RESEARCH ARTICLE

Existence and rotatability of the two-colored Jones–Wenzl projector

Amit Hazi 🄳

Department of Mathematics, University
of York, York, UK

Correspondence
Amit Hazi, Department of Mathematics,
University of York, York, YO10 5DD, UK.
Email: amit.hazi@york.ac.uk

Abstract

The two-colored Temperley–Lieb algebra $2\,\mathrm{TL}_R({}_s n)$ is a generalization of the Temperley–Lieb algebra. The analogous two-colored Jones–Wenzl projector $\mathrm{JW}_R({}_s n) \in 2\,\mathrm{TL}_R({}_s n)$ plays an important role in the Elias–Williamson construction of the diagrammatic Hecke category. We give conditions for the existence and rotatability of $\mathrm{JW}_R({}_s n)$ in terms of the invertibility and vanishing of certain two-colored quantum binomial coefficients. As a consequence, we prove that Abe's category of Soergel bimodules is equivalent to the diagrammatic Hecke category in complete generality.

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1 │ INTRODUCTION

Let R be a commutative ring, and fix two scalars $[2]_s, [2]_t \in R$. The *two-colored Temperley–Lieb algebra* $2\,\mathrm{TL}_R({}_s n) := 2\,\mathrm{TL}_R({}_s n; [2]_s, [2]_t)$ is the R-algebra with generators e_i for $1 \leqslant i \leqslant n-1$, subject to the relations

$$e_i^2 = -[2]_s \qquad\qquad i \text{ odd}, \tag{1}$$

$$e_i^2 = -[2]_t \qquad\qquad i \text{ even}, \tag{2}$$

$$e_i e_j = e_j e_i \qquad\qquad \text{for } |i-j| > 1, \tag{3}$$

$$e_i e_{i \pm 1} e_i = e_i. \tag{4}$$

The algebra $2\,\mathrm{TL}_R(_t n)$ is defined identically, except that the parity conditions on the relations (1) and (2) are swapped. These algebras (introduced by Elias in [3]) form a generalization of the ordinary Temperley–Lieb algebra, which occurs as a special case when $[2]_s = [2]_t$. By a standard argument, there is an R-basis of $2\,\mathrm{TL}_R(_s n)$ consisting of monomials in the generators e_i.

We call a nonzero idempotent $\mathrm{JW}_R(_s n) \in 2\,\mathrm{TL}_R(_s n)$ (and similarly for $_t n$) a *two-colored Jones–Wenzl projector* if $e_i \mathrm{JW}_R(_s n) = 0$ for all $1 \leqslant i \leqslant n-1$ and the coefficient of 1 in $\mathrm{JW}_R(_s n)$ is 1. Such idempotents (if they exist) are unique.

The behavior of $2\,\mathrm{TL}_R(_s n)$ is controlled by certain elements $[n]_s, [n]_t \in R$ for $n \in \mathbb{Z}$ called the *two-colored quantum numbers*. These elements (defined in (5)) are bivariate polynomials in $[2]_s$ and $[2]_t$ that are analogous to ordinary quantum numbers. For an integer $0 \leqslant k \leqslant n$, the *two-colored quantum binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_s = \frac{[n]_s!}{[k]_s![n-k]_s!} = \frac{[n]_s[n-1]_s \cdots [n-k+1]_s}{[k]_s[k-1]_s \cdots [1]_s}$$

can also be shown to be an element of R. Our first main result is the two-colored analogue of the well-known existence theorem for ordinary Jones–Wenzl projectors.

Theorem A. *The two-colored Jones–Wenzl projector $\mathrm{JW}_R(_s n)$ exists if and only if $\begin{bmatrix} n \\ k \end{bmatrix}_s$ is invertible in R for each integer $0 \leqslant k \leqslant n$.*

The terminology for two-colored Temperley–Lieb algebras comes from their presentation as diagram algebras. We associate the labels s and t with the colors red and blue, respectively, writing s and t for emphasis. A *two-colored Temperley Lieb diagram* is a Temperley–Lieb diagram with the planar regions between strands colored with alternating colors. As a diagram, algebra $2\,\mathrm{TL}_R(_s n)$ is spanned by two-colored Temperley–Lieb diagrams with n boundary points on the top and bottom whose leftmost region is colored red. A blue disk inside a red region evaluates to $-[2]_s$, whereas a red disk inside a blue region evaluates to $-[2]_t$. Moreover, we only consider two-colored Temperley–Lieb diagrams up to isotopy. These diagrammatic relations directly correspond to (1)–(4). We draw the two-colored Jones–Wenzl projector as a rectangle labeled $\mathrm{JW}_R(_s n)$:

n odd   *n* even
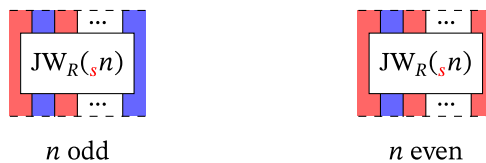
Suppose that both $\mathrm{JW}_R(_s n)$ and $\mathrm{JW}_R(_t n)$ exist. We say that $\mathrm{JW}_R(_s n)$ is *rotatable* if the (clockwise and counterclockwise) rotations of $\mathrm{JW}_R(_s n)$ by one strand are equal to some scalar multiple of $\mathrm{JW}_R(_t n)$:

$$= \lambda \qquad = \qquad (n \text{ odd})$$

$$= \lambda \qquad = \qquad (n \text{ even})$$
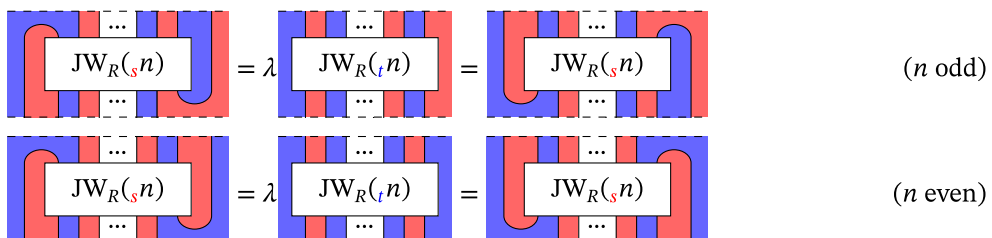
Our second main result gives a combined condition for the existence and rotatability of two-colored Jones–Wenzl projectors.

Theorem B. *The two-colored Jones–Wenzl projectors* $\mathrm{JW}_R({}_s n)$ *and* $\mathrm{JW}_R({}_t n)$ *exist and are rotatable if and only if* $\begin{bmatrix} n+1 \\ k \end{bmatrix}_s = \begin{bmatrix} n+1 \\ k \end{bmatrix}_t = 0$ *in R for all integers* $1 \leqslant k \leqslant n$.

The same algebraic condition was first introduced by Abe in the context of the Hecke category [1, Assumption 1.1], which we discuss below.

The Hecke category

The two-colored Temperley–Lieb algebra lies at the heart of the Elias–Williamson diagrammatic Hecke category [5]. In more detail, the diagrammatic Hecke category is only well defined when certain two-colored Jones–Wenzl projectors exist and are rotatable. Elias–Williamson initially gave an incorrect algebraic condition for rotatability in [5, (3.3)]; they later identified and partially corrected this error in [6, §5]. Our rotatability condition in Theorem B is enough to completely correct this error.

