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Original Paper

An Elliptic Solution of the Classical Yang–Baxter Equation Associated with the Queer Lie Superalgebra

Maxim Nazarov

Abstract. A solution of the classical Yang–Baxter equation associated with the queer Lie superalgebra is constructed in terms of Hermite theta functions.

1. Introduction

Let \mathfrak{g} be any finite-dimensional Lie superalgebra over a complex field \mathbb{C} . Let $r(u, v)$ be a meromorphic function of two complex variables u and v which takes values in $\mathfrak{g} \otimes \mathfrak{g}$. The *classical Yang–Baxter equation* for the function $r(u, v)$ is

$$[r_{12}(u, v), r_{13}(u, w)] + [r_{12}(u, v), r_{23}(v, w)] + [r_{13}(u, w), r_{23}(v, w)] = 0$$

where w is another complex variable and the function of u, v, w at the left-hand side takes values in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. Here we use the standard notation, for example $r_{23}(v, w) = 1 \otimes r(v, w)$. For conventions on the tensor products see Sect. 2.

A solution of the above equation is called *nondegenerate* if not every value of the function $r(u, v)$ is degenerate as a quadratic tensor. For simple Lie algebras \mathfrak{g} the nondegenerate solutions were classified in [4, 5]. In particular, it was shown in [5] that for any nondegenerate solution $r(u, v)$ with a simple Lie algebra \mathfrak{g} one can find a domain $D \subset \mathbb{C}$ and two holomorphic maps

$$\psi : D \rightarrow \mathbb{C} \quad \text{and} \quad \omega : D \rightarrow \text{Aut } \mathfrak{g}$$

where ψ is not constant and the function $(\omega(u) \otimes \omega(v)) r(\psi(u), \psi(v))$ depends only on the difference $u - v$.

This basic result of [5] does not hold for Lie superalgebras. Solutions $r(u, v)$ which depend not only on the difference $u - v$ were constructed in [11].

There \mathfrak{g} is the general linear Lie superalgebra $\mathfrak{gl}_{n|n}$ with any positive integer n . These solutions are *antisymmetric*, that is

$$r_{21}(v, u) = -r(u, v). \tag{1}$$

A general phenomenon underlying this construction was independently described in [1]. See [12] for further discussion of this remarkable phenomenon.

The solutions $r(u, v)$ constructed in [11] are rational functions of variables u and v . Here we construct a solution $r(u, v)$ which is expressed in theta functions of $u - v$ and $u + v$. Our \mathfrak{g} is the quotient of the special linear Lie superalgebra $\mathfrak{sl}_{n|n}$ by its one-dimensional centre. This quotient is denoted by $\mathfrak{psl}_{n|n}$.

Elliptic solutions $r(u, v)$ for the Lie superalgebra $\mathfrak{g} = \mathfrak{psl}_{n|n}$ were constructed in [9]. They have the form $r(u, v) = s(u - v)$ where $s(u)$ is a function such that

$$s_{21}(u) = -s(-u). \tag{2}$$

Hence these solutions are antisymmetric.

Now let η be the involutive automorphism of $\mathfrak{gl}_{n|n}$ defined in Sect. 2. The fixed point subalgebra of $\mathfrak{gl}_{n|n}$ relative to η is the queer Lie superalgebra \mathfrak{q}_n . The automorphism η preserves the subalgebra $\mathfrak{sl}_{n|n}$ and descends to $\mathfrak{psl}_{n|n}$. In our Sect. 5 we show that the function $s(u)$ in [9] can be so chosen that

$$(\eta \otimes \eta) s(u) = s(-u) \tag{3}$$

and that

$$r(u, v) = s(u - v) + (\text{id} \otimes \eta) s(u + v) \tag{4}$$

is another solution of classical Yang–Baxter equation for $\mathfrak{g} = \mathfrak{psl}_{n|n}$.

