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Recursive Constrained Sine Second-Order Error Promoting Adaptive Algorithm

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***Abstract*—This brief proposes a recursive constrained sine second-order error promoting adaptive (RCSSOEPA) algorithm. Compared with classical recursive method, the RCSSOEPA algorithm can achieve better steady state performance in impulsive-noise. In general, the sine second-order error (SSOE) is constructed to devise a new recursive constrained adaptive-filtering within the least-squares framework for solving linear constrained optimization problems. The mean-square (MS) stability of the RCSSOEPA and its theoretical instantaneous MS deviation under Gaussian and non Gaussian noise are analyzed, numerically investigated and discussed in detail. Simulated results are reported to give a comfirmation of the theoretical analysis, and show that the RCSSOEPA outperforms recent developed constrained adaptive filtering algorithms in the estimation misalignment and when used for system identification under impulsive-noise.**

***Index Terms*—mpulsive-noisempulsive-noiseI, Recursive constrained adaptive filtering (RCAF), Sine second-order error (SSOE).**

I. Introduction

IN the field of signal processing in wireless communication, array signal processing, interference suppression and system identification, the parameter vector estimation is often gotten by using adaptive filters [1], [2] and is subject to linear constraints. To solve the above problem, the constrained-adaptive-filtering (CAF) algorithms have been proposed and well developed in recent years [3], [4]. The greatest advantage of a CAF algorithm lies in its error correction property avoiding error accumulation and thus achieving good performance.

The second-order error (SOE) criterion is widely used in adaptive filtering (AF) since it results in low complexity and simple structure algorithms [5], and it is typically an optimal criterion under the Gaussian assumption. The constrained-least-mean-square (CLMS) algorithm, which is a classical CAF constructed based on the SOE criterion [6], can achieve high robustness in Gaussian noise. However, in practice, there are various non-Gaussian noise scenarios, where the behaviors of the CLMS will decrease, especially when it is used in impulsive-noise with heavy-tail distributions (such as alpha-stable noise [7] and Cauchy noise [8]).

In order to effectively combat the adverse effects of non-Gaussian noises, the maximum-correntropy-criterion (MCC) is reported [9], [10], which is a non-linear measure used to maximize the correlation between outputs of the unknown system and the AF algorithm to combat the local similarity in non Gaussian noise. In addition, continuous hybrid p-norm algorithm and constrained minimum M-estimation algorithm are presented in [11], [12]. Using the MCC criterion instead of the SOE criterion, the constrained MCC (CMCC) algorithm [13] achieves high robustness under non-Gaussian noise by maximizing the correlation. Then, using the matrix-inversion theorem [14], the recursive-CMCC (RCMCC) is presented to get a faster convergence and a stronger performance in non-Gaussian noise than the basic CMCC.

Here, we propose a recursive constrained sine second-order error promoting adaptive (RCSSOEPA) algorithm. The sine second-order error (SSOE) criterion used for developing the algorithm helps in suppressing large errors caused by impulsive-noise . The MS stability of the RCSSOEPA and its theoretical instantaneous mean-square-deviation (MSD) under Gaussian and non-Gaussian noises are analyzed. Simulation results illustrated to give a confirmation of the theoretical analysis show that the RCSSOEPA outperforms other recent developed constrained adaptive filtering algorithms in the estimation misalignment when used for system-identification under impulsive-noises.

The brief is structured below. Section II presents the RCSSOEPA whose convergence and instantaneous MSD are analyzed in Section III. In Section IV, we present simulation results and compare them with that of known CAF algorithms. Section V concludes this brief.

Notation: We use plain font letters as scalars, and we also use capital bold and lowercase bold letters to represent matrices and vectors, respectively. All vectors are column-vectors. The inverse, transpose, expectation, and trace operators are expressed as $(\cdot)^{-1}$, $(\cdot)^{T}$, $\mathrm{E}(\cdot)$, and $tr(\cdot)$, respectively.

II. The Proposed RCSSOEPA Algorithm

A. Channel Model

An unknown system output at time l is $d(l) = \mathbf{h}_0^T\mathbf{u}(l) + \eta(l)$, where the system vector is $\mathbf{h}_0 = [h_1, h_2, \cdots, h_M]^T$, $\mathbf{u}(l) = [u_1, u_2, \cdots, u_M]^T$ represents the input-vector at l, $\eta(l)$ is the background noise with variance σ_η^2, and the estimation error is defined as

$$e(l) = d(l) - \mathbf{h}^T(l-1)\,\mathbf{u}(l), \tag{1}$$

where $\mathbf{h}(l-1)$ denotes the system vector estimate at $l-1$.

The SSOE cost function is defined as

$$J(e) = \begin{cases} 4\sin^2\left(\frac{e}{2c}\right) & \text{if } |e| \le \pi c \\ 4 & \text{otherwise} \end{cases}, \tag{2}$$

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where c is a positive constant. The SSOE function is introduced into the CAF scheme to develop a CSSDA algorithm, by solving the following problem:

$$\min_{\mathbf{h}} \mathrm{E}\left[4\sin^2\left(\tfrac{e}{2c}\right)\right] \quad s.t. \quad \mathbf{g}=\mathbf{S}^T\mathbf{h}, \tag{3}$$

where $\mathbf{S}$ represents constraint matrix, having an size of $M\times K$ and $M>K$, and $\mathbf{g}$ represents an $K\times 1$ constraint vector. Utilizing the Lagrange-method, the cost-function for the CSSDA is obtained as

$$J_{\mathrm{CSSDA}}(e)=\mathrm{E}\left[4\sin^2\left(\frac{e}{2c}\right)\right]+\gamma^T\left[\mathbf{g}-\mathbf{S}^T\mathbf{h}\right], \tag{4}$$

where γ is the $K\times 1$ Lagrange multiplier.