Corollary C. *In the absence of parabolic type* H_3 *subgroups (see Remark 5.3), the diagrammatic Hecke category is well defined if and only if the underlying realization is an Abe realization (see [1, Assumption 1.1] or Definition 5.1).*

Recently, Abe has shown that there is a "bimodule-theoretic" category (a modification of the category of classical Soergel bimodules), which under mild conditions is equivalent to the diagrammatic Hecke category when the underlying realization is an Abe realization [1, 2]. An important consequence of Corollary C is that this equivalence essentially always holds.

Corollary D. *Assume that Demazure surjectivity holds (see Remark 5.2), and that the base ring is a Noetherian domain. If the diagrammatic Hecke category is well defined, it is equivalent to Abe's category of Soergel bimodules.*

We find it noteworthy that our result gives the best possible equivalence result for two seemingly distinct categorifications of the Hecke algebra.

2 | PRELIMINARIES

Let $A = \mathbb{Z}[x_s, x_t]$ be the integral polynomial ring in two variables. The *two-colored quantum numbers* are defined as follows. First, set $[1]_s = [1]_t = 1$, $[2]_s = x_s$, and $[2]_t = x_t$ in A. For $n > 1$, we inductively define

$$[n+1]_s = [2]_s[n]_t - [n-1]_s, \qquad\qquad [n+1]_t = [2]_t[n]_s - [n-1]_t. \qquad (5)$$

These formulas can be rearranged to inductively define $[n]_s$ and $[n]_t$ for $n \leqslant 0$. For a commutative A-algebra R, we also define two-colored quantum numbers in R to be the specializations of two-colored quantum numbers in A, which we will write in the same way.

These polynomials are bivariate extensions of the usual (one-colored) quantum numbers, which can be recovered as follows. Let $\overline{A} = A/(x_s - x_t) \cong \mathbb{Z}[x]$, where x is the image of x_s or x_t. Then, the one-colored quantum number $[n]$ is the image of $[n]_s$ or $[n]_t$ in $\overline{A}$. When n is odd, $[n]$ is an even polynomial, so we can formally evaluate $[n]$ at $x = \sqrt{x_s x_t}$ to obtain an element of A. When n is even, $[n]/[2]$ is an even polynomial, which we can similarly formally evaluate at $x = \sqrt{x_s x_t}$. In both cases, it is easy to show by induction that

$$[n]_s = [n](\sqrt{x_s x_t}) = [n]_t \qquad \text{if } n \text{ is odd,}$$

$$\frac{[n]_s}{[2]_s} = \left(\frac{[n]}{[2]}\right)(\sqrt{x_s x_t}) = \frac{[n]_t}{[2]_t} \quad \text{if } n \text{ is even} \tag{6}$$

in A. In other words, two-colored quantum numbers are essentially the same as ordinary quantum numbers up to a factor of $[2]_s$ and $[2]_t$ depending on color.

It is self-evident that the automorphism of A which exchanges x_s and x_t ("color swap") also exchanges $[n]_s$ and $[n]_t$ for all n. For this reason, we will generally write statements only for $[n]_s$ and leave it to the reader to formulate color-swapped analogues. Similarly, we have $2\,\mathrm{TL}(_s n; [2]_s, [2]_t) \cong 2\,\mathrm{TL}(_t n; [2]_t, [2]_s)$, and this isomorphism maps $\mathrm{JW}_R(_s n)$ to $\mathrm{JW}_R(_t n)$ when they exist, so we will only state our results for $2\,\mathrm{TL}_R(_s n)$ and $\mathrm{JW}_R(_s n)$.

Let $D = e_{i_1} e_{i_2} \cdots e_{i_r}$ be a monomial of length r in the generators of $2\,\mathrm{TL}(_s n)$. We say that D is *reduced* if it cannot be rewritten as a monomial $e_{j_1} e_{j_2} \cdots e_{j_s}$ in the generators using (1)–(4) for some $s < r$. As mentioned in Section 1, the two-colored Temperley–Lieb algebra $2\,\mathrm{TL}(_s n)$ has a basis consisting of these reduced monomials. As in the one-colored case, there is a bijection between this basis in $2\,\mathrm{TL}(_s n)$ and (isotopy classes of) two-colored Temperley–Lieb diagrams whose leftmost region is colored red which induces an isomorphism between the algebraic and diagrammatic versions of $2\,\mathrm{TL}(_s n)$. (For a careful proof of this fact in the one-colored case, see for example, [10, Theorem 2.4].) Under this isomorphism, we have
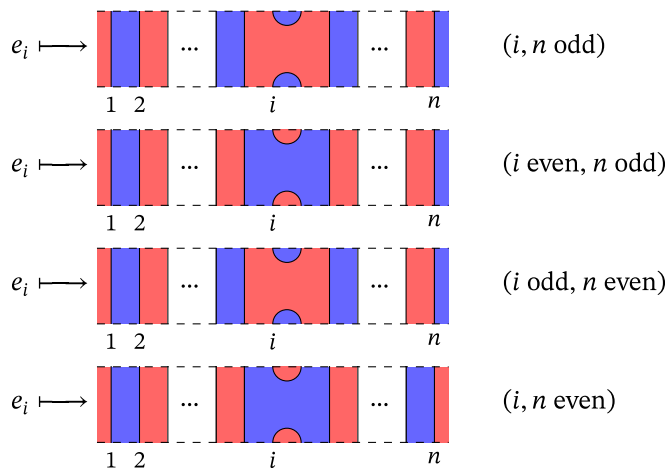
Given an element $f \in 2\,\mathrm{TL}_R(_s n)$ and a two-colored Temperley–Lieb diagram D, we will write

$$\operatorname*{coeff}_{\in f} D$$

for the coefficient of D when f is written in the diagrammatic basis.

If R is a commutative A-algebra for which $\mathrm{JW}_R(_s n)$ exists for all n, then the coefficients of $\mathrm{JW}_R(_s n)$ can be calculated inductively as follows. Suppose that D is a two-colored Temperley–Lieb diagram in $2\,\mathrm{TL}_R(_s(n+1))$. Let $\hat{D}$ be the diagram with $n+2$ bottom boundary points and n top boundary points obtained by folding down the strand connected to the top right boundary point of D. If there is a strand connecting the ith and $(i+1)$th bottom boundary points of $\hat{D}$, let D_i denote the two-colored Temperley–Lieb diagram with n strands so obtained by deleting this cap. For example, if

$$D = \quad$$

then

$$\hat{D} = \quad$$

and

$$D_2 = \quad , \qquad D_5 = \quad .$$
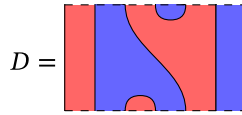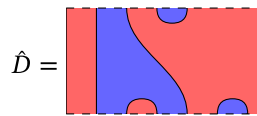
Theorem 2.1. *Suppose that $\mathrm{JW}_R(_s n)$ and $\mathrm{JW}_R(_s(n+1))$ both exist. Then $[n+1]_s$ is invertible, and we have*

$$\operatorname*{coeff}_{\in \mathrm{JW}_R(_s(n+1))} D = \sum_i \frac{[i]_u}{[n+1]_s} \operatorname*{coeff}_{\in \mathrm{JW}_R(_s n)} D_i,$$

where the sum is taken over all positions i where D_i is defined, and u is the color of the deleted cap.

Proof. The argument in the one-color setting (see [9, Proposition 4.1] or [8, Corollary 3.7]) follows essentially unchanged from [6, (6.29)]. □

By a similar computation, it can be shown that $\mathrm{JW}_{\mathrm{Frac}\,A}(_s n)$ exists for all $n \in \mathbb{N}$ (see e.g. [6, Theorem 6.14]). We will carefully show later that this computation is "generic", i.e. if $\mathrm{JW}_R(_s n)$ exists, then its coefficients are specializations of the coefficients of $\mathrm{JW}_{\mathrm{Frac}\,A}(n)$.

