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Chris Bowman, Stephen Doty, Stuart Martin

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CANONICAL BASES AND NEW APPLICATIONS OF INCREASING AND DECREASING SUBSEQUENCES TO INVARIANT THEORY

CHRIS BOWMAN, STEPHEN DOTY, AND STUART MARTIN

Abstract. In 2012 Raghavan, Samuel, and Subrahmanyam showed that the Kazhdan–Lusztig basis for the Iwahori–Hecke algebra in type A provides a "canonical" basis for the centraliser algebra of the Schur algebra acting on tensor space. In 2022 the second author found a similar result for the centraliser of the partition algebra acting on the same tensor space. Each basis is indexed by permutations. We exploit these bases to show that the linear decomposition of an arbitrary invariant (in either centraliser algebra) depends integrally on its entries, and describe combinatorial rules that pick out minimal sets of such entries.

Introduction

We study centraliser algebras for the actions on tensor space of two finite dimensional algebras: the Schur algebra and partition algebra. Thanks to two instances of Schur–Weyl duality, these centraliser algebras are isomorphic to quotients of group algebras of symmetric groups. This paper focuses on the problems:

- (Q1) Find a "canonical" basis for the centraliser algebras.
- (Q2) Given such a basis, calculate the coefficients expressing a given invariant as a linear combination of the basic ones.

Previous results [RSS12, Dot22] solve (Q1) for both centraliser algebras (see Theorem 1.2). The main results of this paper are given in Section 2, where we provide a combinatorial algorithm for computing the coefficients of a given invariant expressed as a linear combination of the canonical basis, solving (Q2) for both algebras.

Thus our work has connections to: invariant theory, symmetric groups, Schur–Weyl dualities, combinatorics (including the RSK correspondence), and computer science. Our motivation comes from the connections to symmetric groups and invariant theory (Schur–Weyl duality), and we hope that the existence of the algorithm will foster subsequent applications to these areas.

In the case of the Schur algebra invariants, if $\Bbbk$ is any infinite field then the first of the aforementioned algebras is

$$\mathrm{End}_{S(n,r)}(\mathbf{V}^{\otimes r}) = \mathrm{End}_{\mathrm{GL}_n(\Bbbk)}(\mathbf{V}^{\otimes r})$$

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where $\mathbf{V}$ is the natural module for $\mathrm{GL}_n(\Bbbk)$ and $S(n,r)$ is the Schur algebra. Thus, over an infinite field our results are applicable to the classical problem of decomposing GL_n-invariants. Furthermore, the above centraliser algebras are related to (generalised) Temperley–Lieb algebras studied in [Fan95, Fan97, Ste96, Här99].

We believe that the use of the word "canonical" to describe the bases in Theorem 1.2 is justified by the origins of the bases in the papers [RSS12, Dot22], where it was shown that they arise as leading terms of certain Kazhdan–Lusztig basis [KL79] elements (for the Iwahori–Hecke algebra of type A) evaluated at $q = 1$. The canonical bases of [RSS12, Dot22] consist of the (image of) permutations satisfying a combinatorial condition formulated in terms of the length of their longest increasing or decreasing subsequences.

The paper is organized as follows. In Section 1 we summarise the aforementioned previous results that underlie this work. The main results are formulated in Section 2 and proved in Sections 3 and 4. Finally, some applications are considered in Section 5, one of which answers a question of Rouquier.

1. Summary of prior literature

Let $\Bbbk$ be a unital commutative ring, and fix a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for a free $\Bbbk$-module $\mathbf{V}$ of rank n. Let $\mathbf{V}^{\otimes r} = \mathbf{V} \otimes \cdots \otimes \mathbf{V}$ (r factors) be the rth tensor power of $\mathbf{V}$. The parameters n, r are independent. *Throughout the rest of the paper, n and r always refer to the rank of $\mathbf{V}$ and the number of tensor factors in $\mathbf{V}^{\otimes r}$, respectively.*

There are natural right and left commuting actions of the two symmetric groups $\mathfrak{S}_r$, W_n on $\mathbf{V}^{\otimes r}$, given respectively by:

$$(1)\qquad (\mathbf{v}_{j_1} \otimes \mathbf{v}_{j_2} \otimes \cdots \otimes \mathbf{v}_{j_r}) \cdot \sigma = \mathbf{v}_{j_{\sigma(1)}} \otimes \mathbf{v}_{j_{\sigma(2)}} \otimes \cdots \otimes \mathbf{v}_{j_{\sigma(r)}}$$

$$(2)\qquad w \cdot (\mathbf{v}_{j_1} \otimes \mathbf{v}_{j_2} \otimes \cdots \otimes \mathbf{v}_{j_r}) = \mathbf{v}_{w(j_1)} \otimes \mathbf{v}_{w(j_2)} \otimes \cdots \otimes \mathbf{v}_{w(j_r)}$$

for any $\sigma \in \mathfrak{S}_r$, $w \in W_n$. The action of $\mathfrak{S}_r$ is by place-permutation and the action of W_n is the diagonal extension of the natural permutation action of W_n on the given basis. We use different symbols for the two symmetric groups to avoid the possibility of confusing one for the other. Even when $n = r$ they act differently on the tensor space $\mathbf{V}^{\otimes r}$. Note that W_n may be identified with the Weyl group of $\mathrm{GL}(\mathbf{V}) \cong \mathrm{GL}_n(\Bbbk)$ and the action of W_n is the one induced by the natural diagonal action of $\mathrm{GL}(\mathbf{V})$.

In Theorems 1.1 and 1.2 below we summarise the recent results underlying the question addressed by this paper. Let

$$\Phi = \Phi_{n,r} : \Bbbk[\mathfrak{S}_r]^{\mathrm{op}} \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r}), \quad \Psi = \Psi_{n,r} : \Bbbk[W_n] \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$$

be the representations afforded by the respective actions (1), (2). Each representation $\Phi = \Phi_{n,r}$ and $\Psi = \Psi_{n,r}$ depends on both n and r.

Theorem 1.1. *Let $\Bbbk$ be a unital commutative ring. Let $n \geq 1$ and $r \geq 0$ be integers. Then*

(i) $\mathrm{im}(\Phi_{n,r}) = \mathrm{End}_{S(n,r)}(\mathbf{V}^{\otimes r})$, and (ii) $\mathrm{im}(\Psi_{n,r}) = \mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$.

Proof sketch, with historical context. We will now outline how these results were proved, including pointers to the literature to aid the reader. As an

$(\mathrm{GL}(\mathbf{V}), \Bbbk[\mathfrak{S}_r]^{\mathrm{op}})$-bimodule, $\mathbf{V}^{\otimes r}$ satisfies Schur–Weyl duality, in the sense that the image of each action generates the centraliser for the other. When $\Bbbk = \mathbb{C}$ this is a classical result of Schur, and over infinite fields it is due to [Gre07, dCP76]. The result was extended to sufficiently large finite fields in [BD09]. The paper [Cru19] showed that it fails in general when $\Bbbk$ is a commutative ring.

However, the Schur algebra $S(n,r)$ is defined over $\mathbb{Z}$ and acts on $\mathbf{V}^{\otimes r}$ as described in [Gre07, §2.6]. That action commutes with the place-permutation action of $\mathfrak{S}_r$, so $\mathbf{V}^{\otimes r}$ is an $(S(n,r), \Bbbk[\mathfrak{S}_r]^{\mathrm{op}})$-bimodule. By extending an argument in [BD09], [Cru19, Thm. 3.4] proved equality (i), for any unital commutative ring $\Bbbk$. Incidentally, this implies that Schur–Weyl duality holds for the $(S(n,r), \Bbbk[\mathfrak{S}_r]^{\mathrm{op}})$-bimodule structure on $\mathbf{V}^{\otimes r}$, in the same generality, thanks to the well known equality $S(n,r) = \mathrm{End}_{\mathfrak{S}_r}(\mathbf{V}^{\otimes r})$, which goes all the way back to Schur. This equality holds for any unital commutative ring $\Bbbk$ (see the proof of [Gre07, (2.6c)]).