B. Optimal Solution

Considering the constraint $\mathbf{g}=\mathbf{S}^T\mathbf{h}$ and taking $\frac{\partial J_{\mathrm{CSSDA}}}{\partial \mathbf{h}}=\mathbf{0}$ into consideration, the CSSDA algorithm for finding the optimized weight-vector $\mathbf{h}_{opt}$ can be derived as follows

$$\begin{aligned}
&\mathrm{E}\left[\tfrac{2}{c}\sin\left(\tfrac{e(l)}{c}\right)\mathbf{u}(l)\right]+\mathbf{S}\gamma=\mathbf{0},\\
&\Rightarrow \mathrm{E}\left[w(e(l))\,e(l)\,\mathbf{u}(l)\right]+\mathbf{S}\gamma=\mathbf{0},\\
&\Rightarrow \mathrm{E}\left[w(e(l))\left(d(l)-\mathbf{h}_{opt}^T(l)\,\mathbf{u}(l)\right)\mathbf{u}(l)\right]+\mathbf{S}\gamma=\mathbf{0},\\
&\Rightarrow \mathrm{E}\left[w(e(l))\,\mathbf{u}(l)\,\mathbf{u}^T(l)\right]\mathbf{h}_{opt}=\mathrm{E}\left[w(e(l))\,d(l)\,\mathbf{u}(l)\right]+\mathbf{S}\gamma,\\
&\Rightarrow \mathbf{R}_w\mathbf{h}_{opt}=\mathbf{p}_w+\mathbf{S}\gamma,\\
&\Rightarrow \mathbf{h}_{opt}=\mathbf{R}_w^{-1}\mathbf{p}_w+\mathbf{R}_w^{-1}\mathbf{S}\gamma,
\end{aligned} \tag{5}$$

where $w(e(l))=\frac{2\sin\left(\frac{e(l)}{c}\right)}{ce(l)}$, $\mathbf{R}_w=\mathrm{E}\left[w(e(l))\,\mathbf{u}(l)\,\mathbf{u}^T(l)\right]$, $\mathbf{p}_w=\mathrm{E}\left[w(e(l))\,d(l)\,\mathbf{u}(l)\right]$. Thereby, the constraint vector $\mathbf{g}=\mathbf{S}^T\mathbf{h}$ is represented as

$$\begin{aligned}
&\mathbf{g}=\mathbf{S}^T\mathbf{h}_{opt},\\
&\Rightarrow \mathbf{g}=\mathbf{S}^T\left[\mathbf{R}_w^{-1}\mathbf{p}_w+\mathbf{R}_w^{-1}\mathbf{S}\gamma\right],\\
&\Rightarrow \gamma=\left(\mathbf{S}^T\mathbf{R}_w^{-1}\mathbf{S}\right)^{-1}\left(\mathbf{g}-\mathbf{S}^T\mathbf{R}_w^{-1}\mathbf{p}_w\right).
\end{aligned} \tag{6}$$

Using (6), $\mathbf{h}_{opt}$ in (5) is updated to

$$\mathbf{h}_{opt}=\mathbf{R}_w^{-1}\mathbf{p}_w+\mathbf{R}_w^{-1}\mathbf{S}\left(\mathbf{S}^T\mathbf{R}_w^{-1}\mathbf{S}\right)^{-1}\left(\mathbf{g}-\mathbf{S}^T\mathbf{R}_w^{-1}\mathbf{p}_w\right). \tag{7}$$

We use the following approximations of $\mathbf{R}_w$ and $\mathbf{p}_w$, respectively:

$$\hat{\mathbf{R}}_w=\frac{1}{l}\sum_{i=1}^{l}w(e(i))\mathbf{u}(i)\,\mathbf{u}^T(i), \tag{8}$$

$$\hat{\mathbf{p}}_w=\frac{1}{l}\sum_{i=1}^{l}w(e(i))d(i)\,\mathbf{u}(i), \tag{9}$$

which can be recursively computed as

$$\begin{aligned}
\hat{\mathbf{R}}_w(l)&=\hat{\mathbf{R}}_w(l-1)+w(e(l))\,\mathbf{u}(l)\,\mathbf{u}^T(l),\\
\hat{\mathbf{p}}_w(l)&=\hat{\mathbf{p}}_w(l-1)+w(e(l))\,d(l)\,\mathbf{u}(l).
\end{aligned} \tag{10}$$

Thus, we finally obtain the system vector estimate

$$\begin{aligned}
\mathbf{h}(l)&=\left(\tfrac{1}{l}\hat{\mathbf{R}}_w(l)\right)^{-1}\tfrac{1}{l}\hat{\mathbf{p}}_w(l)+\left(\tfrac{1}{l}\hat{\mathbf{R}}_w(l)\right)^{-1}\mathbf{S}\\
&\times\left(\mathbf{S}^T\left(\tfrac{1}{l}\hat{\mathbf{R}}_w(l)\right)^{-1}\mathbf{S}\right)^{-1}\left(\mathbf{g}-\mathbf{S}^T\left(\tfrac{1}{l}\hat{\mathbf{R}}_w(l)\right)^{-1}\tfrac{1}{l}\hat{\mathbf{p}}_w(l)\right)\\
&=\hat{\mathbf{R}}_w^{-1}(l)\,\hat{\mathbf{p}}_w(l)+\hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{S}\left(\mathbf{S}^T\hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{S}\right)^{-1}\\
&\times\left(\mathbf{g}-\mathbf{S}^T\hat{\mathbf{R}}_w^{-1}(l)\,\hat{\mathbf{p}}_w(l)\right).
\end{aligned} \tag{11}$$

C. RCSSOEPA Algorithm

Using matrix-inversion lemma [15], from (10), we get

$$\begin{aligned}
\hat{\mathbf{R}}_w^{-1}(l)&=\hat{\mathbf{R}}_w^{-1}(l-1)-\hat{\mathbf{R}}_w^{-1}(l-1)\,\mathbf{u}(l)\left(w^{-1}(e(l))\right.\\
&\left.+\,\mathbf{u}^T(l)\,\hat{\mathbf{R}}_w^{-1}(l-1)\,\mathbf{u}(l)\right)^{-1}\mathbf{u}^T(l)\,\hat{\mathbf{R}}_w^{-1}(l-1).
\end{aligned} \tag{12}$$

Define

$$\begin{aligned}
\mathbf{n}(l)&=\hat{\mathbf{R}}_w^{-1}(l-1)\,\mathbf{u}(l)\left(w^{-1}(e(l))\right.\\
&\left.+\,\mathbf{u}^T(l)\,\hat{\mathbf{R}}_w^{-1}(l-1)\,\mathbf{u}(l)\right)^{-1}.
\end{aligned} \tag{13}$$

Then, we can write

$$\begin{aligned}
\mathbf{n}(l)&=w(e(l))\,\hat{\mathbf{R}}_w^{-1}(l-1)\,\mathbf{u}(l)\\
&\quad-\mathbf{n}(l)\,w(e(l))\,\mathbf{u}^T(l)\,\hat{\mathbf{R}}_w^{-1}(l-1)\,\mathbf{u}(l)\\
&=w(e(l))\left(\hat{\mathbf{R}}_w^{-1}(l-1)-\mathbf{n}(l)\,\mathbf{u}^T(l)\,\hat{\mathbf{R}}_w^{-1}(l-1)\right)\mathbf{u}(l)\\
&=w(e(l))\,\hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{u}(l).
\end{aligned} \tag{14}$$