It immediately follows from (3) and (4) that

$$r(u, v) = s(u - v) + (\eta \otimes \text{id}) s(-u - v). \tag{5}$$

By comparing the definition (4) with (5) and by using (2) we see that our solution $r(u, v)$ is antisymmetric as well.

2. General Conventions

We shall use the following general conventions. Let A and B be any associative \mathbb{Z}_2 -graded algebras. Their tensor product $A \otimes B$ is also an associative \mathbb{Z}_2 -graded algebra such that for any homogeneous elements $X, X' \in A$ and $Y, Y' \in B$

$$\begin{aligned} (X \otimes Y)(X' \otimes Y') &= X X' \otimes Y Y' (-1)^{\deg X' \deg Y}, \\ \deg(X \otimes Y) &= \deg X + \deg Y. \end{aligned}$$

Furthermore, for any two \mathbb{Z}_2 -graded modules U and V over A and B , respectively, the vector space $U \otimes V$ is a \mathbb{Z}_2 -graded module over $A \otimes B$ such that for any homogeneous elements $x \in U$ and $y \in V$

$$\begin{aligned} (X \otimes Y)(x \otimes y) &= X x \otimes Y y (-1)^{\deg x \deg Y}, \\ \deg(x \otimes y) &= \deg x + \deg y. \end{aligned} \tag{6}$$

$$\tag{7}$$

Now let the indices i, j, k, l run through $\pm 1, \dots, \pm n$. Put $\bar{i} = 0$ if $i > 0$ and $\bar{i} = 1$ if $i < 0$. Consider the \mathbb{Z}_2 -graded vector space $\mathbb{C}^{n|n}$. Let $e_i \in \mathbb{C}^{n|n}$ be an element of the standard basis. The \mathbb{Z}_2 -grading on $\mathbb{C}^{n|n}$ is defined by $\deg e_i = \bar{i}$.

Let $E_{ij} \in \text{End } \mathbb{C}^{n|n}$ be the standard matrix unit, defined by $E_{ij} e_k = \delta_{jk} e_i$. The associative algebra $\text{End } \mathbb{C}^{n|n}$ is \mathbb{Z}_2 -graded by setting $\deg E_{ij} = \bar{i} + \bar{j}$. Hence $\mathbb{C}^{n|n}$ is a \mathbb{Z}_2 -graded module over $\text{End } \mathbb{C}^{n|n}$. For any positive integer m we can also identify the tensor product $(\text{End } \mathbb{C}^{n|n})^{\otimes m}$ with the algebra $\text{End}((\mathbb{C}^{n|n})^{\otimes m})$ acting on the vector space $(\mathbb{C}^{n|n})^{\otimes m}$ by repeatedly using conventions (6) and (7).

The *supertrace* str is a linear function $\text{End } \mathbb{C}^{n|n} \rightarrow \mathbb{C}$ defined by setting

$$\text{str}(E_{ij}) = \delta_{ij} (-1)^{\bar{i}}.$$

This definition implies that for any homogeneous elements $X, Y \in \text{End } \mathbb{C}^{n|n}$

$$\text{str}(YX) = \text{str}(XY) (-1)^{\deg X \deg Y}.$$

Further, we can define an involutive automorphism η of $\text{End } \mathbb{C}^{n|n}$ by mapping

$$\eta : E_{ij} \mapsto E_{-i, -j}. \tag{8}$$

This automorphism is the conjugation by the involutive odd element of $\text{End } \mathbb{C}^{n|n}$

$$E_{1,-1} + E_{-1,1} + \dots + E_{n,-n} + E_{-n,n}.$$

We have

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj} (-1)^{(\bar{i} + \bar{j})(\bar{k} + \bar{l})} \tag{9}$$

in $\text{End } \mathbb{C}^{n|n}$. Here the square brackets indicate the supercommutator. We will also consider each E_{ij} as an element of the Lie superalgebra $\mathfrak{gl}_{n|n}$. The special linear Lie superalgebra $\mathfrak{sl}_{n|n}$ is the subalgebra of $\mathfrak{gl}_{n|n}$ defined as the kernel of the function str . The centre of $\mathfrak{gl}_{n|n}$ is spanned by the element