The existence criterion in Theorem A is known to hold in the one-color setting, i.e. when the images of x_s and x_t in R are equal. In these circumstances, we write $\mathrm{TL}_R(n)$ and $\mathrm{JW}_R(n)$ for the one-color Temperley–Lieb algebra and Jones–Wenzl projector.

Theorem 2.2 [4, Theorem A.2]. *Suppose that R is a commutative A-algebra which factors through A. Then $\mathrm{JW}_R(n)$ exists if and only if the one-color quantum binomial coefficients*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = \frac{[n][n-1]\cdots[n-k+1]}{[k][k-1]\cdots[1]}$$

are invertible in R for all integers $0 \leqslant k \leqslant n$.

In light of the "generic" nature of the coefficients of $\mathrm{JW}_R({}_s n)$, we can interpret Theorem 2.2 as description of the denominators of the coefficients of $\mathrm{JW}_{\mathrm{Frac}\,\overline{A}}(n)$. Unfortunately, none of the known proofs of this result (most of which use connections to Lie theory in a crucial way) generalize easily to the two-colored setting.

Finally, we will give an alternative criterion for checking rotatability. For $f \in 2\,\mathrm{TL}_R({}_s n)$, define the *partial trace* of f to be

$$\mathrm{pTr}(f) = \qquad\qquad (n \text{ odd})$$

$$\mathrm{pTr}(f) = \qquad\qquad (n \text{ even})$$

From the definition of the Jones–Wenzl projector, it is easy to see that $\mathrm{JW}_R({}_s n)$ is rotatable if and only if $\mathrm{pTr}(\mathrm{JW}_R({}_s n)) = 0$. Using entirely standard techniques (e.g., [6, §6.6]), one can show that

$$\mathrm{pTr}(\mathrm{JW}_R({}_s n)) = -\frac{[n+1]_s}{[n]_s}\mathrm{JW}_R({}_s(n-1)) \qquad (7)$$

when both $\mathrm{JW}_R({}_s n)$ and $\mathrm{JW}_R({}_s(n-1))$ exist. This gives the following partial rotatability criterion.

Proposition 2.3. *Suppose that both $\mathrm{JW}_R({}_s n)$ and $\mathrm{JW}_R({}_s(n-1))$ exist. Then $\mathrm{JW}_R({}_s n)$ is rotatable if and only if $[n+1]_s = 0$.*

The key to proving the full rotatability criterion will be to interpret (7) generically.

3 | PRINCIPAL IDEALS

In this section, we show that several ideals generated by certain two-colored quantum numbers and binomial coefficients are principal. Recall that for ordinary quantum numbers, one can show that if $d|n$ then $[d]|[n]$. Using (6), it immediately follows that $[d]_s|[n]_s$.

Lemma 3.1 (Quantum Bézout's identity). *Let $m, n \in \mathbb{N}$. There exist polynomials $a, b \in A$ such that*

$$a[m]_s + b[n]_s = [\gcd(m, n)]_s.$$

Proof. Suppose without loss of generality that $m < n$. We will show that the ideal in A generated by $[m]_s$ and $[n]_s$ contains $[n-m]_s$. If m and n are not both odd, then

$$[n-1]_t[m]_s - [m-1]_t[n]_s = ([m+n-2]_s + [m+n-4]_s + \cdots + [-(n-m)+2]_s)$$

$$- ([m+n-2]_s + [m+n-4]_s + \cdots + [n-m+2]_s)$$

$$= [n-m]_s + [n-m-2]_s + \cdots + [-(n-m)+2]_s$$

$$= [n-m]_s$$

by [6, (6.5a)–(6.5c)]. If m and n are both odd, a similar calculation yields

$$[n-1]_s[m]_s - [m-1]_s[n]_s = [n-m]_s.$$

By repeating this step multiple times, we can run Euclid's algorithm, and the result follows. □

Next, we introduce the cyclotomic parts of quantum numbers, which are roughly analogous to cyclotomic polynomials. Recall that the one-color quantum numbers are renormalizations of Chebyshev polynomials of the second kind. More precisely, if we evaluate a quantum number $[n]$ at $x = 2\cos\theta$, we obtain

$$[n](2\cos\theta) = \frac{\sin n\theta}{\sin\theta}.$$

Since $[n]$ is a monic polynomial in x of degree $n-1$, we conclude that

$$[n] = \prod_{k=1}^{n-1}\left(x - 2\cos\frac{k\pi}{n}\right).$$

We define the *cyclotomic part* of the one-color quantum number $[n]$ to be the polynomial

$$\Theta_n = \prod_{\substack{1\leqslant k<n \\ (k,n)=1}}\left(x - 2\cos\frac{k\pi}{n}\right).$$

Lemma 3.2. *Let $n \in \mathbb{N}$. We have*

(i) $\Theta_n \in \mathbb{Z}[x]$, *and* $\deg\Theta_n = \varphi(n)$ *when $n > 1$;*
(ii) $[n] = \prod_{k|n}\Theta_n$;
(iii) $\Theta_n = \prod_{k|n}[k]^{\mu(n/k)}$, *where* $\mu : \mathbb{N} \to \{\pm 1\}$ *is the Möbius function.*

Moreover, if $n > 2$ then we also have $\Theta_n(x) = \Psi_n(x^2)$, *where* $\Psi_n \in \mathbb{Z}[x]$ *is the minimal polynomial of* $4\cos^2(\pi/n)$.

Proof. Both (i) and (ii) follow from the definition and basic properties of cyclotomic fields and algebraic integers. Applying Möbius inversion to (ii) yields (iii). For the final claim, we observe that if $n > 2$ then Θ_n is an even polynomial, so is of the form of $\Psi_n(x^2)$ for some $\Psi_n \in \mathbb{Z}[x]$ of degree $\varphi(n)/2$. By construction, $4\cos^2(\pi/n)$ is a root of Ψ_n. Since

$$4\cos^2\frac{\pi}{n} = 2\cos\frac{2\pi}{n} + 2$$

and $\mathbb{Q}(2\cos(2\pi/n) + 2) = \mathbb{Q}(\cos(2\pi/n))$ is a field extension of $\mathbb{Q}$ of degree $\varphi(n)/2$, Ψ_n must be the minimal polynomial of $4\cos^2(\pi/n)$. □

Definition 3.3. For $n \in \mathbb{N}$, we define the *cyclotomic part* of the two-colored quantum number $[n]_s$ to be

$$\Theta_{n,s} = \begin{cases} \Psi_n(x_s x_t) & \text{if } n > 2, \\ x_s & \text{if } n = 2, \\ 1 & \text{if } n = 1. \end{cases}$$

Using (6) and Lemma 3.2, we similarly obtain $[n]_s = \prod_{k|n} \Theta_{n,s}$ and $\Theta_{n,s} = \prod_{k|n} [n]_s^{\mu(n/k)}$.

