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Admissible extensions of subtheories of second order arithmetic

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ABSTRACT

In this paper we study admissible extensions of several theories T of reverse mathematics. The idea is that in such an extension the structure $\mathfrak{M} = (\mathbb{N}, \mathbb{S}, \in)$ of the natural numbers and collection of sets of natural numbers $\mathbb{S}$ has to obey the axioms of T while simultaneously one also has a set-theoretic world with transfinite levels erected on top of $\mathfrak{M}$ governed by the axioms of Kripke-Platek set theory, KP. In some respects, the admissible extension of T can be viewed as a proof-theoretic analog of Barwise's admissible cover of an arbitrary model of set theory; see [2]. However, by contrast, the admissible extension of T is usually not a conservative extension of T. Owing to the interplay of T and KP, either theory's axioms may force new sets of naturals to exist which in turn may engender yet new sets of naturals on account of the axioms of the other.

The paper discerns a general pattern though. It turns out that for many familiar theories T, the second order part of the admissible cover of T equates to T augmented by transfinite induction over all initial segments of the Bachmann-Howard ordinal. Technically, the paper uses a novel type of ordinal analysis, expanding that for KP to the higher set-theoretic universe while at the same time treating the world of subsets of $\mathbb{N}$ as an unanalyzed class-sized urelement structure. Among the systems of reverse mathematics, for which we determine the admissible extension, are $\Pi_1^1\text{-CA}_0$ and ATR_0 as well as the theory of bar induction, BI.

1. Introduction

In this paper we study admissible extensions of several theories T of reverse mathematics. The idea is that in such an extension the structure $\mathfrak{M} = (\mathbb{N}, \mathbb{S}, \in)$ of the natural numbers and collection of sets of natural numbers $\mathbb{S}$ has to obey the axioms of T while simultaneously one also has a set-theoretic world with transfinite levels erected on top of $\mathfrak{M}$ governed by the axioms of Kripke-Platek set theory, KP.

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In some respects, the admissible extension of T can be viewed as a proof-theoretic analog of Barwise's admissible cover of an arbitrary model of set theory (in his book "Admissible Sets and Structures"). However, by contrast, the admissible extension of T is usually not a conservative extension of T. Owing to the interplay of T and KP, either theory's axioms may force new sets of naturals to exist which in turn may engender yet new sets of naturals on account of the axioms of the other.

This approach will be studied in detail and paradigmatically by combining Π_1^1 comprehension on the natural numbers with Kripke-Platek set theory, the respective theory being called $\mathsf{KP}+(\Pi_1^1\text{-}\mathsf{CA}^*)$. In the next two sections we present the syntactic machinery of this system and then turn to its Tait-style reformulation, which is convenient for the later proof-theoretic analysis. For this proof-theoretic analysis we make use of a more or less standard ordinal notation system, following Buchholz [3], that includes the Bachmann-Howard ordinal $\mathcal{BH}$ and the collapsing functions needed later.

Section 5 introduces the semi-formal system RS^* in which $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$ will be embedded. For the subsequent analysis of RS^* the paper uses a novel type of ordinal analysis, expanding that for KP to the higher set-theoretic universe while at the same time treating the world of subsets of $\mathbb{N}$ as an unanalyzed class-sized urelement structure.

Based on these results we turn to the reduction of $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$ to $\Pi_1^1\text{-}\mathsf{CA} + TI(<\mathcal{BH})$ in Section 6. Main ingredients here are the so-called α-trees - extending Simpson's suitable trees in [20] - and a specific truth definition based on those.

The final section shows how the previous results can be extended to other systems of reverse mathematics such as ATR_0 and BI.

2. KP and $(\Pi_1^1\text{-}\mathsf{CA}^*)$

Let $\mathcal{L}_\in$ be the usual language of set theory with $\in$ as its only non-logical relation symbol plus set constants $\underline{\varnothing}$ (for the empty set) and $\underline{\omega}$ (for the first infinite ordinal). $\mathcal{L}_2^{set}$ is the extension of $\mathcal{L}_\in$ by countably many unary relation variables. As we will see, $\mathcal{L}_2^{set}$ may be regarded as a two-sorted first order language where the first sort is supposed to range over sets and the second sort over collections of natural numbers.

The *set terms* of $\mathcal{L}_2^{set}$ are the set constants and the set variables. The *atomic formulas* of $\mathcal{L}_2^{set}$ are the expressions of the form $(a \in b)$, $(a \notin b)$, $U(a)$, and $\neg U(a)$, where a, b are set terms and U is a relation variable. The *formulas* of $\mathcal{L}_2^{set}$ are built up from these atomic formulas by means of the propositional connectives $\vee, \wedge$, the bounded set quantifiers $(\exists x \in a)$, $(\forall x \in a)$, the unbounded set quantifiers $\exists x, \forall x$, and the relation quantifiers $\exists X, \forall X$. We use as metavariables (possibly with subscripts):

- u, v, w, x, y, z for set variables,
- a, b, c for set terms,
- U,V,W,X,Y,Z for relation variables,
- A, B, C, D for formulas.

Observe that all $\mathcal{L}_2^{set}$ formulas are in negation normal form. We assume classical logic throughout this article. Therefore, the negation $\neg A$ of an $\mathcal{L}_2^{set}$ formula A is defined via de Morgan's laws and the law of double negation. Furthermore, $(A \rightarrow B)$ is defined as $(\neg A \vee B)$ and $(A \leftrightarrow B)$ as $((A \rightarrow B) \wedge (B \rightarrow A))$.

To simplify the notation we often omit parentheses if there is no danger of confusion. Equality is not a basic symbol but defined for sets and relations:

$$(a = b) \; := \; (\forall x \in a)(x \in b) \, \wedge \, (\forall x \in b)(x \in a),$$

$$(U = V) \; := \; (\forall x \in \underline{\omega})(U(x) \leftrightarrow V(x)).$$

Since relations are supposed to range over subcollections of $\underline{\omega}$ – see axiom (Sub $\underline{\omega}$) below – this definition of the equality of relations makes sense.

The vector notation $\vec{a}$ will be used to denote finite sequences of $\mathcal{L}_2^{set}$ terms. $A^{(a)}$ results from A by restricting all unbounded set quantifiers to a; relation quantifiers QX are not affected by this restriction. The Δ_0, Σ, Π, Σ_n, and Π_n formulas of $\mathcal{L}_2^{set}$ are defined as usual where relation variables are permitted as parameters. However, relation quantifiers are not allowed in these formulas. The *bounded* formulas are all those $\mathcal{L}_2^{set}$ formulas which do not contain unbounded set quantifiers; they may contain relation quantifiers.

Moreover, we shall employ the common set-theoretic terminology and the standard notational conventions, for example:

- $Tran[a] := (\forall x \in a)(\forall y \in x)(y \in a)$,
- $Ord[a] := Tran[a] \wedge (\forall x \in a)\, Tran[x]$,
- $Succ[a] := Ord[a] \wedge (\exists x \in a)(a = x \cup \{x\})$,
- $FinOrd[a] := Ord[a] \wedge (a = \underline{\varnothing} \vee Succ[a]) \wedge (\forall x \in a)(x = \underline{\varnothing} \vee Succ[x])$.

Clearly, $Succ[a]$ expresses that a is a successor ordinal and $FinOrd[a]$ says that a is a finite ordinal.

Now we formulate *Kripke-Platek set theory* KP in $\mathcal{L}_2^{set}$, based on classical logic.[1] Its non-logical axioms are:

(Equality)	$a = b \wedge D[a] \rightarrow D[b]$.
(Sub $\underline{\omega}$)	$U(a) \rightarrow a \in \underline{\omega}$.
(Pair)	$\exists z(a \in z \wedge b \in z)$.
(Union)	$\exists z(\forall y \in a)(\forall x \in y)(x \in z)$.
(Empty set)	$(\forall x \in \underline{\varnothing})(x \neq x)$.
(Infinity)	$a \in \underline{\omega} \leftrightarrow FinOrd[a]$.
(Δ_0-Sep)	$\exists z(z = \{x \in a : D[x]\})$.
(Δ_0-Col)	$(\forall x \in a)\exists y D[x, y] \rightarrow \exists z(\forall x \in a)(\exists y \in z)D[x, y]$.
($\in$-Ind)	$\forall x((\forall y \in x)A[y] \rightarrow A[x]) \rightarrow \forall x A[x]$.

The formulas D in the schemas (Equality), (Δ_0-Sep), and (Δ_0-Col) are Δ_0 whereas the formula A in ($\in$-Ind) ranges over arbitrary formulas of $\mathcal{L}_2^{set}$.

It is easy to see that the theory KP is a conservative extension of the usual first order formalization of Kripke-Platek set theory with infinity. This system is called KPω in some texts.

In this article we are interested in the theory that is obtained from KP by adding the following form of Π_1^1 comprehension, where $D[x, Y]$ is a Δ_0 formula of $\mathcal{L}_2^{set}$,

$$\exists Z(\forall x \in \underline{\omega})(Z(x) \leftrightarrow \forall Y D[x, Y]). \tag{Π_1^1-CA*}$$

From (Δ_0-Sep) – and here it is crucial that relations are permitted as parameters – we immediately obtain that the intersection of any relation with $\underline{\omega}$ is extensionally equal to a subset of $\underline{\omega}$. On the other hand, in view of (Π_1^1-CA*) we also know that for every subset of $\underline{\omega}$ there exists a relation with the same elements.

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[1] Observe that $(a = a)$ and $(U = U)$ are provable by logic.

Let L_2 be the usual language of second order arithmetic as presented, for example, in Simpson [20]. A natural translation of L_2 into $\mathcal{L}_2^{set}$ is defined as follows: The number variables of L_2 are interpreted in $\mathcal{L}_2^{set}$ as ranging over $\underline{\omega}$ and the set variables of L_2 are replaced in $\mathcal{L}_2^{set}$ by relation variables. The other symbols and connectives of L_2 are dealt with as in [20, VII.3.9].

It is also clear that our comprehension schema (Π_1^1-CA^*) implies the usual form of Π_1^1 comprehension of second order arithmetic modulo its natural translation into $\mathcal{L}_2^{set}$.

Remark 1.

1. Let $A'[x, Y]$ be the natural translation of an arithmetical $A[x, Y]$. Then $\mathsf{KP} + (\Pi_1^1$-$\mathsf{CA}^*)$ proves that

$$(\exists z \subseteq \underline{\omega})(\forall x \in \underline{\omega})(x \in z \;\leftrightarrow\; (\forall y \subset \underline{\omega})A'[x, y]).$$

2. It is clear that $\mathsf{KP} + (\Pi_1^1$-$\mathsf{CA}^*)$ is a very weak subsystem of Power Kripke-Platek set theory $\mathbf{KP}(\mathcal{P})$ as introduced in Rathjen [17]. It follows from the results of [17] that the Axiom Beta is not provable in $\mathbf{KP}(\mathcal{P})$. Hence it is also not provable in $\mathsf{KP} + (\Pi_1^1$-$\mathsf{CA}^*)$ and, therefore, $\mathsf{KP} + (\Pi_1^1$-$\mathsf{CA}^*)$ does not contain the theory KPi for a recursively inaccessible universe; see, for example, Jäger [11,13].

3. A Tait-style reformulation of $\mathsf{KP} + (\Pi_1^1$-$\mathsf{CA}^*)$

The basic idea of a Tait-style calculus is that it derives finite sets of $\mathcal{L}_2^{set}$ formulas rather than individual formulas. As in one-sided sequent calculi, the intended meaning of such a set of formulas is the disjunction of its elements. The following reformulation of $\mathsf{KP} + (\Pi_1^1$-$\mathsf{CA}^*)$ is for technical reasons only and simplifies the proof of Theorem 21.

The capital Greek letters Γ, Θ, Λ (possibly with subscripts) will act as metavariables for finite sets of $\mathcal{L}_2^{set}$ formulas. Also, we write (for example) $\Gamma, A_1, \ldots, A_n$ for the set $\Gamma \cup \{A_1, \ldots, A_n\}$; similarly for expressions such as Γ, Θ, A.

The Tait-style axioms of $\mathsf{KP} + (\Pi_1^1$-$\mathsf{CA}^*)$

(TnD)	$\Gamma, \neg D, D$.
(Equality)	$\Gamma, a \neq b, \neg D[a], D[b]$
(Sub $\underline{\omega}$)	$\Gamma, \neg U(a), a \in \underline{\omega}$.
(Pair)	$\Gamma, \exists z(a \in z \wedge b \in z)$.
(Union)	$\Gamma, \exists z(\forall y \in a)(\forall x \in y)(x \in z)$.
(Empty set)	$\Gamma, a \notin \underline{\varnothing}$.
(Infinity)	$\Gamma, a \in \underline{\omega} \leftrightarrow FinOrd[a]$.
(Δ_0-Sep)	$\Gamma, \exists z(z = \{x \in a : D[x]\})$.
(Δ_0-Col)	$\Gamma, (\forall x \in a)\exists y D[x, y] \;\rightarrow\; \exists z(\forall x \in a)(\exists y \in z)D[x, y]$.
($\in$-Ind)	$\Gamma, \forall x((\forall y \in x)A[y] \;\rightarrow\; A[x]) \;\rightarrow\; \forall x A[x]$.
(Π_1^1-CA^*)	$\Gamma, \exists Z(\forall x \in \underline{\omega})(Z(x) \;\leftrightarrow\; \forall Y D[x, Y])$.

In these axioms the formulas D in (TnD), (Equality), (Δ_0-Sep), (Δ_0-Col), and (Π_1^1-CA^*) are supposed to be Δ_0. The formula A in ($\in$-Ind) may be an arbitrary $\mathcal{L}_2^{set}$ formula. (TnD) stands for "Tertium non datur".

The Tait-style Inference rules of $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B} \quad (\vee) \qquad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \quad (\wedge)$$

$$\frac{\Gamma, A[b]}{\Gamma, \exists x A[x]} \quad (\exists) \qquad \frac{\Gamma, A[u]}{\Gamma, \forall x A[x]} \quad (\forall)$$

$$\frac{\Gamma, b \in a \,\wedge\, A[b]}{\Gamma, (\exists x \in a) A[x]} \quad (b\exists) \qquad \frac{\Gamma, u \in a \,\rightarrow\, A[u]}{\Gamma, (\forall x \in a) A[x]} \quad (b\forall)$$

$$\frac{\Gamma, A[V]}{\Gamma, \exists X A[X]} \quad (\exists_2) \qquad \frac{\Gamma, A[U]}{\Gamma, \forall X A[X]} \quad (\forall_2)$$

$$\frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma} \quad (\mathsf{Cut})$$

Of course, it is demanded that in $(\forall)$ and $(b\forall)$ the eigenvariable u must not occur in the conclusion; the same is the case for the variable U in $(\forall_2)$.