$$E_{11} + E_{-1,-1} + \dots + E_{nn} + E_{-n,-n} = 1.$$

The subalgebra $\mathfrak{sl}_{n|n}$ contains this element. The quotient of the Lie superalgebra $\mathfrak{sl}_{n|n}$ by the one-dimensional subspace spanned by this element is denoted by $\mathfrak{psl}_{n|n}$. According to [8] the Lie superalgebra $\mathfrak{psl}_{n|n}$ is simple if and only if $n > 1$. By using (9) we obtain that the Lie bracket on $\mathfrak{psl}_{1|1}$ is just zero.

By (9) our η is also an involutive automorphism of the Lie superalgebra $\mathfrak{gl}_{n|n}$. This automorphism preserves its subalgebra $\mathfrak{sl}_{n|n}$. The *queer Lie superalgebra* \mathfrak{q}_n is the fixed point subalgebra of $\mathfrak{gl}_{n|n}$ by η .

3. Theta Functions

Fix any complex number τ with a positive imaginary part. For any real numbers a and b consider the *Hermite theta function*

$$\theta_{a,b}(u) = \sum_{m=-\infty}^{\infty} e^{\pi i \tau (a+m)^2 + 2\pi i (a+m)(b+u)}. \tag{10}$$

The above series converges to a holomorphic function of the complex variable u . All zeroes of this function are simple and form the subset

$$(a + \frac{1}{2} + \mathbb{Z})\tau + (b + \frac{1}{2} + \mathbb{Z}) \subset \mathbb{C},$$

see [7, pp. 196–199]. The numbers a and b here are called *characteristics*. For $a = b = 0$ the series (10) is the *Jacobi theta function*. It follows from (10) that

$$\theta_{a,b}(u + 1) = e^{2\pi ia} \theta_{a,b}(u) \quad \text{and} \quad \theta_{a,b}(u + \tau) = e^{-2\pi i(u+b+\frac{\tau}{2})} \theta_{a,b}(u). \quad (11)$$

By changing m to $m + 1$ in (10) and by using the first equation in (11) we obtain

$$\theta_{a+1,b}(u) = \theta_{a,b}(u) \quad \text{and} \quad \theta_{a,b+1}(u) = e^{2\pi ia} \theta_{a,b}(u) \quad (12)$$

respectively. Further, by changing m to $-m$ in (10) we obtain the parity relation

$$\theta_{a,b}(-u) = \theta_{-a,-b}(u). \quad (13)$$

Now let g and h be any integers not simultaneously divisible by $2n$ and by n , respectively. Consider the function of the complex variable u

$$\varphi_{g,h}(u) = \frac{\theta_{\frac{g}{2n} + \frac{1}{2}, \frac{1}{2} - \frac{h}{n}}(u) \theta'_{\frac{1}{2}, \frac{1}{2}}(0)}{\theta_{\frac{g}{2n} + \frac{1}{2}, \frac{1}{2} - \frac{h}{n}}(0) \theta_{\frac{1}{2}, \frac{1}{2}}(u)}.$$

This function is meromorphic. It has simple poles at every point on the lattice

$$\mathcal{L} = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}.$$

Due to the chosen normalisation the residue of the pole of $\varphi_{g,h}(u)$ at $u = 0$ is 1.

It immediately follows from the relations (12) that

$$\varphi_{g+2n,h}(u) = \varphi_{g,h}(u) \quad \text{and} \quad \varphi_{g,h+n}(u) = \varphi_{g,h}(u).$$

Thus $\varphi_{g,h}(u)$ depends on the integers g and h only modulo $2n$ and n , respectively. From now on g and h will run not through \mathbb{Z} , but through the additive groups $\mathbb{Z}_{2n} = \mathbb{Z}/2n\mathbb{Z}$ and $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, respectively.