Lemma 3.4. *The polynomials $\Theta_{n,s}$ are all irreducible and distinct in A (but note that $\Theta_{n,s} = \Theta_{n,t}$ if $n > 2$).*

Proof. Irreducibility is clear when $n = 2$. When $n > 2$, we have $\Theta_{n,s} = \Theta_{n,t} = \Psi_n(x_s x_t)$, which is irreducible because Ψ_n is (see e.g. [11, (3.3)]). Distinctness follows as well because the polynomials Ψ_n are distinct. $\qquad\square$

Lemma 3.5. *Let $m, n \in \mathbb{N}$ such that $m \nmid n$ and $n \nmid m$. There exist polynomials $a, b \in A$ such that*

$$a\Theta_{m,s} + b\Theta_{n,s} = 1.$$

Proof. Suppose without loss of generality that $m < n$, and let $d = \gcd(m, n)$. By Lemma 3.1, there exist $a', b' \in A$ such that

$$a'[m]_s + b'[n]_s = [d]_s.$$

By assumption $d < m < n$, so we have

$$\frac{[m]_s}{[d]_s} \in \Theta_{m,s}A$$

$$\frac{[n]_s}{[d]_s} \in \Theta_{n,s}A,$$

and thus on dividing by $[d]_s$, we obtain

$$a\Theta_{m,s} + b\Theta_{n,s} = 1.$$

$\qquad\square$

Proposition 3.6. *Let $m_1, m_2, \dots, m_k, n_1, n_2, \dots, n_l \in \mathbb{N}$ such that for all i, j either $m_i = n_j$ or $m_i \nmid n_j$ and $n_j \nmid m_i$. Then the ideal*

$$(\Theta_{m_1,s}\Theta_{m_2,s} \cdots \Theta_{m_k,s}, \Theta_{n_1,s}\Theta_{n_2,s} \cdots \Theta_{n_l,s})$$

in A is principal.

Proof. Let I be the ideal above. We may assume without loss of generality that $m_i \neq n_j$ for all i, j, i.e. the generators of I are coprime in A. For each i, j we can apply Lemma 3.5 to obtain

$a_{i,j}, b_{i,j} \in A$ such that $a_{i,j}\Theta_{m_i,s} + b_{i,j}\Theta_{n_j,s} = 1$. Taking the product over all i and j, we obtain

$$1 = \prod_i \left(\prod_j (a_{i,j}\Theta_{m_i,s} + b_{i,j}\Theta_{n_j,s}) \right) \in \prod_i (\Theta_{m_i,s}, \Theta_{n_1,s}\Theta_{n_2,s} \cdots \Theta_{n_l,s}) \subseteq I,$$

so $I = (1)$ is principal. $\qquad\square$

For $f \in A$ and $l > 1$ an integer, we define the cyclotomic valuation $\nu_{l,s}(f)$ to be the exponent of the highest power of $\Theta_{l,s}$ dividing f. This extends to $\operatorname{Frac} A$ in the obvious way; namely, we define $\nu_{l,s}(f/g) = \nu_{l,s}(f) - \nu_{l,s}(g)$ for $f, g \in A$. If f and g are products of s-colored cyclotomic parts, then

$$\frac{f}{g} = \prod_l \Theta_{l,s}{}^{\nu_{l,s}(f/g)}.$$

For $f \in \overline{A}$, we similarly define $\nu_l(f)$ to be the highest power of Θ_l dividing f, which extends in a completely analogous way to $\operatorname{Frac} \overline{A}$.

Lemma 3.7. *Let n, k be nonnegative integers. For all integers $1 < l \leqslant n$, we have*

$$\nu_{l,s} \begin{bmatrix} n \\ k \end{bmatrix}_s = \left\lfloor \frac{n}{l} \right\rfloor - \left\lfloor \frac{k}{l} \right\rfloor - \left\lfloor \frac{n-k}{l} \right\rfloor.$$

In particular, $\nu_{l,s} \begin{bmatrix} n \\ k \end{bmatrix}_s \in \{0, 1\}$.

Proof. Clearly

$$\nu_{l,s}[m]_s = \begin{cases} 1 & \text{if } l|m, \\ 0 & \text{otherwise,} \end{cases}$$

so $\nu_{l,s}([m]_s!) = \lfloor m/l \rfloor$ and the equation above follows. To show the bound on the valuation, note that $m/l - 1 < \lfloor m/l \rfloor \leqslant m/l$, so

$$-1 = \left(\frac{n}{l} - 1 \right) - \frac{k}{l} - \frac{n-k}{l} < \left\lfloor \frac{n}{l} \right\rfloor - \left\lfloor \frac{k}{l} \right\rfloor - \left\lfloor \frac{n-k}{l} \right\rfloor < \frac{n}{l} - \left(\frac{k}{l} - 1 \right) - \left(\frac{n-k}{l} - 1 \right) = 2.$$

$\qquad\square$

Theorem 3.8. *Let $n \in \mathbb{N}$. The ideal*

$$\left(\begin{bmatrix} n \\ 1 \end{bmatrix}_s, \begin{bmatrix} n \\ 2 \end{bmatrix}_s, \dots, \begin{bmatrix} n \\ n-1 \end{bmatrix}_s \right)$$

in A is principal, generated by $\Theta_{n,s}$.

Proof. We will prove the result by induction. Let

$$I_m = \left(\begin{bmatrix} n \\ 1 \end{bmatrix}_s, \begin{bmatrix} n \\ 2 \end{bmatrix}_s, \dots, \begin{bmatrix} n \\ m \end{bmatrix}_s \right)$$

and write

$$[n]_s^{>m} = \prod_{\substack{k>m \\ k|n}} \Theta_{k,s}.$$

Suppose that we have shown that I_m is principal, generated by $[n]_s^{>m}$. We will show that I_{m+1} is principal, generated by $[n]_s^{>m+1}$. It is enough to show that

$$\left(\begin{bmatrix} n \\ m+1 \end{bmatrix}_s, [n]_s^{>m} \right) = ([n]_s^{>m+1}). \tag{8}$$

Clearly $[n]_s^{>m+1}$ divides $[n]_s^{>m}$. If $k > m+1$ and $k|n$, it is easy to see that

$$\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{m+1}{k} \right\rfloor - \left\lfloor \frac{n-(m+1)}{k} \right\rfloor = 1,$$

so Θ_k divides $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s$ exactly once, and thus, $[n]_s^{>m+1}$ divides $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s$. If $m+1 \nmid n$ then $[n]_s^{>m+1} = [n]_s^{>m}$, (8) follows trivially.

Otherwise suppose $m+1|n$. We claim that if $\Theta_{l,s}$ divides $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s/[n]_s^{>m+1}$, we must have $l \nmid m+1$ and $m+1 \nmid l$. This implies that

$$\left(\frac{\begin{bmatrix} n \\ m+1 \end{bmatrix}_s}{[n]_s^{>m+1}}, \Theta_{m+1,s} \right) = (1)$$

by Proposition 3.6, from which (8) holds and the result follows.

To prove the claim, suppose $l|m+1$. It is straightforward to check that

$$\left\lfloor \frac{n}{l} \right\rfloor - \left\lfloor \frac{m+1}{l} \right\rfloor - \left\lfloor \frac{n-(m+1)}{l} \right\rfloor = 0,$$

so $\Theta_{l,s}$ does not divide $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s$, let alone $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s/[n]_s^{>m+1}$.