By defining the unconstrained optimal solution as

$$\begin{aligned}
\mathbf{h}_{unc}(l-1)&=\hat{\mathbf{R}}_w^{-1}(l-1)\,\hat{\mathbf{p}}_w(l-1),\\
e_{unc}(l)&=d(l)-\mathbf{h}_{unc}^T(l-1)\,\mathbf{u}(l),
\end{aligned} \tag{15}$$

and substituting (10), (12) and (14) into (11), the updated RCSSOEPA recursion is given by

$$\begin{aligned}
\mathbf{h}(l)&=\hat{\mathbf{R}}_w^{-1}(l)\left[\hat{\mathbf{p}}_w(l-1)+w(e(l))\,d(l)\,\mathbf{u}(l)\right]\\
&\quad+\hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{S}\gamma(l)\\
&=\mathbf{h}_{unc}(l-1)+e_{unc}(l)\,\mathbf{n}(l)+\hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{S}\gamma(l),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\gamma(l)&=\left(\mathbf{S}^T\hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{S}\right)^{-1}\left(\mathbf{g}-\mathbf{S}^T\left(\mathbf{h}_{unc}(l-1)\right.\right.\\
&\left.\left.\quad+e_{unc}(l)\,\mathbf{n}(l)\right)\right).
\end{aligned} \tag{17}$$

From the iteration of the proposed RCSSOEPA algorithm in (16), it has a complexity of $o(M^2)$.

III. Theoretical Analysis of Convergence and MSD of the RCSSOEPA Algorithm

A. Assumptions

To obtain theoretical analysis convergence and MSD of the proposed RCSSOEPA, the following assumptions are adopted to simplify the analysis [8], [16].

(1) The noise $\eta(l)$ is an i. i. d. non-Gaussian with zero-mean (ZM), and it is independent of any other signals.

(2) The prior error $e^c(l)$, defining later, is ZM Gaussian.

(3) $\mathbf{u}(l)$ is an i. i. d. ZM Gaussian process with positive definite covariance-matrix $\mathbf{R}_u=\mathrm{E}\left[\mathbf{u}(l)\,\mathbf{u}^T(l)\right]$.

(4) When the system operates in its steady-state, $w(e(l))$ is uncorrelated with $\|\mathbf{u}(l)\|^2$.

B. Mean Square Stability

Based on assumption (4), we obtain an approximation

$$\begin{aligned}
\hat{\mathbf{R}}_w^{-1}(l)&=\left[l\left[\tfrac{1}{l}\sum_{i=1}^{l}w(e(i))\,\mathbf{u}(i)\,\mathbf{u}^T(i)\right]\right]^{-1}\\
&\approx\tfrac{1}{l}\left[\mathrm{E}\left[w(e(l))\,\mathbf{u}(l)\,\mathbf{u}^T(l)\right]\right]^{-1}\\
&=\tfrac{b}{l}\mathbf{R}_u^{-1},
\end{aligned} \tag{18}$$

where $b=1/\mathrm{E}\left[w(e(l))\right]$.

Using the above definitions of $e(l)$ and $e_{unc}(l)$ and substituting (14) and (18) into (16), we obtain

$$\begin{aligned} e_{unc}(l) - e(l) &= \mathbf{u}^T(l)\,\mathbf{h}(l-1) - \mathbf{u}^T(l)\,\mathbf{h}_{unc}(l-1) \\ &= \mathbf{u}^T(l)\left(\mathbf{h}(l-1) - \mathbf{h}_{unc}(l-1)\right) \\ &= \mathbf{u}^T(l)\,\hat{\mathbf{R}}_w^{-1}(l-1)\,\mathbf{S}\gamma(l-1). \end{aligned} \tag{19}$$

Substituting (14) and (19) into (16), we obtain

$$\begin{aligned} &\mathbf{h}(l) = \mathbf{h}(l-1) + w(e(l))\,e(l)\,\hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{u}(l) \\ &+ \hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{S}\left(\gamma(l) - \gamma(l-1)\right) \\ &= \mathbf{h}(l-1) + w(e(l))\,e(l)\,\hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{u}(l) \\ &+ \hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{S}\Big(\mathbf{S}^T\hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{S}\Big)^{-1} \\ &\times\Big(\mathbf{g} - \mathbf{S}^T\left(\mathbf{h}(l-1) + w(e(l))\,e(l)\,\hat{\mathbf{R}}_w^{-1}(l)\,\mathbf{u}(l)\right)\Big). \end{aligned} \tag{20}$$

Combining with (18), we arrive at

$$\begin{aligned} \mathbf{h}(l) &\approx \mathbf{h}(l-1) + \tfrac{b}{l}w(e(l))\,e(l)\,\mathbf{R}_u^{-1}\mathbf{u}(l) \\ &\quad + \mathbf{R}_u^{-1}\mathbf{S}\left(\mathbf{S}^T\mathbf{R}_u^{-1}\mathbf{S}\right)^{-1} \\ &\quad \times\left(\mathbf{g} - \mathbf{S}^T\left(\mathbf{h}(l-1) + \tfrac{b}{l}w(e(l))\,e(l)\,\mathbf{R}_u^{-1}\mathbf{u}(l)\right)\right) \\ &= \mathbf{T}\left[\mathbf{h}(l-1) + \tfrac{b}{l}w(e(l))\,e(l)\,\mathbf{R}_u^{-1}\mathbf{u}(l)\right] + \mathbf{r}, \end{aligned} \tag{21}$$

where $\mathbf{T} = \mathbf{I} - \mathbf{R}_u^{-1}\mathbf{S}\left(\mathbf{S}^T\mathbf{R}_u^{-1}\mathbf{S}\right)^{-1}\mathbf{S}^T$ and $\mathbf{r} = \mathbf{R}_u^{-1}\mathbf{S}\left(\mathbf{S}^T\mathbf{R}_u^{-1}\mathbf{S}\right)^{-1}\mathbf{g}$.