We say that Γ is Tait-style derivable from $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$ iff there exists a finite sequence of finite sets of $\mathcal{L}_2^{set}$ formulas

$$\Theta_0, \ldots, \Theta_h$$

such that Θ_h is the set Γ and for any $i = 0, \ldots, h$ one of the following two conditions is satisfied:

- Θ_i is a Tait-style axiom of $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$.
- Θ_i is the conclusion of an inference of a Tait-style inference rule of $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$ with premise(s) from $\Theta_0, \ldots, \Theta_{i-1}$.

In this case we write $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*) \vdash^h \Gamma$ and say that Γ has a proof of length h. It is an easy exercise to show that a formula A is provable in one of the usual Hilbert-style formalizations of $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$ iff $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*) \vdash^h A$ for some natural number h. Details are left to the reader.

4. Ordinal notations

In the next sections we establish the upper proof-theoretic bound of the theory $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$. Our method of choice is the ordinal analysis of $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$ via a system RS^* of ramified set theory. And in order to build up this system and to control the derivations in RS^* we work with specific ordinal notations.

Buchholz has developed several ordinal notation systems based on so-called collapsing functions; see, for example, Buchholz [3–5]. In the following we will work with a reduced version, which is sufficient for our purposes.

Let On be the collection of all ordinals and let Ω be a sufficiently large ordinal. To simplify matters we set $\Omega := \aleph_1$, but also ω_1^{ck} or even somewhat small ordinals could do the job. The following outline is based on [4].

Definition 2. The set of ordinals $C(\alpha, \beta)$ and the ordinals $\psi(\alpha)$ are defined for all ordinals α and β by induction on α.

(i) $\{0, \Omega\} \cup \beta \subseteq C(\alpha, \beta)$.
(ii) If $\eta, \xi \in C(\alpha, \beta)$, then $\eta + \xi \in C(\alpha, \beta)$ and $\omega^\xi \in C(\alpha, \beta)$.

(iii) If $\xi < \alpha$ and $\xi \in C(\alpha, \beta)$, then $\psi(\xi) \in C(\alpha, \beta)$.
(iv) $\psi(\alpha) := \min(\{\eta \in On : C(\alpha, \eta) \cap \Omega = \eta\})$.

The following lemma summarizes some key properties of the sets $C(\alpha, \beta)$ and the function ψ. For its proof see Buchholz [3].

Lemma 3. *We have for all ordinals* $\alpha, \alpha_1, \alpha_2, \beta$:

(1) $\psi(\alpha) < \Omega$.
(2) $C(\alpha, \psi(\alpha)) \cap \Omega = \psi(\alpha)$.
(3) $\psi(\alpha)$ *is an* ε-*number.*
(4) *If* $\alpha_1 < \alpha_2$ *and* $\alpha_1 \in C(\alpha_2, \psi(\alpha_2))$, *then* $\psi(\alpha_1) < \psi(\alpha_2)$.
(5) *If* $\alpha_1 \leq \alpha_2$, *then* $\psi(\alpha_1) \leq \psi(\alpha_2)$ *and* $C(\alpha_1, \psi(\alpha_1)) \subseteq C(\alpha_2, \psi(\alpha_2))$.
(6) $C(\alpha, 0) = C(\alpha, \psi(\alpha))$.

We write $\varepsilon_{\Omega+1}$ for the least ordinal $\alpha > \Omega$ such that $\omega^\alpha = \alpha$. Its collapse $\mathcal{BH} := \psi(\varepsilon_{\Omega+1})$ is called the *Bachmann-Howard ordinal.* This number gained importance in proof theory since it is the proof-theoretic ordinal of the theory ID_1 of one positive inductive definition and of Kripke-Platek set theory KP; see, for example, Buchholz and Pohlers [7], Jäger [12], and Pohlers [14].

5. The semi-formal ramified system RS^*

In this section we introduce the semi-formal proof system RS^* of ramified set theory. We begin with extending our language $\mathcal{L}_2^{set}$ to the language $\mathcal{L}^\star$ and then present the axioms and rules of inference of RS^*. Afterwards we turn to operator controlled derivations and some basic properties of RS^*. We show that $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$ can be embedded into RS^* and prove cut elimination and collapsing for RS^*. Henceforth, all ordinals used in this section on the metalevel range over the set $C(\varepsilon_{\Omega+1}, 0)$ if not stated otherwise.

5.1. The language $\mathcal{L}^\star$

The basic idea is to extend the language $\mathcal{L}_2^{set}$ to the language $\mathcal{L}^\star$ by adding unary relation symbols M_α and new quantifiers $\exists x^\alpha / \forall x^\alpha$ for all $\alpha < \Omega$. The quantifiers $Q x^\alpha$ are supposed to range over M_α, and later, see Subsection 6.1, an atomic formula $M_\alpha(a)$ will be interpreted as stating that a can be coded by a so-called α-tree, see Definition 36 below.

Definition 4. The *formulas* F of RS^*, their *ranks* $rk(F)$ and *parameter sets* $|F|$ are inductively defined as follows.

1. If a and b are set terms of $\mathcal{L}_2^{set}$, then $(a \in b)$ and $(a \notin b)$ are formulas with

$$rk(a \in b) := rk(a \notin b) := 0 \text{ and } |(a \in b)| := |(a \notin b)| := \{0\}.$$

2. If a is a set term and U a relation variable, then $U(a)$ and $\neg U(a)$ are formulas with

$$rk(U(a)) := rk(\neg U(a)) := 0 \text{ and } |U(a)| := |\neg U(a)| := \{0\}.$$

3. If a is a set term and M_α one of these new relation symbols, then $M_\alpha(a)$ and $\neg M_\alpha(a)$ are formulas with

$$rk(M_\alpha(a)) := rk(\neg M_\alpha(a)) := \omega\alpha \text{ and } |M_\alpha(a)| := |\neg M_\alpha(a)| := \{\alpha\}.$$

4. If F and G are formulas, then $(F \vee G)$ and $(F \wedge G)$ are formulas with

$$rk(F \vee G) := rk(F \wedge G) := \max(rk(F), rk(G)) + 1,$$
$$|F \vee G| := |F \wedge G| := |F| \cup |G|.$$

5. If $F[u]$ is a formula, then $(\exists x \in a)F[x]$ and $(\forall x \in a)F[x]$ are formulas with

$$rk((\exists x \in a)F[x]) := rk((\forall x \in a)F[x]) := rk(u \in a \ \wedge \ F[u]) + 1.$$
$$|(\exists x \in a)F[x]| := |(\forall x \in a)F[x]| := |F[u]|.$$

6. If $F[u]$ is a formula, then $\exists x^\alpha F[x]$ and $\forall x^\alpha F[x]$ are formulas with

$$rk(\exists x^\alpha F[x]) := rk(\forall x^\alpha F[x]) := rk(M_\alpha(u) \ \wedge \ F[u]) + 1.$$
$$|\exists x^\alpha F[x]| := |\forall x^\alpha F[x]| := \{\alpha\} \cup |F[u]|.$$

7. If $F[u]$ is a Δ_0 formula of $\mathcal{L}_2^{set}$, then $\exists x F[x]$ and $\forall x F[x]$ are formulas with

$$rk(\exists x F[x]) := rk(\forall x F[x]) := \Omega,$$
$$|\exists x F[x]| := |\forall x F[x]| := |F[u]|.$$

8. If $F[u]$ is not a Δ_0 formula of $\mathcal{L}_2^{set}$, then $\exists x F[x]$ and $\forall x F[x]$ are also formulas but with

$$rk(\exists x F[x]) := rk(\forall x F[x]) := \max(\Omega + 1, rk(F[u]) + 3),$$
$$|\exists x F[x]| := |\forall x F[x]| := |F[u]|.$$

9. If $F[U]$ is a formula, then $\exists X F[X]$ and $\forall X F[X]$ are formulas with

$$rk(\exists X F[X]) := rk(\forall X F[X]) := rk(F[U]) + 1,$$
$$|\exists X F[X]| := |\forall X F[X]| := |F[U]|.$$

Finally, we define the *level* of a formula as

$$lev(F) \; := \; \begin{cases} \max(|F|) & \text{if } rk(F) < \Omega, \\ \Omega & \text{if } \Omega \le rk(F). \end{cases}$$

To be precise: If F is a formula of RS^*, then $|F|$ collects the levels of the relations symbols M_α and of the quantifiers $Q x^\alpha$ occurring in F plus possibly the number 0.

There are several collections of $\mathcal{L}^\star$ formulas that will play an important role later.

1. $\mathcal{D}$ is the collection of all $\mathcal{L}^\star$ formulas in which unbounded set quantifiers Qx and relation quantifiers QX do not occur.
2. $\mathcal{S}$ is the closure of $\mathcal{D}$ under the propositional connectives, quantifiers $Q x^\alpha$, bounded quantifiers $(Qx \in r)$, and unbounded existential set quantifiers.
3. $\mathcal{S}_0$ is the subclass of $\mathcal{S}$ that contains all Σ formulas of $\mathcal{L}_2^{set}$ that are not Δ_0.
4. $\mathcal{B}$ consists of all $\mathcal{L}^\star$ formulas that do not contain unbounded set quantifiers Qx.

Some important properties of the ranks of RS^* formulas are summarized in the following lemma. Its proof is straightforward and will be omitted.

Lemma 5.

(1) $rk(F) = rk(\neg F) < \omega \cdot lev(F) + \omega \leq \Omega + \omega$.
(2) $rk(F) < \Omega$ iff F belongs to $\mathcal{B}$.
(3) If $rk(\exists x F[x]) = \Omega$, then $F[u]$ is a Δ_0 formula of $\mathcal{L}_2^{set}$.
(4) $rk(M_\alpha(a) \wedge F[a]) < rk(\exists x^\alpha F[x]) < rk(\exists x F[x])$.
(5) $rk(s \in r \wedge F[s]) < rk((\exists x \in r)F[x]) < rk(\exists x F[x])$.
(6) $rk(F[P]) = rk(F[U]) < rk(\exists X F[X])$.

5.2. Axioms and rules of inference of RS^*

We shall use a Tait-style calculus as proof system for RS^*. In the following the Greek capital letters Γ, Θ, Λ (possibly with subscripts) will now act as metavariables for finite sets of $\mathcal{L}^\star$ formulas. If Γ is the set $\{F_1, \ldots, F_k\}$ we define

$$|\Gamma| := |F_1| \cup \ldots \cup |F_k| \quad \text{and} \quad \Gamma^\vee := F_1 \vee \ldots \vee F_k$$

such that $|\Gamma|$ collects the parameter sets of the formulas in Γ and $\Gamma^\vee$ is the disjunction of its elements.

Axioms of RS^*

(1) $\Gamma, \neg F, F$ for $F \in \mathcal{B}$.
(2) $\Gamma, a \neq b, \neg F[a], F[b]$ for $F[u] \in \mathcal{B}$.
(3) $\Gamma, \neg U(a), a \in \underline{\omega}$.
(4) $\Gamma, \neg M_0(a)$.
(5) $\Gamma, M_1(\underline{\varnothing})$.
(6) $\Gamma, a \notin \varnothing$.
(7) $\Gamma, M_{\omega+1}(\underline{\omega})$.
(8) $\Gamma, (a \in \underline{\omega} \leftrightarrow FinOrd[a])$.
(9) $\Gamma, \neg M_\alpha(a), M_\beta(a)$ for $\alpha \leq \beta$.
(10) $\Gamma, \neg M_{\alpha+1}(a), b \notin a, M_\alpha(b)$.
(11) $\Gamma, \neg M_\alpha(a), \neg M_\beta(b), \exists z^{\beta+1}(a \in z \wedge b \in z)$ for $\alpha \leq \beta$.
(12) $\Gamma, \neg M_\alpha(a), \exists z^\alpha(\forall y \in a)(\forall x \in y)(x \in z)$.
(13) $\Gamma, \neg M_\alpha(a), \exists z^{\alpha+1}(z = \{x \in a : D[x]\})$ for $D[u]$ from Δ_0 of $\mathcal{L}_2^{set}$.
(14) $\Gamma, \exists Z(\forall x \in \underline{\omega})(Z(x) \leftrightarrow \forall Y D[x, Y])$ for $D[u, V]$ from Δ_0 of $\mathcal{L}_2^{set}$.

Please observe that the main formulas of all axioms belong to $\mathcal{B}$. This will be important in the later subsections when it comes to the ordinal analysis of the derivations in RS^*.

Inference rules of RS^*

$$(\vee) \quad \frac{\Gamma, F, G}{\Gamma, F \vee G}$$

$$(\wedge) \quad \frac{\Gamma, F \quad \Gamma, G}{\Gamma, F \wedge G}$$

$$(\neg M) \quad \frac{\Gamma, \neg M_\beta(a) \text{ for all } \beta < \lambda}{\Gamma, \neg M_\lambda(a)} \text{ for } \lambda \text{ limit}$$

$$(\exists) \quad \frac{\Gamma, M_\beta(a) \wedge F[a]}{\Gamma, \exists x F[x]}$$

$$(\forall) \qquad \frac{\Gamma, \neg M_\beta(a) \vee F[a] \text{ for all } \beta < \Omega \text{ and all } a}{\Gamma, \forall x F[x]}$$

$$(\exists^\alpha) \qquad \frac{\Gamma, M_\beta(a) \wedge F[a]}{\Gamma, \exists x^\alpha F[x]} \text{ for } \beta \leq \alpha$$

$$(\forall^\alpha) \qquad \frac{\Gamma, \neg M_\beta(a) \vee F[a] \text{ for all } \beta \leq \alpha \text{ and all } a}{\Gamma, \forall x^\alpha F[x]}$$

$$(b\exists) \qquad \frac{\Gamma, b \in a \wedge F[b]}{\Gamma, (\exists x \in a) F[x]}$$

$$(b\forall) \qquad \frac{\Gamma, b \notin a \vee F[b] \text{ for all } b}{\Gamma, (\forall x \in a) F[x]}$$

$$(\exists_2) \qquad \frac{\Gamma, F[U]}{\Gamma, \exists X F[X]}$$

$$(\forall_2) \qquad \frac{\Gamma, F[U] \text{ for all } U}{\Gamma, \forall X F[X]}$$

$$(\mathsf{Cut}) \qquad \frac{\Gamma, F \qquad \Gamma, \neg F}{\Gamma}$$

$$(\mathcal{S}_0\text{-Ref}) \qquad \frac{\Gamma, F}{\Gamma, \exists z F^{(z)}} \text{ for } F \in \mathcal{S}_0$$

$$(\mathsf{BC}) \qquad \frac{\Gamma, F^\beta}{\Gamma, \exists z^{\beta+\omega} F^{(z)}} \text{ for } F \in \mathcal{S}_0$$

If F is from $\mathcal{S}$, we write F^β for the result of replacing each unbounded existential quantifier $\exists x$ by $\exists x^\beta$. This must not be confused with $F^{(a)}$ where each unbounded set quantifier Qx in F is replaced by $(Qx \in a)$.

The meaning of the rules $(\vee) - (\mathcal{S}_0\text{-Ref})$ should be self-explaining. Rule (BC) will be needed in connection with the boundedness and collapsing results in subsection 5.7.