Similarly, suppose $m+1|l$, and take $0 \leqslant r < l$ such that $n-(m+1) = ql+r$. If $\Theta_{l,s}$ divides $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s$ then

$$\left\lfloor \frac{n}{l} \right\rfloor - \left\lfloor \frac{m+1}{l} \right\rfloor - \left\lfloor \frac{n-(m+1)}{l} \right\rfloor = 1,$$

and we must have $r+m+1 \geqslant l$. Now let $d = \gcd(l, n-(m+1))$. As $m+1|n$ and $m+1|l$, we have $m+1|d$, and in particular $m+1 \leqslant d$. We also have $d|r$, so in particular $l-r \geqslant d$. We combine these two equalities to obtain $r+m+1 \leqslant l$, with equality if and only if $l-r = d$ and $m+1 = d$. This immediately implies that $l|n$, so $\Theta_{l,s}$ does not divide $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s/[n]_s^{>m+1}$. □

Theorem 3.9. *Let $n \in \mathbb{N}$. The fractional ideal of A generated by*

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_s^{-1}, \begin{bmatrix} n \\ 1 \end{bmatrix}_s^{-1}, \ldots, \begin{bmatrix} n \\ n \end{bmatrix}_s^{-1}$$

is principal, generated by

$$\left(\prod_{\substack{1 \leqslant k \leqslant n \\ k \nmid n+1}} \Theta_{k,s} \right)^{-1}.$$

Proof. We follow a similar strategy as in the proof of Theorem 3.8. Let I_m denote the fractional ideal generated by

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_s^{-1}, \begin{bmatrix} n \\ 1 \end{bmatrix}_s^{-1}, \dots, \begin{bmatrix} n \\ m \end{bmatrix}_s^{-1},$$

and let

$$g_m = \prod_{\substack{k | n-m+i \text{ for some } 1 \leqslant i \leqslant m \\ k \nmid n+1}} \Theta_{k,s}.$$

Suppose we have shown that I_m is principal, generated by g_m^{-1}. We will show that I_{m+1} is principal, generated by g_{m+1}^{-1}. It is enough to show that the fractional ideal generated by

$$\begin{bmatrix} n \\ m+1 \end{bmatrix}_s^{-1}, g_m^{-1}$$

is equal to the principal fractional ideal generated by g_{m+1}^{-1}. This is equivalent to proving equality of the following (ordinary) ideals

$$\left(g_m, \begin{bmatrix} n \\ m+1 \end{bmatrix}_s \right) = \left(\frac{\begin{bmatrix} n \\ m+1 \end{bmatrix}_s}{h_m} \right) \tag{9}$$

of A, where

$$h_m = \frac{g_{m+1}}{g_m} = \prod_{\substack{k | n-m \\ k \nmid n-m+1, k \nmid n-m+2, \dots, k \nmid n+1}} \Theta_{k,s}.$$

(In particular, this shows that $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s$ divides $g_m h_m = g_{m+1}$.)

We first check that the ideal on the right-hand side of (9) is an ordinary ideal. If $\Theta_{k,s}$ divides h_m (i.e., if $k | n-m$ and $k \nmid n-m+i$ for all $1 \leqslant i \leqslant m+1$), then $k \nmid m+1$ and the fractional part of $(n-(m+1))/k$ is $(k-1)/k$. This implies that

$$\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{m+1}{k} \right\rfloor - \left\lfloor \frac{n-(m+1)}{k} \right\rfloor = 1,$$

so $\Theta_{k,s}$ also divides $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s$.

It is clear that $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s/h_m$ divides $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s$. Suppose that $\Theta_{k,s}$ divides $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s/h_m$. Since we can write

$$\begin{bmatrix} n \\ m+1 \end{bmatrix} = \frac{[n]_s[n-1]_s \cdots [n-m]_s}{[m+1]_s[m]_s \cdots [1]_s},$$

this implies that either $k \nmid n - m$ and $k | n - m + i$ for some $1 \leqslant i \leqslant m$, or $k | n - m$ and $k | n - m + i$ for some $1 \leqslant i \leqslant m+1$. In either case, it is easy to check that $k \nmid n+1$, for otherwise n/k has fractional part $(k-1)/k$, so

$$\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{m+1}{k} \right\rfloor - \left\lfloor \frac{n-(m+1)}{k} \right\rfloor = 0,$$

and $\Theta_{k,s}$ cannot divide $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s$. This shows that $\begin{bmatrix} n \\ m+1 \end{bmatrix}_s / h_m$ divides g_m.

We will now show that

$$\left(\frac{g_{m+1}}{\begin{bmatrix} n \\ m+1 \end{bmatrix}_s}, h_m \right) = (1)$$

using Proposition 3.6, from which (9) holds and the result follows. It is enough to show that for any $l, d > 1$, we do not have $\Theta_{l,s} | g_{m+1} / \begin{bmatrix} n \\ m+1 \end{bmatrix}_s$ and $\Theta_{ld,s} | h_m$, or $\Theta_{ld,s} | g_{m+1} / \begin{bmatrix} n \\ m+1 \end{bmatrix}_s$ and $\Theta_{l,s} | h_m$.

Suppose first that $\Theta_{l,s} | g_{m+1} / \begin{bmatrix} n \\ m+1 \end{bmatrix}_s$ and $\Theta_{ld,s} | h_m$. Then, $l \nmid n+1$ and $ld | n - m$, so $l | n - m$ and $l \nmid m+1$. This shows that the fractional part of $(n - (m+1))/l$ is $(l-1)/l$ and the fractional part of $(m+1)/l$ is nonzero, so

$$\left\lfloor \frac{n}{l} \right\rfloor - \left\lfloor \frac{m+1}{l} \right\rfloor - \left\lfloor \frac{n-(m+1)}{l} \right\rfloor = 1,$$

which contradicts $\Theta_{l,s} | g_{m+1} / \begin{bmatrix} n \\ m+1 \end{bmatrix}_s$.

Similarly, suppose that $\Theta_{ld,s} | g_{m+1} / \begin{bmatrix} n \\ m+1 \end{bmatrix}_s$ and $\Theta_{l,s} | h_m$. Then $l | n - m$ and $l \nmid n - m + i$ for $1 \leqslant i \leqslant m+1$, while $ld | n - m + i$ for some $0 \leqslant i \leqslant m$ and $ld \nmid n+1$. The only way this can happen is if $ld | n - m$. This implies that $ld \nmid m+1$, and we similarly obtain

$$\left\lfloor \frac{n}{ld} \right\rfloor - \left\lfloor \frac{m+1}{ld} \right\rfloor - \left\lfloor \frac{n-(m+1)}{ld} \right\rfloor = 1,$$

which contradicts $\Theta_{ld,s} | g_{m+1} / \begin{bmatrix} n \\ m+1 \end{bmatrix}_s$. □

4 | EXISTENCE AND ROTATABILITY

Let $Q = \text{Frac } A$ and $\overline{Q} = \text{Frac } \overline{A}$. Our goal in this section is to prove Theorem A by showing that the denominators of the coefficients of $\text{JW}_Q(_s n)$ divide

$$\prod_{\substack{1 \leqslant k \leqslant n \\ k \nmid n+1}} \Theta_{k,s}$$

by comparing them with the coefficients of $\text{JW}_{\overline{Q}}(n)$. First, we prove the analogous statement for $\text{JW}_{\overline{Q}}(n)$.

Lemma 4.1. *Let $k > 1$ be an integer, and let D be a one-colored Temperley–Lieb diagram in $\mathrm{TL}_{\overline{Q}}(n)$. Then*

$$\nu_k\left(\operatorname*{coeff}_{\in \mathrm{JW}_{\overline{Q}}(n)} D\right) \geqslant -1,$$

with equality only if $1 < k \leqslant n$ and $k \nmid n + 1$.