Let $\varepsilon = e^{\pi i/n}$ be a primitive root of unity of the $2n$. Direct calculation using (11) yields the periodicity properties

$$\varphi_{g,h}(u + 1) = \varepsilon^g \varphi_{g,h}(u) \quad \text{and} \quad \varphi_{g,h}(u + \tau) = \varepsilon^{2h} \varphi_{g,h}(u). \quad (14)$$

Another direct calculation using (12) and (13) yields the parity relation

$$\varphi_{g,h}(-u) = -\varphi_{-g,-h}(u). \quad (15)$$

4. Commuting Automorphisms

Consider the following two elements of the algebra $\text{End } \mathbb{C}^{n|n}$,

$$\begin{aligned} A &= E_{11} + \varepsilon^2 E_{22} + \dots + \varepsilon^{2(n-1)} E_{nn} \\ &\quad + \varepsilon E_{-1,-1} + \varepsilon^{-1} E_{-2,-2} + \dots + \varepsilon^{3-2n} E_{-n,-n} \end{aligned}$$

and

$$B = E_{12} + \cdots + E_{n-1,n} + E_{n1} \\ + E_{-2,-1} + \cdots + E_{-n,1-n} + \cdots + E_{-1,-n}.$$

These elements are invertible and of \mathbb{Z}_2 -degree zero. They satisfy the relations

$$A^{2n} = B^n = 1 \quad \text{and} \quad BA = \varepsilon^2 AB. \quad (16)$$

By (8) we get

$$\eta(A) = \varepsilon A^{-1} \quad \text{and} \quad \eta(B) = B^{-1}. \quad (17)$$

Let us define two automorphisms α and β of the algebra $\text{End } \mathbb{C}^{n|n}$ by setting

$$\alpha(X) = A^{-1}XA \quad \text{and} \quad \beta(X) = B^{-1}XB$$

for $X \in \text{End } \mathbb{C}^{n|n}$. These automorphisms commute and are of degrees $2n$ and n , respectively. Here $\alpha(X) = \beta(X) = X$ if and only if X is a linear combination of

$$E_{11} + \cdots + E_{nn} \quad \text{and} \quad E_{-1,-1} + \cdots + E_{-n,-n}.$$

Since the elements A and B are of \mathbb{Z}_2 -degree 0, for each $X \in \text{End } \mathbb{C}^{n|n}$ we have

$$\text{str}(\alpha(X)) = \text{str}(X) \quad \text{and} \quad \text{str}(\beta(X)) = \text{str}(X). \quad (18)$$

Let us now regard α and β as automorphisms of the Lie superalgebra $\mathfrak{gl}_{n|n}$. It follows from the first equation in (16) that $\alpha^{2n} = \beta^n = 1$. The eigenvalues of α and β are ε^g and ε^{2h} where g and h range over \mathbb{Z}_{2n} and \mathbb{Z}_n , respectively. Let $\mathfrak{gl}_{n|n}^{g,h}$ be the joint eigenspace of α and β corresponding to ε^g and ε^{2h} . By (18)

$$\mathfrak{gl}_{n|n}^{g,h} \subset \mathfrak{sl}_{n|n} \quad \text{for} \quad (g,h) \neq (0,0). \quad (19)$$

Consider the *Casimir element* of the tensor square $\mathfrak{gl}_{n|n} \otimes \mathfrak{gl}_{n|n}$

$$t = \sum_{i,j} E_{ij} \otimes E_{ji} (-1)^{\bar{j}}.$$

This element is invariant by $\mathfrak{gl}_{n|n}$ as for any indices k, l by using (9) we get

$$[t, E_{kl} \otimes 1 + 1 \otimes E_{kl}] = 0. \quad (20)$$

Note that

$$(\eta \otimes \eta) t = -t. \quad (21)$$

It follows from (20) that the Casimir element t is invariant by both $\alpha \otimes \alpha$ and $\beta \otimes \beta$. Therefore t belongs to the direct sum of subspaces

$$\mathfrak{gl}_{n|n}^{g,h} \otimes \mathfrak{gl}_{n|n}^{-g,-h} \subset \mathfrak{gl}_{n|n} \otimes \mathfrak{gl}_{n|n}. \quad (22)$$

