Fast Linear State Estimation for Unbalanced Distribution Systems Using Hybrid Measurements

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***Abstract*—Distribution system state estimation (DSSE) is traditionally solved iteratively using unsynchronized measurements provided by the SCADA system and/or smart meters. This paper puts forward a decoupled linear state estimation method for unbalanced distribution systems. Contrary to conventional methods, the proposed linear DSSE (LDSSE) method can function with purely unsynchronized or hybrid synchronized/unsynchronized measurements. In the case of purely unsynchronized measurements, the voltage phase angles of the reference bus are acquired through local measurements. In the first stage, the proposed LDSSE method estimates the voltage phase angles in terms of network parameters and available measurements. These are referred to as pseudo-synchronized voltage phase angles, which establish a basis for deriving pseudo-synchronized voltage/current phasors. In the second stage, a set of linear equations are derived for each phase separately. Solving these equations results in estimates for voltage phasors. The linearity and decoupled nature of the proposed LDSSE method significantly reduce the computation time without impacting the accuracy of estimates. The superiority of the proposed LDSSE method over existing methods is verified using extensive simulations conducted on several test feeders, delivering results 20 times faster than the nonlinear DSSE on the 8500-bus test feeder.**

***Index Terms*— Hybrid measurements, linear state estimation, unbalanced distribution system, weighted least squares.**

I. INTRODUCTION

STATE estimation (SE) is an important process aimed at providing an accurate estimate of the power system's state by minimizing the impact of measurement errors [1]. As an integral function regularly runs in the control center, SE enables energy management applications such as contingency analysis and optimal power flow [1, 2]. Despite its potential benefits, SE has been rarely employed in distribution systems until recently, primarily due to the absence of real-time measurements. This is considered sufficient for passive distribution systems thanks to the reliability of load forecasting based on historical data [3].

With the increasing integration of distributed generation resources, battery energy storage systems, and electric vehicle charging stations [4], distribution system state estimation (DSSE) is becoming increasingly important for real-time monitoring and control purposes [5, 6]. Except for pioneering works on DSSE in the 1990s [7, 8], this field has gained most of its popularity mainly in recent years.

SE in distribution systems is more challenging compared to that in transmission systems due to factors such as insufficient observability, unbalanced radial configuration, and a high ratio of R/X [9, 10]. Extensive work is devoted to addressing these challenges [11-20]. DSSE suffers from the scarcity of real-time measurements [10]. However, this is changing thanks to the adoption of advanced metering infrastructures (AMIs) and the proliferation of smart meters and micro phasor measurement units (µPMUs) [11, 20]. The increased availability of metering data and the recent surge in the provision of historical data as pseudo-measurements are leading to a significant escalation in measurement redundancy in distribution systems [16, 21].

Unbalanced operations instigated by single- or double-phase loads and untransposed lines pose significant challenges to DSSE. This motivates the per-phase development of DSSE formulation based on a three-phase model of the system [22, 23]. Predicated on unsynchronized measurements from SCADA or smart meters, DSSE is conventionally formulated in terms of a nonlinear system of equations. Iterative methods such as the Gauss-Newton method [7, 15], the Forward-Backward sweep method [13, 24], or their combination [12, 14] are typically used to solve nonlinear DSSE (NDSSE). NDSSE requires both initialization and iteration, which introduce the risk of divergence and the emergence of multiple solutions.

Linear DSSE (LDSSE) methods are introduced to address some of the challenges associated with NDSSE. To avoid iterative solution processes, a linear DSSE can be formulated based on measurements from µPMUs [25, 26]. However, this requires the installation of a significant number of µPMUs, which is costly and complex. This is why some recent works focus on formulating LDSSE using unsynchronized measurements [27]. This approach can be considered more practical as it relies on unsynchronized measurements. Nonetheless, the simplifications employed in the modeling process could result in imprecise estimations.

To mitigate the computational demands associated with the DSSE process, decoupling techniques are introduced. These techniques, rooted in approximations, aim to estimate voltage magnitudes and phase angles individually [28, 29]. Another approach to decoupling DSSE in unbalanced systems involves handling each phase independently. A prevalent method to break down the three-phase state estimation problem is using modal transformations. To this end, existing methods leverage iterative solutions for fully transposed [30] or untransposed [31, 32] three-phase lines, monitored by µPMUs.

The need for accurate modeling of three-phase voltage phase

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angles for the reference bus further complicates DSSE. A trivial approach for addressing this challenge is the assumption of balanced voltage phase angles for the reference bus. Defining a virtual reference bus is proposed in [33, 34] as another approach to this challenge. The introduction of uncertainties in the modeling process by these approaches may lead to biased or inaccurate results.

This paper proposes an extended version of the idea presented in [35] for SE with unsynchronized data [36, 37]. A two-stage linear method is proposed for DSSE, which can function with unsynchronized or hybrid measurements. In the first stage, the voltage phase angles of the reference bus are obtained at the substation level. This paves the way for the initial estimation of three-phase voltage phase angles in a simple way. In the second stage, the voltage and current phasors are formed using the voltage phase angles calculated in the first stage and available measurements. By forming a decoupled linear system of equations for each phase, voltage phasors are estimated using the weighted least squares (WLS) method. The advancements of the proposed LDSSE method are as follows:

- A simple measurement-based technique to address the challenge of modeling the reference bus in unbalanced distribution systems.
- A linear formulation for DSSE with no need for linearization or iteration, even with purely unsynchronized measurements.
- Natural phase decoupling without resorting to the modal transformation based upon synchrophasors.

The rest of this article is organized as follows. Section II explains the basics of state estimation and associated statistical measures. The proposed LDSSE method is detailed in Section III. Performance evaluation and comparison studies are presented in Section IV. The paper is concluded in Section V.

II. Basic Concepts

A. WLS-based Estimation

At the core of SE, there is a regression model to predict the values of target variables while minimizing the adverse impact of measurement errors. To find the best fit for the states, various regression methods can be applied, such as WLS, least absolute values (LAV), or generalized maximum likelihood (GM) [10]. Within the class of linear unbiased estimators, the least squares estimator is considered one of the frequently used ones as it can provide the lowest variance based on the Gauss–Markov theorem [38]. In the case of a nonlinear formulation, the first-order approximation of the Taylor series can be employed to linearize the equations and solve them iteratively. This is the case while solving linear equations does not involve initialization or iteration.

The DSSE problem can be modeled as a set of linear equations as follows:

$$\boldsymbol{H}\boldsymbol{x} + \boldsymbol{e} = \boldsymbol{z} \tag{1}$$

where $\boldsymbol{H}$ denotes the coefficient matrix and $\boldsymbol{x}$ denotes the vector of states. Also, $\boldsymbol{z}$ and $\boldsymbol{e}$ denote the vectors of measurements and relevant errors, respectively.

The closed-form solution of (1) by WLS is written by [38]:

$$\widehat{\boldsymbol{x}} = \boldsymbol{P}\boldsymbol{H}^{*}\boldsymbol{R}^{-1}\boldsymbol{z} \tag{2}$$

$$\boldsymbol{P} = (\boldsymbol{H}^{*}\boldsymbol{R}^{-1}\boldsymbol{H})^{-1} \tag{3}$$

where $\widehat{\boldsymbol{x}}$ denotes the vector of estimates, and $\boldsymbol{P}$ and $\boldsymbol{R}$ denote the covariance matrices of states and measurements, respectively. When an identity matrix is used for $\boldsymbol{R}$, this is known as ordinary least squares (OLS). The closed-form solution of (1) by OLS is:

$$\widehat{\boldsymbol{x}} = (\boldsymbol{H}^{*}\boldsymbol{H})^{-1}\boldsymbol{H}^{*}\boldsymbol{z} \tag{4}$$

B. Statistical Measures

There are some statistical measures to evaluate the performance of an estimator. Three of the primary statistical measures are introduced here to evaluate the performance of LDSSE. These measures include bias, consistency, and quality tests as described in the following parts [15, 21]:

Bias: When the expected (mean) values of estimations are equal to their corresponding actual value, the estimator is called unbiased. Therefore, the mean error of estimated states is zero in unbiased estimators. This feature is mathematically expressed as:

$$\mathbb{E}(\boldsymbol{x} - \widehat{\boldsymbol{x}}) = \boldsymbol{x} - \mathbb{E}(\widehat{\boldsymbol{x}}) = 0 \tag{5}$$

where $\mathbb{E}(\widehat{\boldsymbol{x}})$ denotes the expected values of states.