5.3. Derivation operators

The general theory of derivation operators and operator controlled derivations has been introduced in Buchholz [4]. In the following we adapt his general approach to the more specific (and simpler) situation with which we have to deal here.

Definition 6. Let $Pow(On)$ denote the collection of all sets of ordinals. A class function

$$\mathcal{H} : Pow(On) \to Pow(On)$$

is called a *derivation operator (d-operator for short)* iff it is a closure operator and satisfies the following conditions for all $X, Y \in Pow(On)$:

(i) $X \subseteq \mathcal{H}(X)$.
(ii) $Y \subseteq \mathcal{H}(X) \implies \mathcal{H}(Y) \subseteq \mathcal{H}(X)$.
(iii) $\{0, \Omega\} \subseteq \mathcal{H}(X)$.
(iv) If α has Cantor normal form $\omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$, then

$$\alpha \in \mathcal{H}(X) \iff \alpha_1, \ldots, \alpha_k \in \mathcal{H}(X).$$

These requirements ensure that every d-operator $\mathcal{H}$ is monotone, inclusive, and idempotent. Every $\mathcal{H}(X)$ is closed under $+$ as well as $\xi \mapsto \omega^\xi$, and under the decomposition of its members into their Cantor normal form components.

Let $\mathcal{H}$ be a d-operator. Then we define for all finite sets of ordinals $\mathfrak{m}$ the operators

$$\mathcal{H}[\mathfrak{m}] \ : \ Pow(On) \to Pow(On)$$

by setting for all $X \in Pow(On)$:

$$\mathcal{H}[\mathfrak{m}](X) \ := \ \mathcal{H}(\mathfrak{m} \cup X).$$

Also, we simply write $\mathcal{H}[\sigma]$ for $\mathcal{H}[\{\sigma\}]$. If $\mathcal{H}$ and $\mathcal{K}$ are d-operators, then we write $\mathcal{H} \subseteq \mathcal{K}$ to state that

$$\mathcal{H}(X) \subseteq \mathcal{K}(X) \ \text{ for all } X \subseteq On.$$

In this case $\mathcal{K}$ is called an *extension* of $\mathcal{H}$. The following observation is immediate from these definitions.

Lemma 7. *If $\mathcal{H}$ is a d-operator, then we have for all finite sets of ordinals $\mathfrak{m}, \mathfrak{n}$:*

(1) $\mathcal{H}[\mathfrak{m}]$ *is a d-operator and an extension of $\mathcal{H}$.*
(2) *If $\mathfrak{m} \subseteq \mathcal{H}(\varnothing)$, then $\mathcal{H}[\mathfrak{m}] = \mathcal{H}$.*
(3) *If $\mathfrak{n} \subseteq \mathcal{H}[\mathfrak{m}](\varnothing)$, then $\mathcal{H}[\mathfrak{n}] \subseteq \mathcal{H}[\mathfrak{m}]$.*

Now we turn to specific operators $\mathcal{H}_\sigma$. They will play a central role in the collapsing process of RS^* derivations; see Theorem 24.

Definition 8. For any σ, the operator

$$\mathcal{H}_\sigma \ : \ Pow(On) \to Pow(On)$$

is defined by setting for all $X \subseteq On$:

$$\mathcal{H}_\sigma(X) \ := \ \bigcap \{C(\alpha, \beta) : X \subseteq C(\alpha, \beta) \ \& \ \sigma < \alpha\}.$$

In the following lemmas we collect those properties of these operators that will be needed later. For their proofs we refer to Buchholz [4], in particular Lemma 4.6 and Lemma 4.7.

Lemma 9. *We have for all ordinals σ, τ and all $X \subseteq On$:*

(1) $\mathcal{H}_\sigma$ *is a d-operator.*
(2) $\mathcal{H}_\sigma(\varnothing) = C(\sigma + 1, 0).$
(3) $\tau \leq \sigma \ \text{ and } \ \tau \in \mathcal{H}_\sigma(X) \ \implies \ \psi(\tau) \in \mathcal{H}_\sigma(X).$
(4) $\sigma < \tau \ \implies \ \mathcal{H}_\sigma \subseteq \mathcal{H}_\tau.$
(5) $\mathcal{H}_\sigma(X) \cap \Omega$ *is an ordinal.*

Lemma 10. *Let $\mathfrak{m}$ be a finite set of ordinals and σ be an ordinal such that the following conditions are satisfied:*

$$\mathfrak{m} \subseteq C(\sigma + 1, \psi(\sigma + 1)) \cap \Omega \quad and \quad \sigma \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing).$$

Given any α and β, we then have for $\widehat{\alpha} := \sigma + \omega^{\Omega + \alpha}$ and $\widehat{\beta} := \sigma + \omega^{\Omega + \beta}$:

(1) $\alpha \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing) \implies \widehat{\alpha} \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing)$ and $\psi(\widehat{\alpha}) \in \mathcal{H}_{\widehat{\alpha}}[\mathfrak{m}](\varnothing)$.

(2) $\alpha \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing)$ and $\alpha < \beta \implies \psi(\widehat{\alpha}) < \psi(\widehat{\beta})$.

(3) $\mathcal{H}_\sigma[\mathfrak{m}](\varnothing) \cap \Omega \subseteq \psi(\sigma + 1)$.

Convention. From now on the letter $\mathcal{H}$ will be used as a metavariable that ranges over d-operators.

5.4. Operator controlled derivations in RS^*

For formulas F, finite formula sets Γ, and d-operators $\mathcal{H}$ we define

$$\mathcal{H}[F] := \mathcal{H}[|F|] \quad \text{and} \quad \mathcal{H}[\Gamma] := \mathcal{H}[|\Gamma|].$$

Likewise, if each $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_k$ is an ordinal, a formula or a finite set of formulas we define

$$\mathcal{H}[\mathfrak{f}_1, \mathfrak{f}_2] := \mathcal{H}[\mathfrak{f}_1][\mathfrak{f}_2], \quad \mathcal{H}[\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3] := \mathcal{H}[\mathfrak{f}_1, \mathfrak{f}_2][\mathfrak{f}_3], \ldots$$

and you can convince yourself immediately that the order of the expressions $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_k$ does not matter.

Definition 11. Given a derivation operator $\mathcal{H}$ and a finite set Θ of RS^* formulas, $\mathcal{H} \vdash^{\sigma}_{\rho} \Theta$ is defined by recursion on σ:

1. If Θ is an axiom of RS^* and $\{\sigma\} \cup |\Theta| \subseteq \mathcal{H}(\varnothing)$, then $\mathcal{H} \vdash^{\sigma}_{\rho} \Theta$.
2. For the inference rules of RS^* we proceed according to the ordinal assignments listed below; in addition, we always demand that $\{\sigma\} \cup |\Theta| \subseteq \mathcal{H}(\varnothing)$ for the conclusions Θ of all rules.

Inference rules of RS^* with ordinal assignments

$$(\vee) \qquad \frac{\mathcal{H} \vdash^{\sigma_0}_{\rho} \Gamma, F, G}{\mathcal{H} \vdash^{\sigma}_{\rho} \Gamma, F \vee G} \qquad\qquad \sigma_0 < \sigma$$

$$(\wedge) \qquad \frac{\mathcal{H} \vdash^{\sigma_0}_{\rho} \Gamma, F \qquad \mathcal{H} \vdash^{\sigma_1}_{\rho} \Gamma, G}{\mathcal{H} \vdash^{\sigma}_{\rho} \Gamma, F \wedge G} \qquad\qquad \sigma_0, \sigma_1 < \sigma$$

$$(\neg M) \qquad \frac{\mathcal{H}[\beta] \vdash^{\sigma_\beta}_{\rho} \Gamma, \neg M_\beta(a) \text{ for all } \beta < \lambda}{\mathcal{H} \vdash^{\sigma}_{\rho} \Gamma, \neg M_\lambda(a)} \qquad \begin{array}{l} \sigma_\beta < \sigma \\ \lambda \text{ limit} \end{array}$$

$$(\exists) \qquad \frac{\mathcal{H} \vdash^{\sigma_0}_{\rho} \Gamma, M_\beta(a) \wedge F[a]}{\mathcal{H} \vdash^{\sigma}_{\rho} \Gamma, \exists x F[x]} \qquad \begin{array}{l} \sigma_0 < \sigma \\ \beta < \sigma \end{array}$$

$$(\forall) \qquad \frac{\mathcal{H}[\beta] \vdash^{\sigma_\beta}_{\rho} \Gamma, \neg M_\beta(a) \vee F[a] \text{ for all } \beta < \Omega \text{ and all } a}{\mathcal{H} \vdash^{\sigma}_{\rho} \Gamma, \forall x F[x]} \qquad \beta < \sigma_\beta < \sigma$$

$$(\exists^\alpha) \qquad \frac{\mathcal{H} \vdash^{\sigma_0}_{\rho} \Gamma, M_\beta(a) \wedge F[a]}{\mathcal{H} \vdash^{\sigma}_{\rho} \Gamma, \exists x^\alpha F[x]} \qquad \begin{array}{l} \sigma_0 < \sigma \\ \beta \leq \alpha, \, \beta < \sigma \end{array}$$

$$(\forall^\alpha) \qquad \frac{\mathcal{H}[\beta] \vdash^{\sigma_\beta}_{\rho} \Gamma, \neg M_\beta(a) \vee F[a] \text{ for all } \beta \leq \alpha \text{ and all } a}{\mathcal{H} \vdash^{\sigma}_{\rho} \Gamma, \forall x^\alpha F[x]} \qquad \beta < \sigma_\beta < \sigma$$

$$(b\exists) \qquad \frac{\mathcal{H} \vdash^{\sigma_0}_{\rho} \Gamma, b \in a \wedge F[b]}{\mathcal{H} \vdash^{\sigma}_{\rho} \Gamma, (\exists x \in a) F[x]} \qquad\qquad \sigma_0 < \sigma$$

$$(b\forall) \qquad \frac{\mathcal{H} \vert\frac{\sigma_0}{\rho}\ \Gamma,\ b \notin a \vee F[b]\ \text{ for all } b}{\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma,\ (\forall x \in a)F[x]} \qquad\qquad \sigma_0 < \sigma$$

$$(\exists_2) \qquad \frac{\mathcal{H} \vert\frac{\sigma_0}{\rho}\ \Gamma,\ F[U]}{\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma,\ \exists X F[X]} \qquad\qquad \sigma_0 < \sigma$$

$$(\forall_2) \qquad \frac{\mathcal{H} \vert\frac{\sigma_0}{\rho}\ \Gamma,\ F[U]\ \text{ for all } U}{\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma,\ \forall X F[X]} \qquad\qquad \sigma_0 < \sigma$$

$$(\mathsf{Cut}) \qquad \frac{\mathcal{H} \vert\frac{\sigma_0}{\rho}\ \Gamma,\ F \qquad \mathcal{H} \vert\frac{\sigma_0}{\rho}\ \Gamma,\ \neg F}{\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma} \qquad\qquad \begin{array}{c} \sigma_0 < \sigma \\ rk(F) < \rho \end{array}$$

$$(\mathcal{S}_0\text{-Ref}) \qquad \frac{\mathcal{H} \vert\frac{\sigma_0}{\rho}\ \Gamma,\ F}{\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma,\ \exists z F^{(z)}} \qquad\qquad \begin{array}{c} \sigma_0, \Omega < \sigma \\ F \in \mathcal{S}_0 \end{array}$$

$$(\mathsf{BC}) \qquad \frac{\mathcal{H} \vert\frac{\sigma_0}{\rho}\ \Gamma,\ F^\beta}{\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma,\ \exists z^{\beta+\omega} F^{(z)}} \qquad\qquad \begin{array}{c} \sigma_0 < \sigma \\ F \in \mathcal{S}_0 \end{array}$$

This concludes the list of the rules of inference of RS^* with their respective ordinal assignments.

The following lemmas collect a few basic properties of the system RS^*. They are proved by straightforward induction on α.

Lemma 12 *(Weakening).*

$$\mathcal{H} \vert\frac{\sigma}{\rho_0}\ \Gamma\ \&\ \sigma \le \tau \in \mathcal{H}(\varnothing)\ \&\ \rho_0 \le \rho_1\ \&\ |\Theta| \subseteq \mathcal{H}(\varnothing) \quad\Longrightarrow\quad \mathcal{H} \vert\frac{\tau}{\rho_1}\ \Gamma, \Theta.$$

Lemma 13 *(Inversion).*

1. $\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, F_1 \vee F_2\ \&\ \Omega \le rk(F_1 \vee F_2) \quad\Longrightarrow\quad \mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, F_1, F_2.$
2. $\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, F_1 \wedge F_2\ \&\ \Omega \le rk(F_1 \wedge F_2)\ \&\ i \in \{1,2\} \quad\Longrightarrow\quad \mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, F_i.$
3. $\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, \forall x F[x]\ \&\ \beta \in \mathcal{H}(\varnothing) \cap \Omega \quad\Longrightarrow\quad \mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, \neg M_\beta(a) \vee F[a].$
4. $\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, \forall x F[x]\ \&\ \beta \in \mathcal{H}(\varnothing) \cap \Omega \quad\Longrightarrow\quad \mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, \forall x^\beta F[x].$
5. $\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, \forall x^\alpha F[x]\ \&\ \beta \in \mathcal{H}(\varnothing)\ \&\ \beta \le \alpha \quad\Longrightarrow\quad \mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, \neg M_\beta(a) \vee F[a].$
6. $\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, (\forall x \in a)F[x]\ \&\ \Omega \le rk(F[u]) \quad\Longrightarrow\quad \mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, b \notin a \vee F[s].$
7. $\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, \forall X F[X]\ \&\ \Omega \le rk(F[U]) \quad\Longrightarrow\quad \mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, F[U].$

Lemma 14 *(Boundedness). Assume $F \in \mathcal{S}$, $\sigma \le \tau < \Omega$, and $\tau \in \mathcal{H}(\varnothing)$. Then we have for all Γ that*

$$\mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, F \quad\Longrightarrow\quad \mathcal{H} \vert\frac{\sigma}{\rho}\ \Gamma, F^\tau.$$

This boundedness lemma is one of the main ingredients of impredicative cut elimination and will play a central role in the proof of the collapsing theorem in Subsection 5.7. However, before dealing with cut elimination we turn to some embedding results.

5.5. Embedding

In a next step we show that $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$ can be embedded into RS^*, and as it will turn out, that finite derivations in $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$ translate into uniform infinitary derivations in RS^*. We begin with showing (TnD).

