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18 **1 Introduction**

19 Exit problems for stochastic processes is a classic topic in applied probability and has
20 received a great deal of attention within the literature. In the continuous setting (time and
21 space), exit problems for so-called upward skip-free processes, known in the literature as
22 ‘spectrally negative Lévy processes’, have been extensively considered in [5] (Chapter VII),
23 [18] (Chapter 8) and references therein, by means of fluctuation theory where semi-explicit
24 expressions are derived in terms of the so-called ‘scale functions’. On the other hand, in the
25 fully discrete setting exit problems for general discrete-time random walks are excellently
26 treated in [10] and [13], among others, by means of probabilistic arguments and include,
27 as particular cases, the corresponding upward skip-free random walks. That is, a random
28 walk for which downward jumps are unrestricted but upward jumps are constrained to a
29 magnitude of at most one, emulating the upward ‘drift’ in continuous-time. More recently,
30 [3] implement the ideas underlying the exit problems for continuous spectrally negative
31 Lévy processes for their discrete random walk counterparts and derive exit problems and
32 other fluctuation identities in terms of analogous ‘discrete scale functions’.

33 A natural generalisation of the above processes are the broad family of Markov Addi-
34 tive Processes (MAPs), which incorporate an externally influencing Markov environment,
35 providing greater flexibility to the characteristics of the underlying process in terms of its
36 claim frequency and severity distributions, see [1] (Chapter XI). Within this generalised
37 framework, the existence of multidimensional scale functions, known as ‘scale matrices’,
38 was first discussed in [19] and were used to derive fluctuation identities and first passage
39 results for continuous-time MAPs. [15] extended the initial findings of [19] by providing
40 the probabilistic construction of the scale matrices, identifying their transforms and con-
41 sidering an extensive study of exit problems including one-sided and two-sided exits, as
42 well as exits for reflected processes via implementation of the occupation density formula.
43 Further studies on MAPs and their exit/passage times can be found in [8], [4], [9], among
44 others. More recently, [17], derive and compare results for continuous-time MAPs with
45 lattice (discrete-space) and non-lattice support. It is worth noting here that the authors
46 in this work do discuss some of the corresponding results for the fully-discrete (time and
47 space) MAP model considered in this paper, however, only a limited number of results are
48 stated and a variety of important steps and proofs were omitted.

49 This paper bridges the gap between the aforementioned works and provides a theoretical
50 framework for fully discrete, upward skip-free MAPs, in terms of ‘discrete scale matrices’,
51 spelling out the differences in results, methodologies and necessary adjustments for deriving
52 fluctuation identities between discrete and continuous MAPs. In particular, we derive
53 results for the first passage theory, including one and two-sided exit problems as well as
54 the under(over)-shoots upon exit via the associated ‘reflected’ process. The motivation for
55 deriving such a framework comes from the discrete set up having known advantages over
56 the continuous-time models. For example, it is known that the Wiener-Hopf factorisation
57 can be replaced by a simple Laurent series (see [3]). Moreover, due to the equivalence

58 between a discrete MAP and a Markov-modulated random walk, this paper provides a
 59 more flexible random walk model and enriches the numerous applications of random walk
 60 theory across a variety of disciplines.

61 The paper is organised as follows: In Section 2 we define the MAP in discrete time
 62 and space and derive the so-called occupation mass matrix formula, from which we obtain
 63 some useful identities to be used in the following sections. In Section 3, we introduce some
 64 fundamental matrices associated to the discrete MAP, identify the first of two discrete scale
 65 matrices and derive matrix expressions for the one and two-sided upward exit problem. In
 66 Section 4, we derive results for the corresponding one and two-sided reflected processes,
 67 including the over-shoot and under-shoot upon exit which are then used in Section 5 to
 68 derive expressions for the one and two-sided downward exit problems of the original (non-
 69 reflected) discrete MAP.