Consistency: The estimator is consistent when the estimation error statistically corresponds to the relevant covariance matrix. One way of examining consistency is the normalized estimation error squared (NEES), defined by [21]:

$$\varepsilon = (\boldsymbol{x} - \widehat{\boldsymbol{x}})^{*}\boldsymbol{P}^{-1}(\boldsymbol{x} - \widehat{\boldsymbol{x}}) \tag{6}$$

where ε denotes the NEES. ε should be within a confidence interval to keep the consistency of the estimator. For multivariate cases, this interval can be obtained from Chi-squared (χ^2) statistics if errors are normally distributed.

The lower and upper bounds of the confidence interval can be obtained from (7)-(8) using Monte Carlo simulation:

$$b_l = \frac{1}{m}\chi^2_{mn}((1+\alpha)/2) \tag{7}$$

$$b_u = \frac{1}{m}\chi^2_{mn}((1-\alpha)/2) \tag{8}$$

where b_l and b_u denote the lower and upper bounds, respectively. m and n denote the numbers of Monte Carlo simulations and system states, respectively. Equations (7) and (8) are based on the Chi-squared distribution with α confidence level and mn degrees of freedom. A 95% confidence level is commonly used in statistical analysis [21]. For a consistent estimator, NEES follows a Chi-squared distribution with mn degrees of freedom, mathematically defined as [15]:

$$\mathbb{E}(\varepsilon) = \frac{1}{m}\chi^2_{mn}(\alpha) \tag{9}$$

Considering numerous Monte Carlo simulations yields $\chi^2_{mn}(\alpha) \approx mn$. Consequently, (9) can be simplified by:

$$\mathbb{E}(\varepsilon) \approx n \tag{10}$$

It follows from (10) that by increasing the number of simulations, the mean of NEES in a consistent estimator approaches the number of states.

Quality: The last statistical measure evaluated here is the quality of the estimator. The quality of an estimator is inversely related to the covariance of estimated states. Considering the direct relationship between state and measurement covariances in (2), the quality declines by increasing the level of measurement errors. This hypothetical test can be defined by the square root of the determinant of $\boldsymbol{P}$. Due to the precision limits of the solver for large-scale systems, calculating the determinant of state error covariance might be complicated. An alternative way to express the quality of an estimator is by disregarding less important data in off-diagonal elements of $\boldsymbol{P}$. The quality test of the estimator can be defined by [21]:

$$Q_{est} = \ln(1/\mathrm{tr}(\boldsymbol{P})) \tag{11}$$

where Q_{est} denotes the quality of the estimator and $\mathrm{tr}(\boldsymbol{P})$ is the trace of the state's covariance matrix.

III. Proposed LDSSE Method

The proposed LDSSE method is detailed in this section. The first challenge is modeling the voltage phase angles of the reference bus. In this paper, a technique is used to obtain the voltage phase angles of the reference bus based on available measurements. Afterward, a linear formulation is developed to calculate the voltage phase angles for the distribution system by employing OLS. The voltage and current phasors are then formed using the voltage phase angles from the first step and available measurements. By employing the calculated voltage/current phasors, the decoupled linear equations are formed in the second stage to estimate the final voltage phasors by WLS. The details of the proposed LDSSE method are elaborated in the following parts.

A. Obtaining Voltage Phase Angles of the Reference Bus

State estimation in distribution systems requires considering three-phase voltage phase angles at the reference bus due to their unbalanced nature. While current solutions typically involve modeling to determine these angles [33, 34], this paper proposes relying on measurements to obtain these variables. Discrete Fourier transform (DFT) or any other phasor estimation method can be applied to the measured signal of three-phase voltages at the reference bus. This enables the estimation of the voltage phasor for each phase of the reference bus. Then, the estimation and transmission of both magnitudes and phase angles of three-phase voltages from the reference substation to the control center would be possible. This measurement-based technique will remove the need to model the voltage phase angle of the reference bus, leading to an improvement in the accuracy of LDSSE.

In the case of having µPMUs in the distribution system, three-phase voltage phase angles will be provided by synchrophasors. Consequently, this step is applicable in cases with only unsynchronized measurements.

B. Initial Estimation of Voltage Phase Angles

The first stage of the proposed LDSSE method is designed to estimate system voltage phase angles. The voltage phasors are considered as the states in the proposed LDSSE method.

Voltage and current phasors are defined as follows:

$$v_k^\alpha = V_k^\alpha \, e^{j\delta_k^\alpha} \tag{12}$$

$$i_{kl}^\alpha = I_{kl}^\alpha \, e^{j\theta_{kl}^\alpha} = (P_{kl}^\alpha - jQ_{kl}^\alpha)/\left(V_k^\alpha \, e^{-j\delta_k^\alpha}\right) \tag{13}$$

$$\theta_{kl}^\alpha = \delta_k^\alpha - \varphi_{kl}^\alpha \tag{14}$$

$$\varphi_{kl}^\alpha = \tan^{-1}(Q_{kl}^\alpha/P_{kl}^\alpha) \tag{15}$$

where v_k^α and V_k^α denote the voltage phasor and its magnitude for phase α at the bus k. δ_k^α denotes the phase angle of v_k^α. i_{kl}^α and I_{kl}^α denote the current phasor and its magnitude for phase α at feeder kl. θ_{kl}^α denotes the phase angle of i_{kl}^α. Also, P_{kl}^α and Q_{kl}^α denote the active and reactive power of the phase α at feeder kl. φ_{kl}^α denotes the phase angle between v_k^α and i_{kl}^α. It should be noted that α is used for the sake of brevity and can take one of the phases a, b, or c.

The three-phase voltage phase angles of the reference bus are obtained using the method suggested in Part A of this section. In the case of purely unsynchronized measurements, v_k^α can only be formed for the reference bus using (12). For any bus equipped with μPMU, the corresponding v_k^α is obtained by (12).

Using the π model of feeder kl, one can write:

$$\boldsymbol{Y}_{kl}^{abc}\left(\boldsymbol{v}_k^{abc} - \boldsymbol{v}_l^{abc}\right) + \boldsymbol{B}_{kl}^{abc}\boldsymbol{v}_k^{abc} = \boldsymbol{i}_{kl}^{abc} \tag{16}$$

where $\boldsymbol{v}_k^{abc}$and $\boldsymbol{v}_l^{abc}$ denote the vectors of three-phase voltage phasors at buses k and l, respectively. $\boldsymbol{i}_{kl}^{abc}$ denotes the vector of three-phase current phasors from bus k to bus l. Also, $\boldsymbol{Y}_{kl}^{abc}$ and $\boldsymbol{B}_{kl}^{abc}$ denote the series admittance and shunt susceptance matrices of feeder kl. In the case of the availability of μPMU at bus k, the current synchrophasor can be directly used in (16). For the unsynchronized measurements in feeders, (16) is reformulated as follows, where the voltage phase angle is omitted from the formulation:

$$\frac{i_{kl}^\alpha}{v_k^\alpha} = \frac{I_{kl}^\alpha \, e^{j\theta_{kl}^\alpha}}{V_k^\alpha e^{j\delta_k^\alpha}} = \frac{I_{kl}^\alpha \, e^{j(\delta_k^\alpha - \varphi_{kl}^\alpha)}}{V_k^\alpha e^{j\delta_k^\alpha}} = \frac{I_{kl}^\alpha \, e^{-j\varphi_{kl}^\alpha}}{V_k^\alpha} \tag{17}$$

