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Research Paper

On the endomorphism algebra of Specht modules in even characteristic ☆

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ABSTRACT

Over fields of characteristic 2, Specht modules may decompose and there is no upper bound for the dimension of their endomorphism algebra. A classification of the (in)decomposable Specht modules and a closed formula for the dimension of their endomorphism algebra remain two important open problems in the area. In this paper, we introduce a novel description of the endomorphism algebra of the Specht modules and provide infinite families of Specht modules with one-dimensional endomorphism algebra.

1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$ and r a positive integer. We write $\mathfrak{S}_r$ for the symmetric group on r letters and $\Bbbk\mathfrak{S}_r$ for its group algebra over $\Bbbk$. For each partition λ of r we have the Specht module $\mathrm{Sp}(\lambda)$ and for each composition

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α of r we have the permutation module $M(\alpha)$. Recall that $\mathrm{Sp}(\lambda)$ may be viewed as a submodule of $M(\lambda)$. One fundamental result by James states that unless the characteristic of $\Bbbk$ is 2 and λ is 2-singular, the space of homomorphisms $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda), M(\lambda))$ is one-dimensional[11, Corollary 13.17]. It follows that the endomorphism algebra of $\mathrm{Sp}(\lambda)$ is one-dimensional and so in particular that $\mathrm{Sp}(\lambda)$ is indecomposable.

In contrast, if the characteristic of $\Bbbk$ is 2 and λ is a 2-singular partition, that is λ has a repeated term, $\mathrm{Sp}(\lambda)$ may certainly decompose. The first example of a decomposable Specht module was discovered by James in the late 70s, thereby setting in motion the investigation of the (in)decomposability of Specht modules; a problem that has attracted a lot of attention over the years [14], [4], [8], [2]. In a recent paper [8], Donkin and the first author considered partitions of the form $\lambda = (a, m-1, m-2, \ldots, 2, 1^b)$ and obtained precise decompositions of $\mathrm{Sp}(\lambda)$ in the case where $a - m$ is even and b is odd. An interesting feature arising in these decompositions is that there is no upper bound for the number of indecomposable summands of $\mathrm{Sp}(\lambda)$ and so in turn for the dimension of its endomorphism algebra [8, Example 6.3]. Almost half a century after James' first example, a classification of the (in)decomposable Specht modules remains to be found and there is no known formula describing the dimension of their endomorphism algebra. In this paper, we provide a new characterisation of $\mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda))$ as a subset of the homomorphism space $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$, where λ' is the transpose partition of λ. Our description allows one to realise an endomorphism of $\mathrm{Sp}(\lambda)$ as an element of the set $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ that satisfies certain concrete relations. In this way, we are able to show that for $\lambda = (a, m-1, \ldots, 2, 1^b)$ with $a - m \equiv b \pmod 2$, the endomorphism algebra of $\mathrm{Sp}(\lambda)$ is one-dimensional.

We do so by taking inspiration from the category of polynomial representations of the general linear groups. More precisely, for a partition λ, we compare two different constructions of the induced module $\nabla(\lambda)$ for $\mathrm{GL}_n(\Bbbk)$: the first introduced by Akin, Buchsbaum, and Weyman [1, Theorem II.2.11] and the second by James[11, Theorem 26.3(ii)]. By applying the Schur functor [10, §6.3], we then obtain two characterisations of the Specht module $\mathrm{Sp}(\lambda)$: first as a quotient of $M(\lambda')$ and then as a submodule of $M(\lambda)$. This leads to a concrete description of the endomorphism algebra of $\mathrm{Sp}(\lambda)$, which we shall then investigate in detail for partitions of the form $\lambda = (a, m-1, \ldots, 2, 1^b)$.

The paper is arranged in the following way. Section 2 provides the necessary background on polynomial representations of $\mathrm{GL}_n(\Bbbk)$ and $\Bbbk\mathfrak{S}_r$-modules. In Section 3 we explore the connection between these two categories via the Schur functor f and its right-inverse g. As a by-product of our considerations, we provide a new short proof of the fact that $g\mathrm{Sp}(\lambda) \cong \nabla(\lambda)$ for $p \neq 2$. Then, we focus on homomorphisms and in Lemma 3.3 we obtain the desired description of $\mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda))$ in characteristic 2. In Section 4 we utilise more tools from the representation theory of $\mathrm{GL}_n(\Bbbk)$ to obtain a reduction technique that will be instrumental to our investigation of the case $\lambda = (a, m-1, \ldots, 2, 1^b)$ in Section 5.

2. Preliminaries

We write $\mathbb{N}$ for the set of non-negative integers.

2.1. Combinatorics

Let ℓ be a positive integer and $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be an ℓ-tuple of non-negative integers. We let $\deg(\alpha) \coloneqq \alpha_1 + \cdots + \alpha_\ell$ and call it the *degree* of α. We define the *length* of α, denoted $\ell(\alpha)$, to be the maximal positive integer l with $1 \leq l \leq \ell$ such that $\alpha_l \neq 0$ if α is non-zero, and we set $\ell(\alpha) \coloneqq 0$ for $\alpha = (0^\ell)$. Now, fix positive integers n and r. We write $\Lambda(n)$ for the set of n-tuples of non-negative integers, and $\Lambda^+(n)$ for the set of partitions with at most n parts. We write $\Lambda(n, r)$ for the subset of $\Lambda(n)$ consisting of those elements of degree r, and $\Lambda^+(n, r)$ for the partitions of r with at most n parts. Given a partition $\lambda \in \Lambda^+(n)$, we write λ' for its transpose partition. For $\alpha \in \Lambda(n)$ and $1 \leq i < j \leq \ell(\alpha)$ with $\alpha_j \neq 0$, and for $0 < k \leq \alpha_j$, we shall denote by $\alpha^{(i,j,k)} = (\alpha_1^{(i,j,k)}, \alpha_2^{(i,j,k)}, \ldots)$ the element of $\Lambda(n)$ with terms $\alpha_l^{(i,j,k)} \coloneqq \alpha_l + k(\delta_{i,l} - \delta_{j,l})$.

2.2. Representations of general linear groups

We consider the general linear group $G \coloneqq \mathrm{GL}_n(\Bbbk)$ and its coordinate algebra $\Bbbk[G] = \Bbbk[c_{11}, \ldots, c_{nn}, \det^{-1}]$, where det is the determinant function. We write $A_\Bbbk(n) \coloneqq \Bbbk[c_{11}, \ldots, c_{nn}]$ for the polynomial subalgebra of $\Bbbk[G]$ generated by the functions c_{ij} with $1 \leq i, j \leq n$. The algebra $A_\Bbbk(n)$ has an $\mathbb{N}$-grading of the form $A_\Bbbk(n) = \bigoplus_{r\in\mathbb{N}} A_\Bbbk(n, r)$ where $A_\Bbbk(n, r)$ consists of the homogeneous degree r polynomials in the c_{ij}. Given a rational G-module V, we shall denote by $\mathrm{cf}(V)$ the *coefficient space* of V, that is the subspace of $\Bbbk[G]$ generated by the *coefficient functions* $f_{vv'} : G \to \Bbbk$ satisfying $g \cdot v' = \sum_{v\in\mathcal{V}} f_{vv'}(g)v$ for $g \in G$, $v, v' \in \mathcal{V}$, where $\mathcal{V}$ is some $\Bbbk$-basis of V. We say that V is a *polynomial representation* of G if $\mathrm{cf}(V) \subseteq A_\Bbbk(n)$ and a *polynomial representation of G of degree r* if $\mathrm{cf}(V) \subseteq A_\Bbbk(n, r)$. We write $M_\Bbbk(n)$ for the category of polynomial representations of G and $M_\Bbbk(n, r)$ for its subcategory of representations of degree r. Recall that the category $M_\Bbbk(n, r)$ is naturally equivalent to the category of $S_\Bbbk(n, r)$-modules, where $S_\Bbbk(n, r) \coloneqq A_\Bbbk(n, r)^*$ is the corresponding *Schur algebra*[10, §2.3, §2.4]. For $V \in M_\Bbbk(n)$ we write V° for its contravariant dual, in the sense of [10, §2.7].

We fix T to be the maximal torus of G consisting of the diagonal matrices in G. An element $\alpha \in \Lambda(n)$ may be identified with the multiplicative character of T that takes an element $t = \mathrm{diag}(t_1, \ldots, t_n) \in T$ to $\alpha(t) \coloneqq t_1^{\alpha_1} \cdots t_n^{\alpha_n} \in \Bbbk$. We denote by $\Bbbk_\alpha$ the one-dimensional rational T-module on which $t \in T$ acts by multiplication by $\alpha(t)$. Then, given $V \in M_\Bbbk(n)$, $\alpha \in \Lambda(n)$, we write $V^\alpha \coloneqq \{v \in V \mid t \cdot v = \alpha(t)v \text{ for all } t \in T\}$ for the *α-weight space* of V. We write $E \coloneqq \Bbbk^{\oplus n}$ for the natural G-module and $S^r E$ (resp. $\Lambda^r E$, $D^r E$) for the corresponding rth-symmetric power (resp. exterior power, divided power) of E. For $\ell \geq 1$ and an ℓ-tuple $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of non-negative integers, we define the polynomial G-modules: $S^\alpha E \coloneqq S^{\alpha_1} E \otimes \cdots \otimes S^{\alpha_\ell} E$, $\Lambda^\alpha E \coloneqq \Lambda^{\alpha_1} E \otimes \cdots \otimes \Lambda^{\alpha_\ell} E$, and

$D^\alpha E := D^{\alpha_1}E \otimes \cdots \otimes D^{\alpha_\ell}E$. If $\deg(\alpha) = r$, then each of these modules lies in $M_{\Bbbk}(n,r)$. For $V \in M_{\Bbbk}(n)$, there is a $\Bbbk$-linear isomorphism $\mathrm{Hom}_G(V, S^\alpha E) \cong V^\alpha$[6, §2.1(8)]. For $\alpha \in \Lambda(n)$, the T-action on $\Bbbk_\alpha$ extends uniquely to a module action of the subgroup $B \subseteq G$ of lower-triangular matrices. For $\lambda \in \Lambda^+(n)$, we write $\nabla(\lambda) := \mathrm{ind}_B^G \Bbbk_\lambda$ for the *induced G-module* corresponding to λ[12, §II.2]. Recall that there is a G-isomorphism $\nabla(\lambda)^\circ \cong \Delta(\lambda)$, where $\Delta(\lambda)$ is the *Weyl module* corresponding to λ[12, §II.2.13(1)].

Here, we shall review a construction of the induced module by Akin, Buchsbaum, and Weyman. In [1, §II.1], the authors associate to a partition λ with $\lambda_1 \le n$, a G-module denoted $L_\lambda(E)$, which they call the *Schur functor of E*. Further, in [1, §II.2] the authors provide a description of $L_\lambda(E)$ by generators and relations. More precisely, in [1, Theorem II.2.16], the authors identify $L_\lambda(E)$ with the cokernel of a G-homomorphism between a pair of (direct sums of) tensor products of exterior powers of E. By [7, §2.7(5)], we have that $L_\lambda(E)$ is isomorphic to an induced module, namely $L_\lambda(E) \cong \nabla(\lambda')$ for $\lambda \in \Lambda^+(n)$ (note that $Y(\lambda)$ is used in place of $\nabla(\lambda)$ in [7]). The construction as a cokernel by Akin, Buchsbaum, and Weyman is as follows. Recall that the exterior algebra $\Lambda(E)$ of E enjoys a Hopf algebra structure [1, §I.2]. We write Δ and μ for the comultiplication and multiplication of $\Lambda(E)$ respectively. Let λ be a partition with $\ell := \ell(\lambda)$. For $1 \le i < \ell$, $1 < j \le \ell$, $t \ge 1$, and $1 \le s \le \lambda_j$, we consider the G-homomorphisms $\Delta_\lambda^{(i,t)} : \Lambda^{\lambda_i + t}E \to \Lambda^{\lambda_i}E \otimes \Lambda^t E$ and $\mu_\lambda^{(j,s)} : \Lambda^s E \otimes \Lambda^{\lambda_j - s}E \to \Lambda^{\lambda_j}E$, coming from Δ and μ respectively. Further, for $1 \le i < j \le \ell$, $1 \le s \le \lambda_j$ we construct the G-homomorphism $\phi_\lambda^{(i,j,s)} : \Lambda^{\lambda^{(i,j,s)}}E \to \Lambda^\lambda E$ as the composition:

$$\Lambda^{\lambda^{(i,j,s)}}E \xrightarrow{1\otimes\cdots\otimes\Delta_\lambda^{(i,s)}\otimes\cdots\otimes 1} \Lambda^{\lambda_1}E \otimes \cdots \otimes \Lambda^{\lambda_i}E \otimes \Lambda^s E \otimes \cdots \otimes \Lambda^{\lambda_j - s}E \otimes \cdots \otimes \Lambda^{\lambda_\ell}E$$

$$\xrightarrow{\sigma} \Lambda^{\lambda_1}E \otimes \cdots \otimes \Lambda^{\lambda_i}E \otimes \cdots \otimes \Lambda^s E \otimes \Lambda^{\lambda_j - s}E \otimes \cdots \otimes \Lambda^{\lambda_\ell}E \xrightarrow{1\otimes\cdots\otimes\mu_\lambda^{(j,s)}\otimes\cdots\otimes 1} \Lambda^\lambda E, \tag{2.1}$$

where σ denotes the isomorphism that permutes the corresponding tensor factors, and each 1 refers to the identity map on the corresponding tensor factor. Now, set:

$$\phi_\lambda^{(i,i+1)} := \sum_{s=1}^{\lambda_{i+1}} \phi_\lambda^{(i,i+1,s)} : \sum_{s=1}^{\lambda_{i+1}} \Lambda^{\lambda^{(i,i+1,s)}}E \to \Lambda^\lambda E, \tag{2.2}$$

$$\phi_\lambda := \sum_{i=1}^{\ell-1} \phi_\lambda^{(i,i+1)} : \sum_{i=1}^{\ell-1}\sum_{s=1}^{\lambda_{i+1}} \Lambda^{\lambda^{(i,i+1,s)}}E \to \Lambda^\lambda E. \tag{2.3}$$

For $\lambda \in \Lambda^+(n)$, we have that $\operatorname{coker}\phi_{\lambda'} \cong L_{\lambda'}(E)$ [1, Theorem II.2.16], and hence $\operatorname{coker}\phi_{\lambda'} \cong \nabla(\lambda)$ [7, §2.7(5)]. We shall refer to this description as the *ABW-construction* of $\nabla(\lambda)$.

Now, we review an alternative description of $\nabla(\lambda)$ due to James [11, §26]. Although James refers to this module as the "Weyl module", it is not to be confused with the usual Weyl module $\Delta(\lambda)$ that we discussed above [10, Theorem (4.8f)]. James' construction is as follows. Recall that the symmetric algebra $S(E)$ of E also has a Hopf algebra structure

[1, §I.2]. As a slight abuse of notation, we shall once again use the symbols Δ and μ for the corresponding comultiplication and multiplication of $S(E)$ respectively. Let λ be a partition with $\ell := \ell(\lambda)$. For $1 \le i < \ell$, $1 < j \le \ell$, $1 \le t \le \lambda_j$, and $s \ge 1$, we consider the G-homomorphisms $\Delta_\lambda^{(j,t)} : S^{\lambda_j}E \to S^tE \otimes S^{\lambda_j - t}E$ and $\mu_\lambda^{(i,s)} : S^{\lambda_i}E \otimes S^sE \to S^{\lambda_i+s}E$ coming from Δ and μ respectively. Further, for $1 \le i < j \le \ell$, $1 \le t \le \lambda_j$, we construct the G-homomorphism $\psi_\lambda^{(i,j,t)} : S^\lambda E \to S^{\lambda^{(i,j,t)}}E$ as the composition:

$$S^\lambda E \xrightarrow{1\otimes\cdots\otimes\Delta_\lambda^{(j,t)}\otimes\cdots\otimes 1} S^{\lambda_1}E \otimes \cdots \otimes S^{\lambda_i}E \otimes \cdots \otimes S^tE \otimes S^{\lambda_j - t}E \otimes \cdots \otimes S^{\lambda_\ell}E \xrightarrow{\bar\sigma}$$
$$S^{\lambda_1}E \otimes \cdots \otimes S^{\lambda_i}E \otimes S^tE \otimes \cdots \otimes S^{\lambda_j-t}E \otimes \cdots \otimes S^{\lambda_\ell}E \xrightarrow{1\otimes\cdots\otimes\mu_\lambda^{(i,t)}\otimes\cdots\otimes 1} S^{\lambda^{(i,j,t)}}E, \tag{2.4}$$

where $\bar\sigma$ denotes the isomorphism that permutes the corresponding tensor factors, and each 1 refers to the identity map on the corresponding tensor factor. Now, set:

$$\psi_\lambda^{(i,i+1)} := \sum_{t=1}^{\lambda_{i+1}} \psi_\lambda^{(i,i+1,t)} : S^\lambda E \to \sum_{t=1}^{\lambda_{i+1}} S^{\lambda^{(i,i+1,t)}}E, \tag{2.5}$$

$$\psi_\lambda := \sum_{i=1}^{\ell-1} \psi_\lambda^{(i,i+1)} : S^\lambda E \to \sum_{i=1}^{\ell-1}\sum_{t=1}^{\lambda_{i+1}} S^{\lambda^{(i,i+1,t)}}E. \tag{2.6}$$

For $\lambda \in \Lambda^+(n)$, we have that $\nabla(\lambda) \cong \ker \psi_\lambda$ [11, Theorem 26.5]. We shall refer to this description as the *James-construction* of $\nabla(\lambda)$.