Using $d(l) = \mathbf{h}_0^T\mathbf{u}(l) + \eta(l)$ and assumption (1), we obtain

$$\begin{aligned} w(e(l))\,d(l)\,\mathbf{u}(l) &= w(e(l))\,\mathbf{u}(l)\,\mathbf{u}^T(l)\,\mathbf{h}_0 \\ &\quad + w(e(l))\,\eta(l)\,\mathbf{u}(l) \\ &\Rightarrow \mathbf{p}_w = \mathbf{R}_w\mathbf{h}_0 \\ &\quad \Rightarrow \mathbf{h}_0 = \mathbf{R}_w^{-1}\mathbf{p}_w. \end{aligned} \tag{22}$$

The following two error measures are introduced to simplify the analysis:

$$\tilde{\mathbf{h}}(l) = \mathbf{h}(l) - \mathbf{h}_{opt}, \qquad e^c(l) = \left(\mathbf{h}_0 - \mathbf{h}(l)\right)^T\mathbf{u}(l), \tag{23}$$

where $e^c(l)$ and $\tilde{\mathbf{h}}(l)$ represent prior error and weighted error vectors, respectively. We also define

$$\zeta_{\mathbf{h}} = \mathbf{h}_0 - \mathbf{h}_{opt}. \tag{24}$$

Based on the above analysis, $\zeta_{\mathbf{h}}$ in (24) can be written as

$$\zeta_{\mathbf{h}} = \mathbf{R}_u^{-1}\mathbf{S}\left(\mathbf{S}^T\mathbf{R}_u^{-1}\mathbf{S}\right)^{-1}\left(\mathbf{S}^T\mathbf{h}_0 - \mathbf{g}\right). \tag{25}$$

Subtracting $\mathbf{h}_{opt}$ from both sides of (21) and using $e(l) = d(l) - \mathbf{u}^T(l)\,\mathbf{h}(l-1)$, we obtain

$$\begin{aligned} \tilde{\mathbf{h}}(l) &= \mathbf{T}\left[\mathbf{h}(l-1) + \tfrac{b}{l}d(l)\,w(e(l))\,\mathbf{R}_u^{-1}\mathbf{u}(l)\right. \\ &\left.-\tfrac{b}{l}w(e(l))\,\mathbf{R}_u^{-1}\mathbf{u}(l)\,\mathbf{u}^T(l)\,\mathbf{h}(l-1)\right] + \mathbf{r} - \mathbf{h}_{opt}, \end{aligned} \tag{26}$$

and due to $\mathbf{T}\mathbf{h}_{opt} - \mathbf{h}_{opt} + \mathbf{r} = \mathbf{0}$, (26) is modified to

$$\begin{aligned} \tilde{\mathbf{h}}(l) &= \mathbf{T}\left[\mathbf{I} - \tfrac{b}{l}w(e(l))\,\mathbf{R}_u^{-1}\mathbf{u}(l)\,\mathbf{u}^T(l)\right]\tilde{\mathbf{h}}(l-1) \\ &+ \tfrac{b}{l}w(e(l))\,\mathbf{T}\left(\mathbf{R}_u^{-1}\mathbf{u}(l)\,\eta(l) + \mathbf{R}_u^{-1}\mathbf{u}(l)\,\mathbf{u}^T(l)\,\zeta_{\mathbf{h}}\right). \end{aligned} \tag{27}$$

Owning to the definition of $\mathbf{T}$, we have

$$\mathbf{T}\tilde{\mathbf{h}}(l) = \tilde{\mathbf{h}}(l). \tag{28}$$

Then, (27) can be rewritten as

$$\begin{aligned} \tilde{\mathbf{h}}(l) &= \left[\mathbf{I} - \tfrac{b}{l}w(e(l))\,\mathbf{T}\mathbf{R}_u^{-1}\mathbf{u}(l)\,\mathbf{u}^T(l)\right]\tilde{\mathbf{h}}(l-1) \\ &+ \tfrac{b}{l}w(e(l))\left(\mathbf{T}\mathbf{R}_u^{-1}\mathbf{u}(l)\,\eta(l) + \mathbf{T}\mathbf{R}_u^{-1}\mathbf{u}(l)\,\mathbf{u}^T(l)\,\zeta_{\mathbf{h}}\right). \end{aligned} \tag{29}$$

Exerting square expectation on both-sides of (29), and utilizing assumptions (1), (2) and (3), we get

$$\begin{aligned} &\mathrm{E}\left[\left\|\tilde{\mathbf{h}}(l)\right\|^2\right] = \mathrm{E}\left[\left\|\tilde{\mathbf{h}}(l-1)\right\|^2_{\mathbf{G}(l)}\right] + \tfrac{b^2}{l^2}\mathrm{E}\left[w^2(e(l))\right]\times \\ &\mathrm{E}\left[v^2(l)\right]\mathrm{E}\left[\mathbf{u}^T(l)\,\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{u}(l)\right] + \tfrac{b^2}{l^2}\mathrm{E}\left[w^2(e(l))\right]\zeta_{\mathbf{h}} \\ &\times\mathrm{E}\left[\mathbf{u}(l)\;\mathbf{u}^T(l)\;\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{u}(l)\,\mathbf{u}^T(l)\right]\zeta_{\mathbf{h}}^T, \end{aligned} \tag{30}$$

where

$$\begin{aligned} \mathbf{G}(l) &= \left[\mathbf{I} - \tfrac{b}{l}w(e(l))\,\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{u}(l)\,\mathbf{u}^T(l)\right]^T \\ &\quad\times\left[\mathbf{I} - \tfrac{b}{l}w(e(l))\,\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{u}(l)\,\mathbf{u}^T(l)\right]. \end{aligned} \tag{31}$$

From (30), utilizing Isserlis' theory [20], we get

$$\begin{aligned} &\mathrm{E}\left[\mathbf{u}(l)\,\mathbf{u}^T(l)\,\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{u}(l)\,\mathbf{u}^T(l)\right] \\ &\quad = 2\mathbf{T}^T\mathbf{T} + tr\left\{\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right\}\mathbf{R}_{\mathbf{u}}. \end{aligned} \tag{32}$$

Then, (31) is modified to

$$\begin{aligned} \mathbf{G}(l) &= \mathbf{I} - \tfrac{b}{l}w(e(l))\left(\mathbf{T}^T + \mathbf{T}\right) + \tfrac{b^2}{l^2}w^2(e(l)) \\ &\quad\times\left(2\mathbf{T}^T\mathbf{T} + tr\left\{\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right\}\mathbf{R}_{\mathbf{u}}\right). \end{aligned} \tag{33}$$