In order to obtain a linear relationship between current and voltage phasors using unsynchronized measurements, (17) can be rewritten as given in (18)-(19):

$$i_{kl}^\alpha = A_{kl}^\alpha \, v_k^\alpha \tag{18}$$

$$A_{kl}^\alpha = \left(I_{kl}^\alpha \, e^{-j\varphi_{kl}^\alpha}\right)/V_k^\alpha = (P_{kl}^\alpha - jQ_{kl}^\alpha)/{V_k^\alpha}^2 \tag{19}$$

Substituting (18) into (16) for unsynchronized measurements yields:

$$\left[-\boldsymbol{A}_{kl}^{abc} + \left(\boldsymbol{Y}_{kl}^{abc} + \boldsymbol{B}_{kl}^{abc}\right)\right]\boldsymbol{v}_k^{abc} - \boldsymbol{Y}_{kl}^{abc}\boldsymbol{v}_l^{abc} = 0 \tag{20}$$

where $\boldsymbol{A}_{kl}^{abc}$ denotes a diagonal matrix obtained based on (19). The feeders with unsynchronized measurements can be formulated using (20). The unsynchronized current or power injection measurements can further be formulated as follows based on (20):

$$\left[-\boldsymbol{A}_k^{abc} + \textstyle\sum_{l\epsilon c}\left(\boldsymbol{Y}_{kl}^{abc} + \boldsymbol{B}_{kl}^{abc}\right)\right]\boldsymbol{v}_k^{abc} - \textstyle\sum_{l\epsilon c}\boldsymbol{Y}_{kl}^{abc}\,\boldsymbol{v}_l^{abc} = 0 \tag{21}$$

where c denotes the set of buses connected to bus k, and $\boldsymbol{A}_k^{abc}$ is calculated based on (19) for injected currents/powers.

To address the challenges associated with the unavailability of measurements in DSSE, zero-injection buses can be taken into account. As such, a virtual measurement is assigned to zero-injection buses. Zero-injection buses are formulated by:

$$\sum_{l\epsilon c}\left(\boldsymbol{Y}_{kl}^{abc}+\boldsymbol{B}_{kl}^{abc}\right)\boldsymbol{v}_{k}^{abc}-\sum_{l\epsilon c}\boldsymbol{Y}_{kl}^{abc}\,\boldsymbol{v}_{l}^{abc}=0 \tag{22}$$

The first stage of LDSSE is formulated using (12) for the reference bus and synchronized voltage measurements, (16) for synchronized current measurements, (20) for unsynchronized current/power flow measurements, (21) for unsynchronized current/power injection measurements, and (22) for modeling zero-injection buses, respectively.

A general formulation of the first stage of the LDSSE is presented in (23) using hybrid measurements for a n-bus system. As can be seen in (23), the measurements are included in both the coefficient matrix and measurement vector in the first stage of the LDSSE. As such, it is difficult to consider the covariance matrix for the measurements in the coefficient matrix. Moreover, the coefficient matrix is not error-free due to the inclusion of measurements. Despite the aforementioned challenges, the first stage can provide an accurate estimation of voltage phase angles. The phase angles obtained from the first stage are used in the second stage to achieve accurate estimates.

C. Final Estimation of Voltage Phasors

After acquiring the initial voltage phase angles from the first stage, the current and voltage phasors can be formed for the second stage. It is worth noting that the voltage phase angles estimated in the first stage are not fully synchronized since they are calculated based on unsynchronized measurements. As such, the estimated voltage phase angles in the first stage are called pseudo-synchronized in this paper. Moreover, the voltage and current phasors formed by these voltage phase angles are called pseudo-synchronized phasors. After forming pseudo-synchronized voltage and current phasors, the measurements are removed from the coefficient matrix in the second stage and included in the measurement vector. This creates an error-free coefficient matrix. Furthermore, the second stage is solved by WLS while considering the covariance of the measurements.

In the second stage, the pseudo-synchronized voltage phasors are given by:

$$\hat{v}_{k}^{\alpha}=V_{k}^{\alpha}\,e^{j\hat{\delta}_{k}^{\alpha}} \tag{24}$$

where $\hat{v}_{k}^{\alpha}$ denotes the pseudo-synchronized voltage phasor of phase α at bus k. $\hat{v}_{k}^{\alpha}$ consists of the voltage magnitude V_{k}^{α} obtained from the measurements and the estimated voltage phase angle $\hat{\delta}_{k}^{\alpha}$ obtained from the first stage of the LDSSE. In the case of synchrophasor availability, $\hat{v}_{k}^{\alpha}$ is directly obtained from μPMUs. Equation (24) is inherently decoupled for each phase and can be directly employed in the second stage of the proposed LDSSE method.

Similar to voltage phasors, pseudo-synchronized current phasors are given in (25):

$$\hat{\imath}_{kl}^{\alpha}=I_{kl}^{\alpha}\,e^{j\left(\hat{\delta}_{k}^{\alpha}-\varphi_{kl}^{\alpha}\right)} \tag{25}$$

where $\hat{\imath}_{kl}^{\alpha}$ denotes pseudo-synchronized current phasor. $\hat{\imath}_{kl}^{\alpha}$ consists of $I_{kl}^{\alpha}\,e^{-j\varphi_{kl}^{\alpha}}$ obtained from the measurements and the estimated voltage phase angle $\hat{\delta}_{k}^{\alpha}$ obtained from the first stage of the proposed LDSSE method. In the case of synchrophasor availability, $\hat{\imath}_{kl}^{\alpha}$ is directly obtained from μPMUs.

By reformulating the KCL equation shown in (16), one can derive (26):

$$\boldsymbol{Z}_{kl}^{abc}\boldsymbol{Y}_{kl}^{abc}\left(\boldsymbol{v}_{k}^{abc}-\boldsymbol{v}_{l}^{abc}\right)+\boldsymbol{Z}_{kl}^{abc}\boldsymbol{B}_{kl}^{abc}\boldsymbol{v}_{k}^{abc}=\boldsymbol{Z}_{kl}^{abc}\boldsymbol{i}_{kl}^{abc} \tag{26}$$

where $\boldsymbol{Z}_{kl}^{abc}$ denotes the series impedance matrix of feeder kl (i.e. $\boldsymbol{Z}_{kl}^{abc}=\boldsymbol{Y}_{kl}^{abc\,-1}$). Thanks to the insignificant shunt susceptance in distribution feeders, it can be disregarded without introducing a considerable error into (26).

Using pseudo-synchronized voltage and current phasors from (24) and (25), and since $\boldsymbol{Z}_{kl}^{abc}\boldsymbol{Y}_{kl}^{abc}$ gives an identity matrix, (26) can be expressed as:

$$\boldsymbol{v}_{k}^{abc}-\boldsymbol{v}_{l}^{abc}=\boldsymbol{Z}_{kl}^{abc}\hat{\boldsymbol{\imath}}_{kl}^{abc}-\boldsymbol{Z}_{kl}^{abc}\boldsymbol{B}_{kl}^{abc}\hat{\boldsymbol{v}}_{k}^{abc} \tag{27}$$

By defining $\boldsymbol{T}_{kl}^{abc}=\boldsymbol{Z}_{kl}^{abc}\boldsymbol{B}_{kl}^{abc}$ and $\hat{u}_{kl}^{\alpha}$ as the estimated voltage drop of phase α across the feeder kl, (27) for each phase can be written in the compact form given in (28):

$$v_{k}^{\alpha}-v_{l}^{\alpha}=\hat{u}_{kl}^{\alpha} \tag{28}$$

$$\hat{u}_{kl}^{\alpha}=z_{kl}^{\alpha a}\hat{\imath}_{kl}^{a}+z_{kl}^{\alpha b}\hat{\imath}_{kl}^{b}+z_{kl}^{\alpha c}\hat{\imath}_{kl}^{c}-\left(t_{kl}^{\alpha a}\hat{v}_{k}^{a}+t_{kl}^{\alpha b}\hat{v}_{k}^{b}+t_{kl}^{\alpha c}\hat{v}_{k}^{c}\right) \tag{29}$$

where $z_{kl}^{\alpha\beta}$ and $t_{kl}^{\alpha\beta}$ denote the elements of matrices $\boldsymbol{Z}_{kl}^{abc}$ and $\boldsymbol{T}_{kl}^{abc}$, respectively. As shown in (28), $\hat{u}_{kl}^{\alpha}$ for each phase can be obtained using network parameters, measurements, and estimated voltage phase angles from the first stage. Then, the linear equations obtained for each feeder will be essentially decoupled from the other two phases. As such, the conventional transformation for developing decoupled equations by symmetrical components is no longer required.