The partition algebra $\mathcal{P}_r(n)$ acts on $\mathbf{V}^{\otimes r}$ as described in [HR05, eq. (3.2)]. As a $(\Bbbk[W_n], \mathcal{P}_r(n))$-bimodule, $\mathbf{V}^{\otimes r}$ also satisfies Schur–Weyl duality over any unital commutative ring $\Bbbk$, by the main result of [BDM22a] (see also [Don22, §6]), so in particular equality (ii) holds. □

Theorem 1.2. *Let* LLIS *(resp.,* LLDS*) denote the length of a longest increasing (resp., decreasing) subsequence of a permutation. Let*

$$\mathcal{B} = \mathcal{B}_{n,r} = \{\sigma \in \mathfrak{S}_r \mid \mathrm{LLDS}(\sigma) \le n\},$$
$$\mathcal{C} = \mathcal{C}_{n,r} = \{w \in W_n \mid \mathrm{LLIS}(w) \ge n - r\}.$$

Then we have

(i) $\{\Phi(\sigma)\}_{\sigma \in \mathcal{B}_{n,r}}$ *is a* $\Bbbk$*-basis of* $\mathrm{End}_{S(n,r)}(\mathbf{V}^{\otimes r})$.
(ii) $\{\Psi(w)\}_{w \in \mathcal{C}_{n,r}}$ *is a* $\Bbbk$*-basis of* $\mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$.

See [RSS12, Thm. 1] for the statement for Φ, and [Dot22, Thm. 1] for the statement for Ψ.

Remark 1.3. Although the statements of both [RSS12, Thm. 1] and [Dot22, Thm. 1] are statements about representations of symmetric groups, the proof in both cases takes place in the corresponding Iwahori–Hecke algebra and relies on deeper results of Geck's paper [Gec06] relating its Kazhdan–Lusztig bases to its Murphy bases. The connection to increasing and decreasing subsequences comes from the Robinson–Schensted correspondence [Sch61]. Furthermore, the proof of [Dot22, Thm. 1] follows from previous joint work [BDM22b] of the authors and results of Donkin [Don22, §8] that obtains a q-analogue of one of the results in [BDM22b].

Combining Theorems 1.1 and 1.2 gives the following immediate consequence. Write $\bar{x} = x + J$ for the image of an element x under the canonical quotient map $A \to A/J$, where A is an algebra and J an ideal.

Corollary 1.4. *Let* $\Bbbk$ *be a unital commutative ring.*

(i) $\overline{\mathcal{B}}_{n,r} = \{\bar{\sigma} \mid \sigma \in \mathcal{B}_{n,r}\}$ *is a* $\Bbbk$*-basis of* $\Bbbk[\mathfrak{S}_r]^{\mathrm{op}}/\ker(\Phi_{n,r})$.
(ii) $\overline{\mathcal{C}}_{n,r} = \{\bar{w} \mid w \in \mathcal{C}_{n,r}\}$ *is a* $\Bbbk$*-basis of* $\Bbbk[W_n]/\ker(\Psi_{n,r})$.

Notice that $\mathcal{B}_{1,r} = \{id\}$ consists of only the identity permutation, $\mathcal{B}_{r-1,r} = \mathfrak{S}_r \setminus \{\sigma_0\}$, where σ_0 is the element of longest Coxeter length, and we have inclusions

$$\mathcal{B}_{1,r} \subset \mathcal{B}_{2,r} \subset \cdots \subset \mathcal{B}_{r,r} = \mathfrak{S}_r.$$

(Clearly, $\mathcal{B}_{n,r} = \mathfrak{S}_r$ stabilises if $n \geq r$). Furthermore, in case (ii), $\mathcal{C}_{n,0} = \{id\}$, $\mathcal{C}_{n,n-2} = W_n \setminus \{w_0\}$, where w_0 is the element of longest Coxeter length, and we have inclusions

$$\mathcal{C}_{n,0} \subset \mathcal{C}_{n,1} \subset \cdots \subset \mathcal{C}_{n,n-1} = W_n.$$

(In this case, $\mathcal{C}_{n,r} = W_n$ stabilises if $r \geq n-1$).

Example 1.5. In this and subsequent examples, we write permutations in the one-line notation, omitting punctuation (commas) between entries. We have:

$$\mathcal{B}_{1,3} = \{(123)\}, \quad \mathcal{B}_{2,3} = \mathfrak{S}_3 \setminus \{(321)\}, \quad \text{and} \quad \mathcal{B}_{3,3} = \mathfrak{S}_3.$$
$$\mathcal{C}_{3,0} = \{(123)\}, \quad \mathcal{C}_{3,1} = W_3 \setminus \{(321)\}, \quad \text{and} \quad \mathcal{C}_{3,2} = W_3.$$
$$\mathcal{B}_{1,4} = \{(1234)\},$$
$$\mathcal{B}_{2,4} = \{(1234), (1243), (1324), (2134), (1342), (1423), (2143),$$
$$(2314), (3124), (2341), (2413), (3142), (4123), (3412)\},$$
$$\mathcal{B}_{3,4} = \mathfrak{S}_4 \setminus \{(4321)\}, \quad \text{and} \quad \mathcal{B}_{4,4} = \mathfrak{S}_4.$$
$$\mathcal{C}_{4,0} = \{(1234)\},$$
$$\mathcal{C}_{4,1} = \{(1234), (1243), (1324), (2134), (1342),$$
$$(1423), (2314), (3124), (2341), (4123)\},$$
$$\mathcal{C}_{4,2} = W_4 \setminus \{(4321)\}, \quad \text{and} \quad \mathcal{C}_{4,3} = W_4.$$

It is known [Dot22, Prop. 4] that $\mathcal{C}_{n,1}$ is the set of "consecutive cycles" in W_n, for any n. (A permutation is a consecutive cycle if it or its inverse is a cycle of the form $i \mapsto i+1 \mapsto \cdots \mapsto i+k-1 \mapsto i$ for some i, k.)

The kernels of the maps $\Phi_{n,r}$ and $\Psi_{n,r}$ are well understood; see [dCP76, Thm. 4.2] for the former and [BDM22b] for the latter. In particular, the maps $\Phi_{n,r}$ and $\Psi_{n,r}$ are injective (the corresponding symmetric groups act faithfully) if and only if $n \geq r$ and $r \geq n-1$, respectively. Cellular bases for each kernel are known; see [Här99] for Φ and [BDM22b] for Ψ. The kernels are cell ideals, so each quotient is a cellular algebra, in the sense of [GL96].

Corollary 1.6. *Let* $\Bbbk$ *be a commutative ring with unit. Define* $\mathbf{B}(n,r) := \Bbbk[\mathfrak{S}_r]^{\mathrm{op}}/\ker(\Phi_{n,r})$ *and* $\mathbf{C}(n,r) := \Bbbk[W_n]/\ker(\Psi_{n,r})$*. These are cellular algebras, and we have the following.*

(i) *Fixing* r *and letting* n *vary from* 1 *to* r *produces a sequence of (surjective) quotient maps*

$$\mathbf{B}(1,r) \leftarrow \mathbf{B}(2,r) \leftarrow \cdots \leftarrow \mathbf{B}(r,r) = \Bbbk[\mathfrak{S}_r]^{\mathrm{op}}$$

such that each σ *in* $\mathfrak{S}_r$ *maps to the corresponding coset in each quotient.*

(ii) *Similarly, fixing n and letting r vary from 0 to $n-1$ produces a sequence of quotient maps*

$$\mathbf{C}(n,0) \leftarrow \mathbf{C}(n,1) \leftarrow \cdots \leftarrow \mathbf{C}(n,n-1) = \Bbbk[W_n]$$

such that each w in W_n maps to the corresponding coset in each quotient.