It is important to point out that the James-construction of $\nabla(\lambda)$ may be derived from Akin, Buchsbaum, and Weyman's construction of the Weyl module $\Delta(\lambda)$ via contravariant duality [1, §II.3]. Similarly to (2.1), (2.2), and (2.3), one may define a G-homomorphism $\theta_\lambda^{(i,j,t)} : D^{\lambda^{(i,j,t)}}E \to D^\lambda E$ for $1 \le i < j \le \ell$, $1 \le t \le \lambda_j$, and then construct the G-homomorphism:

$$\theta_\lambda := \sum_{i=1}^{\ell-1}\sum_{t=1}^{\lambda_{i+1}} \theta_\lambda^{(i,i+1,t)} : \sum_{i=1}^{\ell-1}\sum_{t=1}^{\lambda_{i+1}} D^{\lambda^{(i,i+1,t)}}E \to D^\lambda E. \tag{2.7}$$

For $\lambda \in \Lambda^+(n)$, we have that $\Delta(\lambda) \cong \operatorname{coker}\theta_\lambda$ [1, Theorem II.3.16]. Now, recall that $\Delta(\lambda)^\circ \cong \nabla(\lambda)$ and that $(D^\alpha E)^\circ \cong S^\alpha E$ for $\alpha \in \Lambda(n)$. By taking contravariant duals, it follows that $\nabla(\lambda) \cong \ker {\theta_\lambda}^\circ$ and it is easy to check that we have the identifications ${\theta_\lambda}^\circ = \psi_\lambda$ and $\theta_\lambda^{(i,j,t)\circ} = \psi_\lambda^{(i,j,t)}$ for $1 \le i < j \le \ell$, $1 \le t \le \lambda_j$.

2.3. Connections with the symmetric groups

Recall that for a partition λ of r, we have the Specht module $\mathrm{Sp}(\lambda)$ for $\Bbbk\mathfrak{S}_r$. For $\lambda = (1^r)$, we have that $\mathrm{Sp}(1^r)$ is the sign representation $\mathtt{sgn}_r$ of $\Bbbk\mathfrak{S}_r$. We fix $n \ge r$, and we consider the *Schur functor* $f : M_\Bbbk(n,r) \to \Bbbk\mathfrak{S}_r$-mod, where $fV := V^{(1^r)}$ for $V \in M_\Bbbk(n,r)$ [10, §6.1, §6.3]. For $\lambda \in \Lambda^+(n,r)$ we have the isomorphism $f\nabla(\lambda) \cong \mathrm{Sp}(\lambda)$

[10, (6.3c)], and for $\alpha \in \Lambda(n,r)$ we have the $\Bbbk\mathfrak{S}_r$-isomorphisms $fS^\alpha E \cong M(\alpha)$ and $f\Lambda^\alpha E \cong M(\alpha) \otimes \mathtt{sgn}_r =: M_s(\alpha)$, where $M_s(\alpha)$ denotes the *signed permutation module* corresponding to α [5, Lemma 3.4]. We set $\ell := \ell(\lambda)$. By applying the Schur functor to the maps ϕ_λ and ψ_λ from (2.3) and (2.6) respectively, we obtain the $\Bbbk\mathfrak{S}_r$-homomorphisms:

$$\bar{\phi}_\lambda := f(\phi_\lambda) : \bigoplus_{i=1}^{\ell-1} \bigoplus_{s=1}^{\lambda_{i+1}} M_s(\lambda^{(i,i+1,s)}) \to M_s(\lambda), \tag{2.8}$$

$$\bar{\psi}_\lambda := f(\psi_\lambda) : M(\lambda) \to \bigoplus_{i=1}^{\ell-1} \bigoplus_{t=1}^{\lambda_{i+1}} M(\lambda^{(i,i+1,t)}). \tag{2.9}$$

As a consequence of the exactness of the Schur functor f, it follows that $\mathrm{Sp}(\lambda) \cong \operatorname{coker} \bar{\phi}_{\lambda'}$ and $\mathrm{Sp}(\lambda) \cong \ker \bar{\psi}_\lambda$. This second isomorphism is an alternative realisation of James' Kernel Intersection Theorem [11, Corollary 17.18]. These two descriptions of the Specht module $\mathrm{Sp}(\lambda)$ will be crucial for our considerations in this paper.

We set $S := S_\Bbbk(n,r)$ for the Schur algebra. The group algebra $\Bbbk\mathfrak{S}_r$ may be identified with the algebra eSe for a certain idempotent e of S [10, (6.3)]. Accordingly, the Schur functor f may be identified with the functor $f : S\text{-mod} \to \Bbbk\mathfrak{S}_r\text{-mod}$ with $fV = eV$ [10, §6.2, §6.3]. Now, the Schur functor f has a partial inverse $g : \Bbbk\mathfrak{S}_r\text{-mod} \to S\text{-mod}$ with $gW := Se \otimes_{eSe} W$ for $W \in \Bbbk\mathfrak{S}_r$-mod [10, (6.2c)]. This functor is a right-inverse of f and it is right-exact. Moreover, it is easy to see that g is left-adjoint to f, and so for $V \in M_\Bbbk(n,r)$ and $W \in \Bbbk\mathfrak{S}_r$-mod, there is a $\Bbbk$-linear isomorphism $\mathrm{Hom}_G(gW, V) \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(W, fV)$. For $\alpha \in \Lambda(n,r)$ one has that $gM(\alpha) \cong S^\alpha E$ [9, Appendix A], and for $\lambda \in \Lambda^+(n,r)$ and $p \neq 2$ one has that $g\mathrm{Sp}(\lambda) \cong \nabla(\lambda)$ [5, Proposition 10.6(i)], [13, Theorem 1.1]. Further results related to the properties of g will be proved in Section 3, including a new short proof of the fact that $g\mathrm{Sp}(\lambda) \cong \nabla(\lambda)$ for $p \neq 2$.

3. Endomorphism algebras

3.1. General Results

From now on we fix $n \geq r$. Note that for $\lambda \in \Lambda^+(n,r)$ we have that $\lambda' \in \Lambda^+(n,r)$. First, in Proposition 3.1(i), we point out a new property of the functor g, which we immediately apply in Proposition 3.1(ii) to obtain a new short proof of the fact that $g\mathrm{Sp}(\lambda) \cong \nabla(\lambda)$ when $p \neq 2$. Then, we utilise the two different descriptions of the Specht module $\mathrm{Sp}(\lambda)$ to introduce a new description of its endomorphism algebra.

Proposition 3.1. *Assume that $p \neq 2$. Then:*

(i) *For $\alpha \in \Lambda(n,r)$, we have $gM_s(\alpha) \cong \Lambda^\alpha E$.*
(ii) *For $\lambda \in \Lambda^+(n,r)$, we have $g\mathrm{Sp}(\lambda) \cong \nabla(\lambda)$.*

Proof. (i) Recall that for $\beta \in \Lambda(n,r)$ and $V \in M_{\Bbbk}(n,r)$, we have a $\Bbbk$-isomorphism $\mathrm{Hom}_G(V, S^\beta E) \cong V^\beta$, and so in particular $\dim V^\beta = \dim \mathrm{Hom}_G(V, S^\beta E)$. Moreover, $fS^\alpha E \cong M(\alpha)$ and so it follows that:

$$\mathrm{Hom}_G(gM_s(\alpha), S^\beta E) \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M_s(\alpha), fS^\beta E) \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M_s(\alpha), M(\beta)).$$

Now, since $p \neq 2$, the dimension of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M_s(\alpha), M(\beta))$ does not depend on the value of p [3, Theorem 3.3(ii)], and so in order to calculate the dimension of $gM_s(\alpha)^\beta$, we may assume that $p = 0$. However, in characteristic 0, the functors f and g are inverse equivalences of categories and so $gM_s(\alpha) \cong \Lambda^\alpha E$. Therefore, for $p \neq 2$, we deduce that $\dim gM_s(\alpha)^\beta = \dim \Lambda^\alpha E^\beta$ for all $\beta \in \Lambda(n,r)$. Now, recall that for $V \in M_{\Bbbk}(n,r)$, we have the weight space decomposition $V = \bigoplus_{\beta \in \Lambda(n,r)} V^\beta$ [10, (3.2c)], and so it follows that, for $p \neq 2$, we have $\dim gM_s(\alpha) = \dim \Lambda^\alpha E$.

Now, we have that $M(1^r) \cong eSe$ and so $gM(1^r) \cong Se \otimes_{eSe} eSe \cong Se \cong E^{\otimes r}$ [10, (6.4f)]. For $\alpha \in \Lambda(n,r)$ we have a surjective G-homomorphism $E^{\otimes r} \to \Lambda^\alpha E$ and so via the Schur functor, we get a surjective $\Bbbk\mathfrak{S}_r$-homomorphism $M(1^r) \to M_s(\alpha)$. The functor g, being right-exact, preserves surjections, and so the G-homomorphism $gM(1^r) \to gM_s(\alpha)$ is surjective. We consider the commutative diagram:

$$\begin{array}{ccc} gM(1^r) & \xrightarrow{\cong} & E^{\otimes r} \\ \downarrow & & \downarrow \\ gM_s(\alpha) & \longrightarrow & \Lambda^\alpha E \end{array},$$

where the horizontal maps are induced from the $\Bbbk\mathfrak{S}_r$-inclusions $M(1^r) \cong fE^{\otimes r} \to E^{\otimes r}$ and $M_s(\alpha) \cong f\Lambda^\alpha E \to \Lambda^\alpha E$. The top horizontal map is an isomorphism and the right-hand vertical map is surjective, and so the bottom horizontal map is hence surjective. Since $\dim gM_s(\alpha) = \dim \Lambda^\alpha E$ away from characteristic 2, we obtain $gM_s(\alpha) \cong \Lambda^\alpha E$ for $p \neq 2$.

(ii) Recall that $\nabla(\lambda) \cong \operatorname{coker} \phi_{\lambda'}$, where $\phi_{\lambda'} : K(\lambda') \to \Lambda^{\lambda'} E$ and $K(\lambda')$ is the direct sum of tensor products of exterior powers given in (2.3), where here we replace the partition λ with λ'. By applying the Schur functor f to $\phi_{\lambda'}$, we obtain the $\Bbbk\mathfrak{S}_r$-homomorphism $\bar{\phi}_{\lambda'} : \bar{K}(\lambda') \to M_s(\lambda')$, where $\bar{K}(\lambda')$ is the direct sum of signed permutation modules given in (2.6), again substituting λ with λ'. Also, recall that $\mathrm{Sp}(\lambda) \cong \operatorname{coker} \bar{\phi}_{\lambda'}$. By part (i), we have that $gM_s(\lambda') \cong \Lambda^{\lambda'} E$ and so $g\bar{K}(\lambda') \cong K(\lambda')$. Hence, we obtain the commutative diagram:

$$\begin{array}{ccc} g\bar{K}(\lambda') & \xrightarrow{g(\bar{\phi}_{\lambda'})} & gM_s(\lambda') \\ \cong\downarrow & & \downarrow\cong \\ K(\lambda') & \xrightarrow{\phi_{\lambda'}} & \Lambda^{\lambda'} E \end{array}.$$

The image of $g(\bar{\phi}_{\lambda'})$ is mapped isomorphically onto the image of $\phi_{\lambda'}$, and so in particular $\operatorname{coker}\phi_{\lambda'} \cong \operatorname{coker} g(\bar{\phi}_{\lambda'})$. Finally, g preserves cokernels since it is right-exact, and so we deduce that $\nabla(\lambda) \cong \operatorname{coker}\phi_{\lambda'} \cong \operatorname{coker} g(\bar{\phi}_{\lambda'}) \cong g \operatorname{coker}\bar{\phi}_{\lambda'} \cong g\mathrm{Sp}(\lambda)$. □

Lemma 3.2. *Let $\alpha, \beta \in \Lambda(n,r)$. Then:*

(i) $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \cong \mathrm{Hom}_G(S^\alpha E, S^\beta E) \cong (S^\alpha E)^\beta$.
(ii) *For $p \neq 2$, we have* $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M_s(\alpha), M(\beta)) \cong \mathrm{Hom}_G(\Lambda^\alpha E, S^\beta E) \cong (\Lambda^\alpha E)^\beta$.

Proof. Recall that for $V \in M_\Bbbk(n,r)$ and $W \in \Bbbk\mathfrak{S}_r$-mod, we have a $\Bbbk$-isomorphism of the form $\mathrm{Hom}_G(gW, V) \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(W, fV)$. Parts (i)-(ii) then both follow from our comments in §2.2, §2.3, and Proposition 3.1(i). □

Lemma 3.3. *Let $\lambda \in \Lambda^+(n,r)$. Then:*

(i) *There is a $\Bbbk$-isomorphism:*

$$\mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda)) \cong \{h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M_s(\lambda'), M(\lambda)) \mid h \circ \bar{\phi}_{\lambda'} = 0 \text{ and } \bar{\psi}_\lambda \circ h = 0\}.$$

(ii) *In particular, when $p = 2$, there is a $\Bbbk$-isomorphism:*

$$\mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda)) \cong \{h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda)) \mid h \circ \bar{\phi}_{\lambda'} = 0 \text{ and } \bar{\psi}_\lambda \circ h = 0\}.$$

Proof. Part (i) follows immediately from the two descriptions of the Specht module: $\mathrm{Sp}(\lambda) \cong \operatorname{coker}\bar{\phi}_{\lambda'}$ and $\mathrm{Sp}(\lambda) \cong \ker\bar{\psi}_\lambda$ from §2.3. Part (ii) then follows from part (i) and the fact that the permutation module and the signed permutation module coincide in characteristic 2. □

Recall the G-homomorphisms $\phi_\lambda^{(i,j,s)}$ and $\psi_\lambda^{(i,j,t)}$ from (2.1) and (2.4) respectively.

Lemma 3.4. *Let $\lambda \in \Lambda^+(n)$ with $\ell := \ell(\lambda)$. Then:*

(i) $\operatorname{im}\phi_\lambda^{(i,j,s)} \subseteq \operatorname{im}\phi_\lambda$ *for $1 \leq i < j \leq \ell$, $1 \leq s \leq \lambda_j$.*
(ii) $\ker\psi_\lambda \subseteq \ker\psi_\lambda^{(i,j,t)}$ *for $1 \leq i < j \leq \ell$, $1 \leq t \leq \lambda_j$.*

Proof. For part (i), from [1, Theorem II.2.16], we have that $\operatorname{im}\phi_\lambda = \ker d_\lambda$, where the map $d_\lambda : \Lambda^\lambda E \to S^{\lambda'} E$ is a G-homomorphism that arises as a composition of (tensor products of) comultiplications between exterior powers and (tensor products of) multiplications between symmetric powers [1, Definition II.1.3]. Now, from [1, Lemma II.2.3], we have that for each $1 \leq i < \ell$, the map d_λ may be factored through the G-homomorphism:

$$\Lambda^\lambda E \xrightarrow{1\otimes\cdots\otimes d_{(\lambda_i,\lambda_{i+1})}\otimes\cdots\otimes 1} \Lambda^{\lambda_1} E \otimes \cdots \otimes \Lambda^{\lambda_{i-1}} E \otimes (S^2 E)^{\otimes \lambda_{i+1}}$$

$$\otimes E^{\otimes(\lambda_i-\lambda_{i+1})} \otimes \Lambda^{\lambda_{i+2}} E \otimes \cdots \otimes \Lambda^{\lambda_\ell} E,$$

where $d_{(\lambda_i,\lambda_{i+1})}$ is the corresponding map associated to the partition $(\lambda_i, \lambda_{i+1})$, and each 1 refers to the identity map on the corresponding tensor factor. Now, it is clear that one may replace $i+1$ with any $j > i$ in the statement of [1, Lemma II.2.3] without any harm. Then, part (i) follows by applying [1, Theorem II.2.16] for the partition (λ_i, λ_j).