Lemma 15. *For any $\mathcal{H}$ and any F we have that*

$$\mathcal{H}[F] \, \vdash^{\frac{rk(F)\#rk(F)}{0}} \, \neg F, \, F$$

Proof. We proceed by induction on $rk(F)$. If F is from $\mathcal{B}$, then $\neg F, F$ is an axiom, and we are done. Now suppose that F is of the form $\exists x G[x]$ and observe that, for any $\beta < \Omega$, the rank of the formula $M_\beta(a) \wedge G[a]$ is independent of a. Therefore, we can set

$$\rho_\beta \; := \; rk(M_\beta(a) \wedge G[a]) \quad \text{and} \quad \rho \; := \; rk(\exists x G[x])$$

and immediately obtain $\beta < \rho_\beta \# \rho < \rho \# \rho$. Moreover,

$$\mathcal{H}[M_\beta(a) \wedge G[a]] \; = \; \mathcal{H}[\exists x G[x], \beta].$$

Thus the induction hypothesis yields

$$\mathcal{H}[\exists x G[x], \beta] \, \vdash^{\frac{\rho_\beta \# \rho_\beta}{0}} \, \neg M_\beta(a) \vee \neg G[a], \, M_\beta(a) \wedge G[a]$$

and an application of $(\exists)$ gives us

$$\mathcal{H}[\exists x G[x], \beta] \, \vdash^{\frac{\rho_\beta \# \rho}{0}} \, \neg M_\beta(a) \vee \neg G[a], \, \exists x G[x]].$$

This is so for all $\beta < \Omega$ and all a. Therefore, by means of $(\forall)$ we obtain

$$\mathcal{H}[\exists x G[x]] \, \vdash^{\frac{\rho \# \rho}{0}} \, \forall x \neg G[x], \, \exists x G[x]],$$

as desired. The other cases are similar. $\quad\square$

Lemma 16 *(Lifting). For $\rho := rk(\exists x^\alpha F[x])$ and all $\mathcal{H}$ we have that*

$$\mathcal{H}[\exists x^\alpha F[x]] \, \vdash^{\frac{\rho \# \rho}{0}} \, \neg \exists x^\alpha F[x], \, \exists x F[x].$$

Proof. For $\rho_\beta := rk(M_\beta(a) \wedge F[a])$ we obtain from the previous lemma that

$$\mathcal{H}[F[u], \beta] \, \vdash^{\frac{\rho_\beta \# \rho_\beta}{0}} \, \neg M_\beta(a) \vee \neg F[a], \, M_\beta(a) \wedge F[a]$$

for all $\beta \leq \alpha$. By applying $(\exists)$ and $(\forall^\alpha)$ our assertion follows. $\quad\square$

Lemma 17 *($\in$-induction). For every formula $F[u]$ and all $\mathcal{H}$ we have*

$$\mathcal{H}[F[u]] \, \vdash^{\frac{\sigma+\Omega+1}{\Omega}} \, \forall x((\forall y \in x)F[y] \rightarrow F[x]) \; \rightarrow \; \forall x F[x],$$

where $\sigma := \omega^{rk(\forall x((\forall y \in x)F[y] \rightarrow F[x]))}$.

Proof. We set

$$G \; := \; \forall x((\forall y \in x)F[y] \rightarrow F[x]), \quad \sigma \; := \; \omega^{rk(G)}, \quad \gamma_\alpha \; := \; \sigma \# \omega^\alpha$$

and prove in a first step

$$\mathcal{H}[F, \alpha] \, \vdash^{\frac{\gamma_\alpha}{\Omega}} \, \neg G, \neg M_\alpha(a), \, F[a] \tag{$*$}$$

by induction on α. If $|\alpha| = 0$, then simply apply Axiom (4). If α is a limit, then the assertion is immediate from the induction hypothesis by means of inference rule $(\neg M)$. Now suppose $\alpha = \beta + 1$. Then $\mathcal{H}[F, \beta] = \mathcal{H}[F, \alpha]$ and the induction hypothesis gives us

$$\mathcal{H}[F, \alpha] \vdash^{\gamma_\beta}_{\Omega} \neg G, \neg M_\beta(b), F[b].$$

In view of Axiom (10) we also have

$$\mathcal{H}[F, \alpha] \vdash^0_0 \neg M_\alpha(a), b \notin a, M_\beta(b).$$

Therefore, a cut yields

$$\mathcal{H}[F, \alpha] \vdash^{\gamma_\beta + 1}_{\Omega} \neg G, \neg M_\alpha(a), b \notin a, F[b].$$

Using $(\vee)$ and $(b\forall)$ we can continue with

$$\mathcal{H}[F, \alpha] \vdash^{\gamma_\beta + 3}_{\Omega} \neg G, \neg M_\alpha(a), (\forall x \in a)F[x].$$

On the other hand, the previous lemma also implies that

$$\mathcal{H}[F, \alpha] \vdash^{\sigma}_{\Omega} \neg F[a], F[a].$$

From the previous two assertions we obtain

$$\mathcal{H}[F, \alpha] \vdash^{\gamma_\beta + 4}_{\Omega} \neg G, \neg M_\alpha(a), (\forall x \in a)F[x] \wedge \neg F[a], F[a]$$

via $(\wedge)$ and from the latter

$$\mathcal{H}[F, \alpha] \vdash^{\gamma_\beta + 5}_{\Omega} \neg G, \neg M_\alpha(a), M_\alpha(a) \wedge ((\forall x \in a)F[x] \wedge \neg F[a]), F[a]$$

via Axiom (1) and $(\wedge)$. Thus $(\exists)$ and the definition of G lead to

$$\mathcal{H}[F, \alpha] \vdash^{\gamma_\alpha}_{\Omega} \neg G, \neg M_\alpha(a), F[a].$$

Therefore, (*) is proved.

The rest is simple. The rule $(\vee)$ applied to (*) yields

$$\mathcal{H}[F, \alpha] \vdash^{\gamma_\alpha}_{\Omega} \neg G, \neg M_\alpha(a) \vee F[a]$$

for all $\alpha < \Omega$ and all a. Hence we are in the position to apply $(\forall)$ and deduce

$$\mathcal{H}[F,] \vdash^{\sigma + \Omega}_{\Omega} \neg G, \forall x F[x].$$

From this our assertion follows by applying $(\vee)$. $\quad\square$

Lemma 18. *We have for all $\mathcal{H}$, all a, b, and all α, β:*

(1) $\mathcal{H} \vdash^3_0 M_1(\underline{\varnothing}) \wedge (\forall x \in \underline{\varnothing})(x \neq x)$.
(2) $\mathcal{H} \vdash^0_0 M_{\omega + 1}(\underline{\omega})$.
(3) $\mathcal{H} \vdash^0_0 \neg M_\alpha(a), (a \in \underline{\omega} \leftrightarrow FinOrd[a])$.

(4) $\mathcal{H}[\alpha,\beta] \vdash^{\omega^{\beta+2}}_{\omega\beta+\omega+\omega} \neg M_\alpha(a), \neg M_\beta(b), \exists z(a \in x \wedge b \in z)$ for $\alpha \leq \beta$.

(5) $\mathcal{H}[\alpha] \vdash^{\omega^{\alpha+2}}_{\omega\alpha+\omega} \neg M_\alpha(a), \exists z(\forall y \in a)(\forall x \in y)(x \in z)$.

Proof. These five assertions are immediate consequences of the respective axioms and Lemma 16. □

It $\vec{\alpha} = \alpha_1, \ldots, \alpha_k$ and $\vec{a} = a_1, \ldots, a_k$, then we write $\neg M_{\vec{\alpha}}(\vec{a})$ for the set

$$\{\neg M_{\alpha_1}(a_1), \ldots, \neg M_{\alpha_k}(a_k)\}.$$

Now we turn to Δ_0 separation and our form of Π^1_1 class comprehension. Again, they are basically given by the axioms. However, for (Δ_0-Sep) a cut is needed.

Lemma 19 ((Δ_0-*Sep*) *and* (Π^1_1-CA*))*. *Let* $A[u, \vec{v}]$ *and* $B[u, \vec{v}, W]$ *be* Δ_0 *formulas of* $\mathcal{L}^{set}_2$ *whose free set variables are from the list* $u, \vec{v}$. *Then we have for all* $\mathcal{H}$, *all* $\alpha, \vec{\beta}$, *and all* $a, \vec{b}$:

(1) $\mathcal{H}[\alpha, \vec{\beta}] \vdash^{\rho}_{\Omega} \neg M_\alpha(a), \neg M_{\vec{\beta}}(\vec{b}), \exists z(z = \{x \in a : A[x, \vec{b}]\})$, where $\rho := \omega^{\max(\alpha, \vec{\beta})+2}$.

(2) $\mathcal{H}[\vec{\beta}] \vdash^{0}_{0} \neg M_{\vec{\beta}}(\vec{b}), \exists Z(\forall x \in \underline{\omega})(Z(x) \leftrightarrow \forall Y B[x, \vec{b}, Y])$.

Proof. The first assertion is obtained from Axiom (13) and Lemma 16 by a cut. The second is a direct consequence of Axiom (14). □

Lemma 20 ($\mathcal{S}_0$ *reflection*). *Let* $A[\vec{u}]$ *be a formula from* $\mathcal{S}_0$ *whose free set variables are from the list* $\vec{u}$. *Then we have for all* $\mathcal{H}$, *all* $\vec{\alpha}$, *and all* $\vec{a}$ *that*

$$\mathcal{H}[\vec{\alpha}] \vdash^{\sigma}_{0} \neg M_{\vec{\alpha}}(\vec{a}), A[\vec{a}] \rightarrow \exists z A^{(z)}[\vec{a}],$$

where $\sigma := \omega^{rk(A[\vec{u}])+1}$.

Proof. In view of Lemma 15 we have

$$\mathcal{H}[\vec{\alpha}] \vdash^{\rho}_{0} \neg M_{\vec{\alpha}}(\vec{a}), \neg A[\vec{a}], A[\vec{a}]$$

for $\rho := rk(A[\vec{a}]) \# rk(A[\vec{a}])$. Since $A[\vec{u}]$ is a proper Σ formula, we know that $\Omega \leq \rho$. Now we apply ($\mathcal{S}_0$-Ref) and obtain

$$\mathcal{H}[\vec{\alpha}] \vdash^{\rho+1}_{0} \neg M_{\vec{\alpha}}(\vec{a}), \neg A[\vec{a}], \exists z A^{(z)}[\vec{a}]$$

Thus an application of ($\vee$) yields our assertion. □

Theorem 21 (*Embedding*). *Let* $\Gamma[u_1, \ldots, u_k]$ *be a finite set of* $\mathcal{L}^{set}_2$ *formulas whose free set variables are exactly those indicated and suppose that*

$$\mathsf{KP} + (\Pi^1_1\text{-}\mathsf{CA}^*) \vdash^h \Gamma[u_1, \ldots, u_k].$$

Then there exist $m, n < \omega$ *such that*

$$\mathcal{H}[\vec{\alpha}] \vdash^{\omega^{\Omega+m}}_{\Omega+n} \neg M_{\vec{\alpha}}(\vec{a}), \Gamma[\vec{a}]$$

for all $\mathcal{H}$, *all* $\vec{\alpha} = \alpha_1, \ldots, \alpha_k$, *and all* $\vec{a} = a_1, \ldots, a_k$.

Proof. We proceed by induction on the length h of the derivation of $\Gamma[\vec{u}]$ in the Tait-style formalization of $\mathsf{KP} + (\Pi^1_1\text{-}\mathsf{CA}^*)$. If $\Gamma[\vec{u}]$ is an axiom, then the assertion follows from Lemma 15 – Lemma 20. Observe that Δ_0 collection is a special case of $\mathcal{S}_0$ reflection.

As a first example of an inference rule we treat $(\forall)$. Then $\Gamma[\vec{u}]$ contains a formula of the form $\forall x A[x, \vec{u}]$ and in $\mathsf{KP} + (\Pi^1_1\text{-}\mathsf{CA}^*)$ there is a shorter derivation of

$$\Gamma'[\vec{u}], \, A[v, \vec{u}],$$

where v is a fresh set variable not occurring in $\Gamma[\vec{u}]$ and $\Gamma'[\vec{u}]$ is either $\Gamma[\vec{u}]$ or $\Gamma[\vec{u}] \setminus \{\forall x A[x, \vec{u}]\}$. Thus the induction hypothesis provides us with $m_0, n_0 < \omega$ such that

$$\mathcal{H}[\vec{\alpha}, \beta] \vdash^{\omega^{\Omega+m_0}}_{\Omega+n_0} \neg M_{\vec{\alpha}}(\vec{a}), \, \neg M_\beta(b), \, \Gamma'[\vec{a}], \, A[b, \vec{a}]$$

for all $\vec{\alpha}, \beta$ and all $\vec{a}, b$. Now we apply $(\vee)$ and then $(\forall)$ and obtain

$$\mathcal{H}[\vec{\alpha}] \vdash^{\omega^{\Omega+m}}_{\Omega+n_0} \neg M_{\vec{\alpha}}(\vec{a}), \, \Gamma'[\vec{a}], \, \forall x A[x, \vec{a}]$$

for $m := m_0 + 1$ and arbitrary $\vec{\alpha}$ and $\vec{a}$. Since $\forall x A[x, \vec{a}]$ belongs to $\Gamma[\vec{a}]$ this is our assertion.

As second example we treat the rule $(\exists)$. Then $\Gamma[\vec{u}]$ contains a formula of the form $\exists x A[x, \vec{u}]$ and in $\mathsf{KP} + (\Pi^1_1\text{-}\mathsf{CA}^*)$ there is a shorter derivation of

$$\Gamma'[\vec{u}], \, A[b, \vec{u}],$$

for some set term b and $\Gamma'[\vec{u}]$ is either $\Gamma[\vec{u}]$ or $\Gamma[\vec{u}] \setminus \{\forall x A[b, \vec{u}]\}$. Now we have to distinguish three cases:

(i) b is the term $\underline{\varnothing}$ or the term $\underline{\omega}$. Then the induction hypothesis supplies us with $m_0, n_0 < \omega$ such that

$$\mathcal{H}[\vec{\alpha}] \vdash^{\omega^{\Omega+m_0}}_{\Omega+n_0} \neg M_{\vec{\alpha}}(\vec{a}), \, \Gamma[\vec{a}], \, A[b, \vec{a}]$$

for all $\vec{\alpha}$ and all $\vec{a}$. Together with Axiom (5) or Axiom (7) and $(\wedge)$ we thus obtain

$$\mathcal{H}[\vec{\alpha}] \vdash^{\omega^{\Omega+m_0}+1}_{\Omega+n_0} \neg M_{\vec{\alpha}}(\vec{a}), \, \Gamma[\vec{a}], \, M_\beta(b) \wedge A[b, \vec{a}]$$

(where β is 1 or $\omega + 1$) and thus, by $(\exists)$,

$$\mathcal{H}[\vec{\alpha}] \vdash^{\omega^{\Omega+m}}_{\Omega+n_0} \neg M_{\vec{\alpha}}(\vec{a}), \, \Gamma[\vec{a}], \, \exists x A[x, \vec{a}],$$

for $m := m_0 + 1$ and arbitrary $\vec{\alpha}$ and $\vec{a}$.