70 2 Preliminaries

71 A fully discrete (time and space) MAP, which we will call a *Markov Additive Chain* (MAC),
 72 is defined as a bivariate discrete-time Markov chain $(X, J) = \{(X_n, J_n)\}_{n \geq 0}$, on the product
 73 space $\mathbb{Z} \times E$, where $X_n \in \mathbb{Z}$ describes the *level* of the underlying process, whilst $J_n \in$
 74 $E = \{1, 2, \dots, N\}$ describes the *phase* of some external Markov chain (which affects the
 75 dynamics of X_n) having transition probability matrix \mathbf{P} , such that for $i, j \in E$, $(\mathbf{P})_{ij} =$
 76 $\mathbb{P}(J_1 = j | J_0 = i)$. It is assumed throughout this work, that the Markov chain $\{J_n\}_{n \geq 0}$
 77 is ergodic such that its stationary distribution $\boldsymbol{\pi}^\top = (\pi_1, \dots, \pi_N)$ exists and is unique.
 78 The defining property of the MAC is the conditional independence and stationarity of
 79 law governing X_n , given J_n . That is, given that $\{J_T = i\}$ for some fixed $T \in \mathbb{N}$, the
 80 Markov chain $\{(X_{T+n} - X_T, J_{T+n})\}$ is independent of \mathcal{F}_T (the natural filtration to which
 81 the bivariate process (X, J) is adapted) and $\{(X_{T+n} - X_T, J_{T+n})\} \stackrel{d}{=} \{(X_n - X_0, J_n)\}$,
 82 given $\{J_0 = i\}$ for any phase state $i \in E$. This is known as the Markov additive property,
 83 a consequence of which is that the level process $\{X_n\}_{n \geq 0}$ is translation invariant on the
 84 lattice.

85 Intuitively, the MAC is simply a Markov-modulated random walk where $\{X_n\}_{n \geq 0}$
 86 evolves according to the random walk $X_n = Y_1 + Y_2 + \dots + Y_n$, where $\{Y_k\}_{k \geq 1}$ is a
 87 sequence of conditionally i.i.d. random variables with common, conditional distribution
 88 $\tilde{q}_{ij}(y) = \mathbb{P}(Y_1 = y | J_1 = j, J_0 = i)$, and thus, probability mass matrix $\tilde{\mathbf{Q}}(y)$, with i, j -th
 89 element $(\tilde{\mathbf{Q}}(y))_{ij} = \tilde{q}_{ij}(y)$. As such, and due to the invariance property, the transition
 90 probability matrix of (X, J) has a block-like structure with blocks $\tilde{\mathbf{A}}_m$ which represent the
 91 (one-step) transition matrix for an increase of m levels in $\{X_n\}_{n \geq 0}$ whilst capturing the
 92 phase transitions of $\{J_n\}_{n \geq 0}$, such that

$$93 \quad \tilde{\mathbf{A}}_m = \tilde{\mathbf{Q}}(m) \circ \mathbf{P}, \quad (2.1)$$

94 where \circ denotes entry-wise products (Hadamard multiplication). In the remainder of this

95 paper, we assume that $X = \{X_n\}_{n \geq 0}$ may only increase by at most one level per unit time
 96 whilst experiencing downward jumps of arbitrary size. That is, for all $i, j \in E$, we have
 97 $\tilde{q}_{ij}(m) = \mathbb{P}(Y_1 = m | J_1 = j, J_0 = i) \geq 0$ for $m \leq 1$ and $\tilde{q}_{ij}(m) = 0$ otherwise, which leads
 98 to $\tilde{\mathbf{Q}}(m) = \mathbf{0}$ and thus $\tilde{\mathbf{A}}_m = \mathbf{0}$ for $m = 2, 3, \dots$. In this sense, we say that X possesses
 99 an ‘upward skip-free’ property, an advantage of which is that the value of X is known
 100 at stopping time corresponding to ‘upward’ crossing/hitting of a given level (see below).
 101 This corresponds to the discrete analogue of a ‘spectrally negative’ MAP in the continuous
 102 setting, which have important applications to workload and surplus processes in queuing
 103 and risk theory, respectively (see [1] and [2] for more details).

104 2.1 MAC Matrix Generator

105 It has already been noted that the dynamics of the level process (X) within the MAC
 106 depends on the phase transitions of the external Markov chain (J). As such, the majority
 107 of quantities and results presented in this paper depend on the initial and final states
 108 of $\{J_n\}_{n \geq 0}$ and thus, are given in matrix form. With this in mind, let us define the
 109 expectation matrix operator $\mathbb{E}_x(\cdot; J_n)$ which denotes an $N \times N$ matrix with i, j -th element
 110 $(\mathbb{E}_x(\cdot; J_n))_{ij} = \mathbb{E}(\cdot 1_{(J_n=j)} | X_0 = x, J_0 = i)$, where $1_{(\cdot)}$ represents the indicator function,
 111 and corresponding probability matrix $\mathbb{P}_x(\cdot, J_n)$ with elements $(\mathbb{P}_x(\cdot, J_n))_{ij} = \mathbb{P}(\cdot, J_n =$
 112 $j | X_0 = x, J_0 = i)$. Moreover, we denote $\mathbb{E}(\cdot; J_n) \equiv \mathbb{E}_0(\cdot; J_n)$, having associated probability
 113 measure $\mathbb{P}(\cdot, J_n) \equiv \mathbb{P}_0(\cdot, J_n)$ and thus, we can define the probability generating matrix
 114 (p.g.m.) of the process $\{X_n\}_{n \geq 0}$ with initial level $X_0 = 0$, for at least $|z| \leq 1$ and $z \neq 0$, by
 115 $\mathbb{E}(z^{-X_n}; J_n)$, which satisfies