The final linear system of equations for each phase can be represented by:

$$\begin{bmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\\1&-1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{bmatrix}\begin{bmatrix}v_{1}^{\alpha}\\v_{2}^{\alpha}\\\vdots\\v_{n}^{\alpha}\end{bmatrix}=\begin{bmatrix}\hat{v}_{1}^{\alpha}\\\hat{v}_{2}^{\alpha}\\\vdots\\\hat{v}_{n}^{\alpha}\\\hat{u}_{12}^{\alpha}\\\vdots\\\hat{u}_{nm}^{\alpha}\end{bmatrix} \tag{30}$$

$$\begin{bmatrix}1&0&0&0&0&0&\cdots&0&0&0\\0&1&0&0&0&0&\cdots&0&0&0\\0&0&1&0&0&0&\cdots&0&0&0\\-y_{12}^{aa}&-y_{12}^{ab}&-y_{12}^{ac}&-A_{12}^{a}+y_{12}^{aa}+b_{12}^{aa}&y_{12}^{ab}+b_{12}^{ab}&y_{12}^{ac}+b_{12}^{ac}&\cdots&0&0&0\\-y_{12}^{ba}&-y_{12}^{bb}&-y_{12}^{bc}&y_{12}^{ba}+b_{12}^{ba}&-A_{12}^{b}+y_{12}^{bb}+b_{12}^{bb}&y_{12}^{bc}+b_{12}^{bc}&\cdots&0&0&0\\-y_{12}^{ca}&-y_{12}^{cb}&-y_{12}^{cc}&y_{12}^{ca}+b_{12}^{ca}&y_{12}^{cb}+b_{12}^{cb}&-A_{12}^{c}+y_{12}^{cc}+b_{12}^{cc}&\cdots&0&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&0&0&0&0&0&\cdots&y_{nm}^{aa}+b_{nm}^{aa}&y_{nm}^{ab}+b_{nm}^{ab}&y_{nm}^{ac}+b_{nm}^{ac}\\0&0&0&0&0&0&\cdots&y_{nm}^{ba}+b_{nm}^{ba}&y_{nm}^{bb}+b_{nm}^{bb}&y_{nm}^{bc}+b_{nm}^{bc}\\0&0&0&0&0&0&\cdots&y_{nm}^{ca}+b_{nm}^{ca}&y_{nm}^{cb}+b_{nm}^{cb}&y_{nm}^{cc}+b_{nm}^{cc}\end{bmatrix}\begin{bmatrix}v_{1}^{a}\\v_{1}^{b}\\v_{1}^{c}\\v_{2}^{a}\\v_{2}^{b}\\v_{2}^{c}\\\vdots\\v_{n}^{a}\\v_{n}^{b}\\v_{n}^{c}\end{bmatrix}=\begin{bmatrix}V_{1}^{a}e^{j\delta_{1}^{a}}\\V_{1}^{b}e^{j\delta_{1}^{b}}\\V_{1}^{c}e^{j\delta_{1}^{c}}\\0\\0\\0\\\vdots\\I_{nm}^{a}\,e^{j\theta_{nm}^{a}}\\I_{nm}^{b}\,e^{j\theta_{nm}^{b}}\\I_{nm}^{c}\,e^{j\theta_{nm}^{c}}\end{bmatrix} \tag{23}$$

As can be seen in (30), the coefficient matrix is error-free and contains only 0 and ± 1. Dealing with a smaller system of equations with an error-free coefficient matrix can alleviate both the computational burden and inaccuracy of LDSSE. Equation (30) is established for each phase individually and the final voltage phasors are estimated by WLS.

To solve (30) by WLS, the corresponding variances of existing errors for each equation should be determined. To this end, it is needed to define a real-valued variance for each phasor associated with its magnitude and phase angle errors. The general form of a phasor can be written as follows:

$$\hat{z} = z + e = re^{j\gamma} + e = \hat{r}e^{j\hat{\gamma}} \tag{31}$$

where z and e denote the actual value and error of the measured phasor $\hat{z}$. r and γ denote the actual values of magnitude and phase angle of the phasor. $\hat{r}$ and $\hat{\gamma}$ denote the measured values of the same variables. The errors in both magnitude and phase angle of e are assumed to be independent and have Gaussian distribution $\hat{r} \sim \mathcal{N}(r, \sigma_{\hat{r}}^2)$ and $\hat{\gamma} \sim \mathcal{N}(\gamma, \sigma_{\hat{\gamma}}^2)$. $\sigma_{\hat{r}}^2$ and $\sigma_{\hat{\gamma}}^2$ denote the variance of the magnitude and phase angle errors, respectively. This well-established assumption is made for mathematical tractability and to facilitate comparison with prior research [2, 39]. The variance of $\hat{z}$ associated with its magnitude and phase angle errors is then defined by [39]:

$$\sigma_{\hat{z}}^2 = \hat{r}^2 \left(1 - e^{-\sigma_{\hat{\gamma}}^2}\right) + \sigma_{\hat{r}}^2 \left(2 - e^{-\sigma_{\hat{\gamma}}^2}\right) \tag{32}$$

Using (32) a real-valued variance can be defined for the error of a phasor in polar form. This equation directly applies to synchronized voltage and current phasors, assuming available variances of both magnitude and phase angle.

For pseudo-synchronized measurements, determining the exact variance of the estimated voltage phase angles is challenging. However, the simulation results indicate a satisfactory level of accuracy for the estimated voltage phase angles in the first stage. Consequently, for practical purposes, the variance of the estimated voltage phase angles is assumed to be zero. The variance of pseudo-synchronized voltage phasors can be approximated as:

$$\sigma_{\hat{v}_k^\alpha}^2 \approx \sigma_{V_k^\alpha}^2 \tag{33}$$

where $\sigma_{V_k^\alpha}^2$ denotes the variance of voltage magnitude V_k^α. For pseudo-synchronized current phasors, first, it is needed to calculate the variance of φ_{kl}^α. Having the variances of active and reactive powers as $\sigma_{P_{kl}^\alpha}^2$ and $\sigma_{Q_{kl}^\alpha}^2$, the variance of φ_{kl}^α is [40]:

$$\sigma_{\varphi_{kl}^\alpha}^2 = \frac{{P_{kl}^\alpha}^2 \, \sigma_{Q_{kl}^\alpha}^2 + {Q_{kl}^\alpha}^2 \, \sigma_{P_{kl}^\alpha}^2}{\left({P_{kl}^\alpha}^2 + {Q_{kl}^\alpha}^2\right)^2} \tag{34}$$

Having the variances of current magnitude and its phase angle, the variance for pseudo-synchronized current phasor is:

$$\sigma_{\hat{\imath}_{kl}^\alpha}^2 \approx {I_{kl}^\alpha}^2 \left(1 - e^{-\sigma_{\varphi_{kl}^\alpha}^2}\right) + \sigma_{I_{kl}^\alpha}^2 \left(2 - e^{-\sigma_{\varphi_{kl}^\alpha}^2}\right) \tag{35}$$

Now, just the variance of $\hat{u}_{kl}^\alpha$ should be determined. For multivariable functions such as $g = f(x, y, z, \ldots)$, using the Taylor series expansion, the approximate variance is determined by (36) [41]:

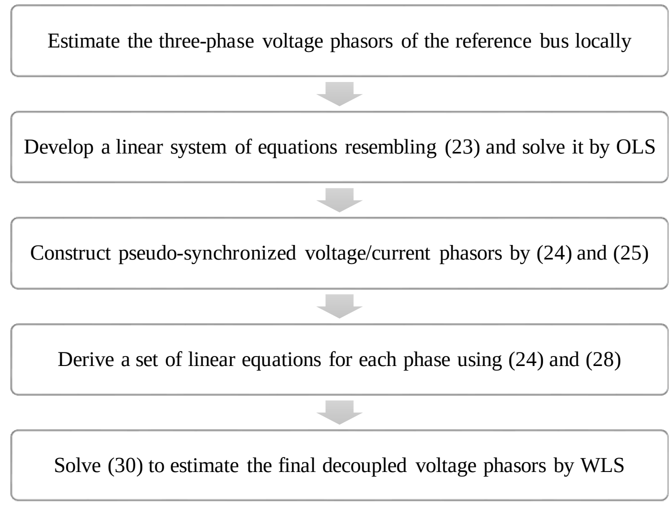

Fig.1. Flowchart of the proposed LDSSE method.

$$\sigma_g^2 \approx \left|\frac{\partial g}{\partial x}\right|^2 \sigma_x^2 + \left|\frac{\partial g}{\partial y}\right|^2 \sigma_y^2 + \left|\frac{\partial g}{\partial z}\right|^2 \sigma_z^2 + \cdots \tag{36}$$

Using (36), the variance of $\hat{u}_{kl}^\alpha$ can be calculated as follows:

$$\begin{aligned} \sigma_{\hat{u}_{kl}^\alpha}^2 \approx & |z_{kl}^{\alpha a}|^2 \sigma_{\hat{\imath}_{kl}^a}^2 + \left|z_{kl}^{\alpha b}\right|^2 \sigma_{\hat{\imath}_{kl}^b}^2 + |z_{kl}^{\alpha c}|^2 \sigma_{\hat{\imath}_{kl}^c}^2 \\ & + |t_{kl}^{\alpha a}|^2 \sigma_{\hat{v}_k^a}^2 + \left|t_{kl}^{\alpha b}\right|^2 \sigma_{\hat{v}_k^b}^2 + |t_{kl}^{\alpha c}|^2 \sigma_{\hat{v}_k^c}^2 \end{aligned} \tag{37}$$

The above equation results in a real-valued variance for $\hat{u}_{kl}^\alpha$. The variances of current and voltage phasors in (37) are determined by (32) for synchronized measurements and determined by (33) and (35) for pseudo-synchronized measurements. Then, the measurement covariance matrix is formed for each phase. Finally, the linear system of equations for each phase in (30) is solved by WLS in (2), and the final voltage phasors are estimated.

Figure 1 presents the stepwise implementation process of the proposed LDSSE method. It comprises five primary stages, commencing with the local estimation of three-phase voltage phasors at the reference bus. Subsequently, a linear equation system akin to (23) is formulated by employing available measurements to estimate initial voltage phase angles through OLS. Upon obtaining estimated voltage phase angles, pseudo-synchronized phasors of voltages and currents are determined using (24) and (25). These pseudo-synchronized phasors are then employed in (24) and (28) to construct linear equation systems for each phase individually, mirroring (30). Finally, the covariance matrix of each set is calculated, and the linear equation is solved using WLS to derive the final decoupled voltage phasors.

IV. Performance Evaluations

Extensive simulation studies are carried out to demonstrate the effectiveness of the proposed LDSSE method and compare it with that of the NDSSE method. The NDSSE is conventionally solved iteratively by the Gauss-Newton algorithm [7]. In contrast to NDSSE, the proposed LDSSE is linear and does not require initialization or iteration. Both NDSSE and LDSSE methods take advantage of WLS to find the best set of states. Different measurement types and variances are considered in the simulation studies. Moreover, measurement errors are assumed to be independent and have a normal distribution with standard deviation σ. Further, 99.73% of the error distribution is within a range of ±3σ based on the three-sigma criterion [41].

A PC with a Core i7 CPU and 32 GB RAM is used to implement the codes developed in MATLAB. To statistically evaluate each case study, Monte Carlo simulations are run, and the obtained results are compared with the reference values. While the actual values of states remain unknown, the results of load flow are treated as reference values in the simulations. The load flow results are gathered from OpenDSS [42].

In this section, various statistical measures of errors (MoE) are employed to assess the accuracy of the methods. The general equation for MoE can be defined as follows:

$$MoE = \left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_i-\hat{x}_i}{x_i{}^{\phi}}\right|^{\lambda}\right)^{\psi} \quad (38)$$

where x_i and $\hat{x}_i$ denote the actual and estimated values of i^{th} variable and n is the number of all states. For root-mean-square error (RMSE), $\phi = 0$, $\lambda = 2$, and $\psi = 0.5$. For mean absolute error (MAE), $\phi = 0$, $\lambda = 1$, and $\psi = 1$. Finally, for mean absolute percentage error (MAPE), $\phi = 1$, $\lambda = 1$, and $\psi = 1$.

A. Evaluating LDSSE by Statistical Tests

Given the statistical measures introduced in Part B, Section II, the bias, consistency, and quality of the proposed LDSSE method are evaluated in this part. All three tests are carried out on the IEEE 13-bus test feeder using purely unsynchronized measurements. To evaluate the bias of LDSSE, 10,000 simulations are run, and the results for voltage magnitude and phase angle errors (i.e. mismatches between estimated and actual values) are shown in Fig. 2. The simulations are carried out for three different levels of measurement errors. The errors in unsynchronized measurements and pseudo-measurements are assumed to be 1% and 10% in the first case, 3% and 30% in the second case, and 5% and 50% in the last case. About 20% of measurements are considered to be pseudo-measurements. Figure 2 shows the accumulative mismatch between the estimated values and actual values in all phases. As shown, the proposed LDSSE method is unbiased as per the estimated voltage magnitudes and phase angles. By decreasing the level of errors in measurements, the inaccuracy of the proposed LDSSE method is reduced.

As the second statistical test, the consistency of the proposed LDSSE method is studied using (10). The level of errors for real-time unsynchronized measurements and pseudo-measurements are set to 5% and 30%, respectively. The number of states (voltage phasors) for phases a, b, and c on the IEEE 13-bus test feeder is 10, 10, and 12, respectively. Conducting up to 500 Monte Carlo simulations, the mean of NEES in each case is calculated and the result for phases b and c are shown in Fig. 3. The mean values of NEES are within a 95% confidence interval, confirming the consistency of LDSSE.

To assess the effectiveness of the proposed LDSSE method, the last statistical measure, as presented in (11), is employed. This measure demonstrates the estimator's accuracy to varying levels of measurement errors. An increase in the error level in measurements is expected to reduce the quality of LDSSE.

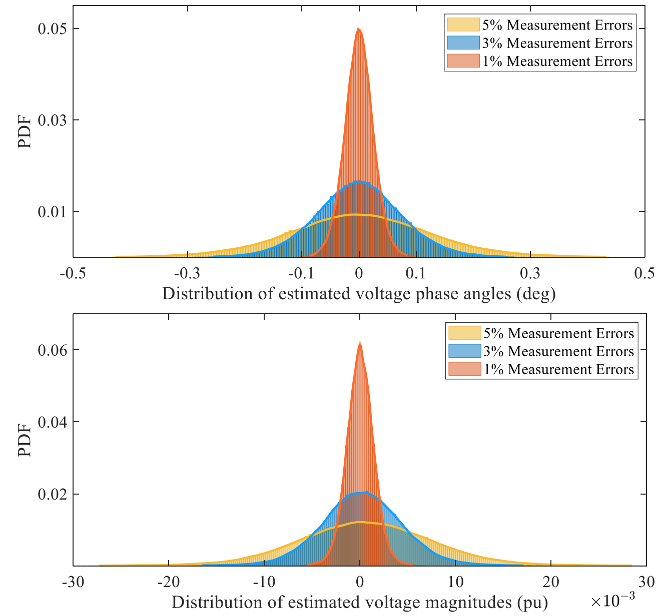

Fig. 2. Unbiasedness of LDSSE for various measurement errors.