Proof. To prove (i), it suffices to observe that there is a descending series of two-sided ideals

$$\ker(\Phi_{1,r}) \supset \ker(\Phi_{2,r}) \supset \cdots \supset \ker(\Phi_{r,r}) = (0).$$

To prove (ii), we observe that there is a descending series of two-sided ideals

$$\ker(\Psi_{n,0}) \supset \ker(\Psi_{n,1}) \supset \cdots \supset \ker(\Psi_{n,n-1}) = (0).$$

These observations follow from the descriptions of the kernels given in the references cited in the remarks preceding the proof. □

In light of Corollary 1.6, it is natural to ask whether or not the bases in Corollary 1.4 are cellular bases. Example 1.5 shows that the answer is negative in general. To see this, recall that König and Xi [KX98] showed that an algebra with a finite basis is cellular if and only if it has a certain chain of ideals known as a cell chain. In particular, if an algebra with an anti-involution is cellular, it would have to contain a cell ideal or be isomorphic to a matrix algebra. But the ideal generated by any of the elements of $\bar{\mathcal{C}}_{3,1}$ is the entire algebra $\mathbf{C}(3,1)$, so it cannot have a proper cell ideal containing an element of $\bar{\mathcal{C}}_{3,1}$. Furthermore, $|\mathcal{C}_{3,1}| = |\bar{\mathcal{C}}_{3,1}| = 5$, so $\mathbf{C}(3,1)$ cannot be isomorphic to a matrix algebra, either. The same argument shows that $\bar{\mathcal{B}}_{2,3}$ is not a cellular basis of $\mathbf{B}(2,3)$. The argument easily extends to other cases.

Remark 1.7. When a permutation is written in the one-line notation, reversing its order interchanges the notions of increasing and decreasing subsequences. This order reversal map is an involution on the ambient symmetric group [Sch61]. It is easy to see that order reversal is the map given by right multiplication[1] by the (unique) element of longest Coxeter length. The kernels of $\Phi_{n,r}$ and $\Psi_{n,r}$ are invariant under such multiplication. It follows that order reversal induces an involution on both $\Bbbk[\mathfrak{S}_r]^{\mathrm{op}}/\ker(\Phi_{n,r})$ and $\Bbbk[W_n]/\ker(\Psi_{n,r})$. So we can interchange the notions of LLIS, LLDS in the definition of the sets $\mathcal{B}_{n,r}$ and $\mathcal{C}_{n,r}$ and all the results of this section remain true. This implies that all the results of this paper have a "dual" form in which such an interchange has been effected.

2. The main results

In this section we fix n and r and write $\Phi = \Phi_{n,r}$, $\Psi = \Psi_{n,r}$, $\mathcal{B} = \mathcal{B}_{n,r}$, $\mathcal{C} = \mathcal{C}_{n,r}$. The problem considered in this paper is the following: given any

[1] This assumes that we compose maps from right-to-left, so that $(f \circ g)(x) = f(g(x))$ If we compose maps from left-to-right then we should replace right multiplication by left multiplication.

invariant X in $\mathrm{End}_{S(n,r)}(\mathbf{V}^{\otimes r})$, $\mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$, find the (unique) scalars $c(\sigma)$, $c(w)$ in $\Bbbk$ such that

$$\sum_{\sigma\in\mathcal{B}} c(\sigma)\Phi(\sigma) = X, \qquad \sum_{w\in\mathcal{C}} c(w)\Psi(w) = X \tag{3}$$

respectively. In other words, we are considering (Q2) for the above centraliser algebras, with respect to the bases of Theorem 1.2.

We need some notation in order to formulate our main result. Set $[n] := \{1,\dots,n\}$. Given any multi-index $\underline{j} = (j_1, j_2, \dots, j_r)$ in $[n]^r$, set

$$\mathbf{v}_{\underline{j}} := \mathbf{v}_{j_1} \otimes \mathbf{v}_{j_2} \otimes \cdots \otimes \mathbf{v}_{j_r}.$$

The set $\{\mathbf{v}_{\underline{j}} \mid \underline{j} \in [n]^r\}$ is a $\Bbbk$-basis of $\mathbf{V}^{\otimes r}$. Let $(\underline{j}, \sigma) \mapsto \underline{j}\sigma$, $(w, \underline{j}) \mapsto w\underline{j}$ denote the actions of $\mathfrak{S}_r$, W_n on $[n]^r$ such that

$$\underline{j}\sigma = (j_{\sigma(1)}, \dots, j_{\sigma(r-1)}, j_{\sigma(r)}), \quad w\underline{j} = (w(j_1), \dots, w(j_{r-1}), w(j_r)).$$

This notation is compatible with equations (1) and (2) in the sense that $\mathbf{v}_{\underline{j}} \cdot \sigma = \mathbf{v}_{\underline{j}\sigma}$ and $w \cdot \mathbf{v}_{\underline{j}} = \mathbf{v}_{w\underline{j}}$. Taking matrix coordinates in equation (3) with respect to this basis, we obtain the (overdetermined, in general) linear systems

$$\sum_{\sigma\in\mathcal{B}} c(\sigma)\Phi(\sigma)_{\underline{i},\underline{j}} = X_{\underline{i},\underline{j}}, \qquad \sum_{w\in\mathcal{C}} c(w)\Psi(w)_{\underline{i},\underline{j}} = X_{\underline{i},\underline{j}} \tag{4}$$

as $(\underline{i}, \underline{j})$ vary over all pairs of multi-indices in $[n]^r \times [n]^r$. By Theorem 1.2, the coefficents in each system are uniquely determined. What is needed is a method of picking out a minimal linear subsystem producing the unique solution in each case. Our contribution is to find combinatorial maps that do this job. Here is our main result.

Theorem 2.1. *Injective maps $\varphi : \mathcal{B} \to [n]^r \times [n]^r$ and $\psi : \mathcal{C} \to [n]^r \times [n]^r$ exist such that:*

(i) *If $\varphi(\sigma) = (\underline{i}, \underline{j})$ then σ is the unique element of $\mathcal{B}$ of minimal Coxeter length satisfying $\underline{i}\sigma = \underline{j}$.*
(ii) *If $\psi(w) = (\underline{i}, \underline{j})$ then w is the unique element of $\mathcal{C}$ of minimal Coxeter length satisfying $w\underline{j} = \underline{i}$.*

Part (i) is proved in Proposition 3.8 and part (ii) in Proposition 4.3 below. Given any σ' in $\mathcal{B}$ and any w' in $\mathcal{C}$ we write

$$\varphi(\sigma') = (\underline{i}(\sigma'), \underline{j}(\sigma')) \quad \text{and} \quad \psi(w') = (\underline{i}(w'), \underline{j}(w')).$$

For $X = [X_{\underline{i},\underline{j}}]_{\underline{i},\underline{j}\in[n]^r}$ in either $\mathrm{End}_{S(n,r)}(\mathbf{V}^{\otimes r})$ or $\mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$, consider the corresponding linear subsystems

$$\begin{aligned} \textstyle\sum_{\sigma\in\mathcal{B}} c(\sigma)\Phi(\sigma)_{\underline{i}(\sigma'),\underline{j}(\sigma')} &= X_{\underline{i}(\sigma'),\underline{j}(\sigma')}, \\ \textstyle\sum_{w\in\mathcal{C}} c(w)\Psi(w)_{\underline{i}(w'),\underline{j}(w')} &= X_{\underline{i}(w'),\underline{j}(w')} \end{aligned} \tag{5}$$

of the systems in (4), as σ' varies over $\mathcal{B}$ and w' varies over $\mathcal{C}$. By Theorem 2.1, these subsystems have unitriangular coefficient matrices

$$A_\Phi = \left[\Phi(\sigma)_{\underline{i}(\sigma'),\underline{j}(\sigma')}\right]_{\sigma,\sigma'\in\mathcal{B}}, \qquad A_\Psi = \left[\Psi(w)_{\underline{i}(w'),\underline{j}(w')}\right]_{w,w'\in\mathcal{C}}$$

if their rows and columns are ordered by any total ordering of $\mathcal{B}$, $\mathcal{C}$ compatible with the Coxeter length function. We observe that the matrices A_Φ and

A_Ψ are $(0,1)$-matrices. To see this, notice that for any $\sigma \in \mathfrak{S}_r$ or $w \in W_n$, we have

$$\Phi(\sigma)_{\underline{i},\underline{j}} = \delta_{i_{\sigma(1)},j_1} \cdots \delta_{i_{\sigma(r)},j_r}, \qquad \Psi(w)_{\underline{i},\underline{j}} = \delta_{i_1,w(j_1)} \cdots \delta_{i_r,w(j_r)} \tag{6}$$

for any $\underline{i}, \underline{j}$ in $[n]^r$, where δ_{ij} is the usual Kronecker delta symbol. It follows that the entries of A_Φ^{-1} and A_Ψ^{-1} are integers. In summary, we have the following.