$$116 \quad \mathbb{E}(z^{-X_n}; J_n) = (\tilde{\mathbf{F}}(z))^n, \quad \tilde{\mathbf{F}}(z) := \mathbb{E}(z^{-X_1}; J_1) = \sum_{m=-1}^{\infty} z^m \tilde{\mathbf{A}}_{-m}, \quad (2.2)$$

117 and for $z = 1$, we have $\tilde{\mathbf{F}}(1) = \mathbf{P} = \sum_{m=-1}^{\infty} \tilde{\mathbf{A}}_{-m}$.

118 **Remark 1.** Note that since the matrices $\tilde{\mathbf{A}}_{-m}$ are probability transition matrices, such that
 119 $\tilde{\mathbf{A}}_{-m} \geq 0$ (non-negative), it follows that for $z > 0$, the matrix $\tilde{\mathbf{F}}(z)$ is also non-negative.
 120 Hence, by the Perron-Frobenius theorem, $\tilde{\mathbf{F}}(z)$ has a (simple) eigenvalue, denoted $\kappa(z)$,
 121 which is greater than or equal in absolute value than all other eigenvalues with correspond-
 122 ing left and right (column) eigenvectors, denoted $\tilde{\mathbf{v}}(z)$ and $\tilde{\mathbf{h}}(z)$, respectively, such that
 123 $\tilde{\mathbf{v}}(z)^\top \tilde{\mathbf{F}}(z) = \kappa(z) \tilde{\mathbf{v}}(z)^\top$ and $\tilde{\mathbf{F}}(z) \tilde{\mathbf{h}}(z) = \kappa(z) \tilde{\mathbf{h}}(z)$. Moreover, since $\tilde{\mathbf{F}}(1) = \mathbf{P}$ is a stochas-
 124 tic matrix, using standard facts from matrix analysis (see [7]) we have $\kappa(1) = 1$ and it
 125 can be shown that $\kappa'(1)$ determines the asymptotic drift of the level process $\{X_n\}_{n \geq 0}$ (see
 126 Section 1.3 in [22] and [11]), given by

$$127 \quad \lim_{n \rightarrow \infty} \frac{X_n}{n} = -\kappa'(1) = -\boldsymbol{\pi}^\top \sum_{m=-1}^{\infty} m \tilde{\mathbf{A}}_{-m} \mathbf{e}.$$

128

129 Within the theory of continuous-time Lévy processes, it is often desirable to analyse the
 130 process prior to some independent exponential ‘killing time’ as this can emulate the role of
 131 Laplace transforms or generating functions within the calculations (see [18]). For a MAP,
 132 this exponential killing time can alternatively be incorporated via an enlargement to the
 133 state space of the Markov chain with the addition of an ‘absorbing’ (killing) state and
 134 analysing the process prior to absorption (see [15] for details).

135 In a similar way, let us enlarge the state space E to $E \cup \{\dagger\}$, where \dagger denotes an
 136 absorbing state, often called the *cemetery* state, and we set $X = \partial$ whenever $J = \dagger$.
 137 Moreover, let us assume that the (one-step) ‘absorption’ probability is the same from all
 138 states and denoted by $1 - v = \mathbb{P}(J_1 = \dagger | J_0 = i)$, for all $i \in E$, such that the corresponding
 139 ‘non-absorption’ probability (survival) is given by $v \in (0, 1]$. Now, due to the addition of
 140 this cemetery state, it is clear that the probability transition matrix for transitions between
 141 the ‘transient’ (when $v < 1$) states of E is dependent on v . Let us define this by $\mathbf{P}(v) \equiv v\mathbf{P}$,
 142 where \mathbf{P} denotes the stochastic probability transition matrix defined in Section 2, in the
 143 absence of an absorbing state or ‘killing’ ($v = 1$). Hence, it follows that $\mathbf{P}(v)\vec{e} = v\mathbf{P}\vec{e} = v\vec{e}$
 144 and thus, for $v < 1$, $\mathbf{P}(v)$ is sub-stochastic and its Perron-Frobenius eigenvalue is less than
 145 1 (see [7]). Finally, it follows that the absorption or ‘killing’ time of the Markov chain,
 146 denoted $g_v = \inf\{n > 0 : J_n = \dagger\}$, is geometrically distributed with parameter $v \in (0, 1]$
 147 and we have