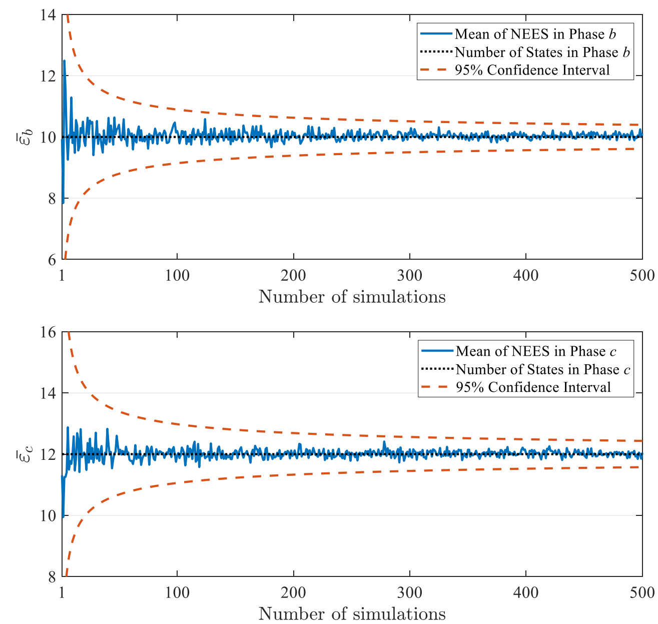

Fig. 3. Consistency of LDSSE on the IEEE 13-bus test feeder.

Different measurement error levels are evaluated for the IEEE 13-bus test feeder. The mean values of quality measures over 10,000 simulations are shown in Fig. 4. The figure shows the accuracy of LDSSE in relation to the variance of measurements. In the first case, the proposed LDSSE method incorporates hybrid measurements, including μPMUs, unsynchronized measurements, and pseudo-measurements with errors of 0.01%, 1%, and 10%, respectively. In this case, μPMUs and pseudo-measurements constitute 20% of the measurements each. The next cases focus on unsynchronized measurements and pseudo-measurements, solely. The errors range from 1% to 5% for unsynchronized measurements and 10% to 50% for pseudo-measurements. Pseudo-measurements make up 20% of all measurements in these cases. As anticipated, the results demonstrate that an increase in the level of errors corresponds to a decrease in the quality of LDSSE.

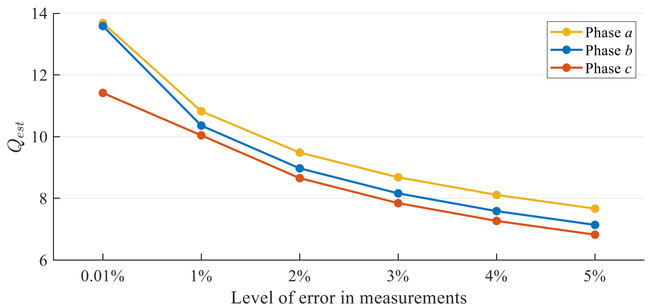

Fig. 4. Quality of estimates using the proposed LDSSE method.

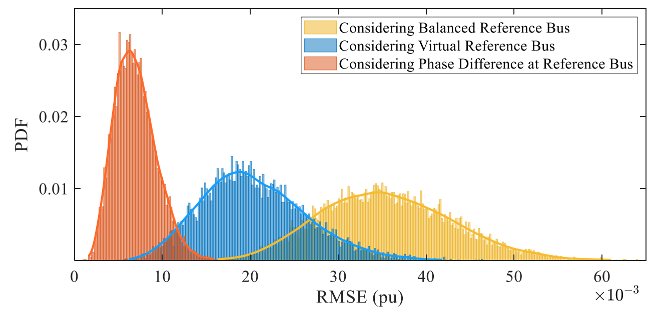

Fig. 5. Impact of the reference bus voltage phase angles on LDSSE accuracy.

B. Dealing with the Reference Bus Voltage Phase Angles

The impact of the accuracy of reference bus phase angles for DSSE using solely unsynchronized measurements is assessed in this part. After changing the loading conditions to create an unbalanced case in the IEEE 13-bus test feeder, the phase angles of the reference bus become 0, -118.7782, and 120.0253 degrees for phases a, b, and c, respectively. Considering balanced voltage phase angles for the reference bus might decrease the accuracy of the estimator. The use of a virtual reference bus is an alternative solution for this problem in the literature [33, 34]. However, as described in Section III, the phase angles of voltage phasors at the reference bus can be directly calculated. Obtaining the phase angles of the reference bus using the proposed technique improves the accuracy of the DSSE compared to the existing methods in the literature.

To distinguish the effectiveness of each method based on linear formulation, 10,000 Monte Carlo simulations are conducted on the test feeder. The errors for real-time and pseudo-measurements are assumed to be 5% and 30%, respectively. The share of pseudo-measurements is 20% of all measurements for all cases. Results for RMSE of estimated voltage phasors using the three mentioned methods are shown in Fig. 5. Considering balanced phase angles at the reference bus reduces the accuracy, as shown in Fig. 5. The mean value of RMSEs for this case is 0.0356 pu, while the estimator is no longer unbiased. The bias stems from the mismatch between the balanced voltage phase angles considered for the reference bus and the actual voltage phase angles. Employing the virtual reference bus in LDSSE can improve the accuracy compared to when the reference bus phase angles are assumed to be balanced. Although RMSE for this particular scenario averages at 0.0205 pu, the estimator exhibits biased characteristics. The bias introduced by the virtual reference bus method can be attributed to the inaccuracies present in the Thevenin equivalent of the virtual reference bus, which is modeled by the short-circuit level. In real-world cases, determining an accurate model for the virtual reference bus might become challenging.

Having the measured signals of three-phase voltages at the reference bus, DFT can be applied to obtain the voltage phasor for each phase of the reference bus. Considering 10 μs timing errors for this case study and by setting the phase angle of phase a to zero, the phase angles of phases b and c in the ABC sequence would be -118.7208 and 120.0348 degrees, respectively. The mean value of RMSEs by employing this method is as small as 0.0069 pu, while the estimation remains unbiased. Not only does obtaining reference bus voltage phase angles become easier using the proposed technique, but also the accuracy increases.

C. Impact of Time Synchronization Errors on LDSSE

This part aims to assess how timing errors affect the accuracy of estimation. As outlined in Section III, voltage phasors can be estimated linearly even with purely unsynchronized measurements. However, the presence of time differences among unsynchronized measurements results in inaccuracies in the final estimations. That is why the estimated voltage phasors under such conditions are referred to as pseudo-synchronized in this paper. This part assesses the effect of different synchronization errors on the accuracy of LDSSE. Additionally, the influence of defined weights for the second stage of LDSSE is examined for each case. The method with defined weights is referred to as LDSSE by WLS, while the one without weights is called LDSSE by OLS. Equations (32)-(37) can be used to form the covariance matrix for WLS, depending on the available measurements.

To evaluate the results, 1,000 Monte Carlo simulations are conducted on the IEEE 123-bus test feeder. Approximately half of the input data are assumed to be pseudo-measurements, while the remaining data are unsynchronized and virtual measurements. No µPMUs are used in the simulations. The variances of unsynchronized measurements and pseudo-measurements are assumed to be 5% and 30%, respectively. Although the measurement variances are constant for all cases, the assumed time synchronization errors range from 0.01 to 100 seconds. The discrepancies between the estimated and actual values of voltage magnitude are depicted in the box charts shown in Fig. 6.

As can be seen in Fig. 6, as the time synchronization error increases, the mean of the estimation error changes, and the DSSE becomes biased. When there is a constant measurement error, the variance for the estimated voltage almost remains similar across all cases. However, the mean shifts away from zero as the synchronization error increases. The results shown in Fig. 6 are derived from the steady-state operation of the distribution system. It is important to note that during contingencies, synchronization errors can lead to significant estimation errors due to the rapid changes of variables. Utilizing synchronized measurements from µPMUs can enhance accuracy in dynamic studies.

Furthermore, the use of defined weights for LDSSE by WLS results in a lower variance compared to LDSSE by OLS, as shown in Fig. 6. The impact of weights is particularly evident in the estimated voltage magnitudes. Due to space constraints, only the results for estimated voltage magnitude are presented here. The inaccuracy of estimated voltage magnitudes using LDSSE with WLS is approximately four times less than that achieved using OLS. Employing WLS based on defined weights for LDSSE can effectively enhance the accuracy of estimates.