For part (ii), we use the ABW-construction of the Weyl module $\Delta(\lambda)$ (2.7). Similarly to part (i), from [1, Theorem II.3.16] and the comment before [1, Definition II.3.4], we deduce that $\operatorname{im}\theta_\lambda^{(i,j,t)} \subseteq \operatorname{im}\theta_\lambda$ for $1 \le i < j \le \ell$ and $1 \le t \le \lambda_j$. Taking contravariant duals, we have that $\ker\theta_\lambda{}^\circ \subseteq \ker\theta_\lambda^{(i,j,t)\circ}$ for all such i, j, t. The result follows by recalling the identifications $\theta_\lambda{}^\circ = \psi_\lambda$ and $\theta_\lambda^{(i,j,t)\circ} = \psi_\lambda^{(i,j,t)}$ from §2.2. □

Let $\lambda \in \Lambda^+(n,r)$. By applying the Schur functor f to the maps $\phi_\lambda^{(i,j,s)}$ and $\psi_\lambda^{(i,j,t)}$ of (2.1) and (2.4) respectively, we obtain the $\Bbbk\mathfrak{S}_r$-homomorphisms:

$$\bar{\phi}_\lambda^{(i,j,s)} : M_s(\lambda^{(i,j,s)}) \to M_s(\lambda), \qquad \bar{\psi}_\lambda^{(i,j,t)} : M(\lambda) \to M(\lambda^{(i,j,t)}).$$

Remark 3.5. We may view any partition $\lambda \in \Lambda^+(n,r)$ as an n-tuple by appending an appropriate number of zeros to λ. Accordingly, we may relax the dependence on $\ell(\lambda)$ of the maps $\bar{\phi}_\lambda$ and $\bar{\psi}_\lambda$. We do so by setting $\bar{\phi}_\lambda^{(i,j,s)} := 0$ and $\bar{\psi}_\lambda^{(i,j,t)} := 0$ if $\ell(\lambda) < j \le n$.

By Lemma 3.3(ii) and Lemma 3.4, we obtain the following Corollary:

Corollary 3.6. *Assume that* char $\Bbbk = 2$ *and let* $\lambda \in \Lambda^+(n,r)$*. Then the endomorphism algebra of* $\mathrm{Sp}(\lambda)$ *may be identified with the* $\Bbbk$*-subspace of* $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ *consisting of those elements* h *that satisfy:*

(i) $h \circ \bar{\phi}_{\lambda'}^{(i,j,s)} = 0$ *for* $1 \le i < j \le n$ *and* $1 \le s \le \lambda'_j$,
(ii) $\bar{\psi}_\lambda^{(i,j,t)} \circ h = 0$ *for* $1 \le i < j \le n$ *and* $1 \le t \le \lambda_j$.

3.2. A concrete description

From now on we shall assume that the underlying field $\Bbbk$ has characteristic 2. We write $[r] := \{1, \dots, r\}$ and as always we assume that $n \ge r$. First, we provide a matrix description of a $\Bbbk$-basis of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ for $\alpha, \beta \in \Lambda(n,r)$, and then we shall utilise this description to obtain some crucial information regarding the endomorphism algebra of $\mathrm{Sp}(\lambda)$.

We write $M_{n\times n}(\mathbb{N})$ for the set of $(n \times n)$-matrices with non-negative integer entries. Let $\{e_i \mid 1 \le i \le n\}$ be the standard basis of column vectors of E. Then, for $\alpha \in \Lambda(n,r)$, we consider the $\Bbbk$-basis $\{e_1^{a_{11}} e_2^{a_{12}} \dots e_n^{a_{1n}} \otimes \cdots \otimes e_1^{a_{n1}} e_2^{a_{n2}} \dots e_n^{a_{nn}} \mid \sum_j a_{ij} = \alpha_i\}$ of $S^\alpha E$, where the ith-tensor factor is defined to be 1 if $\alpha_i = 0$ for some $1 \le i \le n$. We may parametrise this $\Bbbk$-basis by the set of all elements of $M_{n\times n}(\mathbb{N})$ whose sequence of row-sums is equal to α. Accordingly, for $\beta \in \Lambda(n,r)$, the β-weight space $(S^\alpha E)^\beta$ has a

$\Bbbk$-basis parametrised by the set of all matrices in $M_{n\times n}(\mathbb{N})$ whose sequence of row-sums is equal to α, and whose sequence of column-sums is equal to β. On the other hand, the permutation module $M(\alpha)$ has a $\Bbbk$-basis consisting of all ordered sequences of the form $(\boldsymbol{x}_1|\dots|\boldsymbol{x}_n)$, where each $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \dots, x_{i\alpha_i})$ is an unordered sequence with terms from $[r]$, that satisfy the property that for each $k \in [r]$, there is a unique pair (i, j) with $x_{ij} = k$. Here $\boldsymbol{x}_i$ denotes the zero sequence whenever $\alpha_i = 0$.

We set $\mathrm{Tab}(\alpha, \beta) := \{A = (a_{ij})_{i,j} \in M_{n\times n}(\mathbb{N}) \mid \sum_j a_{ij} = \alpha_i, \sum_i a_{ij} = \beta_j\}$. We associate to each $A \in \mathrm{Tab}(\alpha, \beta)$, a homomorphism $\rho[A] \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$. We do so as follows: Given a basis element $\boldsymbol{x} := (\boldsymbol{x}_1|\dots|\boldsymbol{x}_n) \in M(\alpha)$, we set $\rho[A](\boldsymbol{x})$ to be the sum of all basis elements of $M(\beta)$ that may be obtained from $\boldsymbol{x}$ by moving, in concert, a_{ij} entries from its ith-position $\boldsymbol{x}_i$ to its jth-position $\boldsymbol{x}_j$ for every $1 \le i, j \le n$. The set $\{\rho[A] \mid A \in \mathrm{Tab}(\alpha, \beta)\}$ is linearly independent. Indeed, take any linear combination of the $\rho[A]$s, say $h = \sum_A h[A]\rho[A]$ $(h[A] \in \Bbbk)$, along with any basis element $\boldsymbol{x}$ of $M(\alpha)$, and then consider the coefficients of the basis elements of $M(\beta)$ in $h(\boldsymbol{x})$. The linear independence of the $\rho[A]$s along with Lemma 3.2(i) gives that the set $\{\rho[A] \mid A \in \mathrm{Tab}(\alpha, \beta)\}$ forms a $\Bbbk$-basis of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$. Accordingly, for $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ and $A \in \mathrm{Tab}(\alpha, \beta)$, we shall denote by $h[A] \in \Bbbk$ the coefficient of $\rho[A]$ in h so that $h = \sum_{A\in\mathrm{Tab}(\alpha,\beta)} h[A]\rho[A]$.

Examples 3.7. Let $\lambda \in \Lambda^+(n, r)$. For $1 \le i, j \le n$, denote by $E_{ij} \in M_{n\times n}(\mathbb{N})$ the matrix with a 1 in its (i, j)th-position and 0s elsewhere. Notice that:

(i) $\bar{\phi}_\lambda^{(i,j,s)} = \rho[A]$, where $A := \mathrm{diag}(\lambda_1, \dots, \lambda_i, \dots, \lambda_j - s, \dots, \lambda_n) + sE_{ij}$.
(ii) $\bar{\psi}_\lambda^{(i,j,t)} = \rho[B]$, where $B := \mathrm{diag}(\lambda_1, \dots, \lambda_i, \dots, \lambda_j - t, \dots, \lambda_n) + tE_{ji}$.

Remark 3.8. Consider the $\Bbbk$-basis $\{\rho[A] \mid A \in \mathrm{Tab}(\alpha, \beta)\}$ of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$. For $A \in M_{n\times n}(\mathbb{N})$, we write $A' \in M_{n\times n}(\mathbb{N})$ for the transpose matrix of A. If $A \in \mathrm{Tab}(\alpha, \beta)$, then it is clear that $A' \in \mathrm{Tab}(\beta, \alpha)$. Moreover, the set $\{\rho[A'] \mid A \in \mathrm{Tab}(\alpha, \beta)\}$ forms a $\Bbbk$-basis of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$.

Now, for $\alpha \in \Lambda(n, r)$, recall that the permutation module $M(\alpha)$ is self-dual. We write $\delta_\alpha : M(\alpha) \to M(\alpha)^*$ for the $\Bbbk\mathfrak{S}_r$-isomorphism that sends each basis element $\boldsymbol{x}$ of $M(\alpha)$ to the corresponding basis element of $M(\alpha)^*$ dual to $\boldsymbol{x}$. We shall denote by $\zeta_{\alpha,\beta} : \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \to \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta)^*, M(\alpha)^*)$ the natural $\Bbbk$-isomorphism, and by $\eta_{\alpha,\beta} : \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \to \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$ the $\Bbbk$-isomorphism with $\eta_{\alpha,\beta}(h) = \delta_\alpha^{-1} \circ \zeta_{\alpha,\beta}(h) \circ \delta_\beta$ for $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$.

Lemma 3.9. *Let $\alpha, \beta \in \Lambda(n, r)$. Then $\eta_{\alpha,\beta}(\rho[A]) = \rho[A']$ for all $A \in \mathrm{Tab}(\alpha, \beta)$.*

Proof. This is a simple calculation which we leave to the reader. □

Definition 3.10. For $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$, we shall denote by h' the homomorphism $\eta_{\alpha,\beta}(h) \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$ and call it the *transpose homomorphism of h*.

Notice that if $h = \sum_{A\in \mathrm{Tab}(\alpha,\beta)} h[A]\rho[A]$, then $h' = \sum_{A\in \mathrm{Tab}(\alpha,\beta)} h[A]\rho[A']$ by Lemma 3.9.

Lemma 3.11. *Let $\alpha,\beta,\gamma \in \Lambda(n,r)$. Then we have the identity $(h_2 \circ h_1)' = h_1' \circ h_2'$ for all $h_1 \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ and $h_2 \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\gamma))$.*

Proof. Since $\zeta_{\alpha,\gamma}(h_2 \circ h_1) = \zeta_{\alpha,\beta}(h_1) \circ \zeta_{\beta,\gamma}(h_2)$, we have:

$$\begin{aligned}(h_2 \circ h_1)' &= \delta_\alpha^{-1} \circ \zeta_{\alpha,\beta}(h_1) \circ \zeta_{\beta,\gamma}(h_2) \circ \delta_\gamma \\ &= (\delta_\alpha^{-1} \circ \zeta_{\alpha,\beta}(h_1) \circ \delta_\beta) \circ (\delta_\beta^{-1} \circ \zeta_{\beta,\gamma}(h_2) \circ \delta_\gamma) = h_1' \circ h_2'. \quad \square\end{aligned}$$

Lemma 3.12. *Let $\lambda \in \Lambda^+(n,r)$ and $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$. Then:*

(i) $(h \circ \bar{\phi}_{\lambda'}^{(i,j,s)})' = \bar{\psi}_{\lambda'}^{(i,j,s)} \circ h'$.
(ii) $(\bar{\psi}_{\lambda}^{(i,j,t)} \circ h)' = h' \circ \bar{\phi}_{\lambda}^{(i,j,t)}$.
(iii) *The map $\eta_{\lambda',\lambda}$ induces a $\Bbbk$-isomorphism $\bar{\eta}_\lambda : \mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda)) \to \mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda'))$.*

Proof. By Lemma 3.9 and the examples in Examples 3.7, it follows that $(\bar{\phi}_{\lambda}^{(i,j,t)})' = \bar{\psi}_{\lambda}^{(i,j,t)}$. Now, parts (i)-(ii) follow directly from Lemma 3.11. For part (iii), notice that Lemma 3.3 gives that any element $\bar{h} \in \mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda))$ may be identified with a homomorphism $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ such that $h \circ \bar{\phi}_{\lambda'}^{(i,i+1,s)} = 0$ for $1 \le i < n$, $1 \le s \le \lambda'_{i+1}$, and also $\bar{\psi}_{\lambda}^{(i,i+1,t)} \circ h = 0$ for $1 \le i < n$, $1 \le t \le \lambda_{i+1}$. By parts (i)-(ii), we deduce that $\bar{\psi}_{\lambda'}^{(i,i+1,s)} \circ h' = 0$ and $h' \circ \bar{\phi}_{\lambda}^{(i,i+1,t)} = 0$ for all such i, s, t and so h' induces an endomorphism of $\mathrm{Sp}(\lambda')$, $\bar{h}'$ say. Therefore, it follows that the map $\eta_{\lambda',\lambda}$ induces a $\Bbbk$-homomorphism $\bar{\eta}_\lambda : \mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda)) \to \mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda'))$ with $\bar{h} \mapsto \bar{h}'$. By applying the same procedure to the map $\eta_{\lambda,\lambda'}$, we see that $\bar{\eta}_\lambda$ is a $\Bbbk$-isomorphism with inverse $\bar{\eta}_{\lambda'}$ as required. $\square$

For $A = (a_{ij})_{i,j} \in M_{n\times n}(\mathbb{Z})$ and $1 \le k, l \le n$, we shall write $A^{(k,l)}$ for the element of $M_{n\times n}(\mathbb{Z})$ with entries given by $a_{ij}^{(k,l)} := a_{ij} + \delta_{(i,j),(k,l)}$, and $A_{(k,l)}$ for the element of $M_{n\times n}(\mathbb{Z})$ with entries given by $a_{(k,l)ij} := a_{ij} - \delta_{(i,j),(k,l)}$. Let $\alpha, \beta \in \Lambda(n,r)$ with $A \in \mathrm{Tab}(\alpha,\beta)$, and let $1 \le i < j \le n$, $1 \le k, l \le n$. Note that $A_{(j,l)}^{(i,l)} \in \mathrm{Tab}(\alpha^{(i,j,1)}, \beta)$ if $a_{jl} \ne 0$, whilst $A_{(k,j)}^{(k,i)} \in \mathrm{Tab}(\alpha, \beta^{(i,j,1)})$ if $a_{kj} \ne 0$.

Henceforth, we denote by $\mathcal{T}_\lambda$ the set $\mathrm{Tab}(\lambda', \lambda)$ for $\lambda \in \Lambda^+(n,r)$.