Corollary 2.2. *The $c(\sigma)$ or $c(w)$ in (3) are respectively expressible as integral linear combinations of the $X_{\underline{i}(\sigma'),\underline{j}(\sigma')}$ or $X_{\underline{i}(w'),\underline{j}(w')}$, as σ' varies over $\mathcal{B}$ and w' over $\mathcal{C}$.*

Theorem 2.1 can be said to "solve" the problem of computing the $c(\sigma)$, $c(w)$ in equations (3), by reducing the problem to the calculation of the inverse of a unitriangular $(0,1)$-matrix. Such a calculation is relatively easy for a digital computer, as no multiplications or divisions are required.

Remark 2.3. In all the examples known to the authors, it turns out that the entries of A_Φ^{-1} and A_Ψ^{-1} all belong to the set $\{0, 1, -1\}$. We do not know whether or not such a statement holds in general.

3. The map φ

We continue to assume that $\Bbbk$ is a unital commutative ring. We write $\mathcal{B} = \mathcal{B}_{n,r}$, $A = A_\Phi$ in this section. We now define a map $\varphi' : \mathfrak{S}_r \to [r]^r \times [r]^r$ which, for any positive integer n, restricts to a map $\varphi : \mathcal{B} \to [n]^r \times [n]^r$ having the properties needed to complete the proof of Theorem 2.1(i). The map φ' depends on r and is independent of n.

Definition 3.1. The *initial increasing subsequence* $M = (y_1, y_{i_2}, \dots, y_{i_\ell})$ of a given finite sequence $y = (y_i)$ of real numbers is defined as follows:

(a) Set $i_1 = 1$ and $M_1 = (y_{i_1}) = (y_1)$.
(b) For any $a \geq 1$, assuming that $(i_1, \dots, i_a)$ and $M_a = (y_{i_1}, \dots, y_{i_a})$ have been selected so far, let i_{a+1} be the unique smallest index i for which $i_{a+1} > i_a$ and $y_{i_{a+1}} > y_{i_a}$, if possible, and set $M_{a+1} = (y_{i_1}, \dots, y_{i_a}, y_{i_{a+1}})$.
(c) The process terminates when there is no such element in the given sequence. Let $M = M_{\max}$, the final subsequence at time of termination.

Example 3.2. The initial increasing subsequence of $y = (5, 2, 1, 3, 7, 4, 6, 9)$ is $(5, 7, 9)$. Note that this is not a longest increasing subsequence.

Now let $x = (x_1, x_2, \dots, x_r)$, $y = (y_1, y_2, \dots, y_r)$ be given finite sequences of the same length, each consisting of *distinct* real numbers (or elements of any linearly ordered set), such that the first sequence is increasing: $x_1 < x_2 < \cdots < x_r$. We may display the pair of sequences in a two-line notation by writing

$$(x/y) = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ y_1 & y_2 & \cdots & y_r \end{pmatrix}.$$

The standard way of computing the length of the longest decreasing subsequence of y is the tableau-based Robinson–Schensted algorithm [Sch61, Ful97]. But there are other (less efficient) methods of computing that statistic. The map φ is based on one such method. As we show in Proposition 3.5, it computes the length of a longest decreasing subsequence.

Given (x/y) with x increasing, let $(y_1, y_{i_2}, \dots, y_{i_\ell})$ be the initial increasing subsequence of y. Let $(x_1, x_{i_2}, \dots, x_{i_\ell})$ be the corresponding (necessarily increasing) subsequence of x. Write

$$\pi_1(x/y) = \begin{pmatrix} x_1 & x_{i_2} & \cdots & x_{i_\ell} \\ y_1 & y_{i_2} & \cdots & y_{i_\ell} \end{pmatrix}$$

for the resulting pair of increasing subsequences of (x/y). The *complement* $(x'/y') = (x/y) \setminus \pi_1(x/y)$ is obtained by excising the columns of $\pi_1(x/y)$ from (x/y) and reindexing. If y is increasing then the complement is empty and $\pi_1(x/y) = (x/y)$. Otherwise, the complement (x'/y') is a new pair of sequences satisfying the same conditions as (x/y). So we may apply π_1 again to the pair (x'/y'), obtaining $\pi_2(x/y) = \pi_1(x'/y')$, a second pair of increasing subsequences of (x/y). Continuing in this manner, we obtain the *canonical factorisation* of the given pair (x/y), as the configuration

$$\pi(x/y) := \big(\pi_1(x/y) \mid \pi_2(x/y) \mid \cdots \mid \pi_k(x/y)\big)$$

consisting of pairs of increasing subsequences of (x/y). The integer k is the *length* of the canonical factorisation. (See [Gre74] for a similar construction.)

Example 3.3. If (x/y) is the pair of sequences given by

$$(x/y) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 2 & 1 & 7 & 3 & 8 & 9 & 6 & 5 \end{pmatrix}$$

then we have

$$\pi_1(x/y) = \begin{pmatrix} 1 & 4 & 6 & 7 \\ 4 & 7 & 8 & 9 \end{pmatrix}, \quad \pi_2(x/y) = \begin{pmatrix} 2 & 5 & 8 \\ 2 & 3 & 6 \end{pmatrix}, \quad \pi_3(x/y) = \begin{pmatrix} 3 & 9 \\ 1 & 5 \end{pmatrix}.$$

We write this succinctly in the form

$$\pi(x/y) = \left(\begin{array}{cccc|ccc|cc} 1 & 4 & 6 & 7 & 2 & 5 & 8 & 3 & 9 \\ 4 & 7 & 8 & 9 & 2 & 3 & 6 & 1 & 5 \end{array} \right)$$

in which vertical lines are used to separate the various $\pi_i(x/y)$ in their natural order $\pi_1(x/y), \dots, \pi_k(x/y)$.

We apply the above observations to permutations. By writing $\sigma \in \mathfrak{S}_r$ in the usual two-line notation, we obtain a pair of sequences (x/y) as above (with the first-line sequence $x = (1, 2, \dots, r)$ increasing). For instance, the pair of sequences (x/y) written in Example 3.3 is the two-line notation for the permutation $\sigma = (421738965)$ in the one-line notation. Write

$$\pi(\sigma) = \big(\pi_1(\sigma) \mid \pi_2(\sigma) \mid \cdots \mid \pi_k(\sigma)\big)$$

with the obvious interpretation, namely, that $\pi_i(\sigma) = \pi_i(x/y)$, for all $i = 1, \dots, k$, where k is the length of the canonical factorisation.

We now define the promised map $\varphi' : \mathfrak{S}_r \to [r]^r \times [r]^r$, which will soon be used to define the map φ. Given $\sigma \in \mathfrak{S}_r$, let k be the length of its canonical factorisation, and set $[k] := \{1, \dots, k\}$. Let $i = (i_1, i_2, \dots, i_r)$, $j = (j_1, j_2, \dots, j_r)$ be the sequences in $[k]^r$ defined by the conditions

$$i_\alpha = c \iff \alpha \text{ appears in the first line of } \pi_c(\sigma)$$

$$j_\alpha = c \iff \alpha \text{ appears in the second line of } \pi_c(\sigma).$$

In other words, we colour each number in $\pi_c(\sigma)$ by the colour c, for each c in $[k]$, and let $\underline{i}$, $\underline{j}$ be the corresponding sequences of colours in the first and second line of $\pi(\sigma)$, resp. Let $\varphi'(\sigma) = (\underline{i}, \underline{j})$. Evidently, $\varphi'(\sigma)$ is in $[r]^r \times [r]^r$.