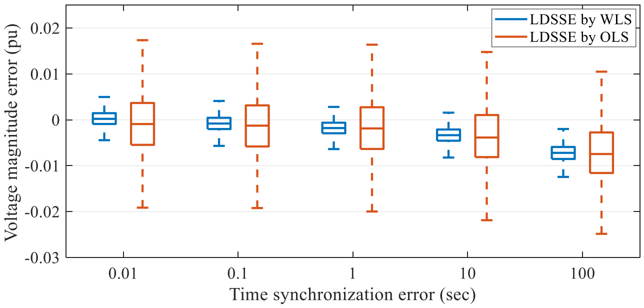

Fig. 6. Time synchronization error on the IEEE 123-bus test feeder.

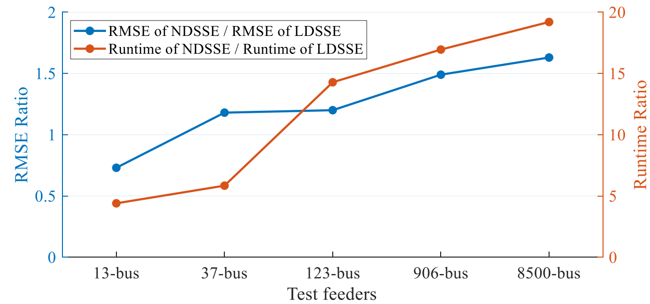

Fig. 7. Relative performance of NDSSE and LDSSE.

D. Comparative Study Considering Various Test Feeders

The proposed LDSSE method's performance is compared with that of the NDSSE method. Several test feeders, namely the IEEE 13-bus, 37-bus, 123-bus, 906-bus (European low voltage test feeder), and 8500-bus test feeders are used for this purpose [43].

Results obtained using unsynchronized data by both NDSSE and LDSSE methods are summarized in Table I. Both methods employ the proposed technique for acquiring the voltage phase angles of the reference bus. The level of errors for unsynchronized measurements and pseudo-measurements are 5% and 30%, while almost half of the input data is based on pseudo-measurements. For zero-injection buses, virtual measurements are assumed to have a variance of 10^{-6} pu. Monte Carlo simulations are executed 10,000 times for small test feeders (i.e. 13-bus, 37-bus, and 123-bus systems) and 1,000 times for large-scale test feeders (i.e. 906-bus and 8500-bus systems). The mean values of RMSE, MAE, and MAPE for voltage phasors further to the maximum, mean, and minimum computation times on various test feeders are reported in Table I. As summarized in Table I, the results obtained using the proposed LDSSE method outperform that of the NDSSE method. In terms of accuracy, the results of both methods are roughly close to each other. In the 13-bus test feeder, the inaccuracy of NDSSE is less than that of LDSSE. For other test feeders, LDSSE presents more accuracy with less RMSE, MAE, and MAPE compared to the results provided by NDSSE. The proposed LDSSE method highly outperforms the NDSSE method in terms of computation time. The mean runtime values for LDSSE are not only lower than those for NDSSE, but the maximum runtime for LDSSE is also lower than the minimum runtime for NDSSE across all cases.

The ratios of RMSEs and computation times obtained for NDSSE over LDSSE are shown in Fig. 7. NDSSE demands an iterative process to solve nonlinear equations, resulting in prolonged computation times, especially for larger systems [12-14]. LDSSE, however, instantly solves the equations in one single iteration. As system size grows, NDSSE experiences a greater rise in computational burden compared to LDSSE. LDSSE is more efficient computationally in comparison with NDSSE. In the case of a large-scale system such as the 8500-bus test feeder, LDSSE is approximately 20 times faster than NDSSE. The accelerated performance of the proposed LDSSE method is attributed to its linearity and decoupled nature, showcasing potential benefits for unbalanced distribution systems.

V. Conclusion

A linear distribution system state estimation (LDSSE) method is proposed in this paper for unbalanced distribution systems using hybrid synchronized/unsynchronized measurements. A simple yet effective technique is presented for obtaining the unbalanced voltage phase angles of the reference bus using available measurements, ensuring unbiased and accurate estimates. The superior performance of the proposed LDSSE method is particularly noticeable in large-scale distribution systems, where conventional nonlinear DSSE (NDSSE) faces prolonged convergence times. Thanks to the linearity of the proposed method, challenges associated with NDSSE, such as the prolonged iterative process and the risk of divergence, are overcome. Simulation results on various test feeders verify the effectiveness of the proposed LDSSE method in terms of accuracy, consistency, and speed.

The proposed method stands out for its expeditious solution delivery coupled with accuracy, e.g., about 20 times faster on the 8500-bus test feeder. This can be attributed to the linear formulation of the problem, which eliminates the need for linearization or iterative processes. The proposed method further decouples unbalanced phases without relying on symmetrical components, which further reduces its computational burden. The proposed LDSSE method emerges as a compelling choice for distribution system operators, both in anticipation of and during the transition toward fully synchronized monitoring systems. Building on the derivations of the proposed LDSSE, the authors intend to extend the method for dynamic DSSE, prioritizing efficient calculations with hybrid measurements.

Table I
Comparison Between The Performance Of NDSSE And LDSSE

Test Feeders	RMSE (pu) × 10^{-3}		MAE (pu) × 10^{-3}		MAPE (%)		Max Runtime (ms)		Mean Runtime (ms)		Min Runtime (ms)	
	NDSSE	LDSSE	NDSSE	LDSSE	NDSSE	LDSSE	NDSSE	LDSSE	NDSSE	LDSSE	NDSSE	LDSSE
13-bus	7.64	10.46	6.09	7.48	0.59	0.76	10.71	1.26	2.42	0.55	1.53	0.27
37-bus	4.28	3.61	2.82	1.87	0.27	0.19	34.16	5.42	10.99	1.88	6.02	0.83
123-bus	5.77	4.80	4.87	3.59	0.46	0.34	337.29	29.06	165.69	11.06	82.01	6.54
906-bus	16.19	10.83	14.22	8.12	1.34	0.79	68008.98	4213.15	44683.96	2636.51	8733.04	1586.63
8500-bus	17.91	10.98	14.71	8.73	1.40	0.85	234779.51	12879.73	175789.82	8897.23	147663.44	7672.37

REFERENCES

[1] F. F. Wu, K. Moslehi, and A. Bose, "Power system control centers: past, present, and future," *Proceedings of the IEEE,* vol. 93, no. 11, pp. 1890-1908, 2005.

[2] A. Abur, and A. G. Exposito, *Power system state estimation: theory and implementation*: CRC press, 2004.

[3] J. Y. Fan, and J. D. McDonald, "A real-time implementation of short-term load forecasting for distribution power systems," *IEEE Transactions on Power Systems,* vol. 9, no. 2, pp. 988-994, 1994.

[4] I. Series, *Microgrids and active distribution networks*: The institution of Engineering and Technology, 2009.

[5] A. Angioni, J. Shang, F. Ponci, and A. Monti, "Real-time monitoring of distribution system based on state estimation," *IEEE Transactions on Instrumentation and Measurement,* vol. 65, no. 10, pp. 2234-2243, 2016.

[6] F. Ahmad, A. Rasool, E. Ozsoy, R. Sekar, A. Sabanovic, and M. Elitaş, "Distribution system state estimation-A step towards smart grid," *Renewable and Sustainable Energy Reviews,* vol. 81, pp. 2659-2671, 2018.

[7] M. E. Baran, and A. W. Kelley, "State estimation for real-time monitoring of distribution systems," *IEEE Transactions on Power Systems,* vol. 9, no. 3, pp. 1601-1609, 1994.

[8] M. E. Baran, and A. W. Kelley, "A branch-current-based state estimation method for distribution systems," *IEEE Transactions on Power Systems,* vol. 10, no. 1, pp. 483-491, 1995.