Lemma 3.13. *Let $\lambda \in \Lambda^+(n,r)$ and $1 \le i < j \le n$. For $A \in \mathcal{T}_\lambda$ we have:*

(i) $\rho[A] \circ \bar{\phi}_{\lambda'}^{(i,j,1)} = \sum_l (a_{il} + 1)\rho[A_{(j,l)}^{(i,l)}]$, *where the sum is over all l such that $a_{jl} \ne 0$.*
(ii) $\bar{\psi}_{\lambda}^{(i,j,1)} \circ \rho[A] = \sum_k (a_{ki} + 1)\rho[A_{(k,j)}^{(k,i)}]$, *where the sum is over all k such that $a_{kj} \ne 0$.*

Proof. We shall only prove part (i) since part (ii) is similar. We may assume that $j \le \ell(\lambda')$. Fix $1 \le i < j \le \ell(\lambda')$, and we denote by $\boldsymbol{x} := (\boldsymbol{x}_1|\dots|\boldsymbol{x}_i|\dots|\boldsymbol{x}_j|\dots|\boldsymbol{x}_n)$ a basis element of $M(\lambda'^{(i,j,1)})$, where $\boldsymbol{x}_i = (x_{i1}, \dots, x_{i(\lambda'_i+1)})$ say. Then $\bar{\phi}^{(i,j,1)}_{\lambda'}(\boldsymbol{x}) = \sum_{k=1}^{\lambda'_i+1} \boldsymbol{x}^k$, where $\boldsymbol{x}^k$ denotes the basis element of $M(\lambda')$ that is obtained from $\boldsymbol{x}$ by omitting the entry x_{ik} from the sequence $\boldsymbol{x}_i$ and placing it in the (unordered) sequence $\boldsymbol{x}_j$. For $1 \le k \le \lambda'_i + 1$, we have $\rho[A](\boldsymbol{x}^k) = \sum_t c_{kt}\boldsymbol{z}[t]$, where the $\boldsymbol{z}[t]$ are the basis elements of $M(\lambda)$ and the c_{kt} are constants with $c_{kt} \in \{0, 1\}$. Then $\rho[A] \circ \bar{\phi}^{(i,j,1)}_{\lambda'}(\boldsymbol{x}) = \sum_t c_t\boldsymbol{z}[t]$ where $c_t \coloneqq \sum_{k=1}^{\lambda'_i+1} c_{kt}$. Now, fix $1 \le k \le \lambda'_i + 1$ and some s with $c_{ks} = 1$. Then, suppose that the entry x_{ik} appears in the lth-position $\boldsymbol{z}[s]_l$ of $\boldsymbol{z}[s]$ and hence $a_{jl} \neq 0$. Note that the sequence $\boldsymbol{z}[s]_l$ contains a_{il} entries from $\{x_{i1}, \dots, x_{i(k-1)}, x_{i(k+1)}, \dots, x_{i(\lambda'_i+1)}\}$. If x_{iv} is such an entry with $v \neq k$, then $c_{vs} = 1$. On the other hand, given $1 \le q \le \lambda'_i + 1$, if x_{iq} does not appear as an entry in $\boldsymbol{z}[s]_l$, then $c_{qs} = 0$. It follows that $c_s = a_{il} + 1$. Meanwhile, given $1 \le l' \le n$, $\boldsymbol{z}[s]$ appears in $\rho\left[A^{(i,l')}_{(j,l')}\right](\boldsymbol{x})$ if and only if $l' = l$, in which case it appears with a coefficient of 1. The result follows. □

Lemma 3.14. *Let $\lambda \in \Lambda^+(n, r)$ and consider a homomorphism $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ with $h = \sum_{A\in\mathcal{T}_\lambda} h[A]\rho[A]$. Then for $1 \le i < j \le n$, we have:*

(i) *$h \circ \bar{\phi}^{(i,j,1)}_{\lambda'} = 0$ if and only if $\sum_l b_{il}h\left[B^{(j,l)}_{(i,l)}\right] = 0$ for all $B \in \mathrm{Tab}(\lambda'^{(i,j,1)}, \lambda)$.*

(ii) *$\bar{\psi}^{(i,j,1)}_{\lambda} \circ h = 0$ if and only if $\sum_k d_{ki}h\left[D^{(k,j)}_{(k,i)}\right] = 0$ for all $D \in \mathrm{Tab}(\lambda', \lambda^{(i,j,1)})$.*

Proof. We shall only prove part (i) since part (ii) is similar. By Lemma 3.13 we have:

$$h \circ \bar{\phi}^{(i,j,1)}_{\lambda'} = \sum_{A\in\mathcal{T}_\lambda} h[A](\rho[A] \circ \bar{\phi}^{(i,j,1)}_{\lambda'}) = \sum_{A\in\mathcal{T}_\lambda} h[A]\left(\sum_l (a_{il} + 1)\rho\left[A^{(i,l)}_{(j,l)}\right]\right)$$

$$= \sum_{A\in\mathcal{T}_\lambda}\sum_l (a_{il} + 1)h[A]\rho\left[A^{(i,l)}_{(j,l)}\right] = \sum_{B\in\mathrm{Tab}(\lambda'^{(i,j,1)},\lambda)} \left(\sum_l b_{il}h\left[B^{(j,l)}_{(i,l)}\right]\right)\rho[B].$$

The result now follows from the linear independence of $\{\rho[B] \mid B \in \mathrm{Tab}(\lambda'^{(i,j,1)}, \lambda)\}$. □

Definition 3.15. Let $\lambda \in \Lambda^+(n, r)$. We say that an element $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ is *relevant* if $h \circ \bar{\phi}^{(i,j,1)}_{\lambda'} = 0$ and $\bar{\psi}^{(i,j,1)}_{\lambda} \circ h = 0$ for all $1 \le i < j \le n$.

Denote by $\mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ the $\Bbbk$-subspace of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ consisting of the relevant homomorphisms $M(\lambda') \to M(\lambda)$. The following Remark is clear:

Remark 3.16. Let $\lambda \in \Lambda^+(n, r)$. Note that there is a $\Bbbk$-embedding of the endomorphism algebra of $\mathrm{Sp}(\lambda)$ into the $\Bbbk$-space $\mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$.

Now, by Lemma 3.14, we deduce the following Corollary:

Corollary 3.17. *Let $\lambda \in \Lambda^+(n, r)$ and $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$. Then we have that $h \in \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ if and only if the coefficients $h[A]$ of the $\rho[A]$ in h satisfy:*

(i) *For all* $1 \leq i < j \leq n$, $1 \leq k \leq n$, *and all* $A \in \mathcal{T}_\lambda$ *with* $a_{jk} \neq 0$, *we have:*

$$(a_{ik}+1)h[A] = \sum_{l \neq k} a_{il} h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right], \qquad (R^k_{i,j}(A))$$

(ii) *For all* $1 \leq i < j \leq n$, $1 \leq k \leq n$, *and all* $A \in \mathcal{T}_\lambda$ *with* $a_{kj} \neq 0$, *we have:*

$$(a_{ki}+1)h[A] = \sum_{l \neq k} a_{li} h\left[A^{(k,i)(l,j)}_{(k,j)(l,i)}\right]. \qquad (C^k_{i,j}(A))$$

4. A reduction trick

4.1. Flattening the partition

Now, we fix integers a, b, m with $a \geq m \geq 2$, and we write $a' := b+m-1$, $b' := a-m+1$. We denote by λ the partition $(a, m-1, \dots, 2, 1^b)$, and we fix $r := \deg(\lambda)$. Note that the transpose partition λ' of λ is given by $\lambda' = (a', m-1, \dots, 2, 1^{b'})$.

Recall that through the ABW-construction of the induced module, we see that $\nabla(\lambda)$ is isomorphic to a G-quotient of $\Lambda^{\lambda'}E = \Lambda^{a'}E \otimes \Lambda^{m-1}E \otimes \cdots \otimes \Lambda^2 E \otimes E^{\otimes b'}$, namely by the submodule $\operatorname{im}\phi_{\lambda'}$ (2.3). We claim that we can replace the factor $E^{\otimes b'}$ with the symmetric power $S^{b'}E$. This process is in fact independent of the characteristic of the field $\Bbbk$. To this end, we construct from the multiplication map $\mu : E^{\otimes b'} \to S^{b'}E$, the surjective G-homomorphism $1 \otimes \mu : \Lambda^{\lambda'}E \to \Lambda^{a'}E \otimes \Lambda^{m-1}E \otimes \cdots \otimes \Lambda^2 E \otimes S^{b'}E$.

Lemma 4.1. *For* $m \geq 2$ *and* $\lambda = (a, m-1, m-2, \dots, 2, 1^b)$, *we have:*

(i) $\ker(1 \otimes \mu) = \sum_{k=1}^{b'-1} \operatorname{im}\phi_{\lambda'}^{(m+k-1,m+k,1)} \subseteq \operatorname{im}\phi_{\lambda'}$.
(ii) $\nabla(\lambda) \cong \operatorname{coker}((1 \otimes \mu) \circ \phi_{\lambda'})$ *as* G*-modules.*

Proof. (i) Firstly, that $\operatorname{im}\phi_{\lambda'}^{(m+k-1,m+k,1)} \subseteq \operatorname{im}\phi_{\lambda'}$ for $1 \leq k < b'$ follows from the definition of $\phi_{\lambda'}$. Then, note that by the definition of the symmetric power $S^{b'}E$, the $\Bbbk$-space $\ker\mu$ is generated by elements of the form $e_{\boldsymbol{i}}^{[k]}$ for $1 \leq k < b'$ and sequences $\boldsymbol{i} := (i_1, \dots, i_{b'})$ with terms in $[n]$, where $e_{\boldsymbol{i}}^{[k]} := (e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e_{i_{k+1}} \otimes \cdots \otimes e_{i_{b'}}) - (e_{i_1} \otimes \cdots \otimes e_{i_{k+1}} \otimes e_{i_k} \otimes \cdots \otimes e_{i_{b'}})$. Then, it follows that the $\Bbbk$-space $\ker(1 \otimes \mu)$ is generated by elements of the form $x \otimes e_{\boldsymbol{i}}^{[k]}$ for $x \in \Lambda^{a'}E \otimes \Lambda^{m-1}E \otimes \cdots \otimes \Lambda^2 E$, and such k and $\boldsymbol{i}$. But given such x, k and $\boldsymbol{i}$, the image of the element $x \otimes e_{i_1} \otimes \cdots \otimes (e_{i_k} \wedge e_{i_{k+1}}) \otimes \cdots \otimes e_{i_{b'}}$ under $\phi_{\lambda'}^{(m+k-1,m+k,1)}$ is precisely $x \otimes e_{\boldsymbol{i}}^{[k]}$, and so $x \otimes e_{\boldsymbol{i}}^{[k]} \in \operatorname{im}\phi_{\lambda'}^{(m+k-1,m+k,1)}$. On the other hand, it is clear that the elements of the form $x \otimes e_{\boldsymbol{i}}^{[k]}$ generate the $\Bbbk$-space $\operatorname{im}\phi_{\lambda'}^{(m+k-1,m+k,1)}$, from which part (i) follows.

(ii) Now, the map $1 \otimes \mu : \Lambda^{\lambda'}E \to \Lambda^{a'}E \otimes \Lambda^{m-1}E \otimes \cdots \otimes \Lambda^2 E \otimes S^{b'}E$ induces a surjective G-homomorphism:

$$\pi : \frac{\Lambda^{\lambda'} E}{\ker(1 \otimes \mu)} \to \frac{\Lambda^{a'} E \otimes \Lambda^{m-1} E \otimes \cdots \otimes \Lambda^2 E \otimes S^{b'} E}{\operatorname{im}((1 \otimes \mu) \circ \phi_{\lambda'})}.$$

Moreover, it follows from part (i) that $\ker \pi = \operatorname{im} \phi_{\lambda'} / \ker(1 \otimes \mu)$, and so we deduce that $\nabla(\lambda) \cong \operatorname{coker}((1 \otimes \mu) \circ \phi_{\lambda'})$. □

On the other hand, recall that through the James-construction of the induced module, we see that $\nabla(\lambda)$ is isomorphic to a submodule of $S^\lambda E$, namely as the kernel of the G-homomorphism ψ_λ (2.6). We claim that we may replace the factor $E^{\otimes b}$ with the exterior power $\Lambda^b E$. Once again, this process is independent of the characteristic of $\Bbbk$. For this, we construct from the comultiplication map $\Delta : \Lambda^b E \to E^{\otimes b}$, the injective G-homomorphism $1 \otimes \Delta : S^a E \otimes S^{m-1} E \otimes \cdots \otimes S^2 E \otimes \Lambda^b E \to S^\lambda E$.

Lemma 4.2. *For $m \geq 2$ and $\lambda = (a, m-1, m-2, \ldots, 2, 1^b)$, we have:*

(i) $\ker \psi_\lambda \subseteq \bigcap_{k=1}^{b-1} \ker \psi_\lambda^{(m+k-1,m+k,1)} = \operatorname{im}(1 \otimes \Delta)$.
(ii) $\nabla(\lambda) \cong \ker(\psi_\lambda \circ (1 \otimes \Delta))$ *as G-modules.*

Proof. (i) Firstly, it follows from the definition of ψ_λ that $\ker \psi_\lambda \subseteq \ker \psi_\lambda^{(m+k-1,m+k,1)}$ for $1 \leq k < b$. Then, the $\Bbbk$-space $\ker \psi_\lambda^{(m+k-1,m+k,1)}$ is generated by elements of the form $x \otimes e_{\boldsymbol{i}}^{[k]}$ for $x \in S^a E \otimes S^{m-1} E \otimes \cdots \otimes S^2 E$, $1 \leq k < b$, and sequences $\boldsymbol{i} := (i_1, \ldots, i_b)$ with terms in $[n]$, where $e_{\boldsymbol{i}}^{[k]} := (e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e_{i_{k+1}} \otimes \cdots \otimes e_{i_b}) - (e_{i_1} \otimes \cdots \otimes e_{i_{k+1}} \otimes e_{i_k} \otimes \cdots \otimes e_{i_b})$. It follows that the $\Bbbk$-space $\bigcap_{k=1}^{b-1} \ker \psi_\lambda^{(m+k-1,m+k,1)}$ is generated by elements of the form:

$$\sum_{\sigma \in \mathfrak{S}_b} \mathtt{sgn}(\sigma) \left(x \otimes e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(b)}} \right) = x \otimes \Delta(e_{i_1} \wedge \cdots \wedge e_{i_b}) \in \operatorname{im}(1 \otimes \Delta).$$

Moreover, it is clear that elements of the form $x \otimes \Delta(e_{i_1} \wedge \cdots \wedge e_{i_b})$ generate the $\Bbbk$-space $\operatorname{im}(1 \otimes \Delta)$, from which part (i) follows.

(ii) Now, the map $1 \otimes \Delta : S^a E \otimes S^{m-1} E \otimes \cdots \otimes S^2 E \otimes \Lambda^b E \to S^\lambda E$ induces an injective G-homomorphism $\nu : \ker(\psi_\lambda \circ (1 \otimes \Delta)) \to \ker \psi_\lambda$. Moreover, it follows from part (i) that ν is surjective, and so we have a G-isomorphism $\ker(\psi_\lambda \circ (1 \otimes \Delta)) \cong \ker \psi_\lambda \cong \nabla(\lambda)$. □

Now, we shall return to the situation where the underlying field $\Bbbk$ has characteristic 2. We fix the sequences $\alpha := (a', m-1, \ldots, 2, b')$ and $\beta := (a, m-1, \ldots, 2, b)$.

Remark 4.3. We shall consider the constructions of this section from the perspective of the Specht module $\operatorname{Sp}(\lambda)$.

(i) By Lemma 4.1(ii) we have that $\nabla(\lambda) \cong \operatorname{coker}((1 \otimes \mu) \circ \phi_{\lambda'})$. By applying the Schur functor f, we obtain that $\operatorname{Sp}(\lambda) \cong \operatorname{coker}(f(1 \otimes \mu) \circ \bar{\phi}_{\lambda'})$. Now, since we are in characteristic 2, we have that $f(\Lambda^{a'} E \otimes \Lambda^{m-1} E \otimes \cdots \otimes \Lambda^2 E \otimes S^{b'} E)$ is identified with $f(S^{a'} E \otimes S^{m-1} E \otimes \cdots \otimes S^2 E \otimes S^{b'} E)$ which in turn is isomorphic to $M(\alpha)$. We write $\pi_\alpha : M(\lambda') \to M(\alpha)$ for the surjective $\Bbbk\mathfrak{S}_r$-homomorphism that is obtained from $f(1 \otimes \mu)$ under these identifications. We set $\bar{\phi}_\alpha := \pi_\alpha \circ \bar{\phi}_{\lambda'}$ and we deduce that $\mathrm{Sp}(\lambda) \cong \mathrm{coker}\,\bar{\phi}_\alpha$.

(ii) On the other hand, by Lemma 4.2(ii) we have that $\nabla(\lambda) \cong \ker(\psi_\lambda \circ (1 \otimes \Delta))$. By applying the Schur functor f, we deduce that $\mathrm{Sp}(\lambda) \cong \ker(\bar{\psi}_\lambda \circ f(1 \otimes \Delta))$. But once again, since we are in characteristic 2, $f(S^a E \otimes S^{m-1} E \otimes \cdots \otimes S^2 E \otimes \Lambda^b E)$ is identified with $f(S^a E \otimes S^{m-1} E \otimes \cdots \otimes S^2 E \otimes S^b E)$ which in turn is isomorphic to $M(\beta)$. We write $\iota_\beta : M(\beta) \to M(\lambda)$ for the injective $\Bbbk\mathfrak{S}_r$-homomorphism that is obtained from $f(1 \otimes \Delta)$ under these identifications. We set $\bar{\psi}_\beta := \bar{\psi}_\lambda \circ \iota_\beta$ and we deduce that $\mathrm{Sp}(\lambda) \cong \ker \bar{\psi}_\beta$.