Let $t_{g,h}$ be the projection of element t to the direct summand (22). By (17), (21)

$$(\eta \otimes \eta) t_{g,h} = -t_{-g,-h}. \quad (23)$$

Further, let σ be the linear transformation of $\mathfrak{gl}_{n|n} \otimes \mathfrak{gl}_{n|n}$ defined by setting

$$\sigma(X \otimes Y) = Y \otimes X (-1)^{\deg X \deg Y}$$

for homogeneous elements X and Y . By the definition of t we have $\sigma(t) = t$. So

$$\sigma(t_{g,h}) = t_{-g,-h}. \tag{24}$$

We do not need explicit formulas for every projection $t_{g,h}$. We only note that

$$t_{0,0} = \frac{1}{2n} (J \otimes 1 + J \otimes 1) \tag{25}$$

where

$$J = E_{11} - E_{-1,-1} + \dots + E_{nn} - E_{-n,-n}.$$

5. Classical Yang–Baxter Equation

The central element 1 of the Lie superalgebra $\mathfrak{gl}_{n|n}$ is contained in the eigenspace $\mathfrak{gl}_{n|n}^{0,0}$. Therefore for any $(g, h) \neq (0, 0)$ the element $t_{g,h}$ can be identified with its image in $\mathfrak{pgl}_{n|n} \otimes \mathfrak{pgl}_{n|n}$. By using this identification, introduce a function of u

$$s(u) = \sum_{(g,h) \neq (0,0)} \varphi_{g,h}(u) t_{g,h}.$$

Observe that the function $s(u)$ takes all its values in the subspace

$$\mathfrak{psl}_{n|n} \otimes \mathfrak{psl}_{n|n} \subset \mathfrak{pgl}_{n|n} \otimes \mathfrak{pgl}_{n|n}.$$

Changing the summation indices g, h to $-g, -h$, respectively, and then using (15), (24) proves that $s(u)$ indeed satisfies the condition (2) for $\mathfrak{g} = \mathfrak{psl}_{n|n}$. Using (15), (23) proves that $s(u)$ satisfies (3). Here we regard η as an automorphism of the Lie algebra $\mathfrak{psl}_{n|n}$. The automorphisms α and β of $\mathfrak{gl}_{n|n}$ preserve $\mathfrak{sl}_{n|n}$ and descend to $\mathfrak{psl}_{n|n}$ too. By the periodicity properties (14) of the function $\varphi_{g,h}(u)$

$$s(u + 1) = (\alpha \otimes \text{id}) s(u) = (\text{id} \otimes \alpha^{-1}) s(u) \tag{26}$$

and

$$s(u + \tau) = (\beta \otimes \text{id}) s(u) = (\text{id} \otimes \beta^{-1}) s(u). \tag{27}$$

By (25) the image in $\mathfrak{pgl}_{n|n} \otimes \mathfrak{pgl}_{n|n}$ of the element $t_{0,0}$ is zero. Hence the image of the element t is

$$\sum_{(g,h) \neq (0,0)} t_{g,h}.$$

Denote this image by p . The residue of the function $s(u)$ at $u = 0$ equals p .

Now consider the function $r(u, v)$ as defined by (4). We already observed in Sect. 1 that (2) and (3) imply the antisymmetry property (1). Let us show that $r(u, v)$ is indeed a solution of the classical Yang–Baxter equation for $\mathfrak{g} = \mathfrak{psl}_{n|n}$. We will employ general arguments from [4, Sect. 5].