If σ_0 is the element of $\mathfrak{S}_r$ of longest Coxeter length, then

$$\varphi'(\sigma_0) = ((1, 2, \dots, r), (r, r-1, \dots, 1)),$$

which shows that the image of φ' is not contained in any $[n]^r \times [n]^r$, for $n < r$. At the opposite extreme, if id is the identity permutation then

$$\varphi'(id) = ((1, 1, \dots, 1), (1, 1, \dots, 1)).$$

For the permutation $\sigma = (421738965)$ of Example 3.3, we have

$$\varphi'(\sigma) = ((1, 2, 3, 1, 2, 1, 1, 2, 3), (3, 2, 2, 1, 3, 2, 1, 1, 1)).$$

The following lemma is similar to Lemma 5 in [Sch61].

Lemma 3.4. *Given any term of the second line of any $\pi_c(\sigma)$, with $c > 1$, there exists a term of the second line of $\pi_{c-1}(\sigma)$ which is larger and appears further to the left in σ.*

Proof. If not, then the term in question would have already been included in the second line of $\pi_{c-1}(\sigma)$. □

In the combinatorics literature, it is customary to write permutations in one-line notation, as sequences $(\sigma_1, \dots, \sigma_r)$ with $\sigma_i = \sigma(i)$ for $i \in [r]$. Such sequences are often called *words*. Increasing and decreasing subsequences of a permutation are always interpreted with respect to the one-line notation (equivalently, the second line of the two-line notation).

Proposition 3.5. *The length k of the canonical factorisation of a permutation σ in $\mathfrak{S}_r$ is always equal to the length of a longest decreasing subsequence of σ.*

Proof. From Lemma 3.4 it follows that there is a decreasing subsequence with at least one term taken from each of the second lines of $\pi_1(\sigma), \dots, \pi_k(\sigma)$. This is necessarily of maximal length, as no decreasing subsequence can be formed using two terms from the second line of the same $\pi_c(\sigma)$. □

Lemma 3.6. *If the terms of any increasing subsequence of a permutation are permuted in a non-trivial way, the Coxeter length of the permutation strictly increases.*

Proof. This follows from the fact that the Coxeter length of a permutation is equal to the number of its inversions. Since any increasing subsequence is contained in a maximal one and a permutation of the original sequence is also a permutation of any maximal one in which it is contained, it suffices to prove the result for maximal increasing subsequences. So from now on we assume that our increasing subsequence is maximal.

Next, we observe that it suffices to prove the result for a swap of consecutive terms of the given subsequence, as any permutation of the subsequence can be achieved by a succession of such swaps.

Recall that an inversion in a permutation σ is a pair (σ_i, σ_j) such that $i < j$ and $\sigma_i > \sigma_j$. Furthermore, recall [Knu98, §5.1.1] that the *inversion table* of a permutation $\sigma \in \mathfrak{S}_r$ is the sequence $(b_1, \dots, b_r)$ such that b_j is the number of σ_i to the left of j which exceed j. In other words, b_j is the number of inversions with second term j. Then $\ell(\sigma) = b_1 + \cdots + b_r$, where $\ell(\sigma)$ is the Coxeter length of σ.

Finally, consider two consecutive terms $\sigma_i < \sigma_j$ (where $i < j$) of our maximal increasing subsequence. Maximality means that every intermediate term in $(\sigma_{i+1}, \dots, \sigma_{j-1})$ lies outside of the closed interval $[\sigma_i, \sigma_j]$. Swapping σ_i and σ_j increases the inversion number b_{σ_i} by $1 + N$ and decreases the inversion number b_{σ_j} by N, where N is the number of intermediate terms exceeding σ_i. All other inversion numbers in the inversion table remain unchanged, so the net effect is an increase of 1 in the total number of inversions. Thus, $\ell(w)$ is incremented by 1, and the proof is complete. □

Example 3.7. The following examples illustrate the proof of Lemma 3.6. Take $\sigma = (539746182)$ in $\mathfrak{S}_9$. Its inversion table is $b = (6, 7, 1, 3, 0, 2, 1, 1, 0)$ so $\ell(\sigma) = 21$.

(i) Consider the maximal increasing subsequence (3468) in σ. Swapping its first two terms produces the permutation $\sigma' = (549736182)$, with inversion table $b' = (6, 7, 4, 1, 0, 2, 1, 1, 0)$. Notice that b_3 increased by 3 and b_4 decreased by 2, so $\ell(\sigma') = 22$.

(ii) Swapping the last two terms of (3468) gives $\sigma'' = (539748162)$. Its inversion table is $b'' = (6, 7, 1, 3, 0, 3, 1, 1, 0)$. This time b_6 increased by 1 and all other terms of b remained the same, so again $\ell(\sigma'') = 22$.

The *weight* of a sequence $\underline{i} = (i_1, i_2, \dots, i_r)$ in $[k]^r$ is the composition $\mu = (\mu_1, \dots, \mu_k)$, where μ_c counts the number of terms in $\underline{i}$ equal to c, for each c in $[k]$. It is clear by construction that each of the sequences $\underline{i}$, $\underline{j}$ such that $\varphi'(\sigma) = (\underline{i}, \underline{j})$ is of the same weight μ.

The following result gives the main properties of the map φ'.

Proposition 3.8. *The map $\varphi' : \mathfrak{S}_r \to [r]^r \times [r]^r$ is injective. Given any σ in $\mathfrak{S}_r$, let $\varphi'(\sigma) = (\underline{i}, \underline{j})$. Then σ is the unique permutation in $\mathfrak{S}_r$ of minimal length such that $\underline{i}\sigma = \underline{j}$.*

Proof. To prove the injectivity, let $(\underline{i}, \underline{j})$ be a pair of multi-indices in $[r]^r \times [r]^r$ of the same weight $\mu = (\mu_1, \dots, \mu_k)$. For each c in $[k]$, let $p(c) = (x(c)/(y(c))$ be the pair of increasing sequences defined by letting $x(c)$ (resp., $y(c)$) be the positions in the sequence $\underline{i}$ (resp., $\underline{j}$) having value c, in order. Then the configuration $(p(1) \mid p(2) \mid \cdots \mid p(k))$ defines, up to column reordering, the original permutation σ that produced the pair $(\underline{i}, \underline{j})$. Sending $(\underline{i}, \underline{j}) \mapsto \sigma$ defines a left inverse to φ', and thus φ' is injective. For instance, if

$$(\underline{i}, \underline{j}) = ((1, 2, 3, 1, 2, 1, 1, 2, 3), (3, 2, 2, 1, 3, 2, 1, 1, 1))$$

then we have

$$(p(1) \mid p(2) \mid p(3)) = \left(\begin{array}{cccc|ccc|cc} 1 & 4 & 6 & 7 & 2 & 5 & 8 & 3 & 9 \\ 4 & 7 & 8 & 9 & 2 & 3 & 6 & 1 & 5 \end{array}\right)$$

which recovers the permutation σ of Example 1.

Now we prove the second claim. Suppose that $\underline{i}\sigma' = \underline{j}$ for some $\sigma' \neq \sigma$ in $\mathfrak{S}_r$. Then σ' is obtainable from σ by permuting the first (or, equivalently, second) line of one or more parts of its canonical factorisation. If more than one part is involved, they can be permuted independently, so it suffices to consider such a permutation of a single part of the canonical factorisation. This amounts to a reordering of the terms of an increasing subsequence of σ, which by Lemma 3.6 necessarily increases the Coxeter length. □

Now we can define the map φ. If $n \geq r$, we have $\mathcal{B} = \mathfrak{S}_r$ since Φ is faithful. If $n < r$, $\mathcal{B} = \{\sigma \in \mathfrak{S}_r \mid \mathrm{LLDS}(\sigma) \leq n\}$ is a proper subset of $\mathfrak{S}_r$, and we define

$$\varphi = \text{ the restriction of } \varphi' \text{ to } \mathcal{B}.$$

If $n \geq r$ then we define $\varphi = \varphi'$. The properties of Proposition 3.8 are invariant under restriction, thus are applicable to φ in all cases.