We summarise the content of Remark 4.3 in the following Lemma:

Lemma 4.4. *For $m \geq 2$ and $\lambda = (a, m-1, m-2, \ldots, 2, 1^b)$, we have:*

(i) $\mathrm{Sp}(\lambda) \cong \mathrm{coker}\,\bar{\phi}_\alpha$ *as $\Bbbk\mathfrak{S}_r$-modules.*
(ii) $\mathrm{Sp}(\lambda) \cong \ker \bar{\psi}_\beta$ *as $\Bbbk\mathfrak{S}_r$-modules.*

We define the following $\Bbbk\mathfrak{S}_r$-homomorphisms:

$$\bar{\phi}_\alpha^{(i,j,s)} := \pi_\alpha \circ \bar{\phi}_{\lambda'}^{(i,j,s)} : M({\lambda'}^{(i,j,s)}) \to M(\alpha), \quad \bar{\psi}_\beta^{(i,j,t)} := \bar{\psi}_\lambda^{(i,j,t)} \circ \iota_\beta : M(\beta) \to M(\lambda^{(i,j,t)}),$$

where π_α and ι_β are as defined in Remark 4.3.

Lemma 4.5. *For $m \geq 2$ and $\lambda = (a, m-1, m-2, \ldots, 2, 1^b)$, we have:*

(i) $\bar{\phi}_\alpha^{(i,j,1)} = 0$ *for $m \leq i < j \leq n$.*
(ii) $\bar{\psi}_\beta^{(i,j,1)} = 0$ *for $m \leq i < j \leq n$.*
(iii) $\bar{\phi}_\alpha = \sum_{i=1}^{m-1} \sum_{s=1}^{\lambda'_{i+1}} \bar{\phi}_\alpha^{(i,i+1,s)}$.
(iv) $\bar{\psi}_\beta = \sum_{i=1}^{m-1} \sum_{t=1}^{\lambda_{i+1}} \bar{\psi}_\beta^{(i,i+1,t)}$.

Proof. Parts (i)-(ii) follow from Lemma 4.1(i) and Lemma 4.2(i) respectively. Then, parts (iii)-(iv) follow immediately from parts (i)-(ii). □

Now, the following Lemma provides an analogue of Lemma 3.4:

Lemma 4.6. *For $m \geq 2$ and $\lambda = (a, m-1, m-2, \ldots, 2, 1^b)$, we have:*

(i) $\mathrm{im}\,\bar{\phi}_\alpha^{(i,j,s)} \subseteq \mathrm{im}\,\bar{\phi}_\alpha$ *for $1 \leq i < j \leq m$, $1 \leq s \leq \lambda'_j$.*
(ii) $\ker \bar{\psi}_\beta \subseteq \ker \bar{\psi}_\beta^{(i,j,t)}$ *for $1 \leq i < j \leq m$, $1 \leq t \leq \lambda_j$.*

Proof. Firstly, recall the $\Bbbk\mathfrak{S}_r$-homomorphisms π_α and ι_β defined within Remark 4.3. Then, part (i) follows from Lemma 3.4(i) by applying the Schur functor and postcomposing by π_α. Similarly, we see that part (ii) follows from Lemma 3.4(ii) by applying the Schur functor and pre-composing by ι_β. □

Then, by combining the results of Lemma 4.4, Lemma 4.5, and Lemma 4.6, we obtain the following description of the endomorphism algebra of $\mathrm{Sp}(\lambda)$:

Corollary 4.7. *The endomorphism algebra of* $\mathrm{Sp}(\lambda)$ *may be identified with the* $\Bbbk$*-subspace of* $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ *consisting of those elements* h *that satisfy:*

(i) $h \circ \bar{\phi}_\alpha^{(i,j,s)} = 0$ *for* $1 \le i < j \le m$ *and* $1 \le s \le \lambda'_j$,
(ii) $\bar{\psi}_\beta^{(i,j,t)} \circ h = 0$ *for* $1 \le i < j \le m$ *and* $1 \le t \le \lambda_j$.

Definition 4.8. Let $m \ge 2$, $\lambda = (a, m-1, m-2, \dots, 2, 1^b)$, $\alpha = (a', m-1, \dots, 2, b')$, and $\beta = (a, m-1, \dots, 2, b)$. Then:

(i) We say that an element $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ is *semirelevant* if $h \circ \bar{\phi}_{\lambda'}^{(i,j,1)} = 0$ and $\bar{\psi}_\lambda^{(i,j,1)} \circ h = 0$ for all $m \le i < j \le n$.
(ii) We say that an element $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ is *relevant* if $h \circ \bar{\phi}_\alpha^{(i,j,1)} = 0$ and $\bar{\psi}_\beta^{(i,j,1)} \circ h = 0$ for all $1 \le i < j \le m$.

Denote by $\mathrm{SRel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ the $\Bbbk$-subspace of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ consisting of the semirelevant homomorphisms $M(\lambda') \to M(\lambda)$, and then, we shall also denote by $\mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ the $\Bbbk$-subspace of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ consisting of the relevant homomorphisms $M(\alpha) \to M(\beta)$.

Lemma 4.9. *Denote by* $\omega : \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \to \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ *the* $\Bbbk$*-linear homomorphism with* $\omega(h) := \iota_\beta \circ h \circ \pi_\alpha$. *Then* ω *induces the following* $\Bbbk$*-linear isomorphisms:*

(i) $\hat{\omega} : \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \to \mathrm{SRel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$.
(ii) $\bar{\omega} : \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \to \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$.

Proof. Firstly, notice that Lemma 4.5(i) and Lemma 4.5(ii) justify the stated codomains of the maps $\hat{\omega}$ and $\bar{\omega}$ respectively. Moreover, $\hat{\omega}$ and $\bar{\omega}$ are clearly injective. Now, Lemma 4.1(i) and Lemma 4.2(i) give that both maps are surjective. □

Remark 4.10. Let $\gamma \in \Lambda(n, r)$ with $\ell := \ell(\gamma)$. Then:

(i) Fix $B \in \mathrm{Tab}(\alpha, \gamma)$. Then $\rho[B] \circ \pi_\alpha \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\gamma))$ and one can easily check that $\rho[B] \circ \pi_\alpha = \sum_A \rho[A]$, where the sum is over those $A \in \mathrm{Tab}(\lambda', \gamma)$ whose first $(m-1)$ rows agree with those of B, and also $\sum_{i=m}^{a} a_{ij} = b_{mj}$ for $1 \le j \le \ell$. Informally, these A are obtained from B by distributing, along columns, each non-zero entry within the mth-row of B into rows m through a of A such that these rows of A contain exactly one non-zero, and hence equal to 1, entry.

(ii) Now, let $B \in \mathrm{Tab}(\gamma, \beta)$. Then $\iota_\beta \circ \rho[B] \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\gamma), M(\lambda))$ and one can easily check that $\iota_\beta \circ \rho[B] = \sum_A \rho[A]$, where the sum is over those $A \in \mathrm{Tab}(\gamma, \lambda)$ whose first $(m-1)$ columns agree with those of B, and also $\sum_{j=m}^{a'} a_{ij} = b_{im}$ for $1 \le i \le \ell$. Informally, these A are obtained from B by distributing, along rows, each non-zero entry within the mth-column of B into columns m through a' of A such that these columns of A contain exactly one non-zero, and hence equal to 1, entry.

The following Example details the forms of the compositions of maps discussed in Remark 4.10.

Example 4.11. For $\lambda = (3, 1^3)$, we have:

$$\rho\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \circ \pi_{(4,2)} = \rho\begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + \rho\begin{bmatrix} 2 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\iota_{(3,3)} \circ \rho\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \rho\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} + \rho\begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} + \rho\begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \iota_{(3,3)} \circ \rho\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \circ \pi_{(4,2)} &= \rho\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \rho\begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \rho\begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &+ \rho\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \rho\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \rho\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The following Lemma provides an analogue of Corollary 3.17:

Lemma 4.12. *Let $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$. Then $h \in \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ if and only if the coefficients $h[B]$ of the $\rho[B]$ in h satisfy:*

(i) *For all $1 \le i < j \le m$, $1 \le k \le m$, and all $B \in \mathrm{Tab}(\alpha, \beta)$ with $b_{jk} \ne 0$, we have:*

$$(b_{ik} + 1)h[B] = \sum_{l \ne k} b_{il} h\left[B^{(i,k)(j,l)}_{(j,k)(i,l)}\right], \qquad (R^k_{i,j}(B))$$

(ii) *For all $1 \le i < j \le m$, $1 \le k \le m$, and all $B \in \mathrm{Tab}(\alpha, \beta)$ with $b_{kj} \ne 0$, we have:*

$$(b_{ki} + 1)h[B] = \sum_{l \ne k} b_{li} h\left[B^{(k,i)(l,j)}_{(k,j)(l,i)}\right]. \qquad (C^k_{i,j}(B))$$

Proof. For $B \in \mathrm{Tab}(\alpha, \beta)$, we denote by $\Omega(B)$ the subset of matrices in $\mathrm{Tab}(\lambda', \lambda)$ with:

$$\omega(\rho[B]) = \iota_\beta \circ \rho[B] \circ \pi_\alpha = \sum_{A \in \Omega(B)} \rho[A]. \qquad (4.13)$$

Clearly, given $B \neq B' \in \mathrm{Tab}(\alpha,\beta)$, we have that $\Omega(B) \cap \Omega(B') = \varnothing$. Now, we fix $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ with $h = \sum_{B\in\mathrm{Tab}(\alpha,\beta)} h[B]\rho[B]$, and we shall fix the notation $\tilde{h} := \omega(h) = \iota_\beta \circ h \circ \pi_\alpha$. Then, it follows from Remark 4.10 that the coefficients $\tilde{h}[A]$ of the $\rho[A]$ in $\tilde{h}$ satisfy:

$$\tilde{h}[A] = \begin{cases} h[B], & \text{if } A \in \Omega(B) \text{ for some } B \in \mathrm{Tab}(\alpha,\beta), \\ 0, & \text{otherwise.} \end{cases} \tag{4.14}$$

Now, suppose that h is relevant and we shall show that the coefficients $h[B]$ of the $\rho[B]$ in h satisfy the relations stated in (i), and it may be shown in a similar manner that they also satisfy the relations stated in (ii). Firstly, note that $\tilde{h}$ is relevant by Lemma 4.9(ii). We fix $1 \le i < j \le m$, $1 \le k \le m$, and $B \in \mathrm{Tab}(\alpha,\beta)$ with $b_{jk} \neq 0$. Then, there exists $A \in \Omega(B)$ with $a_{jk} \neq 0$. For such an A, since $\tilde{h}$ is relevant, the relation $R^k_{i,j}(A)$ of Corollary 3.17(ii) gives that:

$$(a_{ik}+1)\tilde{h}[A] = \sum_{l\neq k} a_{il}\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big]. \tag{4.15}$$

Now, take any $1 \le l \le n$ with $l \neq k$ such that $a_{il} \neq 0$. If $l < m$, then $a_{il} = b_{il}$ and $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,k)(j,l)}_{(j,k)(i,l)})$, so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big]$. On the other hand, if $l \ge m$, then $a_{il} = 1$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,k)(j,m)}_{(j,k)(i,m)})$ so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h\Big[B^{(i,k)(j,m)}_{(j,k)(i,m)}\Big]$. Therefore, we may rewrite (4.15) as:

$$(a_{ik}+1)h[B] = \sum_{\substack{l<m\\ l\neq k}} b_{il}h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] + \Big(\sum_{\substack{l\ge m\\ l\neq k}} a_{il}\Big)h\Big[B^{(i,k)(j,m)}_{(j,k)(i,m)}\Big]. \tag{4.16}$$

Now, if $k < m$, then $a_{ik} = b_{ik}$ and $\sum_{l\ge m} a_{il} = b_{im}$. Thus, (4.16) becomes:

$$(b_{ik}+1)h[B] = \sum_{\substack{l<m\\ l\neq k}} b_{il}h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] + b_{im}\Big[B^{(i,k)(j,m)}_{(j,k)(i,m)}\Big] = \sum_{l\neq k} b_{il}h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big],$$

which is precisely the relation $R^k_{i,j}(B)$.

On the other hand, if $k = m$, then $a_{im} = 0$, since $a_{jm} \neq 0$, and so $\sum_{l>m} a_{il} = b_{im}$. Moreover, $B^{(i,k)(j,m)}_{(j,k)(i,m)} = B$, and so (4.16) becomes:

$$h[B] = \sum_{l<m} b_{il}h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] + b_{im}h[B],$$

which in turn gives the relation $R^m_{i,j}(B)$:

$$(b_{im}+1)h[B] = \sum_{l\neq m} b_{il}h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big].$$

Conversely, suppose that the coefficients $h[B]$ of the $\rho[B]$ in h satisfy the relations stated in the Lemma. Note that by Lemma 4.9(ii), in order to show that h is relevant, it suffices to show that $\tilde{h}$ is relevant. To this end, we shall show that $\tilde{h} \circ \bar{\phi}_{\lambda'}^{(i,j,1)} = 0$ for $1 \le i < j \le n$, and it shall follow similarly that $\bar{\psi}_{\lambda}^{(i,j,1)} \circ \tilde{h} = 0$ for such i, j. Note that $\tilde{h}$ is semirelevant by Lemma 4.9(i) and so $\tilde{h} \circ \bar{\phi}_{\lambda'}^{(i,j,1)} = 0$ for $i \ge m$. Therefore, we may assume that $i < m$. Accordingly, fix some $1 \le i < j \le n$ with $i < m$. Then, as in the proof of Lemma 3.14, we have:

$$\tilde{h} \circ \bar{\phi}_{\lambda'}^{(i,j,1)} = \sum_{C \in \mathrm{Tab}(\lambda'^{(i,j,1)}, \lambda)} \left(\sum_{1 \le l \le n} c_{il} \tilde{h}\left[C_{(i,l)}^{(j,l)}\right] \right) \rho[C]. \tag{4.17}$$

Let $C \in \mathrm{Tab}(\lambda'^{(i,j,1)}, \lambda)$, and we wish to show that the coefficient of $\rho[C]$ in $\tilde{h} \circ \bar{\phi}_{\lambda'}^{(i,j,1)}$ is equal to 0. According to (4.14) and (4.17), we may assume that there exists some $1 \le k \le n$ with $c_{ik} \ne 0$ such that $A := C_{(i,k)}^{(j,k)} \in \Omega(B)$ for some $B \in \mathrm{Tab}(\alpha, \beta)$, where $\Omega(B)$ is as in (4.13), since otherwise, each summand $c_{il}\tilde{h}\left[C_{(i,l)}^{(j,l)}\right]$ appearing in the coefficient of $\rho[C]$ in (4.17) is equal to zero. Then, it follows from (4.17) that the coefficient of $\rho[C]$ in $\tilde{h} \circ \bar{\phi}_{\lambda'}^{(i,j,1)}$ is:

$$c_{ik} h[B] + \sum_{\substack{1 \le l \le n \\ l \ne k}} c_{il} \tilde{h}\left[A_{(j,k)(i,l)}^{(i,k)(j,l)}\right]. \tag{4.18}$$

We split our consideration into the following cases:

(i) ($j < m$; $k < m$): We have $c_{ik} = a_{ik} + 1 = b_{ik} + 1$. Now, if $1 \le l < m$ with $l \ne k$, then $c_{il} = a_{il} = b_{il}$ with $A_{(j,k)(i,l)}^{(i,k)(j,l)} \in \Omega(B_{(j,k)(i,l)}^{(i,k)(j,l)})$ so that $\tilde{h}\left[A_{(j,k)(i,l)}^{(i,k)(j,l)}\right] = h\left[B_{(j,k)(i,l)}^{(i,k)(j,l)}\right]$. On the other hand, if $l \ge m$ with $c_{il} \ne 0$, then $c_{il} = a_{il} = 1$ with $A_{(j,k)(i,l)}^{(i,k)(j,l)} \in \Omega(B_{(j,k)(i,m)}^{(i,k)(j,m)})$ so that $\tilde{h}\left[A_{(j,k)(i,l)}^{(i,k)(j,l)}\right] = h\left[B_{(j,k)(i,m)}^{(i,k)(j,m)}\right]$. Note that there are precisely b_{im} such values of l. Hence, we may rewrite (4.18) as:

$$(b_{ik} + 1)h[B] + \sum_{\substack{1 \le l < m \\ l \ne k}} b_{il} h\left[B_{(j,k)(i,l)}^{(i,k)(j,l)}\right] + b_{im} h\left[B_{(j,k)(i,m)}^{(i,k)(j,m)}\right] = 0,$$

since the coefficient $h[B]$ satisfies the relation $R_{i,j}^{k}(B)$.