(ii) b is the variable u_i, $1 \leq i \leq n$. Then the induction hypothesis yields

$$\mathcal{H}[\vec{\alpha}] \vdash^{\omega^{\Omega+m_0}}_{\Omega+n_0} \neg M_{\vec{\alpha}}(\vec{a}), \, \Gamma[\vec{a}], \, A[a_i, \vec{a}]$$

for suitable $m_0, n_0 < \omega$, all $\vec{\alpha}$ and all $\vec{a}$. Now Axiom (1) and $(\wedge)$ give us

$$\mathcal{H}[\vec{\alpha}] \vdash^{\omega^{\Omega+m_0}+1}_{\Omega+n_0} \neg M_{\vec{\alpha}}(\vec{a}), \, \Gamma[\vec{a}], \, M_{\alpha_i}(a_i) \wedge A[a_i, \vec{a}]$$

and from there we proceed as in (i).

(iii) b is a set variable not occurring in $\Gamma[\vec{u}]$. Then the induction hypothesis supplies us with $m_0, n_0 < \omega$ such that

$$\mathcal{H}[\vec{\alpha}] \vdash^{\omega^{\Omega+m_0}}_{\Omega+n_0} \neg M_{\vec{\alpha}}(\vec{a}),\ \neg M_1(\underline{\varnothing}),\ \Gamma[\vec{a}],\ A[\underline{\varnothing}, \vec{a}]$$

for all $\vec{\alpha}$ and all $\vec{a}$. Now we make first use of Axiom (5) and a cut and then of Axiom (5) and ($\wedge$) to derive

$$\mathcal{H}[\vec{\alpha}] \vdash^{\omega^{\Omega+m_0}+2}_{\Omega+n_0} \neg M_{\vec{\alpha}}(\vec{a}),\ \Gamma[\vec{a}],\ M_1(\underline{\varnothing}) \wedge A[\underline{\varnothing}, \vec{a}]$$

from where we can proceed again as in (i).

In all three cases we obtain

$$\mathcal{H}[\vec{\alpha}] \vdash^{\omega^{\Omega+m}}_{\Omega+n_0} \neg M_{\vec{\alpha}}(\vec{a}),\ \Gamma[\vec{a}],\ \exists x A[x, \vec{a}]$$

for $m := m_0 + 1$ and arbitrary $\vec{\alpha}$ and $\vec{a}$. Since $\exists x A[x, \vec{a}]$ belongs to $\Gamma[\vec{a}]$ this finishes ($\exists$).

All other cases are straightforward from the induction hypothesis or can be treated in the same vein. $\square$

5.6. Predicative cut elimination

The rules of inference of RS^* can be divided into two classes: (i) In all rules except ($\mathcal{S}_0$-Ref) the principal formula is more complex than the corresponding formula(s) in the premise(s). We may, therefore, consider these rules as *predicative* rules. (ii) The rule ($\mathcal{S}_0$-Ref), on the other hand, transforms (in the general case) a formula of rank greater than Ω into one of rank Ω. This is a sort of impredicativity and we consider ($\mathcal{S}_0$-Ref) as an *impredicative* inference.

Also remember that the principal formulas of the axioms of RS^* belong to $\mathcal{B}$ and, therefore, have rank less than Ω. These formulas are an obstacle for cut elimination below Ω. However, above Ω we can follow the pattern of the usual (predicative) cut elimination.

In this section we sketch how all cuts with cut formulas of ranks greater than Ω can be eliminated by standard methods as presented, for example, in Schütte [19] or Buchholz [4].

Lemma 22. *Let F be a formula of the form $G_1 \vee G_2$, $(\exists x \in a)G[x]$, $\exists x^\alpha G[x]$, $\exists x G[x]$ or $\exists X G[X]$ and assume that the rank ρ of F is greater than Ω. Then we have:*

$$\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \neg F \quad and \quad \mathcal{H} \vdash^{\beta}_{\rho} \Theta, F \quad \Longrightarrow \quad \mathcal{H} \vdash^{\alpha+\beta}_{\rho} \Gamma, \Theta.$$

Proof. By straightforward induction on β. $\square$

Theorem 23 (Predicative cut elimination). *We have the following two elimination results, where the second is an immediate consequence of the first. For all $\mathcal{H}$, Γ, α, and all $n < \omega$:*

(1) $\mathcal{H} \vdash^{\alpha}_{\Omega+n+2} \Gamma \quad \Longrightarrow \quad \mathcal{H} \vdash^{\omega^\alpha}_{\Omega+n+1} \Gamma.$
(2) $\mathcal{H} \vdash^{\alpha}_{\Omega+n+1} \Gamma \quad \Longrightarrow \quad \mathcal{H} \vdash^{\omega_n[\alpha]}_{\Omega+1} \Gamma.$

Recall that $\omega_0[\alpha] := \alpha$ and $\omega_{n+1}[\alpha] := \omega^{\omega_n[\alpha]}$.

Proof. The proof of the first assertion is standard. For details see, for example, Buchholz [4]. The second assertion is an immediate consequence of the first. $\square$

5.7. Collapsing

We begin this subsection by showing that specific operator controlled derivations of finite sets of formulas from $\mathcal{S} \cup \mathcal{B}$ in which all cut formulas have ranks $\leq \Omega$ can be collapsed into derivations of depth and cut rank

less than Ω. This technique – called collapsing technique – is a corner stone of *impredicative proof theory*. See Buchholz [4] for more on general collapsings.

Together with the results of the previous subsection this collapsing theorem will then lead to our main theorem about the $\mathcal{S} \cup \mathcal{B}$ fragment of RS^* derivations and the Σ formulas provable in $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$.

For the collapsing theorem we work with the derivation operators $\mathcal{H}_\sigma$ introduced in Definition 8.

Theorem 24 *(Collapsing). Let Γ be a finite set of formulas from $\mathcal{S} \cup \mathcal{B}$ and suppose that*

$$|\Gamma| \subseteq C(\sigma + 1, \psi(\sigma + 1)) \quad and \quad \sigma \in \mathcal{H}_\sigma[\Gamma](\varnothing).$$

Then we have, for all α,

$$\mathcal{H}_\sigma[\Gamma] \vdash^{\alpha}_{\Omega+1} \Gamma \quad \Longrightarrow \quad \mathcal{H}_{\widehat{\alpha}}[\Gamma] \vdash^{\psi(\widehat{\alpha})}_{\psi(\widehat{\alpha})} \Gamma,$$

where $\widehat{\alpha} := \sigma + \omega^{\Omega + \alpha}$.

Proof. By induction on α. Assume $\mathcal{H}_\sigma[\Gamma] \vdash^{\alpha}_{\Omega+1} \Gamma$. Then we certainly have (see Lemma 9 and Lemma 10):

$$|\Gamma| \cup \{\alpha\} \subseteq \mathcal{H}_\sigma[\Gamma](\varnothing) \subseteq \mathcal{H}_{\widehat{\alpha}}[\Gamma](\varnothing), \tag{1}$$

$$\widehat{\alpha} \in \mathcal{H}_\sigma[\Gamma](\varnothing) \quad and \quad \psi(\widehat{\alpha}) \in \mathcal{H}_{\widehat{\alpha}}[\Gamma](\varnothing). \tag{2}$$

Now we distinguish cases according to the last inference of $\mathcal{H}_\sigma[\Gamma] \vdash^{\alpha}_{\Omega+1} \Gamma$ and note that this cannot be $(\forall)$. In the following we confine our attention to the interesting cases; all others can be dealt with in a similar manner.

(i) Γ is an axiom. Then $\mathcal{H}_{\widehat{\alpha}}[\Gamma] \vdash^{\psi(\widehat{\alpha})}_{\psi(\widehat{\alpha})} \Gamma$ follows from (1) and (2).

(ii) The last inference was $(\forall^\tau)$. Then Γ contains a formula of the form $\forall x^\tau F[x]$ and we have

$$\mathcal{H}_\sigma[\Gamma, \xi] \vdash^{\alpha_\xi}_{\Omega+1} \Gamma, \neg M_\xi(a) \vee F[a] \tag{3}$$

with $\alpha_\xi < \alpha$ for all $\xi \leq \tau$ and all a. We know $\tau \in \mathcal{H}_\sigma[\Gamma](\varnothing)$. Lemma 7, Lemma 9(5), and (1) yield

$$\xi \in \mathcal{H}_\sigma[\Gamma](\varnothing) \subseteq \mathcal{H}_{\widehat{\alpha}}[\Gamma](\varnothing), \quad \mathcal{H}_\sigma[\Gamma, \xi] = \mathcal{H}_\sigma[\Gamma], \quad \mathcal{H}_{\widehat{\alpha}}[\Gamma, \xi] = \mathcal{H}_{\widehat{\alpha}}[\Gamma]. \tag{4}$$

Hence (3) gives us

$$\mathcal{H}_\sigma[\Gamma] \vdash^{\alpha_\xi}_{\Omega+1} \Gamma, \neg M_\xi(a) \vee F[a] \quad and \quad \alpha_\xi \in \mathcal{H}_\sigma[\Gamma].$$

By the induction hypothesis we obtain that

$$\mathcal{H}_{\widehat{\alpha_\xi}}[\Gamma] \vdash^{\psi(\widehat{\alpha_\xi})}_{\psi(\widehat{\alpha_\xi})} \Gamma, \neg M_\xi(a) \vee F[a]$$

and, therefore,

$$\mathcal{H}_{\widehat{\alpha}}[\Gamma] \vdash^{\psi(\widehat{\alpha_\xi})}_{\psi(\widehat{\alpha_\xi})} \Gamma, \neg M_\xi(a) \vee F[a], \tag{5}$$

always for all $\xi \leq \tau$ and all a. For $\xi \leq \tau$ we have $\alpha_\xi < \alpha$ and $\alpha_\xi \in \mathcal{H}_\sigma[\Gamma]$, hence Lemma 10 yields $\psi(\alpha_\xi) < \psi(\alpha)$. Furthermore, since $\tau \in |\Gamma| \subseteq C(\sigma + 1, \psi(\sigma + 1))$ we also have

$$\xi \le \tau < \psi(\sigma + 1) \le \psi(\widehat{\alpha_\xi}).$$

Then $\mathcal{H}_{\widehat{\alpha}}[\Gamma] \vdash^{\psi(\widehat{\alpha})}_{\psi(\widehat{\alpha})} \Gamma$ follows immediately by an application of $(\forall^\tau)$.

(iii) The last inference was $(\exists)$. Then Γ contains a formula of the form $\exists x F[x]$ and we have

$$\mathcal{H}_\sigma[\Gamma] \vdash^{\alpha_0}_{\Omega+1} \Gamma, M_\beta(a) \wedge F[a] \tag{6}$$

for some $\alpha_0 < \alpha$ and some $\beta < \alpha$. In view of (6) we have $\alpha_0, \beta \in \mathcal{H}_\sigma[\Gamma](\varnothing)$. The induction hypothesis yields

$$\mathcal{H}_{\widehat{\alpha_0}}[\Gamma] \vdash^{\psi(\widehat{\alpha_0})}_{\psi(\widehat{\alpha_0})} \Gamma, M_\beta(a) \wedge F[a],$$

thus also

$$\mathcal{H}_{\widehat{\alpha}}[\Gamma] \vdash^{\psi(\widehat{\alpha_0})}_{\psi(\widehat{\alpha_0})} \Gamma, M_\beta(a) \wedge F[a].$$

Now recall from Lemma 10(3) that

$$\mathcal{H}_\sigma[\Gamma](\varnothing) \cap \Omega \subseteq \psi(\sigma+1) \le \psi(\widehat{\alpha})$$

and conclude that $\beta < \psi(\widehat{\alpha})$. In addition, $\psi(\widehat{\alpha_0}) < \psi(\widehat{\alpha})$ follows from Lemma 10(2). Therefore, an application of $(\exists)$ yields $\mathcal{H}_\sigma[\Gamma] \vdash^{\psi(\widehat{\alpha})}_{\psi(\widehat{\alpha})} \Gamma$.

(iv) The last inference was $(\mathcal{S}_0\text{-Ref})$. Then there exist an $F \in \mathcal{S}_0$ and an $\alpha_0 < \alpha$ such that $\exists z F^{(z)}$ belongs to Γ and

$$\mathcal{H}_\sigma[\Gamma] \vdash^{\alpha_0}_{\Omega+1} \Gamma, F. \tag{7}$$

The induction hypothesis immediately yields

$$\mathcal{H}_{\widehat{\alpha_0}}[\Gamma] \vdash^{\psi(\widehat{\alpha_0})}_{\psi(\widehat{\alpha_0})} \Gamma, F.$$

Since $F \in \mathcal{S}_0$, we are in the position to make use of Lemma 14 and obtain

$$\mathcal{H}_{\widehat{\alpha_0}}[\Gamma] \vdash^{\psi(\widehat{\alpha_0})}_{\psi(\widehat{\alpha_0})} \Gamma, F^{\psi(\widehat{\alpha_0})}.$$

Now we apply (BC) to derive

$$\mathcal{H}_{\widehat{\alpha_0}}[\Gamma] \vdash^{\psi(\widehat{\alpha_0})+1}_{\psi(\widehat{\alpha_0})} \Gamma, \exists z^\beta F^{(z)} \tag{8}$$

for $\beta := \psi(\widehat{\alpha_0}) + \omega$. On the other hand, Lemma 16 yields

$$\mathcal{H}_{\widehat{\alpha_0}}[\Gamma] \vdash^{\rho \# \rho}_0 \neg \exists z^\beta F^{(z)}, \exists z F^{(z)} \tag{9}$$

for $\rho := rk(\exists z^\beta F^{(z)})$. Furthermore, since F is from $\mathcal{S}_0$ we can easily see that the rank of $\exists z^\beta F^{(z)}$ is smaller than $\omega\beta + \omega$. Hence some trivial ordinal computations yield that $\psi(\widehat{\alpha_0}) + 1, \rho \# \rho < \psi(\widehat{\alpha})$. Therefore, a cut applied to (8) and (9) implies $\mathcal{H}_{\widehat{\alpha}}[\Gamma] \vdash^{\psi(\widehat{\alpha})}_{\psi(\widehat{\alpha})} \Gamma$.

(v) The last inference was (Cut). Then there exist a formula F and an ordinal $\alpha_0 < \alpha$ such that $rk(F) < \Omega+1$ and

$$\mathcal{H}_\sigma[\Gamma] \,\big|\!\!\frac{\alpha_0}{\Omega+1}\, \Gamma, F \quad \text{and} \quad \mathcal{H}_\sigma[\Gamma] \,\big|\!\!\frac{\alpha_0}{\Omega+1}\, \Gamma, \neg F. \tag{10}$$

Therefore, $\alpha_0 \in \mathcal{H}_\sigma[\Gamma](\varnothing)$, $|F| \subseteq \mathcal{H}_\sigma[\Gamma](\varnothing) \cap \Omega$ and, by Lemma 10(3),

$$|F| \subseteq \mathcal{H}_\sigma[\Gamma](\varnothing) \cap \Omega \subseteq \psi(\sigma+1) \le \psi(\widehat{\alpha}).$$

We distinguish two cases.

