set of idempotents of S, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *there exists an $\mathcal{L}$-chain of idempotents of length n in S;*
(2) *there exists an $\mathcal{R}$-chain of idempotents of length n in S;*
(3) *there exists an $\mathcal{H}$-chain of idempotents of length n in S;*
(4) *there exists a chain of idempotents of length n in $(E, \leq)$.*

Consequently, we have $H_E(S) \leq \min(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S), H_{\mathcal{H}}(S))$.

A semigroup S is said to be *regular* if for every $a \in S$ there exists some $b \in S$ such that $a = aba$ and $b = bab$; the element b is called an *inverse* of a. A semigroup is *inverse* if each of its elements has a unique inverse.

A semigroup S is regular (resp. inverse) if and only if every $\mathcal{L}$-class and every $\mathcal{R}$-class of S contain at least (resp. exactly) one idempotent [11, Theorem 6]. It follows that for any regular semigroup S we have $H_E(S) \geq \max(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)$. This fact, together with Corollary 3.6, Proposition 7.1 and Lemma 7.2, yields:

Proposition 7.3. *If S is a regular semigroup, then*

$$H_{\mathcal{L}}(S) = H_{\mathcal{R}}(S) = H_{\mathcal{H}}(S) = H_E(S) \geq H_{\mathcal{J}}(S).$$

The following example shows that the inequality in Proposition 7.3 can be strict, even for inverse semigroups.

Example 7.4. The *bicyclic monoid*, denoted by B, is the monoid defined by the presentation $\langle a, b \mid ab = 1 \rangle$. The bicyclic monoid is inverse and simple (so $H_{\mathcal{J}}(B) = 1$), but its set E of idempotents forms an infinite chain $1 > ba > b^2a^2 > \cdots$ [4, Theorem 2.53], so $H_E(B)\,(= H_{\mathcal{L}}(B) = H_{\mathcal{R}}(B) = H_{\mathcal{H}}(B))$ is infinite.

We now collect some equivalent characterisations for a semigroup to be both regular and stable, and deduce that for such a semigroup all the $\mathcal{K}$-heights coincide. A semigroup is said to be *completely semisimple* if each of its principal factors is either completely simple or completely 0-simple.

Proposition 7.5. *For a semigroup S, the following are equivalent:*

(1) *S is regular and stable;*
(2) *S is regular and either left stable or right stable;*
(3) *S is completely semisimple;*
(4) *S is semisimple and stable;*
(5) *S is regular and does not contain a copy of the bicyclic monoid.*

Moreover, if any (and hence all) of the conditions (1)-(5) hold, then

$$H_{\mathcal{L}}(S) = H_{\mathcal{R}}(S) = H_{\mathcal{H}}(S) = H_E(S) = H_{\mathcal{J}}(S).$$

Proof. The final part of the statement follows from Propositions 3.5(3) and 7.3, so we only need to verify that (1)-(5) are equivalent.

First we remark that (2), (3) and (4) are equivalent by [5, Theorems 6.45 and 6.48], and that (1) $\Rightarrow$ (5) follows from [1, Corollary 2.2].

Next we note that (1) $\Rightarrow$ (2) is obvious, and that (3) $\Rightarrow$ (1) follows from the fact that completely (0-)simple semigroups are regular and stable. It remains to show that (5) $\Rightarrow$ (3), so suppose that (5) holds. Then each principal factor of S contains an idempotent (since S is regular) but no copy of the bicyclic monoid, so is either completely simple or completely 0-simple by [4, Theorem 2.54]. Thus (3) holds, and the proof is complete. $\square$

By Lemma 3.3 and Proposition 7.5, we have:

Corollary 7.6. *If S is a regular semigroup with finite $\mathcal{J}$-height, then*

$$H_{\mathcal{L}}(S) = H_{\mathcal{R}}(S) = H_{\mathcal{H}}(S) = H_E(S) = H_{\mathcal{J}}(S) \iff S \textit{ is stable.}$$

Data availability

No data was used for the research described in the article.

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