(ii) ($j < m$; $k \ge m$): Here, we have $c_{ik} = 1$ and also $b_{jm} \ne 0$ since $A \in \Omega(B)$. Now, if $1 \le l < m$, then $c_{il} = a_{il} = b_{il}$ with $A_{(j,k)(i,l)}^{(i,k)(j,l)} \in \Omega(B_{(j,m)(i,l)}^{(i,m)(j,l)})$ so that $\tilde{h}\left[A_{(j,k)(i,l)}^{(i,k)(j,l)}\right] = h\left[B_{(j,m)(i,l)}^{(i,m)(j,l)}\right]$. On the other hand, if $l \ge m$ with $l \ne k$ and $c_{il} \ne 0$, then $c_{il} = a_{il} = 1$ with $A_{(j,k)(i,l)}^{(i,k)(j,l)} \in \Omega(B)$ so that $\tilde{h}\left[A_{(j,k)(i,l)}^{(i,k)(j,l)}\right] = h[B]$. Note that there are precisely b_{im} such values of l. Hence, we may rewrite (4.18) as:

$$h[B] + \sum_{1 \le l < m} b_{il} h\left[B^{(i,m)(j,l)}_{(j,m)(i,l)}\right] + b_{im} h[B] = 0,$$

since the coefficient $h[B]$ satisfies the relation $R^m_{i,j}(B)$.

(iii) $(j \ge m;\ k < m)$: Now, we have $c_{ik} = a_{ik} + 1 = b_{ik} + 1$ and also $b_{mk} \ne 0$ since $A \in \Omega(B)$. Now, if $1 \le l < m$ with $l \ne k$, then $c_{il} = a_{il} = b_{il}$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,k)(m,l)}_{(m,k)(i,l)})$ so that $\tilde{h}\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right] = h\left[B^{(i,k)(m,l)}_{(m,k)(i,l)}\right]$. On the other hand, if $l \ge m$ with $c_{il} \ne 0$, then $c_{il} = a_{il} = 1$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,k)(m,m)}_{(m,k)(i,m)})$ so that $\tilde{h}\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right] = h\left[B^{(i,k)(m,m)}_{(m,k)(i,m)}\right]$. Note that there are precisely b_{im} such values of l. Hence, we may rewrite (4.18) as:

$$(b_{ik} + 1) h[B] + \sum_{\substack{1 \le l < m \\ l \ne k}} b_{il} h\left[B^{(i,k)(m,l)}_{(m,k)(i,l)}\right] + b_{im} h\left[B^{(i,k)(m,m)}_{(m,k)(i,m)}\right] = 0,$$

since the coefficient $h[B]$ satisfies the relation $R^k_{i,m}(B)$.

(iv) $(j \ge m;\ k \ge m)$: Finally, in this case, we have $c_{ik} = 1$ and also $b_{mm} \ne 0$ since $A \in \Omega(B)$. Now, if $1 \le l < m$, then $c_{il} = a_{il} = b_{il}$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,m)(m,l)}_{(m,m)(i,l)})$ so that $\tilde{h}\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right] = h\left[B^{(i,m)(m,l)}_{(m,m)(i,l)}\right]$. On the other hand, if $l \ge m$ with $l \ne k$ and $c_{il} \ne 0$, then $c_{il} = a_{il} = 1$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B)$ so that $\tilde{h}\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right] = h[B]$. Note that there are precisely b_{im} such values of l. Hence, we may rewrite (4.18) as:

$$h[B] + \sum_{1 \le l < m} b_{il} h\left[B^{(i,m)(m,l)}_{(m,m)(i,l)}\right] + b_{im} h[B] = 0,$$

since the coefficient $h[B]$ satisfies the relation $R^m_{i,m}(B)$.

Thus, we have shown that the coefficient of $\rho[C]$ in $\tilde{h} \circ \bar{\phi}^{(i,j,1)}_{\lambda'}$ is zero in all possible cases, and so we are done. □

Now, since α and β both have length m, we may ignore the final $(n - m)$ rows and columns of each matrix in $\mathrm{Tab}(\alpha, \beta)$ and $\mathrm{Tab}(\beta, \alpha)$. Accordingly, we identify $\mathrm{Tab}(\alpha, \beta)$ with the set $\mathcal{T} := \{A \in M_{m\times m}(\mathbb{N}) \mid \sum_j a_{ij} = \alpha_i \text{ and } \sum_i a_{ij} = \beta_j\}$, and $\mathrm{Tab}(\beta, \alpha)$ with the set $\mathcal{T}' := \{A \in M_{m\times m}(\mathbb{N}) \mid \sum_j a_{ij} = \beta_i \text{ and } \sum_i a_{ij} = \alpha_j\}$.

Remark 4.19. Note that λ and its transpose λ' are of the same form. That is to say, the swap $\lambda \leftrightarrow \lambda'$ is equivalent to the swap $(a, b) \leftrightarrow (a', b')$, where $a' = b + m - 1$, $b' = a - m + 1$ respectively, which in turn is equivalent to the swap $\alpha \leftrightarrow \beta$. Therefore, after defining the notion of *relevance* for elements $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$, similarly to Definition 4.8(ii), and also swapping $\mathcal{T}$ with $\mathcal{T}'$, we obtain the following analogue of Lemma 4.12:

Corollary 4.20. *Let $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$. Then $h \in \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$ if and only if the coefficients $h[B]$ of the $\rho[B]$ in h satisfy:*

(i) $R^k_{i,j}(B)$ *for all* $1 \leq i < j \leq m$, $1 \leq k \leq m$, *and* $B \in \mathcal{T}'$ *with* $b_{jk} \neq 0$,
(ii) $C^k_{i,j}(B)$ *for all* $1 \leq i < j \leq m$, $1 \leq k \leq m$, *and* $B \in \mathcal{T}'$ *with* $b_{kj} \neq 0$.

The following Remark is clear:

Remark 4.21. Let $m \geq 2$ and $\lambda = (a, m-1, m-2, \ldots, 2, 1^b)$. Then:

(i) We have a $\Bbbk$-linear embedding of the endomorphism algebra of $\mathrm{Sp}(\lambda)$ into the $\Bbbk$-space $\mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$.
(ii) We have a $\Bbbk$-linear embedding of the endomorphism algebra of $\mathrm{Sp}(\lambda')$ into the $\Bbbk$-space $\mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$.

Remark 4.22. Let $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ and consider its transpose homomorphism $h' \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$. We have:

(i) For $1 \leq i < j \leq m$, $1 \leq k \leq m$, and $A \in \mathcal{T}$ with $a_{jk} \neq 0$, the relation $R^k_{i,j}(A)$ concerning the coefficient of $\rho[A]$ in h coincides with the relation $C^k_{i,j}(A')$ concerning the coefficient of $\rho[A']$ in h'.
(ii) For $1 \leq i < j \leq m$, $1 \leq k \leq m$, and $A \in \mathcal{T}$ with $a_{kj} \neq 0$, the relation $C^k_{i,j}(A)$ concerning the coefficient of $\rho[A]$ in h coincides with the relation $R^k_{i,j}(A')$ concerning the coefficient of $\rho[A']$ in h'.
(iii) The transpose homomorphism h' is relevant if and only if h is relevant.

4.2. A critical relation

Here, we shall highlight a new relation that occurs as a combination of the relations $R^k_{i,j}(A)$ and $C^k_{i,j}(A)$ of Lemma 4.12 that will play an important role in our considerations below.

Lemma 4.23. *Suppose that* $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ *is a relevant homomorphism. Then the coefficients* $h[A]$ *of the* $\rho[A]$ *in* h *satisfy the relations:*

$$z_{j,k}(A)h[A] = \sum_{\substack{i<j \\ l>k}} a_{il} h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right] + \sum_{\substack{i>j \\ l<k}} a_{il} h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right], \qquad (Z_{j,k}(A))$$

for all $1 \leq j, k \leq m$ *and* $A \in \mathcal{T}$ *with* $a_{jk} \neq 0$, *where* $z_{j,k}(A) := \sum_{i<j} a_{ik} + \sum_{l<k} a_{jl} + j + k \in \Bbbk$.

Proof. Since h is relevant, the coefficients $h[A]$ of the $\rho[A]$ in h satisfy the relations of Lemma 4.12, and so in particular, given $1 \leq j, k \leq m$, the coefficients satisfy the relation $\sum_{i<j} R^k_{i,j}(A) + \sum_{l<k} C^j_{l,k}(A)$ for all $A \in \mathcal{T}$ with $a_{jk} \neq 0$. But, the left-hand side of this relation is given by:

$$\sum_{i<j}(a_{ik}+1)h[A]+\sum_{l<k}(a_{jl}+1)h[A]=z_{j,k}(A)h[A], \tag{4.24}$$

by definition of $z_{j,k}(A)$. On the other hand, the right-hand side of this relation is:

$$\sum_{\substack{i<j\\ l\neq k}} a_{il}h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right]+\sum_{\substack{l<k\\ i\neq j}} a_{il}h\left[A^{(j,l)(i,k)}_{(j,k)(i,l)}\right]. \tag{4.25}$$

Now, notice that for $i<j$, $l<k$ we have $A^{(j,l)(i,k)}_{(j,k)(i,l)}=A^{(i,k)(j,l)}_{(j,k)(i,l)}$ and so after cancelling those terms that appear twice, we may rewrite (4.25) as:

$$\sum_{\substack{i<j\\ l\neq k}} a_{il}h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right]+\sum_{\substack{l<k\\ i\neq j}} a_{il}h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right]=\sum_{\substack{i<j\\ l>k}} a_{il}h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right]+\sum_{\substack{i>j\\ l<k}} a_{il}h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right],$$

which, along with (4.24), gives the required expression. □

5. One-dimensional endomorphism algebra

Given integers s, t, we write $s\equiv t$ to mean that s is congruent to t modulo 2, and so in particular, are equal as elements of the field $\Bbbk$. From here, we shall assume that the parameters a, b, and m satisfy the parity condition: $a-m\equiv b \pmod 2$. Note that this condition is preserved by the swap $(a,b)\leftrightarrow(a',b')$, where $a'=b+m-1$, $b'=a-m+1$.

Firstly, we highlight some basic properties of the coefficients $z_{j,k}(A)$ from Lemma 4.23.

Lemma 5.1. *Let $A\in\mathcal{T}$. Then:*

(i) $z_{j,k}(A)=\sum_{i>j}a_{ik}+\sum_{l>k}a_{jl}+\alpha_j+\beta_k+j+k$ *for* $1\le j,k\le m$.
(ii) $z_{j,k}(A)=\sum_{i>j}a_{ik}+\sum_{l>k}a_{jl}$ *for* $1<j,k<m$.
(iii) $z_{j,m}(A)=b+1+\sum_{i>j}a_{im}$ *and* $z_{m,k}(A)=a+m+\sum_{i>k}a_{mi}$ *for* $1<j,k<m$.
(iv) $z_{m,m}(A)=1$.
(v) $z_{1,m}(A)=\sum_{i>1}a_{im}$ *and* $z_{m,1}(A)=\sum_{i>1}a_{mi}$.

Proof. Part (i) follows from substituting the two expressions: $\sum_{i<j}a_{ik}=\beta_k-\sum_{i\ge j}a_{ik}$ and $\sum_{l<k}a_{jl}=\alpha_j-\sum_{l\ge k}a_{ji}$ into the definition of $z_{i,j}(A)$. Parts (ii)-(v) then follow immediately from part (i) along with the forms of α and β. □

Definition 5.2. Let $A,B\in\mathcal{T}$. Then:

(i) We write $A<_R B$ to mean that B follows A under the induced lexicographical order on rows, reading left to right and bottom to top. This is a total order and we call it the *row-order*.

(ii) We write $A <_C B$ to mean that B follows A under the induced lexicographical order on columns, reading top to bottom and right to left. This is a total order and we call it the *column-order*.

Remark 5.3. Let $1 \leq j, k \leq m$ and let $A \in \mathcal{T}$ with $a_{jk} \neq 0$. Then any $B = A^{(i,k)(j,l)}_{(j,k)(i,l)}$ that appears in the relation $Z_{j,k}(A)$ of Lemma 4.23 satisfies both $B <_R A$ and $B <_C A$.

From now on, we fix a relevant homomorphism $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$.

Lemma 5.4. *Let $A \in \mathcal{T}$ and suppose that $a_{mm} \neq 0$. Then $h[A] = 0$.*

Proof. Firstly, $z_{m,m}(A) = 1$ by Lemma 5.1(iv), and the result follows by $Z_{m,m}(A)$. □

Remark 5.5. Assume that $m = 2$, where then $\alpha = (b + 1, a - 1)$ and $\beta = (a, b)$. Suppose that $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ is a non-zero relevant homomorphism, and suppose that $A \in \mathcal{T}$ is such that $h[A] \neq 0$. We may assume that $a_{22} = 0$ by Lemma 5.4. Now, since $a_{12} + a_{22} = b$ and $a_{21} + a_{22} = a - 1$, we deduce that $a_{12} = b$ and $a_{21} = a - 1$. Moreover, since $a_{11} + a_{12} = b + 1$, we have that $a_{11} = 1$. Hence, there is a unique matrix A for which $h[A] \neq 0$, namely:

$$A = \begin{array}{|c|c|}\hline 1 & b \\ \hline a-1 & 0 \\ \hline \end{array}\,.$$

Hence for $\lambda = (a, 1^b)$ with $a \equiv b \pmod 2$, we deduce that $\mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda)) \cong \Bbbk$, and in this way we recover Murphy's result [14, Theorem 4.1].

Lemma 5.6. *Let $A \in \mathcal{T}$ and suppose that there exist some $1 < j, k < m$ such that $a_{jm} \neq 0$ and $a_{mk} \neq 0$. Then $h[A] = 0$.*

Proof. Suppose for contradiction that the claim is false and let $A \in \mathcal{T}$ be a counterexample that is minimal with respect to the column-order $<_C$. We choose $1 < j, k < m$ to be maximal such that $a_{jm}, a_{mk} \neq 0$. We may assume that $a_{mm} = 0$ by Lemma 5.4. Now, by Lemma 5.1(iii) we have $z_{j,m}(A) + z_{m,k}(A) = 1$ and so the relation $Z_{j,m}(A) + Z_{m,k}(A)$ gives:

$$h[A] = \sum_{\substack{i>j \\ l<m}} a_{il} h[B^{[i,l]}] + \sum_{\substack{i<m \\ l>k}} a_{il} h[D^{[i,l]}],$$

where $B^{[i,l]} := A^{(i,m)(j,l)}_{(j,m)(i,l)}$ for $i > j$, $l < m$ with $a_{il} \not\equiv 0$, and $D^{[i,l]} := A^{(i,k)(m,l)}_{(m,k)(i,l)}$ for $i < m$, $l > k$ with $a_{il} \not\equiv 0$.

Suppose that $i > j$, $l < m$ are such that $a_{il} \not\equiv 0$, and consider the matrix $B^{[i,l]}$. If $i = m$, then $b^{[m,l]}_{mm} \neq 0$ and so $h[B^{[m,l]}] = 0$ by Lemma 5.4. On the other hand, if $i < m$ then $b^{[i,l]}_{im}, b^{[i,l]}_{mk} \neq 0$, and notice also that $B^{[i,l]} <_C A$ by Remark 5.3. Therefore, by minimality of A, we have that $h[B^{[i,l]}] = 0$. Similarly, one may show that $h[D^{[i,l]}] = 0$ for $i < m$, $l > k$ with $a_{il} \not\equiv 0$, and so we deduce that $h[A] = 0$. □

Definition 5.7. We define the sets:

(i) $\mathcal{TR} := \{A \in \mathcal{T} \mid a_{i1} = 1 \text{ for } 1 \leq i < m, \text{ and } a_{mk} = 0 \text{ for } 1 < k \leq m\}$.
(ii) $\mathcal{TC} := \{A \in \mathcal{T} \mid a_{1k} = 1 \text{ for } 1 \leq k < m, \text{ and } a_{im} = 0 \text{ for } 1 < i \leq m\}$.