Proof. We proceed by induction. Suppose that the result holds for $n = m$, and let D be a one-colored Temperley–Lieb diagram in $\mathrm{TL}_{\overline{Q}}(m + 1)$. By the one-color version of Theorem 2.1, we have

$$\nu_k\left(\operatorname*{coeff}_{\in \mathrm{JW}_{\overline{Q}}(m+1)} D\right) = \nu_k\left(\sum_{\{i\}} \frac{[i]}{[m+1]} \operatorname*{coeff}_{\in \mathrm{JW}_{\overline{Q}}(m)} D_i\right) \tag{10}$$

$$\geqslant \min_{\{i\}}\left(\nu_k\left(\frac{[i]}{[m+1]}\right) + \nu_k\left(\operatorname*{coeff}_{\in \mathrm{JW}_{\overline{Q}}(m)} D_i\right)\right).$$

If $k \nmid m + 1$, then $\nu_k([i]/[m+1]) \geqslant 0$ for any i and $\nu_k(\operatorname{coeff}_{\in \mathrm{JW}_{\overline{Q}}(m)} D_i) \geqslant -1$. On the other hand, if $k | m + 1$, then $\nu_k([i]/[m+1]) \geqslant -1$ while $\nu_k(\operatorname{coeff}_{\in \mathrm{JW}_{\overline{Q}}(m)} D_i) \geqslant 0$. In either case, the sum of the two valuations is at least -1, so the right-hand side of (10) is at least -1.

Now suppose that we have equality. By Theorem 2.2, the one-color Jones–Wenzl projector exists over the subring

$$\overline{Q}_{\mathrm{binom}} = \mathbb{Z}[x]\left[\begin{bmatrix} m+1 \\ 0 \end{bmatrix}^{-1}, \begin{bmatrix} m+1 \\ 1 \end{bmatrix}^{-1}, \dots, \begin{bmatrix} m+1 \\ m+1 \end{bmatrix}^{-1}\right].$$

The natural embedding $\overline{Q}_{\mathrm{binom}} \subset \overline{Q}$ induces an embedding $\mathrm{TL}_{\overline{Q}_{\mathrm{binom}}}(m+1) \to \mathrm{TL}_{\overline{Q}}(m+1)$, and the image of $\mathrm{JW}_{\overline{Q}_{\mathrm{binom}}}(m+1)$ in $\mathrm{TL}_{\overline{Q}}(m+1)$ is clearly a Jones–Wenzl projector. Since Jones–Wenzl projectors are unique, we conclude that the coefficients of $\mathrm{JW}_{\overline{Q}}(m+1)$ lie in $\overline{Q}_{\mathrm{binom}}$. In particular, if the k-valuation of any given coefficient is negative, then Θ_k must divide $\begin{bmatrix} m+1 \\ r \end{bmatrix}$ for some $0 \leqslant r \leqslant m + 1$, so it must divide the least common multiple of $\begin{bmatrix} m+1 \\ 0 \end{bmatrix}, \begin{bmatrix} m+1 \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} m+1 \\ m+1 \end{bmatrix}$. But a consequence of the one-color version of Theorem 3.9 is that this least common multiple is

$$g_{m+1} = \prod_{\substack{1 \leqslant r \leqslant m+1 \\ r \nmid m+2}} \Theta_r.$$

So, we must have $(\Theta_k, g_{m+1}) = (\Theta_k)$, so $1 < k \leqslant m + 1$ and $k \nmid m + 2$ as required. $\qquad \square$

Now let $A' = A[x]/(x^2 - x_s x_t)$. We view A' as both an A-algebra and an $\overline{A}$-algebra in the obvious way. Writing $Q' = \operatorname{Frac} A'$, we have an isomorphism

$$\mathrm{TL}_{Q'}(n) \longrightarrow 2\,\mathrm{TL}_{Q'}({}_s n)$$

$$e_i \longmapsto \begin{cases} \dfrac{x}{x_s} e_i & i \text{ odd}, \\ \dfrac{x}{x_t} e_i & i \text{ even}, \end{cases}$$

which maps $\mathrm{JW}_{Q'}(n) \mapsto \mathrm{JW}_{Q'}({}_s n)$. So, for any two-colored Temperley–Lieb diagram D, we have

$$\operatorname*{coeff}_{\in \mathrm{JW}_Q({}_s n)} D = \operatorname*{coeff}_{\in \mathrm{JW}_{Q'}({}_s n)} D = x^a x_s{}^b x_t{}^c \operatorname*{coeff}_{\in \mathrm{JW}_{Q'}(n)} \overline{D} = x^a x_s{}^b x_t{}^c \operatorname*{coeff}_{\in \mathrm{JW}_{\overline{Q}}(n)} \overline{D}$$

for some integers a, b, c for which $a + b + c = 0$, where $\overline{D}$ denotes the one-color diagram obtained from D by forgetting the coloring. It follows that when $k > 2$, we have

$$\nu_{k,s}\left(\operatorname*{coeff}_{\in \mathrm{JW}_Q({}_s n)} D\right) = \nu_{k,t}\left(\operatorname*{coeff}_{\in \mathrm{JW}_Q({}_s n)} D\right) = \nu_k\left(\operatorname*{coeff}_{\in \mathrm{JW}_{\overline{Q}}(n)} \overline{D}\right), \tag{11}$$

and

$$\nu_{2,s}\left(\operatorname*{coeff}_{\in \mathrm{JW}_Q({}_s n)} D\right) + \nu_{2,t}\left(\operatorname*{coeff}_{\in \mathrm{JW}_Q({}_s n)} D\right) = \nu_k\left(\operatorname*{coeff}_{\in \mathrm{JW}_{\overline{Q}}(n)} \overline{D}\right). \tag{12}$$

Lemma 4.2. *Let $k > 1$ be an integer, and let D be a two-colored Temperley–Lieb diagram in $2\,\mathrm{TL}_Q({}_s n)$. Then*

$$\nu_{k,u}\left(\operatorname*{coeff}_{\in \mathrm{JW}_Q({}_s n)} D\right) \geqslant -1,$$

and if we have equality, then $1 < k \leqslant n$ and $k \nmid n + 1$, and $u = s$ if $k = 2$.

Proof. By Lemma 4.1 and (11), we need only concern ourselves with the case where $k = 2$. We proceed by induction as in the proof of Lemma 4.1. Suppose that the result holds for $n = m$, and let D be a two-colored Temperley–Lieb diagram in $\mathrm{TL}_Q({}_s(m + 1))$. By Theorem 2.1, we have

$$\nu_{2,u}\left(\operatorname*{coeff}_{\in \mathrm{JW}_Q({}_s(m+1))} D\right) = \nu_{2,u}\left(\sum_{\{i\}} \frac{[i]_\upsilon}{[m+1]_s} \operatorname*{coeff}_{\in \mathrm{JW}_Q({}_s m)} D_i\right)$$

$$\geqslant \min_{\{i\}}\left(\nu_{2,u}\left(\frac{[i]_\upsilon}{[m+1]_s}\right) + \nu_{2,u}\left(\operatorname*{coeff}_{\in \mathrm{JW}_Q({}_s m)} D_i\right)\right). \tag{13}$$

If m is even, then for all i

$$\nu_{2,u}\left(\frac{[i]_\upsilon}{[m+1]_s}\right) \geqslant 0 \qquad \text{and} \qquad \nu_{2,u}\left(\operatorname*{coeff}_{\in \mathrm{JW}_Q({}_s m)} D_i\right) \geqslant -1$$

by induction. If $u = t$, then for all i

$$\nu_{2,t}\left(\frac{[i]_\upsilon}{[m+1]_s}\right) \geqslant 0 \qquad \text{and} \qquad \nu_{2,t}\left(\operatorname*{coeff}_{\in \mathrm{JW}_Q({}_s m)} D_i\right) \geqslant 0$$

since $\nu_{2,t}[m+1]_s = 0$ for all m and because the t-colored valuation is nonnegative by induction. On the other hand, if m is odd and $u = s$, then for all i

$$\nu_{2,u}\left(\frac{[i]_\upsilon}{[m+1]_s}\right) \geqslant -1 \qquad \text{and} \qquad \nu_{2,s}\left(\operatorname*{coeff}_{\in \mathrm{JW}_{\overline{Q}}({}_s m)} D_i\right) \geqslant 0.$$

In all cases, the sum of the two valuations is at least -1 (and at least 0 in the case where $u = t$), so the right-hand side of (13) is at least -1.