Remark 3.9. In the stable case $n \geq r$, we can do better than the unitriangular subsystem provided by $\varphi = \varphi'$. Namely, for each σ, simply set $\underline{i} = (1, 2, \dots, r)$ and $\underline{j} = \underline{i}\sigma$. Since $n \geq r$, the values in $\underline{j}$ are all distinct, so σ is the unique element of $\mathfrak{S}_r$ such that $\underline{i}\sigma = \underline{j}$. Sending σ to $(\underline{i}, \underline{j})$ instead of using $\varphi = \varphi'$ picks out a subsystem of (4) having a diagonal coefficient matrix A. In fact, as it is a $(0,1)$-matrix, $A = I$ is the identity matrix.

Example 3.10. (i) Suppose that $r = 3$. The pairs $(\underline{i}, \underline{j})$ corresponding to each σ in $\mathfrak{S}_3$ under φ' are tabulated below:

σ	(123)	(213)	(132)	(231)	(312)	(321)
$\underline{i}$	$(1,1,1)$	$(1,2,1)$	$(1,1,2)$	$(1,1,2)$	$(1,2,2)$	$(1,2,3)$
$\underline{j}$	$(1,1,1)$	$(2,1,1)$	$(1,2,1)$	$(2,1,1)$	$(2,2,1)$	$(3,2,1)$

Order the elements of $\mathfrak{S}_3$ as indicated in the table; this is a linear order compatible with Coxeter length. The map φ' picks out the subsystem of (4) given by

$$\begin{aligned}
c(123) + c(213) + \cdots + c(321) &= X_{(1,1,1),(1,1,1)} \\
c(213) + c(312) &= X_{(1,2,1),(2,1,1)} \\
c(132) + c(312) &= X_{(1,1,2),(1,2,1)} \\
c(231) + c(321) &= X_{(1,1,2),(2,1,1)} \\
c(312) + c(321) &= X_{(1,2,2),(2,2,1)} \\
c(321) &= X_{(1,2,3),(3,2,1)}
\end{aligned}$$

with coefficient matrix A of the form

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & \cdot & \cdot & 1 & \cdot \\ & & 1 & \cdot & 1 & \cdot \\ & & & 1 & \cdot & 1 \\ & & & & 1 & 1 \\ & & & & & 1 \end{bmatrix}.$$

By inverting the matrix A, we conclude that, for any X in $\mathrm{End}_{S(n,3)}(\mathbf{V}^{\otimes r})$, with $n \geq 3$, we have

$$c(123) = X_{(1,1,1),(1,1,1)} - X_{(1,2,1),(2,1,1)} - X_{(1,1,2),(1,2,1)}$$

$$\begin{aligned}
&\quad - X_{(1,1,2),(2,1,1)} + X_{(1,2,2),(2,2,1)} - X_{(1,2,3),(3,2,1)}\\
c(213) &= X_{(1,2,1),(2,1,1)} - X_{(1,2,2),(2,2,1)} + X_{(1,2,3),(3,2,1)}\\
c(132) &= X_{(1,1,2),(1,2,1)} - X_{(1,2,2),(2,2,1)} + X_{(1,2,3),(3,2,1)}\\
c(231) &= X_{(1,1,2),(2,1,1)} - X_{(1,2,3),(3,2,1)}\\
c(312) &= X_{(1,2,2),(2,2,1)} - X_{(1,2,3),(3,2,1)}\\
c(321) &= X_{(1,2,3),(3,2,1)}.
\end{aligned}$$

(ii) If $(n, r) = (2, 3)$ then restricting the above to φ picks out the subsystem of (4) given by

$$\begin{aligned}
c(123) + c(213) + \cdots + c(312) &= X_{(1,1,1),(1,1,1)}\\
c(213) + c(312) &= X_{(1,2,1),(2,1,1)}\\
c(132) + c(312) &= X_{(1,1,2),(1,2,1)}\\
c(231) &= X_{(1,1,2),(2,1,1)}\\
c(312) &= X_{(1,2,2),(2,2,1)}
\end{aligned}$$

with coefficient matrix A of the form

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ & 1 & \cdot & \cdot & 1\\ & & 1 & \cdot & 1\\ & & & 1 & \cdot\\ & & & & 1 \end{bmatrix}.$$

Notice that A is a submatrix of the matrix in part (i). By inverting A, we conclude that, for any X in $\mathrm{End}_{S(2,3)}(\mathbf{V}^{\otimes r})$,

$$\begin{aligned}
c(123) &= X_{(1,1,1),(1,1,1)} - X_{(1,2,1),(2,1,1)} - X_{(1,1,2),(1,2,1)}\\
&\quad - X_{(1,1,2),(2,1,1)} + X_{(1,2,2),(2,2,1)}\\
c(213) &= X_{(1,2,1),(2,1,1)} - X_{(1,2,2),(2,2,1)}\\
c(132) &= X_{(1,1,2),(1,2,1)} - X_{(1,2,2),(2,2,1)}\\
c(231) &= X_{(1,1,2),(2,1,1)}\\
c(312) &= X_{(1,2,2),(2,2,1)}.
\end{aligned}$$

Remark 3.11. In general, if $n < r$ then we can always obtain a solution from one in the stable case $n \geq r$ (we may as well take $n = r$) by simply putting $X_{\underline{i}\sigma,\underline{j}\sigma} = 0$ for all σ not in $\mathcal{B} = \mathcal{B}_{n,r}$. This is a consequence of the fact that φ is defined to be the restriction of φ'. Example 3.10 illustrates this phenomenon.

4. The map ψ

We continue to assume that $\Bbbk$ is a unital commutative ring. Now we consider the other symmetric group W_n. We write $\mathcal{C} = \mathcal{C}_{n,r}$, $A = A_\Psi$ in this section. Our goal is to define the combinatorial map ψ and show that it has the properties needed to prove Theorem 2.1(ii).

The defining condition in the set $\mathcal{C} = \{w \in W_n \mid \mathrm{LLIS}(w) \geq n - r\}$ is vacuous if $n - r \leq 1$, equivalently, if $r \geq n - 1$. The condition $r \geq n - 1$ determines the stable case in this situation. Note that (by e.g., [BDM22b, BDM22a]) the representation $\Psi : \Bbbk[W_n] \to \mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$ is faithful if and only if $r \ge n-1$, in which case $\mathcal{C} = W_n$.

Definition 4.1. Unlike φ, which depended only on r, the map ψ always depends on both n, r.

(a) Suppose that $r \le n-1$. If $w = (w_1, w_2, \dots, w_n)$ is in $\mathcal{C}$, where $w_j = w(j)$ for all j in $[n]$, we choose (any way we like) an increasing subsequence of w of length $n-r$, and let

$$(w_{j_1}, w_{j_2}, \dots, w_{j_r}), \quad j_1 < j_2 < \cdots < j_r$$

be its *complement* in w. Set $i_\alpha = w_{j_\alpha} = w(j_\alpha)$ for each $\alpha = 1, \dots, r$. If w is written in two-line notation then $(i_1, \dots, i_r)$ is the complement of the chosen increasing subsequence in the second line and $(j_1, \dots, j_r)$ is the corresponding subsequence in the first line. We then define $\psi(w) = (\underline{i}, \underline{j})$ where $\underline{i} = (i_1, \dots, i_r)$ and $\underline{j} = (j_1, \dots, j_r)$.

(b) Notice that $\mathcal{C} = W_n$ if $r = n-1$, and the map ψ sends W_n into $[n]^{n-1} \times [n]^{n-1}$. If $r > n-1$ we extend this to a map $W_n \to [n]^r \times [n]^r$ by appending an additional $r-(n-1)$ values of 1, $w_1 = w(1)$ to the end of each $\underline{j}$, $\underline{i}$ respectively. The choice of 1 here is purely arbitrary, and it can be replaced by any other number in $[n]$.