Since

$$\begin{aligned} \mathbf{T}\mathbf{R}_{\mathbf{u}}\zeta_{\mathbf{h}} &= \left(\mathbf{I} - \mathbf{R}_{\mathbf{u}}^{-1}\mathbf{S}\left(\mathbf{S}^T\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{S}\right)^{-1}\mathbf{S}^T\right) \\ &\quad\times\mathbf{R}_{\mathbf{u}}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{S}\left(\mathbf{S}^T\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{S}\right)^{-1}\left(\mathbf{S}^T\mathbf{h}_0 - \mathbf{g}\right) = \mathbf{0}, \end{aligned} \tag{34}$$

we get

$$\begin{aligned} &\mathrm{E}\left[\left\|\tilde{\mathbf{h}}(l)\right\|^2\right] = \mathrm{E}\left[\left\|\tilde{\mathbf{h}}(l-1)\right\|^2_{\mathbf{G}(l)}\right] \\ &+ \tfrac{b^2}{l^2}\mathrm{E}\left[w^2(e(l))\right]\mathrm{E}\left[v^2(l)\right]tr\left\{\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right\} \\ &+ \tfrac{b^2}{l^2}\mathrm{E}\left[w^2(e(l))\right]\zeta_{\mathbf{h}}^T\mathrm{E}\left[tr\left\{\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right\}\mathbf{R}_{\mathbf{u}}\right]\zeta_{\mathbf{h}}. \end{aligned} \tag{35}$$

Let t_j and $r_j\,(1 \le j \le M)$ be eigenvalues of $\mathbf{T}$ and $\mathbf{R}_{\mathbf{u}}$, respectively. Based on the above analysis, the mean square stability condition is satisfied if

$$\begin{aligned} &\Big|1 - \tfrac{2b}{l}w(e(l))\,t_j + \tfrac{b^2}{l^2}w^2(e(l)) \\ &\qquad\qquad\times\left(2t_j^2 + tr\left\{\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right\}r_j\right)\Big| < 1. \end{aligned} \tag{36}$$

Therefore, the stability condition is to be

$$l > \max_j \frac{b\mathrm{E}\left[w^2(e(l))\right]\left(2t_j^2 + tr\left\{\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right\}r_j\right)}{2t_j\mathrm{E}\left[w(e(l))\right]}. \tag{37}$$

C. MSD analysis

Consider (34), and assume that $\mathbf{Y}$ is an arbitrary $M \times M$-symmetric non-negative definite-matrix, then we get

$$\begin{aligned} &\mathrm{E}\left[\left\|\tilde{\mathbf{h}}(l)\right\|^2_{\mathbf{Y}}\right] = \mathrm{E}\left[\left\|\tilde{\mathbf{h}}(l-1)\right\|^2_{\mathbf{Q}(l)}\right] \\ &+ \tfrac{b^2}{l^2}\mathrm{E}\left[w^2(e(l))\right]\mathrm{E}\left[v^2(l)\right]tr\left\{\mathbf{Y}\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right\} \\ &+ \tfrac{b^2}{l^2}\mathrm{E}\left[w^2(e(l))\right]\zeta_{\mathbf{h}}^T\mathrm{E}\left[tr\left\{\mathbf{Y}\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right\}\mathbf{R}_{\mathbf{u}}\right]\zeta_{\mathbf{h}}, \end{aligned} \tag{38}$$

where

$$\begin{aligned} &\mathbf{Q}(l) = \left(\mathbf{I} - \tfrac{b}{l}\mathrm{E}\left[w(e(l))\right]\mathbf{T}\right)^T\mathbf{Y}\left(\mathbf{I} - \tfrac{b}{l}\mathrm{E}\left[w(e(l))\right]\mathbf{T}\right) \\ &+ \tfrac{b^2}{l^2}\mathrm{E}\left[w^2(e(l))\right]\left(2\mathbf{T}^T\mathbf{Y}\mathbf{T} + tr\left\{\mathbf{Y}\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right\}\mathbf{R}_{\mathbf{u}}\right) \\ &- \tfrac{b^2}{l^2}\mathrm{E}^2\left[w(e(l))\right]\mathbf{T}^T\mathbf{Y}\mathbf{T}. \end{aligned} \tag{39}$$

Then, we use the vector correlation property $\mathbf{vec}(\mathbf{XYZ}) = \left(\mathbf{Z}^T \otimes \mathbf{X}\right)\mathbf{vec}(\mathbf{Y})$ and $tr\left(\mathbf{X}^T\mathbf{Y}\right) = \mathbf{vec}^T(\mathbf{Y})\,\mathbf{vec}(\mathbf{X})$, where $\mathbf{vec}(\cdot)$ is the vectorization operating and $\otimes$ is Kronecker-product. Utilizing these properties, we get

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$$\begin{aligned}&\mathrm{E}\left[\left\|\tilde{\mathbf{h}}\left(l\right)\right\|_{\mathbf{y}}^{2}\right]=\mathrm{E}\left[\left\|\tilde{\mathbf{h}}\left(l-1\right)\right\|_{\mathbf{F}(l)\mathbf{y}}^{2}\right]\\&+\tfrac{b^2}{l^2}\mathrm{E}\left[w^2\left(e\left(l\right)\right)\right]\mathrm{E}\left[v^2\left(l\right)\right]\mathbf{vec}^T\left(\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right)\mathbf{y}\\&+\tfrac{b^2}{l^2}\mathrm{E}\left[w^2\left(e\left(l\right)\right)\right]\zeta_{\mathbf{h}}^T\mathbf{R}_{\mathbf{u}}\zeta_{\mathbf{h}}\mathbf{vec}^T\left(\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right)\mathbf{y},\end{aligned}\tag{40}$$

where $\mathrm{E}\left[\left\|\tilde{\mathbf{h}}\left(l\right)\right\|_{\mathbf{Y}}^{2}\right]=\mathrm{E}\left[\left\|\tilde{\mathbf{h}}\left(l\right)\right\|_{\mathbf{y}}^{2}\right]$, and

$$\begin{aligned}\mathbf{F}\left(l\right)=&\left(\mathbf{I}-\tfrac{b}{l}\mathrm{E}\left[w\left(e\left(l\right)\right)\right]\mathbf{T}\right)^T\otimes\left(\mathbf{I}-\tfrac{b}{l}\mathrm{E}\left[w\left(e\left(l\right)\right)\right]\mathbf{T}\right)\\&+\tfrac{b^2}{l^2}\mathrm{E}\left[w^2\left(e\left(l\right)\right)\right]\\&\times\left(2\left(\mathbf{T}^T\otimes\mathbf{T}\right)+\mathbf{vec}\left(\mathbf{R}_{\mathbf{u}}\right)\mathbf{vec}^T\left(\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right)\right)\\&-\tfrac{b^2}{l^2}\mathrm{E}^2\left[w\left(e\left(l\right)\right)\right]\left(\mathbf{T}^T\otimes\mathbf{T}\right).\end{aligned}\tag{41}$$