Now suppose $u = s$ and the left-hand side of (13) is -1 but m is even. There is an involution of $\mathbb{Z}$-algebras (or a "color-swap-twisted" R-algebra involution)

$$\tau : 2\,\mathrm{TL}_Q(_s(m+1); [2]_s, [2]_t) \longrightarrow 2\,\mathrm{TL}_Q(_s(m+1); [2]_s, [2]_t)$$

$$[2]_s \longmapsto [2]_t$$

$$[2]_t \longmapsto [2]_s$$

$$e_i \longmapsto e_{(m+1)-i}.$$

For a diagram D, $\tau(D)$ is the diagram obtained by reflecting D about a vertical axis and swapping colors. Clearly this involution fixes $\mathrm{JW}_Q(_s n)$, so we have

$$v_{2,s}\left(\operatorname*{coeff}_{\in \mathrm{JW}_Q(_s(m+1))} D\right) = v_{2,t}\left(\operatorname*{coeff}_{\in \mathrm{JW}_Q(_s(m+1))} \tau(D)\right) \geqslant 0,$$

which is a contradiction, and completes the proof. $\qquad\square$

Lemma 4.3. *Let n, k be integers with $0 \leqslant k \leqslant n$. There exists a two-colored diagram D such that* $\operatorname{coeff}_{\in \mathrm{JW}_Q(_s n)} D = \begin{bmatrix} n \\ k \end{bmatrix}_s^{-1}$.

Proof. Take D to be the diagram with k nested caps on the bottom left, k nested cups on the top right, and all other strands connected from bottom to top. For example, if $n = 5$ and $k = 2$, we set

The result follows by Theorem 2.1 and induction on n. $\qquad\square$

Proof of Theorem A. Let

$$T_R = \{f \in 2\,\mathrm{TL}_R(_s n) : e_i f = 0 \text{ for all } 1 \leqslant i \leqslant n - 1\}.$$

In other words, T_R is the (right) annihilator of the generators $e_1, \dots, e_{n-1}$. One can show that $\mathrm{JW}_R(_s n)$ exists if and only if there exists $f \in T_R$ for which $\operatorname{coeff}_{\in f} 1$ is invertible in R (see e.g. [7, Exercise 9.25] for the one-colored case). When this happens, $T_R = R\mathrm{JW}_R(_s n)$.

Clearly $\mathrm{JW}_Q(_s n)$ exists, so $T_Q = Q\mathrm{JW}_Q(_s n)$. Thus T_A is a free A-module of rank 1, generated by $c\mathrm{JW}_Q(_s n) \in \mathrm{TL}_A(_s n)$, where c is the least common multiple of the denominators of the coefficients of $\mathrm{JW}_Q(_s n)$. Lemma 4.2 implies that c divides

$$g_n = \prod_{\substack{1 \leqslant k \leqslant n \\ k \nmid n+1}} \Theta_{k,s},$$

while Lemma 4.3 and Theorem 3.9 give $c = g_n$.

Suppose that $\begin{bmatrix} n \\ k \end{bmatrix}_s$ is invertible in R for all $0 \leqslant k \leqslant n$. Then g_n is invertible in R too by Theorem 3.9. Thus $T_R \geqslant R \otimes_A g_n \mathrm{JW}_Q(_s n)$ contains an element $f = 1 \otimes g_n \mathrm{JW}_Q(_s n)$ for which $\mathrm{coeff}_{\in f} 1 = g_n$ is invertible, so $\mathrm{JW}_R(_s n)$ exists.

Conversely, suppose that $\mathrm{JW}_R(_s n)$ exists. We have $T_R = R\mathrm{JW}_R(_s n) \geqslant R \otimes_A g_n \mathrm{JW}_Q(_s n)$, and thus $g_n \mathrm{JW}_R(_s n) = 1 \otimes_A g_n \mathrm{JW}_Q(_s n)$. But the coefficients of $g_n \mathrm{JW}_Q(_s n)$ (which lie in A) generate (1) as an ideal of A (again by Theorem 3.9 and Lemma 4.3), which directly implies that g_n is invertible in R, and $\mathrm{JW}_R(_s n) = g_n^{-1} \otimes_A g_n \mathrm{JW}_Q(_s n)$. ☐

A consequence of the above computation is the aforementioned "generic computation" of coefficients of $\mathrm{JW}_R(_s n)$.

Corollary 4.4. *For each two-colored Temperley–Lieb diagram D, there are coprime elements $f_D, g_D \in A$ such that if $\mathrm{JW}_R(_s n)$ exists, the specialization of g_D in R is invertible for all D and*

$$\underset{\in \mathrm{JW}_R(_s n)}{\mathrm{coeff}} D$$

is the specialization of f_D / g_D in R.

Remark 4.5. We consider generic computation of the coefficients of one-colored Jones–Wenzl projectors (at least for subrings of $\mathbb{C}$) to be mathematical folklore, that is, a "known" result without a published proof. In [6, Theorem 6.13], Elias–Williamson carefully prove an analogous result under the assumption that R is both an integral domain and a henselian local ring. Our proof does not require any restrictions on R but is essentially equivalent to Theorem A.

Proof. From the proof of Theorem A, if $\mathrm{JW}_R(_s n)$ exists then g_n is invertible in R, and

$$\underset{\in \mathrm{JW}_R(_s n)}{\mathrm{coeff}} D = \underset{\in g_n^{-1} \otimes_A g_n \mathrm{JW}_Q(_s n)}{\mathrm{coeff}} D.$$

Set $f = \mathrm{coeff}_{\in g_n \mathrm{JW}_Q(_s n)}$, and take $f_D = f / \gcd(f, g_n)$ and $g_D = \gcd(f, g_n)$. ☐

For $f \in Q$, say that f exists in R if there are $a, b \in A$ with $f = a/b$ and b invertible in R.

Lemma 4.6. *Suppose that $\mathrm{JW}_R(_s n)$ exists. Then $\frac{[n+1]_s}{[k]_s}$ exists in R for any integer $1 \leqslant k \leqslant n + 1$.*

Proof. We have

$$\frac{[n+1]_s}{[k]_s} = \frac{\prod_{l|n+1} \Theta_{l,s}}{\prod_{l|k} \Theta_{l,s}} = \frac{\prod_{\substack{l|n+1 \\ l \nmid k}} \Theta_{l,s}}{\prod_{\substack{l|k \\ l \nmid n+1}} \Theta_{l,s}},$$

and the denominator of the right-hand side divides

$$\prod_{\substack{1 < l \leqslant n \\ l \nmid n+1}} \Theta_{l,s},$$

which is invertible by Theorem A and Theorem 3.9. ☐

Proposition 4.7. *Suppose that the two-colored Jones–Wenzl projectors* $\mathrm{JW}_R({}_s n)$ *and* $\mathrm{JW}_R({}_t n)$ *exist.* *Then* $\mathrm{JW}_R({}_s n)$ *is rotatable if and only if* $\frac{[n+1]_s}{[k]_s} = 0$ *for all integers* $1 \leqslant k \leqslant n$.