Lemma 5.8. *Let $A \in \mathcal{T}$ and suppose that $A \notin \mathcal{TR} \cup \mathcal{TC}$. Then $h[A] = 0$.*

Proof. By Lemma 5.4 we may assume that $a_{mm} = 0$. Suppose that $a_{mk} \neq 0$ for some k with $1 < k < m$. Then, by Lemma 5.6, we may assume that $a_{jm} = 0$ for $1 < j < m$. But then $a_{1m} = b$ and so $\sum_{l<m} a_{1l} = m - 1$. Since $A \notin \mathcal{TC}$ we deduce that there exists some $1 \leq l < m$ with $a_{1l} = 0$. Now, the relation $C^1_{l,m}(A)$ gives that $h[A] = \sum_{j>1} a_{jl} h[B^{[j]}]$ where $B^{[j]} := A^{(1,l)(j,m)}_{(1,m)(j,l)}$ for $j > 1$ with $a_{jl} \not\equiv 0$. Suppose that $j > 1$ is such that $a_{jl} \not\equiv 0$. If $j = m$ then $b^{[m]}_{mm} \neq 0$ and so $h[B^{[m]}] = 0$ by Lemma 5.4. Moreover, for $1 < j < m$ we have that $b^{[j]}_{mk}, b^{[j]}_{jm} \neq 0$ and so $h[B^{[j]}] = 0$ by Lemma 5.6. Therefore, we deduce that $h[A] = 0$.

Hence, we may assume that $a_{mk} = 0$ for all $1 < k \leq m$ and so it follows that $a_{m1} = a - m + 1$ and that $\sum_{j<m} a_{j1} = m - 1$. However, since $A \notin \mathcal{TR}$ we must have that $a_{j1} = 0$ for some j with $1 \leq j < m$. Now, the relation $R^1_{j,m}(A)$ gives $h[A] = \sum_{l>1} a_{jl} h[D^{[l]}]$ where $D^{[l]} := A^{(j,1)(m,l)}_{(m,1)(j,l)}$ for $l > 1$ with $a_{jl} \not\equiv 0$. Suppose that $l > 1$ is such that $a_{jl} \not\equiv 0$. If $l = m$, then $d^{[m]}_{mm} \neq 0$ and so $h[D^{[m]}] = 0$ by Lemma 5.4. On the other hand, if $1 < l < m$ then $d^{[l]}_{ml} \neq 0$. Now, if $d^{[l]}_{um} \neq 0$ for some $1 < u < m$, then $h[D^{[l]}] = 0$ by Lemma 5.6. Hence, we may assume that $d^{[l]}_{um} = 0$ for all $1 < u < m$ and so we deduce that $d^{[l]}_{1m} = a_{1m} = b$. Since $A \notin \mathcal{TC}$ we have that there exists some $1 \leq k < m$ with $a_{1k} = 0$ and hence $d^{[l]}_{1k} = 0$. Then, the relation $C^1_{k,m}(D^{[l]})$ expresses $h[D^{[l]}]$ as a linear combination of $h[F]$s where either $f_{mm} \neq 0$, or $f_{ml} \neq 0$ and $f_{vm} \neq 0$ for some v with $1 < v < m$. Once again, Lemma 5.4 and Lemma 5.6 give that $h[F] = 0$ for all such F and so $h[D^{[l]}] = 0$. Hence $h[A] = 0$. □

Definition 5.9. We shall require some additional notation that we shall introduce here:

(i) In order to assist with counting in reverse, set $\tau(i) := m - (i - 1)$ for $1 \leq i \leq m$.
(ii) For $1 < i < m$, we define:

$$\mathcal{TR}_i := \{A \in \mathcal{TR} \mid \text{the } \tau(j)\text{th-row of } A \text{ contains } j \text{ odd entries for } 1 < j \leq i\}.$$

(iii) For $1 < i < m$, we define $\overline{\mathcal{TR}}_i := \mathcal{TR}_i \setminus \mathcal{TR}_{i+1}$, where we set $\mathcal{TR}_m := \varnothing$.

Remark 5.10. Let $A \in \mathcal{T}$. Recall that $\sum_l a_{\tau(i)l} = i$ for $1 < i < m$. Therefore, if $A \in \mathcal{TR}_i$ for some $1 < i < m$, then the $\tau(j)$th-row of A consists entirely of ones and zeros for all $1 < j \leq i$.

Definition 5.11. Let $1 < i < m$ and $A \in \overline{\mathcal{TR}}_i$. Then:

(i) We set $\mathcal{K}_A := \{2 \leq k \leq i \mid a_{uk} = 1 \text{ for } \tau(i) \leq u \leq \tau(k)\}$.
(ii) We set $k_A := \min\{2 \leq k \leq i+1 \mid k \notin \mathcal{K}_A\}$.
(iii) If $k_A \leq i$, we set $j_A := \min\{k_A \leq j \leq i \mid a_{\tau(j)k_A} = 0\}$.
(iv) If $k_A \leq i$ and $k_A \leq j \leq i$, we denote by $w^j(A) := (w_1^j(A), w_2^j(A), \ldots)$ the decreasing sequence of column-indices within the final $\tau(k_A)$ columns of A that satisfy $a_{\tau(j)w_s^j(A)} = 1$ for $s \geq 1$.

Notice that the sequence $w^j(A)$ has $j - k_A + 1$ terms.

Example 5.12. We have $k_A = 4$, $j_A = 4$, and $w^5(A) = (7, 5)$, where:

$A :=$

1	·	·	·	·	·	·	·	·
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	1	1	1	1	2	0	0	0
1	1	1	0	1	0	1	0	0
1	1	1	0	1	0	0	0	0
1	1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
$a - m + 1$	0	0	0	0	0	0	0	0

$\in \overline{\mathcal{TR}}_5$.

Lemma 5.13. *Let $2 < i < m$ and let $A \in \overline{\mathcal{TR}}_i$ with $k_A \leq i$. Suppose that there exists some index k with $k_A < k \leq i$ such that $w_t^j(A) = w_{t-1}^{j-1}(A)$ for all $k_A < j \leq k$ and all even t. Then for $l \geq k_A$, $k_A \leq j \leq k$, we have $\sum_{u \geq \tau(j)} a_{ul} \equiv 1$ if and only if $l = w_s^j(A)$ for some odd s.*

Proof. We proceed by induction on j. The case $j = k_A$ is clear and so we may assume that $j > k_A$ and that the claim holds for all smaller values of j in the given range. Let $l \geq k_A$ and suppose that $\sum_{u \geq \tau(j)} a_{ul} \equiv 1$. Suppose, for the moment, that $a_{\tau(j)l} = 0$. Then $\sum_{u \geq \tau(j)} a_{ul} = \sum_{u \geq \tau(j-1)} a_{ul}$, and so $l = w_s^{j-1}(A)$ for some odd s by the inductive hypothesis. However, $w_{s+1}^j(A) = w_s^{j-1}(A) = l$ and so $a_{\tau(j)l} = 1$, contradicting that $a_{\tau(j)l} = 0$. Hence, $a_{\tau(j)l} = 1$ and so $l = w_s^j(A)$ for some s. Moreover, $\sum_{u \geq \tau(j)} a_{ul} \equiv 1$ if and only if $\sum_{u \geq \tau(j-1)} a_{ul} \equiv 0$ and so by the inductive hypothesis $l \neq w_{s'}^{j-1}(A)$ for any odd s'. Now, if s is even then $w_s^j(A) = w_{s-1}^{j-1}(A)$, leading to a contradiction. Hence, s must be odd. Conversely, suppose that $l = w_s^j(A)$ for some odd s, and suppose, for the sake of contradiction, that $\sum_{u \geq \tau(j)} a_{ul} \equiv 0$. Then, there exists some $k_A \leq j' < j$ such that $a_{\tau(j')l} = 1$, and we choose j' to be maximal with this property. Therefore, $a_{ul} = 0$ for $\tau(j) < u < \tau(j')$ and $\sum_{u \geq \tau(j')} a_{ul} \equiv 1$. Then, by the inductive hypothesis, $l = w_{s'}^{j'}(A)$ for some odd s'. But then $w_{s'+1}^{j'+1}(A) = w_{s'}^{j'}(A) = l$, by our assumption, and so $a_{\tau(j'+1)l} = 1$. Now, by the maximality of j', we must have $j' + 1 = j$. Thus,

$l = w_{s'+1}^{j'+1}(A) = w_{s'+1}^{j}(A) = w_s^j(A)$ and so $s' + 1 = s$, which is impossible since s' and s are both odd. Hence $\sum_{u \geq \tau(j)} a_{ul} \equiv 1$, and so we are done. □

Lemma 5.14. *Let $2 < i < m$ and let $A \in \overline{\mathcal{TR}}_i$ with $k_A \leq i$. Suppose that $z_{\tau(j),l}(A) = 0$ for all $k_A \leq j \leq i$, $k_A \leq l < m$ with $a_{\tau(j)l} = 1$. Then $w_s^j(A) = w_{s-1}^{j-1}(A)$ for $k_A < j \leq i$ and even s with $s \leq j - k_A + 1$.*

Proof. We fix i and we proceed by induction on j, with the base case being $j = k_A + 1$. Here $w^j(A) = (w_1^j(A), w_2^j(A))$ and for $w := w_2^j(A)$ we have $z_{\tau(j),w}(A) = 0$. Now, by Lemma 5.1(ii) we have $z_{\tau(j),w}(A) = \sum_{u > \tau(j)} a_{uw} + \sum_{v > w} a_{\tau(j)v} = a_{\tau(j-1)w} + 1$. Therefore, the entry $a_{\tau(j-1)w}$ is odd and so $w = w_1^{k_A}(A)$ as required. Suppose now that $k_A + 1 < j \leq i$ and that the claim holds for smaller values of j in the given range. Note that this implies that the hypotheses of Lemma 5.13 are met for $k = j - 1$.

Suppose that s is even and set $l := w_s^j(A)$. Then $\sum_{u > \tau(j)} a_{ul} + s - 1 \equiv 0$ by Lemma 5.1(ii) since $z_{\tau(j),l}(A) = 0$. Therefore, $\sum_{u \geq \tau(j-1)} a_{ul} \equiv 1$ and so by Lemma 5.13 we deduce that $l = w_{s'}^{j-1}(A)$ for some odd s' with $s' \leq j - k_A$. Now, the sequence $w^j(A)$ has exactly one extra term compared to $w^{j-1}(A)$ and so the number of even indices in $w^j(A)$ equals the number of odd indices in $w^{j-1}(A)$. It follows that $s' = s - 1$ and so we are done. □

Lemma 5.15. *Let $1 < i < m$ and let $A \in \overline{\mathcal{TR}}_i$ with $k_A \leq i$. Suppose that $w_1^j(A) > w_1^{j-1}(A)$ for all $j_A < j \leq i$. Then we may express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ where either:*

(i) *$B \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$,*
(ii) *$B \in \overline{\mathcal{TR}}_i$ with $k_B > k_A$,*
(iii) *$B \in \overline{\mathcal{TR}}_i$ with $k_B = k_A$ and $B <_C A$, which is witnessed within the final $\tau(w_1^{j_A}(A))$ columns of A and B.*

Moreover, if $A \notin \mathcal{TC}$ then $B \notin \mathcal{TC}$ for all such B listed above.

Proof. To ease notation we set $u := \tau(j_A) > 1$, $k := k_A$, and $w := w_1^{j_A}(A)$. Notice that $w > k$, and that $a_{uk} = 0$ and $a_{uw} = 1$. The relation $C_{k,w}^u(A)$ gives $h[A] = \sum_{l \neq u} a_{lk} h[B^{[l]}]$ where $B^{[l]} := A_{(u,w)(l,k)}^{(u,k)(l,w)}$ for $l \neq u$ with $a_{lk} \not\equiv 0$. Let $l \neq u$ be such that $a_{lk} \not\equiv 0$, and let $k^{[l]} := k_{B^{[l]}}$, $j^{[l]} := j_{B^{[l]}}$, and $w^{[l]} := w_1^{j^{[l]}}(B^{[l]})$. We shall proceed by induction on j_A, decreasing from $j_A = i$.

Firstly, suppose that $j_A = i$. If $l > u$ and $a_{lw} \neq 0$, then $B^{[l]} \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$, and so $B^{[l]}$ is as described in case (i). Now, if $l > u$ with $a_{lw} = 0$, then $k^{[l]} = k$, $B^{[l]} <_C A$, and the final column in which $B^{[l]}$ and A differ is the wth-column. Hence, here $B^{[l]}$ is as described in case (iii). On the other hand, if $l < u$, then $k^{[l]} > k$ and $B^{[l]}$ is as described in case (ii).

Now, suppose that $j_A < i$ and that the claim holds for all $D \in \overline{\mathcal{TR}}_i$ with $j_A < j_D \leq i$. We split our consideration into steps:

Step 1: If $l > u$ and $a_{lw} \neq 0$, then $B^{[l]} \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$, and so $B^{[l]}$ is as described in case (i). On the other hand, if $l > u$ and $a_{lw} = 0$, then $B^{[l]} \in \overline{\mathcal{TR}}_i$ with $k^{[l]} = k$ and $B^{[l]} <_C A$. Moreover, the final column in which $B^{[l]}$ and A differ in this case is the wth-column and so $B^{[l]}$ is as described in case (iii).

Step 2: Now, if $\tau(i) \leq l < u$ with $a_{lw} \neq 0$. Then $B^{[l]} \in \overline{\mathcal{TR}}_{m-l}$ with $m - l < i$ since $l \geq \tau(i) = m - i + 1$, and so $B^{[l]}$ is as described as in case (i).

Step 3: On the other hand, if $\tau(i) \leq l < u$ and $a_{lw} = 0$, then $B^{[l]} \in \overline{\mathcal{TR}}_i$ with $k^{[l]} = k$ and $j^{[l]} > j_A$. Moreover, the final column in which A and B differ is the wth-column, and so $w_1^j(B^{[l]}) = w_1^j(A)$ for all $j_A < j \leq i$, since $w_1^j(A) > w_1^{j-1}(A)$ for all $j_A < j \leq i$, and so in particular $w_1^j(B^{[l]}) > w_1^{j-1}(B^{[l]})$ for each $j^{[l]} < j \leq i$. Hence, by the inductive hypothesis, $B^{[l]}$ must satisfy the claim, and so $h[B^{[l]}]$ may be written as a linear combination of $h[D]$s for some $D \in \mathcal{T}$ where either:

(iv) $D \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$,
(v) $D \in \overline{\mathcal{TR}}_i$ with $k_D > k^{[l]}$,
(vi) $D \in \overline{\mathcal{TR}}_i$ with $k_D = k^{[l]}$ and $D <_C B^{[l]}$, which is witnessed within the final $\tau(w^{[l]})$ columns of $B^{[l]}$ and D.

Any such D as in (iv) is as described in case (i), whereas any such D as in (v) is as described in case (ii) since $k^{[l]} = k_A$. Now, notice that the final $\tau(w^{[l]})$ columns of A and $B^{[l]}$ match since $w^{[l]} > w$, and so any such D as in (vi) also satisfies $D <_C A$ (witnessed within the final $\tau(w)$ columns of A and D), and so is as described in case (iii).

Step 4: Finally, if $l < \tau(i)$, then $B^{[l]} \in \overline{\mathcal{TR}}_i$. Moreover, if $a_{tk} = 1$ for all $\tau(i) \leq t < \tau(j_A)$, then $k^{[l]} > k$ and so $B^{[l]}$ is as described in case (ii). On the other hand, if $a_{tk} = 0$ for some t in this range, then $k^{[l]} = k$ with $j^{[l]} > j_A$ and then one may proceed as in Step 3 above.

Now, suppose that $A \notin \mathcal{TC}$ but $B^{[l]} \in \mathcal{TC}$ for some $l \neq u$ with $a_{lk} \not\equiv 0$. Notice that this forces $l = 1$ and $a_{lk} = 2$, which contradicts that $a_{lk} \not\equiv 0$. Hence if $A \notin \mathcal{TC}$, then $B^{[l]} \notin \mathcal{TC}$ for all $l \neq u$ with $a_{lk} \not\equiv 0$. By applying this argument recursively, it follows that if $A \notin \mathcal{TC}$, then all such B produced by this procedure satisfy $B \notin \mathcal{TC}$ as well. □

Lemma 5.16. *Let $1 < i < m - 1$ and let $A \in \overline{\mathcal{TR}}_i$ with $k_A = i + 1$. Then we may express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ where either:*

(i) *$B \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$,*
(ii) *$B \notin \mathcal{TR}$.*

Moreover, if $A \notin \mathcal{TC}$ then $B \notin \mathcal{TC}$ for all such B listed above.