From [21], $\mathrm{MSD}\left(l\right)$ is written as $\mathrm{E}\left[\left\|\tilde{\mathbf{h}}\left(l\right)\right\|^{2}\right]$, and the relationship between $\mathrm{MSD}\left(l-1\right)$ and $\mathrm{MSD}\left(l\right)$ is given by [14]

$$\frac{\mathrm{E}\left[\left\|\tilde{\mathbf{h}}\left(l\right)\right\|^{2}\right]}{\mathrm{E}\left[\left\|\tilde{\mathbf{h}}\left(l-1\right)\right\|^{2}\right]}=\frac{l-1}{l}.\tag{42}$$

Utilizing (41), we rewrite (39) as

$$\begin{aligned}&\mathrm{E}\left[\left\|\tilde{\mathbf{h}}\left(l\right)\right\|_{\left(\mathbf{I}-\frac{l}{l-1}\mathbf{F}(l)\right)\mathbf{y}}^{2}\right]=\tfrac{b^2}{l^2}\mathrm{E}\left[w^2\left(e\left(l\right)\right)\right]\mathrm{E}\left[v^2\left(l\right)\right]\\&\times\mathbf{vec}^T\left(\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right)\mathbf{y}+\tfrac{b^2}{l^2}\mathrm{E}\left[w^2\left(e\left(l\right)\right)\right]\zeta_{\mathbf{h}}^T\mathbf{R}_{\mathbf{u}}\zeta_{\mathbf{h}}\\&\times\mathbf{vec}^T\left(\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right)\mathbf{y}.\end{aligned}\tag{43}$$

With $\mathbf{y}=\left(\mathbf{I}-\frac{l}{l-1}\mathbf{F}\left(l\right)\right)^{-1}\mathbf{vec}\left(\mathbf{I}\right)$, the MSD is given by

$$\begin{aligned}&\mathrm{E}\left[\left\|\tilde{\mathbf{h}}\left(l\right)\right\|^{2}\right]=\tfrac{b^2}{l^2}\mathrm{E}\left[w^2\left(e\left(l\right)\right)\right]\left(\mathrm{E}\left[v^2\left(l\right)\right]+\zeta_{\mathbf{h}}^T\mathbf{R}_{\mathbf{u}}\zeta_{\mathbf{h}}\right)\\&\times\mathbf{vec}^T\left(\mathbf{T}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{T}^T\right)\left(\mathbf{I}-\tfrac{l}{l-1}\mathbf{F}\left(l\right)\right)^{-1}\mathbf{vec}\left(\mathbf{I}\right).\end{aligned}\tag{44}$$

In the Gaussian-noise case, let $e\left(l\right)=e^c\left(l\right)+v\left(l\right)$. If $v\left(l\right)$ is a ZM Gaussian random number with variance σ_v^2, then $e\left(l\right)$ is also a ZM Gaussian with a variance of σ_e^2. Using approximation $\mathbf{h}\left(l\right)\approx\mathbf{h}_{opt}$ in steady state, we obtain

$$e^c\left(l\right)\approx\left(\mathbf{h}_0-\mathbf{h}_{opt}\right)^T\mathbf{u}\left(l\right)=\zeta_{\mathbf{h}}^T\mathbf{u}\left(l\right).\tag{45}$$

Based on (44), we have

$$\sigma_e^2\approx\zeta_{\mathbf{h}}^T\mathbf{R}_{\mathbf{u}}\zeta_{\mathbf{h}}+\sigma_v^2.\tag{46}$$

Under the steady-state, $w\left(e\left(l\right)\right)$ is approximated as $w\left(e\left(l\right)\right)\approx\frac{2}{c^2}-\frac{e^2(l)}{3c^4}$ using the Taylor expansion. Therefore, $\mathrm{E}\left[w\left(e\left(l\right)\right)\right]$ and $\mathrm{E}\left[w^2\left(e\left(l\right)\right)\right]$ are respectively represented by

$$\mathrm{E}\left[w\left(e\left(l\right)\right)\right]=\frac{2}{c^2}-\frac{1}{3c^4}\left(\zeta_{\mathbf{h}}^T\mathbf{R}_{\mathbf{u}}\zeta_{\mathbf{h}}+\sigma_v^2\right),\tag{47}$$

and

$$\begin{aligned}&\mathrm{E}\left[w^2\left(e\left(l\right)\right)\right]=\tfrac{4}{c^4}+\tfrac{e^4(l)}{9c^8}-\tfrac{4}{3c^6}e^2\left(l\right)\\&=\tfrac{4}{c^4}+\tfrac{1}{9c^8}\left(\zeta_{\mathbf{h}}^T\mathbf{R}_{\mathbf{u}}\zeta_{\mathbf{h}}+\sigma_v^2\right)^2-\tfrac{4}{3c^6}\left(\zeta_{\mathbf{h}}^T\mathbf{R}_{\mathbf{u}}\zeta_{\mathbf{h}}+\sigma_v^2\right).\end{aligned}\tag{48}$$

In a non-Gaussian noise case, considering Taylor expansions of $w\left(e\left(l\right)\right)$ about $v\left(l\right)$ and $e^c\left(l\right)$, we obtain

$$\begin{aligned}w\left(e\left(l\right)\right)&=w\left(e^c\left(l\right)+v\left(l\right)\right)=w\left(v\left(l\right)\right)+w'\left(v\left(l\right)\right)e^c\left(l\right)\\&+\tfrac{1}{2}w''\left(v\left(l\right)\right)\left(e^c\left(l\right)\right)^2+o\left(\left(e^c\left(l\right)\right)^2\right).\end{aligned}\tag{49}$$