Proof. Calculating generically, we have

$$\mathrm{pTr}(\mathrm{JW}_Q({}_s n)) = -\frac{[n+1]_s}{[n]_s} \mathrm{JW}_Q({}_s(n-1))$$

by (7). From the proof of Corollary 4.4, the coefficients of $\mathrm{JW}_Q({}_s(n-1))$ can be written as sums of fractions of the form $a {\begin{bmatrix} n-1 \\ k \end{bmatrix}_s}^{-1}$ for some $a \in A$ and some integer $0 \leqslant k \leqslant n-1$. Now observe that

$$-\frac{[n+1]_s}{[n]_s} \frac{a}{\begin{bmatrix} n-1 \\ k \end{bmatrix}_s} = -\frac{[n+1]_s [k]_s! a}{[n]_s [n-1]_s \cdots [n-k]_s} = -\frac{[n+1]_s}{[k+1]_s} \frac{a}{\begin{bmatrix} n \\ k+1 \end{bmatrix}_s},$$

noting that since $\mathrm{JW}_R({}_s n)$ exists, $\begin{bmatrix} n \\ k+1 \end{bmatrix}_s$ is invertible. Thus $\mathrm{JW}_R({}_s n)$ is rotatable if $\frac{[n+1]_s}{[k+1]_s} = 0$. Conversely, by Lemma 4.3 there is a a diagram whose coefficient in $\mathrm{JW}_Q({}_s(n-1))$ is exactly ${\begin{bmatrix} n-1 \\ k \end{bmatrix}_s}^{-1}$, so the above calculation shows that rotatability implies $\frac{[n+1]_s}{[k+1]_s} = 0$. □

Proof of Theorem B. The condition on quantum binomial coefficients is the same as [1, Assumption 1.1]. By [1, Proposition 3.4], this implies that the quantum binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_s$ and $\begin{bmatrix} n \\ k \end{bmatrix}_t$ are all invertible. Since

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_s = \frac{[n+1]_s}{[k]_s} \begin{bmatrix} n \\ k-1 \end{bmatrix}_s, \tag{14}$$

and similarly for t, we conclude that $\frac{[n+1]_s}{[k]_s} = \frac{[n+1]_t}{[k]_t} = 0$ for all integers $1 \leqslant k \leqslant n$. Conversely, if the two-colored Jones–Wenzl projectors exist and are rotatable, then (14) combined with Proposition 4.7 shows that $\begin{bmatrix} n+1 \\ k \end{bmatrix}_s$ and $\begin{bmatrix} n+1 \\ k \end{bmatrix}_t$ vanish for all integers $1 \leqslant k \leqslant n$. □

5 | APPLICATIONS TO THE HECKE CATEGORY

The diagrammatic Hecke category $\mathcal{H}$ of Elias–Williamson is constructed from a reflection representation of a Coxeter group called a *realization*. For each finite parabolic dihedral subgroup they identify a corresponding two-colored Temperley–Lieb algebra, whose defining parameters depend on the realization [5, §5.2]. In [6, §5], Elias–Williamson highlight some hidden assumptions about their realizations from [5]. Their most basic assumption (without which the diagrammatic Hecke category is not well defined) is that certain two-colored Jones–Wenzl projectors exist and are rotatable. For the benefit of future work, we give a corrected definition of a realization (which we call an *Abe realization*) that ensures the existence and rotatability of these Jones–Wenzl projectors.

Definition 5.1. Let $\Bbbk$ be an integral domain. An *Abe realization* of a Coxeter system (W, S) over $\Bbbk$ consists of a free, finite rank $\Bbbk$-module V along with subsets

$$\{\alpha_s : s \in S\} \subset V \qquad\qquad \{\alpha_s^\vee : s \in S\} \subset V^* = \mathrm{Hom}_\Bbbk(V, \Bbbk)$$

such that

(i) $\langle \alpha_s^\vee, \alpha_s \rangle = 2$ for all $s \in S$;
(ii) the assignment

$$s(\beta) = \beta - \langle \alpha_s^\vee, \beta \rangle \alpha_s$$

for all $s \in S$ and $\beta \in V$ defines a representation of the Coxeter group W on V;
(iii) for all distinct $s, t \in S$ such that st has order $m_{st} < \infty$, we have

$$\begin{bmatrix} m_{st} \\ k \end{bmatrix}_s (\langle \alpha_s^\vee, \alpha_t \rangle, \langle \alpha_t^\vee, \alpha_s \rangle) = \begin{bmatrix} m_{st} \\ k \end{bmatrix}_t (\langle \alpha_s^\vee, \alpha_t \rangle, \langle \alpha_t^\vee, \alpha_s \rangle) = 0$$

for all integers $1 \leqslant k \leqslant m_{st} - 1$.

By Theorem B, condition (iii) above is equivalent to the existence and rotatability of $\mathrm{JW}_{\Bbbk}(_s(m_{st} - 1))$ and $\mathrm{JW}_{\Bbbk}(_t(m_{st} - 1))$ for $[2]_s = \langle \alpha_s^\vee, \alpha_t \rangle$ and $[2]_t = \langle \alpha_t^\vee, \alpha_s \rangle$. Thus Corollary C follows from the discussion in [6, §5.1]. Moreover, condition (iii) is exactly Abe's assumption [1, Assumption 1.1], so Corollary D immediately follows by Abe's results [1, Theorem 3.9] and [2, Theorem 5.9]. It is also equivalent to

$$\begin{aligned} \Psi_{m_{st}}(\langle \alpha_s^\vee, \alpha_t \rangle \langle \alpha_t^\vee, \alpha_s \rangle) = 0 & \quad \text{if } m_{st} > 2, \\ \langle \alpha_s^\vee, \alpha_t \rangle = \langle \alpha_t^\vee, \alpha_s^\vee \rangle = 0 & \quad \text{if } m_{st} = 2, \end{aligned} \tag{15}$$

by Theorem 3.8.

Remark 5.2. Abe's results assume that $\mathcal{H}$ categorifies the Hecke algebra. This only necessitates the additional assumption in Corollary D of *Demazure surjectivity* [5, Assumption 3.9] for the realization V [6, §§5.2–5.3]. This is a mild condition, and in particular holds if $2 \in \Bbbk^\times$.

Remark 5.3. There is a longstanding gap in the literature in defining the diagrammatic Hecke category for Coxeter groups containing a parabolic subgroup of type H_3. The diagrammatic Hecke category is currently not defined in such cases, because a crucial relation (the H_3 Zamolodchikov relation [5, (5.12)]) is incomplete. One can argue that such a relation must exist in Abe's category, but explicitly determining this relation seems to be beyond current computational capabilities — for further discussion, see [5, Remark 5.4] and [6, §3.6]. Assuming such a relation can be found, it seems likely that Corollary C would hold in this case as well.

Remark 5.4. In [5], Elias–Williamson incorrectly state that

$$[m_{st}]_s(\langle \alpha_s^\vee, \alpha_t \rangle, \langle \alpha_t^\vee, \alpha_s \rangle) = [m_{st}]_t(\langle \alpha_s^\vee, \alpha_t \rangle, \langle \alpha_t^\vee, \alpha_s \rangle) = 0 \tag{16}$$

is enough to ensure the existence and rotatability of $\mathrm{JW}_{\Bbbk}(_s(m_{st} - 1))$. (This error was identified in [6] but only partially resolved there.) In the same paper, Elias–Williamson also incorrectly state that (16) is equivalent to (15). Amusingly, when these two statements are combined these errors accidentally cancel and the resulting statement is equivalent to Corollary C!

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ORCID

Amit Hazi https://orcid.org/0000-0001-7264-2211

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