Example 4.2. Suppose that $w = (421738965)$ in W_9. Then we have $\mathrm{LLIS}(w) = 4$. Since the condition $\mathrm{LLIS}(w) \ge n-r$ is equivalent to $r \ge n - \mathrm{LLIS}(w)$, we may take any $r \ge 5$, up to and including $r = 8$.

(i) The choice $r = 5$ with the increasing subsequence (2389) yields the pair $(\underline{i}, \underline{j}) = ((4,1,7,6,5), (1,3,4,8,9))$.

(ii) The choice $r = 6$ with the increasing subsequence (238) yields the pair $(\underline{i}, \underline{j}) = ((4,1,7,9,6,5), (1,3,4,7,8,9))$.

The pair $(\underline{i}, \underline{j})$ clearly depends on the chosen increasing subsequence of w. Notice that $\underline{j}$ is necessarily increasing, and that is always the case.

The following result completes the proof of Theorem 2.1.

Proposition 4.3. *The map $\psi : \mathcal{C} \to [n]^r \times [n]^r$ is injective. Given any w in W_n, let $\psi(w) = (\underline{i}, \underline{j})$. Then w is the unique permutation in W_n of minimal length such that $w\underline{j} = \underline{i}$.*

Proof. To prove the injectivity claim, we need to show that for any given pair $(\underline{i}, \underline{j})$ in the image of ψ, there is precisely one $w \in \mathcal{C}$ for which $\psi(w) = (\underline{i}, \underline{j})$. In the stable case $r \ge n-1$, this is clear because any permutation of n items is determined once its values on any $n-1$ of them is specified. If $r < n-1$ this still holds, because the given pair $(\underline{i}, \underline{j})$ determines w on the numbers in $\underline{j}$, and w is determined on all other numbers in $[n]$ by the fact that it must be increasing on those numbers. Hence ψ is injective.

Now consider the second claim. It is vacuous in the stable range $r \ge n-1$, in which case there is only one w such that $w\underline{j} = \underline{i}$. If $r < n-1$, let $w' \ne w$ be any permutation in W_n such that $w'\underline{j} = \underline{i}$. Then w', w necessarily agree on $\{j_1, j_2, \cdots, j_r\}$. This means that w' may be obtained from w by permuting its chosen increasing subsequence of length $n-r$, indexed by the complement $[n] \setminus \{j_1, \dots, j_r\}$. Such a permutation necessarily increases the total number of inversions, by Lemma 3.6, and thus the Coxeter length. □

Example 4.4. (i) Take $(n, r) = (3, 2)$. If we always choose the first term of w as the increasing subsequence s of length $n - r = 1$, the pairs $(\underline{i}, \underline{j})$ such that $w\underline{j} = \underline{i}$ corresponding to each w in $\mathcal{C} = W_3$ under ψ are:

w	(123)	(213)	(132)	(231)	(312)	(321)
s	(1)	(2)	(1)	(2)	(3)	(3)
$\underline{i}$	$(2,3)$	$(1,3)$	$(3,2)$	$(3,1)$	$(1,2)$	$(2,1)$
$\underline{j}$	$(2,3)$	$(2,3)$	$(2,3)$	$(2,3)$	$(2,3)$	$(2,3)$

Order the elements of W_3 in the reading order across the table. This is a linear order compatible with Coxeter length. Then the matrix A produced by φ is just $A = I$, the identity matrix.

(ii) Take $(n, r) = (3, 1)$. We choose s to be the first increasing subsequence of w of length $n - r = 2$. Then the pairs $(\underline{i}, \underline{j})$ such that $w\underline{j} = \underline{i}$ corresponding to each w in $\mathcal{C}$ under ψ are:

w	(123)	(213)	(132)	(231)	(312)
s	(12)	(23)	(13)	(23)	(12)
$\underline{i}$	3	1	2	1	3
$\underline{j}$	3	2	3	3	1

We again order the elements of $\mathcal{C}$ by the reading order across the table. The subsystem of (4) picked out by ψ is

$$\begin{aligned} c(123) + c(213) &= X_{3,3} \\ c(213) + c(312) &= X_{1,2} \\ c(132) + c(312) &= X_{2,3} \\ c(231) &= X_{1,3} \\ c(312) &= X_{3,1} \end{aligned}$$

with coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & \cdot & \cdot & \cdot \\ & 1 & \cdot & \cdot & 1 \\ & & 1 & \cdot & 1 \\ & & & 1 & \cdot \\ & & & & 1 \end{bmatrix}.$$

Inverting A yields the equations

$$\begin{aligned} &c(123) = X_{3,3} - X_{1,2} + X_{3,1} \\ &c(213) = X_{1,2} - X_{3,1} \qquad && c(132) = X_{3,3} - X_{3,1} \\ &c(231) = X_{1,3} && c(312) = X_{3,1}. \end{aligned}$$

5. Further remarks

We continue to assume that $\Bbbk$ is a commutative ring with 1. Fix n and r and let $\Phi = \Phi_{n,r}$ and $\Psi = \Psi_{n,r}$.

Let $\tau \in \mathfrak{S}_r$ and $y \in W_n$. If we take $X = \Phi(\tau)$ or $\Psi(y)$ in the algebra $\mathbf{B}(n, r) = \Bbbk[\mathfrak{S}_r]^{\mathrm{op}}/\ker(\Phi)$ or $\mathbf{C}(n, r) = \Bbbk[W_n]/\ker(\Psi)$ then we have an algorithm that computes the coefficents $c(\sigma)$ for all $\sigma \in \mathcal{B}_{n,r}$ or $c(w)$ for all

$w \in \mathcal{C}_{n,r}$ in equation (3). By running the algorithm with the indicated value of X we obtain a relation of the form

$$\bar{\tau} = \sum_{\sigma \in \mathcal{B}_{n,r}} c(\sigma)\bar{\sigma} \quad \text{or} \quad \bar{y} = \sum_{w \in \mathcal{C}_{n,r}} c(w)\bar{w}. \tag{7}$$

We call such a linear combination the *normal form* of $\bar{\tau}$ or $\bar{y}$ with respect to the basis $\bar{\mathcal{B}}_{n,r}$ or $\bar{\mathcal{C}}_{n,r}$, respectively. Obviously, the above relation is non-trivial if and only if $\bar{\tau}$ is not in $\bar{\mathcal{B}}_{n,r}$ or $\bar{y}$ is not in $\bar{\mathcal{C}}_{n,r}$. The following is clear.

Proposition 5.1. *The algebra $\mathbf{B}(n,r)$ is isomorphic to the algebra generated by the $\bar{\sigma}$ for $\sigma \in \mathcal{B}_{n,r}$ subject to the relations given by multiplication in $\mathfrak{S}_r$ along with the normal form equations in (7) imposed on all products of elements of $\mathcal{B}_{n,r}$. The algebra $\mathbf{C}(n,r)$ is isomorphic to the algebra generated by the $\bar{w}$ for $w \in \mathcal{C}_{n,r}$ subject to the relations given by multiplication in W_n along with the normal form equations in (7) imposed on all products of elements of $\mathcal{C}_{n,r}$.*

These presentations are in general far from being minimal. They are of course equivalent to the presentations obtained by imposing the group multiplication on the ambient symmetric group along with any set of relations generating $\ker(\Phi)$ or $\ker(\Psi)$.

Example 5.2. In these examples, we express normal form equations as congruences modulo the appropriate kernel in the corresponding quotient of the group algebra.

(i) For $\mathbf{B}(1,r)$ or $\mathbf{C}(n,0)$ the normal form of every permutation is just the identity. That is, every permutation is congruent to the identity.