$$148 \quad \mathbb{E}(z^{-X_n}; n < g_v, J_n) = v^n \tilde{\mathbf{F}}(z)^n = (v\tilde{\mathbf{F}}(z))^n = (\tilde{\mathbf{F}}^v(z))^n \quad (2.3)$$

149 where $\tilde{\mathbf{F}}^v(z) := \mathbb{E}(z^{-X_1}; 1 < g_v, J_1) = v\tilde{\mathbf{F}}(z)$ with $\tilde{\mathbf{F}}(z)$ denoting the matrix generator of
 150 the MAC in the absence of killing, as defined as in Eq. (2.2). The connection between
 151 the killed process and transforms/generating functions of the non-killed process is evident
 152 when we note that Eq. (2.3) is equivalent to $\mathbb{E}(v^n z^{-X_n}; J_n)$ for a ‘non-killed’ MAC. Further
 153 advantages of working with the killed process are discussed in more details in later sections.
 154 Throughout the remainder of this paper, we generally suppress the explicit notation that
 155 absorption has not yet occurred but point out that it is assumed implicitly. As such, the
 156 results derived in the following are, in fact, much more general than they appear, with only
 157 a handful of these generalisations being stated explicitly.

158 2.2 Occupation Times

159 It is well known that occupation times and their densities play an important role within
 160 the theory of Lévy processes and their fluctuations. In a continuous environment, the
 161 definition of the occupation density/time of a process at a given level has to be treated
 162 with some care and detail (see [5], [15]) however, in the fully discrete model considered in
 163 this paper, the mathematical definition is intuitive.

164 Let us define by $\tilde{L}(x, j, n)$, the *occupation mass* denoting the number of periods the

165 process $\{(X_n, J_n)\}_{n \geq 0}$ is in state $(x, j) \in \mathbb{Z} \times E$, up to and including time $n \geq 0$, such that

$$166 \quad \tilde{L}(x, j, n) = \sum_{k=0}^n \mathbf{1}_{(X_k=x, J_k=j)}. \quad (2.4)$$

167 Then, for some measurable non-negative function f , we have the so-called discrete *occupa-*
 168 *tion mass formula*

$$169 \quad \sum_{k=0}^n f(X_k) \mathbf{1}_{(J_k=j)} = \sum_{x \in \mathbb{Z}} f(x) \tilde{L}(x, j, n). \quad (2.5)$$

171 From the above definition, it is clear that $\tilde{L}(x, j, n)$ is a non-decreasing (monotone) process
 172 in $n \geq 0$, which is adapted to the natural filtration \mathcal{F}_n . Therefore, if we further define
 173 the N -dimensional square occupation mass matrix, denoted $\tilde{\mathbf{L}}(x, n)$, with i, j -th element
 174 given by $(\tilde{\mathbf{L}}(x, n))_{ij} = \mathbb{E}(\tilde{L}(x, j, n) | J_0 = i)$. Then, by application of the strong Markov
 175 property, analogously to Proposition 8 in [15], we have the following proposition.

176 **Proposition 1.** *Let*

$$\tau_x = \inf\{n \geq 0 : X_n = x\},$$

177 *denote the first ‘hitting’ time of the level $x \in \mathbb{Z}$. Then, for the occupation mass matrix*
 178 $\tilde{\mathbf{L}}(x, n)$, *it follows that*

$$179 \quad \tilde{\mathbf{L}}(x, \infty) = \mathbb{P}(\tau_x < \infty, J_{\tau_x}) \tilde{\mathbf{L}}, \quad (2.6)$$

180 *where $(\mathbb{P}(\tau_x < \infty, J_{\tau_x}))_{ij} = \mathbb{P}(\tau_x < \infty, J_{\tau_x} = j | J_0 = i)$ and $\tilde{\mathbf{L}} := \tilde{\mathbf{L}}(0, \infty)$ is the occupation*
 181 *mass matrix at the level 0 over an infinite-time horizon, which has strictly positive entries.*