Proof. Firstly, recall that the sum of the entries in the $\tau(i+1)$th-row of A is $i+1$. Now, since $A \notin \mathcal{TR}_{i+1}$, we deduce that the $\tau(i+1)$th-row of A contains at most $i-1$ odd entries. Hence, there exists some $1 < s \leq i$ such that $a_{\tau(i+1)s}$ is even and we choose s be minimal with this property. To ease notation, we set $q := \tau(i+1)$ and $u := \tau(s)$. Note that $a_{us} = 1$. The relation $R^s_{q,u}(A)$ gives that $h[A] = \sum_{l \neq s} a_{ql} h[B^{[l]}]$ where $B^{[l]} := A^{(q,s)(u,l)}_{(u,s)(q,l)}$ for $l \neq s$ with $a_{ql} \not\equiv 0$.

If $l = 1$, then $B^{[1]} \notin \mathcal{TR}$, and so $B^{[1]}$ is as described in case (ii). Now, if $1 < l < s$, then $B^{[l]} \in \overline{\mathcal{TR}}_{s-1}$ with $s-1 < i$, and so $B^{[l]}$ is as described in case (i). Meanwhile, if $l > s$, then $B^{[l]} \in \overline{\mathcal{TR}}_i$ and, as in the previous paragraph, we may find some $s < t \leq i$ (depending on l) such that $b^{[l]}_{qt}$ is even, and we take t to be minimal with this property. The relation $R^t_{q,\tau(t)}(B^{[l]})$ expresses $h[B^{[l]}]$ as a linear combination of $h[D]$s for some $D \in \mathcal{T}$ that must either fit into one of the cases described in the statement of the claim, or otherwise once again $D \in \overline{\mathcal{TR}}_i$ and there exists some $t < v \leq i$ such that d_{qv} is even, and we take v to be minimal with this property. Noting that $v > t > s$, it is clear that this process must terminate, hence providing the desired expression for $h[A]$.

Now, suppose that $A \notin \mathcal{TC}$ but $B^{[l]} \in \mathcal{TC}$ for some $l \neq s$ with $a_{ql} \not\equiv 0$. Then, notice that $B^{[l]}$ agrees with A outside the $\tau(i+1)$th-row and $\tau(s)$th-row, and so in particular they agree in the first row since $i < m-1$. Hence $a_{1v} = b^{[l]}_{1v} = 1$ for $1 \leq v < m$ since $B^{[l]} \in \mathcal{TC}$. Now, by considering the first row-sum and the last column-sum of A, we deduce that $a_{1m} = b$ and $a_{vm} = 0$ for $1 < v \leq m$. However, this implies that $A \in \mathcal{TC}$, which is a contradiction. Once again, by applying this argument recursively, it follows that if $A \notin \mathcal{TC}$, then all such B produced by this procedure satisfy $B \notin \mathcal{TC}$ as well. □

Lemma 5.17. *Let $1 < i < m-1$ and let $A \in \overline{\mathcal{TR}}_i$. Then we may express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T} \setminus \mathcal{TR}$. Moreover, if $A \notin \mathcal{TC}$ then all such B satisfy $B \notin \mathcal{TR} \cup \mathcal{TC}$.*

Proof. We proceed by induction on $i \geq 2$. Firstly, suppose that $i = 2$. Since $A \notin \mathcal{TR}_3$ with $\sum_l a_{(m-2)l} = 3$, the $(m-2)$th-row of A must contain a single odd entry, which must then be equal to 1, and be located in the first column of A. On the other hand, since $A \in \mathcal{TR}_2$, there exists a unique $l > 1$ with $a_{(m-1)l} = 1$. The relation $R^l_{m-2,m-1}(A)$ gives $h[A] = h[B]$ for $B := A^{(m-2,l)(m-1,1)}_{(m-1,l)(m-2,1)}$. Evidently, $B \notin \mathcal{TR}$, and so the claim holds for $i = 2$.

Now, we suppose that $i > 2$ and that the claim holds for all $B \in \mathcal{T}$ such that $B \in \overline{\mathcal{TR}}_{i'}$ for some $2 \leq i' < i$. Suppose, for the sake of contradiction, that the claim fails for this particular value of i and consider the set of counterexamples $A \in \overline{\mathcal{TR}}_i$ whose value of k_A is maximal amongst all counterexamples. Now, we choose A to be the element of this set that is minimal with respect to the column-ordering. In other words, if $D \in \overline{\mathcal{TR}}_i$ is a counterexample to the claim, then either $k_D < k_A$, or $k_D = k_A$ and $D \geq_C A$.

Now if $k_A = i+1$, then Lemma 5.16 states that we may express $h[A]$ as a linear combination of some $h[B]$s for some $B \in \mathcal{T}$ where either $B \in \overline{\mathcal{TR}}_{i'}$ with $i' < i$, or $B \notin \mathcal{TR}$. In the first case the inductive hypothesis states that $h[B]$ can be expressed as a linear combination of some $h[D]$s with $D \notin \mathcal{TR}$, whilst in the second case we have $B \in \mathcal{T} \setminus \mathcal{TR}$. Thus, $h[A]$ satisfies the statement of the claim which contradicts that A was chosen to be a counterexample.

Hence, we may assume that $k_A \leq i$. Suppose, for the sake of contradiction, that there exists $k_A \leq j \leq i$, $k_A \leq k < m$ such that $a_{\tau(j)k} = 1$ and $z_{\tau(j),k}(A) = 1$. The relation $Z_{\tau(j),k}(A)$ gives the expression:

$$h[A] = \sum_{\substack{u<\tau(j) \\ l>k}} a_{ul} h[B^{[u,l]}] + \sum_{\substack{u>\tau(j) \\ l<k}} a_{ul} h[B^{[u,l]}], \tag{5.18}$$

where $B^{[u,l]} := A^{(u,k)(\tau(j),l)}_{(\tau(j),k)(u,l)}$ for all such (u,l) satisfying $a_{ul} \not\equiv 0$.

Now, set $B := B^{[u,l]}$ where (u,l) is as in (5.18) with $a_{ul} \not\equiv 0$. We claim that B fits into one of the following cases: $B \notin \mathcal{TR}$, $B \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$, or $B \in \overline{\mathcal{TR}}_i$ with $k_B = k_A$ and $B <_C A$. We provide full details for the case where $u > \tau(j)$, $l < k$ with the other case, that is $u < \tau(j)$, $l > k$, being similar.

If $l = 1$ then $B \notin \mathcal{TR}$ and so B is of the desired form. Now, if $1 < l < k_A$, then either $u \geq \tau(k_A)$ or $\tau(j) < u < \tau(k_A)$. In the first case, we have $B \in \overline{\mathcal{TR}}_{j-1}$, whilst in the second case we have $B \in \overline{\mathcal{TR}}_{\tau(u)-1}$ if $a_{uk} = 1$ and $B \in \overline{\mathcal{TR}}_{j-1}$ if $a_{uk} = 0$. Hence, in either case, we deduce that $B \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$. Suppose now that $k_A \leq l < k$, then we must have $\tau(j) < u \leq \tau(k_A)$ since $a_{ul} \not\equiv 0$. Now, if $a_{uk} = 1$ then $B \in \overline{\mathcal{TR}}_{\tau(u)-1}$, whilst if $a_{uk} = 0$ and $a_{\tau(j)l} = 1$, then $B \in \overline{\mathcal{TR}}_{j-1}$. Finally, if $a_{uk} = 0$ and $a_{\tau(j)l} = 0$, then $B \in \overline{\mathcal{TR}}_i$ with $k_B = k_A$ and $B <_C A$. But then, either by the inductive hypothesis on i, or by the minimality of A, all such B produced in this procedure must satisfy the statement of the claim, and hence so must A, which contradicts that A was chosen to be a counterexample.

Therefore, we may assume that that $z_{\tau(j),k}(A) = 0$ for all $k_A \leq j \leq i$, $k_A \leq k < m$ such that $a_{\tau(j)k} = 1$. Then, by Lemma 5.14 and Lemma 5.15, we may express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ where either: $B \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$, $B \in \overline{\mathcal{TR}}_i$ with $k_B > k_A$, or $B \in \overline{\mathcal{TR}}_i$ with $k_B = k_A$ and $B <_C A$. But then, either by the inductive hypothesis on i, maximality of k_A, or minimality of A, each such B must satisfy the statement of the claim, and hence so must A, which contradicts that A was chosen to be a counterexample. Thus, no such counterexample may exist. Finally, once again, it is clear to see from the steps taken above that if $A \notin \mathcal{TC}$, then all such B produced by this procedure satisfy $B \notin \mathcal{TC}$ as well. $\square$

Corollary 5.19. *Let $1 < i < m - 1$ and let $A \in \overline{\mathcal{TR}}_i$ with $A \notin \mathcal{TC}$. Then $h[A] = 0$.*

Proof. By Lemma 5.17, we may express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ with $B \notin \mathcal{TR} \cup \mathcal{TC}$. But $h[B] = 0$ for all such B by Lemma 5.8, and so the result follows. $\square$

Lemma 5.20. *Let $A \in \mathcal{TR} \setminus \mathcal{TC}$. Then $h[A] = 0$.*

Proof. Suppose, for the sake of contradiction, that the claim is false, and let $A \in \mathcal{T}$ be a counterexample that is minimal with respect to the column-ordering of Definition 5.2(ii). By Corollary 5.19, we may assume that $A \notin \overline{\mathcal{TR}_i}$ for any $i < m-1$, and so we must have that $A \in \mathcal{TR}_{m-1} \setminus \mathcal{TC}$ since $A \in \mathcal{TR}$. Hence, for each $1 < u < m$, either $a_{um} = 0$ or $a_{um} = 1$, and we claim that there exists at least one u in this range with $a_{um} = 1$. Indeed, suppose otherwise, then there exists some $1 < v < m$ with a_{1v} even since $A \notin \mathcal{TC}$. But then the relation $C^1_{vm}(A)$ expresses $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ with $B <_C A$ and $B \in \mathcal{TR} \setminus \mathcal{TC}$. But $h[B] = 0$ for all such B by minimality of A, which contradicts that A was chosen to be a counterexample. We hence write $(u_1, \ldots, u_s)$ for the increasing sequence whose terms are given by all u in the range $1 < u < m$ with $a_{um} = 1$. Firstly, suppose that $s > 1$ and set $u := u_{s-1}$ and $u' := u_s$. By Lemma 5.1(iii), we have that $z_{u,m}(A) + z_{u',m}(A) = 1$ and so the relation $Z_{u,m}(A) + Z_{u',m}(A)$ is given by:

$$h[A] = \sum_{\substack{v>u \\ l<m}} a_{vl} h[B^{[v,l]}] + \sum_{\substack{v>u' \\ l<m}} a_{vl} h[D^{[v,l]}], \tag{5.21}$$

where $B^{[v,l]} := A^{(v,m)(u,l)}_{(u,m)(v,l)}$ and $D^{[v,l]} := A^{(v,m)(u',l)}_{(u',m)(v,l)}$ for all such (v,l) with $a_{vl} \not\equiv 0$. Now, let (v,l) be as in (5.21) with $a_{vl} \not\equiv 0$.

If $l = 1$, then $B^{[v,1]}, D^{[v,1]} \notin \mathcal{TR} \cup \mathcal{TC}$ and so $h[B^{[v,1]}] = h[D^{[v,1]}] = 0$ by Lemma 5.8. On the other hand, if $l > 1$, then $B^{[v,l]}, D^{[v,l]} \in \mathcal{TR} \setminus \mathcal{TC}$ and $A <_C B^{[v,l]}, D^{[v,l]}$. Hence, by the minimality of A, once again we deduce that $h[B^{[v,l]}] = h[D^{[v,l]}] = 0$. Thus $h[A] = 0$, which contradicts that A was chosen to be a counterexample.

Hence we may assume that $s = 1$, or in other words that there exists a unique u in the range $1 < u < m$ such that $a_{um} = 1$, and so then $z_{1,m}(A) = 1$ by Lemma 5.1(v). By applying similar considerations to the above to the relation $Z_{1,m}(A)$, we once again reach a contradiction, and so no such counterexample may exist. □

Definition 5.22. For $1 < i < m$, similarly to $\mathcal{TR}_i$ of Definition 5.9(ii), we define:

$$\mathcal{TC}_i := \{A \in \mathcal{TC} \mid \text{the } \tau(j)\text{th-column of } A \text{ contains } j \text{ odd entries for } 1 < j \leq i\}.$$

Remark 5.23. Firstly, note that by Remark 4.22, we see that the transpose homomorphism $h' \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$ of h is relevant. Now, the results proven above are independent of the values of a and b, provided that they satisfy the parity condition $a - m \equiv b$. In particular, note that this condition is preserved under the swap $(a, b) \leftrightarrow (a', b')$, where $a' := b + m - 1$, $b' := a - m + 1$. But, as in Remark 4.19, this swap is equivalent to the swap $\lambda \leftrightarrow \lambda'$ and accordingly $\alpha \leftrightarrow \beta$ and $\mathcal{T} \leftrightarrow \mathcal{T}'$. Therefore, by defining the subsets $\mathcal{TR}', \mathcal{TC}' \subseteq \mathcal{T}'$ analogously to $\mathcal{TR}, \mathcal{TC} \subseteq \mathcal{T}$, we obtain the analogous results to those shown in this section for the coefficients $h'[A']$ of the $\rho[A']$ in h'.

Proposition 5.24. *Let $A \in \mathcal{T}$ and suppose that $A \notin \mathcal{TR}_{m-1} \cap \mathcal{TC}_{m-1}$. Then $h[A] = 0$.*

Proof. Suppose that $D \in \mathcal{T}$ is such that $h[D] \neq 0$. Then, we may assume that we have $D \in \mathcal{TR} \cup \mathcal{TC}$ since otherwise $h[D] = 0$ by Lemma 5.8. Moreover, we may assume that $D \notin \mathcal{TR} \setminus \mathcal{TC}$ since otherwise $h[D] = 0$ by Lemma 5.20. On the other hand, if $D \in \mathcal{TC} \setminus \mathcal{TR}$, then $D' \in \mathcal{TR}' \setminus \mathcal{TC}'$, where $\mathcal{TR}', \mathcal{TC}' \subseteq \mathcal{T}'$ are as defined in Remark 5.23. But then we have $h[D] = h'[D'] = 0$ á la Lemma 5.20, which contradicts our choice of D, and so we may assume that $D \notin \mathcal{TC} \setminus \mathcal{TR}$. In sum, we have shown that $h[D] = 0$ for all $D \in \mathcal{T}$ with $D \notin \mathcal{TR} \cap \mathcal{TC}$. In particular, to prove the Proposition, we may assume that $A \in \mathcal{TR} \cap \mathcal{TC}$. Now, if $A \notin \mathcal{TR}_{m-1}$, then there exists some i with $1 < i < m-1$ such that $A \in \overline{\mathcal{TR}_i}$. But then Lemma 5.17 allows one to express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ with $B \notin \mathcal{TR}$. But then every such B satisfies $B \notin \mathcal{TR} \cap \mathcal{TC}$ and hence that $h[B] = 0$ as shown above, and so $h[A] = 0$. On the other hand, if $A \notin \mathcal{TC}_{m-1}$, then $A' \notin \mathcal{TR}'_{m-1}$ where $\mathcal{TR}'_{m-1} \subseteq \mathcal{T}'$ is defined analogously to $\mathcal{TR}_{m-1} \subseteq \mathcal{T}$. But then $h[A] = h'[A'] = 0$ by the $'$-decorated analogue to the argument outlined above, and so we are done. □

Theorem 5.25. *Let* $\lambda = (a, m-1, \ldots, 2, 1^b)$ *with* $a \geq m \geq 2$, $b \geq 1$, *where* $r := \deg(\lambda)$, *and suppose that the parameters* a, b, *and* m *satisfy the parity condition:* $a - m \equiv b \pmod 2$. *Then* $\mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda)) \cong \Bbbk$.

Proof. Let $\bar{h}$ be a non-zero endomorphism of $\mathrm{Sp}(\lambda)$, which we identify with a relevant homomorphism $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ as in Remark 4.21. If $A \in \mathcal{T}$ with $h[A] \neq 0$, then $A \in \mathcal{TR}_{m-1} \cap \mathcal{TC}_{m-1}$ by Proposition 5.24. But since $\sum_v a_{\tau(i)v} = i$, $\sum_u a_{u\tau(j)} = j$ for $1 < i, j < m$, this set consists solely of the matrix:

$$A_0 := \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \cdots & 1 & 1 & b \\ \hline 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \hline a-m+1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline \end{array}.$$

Therefore, we have $h = h[A_0]\rho[A_0]$, and so we are done. □

Data availability

No data was used for the research described in the article.

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