(ii) For $\mathbf{B}(r-1,r)$ or $\mathbf{C}(n,n-2)$ the normal form of the permutation of longest Coxeter length is given by the alternating sum of all the the other permutations. For instance, in $\mathbf{C}(n,n-2)$ we always have the normal form congruence: $w_0 \equiv \sum_{w \neq w_0} (-1)^{\ell(w)+1} w$. All other normal forms are trivial. The analogous formula holds in the other case. This is clear, as $\sum_w (-1)^{\ell(w)} w$ generates the kernel of $\Psi_{n,n-2}$ and similarly in the other case.

(iii) This leaves $\mathbf{B}(2,4)$ as the sole remaining case of interest when $r = 4$. See Example 1.5 for a list of the elements of $\mathcal{B}_{2,4}$. In that case we have the following non-trivial normal form congruences, where as usual permutations are written in one-line notation:

$$\begin{aligned}
(4321) &\equiv -(1234) + (1243) + 2(1324) - (1342) - (1423) + (2134) \\
&\quad - (2143) - (2314) + (2341) - (3124) + (3412) + (4123). \\
(4312) &\equiv -(1234) + (1243) + (1324) - (1423) + (2134) - (2143) \\
&\quad - (3124) + (3412) + (4123). \\
(4231) &\equiv (1243) + (1324) - (1342) - (1423) + (2134) - 2(2143) \\
&\quad - (2314) + (2341) + (2413) - (3124) + (3142) + (4123). \\
(3421) &\equiv -(1234) + (1243) + (1324) - (1342) + (2134) - (2143) \\
&\quad - (2314) + (2341) + (3412). \\
(4213) &\equiv (1243) - (1423) - (2143) + (2413) + (4123).
\end{aligned}$$

$(4132) \equiv (2134) - (2143) - (3124) + (3142) + (4123).$
$(3241) \equiv (1243) - (1342) - (2143) + (2341) + (3142).$
$(2431) \equiv (2134) - (2143) - (2314) + (2341) + (2413).$
$(3214) \equiv (1234) - (1324) - (2134) + (2314) + (3124).$
$(1432) \equiv (1234) - (1243) - (1324) + (1342) + (1423).$

(iv) In case $n = 4$, the sole interesting case is $\mathbf{C}(4, 1)$. In that case the non-trivial normal form congruences are as follows:

$(4321) \equiv (1234) - (1243) - (2134) + (2341) + (4123).$
$(4312) \equiv (1234) - (1243) - (1324) + (1342) - (2134) + (2314) + (4123).$
$(4231) \equiv 2(1234) - (1243) - (1324) - (2134) + (2341) + (4123).$
$(3421) \equiv (1234) - (1243) - (1324) + (1423) - (2134) + (2341) + (3124).$
$(4213) \equiv (1234) - (1324) - (2134) + (2314) + (4123).$
$(4132) \equiv (1234) - (1243) - (1324) + (1342) + (4123).$
$(3412) \equiv (1234) - (1243) - 2(1324) + (1342) + (1423)$
$\quad - (2134) + (2314) + (3124).$
$(3241) \equiv (1234) - (1324) - (2134) + (2341) + (3124).$
$(2431) \equiv (1234) - (1243) - (1324) + (1423) + (2341).$
$(3214) \equiv (1234) - (1324) - (2134) + (2314) + (3124).$
$(3142) \equiv -(1324) + (1342) + (3124).$
$(2413) \equiv -(1324) + (1423) + (2314).$
$(1432) \equiv (1234) - (1243) - (1324) + (1342) + (1423).$
$(2143) \equiv -(1234) + (1243) + (2134).$

This is the end of the examples.

The normal form congruences determine the structure constants for each algebra with respect to its canonical basis.

Proposition 5.3. *Let $b^{\pi}_{\sigma,\tau}$ be the structure constants of the algebra $\mathbf{B}(n, r)$ with respect to its canonical basis, defined by the equations*

$$\overline{\sigma\tau} = \bar{\sigma}\bar{\tau} = \sum_{\pi \in \mathcal{B}_{n,r}} b^{\pi}_{\sigma,\tau}\, \bar{\pi}$$

for any σ, τ in $\mathcal{B}_{n,r}$. The right hand side of the above is given by the normal form of the product $\sigma\tau$ in $\mathfrak{S}_r$, hence the structure constant $b^{\pi}_{\sigma,\tau}$ depends only on $\sigma\tau$ and π. The analogous statements apply to the structure constants of $\mathbf{C}(n, r)$ with respect to its canonical basis.

Recall that $\mathfrak{S}_r$ is a subgroup of the multiplicative monoid $\mathcal{P}_r(n)^{\times}$ of invertible elements in the partition algebra $\mathcal{P}_r(n)$. This immediately implies that

$$\mathbf{C}(n, r) = \mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r}) \subset \mathrm{End}_{\mathfrak{S}_r}(\mathbf{V}^{\otimes r}) = S(n, r). \tag{8}$$

In other words, the quotient algebra $\Bbbk[W_n]/\ker(\Psi_{n,r})$ is isomorphic to a subalgebra of the Schur algebra.

Recall that $\Lambda(n,r)$ is defined in [Gre07, §3.1] as the set of orbits of the action of $\mathfrak{S}_r$ on $I(n,r) = [n]^r$. Elements of $\Lambda(n,r)$ are called *weights*; we already encountered this notion in the paragraph following Lemma 3.6. A weight α is specified by the vector $\alpha = (\alpha_1, \dots, \alpha_n)$ such that α_j counts the number of times that j appears in $\underline{i} = (i_1, \dots, i_r)$, for any $\underline{i} \in I(n,r)$, where j varies over $[n]$. The action of W_n on $\Lambda(n,r)$ is given by

$$w^{-1}\alpha = (\alpha_{w(1)}, \dots, \alpha_{w(n)}). \tag{9}$$

The following result answers a question of R. Rouquier.

Proposition 5.4. *In terms of Green's basis $\xi_{\underline{i},\underline{j}}$ of the Schur algebra $S(n,r)$, the canonical basis elements $\bar{w}$ for $w \in \mathcal{C}_{n,r}$ satisfy the formula*

$$\bar{w} = \textstyle\sum_{\underline{i}} \xi_{\underline{i},w(\underline{i})}$$

where the sum is carried out over any set $\mathcal{O}(n,r)$ of orbit representatives of $\Lambda(n,r)$.

Proof. By [Gre07, §2.6], the matrix of $\xi_{\underline{i},\underline{j}}$ with respect to the basis $\mathbf{v}_{\underline{j}} = \mathbf{v}_{j_1} \otimes \cdots \otimes \mathbf{v}_{j_r}$ is given by the $n^r \times n^r$ matrix $T(\omega) = [T(\omega)_{\underline{p},\underline{q}}]_{\underline{p},\underline{q}}$ defined by

$$T(\omega)_{\underline{p},\underline{q}} = \begin{cases} 1 & \text{if } (\underline{p},\underline{q}) \sim (\underline{i},\underline{j}) \\ 0 & \text{otherwise,} \end{cases}$$

where $\omega \in \Omega(n,r)$ is the $\mathfrak{S}_r$-orbit containing the pair $(\underline{i},\underline{j})$. On the other hand, the matrix of $\Psi(w)$ with respect to the same basis (for any $w \in W_n$) is the Kronecker power $P(w)^{\otimes r}$, where $P(w) = [\delta_{i,w(j)}]_{i,j\in[n]}$ is the permutation matrix representing w with respect to the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Thus, writing $w(\underline{j}) = (w(j_1), \dots, w(j_r))$ for any $\underline{j} \in [n]^r$, we have

$$\Psi(w) = P(w)^{\otimes r} = [\delta_{\underline{i},w(\underline{j})}]_{\underline{i},\underline{j}\in[n]^r}.$$

By expressing the matrices of $\Psi(w)$ and $\xi_{\underline{i},\underline{j}}$ in terms of the matrix units, one deduces the result. □

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Department of Mathematics, University of York, Heslington, York, YO10 5DD, UK

Email address: Chris.Bowman-Scargill@york.ac.uk

Department of Mathematics and Statistics, Loyola University Chicago, Chicago, IL 60660 USA

Email address: doty@math.luc.edu

DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, UK

Email address: sm137@cam.ac.uk