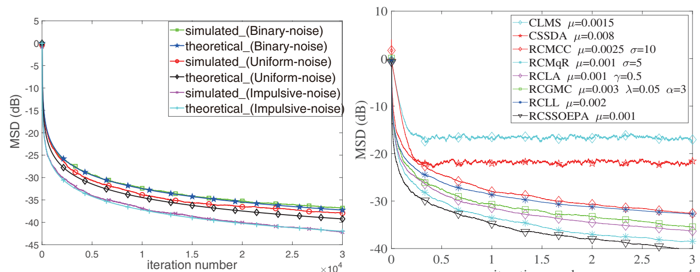
Accordingly, we have

$$\mathrm{E}\left[w\left(e\left(l\right)\right)\right]=\frac{2}{c^2}-\frac{1}{3c^4}\mathrm{E}\left[v^2\left(l\right)\right]-\frac{1}{3c^4}\zeta_{\mathbf{h}}^T\mathbf{R}_{\mathbf{u}}\zeta_{\mathbf{h}},\tag{50}$$

$$\begin{aligned}&\mathrm{E}\left[w^2\left(e\left(l\right)\right)\right]=\tfrac{4}{c^4}+\tfrac{1}{9c^8}\mathrm{E}\left[v^4\left(l\right)\right]-\tfrac{4}{3c^6}\mathrm{E}\left[v^2\left(l\right)\right]\\&+\zeta_{\mathbf{h}}^T\mathbf{R}_{\mathbf{u}}\zeta_{\mathbf{h}}\left[-\left(\tfrac{2}{c^2}-\tfrac{1}{3c^4}\mathrm{E}\left[v^2\left(l\right)\right]\right)\times\tfrac{2}{3c^4}+\tfrac{4}{9c^8}\mathrm{E}\left[v^2\left(l\right)\right]\right].\end{aligned}\tag{51}$$

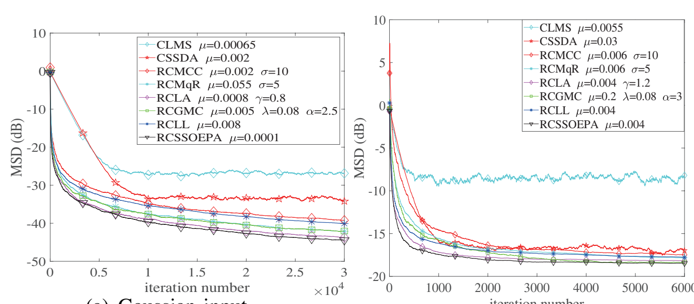
IV. Simulation Results

In Fig. 1(a), three different non-Gaussian noise distributions, namely, uniform-noise, binary-noise and impulsive-noise, are considered to verify the analysis under $c=1.4$. It is seen that theoretical analysis and simulation results match well, which shows the accuracy of the analysis. Here, the probability for the uniform noise is $f(a)=\begin{cases}0.5x,-x\le a\le x\\0,\text{others}\end{cases}$ with a zero mean and variance of 0.86, while it for the binary noise is $\Pr\{a=x\}=\Pr\{a=-x\}=0.5$ with zero mean and variance of 0.9. The impulsive-noise model is given by $v_j\left(l\right)=p\left(l\right)+m\left(l\right)\phi\left(l\right)$, where $p\left(l\right)$ represents white-Gaussian-noise, and its mean equals to zero, and its variance is σ_p^2, respectively. $m\left(l\right)\phi\left(l\right)$ is a Bernoulli Gaussian process that has a success probability $\mathrm{P}\left[m\left(l\right)=1\right]=\mathrm{Pr},\mathrm{P}\left[m\left(l\right)=0\right]=1-\mathrm{Pr}$, and



(a) (b)

Fig. 1: (a) Theoretical and simulated transient MSD of the proposed RCSSOEPA algorithm under binary noise, uniform noise and impulsive noise; (b) Convergence of the CSSDA, RCSSOEPA and mentioned recursive algorithms under Laplace noise.



(a) Gaussian input (b) AR(1) input

Fig. 2: Convergence of the CSSDA, RCSSOEPA and mentioned recursive algorithms under impulsive-noise with two different input signals: (a) Gaussian; (b) AR(1).

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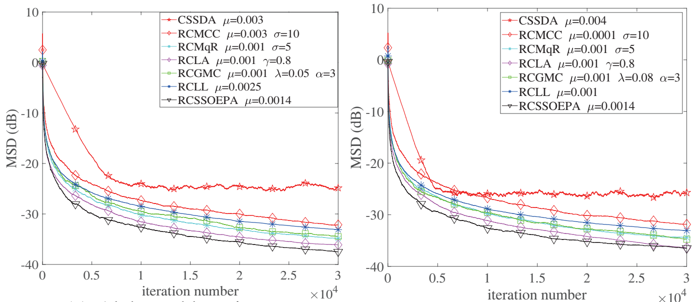

(a) Alpha-stable noise (b) Cauchy noise

Fig. 3: Convergence of the CSSDA, RCSSOEPA and mentioned recursive algorithms under alpha-stable and Cauchy noises.

$\phi(l)$ is a ZM-Gaussian process with $\sigma_\phi^2 = d\sigma_p^2\,(d \gg 1)$ [22], [23]. The parameters $\mathbf{S}$, $\mathbf{g}$ and $\mathbf{h}_0$ are obtained by referring to [6]. Herein, Pr equals to 0.3, and variances of the additive Gaussian noise and $\phi(l)$ are 0.08 and 1, respectively. From the simulation results, the theoretical analysis and simulation results match very well thus verifying these analysis. In Fig.1(b), the convergence of the proposed RCSSOEPA algorithm under Laplace noise [1] is given, where we can see that the proposed RCSSOEPA is still superior to mentioned algorithms.

Fig. 2 compares the performance of RCSSOEPA and RCMCC [14], RCLL [16], RCMqR [17], RCLA [18], RCGMC [19] and CSSDA (CSSDA is obtained by minimizing (4)) under impulsive-noise with Gaussian and AR(1) input signals. Variances of the additive Gaussian noise and $\phi(l)$ are 0.2 and 9, respectively, and $\Pr = 0.3$. Here, AR(1) signal is obtained from a first-order process $Q(Z) = \frac{1}{1-0.8Z^{-1}}$, and the kernel-width for RCMCC and CMCC are 10. Adjust the step-sizes to make the initial convergence for the CMCC, CLMS and CSSDA the same. Simulation results show that the RCSSOEPA algorithm can get faster convergence and achieve lower MSD than mentioned algorithms.