First observe that due to (4) and to the first equalities in (26) and (27) we get

$$r(u+1, v) = (\alpha \otimes \text{id}) r(u, v) \quad \text{and} \quad r(u+\tau, v) = (\beta \otimes \text{id}) r(u, v). \quad (28)$$

Similarly, due to (5) and to the second equalities in (26) and (27) we get

$$r(u, v+1) = (\text{id} \otimes \alpha) r(u, v) \quad \text{and} \quad r(u, v+\tau) = (\text{id} \otimes \beta) r(u, v).$$

Let $f(u, v, w)$ be the left-hand side of the classical Yang–Baxter equation for our $r(u, v)$ with $\mathfrak{g} = \mathfrak{psl}_{n|n}$. Since α and β are Lie algebra automorphisms, we have

$$f(u+1, v, w) = (\alpha \otimes \text{id} \otimes \text{id}) f(u, v, w) \quad (29)$$

and

$$f(u+\tau, v, w) = (\beta \otimes \text{id} \otimes \text{id}) f(u, v, w). \quad (30)$$

Choose any values of v and w such that $v-w, v+w \notin \mathcal{L}$. We will prove that then $f(u, v, w)$ is a holomorphic function of u in the whole \mathbb{C} . By (29) and (30) this function is bounded and hence a constant. This constant is then an element of $\mathfrak{psl}_{n|n} \otimes \mathfrak{psl}_{n|n} \otimes \mathfrak{psl}_{n|n}$ invariant by α and β applied in the first tensor factor. However the only element of $\mathfrak{psl}_{n|n}$ invariant by both α and β is zero. Therefore our function must be zero.

If $u \pm v \in \mathcal{L}$, then $u \pm w \notin \mathcal{L}$ since $v \pm w \notin \mathcal{L}$. Then the function $r_{13}(u, w)$ has no pole. Consider the first two of the three summands of $f(u, v, w)$. By the definition (4) their sum can be written as

$$[s_{12}(u-v), r_{13}(u, w) + r_{23}(v, w)] + \quad (31)$$

$$[(\text{id} \otimes \eta \otimes \text{id}) s_{12}(u+v), r_{13}(u, w) + r_{23}(v, w)]. \quad (32)$$

By multiplying (31) by $u-v$ and then setting $u=v$ we get

$$[p, r_{13}(v, w) + r_{23}(v, w)] = 0.$$

Here we employ (20) and the definition of p as the image of t in $\mathfrak{pgl}_{n|n} \otimes \mathfrak{pgl}_{n|n}$. Therefore (31) has no pole at $u-v=0$. It now follows from (26), (27), (28) that the summand (31) has no pole whenever $u-v \in \mathcal{L}$.

By multiplying the summand (32) by $u+v$ and then setting $u=-v$ we get

$$\begin{aligned} & [(\text{id} \otimes \eta \otimes \text{id}) p, r_{13}(-v, w) + r_{23}(v, w)] \\ &= (\text{id} \otimes \eta \otimes \text{id}) [p, r_{13}(-v, w) + (\text{id} \otimes \eta \otimes \text{id}) r_{23}(v, w)] \\ &= (\text{id} \otimes \eta \otimes \text{id}) [p, r_{13}(-v, w) + r_{23}(-v, w)] = 0. \end{aligned}$$

So (32) has no pole at $u+v=0$. It follows from (26), (27), (28) that (32) has no pole whenever $u+v \in \mathcal{L}$.

Thus the function $f(u, v, w)$ has no pole whenever $u \pm v \in \mathcal{L}$. Further, by using the antisymmetry property (1) the function $-f(u, v, w)$ can be written as

$$[r_{13}(u, w), r_{12}(u, v)] + [r_{13}(u, w), r_{32}(w, v)] + [r_{12}(u, v), r_{32}(w, v)].$$

Similarly to the above argument we can show that the latter function has no pole when $u \pm w \in \mathcal{L}$. So $r(u, v)$ is a solution of the classical Yang–Baxter equation.

Following [2, 3, 6, 10] it would be interesting to find a solution of the quantum Yang–Baxter equation corresponding to our $r(u, v)$. For the rational solutions of the classical equation constructed in [11] this was already done in the same work.

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