[9] D. D. Giustina, M. Pau, P. A. Pegoraro, F. Ponci, and S. Sulis, "Electrical distribution system state estimation: measurement issues and challenges," *IEEE Instrumentation & Measurement Magazine,* vol. 17, no. 6, pp. 36-42, 2014.

[10] K. Dehghanpour, Z. Wang, J. Wang, Y. Yuan, and F. Bu, "A survey on state estimation techniques and challenges in smart distribution systems," *IEEE Transactions on Smart Grid,* vol. 10, no. 2, pp. 2312-2322, 2019.

[11] A. Primadianto, and C. Lu, "A review on distribution system state estimation," *IEEE Transactions on Power Systems,* vol. 32, no. 5, pp. 3875-3883, 2017.

[12] L. Whei-Min, T. Jen-Hao, and C. Shi-Jaw, "A highly efficient algorithm in treating current measurements for the branch-current-based distribution state estimation," *IEEE Transactions on Power Delivery,* vol. 16, no. 3, pp. 433-439, 2001.

[13] D. Youman, H. Ying, and Z. Boming, "A branch-estimation-based state estimation method for radial distribution systems," *IEEE Transactions on Power Delivery,* vol. 17, no. 4, pp. 1057-1062, 2002.

[14] W. Haibin, and N. N. Schulz, "A revised branch current-based distribution system state estimation algorithm and meter placement impact," *IEEE Transactions on Power Systems,* vol. 19, no. 1, pp. 207-213, 2004.

[15] R. Singh, B. C. Pal, and R. A. Jabr, "Choice of estimator for distribution system state estimation," *IET generation, transmission & distribution,* vol. 3, no. 7, pp. 666-678, 2009.

[16] E. Manitsas, R. Singh, B. C. Pal, and G. Strbac, "Distribution system state estimation using an artificial neural network approach for pseudo measurement modeling," *IEEE Transactions on power systems,* vol. 27, no. 4, pp. 1888-1896, 2012.

[17] C. Gomez-Quiles, A. Gomez-Exposito, and A. d. l. V. Jaen, "State estimation for smart distribution substations," *IEEE Transactions on Smart Grid,* vol. 3, no. 2, pp. 986-995, 2012.

[18] M. Pau, P. A. Pegoraro, and S. Sulis, "Efficient branch-current-based distribution system state estimation including synchronized measurements," *IEEE Transactions on Instrumentation and Measurement,* vol. 62, no. 9, pp. 2419-2429, 2013.

[19] C. Muscas, M. Pau, P. A. Pegoraro, and S. Sulis, "Effects of measurements and pseudomeasurements correlation in distribution system state estimation," *IEEE Transactions on Instrumentation and Measurement,* vol. 63, no. 12, pp. 2813-2823, 2014.

[20] M. Kabiri, and N. Amjady, "A new hybrid state estimation considering different accuracy levels of PMU and SCADA measurements," *IEEE Transactions on Instrumentation and Measurement,* vol. 68, no. 9, pp. 3078-3089, 2019.

[21] R. Singh, B. C. Pal, and R. A. Jabr, "Distribution system state estimation through Gaussian mixture model of the load as pseudo-measurement," *IET generation, transmission & distribution,* vol. 4, no. 1, pp. 50-59, 2010.

[22] W. H. Kersting, "Distribution system modeling and analysis," *Electric Power Generation, Transmission, and Distribution: The Electric Power Engineering Handbook,* pp. 26-1: CRC press, 2018.

[23] M. Bazrafshan, and N. Gatsis, "Comprehensive modeling of three-phase distribution systems via the bus admittance matrix," *IEEE Transactions on Power Systems,* vol. 33, no. 2, pp. 2015-2029, 2018.

[24] M. Pau, and Z. Tamim, "A backward–forward sweep algorithm for distribution system state estimation," *IEEE Transactions on Instrumentation and Measurement,* vol. 72, pp. 1-11, 2023.

[25] D. A. Haughton, and G. T. Heydt, "A linear state estimation formulation for smart distribution systems," *IEEE Transactions on Power Systems,* vol. 28, no. 2, pp. 1187-1195, 2013.

[26] A. Salehi, M. Fotuhi-Firuzabad, S. Fattaheian-Dehkordi, M. Gholami, and M. Lehtonen, "Developing an optimal framework for PMU placement based on active distribution system state estimation considering cost-worth analysis," *IEEE Access,* vol. 11, pp. 12088-12099, 2023.

[27] Y. Zhang, and J. Wang, "Towards highly efficient state estimation with nonlinear measurements in distribution systems," *IEEE Transactions on Power Systems,* vol. 35, no. 3, pp. 2471-2474, 2020.

[28] W. M. Lin, and J. H. Teng, "Distribution fast decoupled state estimation by measurement pairing," *IEE Proceedings-Generation, Transmission and Distribution,* vol. 143, no. 1, pp. 43-48, 1996.

[29] Y. Ju, W. Wu, F. Ge, K. Ma, Y. Lin, and L. Ye, "Fast decoupled state estimation for distribution networks considering branch ampere measurements," *IEEE Transactions on Smart Grid,* vol. 9, no. 6, pp. 6338-6347, 2018.

[30] M. Göl, and A. Abur, "A Robust PMU Based Three-phase state estimator using modal decoupling," *IEEE Transactions on Power Systems,* vol. 29, no. 5, pp. 2292-2299, 2014.

[31] T. R. Fernandes, B. Venkatesh, and M. C. d. Almeida, "Symmetrical components based state estimator for power distribution systems," *IEEE Transactions on Power Systems,* vol. 36, no. 3, pp. 2035-2045, 2021.

[32] R. Khalili, and A. Abur, "PMU-based decoupled state estimation for unsymmetrical power systems," *IEEE Transactions on Power Systems,* vol. 36, no. 6, pp. 5359-5368, 2021.

[33] R. Schincariol da Silva, T. R. Fernandes, and M. C. de Almeida, "Specifying angular reference for three-phase distribution system state estimators," *IET Generation, Transmission & Distribution,* vol. 12, no. 7, pp. 1655-1663, 2018.

[34] A. L. Langner, and A. Abur, "Formulation of three-phase state estimation problem using a virtual reference," *IEEE Transactions on Power Systems,* vol. 36, no. 1, pp. 214-223, 2021.

[35] A. Nassaj, A. A. Jahromi, K. Li, V. Terzija, and S. Azizi, "Linear distribution system state estimation by hybrid synchronized/unsynchronized measurements," in 2023 IEEE 13th International Workshop on Applied Measurements for Power Systems (AMPS), 2023, pp. 01-06.

[36] A. S. Dobakhshari, S. Azizi, M. Paolone, and V. Terzija, "Ultra fast linear state estimation utilizing SCADA measurements," *IEEE Transactions on Power Systems,* vol. 34, no. 4, pp. 2622-2631, 2019.

[37] A. Nassaj, M. R. Jegarluei, A. A. Jahromi, V. Terzija, and S. Azizi, "Enhanced linear state estimation for power systems using purely SCADA measurements," in 2023 IEEE Belgrade PowerTech, 2023, pp. 1-6.

[38] K. S. Miller, "Complex linear least squares," *SIAM Review,* vol. 15, no. 4, pp. 706-726, 1973.

[39] A. S. Dobakhshari, M. Abdolmaleki, V. Terzija, and S. Azizi, "Online non-iterative estimation of transmission line and transformer parameters by SCADA data," *IEEE Transactions on Power Systems,* vol. 36, no. 3, pp. 2632-2641, 2021.

[40] A. S. Dobakhshari, M. Abdolmaleki, V. Terzija, and S. Azizi, "Robust hybrid linear state estimator utilizing SCADA and PMU measurements," *IEEE Transactions on Power Systems,* vol. 36, no. 2, pp. 1264-1273, 2020.

[41] I. Hughes, and T. Hase, *Measurements and their uncertainties: a practical guide to modern error analysis*: OUP Oxford, 2010.

[42] "OpenDSS: Distribution system simulator," Available: https://www.epri.com/pages/sa/opendss.

[43] "IEEE PES test feeder," Available: https://cmte.ieee.org/pes-testfeeders/resources/.