We now consider two nonGaussian noise (Cauchy and Alpha-stable) with the probability density functions of $f(v) = \frac{1}{\pi(1+v^2)}$ and $q(v) = \exp\{j\delta v - \rho|v|^\alpha [1 + j\beta \mathrm{sgn}(v) U(v,\alpha)]\}$, respectively [7], [8], where $U(v,\alpha) = \begin{cases} \tan\left(\frac{\alpha\pi}{2}\right), & \alpha \neq 1, \\ \frac{2}{\pi}\log|v|, & \alpha = 1. \end{cases}$, and we define the four parameters α, β, ρ and δ as parameter vector $\tau = (\alpha, \beta, \rho, \delta)$, where α is characteristics factor, β is symmetry factor, ρ is dispersion factor and δ is location parameter. Since the CLMS algorithm is not suitable for processing Cauchy and Alpha-stable noises, only CMCC, CSSDA, RCMCC and RCSSOEPA algorithms are compared in Fig. 3. In this simulation, the Cauchy noise parameter v is 0.5, Alpha-stable noise parameter vector is written as $\tau = (1, 0, 0.5, 0)$, and $c = 2.5$ for CSSDA and RCSSOEPA. From the simulated results, the proposed RCSSOEPA algorithm provides the lowest MSD.

V. Conclusion

This brief has proposed a recursive constrained sine second-order error promoting adaptive (RCSSOEPA) algorithm by constructing the SSOE cost function. The convergence and MSD have been theoretically analyzed and numerically investigated to verify the analysis. The simulated results have shown that the analysis matches very well to simulations, and the devised RCSSOEPA has a smaller MSD and faster convergence compared to other algorithms under various noises.

References

[1] K. Kumar, R. Pandey, and S. S. Bhattacharjee, and N. V. George, "Exponential hyperbolic cosine robust adaptive filters for audio signal processing,"*IEEE Signal Processing Letters*,vol.28,pp.1410-1414,2021.

[2] K. Kumar, R. Pandey, M.L.N.S. Karthik, S. S. Bhattacharjee, and N. V. George, Robust and sparsity-aware adaptive filters: A Review, Signal Processing, vol.189, ID:108276, 2021.

[3] M. L. R. de Campos, S. Werner, and J. A. Apolinário, "Constrained adaptive filters," in *Adaptive Antenna Arrays: Trends and Applications*(Signals and Communication Technology), S. Chandran, Ed. Berlin, Germany: Springer, 2004.

[4] V. C. Gogineni and S. Mula, "Logarithmic cost based constrained adaptive filtering algorithms for sensor array beamforming", in *IEEE Sensors Journal.*, vol. 18, no. 14, pp. 5897-5905, 2018.

[5] R. Arablouei and K. Dogancay, "Linearly-constrained line-search algorithm for adaptive filtering," *Electron. Lett.*, vol.48, pp.1208–1209,2012.

[6] R. Arabloueia, S. Werner, and K. Dogancay, "On the mean-square performance of the constrained LMS algorithm," *Signal Process.*. vol. 117, pp. 192–197, Dec. 2015.

[7] X. Huang, Y. Li, and F. Albu, "A norm constraint Lorentzian algorithm under alpha-stable measurement noise," in *2019 Asia-Pacific Signal and Information Processing Association Annual Summit and Conference (APSIPA ASC).*, Lanzhou, China, 2019, pp. 1076–1079.

[8] B. Chen, L. Xing, J. Liang, N. Zheng, and J. Principe, "Steady-state mean-square error analysis for adaptive filtering under the maximum correntropy criterion," in *IEEE Signal Process. Lett.*, vol. 21, no. 7, pp. 880–884, 2014.

[9] S. Zhao, B. Chen, and J. Principe, "Kernel adaptive filtering with maximum correntropy criterion," in *Proc. Int. Joint Conf. Neural Netw. (IJCNN),* 2011, pp. 2012–2017.

[10] B. Weng and K. E. Barner, "Nonlinear system identification in impulsive environments," *IEEE Trans. Signal Process,*vol.53, pp.2588–2594,2005.

[11] H. Zayyani, "Continuous mixed p-norm adaptive algorithm for system identification," *IEEE Signal Process. Lett.*, vol. 21, pp. 1108–1110,2014.

[12] Z. Wang, H. Zhao, and X. Zeng, "Constrained least mean M-estimation adaptive filtering algorithm," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, early access, Sep. 7, 2020.

[13] S. Peng and B. Chen, "Constrained maximum correntropy adaptive filtering," *Signal Process.*, vol. 140, pp. 116–126, Nov. 2017.

[14] G. Qian, X. Ning, and S. Wang, "Recursive constrained maximum correntropy criterion algorithm for adaptive filtering," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 67, no. 10, pp. 2229–2233, Oct. 2020.

[15] X. Zhang, *Matrix Analysis and Application*, 2th ed. Beijing, China: Tsinghua Univ. Press, 2013.

[16] T. Liang, Y. Li, W. Xue, Y. Li, and T. Jiang, "Performance and analysis of recursive constrained least Lncosh algorithm under impulsive noises," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol.68, pp.2217-2221, 2020.

[17] T. Liang, Y. Li, Y. Xia, Recursive constrained adaptive algorithm under q -R´enyi kernel function, IEEE Trans.Circuits Syst. II, vol.68, no.6, pp.2227-2231, 2020..

[18] Jia W, Feng Z, Cai T, et al, Recursive Constrained Adaptive Filtering Algorithm Based on Arctangent Framework, IEEE Transactions on Circuits and Systems II: Express Briefs, 2022.

[19] Zhao J, Zhang J A, Li Q, et al, Recursive constrained generalized maximum correntropy algorithms for adaptive filtering, Signal Processing, vol.199, 2022.

[20] L. Isserlis, "On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables," *Biometrika*, vol. 12, nos. 1–2, pp. 134–139, 1918.

[21] W. Liu, P. P. Pokharel, and J. C. Principe, "Correntropy: Properties and applications in non-Gaussian signal processing," *IEEE Trans. Signal Process.*, vol. 55, no. 11, pp. 5286–5298, Nov. 2007.

[22] H. Zhao and W. Wang, "Volterra system identification under maximum correntropy criteria in impulsive environments," in *Proc. 35th Chin. Control Conf. (CCC)*, Chengdu, China, 2016, pp. 5067–5070.

[23] R. L. Das and M. Narwaria, "Lorentzian based adaptive filters for impulsive noise environments," *IEEE Trans. Circuits Syst. I, Reg. Papers,* vol. 64, no. 6, pp. 1529–1539, Jun. 2017.