182 **Remark 2.** *Let us point out some of the advantages of working with the killed process at*
 183 *this point:*

184 (i) *If we include the implicit killing in the calculations explicitly, then for $v \in (0, 1]$, the*
 185 *probability $\mathbb{P}(\tau_x < \infty, J_{\tau_x})$ becomes*

$$186 \quad \mathbb{P}(\tau_x < g_v, J_{\tau_x}) = \sum_{n=0}^{\infty} \mathbb{P}(\tau_x = n, n < g_v, J_n) = \sum_{n=0}^{\infty} v^n \mathbb{P}(\tau_x = n, J_n) = \mathbb{E}(v^{\tau_x}; J_{\tau_x}),$$

188 *where in the second equality we have used the fact that $\mathbf{P}(v) = v\mathbf{P}$. That is, the*
 189 *probability matrix $\mathbb{P}(\tau < \infty, J_{\tau_x})$ becomes the generator matrix $\mathbb{E}(v^{\tau_x}; J_{\tau_x})$, if one*
 190 *imposes ‘killing’ explicitly. As mentioned above, throughout this work we will keep*
 191 *killing implicit as it greatly simplifies the presentation but highlight that the above*
 192 *idea holds for all results.*

193 (ii) Similarly, by superimposing killing in Proposition 1, we have that

$$194 \quad \tilde{\mathbf{L}}_v(x, \infty) = \mathbb{E}(v^{\tau_x}; J_{\tau_x}) \tilde{\mathbf{L}}_v,$$

195 with $\tilde{\mathbf{L}}_v := \tilde{\mathbf{L}}_v(0, \infty)$, such that the i, j -th element of $\tilde{\mathbf{L}}_v(x, n)$ is given by $(\tilde{\mathbf{L}}_v(x, n))_{ij} =$
 196 $\mathbb{E}(\tilde{L}_v(x, j, n) | J_0 = i)$, where $\tilde{L}_v(x, j, n) = \sum_{k=0}^n \mathbf{1}_{(X_k=x, J_k=j, k < g_v)}$. Note that since \mathbf{P}
 197 is sub-stochastic, then $\{X_k = x\}$ implies that $\{k < g_v\}$ and thus $\tilde{L}_v(x, j, n)$ coincides
 198 with $\tilde{L}(x, j, n)$.

199 The main reason for introducing the theory of occupation times and their associated mass
 200 matrices, is due to their relationship with the one step p.g.m., namely $\tilde{\mathbf{F}}(z)$. This connection
 201 is highlighted in the following auxiliary theorem which provides the foundations for many
 202 of the results in the following sections.

203 **Theorem 1.** For all $z \in (0, 1]$ such that $\mathbf{I} - \tilde{\mathbf{F}}(z)$ is non-singular, it follows that

$$204 \quad \sum_{x \in \mathbb{Z}} z^{-x} \mathbb{P}(\tau_x < \infty, J_{\tau_x}) \tilde{\mathbf{L}} = (\mathbf{I} - \tilde{\mathbf{F}}(z))^{-1}, \quad (2.7)$$

205 where τ_x is the first hitting time of the level $x \in \mathbb{Z}$.

206 *Proof.* First note by the occupation mass formula, that for any $j \in E$, we have

$$207 \quad \sum_{k=0}^n z^{-X_k} \mathbf{1}_{(J_k=j)} = \sum_{x \in \mathbb{Z}} z^{-x} \tilde{L}(x, j, n).$$

208 Taking expectations in the above equation and considering the limit as $n \rightarrow \infty$, yields

$$209 \quad \lim_{n \rightarrow \infty} \sum_{k=0}^n \mathbb{E}(z^{-X_k} \mathbf{1}_{(J_k=j)} | J_0 = i) = \lim_{n \rightarrow \infty} \sum_{x \in \mathbb{Z}} z^{-x} \mathbb{E}(\tilde{L}(x, j, n) | J_0 = i),$$

210 from which, since z^{-x} is non-negative for $z > 0$, we can apply the monotone convergence
 211 theorem to obtain

$$212 \quad \sum_{k=0}^{\infty} \mathbb{E}(z^{-X_k} \mathbf{1}_{(J_k=j)} | J_0 = i) = \sum_{x \in \mathbb{Z}} z^{-x} \mathbb{E}(\tilde{L}(x, j, \infty) | J_0 = i).$$