(v.1) $rk(F) < \Omega$. Then $rk(F) < \psi(\widehat{\alpha})$ according to the line above. Furthermore, $\Gamma \cup \{F, \neg F\} \subseteq \mathcal{S} \cup \mathcal{B}$ and thus the induction hypothesis yields

$$\mathcal{H}_{\widehat{\alpha_0}}[\Gamma] \,\big|\!\!\frac{\psi(\widehat{\alpha_0})}{\psi(\widehat{\alpha_0})}\, \Gamma, F \quad \text{and} \quad \mathcal{H}_{\widehat{\alpha_0}}[\Gamma] \,\big|\!\!\frac{\psi(\widehat{\alpha_0})}{\psi(\widehat{\alpha_0})}\, \Gamma, \neg F.$$

As above, we can easily convince ourselves that $\psi(\widehat{\alpha_0}) < \psi(\widehat{\alpha})$. Hence, (Cut) yields $\mathcal{H}_{\widehat{\alpha}}[\Gamma] \,\big|\!\!\frac{\psi(\widehat{\alpha})}{\psi(\widehat{\alpha})}\, \Gamma$.

(v.2) $rk(F) = \Omega$. Then F is of the form $\exists x G[x]$ or $\forall x G[x]$ and $G[u]$ is a Δ_0 formula of $\mathcal{L}_2^{set}$. We assume that F is $\exists x G[x]$. Clearly, $\exists x G[x]$ belongs to $\mathcal{S}_0$. Hence, the induction hypothesis applied to the left hand side of (10) yields

$$\mathcal{H}_{\widehat{\alpha_0}}[\Gamma] \,\big|\!\!\frac{\beta}{\beta}\, \Gamma, \exists x G[x]$$

for $\beta := \psi(\widehat{\alpha_0})$. Since $\beta \in \mathcal{H}_{\widehat{\alpha_0}}[\Gamma](\varnothing)$ (see Lemma 9), we can apply Lemma 14 and obtain

$$\mathcal{H}_{\widehat{\alpha_0}}[\Gamma] \,\big|\!\!\frac{\beta}{\beta}\, \Gamma, \exists x^\beta G[x]. \tag{11}$$

Furthermore, applying Lemma 13 to the right hand side of (10) yields

$$\mathcal{H}_{\widehat{\alpha_0}}[\Gamma] \,\big|\!\!\frac{\alpha_0}{\Omega+1}\, \Gamma, \forall x^\beta \neg G[x]. \tag{12}$$

Now we exploit the fact that $\forall x^\beta \neg G[x]$ is from $\mathcal{B}$ and convince ourselves that the induction hypothesis can be applied to (12). So we obtain

$$\mathcal{H}_\gamma[\Gamma] \,\big|\!\!\frac{\psi(\gamma)}{\psi(\gamma)}\, \Gamma, \forall x^\beta \neg G[x]$$

with $\gamma = \widehat{\alpha_0} + \omega^{\Omega+\alpha_0} = \sigma + \omega^{\Omega+\alpha_0} + \omega^{\Omega+\alpha_0} < \sigma + \omega^{\Omega+\alpha} = \widehat{\alpha}$. Moreover, it is easy to check that $\gamma \in \mathcal{H}_\sigma[\Gamma](\varnothing)$, hence $\psi(\gamma) < \psi(\widehat{\alpha})$ according to Lemma 10. An easy computation, similar to above, also shows that

$$rk(\exists x^\beta G[x]) < \psi(\widehat{\alpha}).$$

Therefore, (Cut) applied to (11) and (12) establishes $\mathcal{H}_{\widehat{\alpha}}[\Gamma] \,\big|\!\!\frac{\psi(\widehat{\alpha})}{\psi(\widehat{\alpha})}\, \Gamma$.

(vi) All other cases are trivial or can be dealt with similarly. This completes the proof of collapsing. $\quad\square$

5.8. Finite bounds for the lengths of formulas in infinitary derivations

Clearly, every proof $\mathcal{P}$ in the theory $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$ is finite and, therefore, there exists a natural number p such that every formula in $\mathcal{P}$ has a length less than p. We sketch now how this bound p carries over to the "proof-theoretically treated" derivation stemming from $\mathcal{P}$. First we introduce a suitable definition of length of an $\mathcal{L}^\star$ formula.

Definition 25. The *length* $\ell(F)$ of an $\mathcal{L}^{\star}$ formula F is inductively defined as follows:

1. If F is of the form $(a \in b)$, $(a \notin b)$, $U(a)$, $\neg U(a)$, $M_\alpha(a)$ or $\neg M_\alpha(a)$, then $\ell(F) := 0$.
2. If F is of the form $(G \vee H)$ or $(G \wedge H)$, then $\ell(F) := \max(\ell(G), \ell(H)) + 1$.
3. If F is of the form $\exists x G[x]$, $\forall x G[x]$, $\exists x^\alpha G[x]$, $\forall x^\alpha G[x]$, $(\exists x \in a)G[x]$ or $(\forall x \in a)G[x]$, then $\ell(F) := \ell(G[u]) + 1$.
4. If F is of the form $\exists X G[X]$ or $\forall X G[X]$, then $\ell(F) := \ell(G[U]) + 1$.

Based on this definition we now introduce a refined notion of derivability in RS^*. Given a natural number p we let

$$\mathcal{H} \vdash\!\frac{\alpha}{\rho,\, p} \ \Gamma$$

be defined as $\mathcal{H} \vdash\!\frac{\alpha}{\rho} \ \Gamma$ in Definition 11 with the additional requirements that $\ell(F) < p$ for every element F of Γ.

The proofs of the following three theorems do not raise any questions of principle. We simply follow the original proofs and check by a tedious case to case analysis that the additional requirements are satisfied. In the infinitary derivations new formulas may occur; however, their lengths are always bounded by the lengths of formulas in the finite derivation in $\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*)$.

Theorem 26 (*Refined embedding*)*. Let* $\Gamma[u_1, \ldots, u_k]$ *be a finite set of* $\mathcal{L}_2^{set}$ *formulas whose free set variables are exactly those indicated and suppose that*

$$\mathsf{KP} + (\Pi_1^1\text{-}\mathsf{CA}^*) \vdash^h \Gamma[u_1, \ldots, u_k]$$

such that $\ell(F) < p$ *for all formulas occurring in this proof. Then there exist* $m, n < \omega$ *such that*

$$\mathcal{H}[\vec{\alpha}] \vdash\!\frac{\omega^{\Omega + m}}{\Omega + n,\, p} \ \neg M_{\vec{\alpha}}(\vec{a}), \Gamma[\vec{a}]$$

for all $\mathcal{H}$*, all* $\vec{\alpha} = \alpha_1, \ldots, \alpha_k$*, and all* $\vec{a} = a_1, \ldots, a_k$*.*

Theorem 27 (*Refined predicative cut elimination*)*. For all* $\mathcal{H}$*,* Γ*,* α*, and all* $n, p < \omega$*,*

$$\mathcal{H} \vdash\!\frac{\alpha}{\Omega + n + 1,\, p} \ \Gamma \quad \Longrightarrow \quad \mathcal{H} \vdash\!\frac{\omega_n(\alpha)}{\Omega + 1,\, p} \ \Gamma.$$

Theorem 28 (*Refined collapsing*)*. Let* Γ *be a finite set of formulas from* $\mathcal{S} \cup \mathcal{B}$ *and suppose that*

$$|\Gamma| \subseteq C(\sigma + 1, \psi(\sigma + 1)) \quad \text{and} \quad \sigma \in \mathcal{H}_\sigma[\Gamma](\varnothing).$$

Then we have, for all α *and all* $p < \omega$*,*

$$\mathcal{H}_\sigma[\Gamma] \vdash\!\frac{\alpha}{\Omega + 1,\, p} \ \Gamma \quad \Longrightarrow \quad \mathcal{H}_{\widehat{\alpha}}[\Gamma] \vdash\!\frac{\psi(\widehat{\alpha})}{\psi(\widehat{\alpha}),\, p} \ \Gamma,$$

where $\widehat{\alpha} := \sigma + \omega^{\Omega + \alpha}$*.*

6. Reduction to $\Pi_1^1\text{-}\mathsf{CA} + TI(<\mathcal{BH})$

The goal of this section is to interpret certain RS^*-derivations of length and cut rank $< \Omega$ in the theory $\Pi_1^1\text{-}\mathsf{CA} + TI(<\mathcal{BH})$. Here $\Pi_1^1\text{-}\mathsf{CA}$ stands for the usual system of second order arithmetic with Π_1^1 comprehension and full induction on the natural numbers; it is formulated in the language L_2. This is a familiar

theory in proof theory and, therefore, we refrain from saying more about this theory. If you are interested in all details you may consult, for example Simpson [20].

In the sequel – when arguing in the framework of second order arithmetic – we assume an arithmetization of the notation system $C(\varepsilon_{\Omega+1}, 0)$ such that all relevant ordinal sets, functions and relations become primitive recursive. Also, we will identify the ordinal notations with their arithmetical codes. In addition, we write $\prec$ for the primitive recursive relation defined on $C(\varepsilon_{\Omega+1}, 0) \cap \Omega$ by

$$\alpha \prec \beta \ := \ \alpha, \beta \in C(\varepsilon_{\Omega+1}, 0) \wedge \alpha < \beta < \Omega.$$

Recall that (the codes for) the elements of $C(\varepsilon_{\Omega+1}, 0) \cap \Omega$ denote exactly all ordinals less than the Bachmann-Howard ordinal $\mathcal{BH}$. In the context of L_2, the quantifiers $\exists \alpha, \forall \beta$ range over the latter set.

By $TI(<\mathcal{BH})$ we mean the scheme of transfinite induction over all initial segments (indexed externally) below the Bachmann-Howard ordinal $\mathcal{BH}$. More precisely, $TI(<\mathcal{BH})$ is the collection of all formulas

$$\forall \alpha((\forall \beta \prec \alpha)\mathcal{F}[\beta] \to \mathcal{F}[\alpha]) \ \to \ (\forall \alpha \prec \psi(\omega_n[\Omega + 1]))\mathcal{F}[\alpha]$$

where $\mathcal{F}[u]$ is any formula of L_2 and n is any natural number (or rather the nth numeral).

6.1. α-trees

To interpret set theory in Π^1_1-CA $+ TI(<\mathcal{BH})$ we use well-founded trees, also called *suitable* trees. We will mostly follow the terminology and presentation in Simpson [20, VII.3].

Definition 29. Let variables $\sigma, \tau, \rho, \sigma_0, \sigma_1, \ldots$ range over finite sequences of naturals, i.e., elements of $\mathbb{N}^{<\mathbb{N}}$.

1. $\langle\rangle$ will denote the empty sequence of $\mathbb{N}^{<\mathbb{N}}$ and $\langle n_0, \ldots, n_r \rangle$ the sequence of numbers $n_0, \ldots, n_r$ coded as a single number (see [20, II.2.6]).
2. The *length* of a sequence is defined by $\mathrm{lh}(\langle\rangle) = 0$ and $\mathrm{lh}(\langle n_0, \ldots, n_r \rangle) = r + 1$.
3. The *concatenation* operation on $\mathbb{N}^{<\mathbb{N}}$ will be denoted by $\sigma \star \tau$. This means that $\sigma \star \langle\rangle = \langle\rangle \star \sigma = \sigma$ and $\langle n_0, \ldots, n_r \rangle \star \langle m_0, \ldots, m_s \rangle = \langle n_0, \ldots, n_r, m_0, \ldots, m_s \rangle$.
4. We write $\sigma \subset \tau$ to mean that σ is a proper *initial segment* of τ, i.e., $\tau = \sigma \star \rho$ for some $\rho \neq \langle\rangle$.
5. For a function $f \colon \mathbb{N} \to \mathbb{N}$ let $f[0] := \langle\rangle$ and $f[n + 1] := \langle f(0), \ldots, f(n) \rangle$.

Definition 30. $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is said to be a *tree* if T is nonempty and closed under initial segments, i.e., $(\forall \sigma \in T)(\forall \tau \subset \sigma)(\tau \in T)$. T is a *suitable tree* if T is also well-founded, i.e.,

$$(\forall f : \mathbb{N} \to \mathbb{N})\exists n(f[n] \notin T).$$

Remark 31. If T is a tree and $\sigma \in T$, we put

$$T^\sigma = \{\tau \mid \sigma \star \tau \in T\}$$

and note that if T is suitable tree then so is T^σ.

Suitable trees furnish a way of talking about sets in the language of second order arithmetic. The following definition, also taken from Simpson [20, VII.3], tells us how to define the equality relation and element relation on suitable trees.

Definition 32 $(=^*$ *and* $\in^*)$.

1. Let T be a tree. We write $Iso(X, T)$ to mean that $X \subseteq T \times T$ and, for all $\sigma, \tau \in T$, $(\sigma, \tau) \in X$ if and only if

$$\forall n(\sigma \star \langle n \rangle \in T \rightarrow \exists m((\sigma \star \langle n \rangle, \tau \star \langle m \rangle) \in X))$$

and

$$\forall m(\tau \star \langle m \rangle \in T \rightarrow \exists n((\sigma \star \langle n \rangle, \tau \star \langle m \rangle) \in X)).$$

2. For trees S and T, we define

$$S \oplus T \; := \; \{\langle\rangle\} \cup \{\langle 0 \rangle \star \sigma \mid \sigma \in S\} \cup \{\langle 1 \rangle \star \tau \mid \tau \in T\}$$

and set

$$S =^* T \; \text{ iff } \; \exists X(Iso(X, S \oplus T) \wedge (\langle 0 \rangle, \langle 1 \rangle) \in X),$$
$$S \in^* T \; \text{ iff } \; \exists X(Iso(X, S \oplus T) \wedge \exists n((\langle 0 \rangle, \langle 1, n \rangle) \in X)).$$

Lemma 33 (ATR_0). *Let T be a suitable tree. Then there exists a unique set X such that $Iso(X, T)$. Moreover, for all $\sigma, \tau \in T$,*

$$T^\sigma =^* T^\tau \; \text{ iff } \; (\sigma, \tau) \in X,$$
$$T^\sigma \in^* T^\tau \; \text{ iff } \; \exists n((\sigma, \tau \star \langle n \rangle) \in X).$$

In particular, X is an equivalence relation on T.

Proof. See [20, Lemma VII.3.17]. □

Corollary 34 (ATR_0). *Given suitable trees S and T, one has*

$$S \in^* T \; \text{ iff } \; \exists n(S =^* T^{\langle n \rangle}).$$

Definition 35. In order to model the set-theoretic naturals and ω via suitable trees we introduce trees n^* for $n \in \mathbb{N}$. Let

$$0^* = \{\langle\rangle\}$$
$$(n+1)^* = n^* \cup \{\langle n \rangle \star \sigma \mid \sigma \in n^*\}$$
$$\omega^* = \{\langle\rangle\} \cup \{\langle n \rangle \star \sigma \mid \sigma \in n^*, \; n \in \mathbb{N}\}.$$

Note that $n^* \in^* \omega^*$ and $m^* \in^* n^*$ whenever $m < n$. Since the sequences in n^*, and thus in ω^*, are strictly descending sequences of naturals it is clear that n^* and ω^* are suitable trees. Moreover, there is a map $f\colon \omega^* \to \omega+1$ such that

$$\tau \subset \sigma \wedge \sigma \in \omega^* \implies f(\sigma) \prec f(\tau).$$

To see this, let $f(\langle\rangle) = \omega$ and $f(\langle n_1, \ldots, n_r \rangle) = n_r$. This means that ω^* is an $\omega+1$-tree in the sense of Definition 36 below.