213 Equivalently, in matrix form the above expression can be written as

$$214 \quad \sum_{k=0}^{\infty} \tilde{\mathbf{F}}(z)^k = \sum_{x \in \mathbb{Z}} z^{-x} \tilde{\mathbf{L}}(x, \infty) = \sum_{x \in \mathbb{Z}} z^{-x} \mathbb{P}(\tau_x < \infty, J_{\tau_x}) \tilde{\mathbf{L}} \quad (2.8)$$

216 where the last equality comes from the result of Proposition 1. Finally, we note that the
 217 geometric series on the l.h.s. converges to $(\mathbf{I} - \tilde{\mathbf{F}}(z))^{-1}$ as long as $\kappa(z) < 1$ and the result
 218 follows using analytic continuation to extend the domain of convergence to all $z \in (0, 1]$
 219 such that $(\mathbf{I} - \tilde{\mathbf{F}}(z))^{-1}$ exists. \square

220 **Remark 3.** Note that the result of Theorem 1 holds in the presence of killing ($v < 1$), since
 221 $\mathbf{P}(v)$ is sub-stochastic and thus $\kappa^v(1) < 1$, where $\kappa^v(z)$ is the Perron-Frobenius eigenvalue
 222 of $\mathbf{F}^v(z)$. Hence, by continuity of $\kappa^v(z)$, there exists a small interval around $z = 1$ for
 223 which $\kappa^v(z) < 1$. In addition, \mathbf{L} must have finite entries as under killing the Markov chain
 224 is transient and the expected number of visits to any state is finite.

225 3 Upward Exit Problems

226 In this section we discuss and derive results on exit problems for upward skip-free MACs
 227 above and below a fixed level or strip. In the first instance, we will utilise the upward
 228 skip-free property of the level process, $\{X_n\}_{n \geq 0}$, to determine expressions for upward exit
 229 times (one and two-sided), then extend the theory to consider downward exit problems.
 230 These expressions are given in terms of so-called *fundamental* and *scale matrices* associated
 231 to the MAC, where the existence of the latter were first discussed in [19] and extend the
 232 notion of scale functions associated to Lévy processes (see [18] and [3] for more details).

233 All the results given in this section are stated from an initial level $X_0 = 0$ which, due
 234 to the invariance property, can be generalised to an arbitrary level, say $x_0 \in \mathbb{Z}$, via an
 235 appropriate shift.

236 Let us denote by τ_x^\pm , the first time the level process $\{X_n\}_{n \geq 0}$ up(down)-crosses the
 237 level $x \in \mathbb{Z}$, such that

$$238 \quad \tau_x^+ = \inf\{n \geq 0 : X_n \geq x\} \quad \text{and} \quad \tau_x^- = \inf\{n \geq 0 : X_n \leq x\}. \quad (3.1)$$

239 We note that in a ‘spectrally negative’ MAC with upward drift of one per unit time, for
 240 $x \geq X_0$ the random stopping times τ_x^+ (crossing time) and τ_x (hitting time) coincide.
 241 Moreover, we have that $X_{\tau_x^+} = X_{\tau_x} = x$.

242 3.1 One-Sided Exit Upward

243 The key observation for the first passage upwards, is that the stationary and independent
 244 increments as well as the skip-free property provide an embedded Markov structure. To
 245 see this, recall that $X_{\tau_1^+} = X_{\tau_1} = 1$, which together with the strong Markov and Markov
 246 additive properties, imply that the process $\{J_{\tau_n}\}_{n \geq 0}$ is a (time-homogeneous) discrete-
 247 time Markov chain, given $X_0 = 0$, with some probability transition matrix $\tilde{\mathbf{G}}$, such that
 248 for $a \geq 0$,

$$249 \quad \mathbb{P}(\tau_a < \infty, J_{\tau_a}) = \tilde{\mathbf{G}}^a, \quad \tilde{\mathbf{G}} = \mathbb{P}(\tau_1 < \infty, J_{\tau_1}), \quad (3.2)$$

250 with i, j -th element given by $(\tilde{\mathbf{G}})_{ij} = \mathbb{P}(\tau_1 < \infty, J_{\tau_1} = j \mid J_0 = i)$ for $i, j \in E$.

251 **Remark 4.** In the case of no killing, i.e., $v = 1$ and $\kappa'(1) \leq 0$ (non-negative drift), the
 252 matrix $\tilde{\mathbf{G}}$ is a stochastic matrix and sub-stochastic otherwise.