The natural translation of the set-theoretic language into that of second order arithmetic, L_2, proceeds by letting quantifiers range over suitable trees and interpreting $\in$ and $=$ as $\in^*$ and $=^*$, respectively. In [20, Theorem VII.3.22] it is shown that this yields an interpretation of the axioms of ATR_0^{set} in ATR_0.

In what follows it is our goal to interpret the richer language of RS^* in L_2. For the predicates M_α and the quantifiers $\forall x^\alpha$ and $\exists x^\alpha$ we use the notion of an α-tree.

Definition 36. For the notation α of an ordinal less than $\mathcal{BH}$, we say that a tree T is an α-*tree* if there exists a function $f : T \to \{\beta \in \mathcal{BH} : \beta \prec \alpha\}$ such that

$$(\forall \sigma \in T)\forall \tau(\tau \subset \sigma \to f(\sigma) \prec f(\tau)].$$

The interpretation of the predicate M_α for trees T will be that $M_\alpha(T)$ means that T is an α-tree. Quantifiers $\forall x^\alpha(\ldots x \ldots)$ and $\exists x^\alpha(\ldots x \ldots)$ will then be interpreted as $\forall T(T\ \alpha\text{-tree} \to \ldots T \ldots)$ and $\exists T(T\ \alpha\text{-tree} \wedge \ldots T \ldots)$, respectively.

Relation variables $U, V X, Y, \ldots$ will be interpreted to range over $\omega+1$-trees T such that $T \subseteq^* \omega^*$, i.e., $(\forall \langle n \rangle \in T)(T^{\langle n \rangle} \in^* \omega^*)$. It is perhaps worth mentioning that any suitable tree satisfying $T \subseteq^* \omega^*$ can be seen to be an $\omega+1$-tree.

Note that if we have the notation α of an ordinal less than Ω (given externally), then this ordinal is less than $\mathcal{BH}$ and $\Pi_1^1\text{-}\mathsf{CA} + TI(<\mathcal{BH})$ proves transfinite induction along the ordinal notations $\prec \alpha$, and consequently the notion of an α-tree becomes Δ_1^1 in $\Pi_1^1\text{-}\mathsf{CA} + TI(<\mathcal{BH})$. Even a slightly stronger assertion can be proved.

Lemma 37. *The notion of an α-tree is Δ_1^1 in the theory $\mathsf{ATR}_0 + WO(\alpha)$, where $WO(\alpha)$ expresses that $\prec$ restricted to $\{\beta : \beta \prec \alpha\}$ is a wellordering.*

Proof. For a tree T, define a function g with domain $\{\beta : \beta \prec \alpha\}$ by arithmetical transfinite recursion as follows

$$g(\beta) \;=\; \{\sigma \in T : (\forall \tau \in T)(\sigma \subset \tau \to \tau \in \bigcup_{\xi \prec \beta} g(\xi))\}.$$

Note that g is uniquely determined by T. One can then show that:

$$T \text{ is an } \alpha\text{-tree} \;\Leftrightarrow\; T \text{ is the image of } g.$$

If T is an α-tree witnessed by a function $f : T \to \{\beta : \beta \prec \alpha\}$, then one shows, using $\prec$-induction on $f(\sigma)$, that $\sigma \in g(f(\sigma))$ for all $\sigma \in T$, hence T is the image of g.

Conversely, if T is the image of g, then a function $f : T \to \{\beta : \beta \prec \alpha\}$ witnessing the α-treeness of T is obtained by letting $f(\sigma)$ be the least $\beta \prec \alpha$ such that $\sigma \in g(\beta)$. $\square$

Remark 38. In $\mathsf{ATR}_0 + WO(\alpha)$ every β-tree for $\beta \preceq \alpha$ is suitable.

The next task, which presents itself, is to translate the formulas of $\mathcal{L}^\star$ that belong to the collection $\mathcal{B}$ into the language of second order arithmetic, L_2.

Definition 39. For convenience assume that we have injections $x \mapsto T_x$ and $X \mapsto T_X$ from set variables x and relation variables X of $\mathcal{L}^\star$ to second order variables of L_2 in such a way that T_x is always different from any T_X. The latter provides a translation of the terms of $\mathcal{L}^\star$ except for the constants $\underline{\emptyset}$ and $\underline{\omega}$. These can just be translated as $\{\langle\rangle\}$ and ω^*, respectively (see Definition 35).

Since the relation $\in^*$ of Definition 32 has only been defined for trees, let us agree that henceforth a formula $S \in^* T$ will be considered a shorthand for S, T trees $\wedge\ S \in^* T$.

The translation * from $\mathcal{B}$ to L_2 is then effected as follows.

1. $(a \in \underline{\omega})^*$ is $T_a \in^* \omega^*$. $(\neg a \in \underline{\omega})^*$ is the negation of $(a \in \underline{\omega})^*$. $(a \in \underline{\emptyset})^*$ is $0 = 1$ while $(\neg a \in \underline{\emptyset})^*$ is $0 = 0$.
2. $(a \in x)^*$ is $T_a \in^* T_x$. $(a \notin x)^*$ is the negation of $(a \in x)^*$.
3. $(U(a))^*$ is $T_a \in^* T_U \wedge T_U \subseteq^* \omega^*$. $(\neg U(a))^*$ is the negation of $(U(a))^*$.
4. $(M_\alpha(a))^*$ is T_a is an α-tree. $(\neg M_\alpha(a))^*$ is T_a is not an α-tree.
5. $(F \vee G)^*$ and $(F \wedge G)^*$ are $F^* \vee G^*$ and $F^* \wedge G^*$, respectively.
6. $((\exists x \in a)F[x])^*$ and $((\forall x \in a)F[x])^*$ are

$$\exists T_x (T_x \in^* T_a \wedge F^*[T_x]) \quad \text{and} \quad \forall T_x (T_x \in^* T_a \rightarrow F^*[T_x])$$

respectively, where $F^*[T_x]$ stands for $(F[x])^*$; note that * replaces x by T_x.
7. $(\exists x^\alpha F[x])^*$ and $(\forall x^\alpha F[x])^*$ are

$$\exists T_x (T_x\ \alpha\text{-tree} \wedge F^*[T_x]) \quad \text{and} \quad \forall T_x (T_x\ \alpha\text{-tree} \rightarrow F^*[T_x]),$$

respectively.
8. $(\exists X F[X])^*$ and $(\forall X F[X])^*$ are

$$\exists T_X (T_X\ \omega{+}1\text{-tree} \wedge T_X \subseteq^* \omega^* \wedge F^*[T_X])$$

and

$$\forall T_X (T_X\ \omega{+}1\text{-tree} \wedge T_X \subseteq^* \omega^* \rightarrow F^*[T_X]),$$

respectively. Here $F^*[T_X]$ stands for $(F[X])^*$, noting that * replaces X by T_X.

In order to show that the collapsed derivations of Theorem 28 prove true statements when subjected to the interpretation of the previous, we need to show that the rule (BC) preserves truth. This will mainly be a consequence of Σ^1_1-AC.

Lemma 40. *Let $F[x_0, \ldots, x_q, U_0, \ldots, U_r]$ be a Σ-formula of $\mathcal{L}_2^{set}$ with all free variables indicated, and ℓ_F be the length of the latter formula as a string of symbols. Let*

$$F^\alpha[x_0, \ldots, x_q, U_0, \ldots, U_r] \quad \text{and} \quad F^{(a)}[x_0, \ldots, x_q, U_0, \ldots, U_r]$$

be the results of replacing the unbounded existential set quantifiers $\exists x$ in the formula by $\exists x^\alpha$ and $\exists x \in a$, respectively.

Arguing in the theory Π^1_1-CA + $TI(\alpha{+}\omega)$, we have that whenever $T_0, \ldots, T_q$ are α-trees and $Q_0, \ldots, Q_r$ are trees such that $Q_i \subseteq^* \omega^*$ (so actually $\omega{+}1$-trees), and the $*$-translation of

$$F^\alpha[x_0, \ldots, x_q, U_0, \ldots, U_r]$$

with x_i and U_j replaced by T_i and Q_j, respectively, is true, then the $*$-translation of

$$\exists y^{\alpha+\ell_F} F^{(y)}[x_0, \ldots, x_q, U_0, \ldots, U_r],$$

again with x_i replaced by T_i and U_j replaced by Q_j, holds true as well.

Proof. We proceed by (meta) induction on ℓ_F. The most interesting case arises when this formula starts with a bounded universal quantifier, i.e., if it is of the form

$$\forall v \in x_0 \, F_0^\alpha[x_0, \ldots, x_q, v, U_0, \ldots, U_r].$$

So assume that $T_0, \ldots, T_r$ are α-trees and $Q_0, \ldots, Q_r$ are trees such that $Q_i \subseteq^* \omega^*$ and

$$(\forall\langle n\rangle \in T_0)G_0^\alpha[T_0, \ldots, T_q, T_0^{\langle n\rangle}, Q_0, \ldots, Q_r]$$

holds, where G_0^α is the $*$-translation of F_0^α. Inductively we then have

$$(\forall\langle n\rangle \in T_0)\exists S(S \text{ an } \alpha+\ell_{F_0}\text{-tree} \,\wedge\, G_0^{(S)}[T_0, \ldots, T_q, T_0^{\langle n\rangle}, Q_0, \ldots, Q_r]), \tag{1}$$

where $G_0^{(S)}[T_0, \ldots, T_q, T_0^{\langle n\rangle}, Q_0, \ldots, Q_r]$ results from $G_0^\alpha[T_0, \ldots, T_q, T_0^{\langle n\rangle}, Q_0, \ldots, Q_r]$ by replacing any part of the form $\exists Q[Q \text{ an } \alpha\text{-tree} \wedge \ldots Q \ldots]$ by $(\exists\langle n\rangle \in \mathcal{S})(\ldots \mathcal{S}^{\langle n\rangle} \ldots)$.

Now it follows from Lemma 37 that the part $(\ldots)$ in (1) is Δ_1^1. As a result, we may apply Σ_1^1-AC (which is provable in Π_1^1-CA$_0$) to conclude that there exists a set Z such that for all $\langle n\rangle \in T_0$,

$$Z_{(n)} \text{ an } \alpha+\ell_{F_0}\text{-tree} \,\wedge\, G_0^{(Z_{(n)})}[T_0, \ldots, T_q, T_i^{\langle n\rangle}, Q_0, \ldots, Q_r] \tag{2}$$

where $Z_{(n)} = \{k : (n, k) \in Z\}$.

Using Σ_1^1-AC again, we can also single out a sequence of functions

$$f_{2^n 3^k} : Z_{(n)}^{\langle k\rangle} \to \alpha+\ell_{F_0}$$

such that $f_{2^n 3^k}$ witnesses that $Z_{(n)}^{\langle k\rangle}$ is an $\alpha+\ell_{F_0}$-tree whenever $\langle n\rangle \in T_0$ and $\langle k\rangle \in Z_{(n)}$.

Now define a tree $\mathcal{S}$ by

$$\mathcal{S} := \{\langle\rangle\} \cup \{\langle 2^n 3^k\rangle \star \sigma : \langle n\rangle \in T_0 \wedge \langle k\rangle \in Z_{(n)} \wedge \sigma \in Z_{(n)}^{\langle k\rangle}\}.$$

By design, S is a tree and, moreover, S is an $\alpha+\ell_F$-tree as witnessed by the function $f \colon S \to \alpha+\ell_F$ defined by

$$f(\langle\rangle) := \alpha+\ell_{F_0} \quad \text{and} \quad f(\langle 2^n 3^k\rangle \star \sigma) := f_{2^n 3^k}(\sigma).$$

As a consequence of (2) and the fact that $F[x_0, \ldots, x_q, U_0, \ldots, U_r]$ is a Σ-formula, we infer that

$$(\forall\langle n\rangle \in T_0)G_0^{\mathcal{S}}[T_0, \ldots, T_q, T_0^{\langle n\rangle}, Q_0, \ldots, Q_r]$$

and thus, as desired, the truth of the $*$-translation of $\exists y^{\alpha+\ell_F} F^{(y)}(x_0, \ldots, x_q, U_0, \ldots, U_r)$ with the substitutions $x_i \mapsto T_i$ and $U_j \mapsto Q_j$ follows. $\quad\square$

6.2. Conservativity

It remains to ascertain that KP + $(\Pi_1^1$-CA$^*)$ is conservative over Π_1^1-CA + $TI(<\mathcal{BH})$ for formulas in the language of second order arithmetic. More precisely, if $\mathcal{F}$ is a formula of second order arithmetic (as for instance defined in [20], I.2) and $\mathcal{F}_0$ denotes its natural translation into the language $\mathcal{L}_2^{set}$, then we aim to show that

$$\text{KP} + (\Pi_1^1\text{-CA}^*) \vdash \mathcal{F}_0 \implies \Pi_1^1\text{-CA} + TI(<\mathcal{BH}) \vdash \mathcal{F}.$$

Now assume $\mathsf{KP} + (\Pi^1_1\text{-}\mathsf{CA}^*) \vdash \mathcal{F}_0$. The latter being a finite deduction, it follows from what we have established in the previous sections that we can determine fixed naturals n, p such that $\mathcal{H} \vdash^{\beta}_{\eta,p} \mathcal{F}_0$ for some derivation operator $\mathcal{H}$ and $\beta, \eta \in C(\omega_n[\Omega + 1], 0)$. This follows from Theorems 26, 27, and 28. To dilate on this, note that because of the finiteness of the deduction in $\mathsf{KP} + (\Pi^1_1\text{-}\mathsf{CA}^*)$ we can find this a priori bound $\omega_n[\Omega + 1]$ so that the entire ordinal analysis, commencing with the embedding of this finite deduction into the infinitary proof system, solely uses ordinals from $C(\omega_n[\Omega + 1], 0)$.

Moreover, we can carry this ordinal analysis out in the background theory $\Pi^1_1\text{-}\mathsf{CA} + TI(<\mathcal{BH})$ as there exists an order preserving map from $C(\omega_n[\Omega+1], 0)$ into the segment of ordinals $< \psi(\omega_{n+1}[\Omega+1])$ for which transfinite induction is available in this theory. Such a map exists, for if $\delta, \eta \in C(\omega_n[\Omega + 1], 0)$ with $\delta < \eta$ one has

$$\psi(\omega_n[\Omega + 1] + \omega^{\Omega+\delta}) < \psi(\omega_n[\Omega + 1] + \omega^{\Omega+\eta}) < \psi(\omega_{n+1}[\Omega + 1])$$

by Lemma 10(2).