253 The transition probability matrix $\tilde{\mathbf{G}}$ is widely known as the *fundamental matrix* of the
 254 MAC and contains the probabilistic characteristics to determine upward passage times
 255 and the corresponding phase state at passage. That is, determining the matrix $\tilde{\mathbf{G}}$ provides
 256 the probability of hitting any upper level $a \geq 0$ and the phase of $\{J_n\}_{n \geq 0}$ at this hitting
 257 time.

258 The matrix $\tilde{\mathbf{G}}$ has a long history in the theory of structured stochastic matrices (see
 259 for e.g., Lemma 4.2 in [7]) and can be computed by conditioning on the first time period,
 260 i.e.,

$$261 \quad \tilde{\mathbf{G}} = \mathbb{P}(\tau_1 < \infty, J_{\tau_1}) = \sum_{m=-1}^{\infty} \tilde{\mathbf{A}}_{-m} \tilde{\mathbf{G}}^{m+1} = \left(\sum_{m=-1}^{\infty} \tilde{\mathbf{A}}_{-m} \tilde{\mathbf{G}}^m \right) \tilde{\mathbf{G}}.$$

263 Multiplying on the right by $\tilde{\mathbf{G}}^{-1}$, assuming it exists (see Remark 7), and using the definition
 264 of $\tilde{\mathbf{F}}(z)$ given in Eq. (2.2), it follows that the fundamental matrix, $\tilde{\mathbf{G}}$, is the right solution
 265 of $\tilde{\mathbf{F}}(\cdot) = \mathbf{I}$, which is a well known equation established in [21] and further studied in [7],
 266 [22], [11] and [12], among others.

267 **Remark 5.** *Let us discuss a few important observations about the fundamental matrix $\tilde{\mathbf{G}}$*
 268 *and its significance within applied probability:*

269 (i) *For the continuous-time (scalar) spectrally negative Lévy process, the fundamental*
 270 *matrix $\tilde{\mathbf{G}}$, corresponds to the of inverse Laplace exponent at zero, namely $\Phi(0)$, i.e.,*
 271 *the solution to $\psi(\beta) = 0$, where $\psi(\beta)$ denotes the Laplace exponent of the Lévy process*
 272 *(see [18]).*

273 (ii) *It follows by definition that $\mathbb{E}(\tilde{\mathbf{G}}^{-X_n}; J_n)$ is a martingale. In fact, it is clear that in*
 274 *the matrix setting, there exists another solution (left solution) to $\tilde{\mathbf{F}}(\cdot) = \mathbf{I}$, say $\tilde{\mathbf{R}}$,*
 275 *which would also result in the martingale $\mathbb{E}(\tilde{\mathbf{R}}^{-X_n}; J_n)$. It turns out that the matrix*
 276 *$\tilde{\mathbf{R}}$ is actually the counterpart of $\tilde{\mathbf{G}}$ for the ‘time-reversed MAC’ and is considered*
 277 *another fundamental matrix. The time-reversed MAP and the corresponding matrix*
 278 *$\tilde{\mathbf{R}}$ are considered in [16] for the continuous-time (lattice) case and we direct the reader*
 279 *to this paper for more details.*

280 (iii) *Superimposing killing in the above produces the transform of the first passage time,*
 281 *namely $\mathbb{E}(v^{\tau_a}; J_{\tau_a})$, such that*

$$282 \quad \mathbb{E}(v^{\tau_a}; J_{\tau_a}) = \tilde{\mathbf{G}}_v^a, \quad \tilde{\mathbf{G}}_v = \mathbb{E}(v^{\tau_1}; J_{\tau_1}), \quad (3.3)$$

283 and $\tilde{\mathbf{G}}_v$ is the right solution of $\tilde{\mathbf{F}}(\cdot) = v^{-1}\mathbf{I}$.

284 (iv) *As discussed in [16], the right solutions of the above equations cannot be determined*
 285 *analytically except in some special cases. However, there exists a number of numerical*
 286 *algorithms which can be employed, e.g., the iterative algorithm [21], logarithmic*

287 reduction [20] and the cyclic reduction [6], to name a few. For further details on
 288 the variety of algorithms available for solving such equations, see [7] and references
 289 therein.