In light of the previous, the next step will be to show that $\mathcal{H} \vdash^{\beta}_{\eta,p} \mathcal{F}_0$ entails that $\mathcal{F}$ is true, all the while working in our background theory $\Pi^1_1\text{-}\mathsf{CA} + TI(<\mathcal{BH})$. This is where the up to now neglected parameter p comes into its own in that the idea is to employ a formal truth predicate for formulas of length $< p$. However, it is not possible to just focus on formulas $\mathcal{F}_0$, where $\mathcal{F}$ resides in the language of second order arithmetic, since in showing that $\mathcal{H} \vdash^{\beta}_{\eta,p} \mathcal{F}_0$ implies the truth of $\mathcal{F}$ we shall induct on β and the pertaining derivation is usually not cut-free, so we have to take formulas from $\mathcal{B}$ of length $< p$ into account.

It is perhaps noteworthy that if $\mathcal{H} \vdash^{\beta}_{\eta,p} \Gamma$ with Γ a set of formulas in $\mathcal{B}$ and $\beta, \eta < \Omega$, then all formulas occurring in the derivation must be in $\mathcal{B}$, too, as there can be no cuts in it involving formulas with unbounded set quantifiers since their ranks would be $\geq \Omega$ (see Definition 4 items (7) and (8)).

The translation of the formulas from $\mathcal{B}$ into L_2 formulas has been introduced in Definition 39. The purpose of the following definition is to fix an arithmetized truth definition for $\mathcal{B}$-formulas of length $< p$ in $\Pi^1_1\text{-}\mathsf{CA} + TI(<\mathcal{BH})$.

Definition 41. Let Z be a set of naturals such $(Z)_k$ is an α-tree for all k. We'd like to engineer a formula $\mathcal{T}_p(x, X)$ of L_2 such that for all $\mathcal{B}$-formulas $F[x_1, \ldots, x_r, U_1, \ldots, U_s]$ of length $< p$ with all free variables exhibited,

$$\mathcal{T}_p(\ulcorner F[x_1, \ldots, x_r, U_1, \ldots, U_s] \urcorner, Z) \ \leftrightarrow \ \tilde{F}[Z_{(1)}, \ldots Z_{(r)}, Z_{(r+1)}, \ldots, Z_{(r+s)}] \tag{*}$$

where $\ulcorner F[x_1, \ldots, x_r, U_1, \ldots, U_s] \urcorner$ stands for the Gödel number of $F[x_1, \ldots, x_r, U_1, \ldots, U_s]$ whilst

$$\tilde{F}[Z_{(1)}, \ldots Z_{(r)}, Z_{(r+1)}, \ldots, Z_{(r+s)}]$$

denotes the formula obtained from $(F[x_1, \ldots, x_r, U_1, \ldots, U_s])^*$ by replacing T_{x_i} by $(Z)_i$ and T_{U_j} by $(Z)_{r+j}$. Thus the sections $(Z)_k$ of Z furnish an assignment of α-trees to the free variables of $(F[x_1, \ldots, x_r, U_1, \ldots, U_s])^*$.

The formal definition of such a truth predicate is a standard but cumbersome procedure. A place in the literature, where one finds this carried out in detail, is Takeuti [22, CH. 3,19], and another is Troelstra [23, 1.5.4].

Theorem 42. *Fix n and p. Let $TV(Z, \alpha)$ be short for "$\forall k \, (Z)_k$ is an α-tree". Then $\Pi^1_1\text{-}\mathsf{CA} + TI(<\mathcal{BH})$ proves for all sequents of formulas Γ in $\mathcal{B}$ consisting of formulas of length $< p$ and $\alpha, \beta, \rho \in C(\omega_n[\Omega + 1], 0) \cap \Omega$ that $\mathcal{H} \vdash^{\beta}_{\eta,p} \Gamma$ implies*

$$\forall Z(\mathsf{TV}(Z, \alpha) \to \mathcal{T}_p(\ulcorner \bigvee \Gamma \urcorner, Z))$$

where $\bigvee \Gamma$ stands for the disjunction of all formulas of Γ, but if Γ is empty let $\bigvee \Gamma$ be $\underline{\emptyset} \in \underline{\emptyset}$.

Proof. We reason in $\Pi_1^1\text{-CA} + TI(<\mathcal{BH})$ by induction on β. The axiom cases are obvious. If $\mathcal{H} \vdash^{\beta}_{\eta, p} \Gamma$ is the result of an inference of a form other than (BC), then this follows immediately from the induction hypothesis applied to the immediate subderivations. Note also that in case of a cut, the cut formulas belong to $\mathcal{B}$ and have lengths $< p$. Observe also that the derivation cannot contain ($\mathcal{S}_0$-Ref) inferences since $\beta < \Omega$.

So it remains to deal with the case where the last inference is an instance of (BC). Fortunately, this is what Lemma 40 is really about. The latter shows that if the premise Θ of an instance of (BC) is true under an assignment Z, then so is the conclusion. $\square$

Theorem 43. $\mathsf{KP} + (\Pi_1^1\text{-CA}^*)$ is conservative over $\Pi_1^1\text{-CA} + TI(<\mathcal{BH})$ for formulas of second order arithmetic. More precisely, if $\mathcal{F}$ is sentence of second order arithmetic (i.e. L_2) and $\mathcal{F}_0$ denotes its natural translation into the language $\mathcal{L}_2^{set}$, then

$$\mathsf{KP} + (\Pi_1^1\text{-CA}^*) \vdash \mathcal{F}_0 \quad \Longrightarrow \quad \Pi_1^1\text{-CA} + TI(<\mathcal{BH}) \vdash \mathcal{F}.$$

Proof. Assume $\mathsf{KP} + (\Pi_1^1\text{-CA}^*) \vdash \mathcal{F}_0$. As elaborated on before, with the help of the results in subsection 5.8 and Theorem 42, it follows that

$$\Pi_1^1\text{-CA} + TI(<\mathcal{BH}) \vdash \forall Z(\mathsf{TV}(Z) \to \mathcal{T}_p(\ulcorner \mathcal{F}_0 \urcorner, Z)).$$

Thus, in light of (*) and noting that $\mathcal{F}_0$ has no free variables,

$$\Pi_1^1\text{-CA} + TI(<\mathcal{BH}) \vdash \mathcal{F}_0^*$$

where $\mathcal{F}_0^*$ is the translation of $\mathcal{F}_0$ according to Definition 41. It remains to establish the relationship between $\mathcal{F}_0^*$ and $\mathcal{F}$.

$\mathcal{F}_0^*$ arises from $\mathcal{F}$ by translating numerical quantifiers Qn as ranging over the immediate subtrees of ω^* and second order quantifiers as ranging over $\omega+1$-trees T such that $T \subseteq^* \omega^*$. Now, as the naturals with their ordering are isomorphic to the immediate subtrees of ω^* ordered via $\in^*$ and also the collection of sets of natural numbers is naturally isomorphic to the collection of $\omega+1$-trees T such that $T \subseteq^* \omega^*$, it follows that $\mathcal{F}_0^*$ implies $\mathcal{F}$, completing the proof. Anyone insisting on a more formal proof is invited to proceed by induction on the buildup of $\mathcal{F}$. $\square$

Remark 44. In this article we do not compute the proof-theoretic ordinal of the system $\Pi_1^1\text{-CA} + TI(<\mathcal{BH})$. However, it seems that the ordinal analysis of $\Pi_1^1\text{-CA} + TI(<\mathcal{BH})$ simply has to follow that of $\Pi_1^1\text{-CA}$ with ε_0 replaced by $\mathcal{BH}$. So we conjecture that it is the ordinal $\psi_0(\Omega_\omega \cdot \mathcal{BH})$ in the terminology of Buchholz and Schütte [8].

For the ordinal analysis of $\Pi_1^1\text{-CA}$ see, for example, the relevant chapters in Buchholz, Feferman, Pohlers and Sieg [6], Buchholz and Schütte [8], and Jäger [11].

7. Extensions

Thus far we have investigated what happens if one adds Kripke-Platek set theory to the subtheory of second order arithmetic based on Π_1^1-comprehension. There one obtains a certain analogy to what Barwise

[2] called the *Admissible Cover*, $\mathbb{C}ov_{\mathfrak{M}}$, of a basic structure $\mathfrak{M}$. In his case, the basic structures $\mathfrak{M}$ were models of set theory. $\mathbb{C}ov_{\mathfrak{M}}$ is the intersection of all admissible sets which cover $\mathfrak{M}$ (see [2], Appendix 2.1).[2]

In our context, the admissible cover amounts to grafting the theory KP onto a subsystem of second order arithmetic, T. This could be called the *proof-theoretic admissible cover* of T. There is, however, a crucial difference between Barwise's model-theoretic construction and the proof-theoretic one employed in this paper. In the former the basic structure one starts from remains unchanged in a strong sense when building its admissible cover in that no new subsets of $\mathfrak{M}$ become available in $\mathbb{C}ov_{\mathfrak{M}}$ (see [2] Appendix Corollary 2.4), whereas in the proof-theoretic case the axioms of the basic theory T will interact with the axioms of the set theory KP, witnessed by the fact that, in general, more theorems of second order arithmetic become deducible in $T + \mathsf{KP}$ than in T alone.

It is also interesting to investigate how the proof-theoretic admissible cover plays out in the case of other well-known subsystems of second order arithmetic. It turns out that a certain pattern emerges. Moreover, the techniques developed in this paper, when combined with insights from the literature, suffice to get these additional results.

Definition 45. We will focus on two well-studied theories.

1. ATR_0 with its signature axiom of *arithmetical transfinite recursion* is the fourth system of the "big" five of reverse mathematics (see [20, V]).
2. The second system we will consider is traditionally called the theory of *bar induction*, BI, by proof theorists. In [20, VII.2.14] it is denoted by Π_∞^1-TI_0. The axioms of BI are those of ACA_0 plus the scheme of transfinite induction

$$\forall X(WO(X) \to TI(X, A))$$

for every L_2 formula $A[x]$, where $TI(X, A)$ stands for the formula

$$\forall u((\forall v <_X u)A[v] \to A[u]) \to \forall u A[u]$$

with $WO(X)$ expressing that the ordering $<_X$ defined by $v <_X u :\Leftrightarrow 2^v \cdot 3^u \in X$ is a well ordering.

(ATR_0^*) and (BI^*) are the axiom schemas – formulated in the language $\mathcal{L}_2^{set}$ – that comprise the natural translations of all instances of arithmetical transfinite recursion and transfinite induction, respectively. If $\mathcal{C}$ is a collection of sentences of L_2 and T_1, T_2 are theories of the language L_2 or $\mathcal{L}_2^{set}$, we write $T_1 \equiv_{\mathcal{C}} T_2$ to convey that T_1 and T_2 possess the same $\mathcal{C}$-theorems (perhaps modulo the translation of $\mathcal{C}$ into $\mathcal{L}_2^{set}$).

Theorem 46. $\mathsf{KP} + (\mathsf{ATR}_0^*) \equiv_{\Pi_\infty^1} \mathsf{ATR}_0 + TI(<\mathcal{BH})$.

Proof. The proof of Theorem 43 essentially carries over with (ATR_0^*) replacing $(\Pi_1^1\text{-}\mathsf{CA}^*)$ since the crucial Lemma 40 is also provable in ATR_0 as the latter theory proves Σ_1^1-AC, too (this is an old result, see [20, Theorem V.8.3]). $\quad\square$

Theorem 47.

(i) $\mathsf{KP} + (\mathsf{BI}^*) \equiv_{\Pi_\infty^1} \mathsf{BI}$.
(ii) $\mathsf{KP} + (\mathsf{BI}^*) \equiv_{\Pi_1^1} \mathsf{KP}$.

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[2] The proper proof-theoretic counterpart of the admissible cover was developed in Jäger [13].

(iii) $\mathsf{KP} + (\mathsf{ATR}_0^*) \equiv_{\Pi_1^1} \mathsf{KP}$.

Proof. (i) The proof of Theorem 43 also works with BI in lieu of Π_1^1-CA since ATR_0 is a consequence of BI (see [20, Corollary VII.2.19]. So we infer that $\mathsf{KP} + (\mathsf{BI}^*) \equiv_{\Pi_\infty^1} \mathsf{BI} + TI(<\mathcal{BH})$. However, it is well known from the proof-theoretic literature that BI already proves $TI(<\mathcal{BH})$, yielding (i) (for more details see [9,6,8]).

(ii) By Jäger [12] and the papers cited in the previous line, it is known that KP and BI share the same proof-theoretic ordinal. Moreover, from the ordinal analyses of these two theories it can be inferred that they prove the same Π_1^1-theorems.

The degree of conservativity, though, cannot be improved much beyond this level as ATR_0 has a Π_2^1-axiomatization and KP does not prove ATR_0. To see this, first note that KP has a model $\mathbb{HYP}_\mathfrak{N}$ which is the intersection of all admissible sets that contain the standard structure $\mathfrak{N}$ of the naturals as a set (see [2, Theorem 5.9]). The subsets of $\mathbb{N}$ in $\mathbb{HYP}_\mathfrak{N}$ are the hyperarithmetical sets. Then we can proceed, for example, in one of the following two ways:

- The collection of the hyperarithmetical sets, as apparently proved by Kreisel, constitutes the smallest ω-model of Σ_1^1-AC, but it cannot be a model of ATR_0 as it would have to have a proper ω-submodel that is again an ω-model of ATR_0, and thus of Σ_1^1-AC. The result about ω-models of ATR_0 is due to Quinsey [16, pages 93–96] (for more details see [20] Theorem VIII.6.12 and the Notes for §VIII.6).
- Alternatively, observe that the theory ATR_0 is equivalent to the theory FP_0 which extends ACA_0 by axioms stating that any positive arithmetic operator has a fixed point; see Avigad [1]. But the hyperarithmetical sets do not constitute a model of FP_0 according to Probst [15, II.2.4] and Gregoriades [10].

(iii) By Theorem 46 we have $\mathsf{ATR}_0^* + \mathsf{KP} \equiv_{\Pi_\infty^1} \mathsf{ATR}_0 + TI(<\mathcal{BH})$. Since $\mathsf{ATR}_0 + TI(<\mathcal{BH})$ is a subtheory of BI and $\mathsf{BI}^* + \mathsf{KP} \equiv_{\Pi_1^1} \mathsf{KP}$, it follows that $\mathsf{ATR}_0^* + \mathsf{KP} \equiv_{\Pi_1^1} \mathsf{KP}$. □

In connection with the previous results we would also like to mention Sato [18]. Among other things, he states there a special case of the previous Theorem 47(ii), namely that the addition of a Π_2^1 theorem of BI to KP does not increase the consistency strength of the augmented theory.[3] Further interesting work about the relationship between Kripke-Platek set theory and Π_1^1 comprehension on the natural numbers is presented in Simpson [21]. There the interplay between KP and Π_1^1-CA is studied from a recursion- and model-theoretic perspective. However, it does not provide the exact proof-theoretic strength of $\mathsf{KP} + (\Pi_1^1$-$\mathsf{CA}^*)$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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[3] What we call KP is called $\mathsf{KP}\omega$ in [18].

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