290 3.2 Two-Sided Exit Upward - $\{\tau_a^+ < \tau_{-b}^-\}$

291 Within the literature of spectrally negative Lévy processes and their fully discrete counter-
 292 parts [3], the common approach to solving two-sided exit problems relies on the introduction
 293 of a family of functions, W^q and Z^q , known as the q -scale functions (see [18] for details).
 294 The extension of these auxiliary, one dimensional scale functions to the multidimensional
 295 MAP setting was first proposed in [19], where the existence of the corresponding ‘scale
 296 matrices’ was shown and were further investigated in [15] who derived their probabilistic
 297 interpretation within the continuous setting.

298 For $v \in (0, 1]$, the discrete $\widetilde{\mathbf{W}}_v$ scale matrix is defined as the mapping $\widetilde{\mathbf{W}}_v : \mathbb{N} \rightarrow \mathbb{R}^{N \times N}$,
 299 with $\widetilde{\mathbf{W}}_v(0) = \mathbf{0}$ (the matrix of zeros), such that

$$300 \quad \widetilde{\mathbf{W}}_v(n) = \left[\widetilde{\mathbf{G}}_v^{-n} - \mathbb{E}(v^{\tau_{-n}}; J_{\tau_{-n}}) \right] \widetilde{\mathbf{L}}_v, \quad (3.4)$$

301 where we write $\widetilde{\mathbf{W}}_1(n) =: \widetilde{\mathbf{W}}(n)$ for the 1-scale matrix. The definition of the scale matrix
 302 above is only unique up to a multiplicative constant and the presence of the infinite-time
 303 occupation matrix, $\widetilde{\mathbf{L}}_v$, is somewhat arbitrary here but is included in order to obtain the
 304 most concise form for the p.g.m. of $\widetilde{\mathbf{W}}_v(\cdot)$, which is derived in Theorem 2 (see also [15]).

305 In the two-sided exit problem, we are interested in the time of exiting a (fixed) ‘strip’,
 306 $[-b, a]$, consisting of an upper and lower level denoted by a and $-b$, respectively, such that
 307 $a > 0 > -b$. More formally, we are interested in the events $\{\tau_a^+ < \tau_{-b}^-\}$ and $\{\tau_a^+ > \tau_{-b}^-\}$,
 308 which correspond to the upward and downward exits from the strip $[-b, a]$, respectively.
 309 In this section, we are concerned with the former (upward exit). The the latter (downward
 310 exit) will be discussed in a later section as its derivation depends on alternative methods.

311 Let us denote by $\rho(\cdot)$, the spectral radius of a matrix. That is, if $\Lambda(\mathbf{A})$ denotes the
 312 spectrum of a matrix \mathbf{A} , then $\rho(\mathbf{A}) = \max\{|\lambda_i| : \lambda_i \in \Lambda(\mathbf{A})\}$.

313 3.2.1 Two-sided exit theory for non-singular $\widetilde{\mathbf{A}}_1$

314 **Theorem 2.** Assume that $\widetilde{\mathbf{A}}_1$ is non-singular. Then, there exists a matrix $\widetilde{\mathbf{W}} : \mathbb{N} \rightarrow \mathbb{R}^{N \times N}$
 315 with $\widetilde{\mathbf{W}}(0) = \mathbf{0}$, which is invertible and satisfies

$$316 \quad \mathbb{P}(\tau_a^+ < \tau_{-b}^-, J_{\tau_a^+}) = \widetilde{\mathbf{W}}(b) \widetilde{\mathbf{W}}(a+b)^{-1}, \quad (3.5)$$

317 where $(\mathbb{P}(\tau_a^+ < \tau_{-b}^-, J_{\tau_a^+}))_{ij} = \mathbb{P}(\tau_a^+ < \tau_{-b}^-, J_{\tau_a^+} = j | J_0 = i)$ and $\widetilde{\mathbf{W}}(\cdot)$ has representation

$$318 \quad \widetilde{\mathbf{W}}(n) = \left(\widetilde{\mathbf{G}}^{-n} - \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \right) \widetilde{\mathbf{L}}. \quad (3.6)$$

319 Furthermore, it holds that

$$320 \quad \sum_{n=0}^{\infty} z^n \widetilde{\mathbf{W}}(n) = \left(\widetilde{\mathbf{F}}(z) - \mathbf{I} \right)^{-1}, \quad (3.7)$$

321 for $z \in (0, 1]$ such that $z \notin \Lambda(\widetilde{\mathbf{G}})$, and

$$322 \quad \widetilde{\mathbf{W}}(n) = \widetilde{\mathbf{G}}^{-n} \widetilde{\mathbf{L}}^+(n), \quad (3.8)$$

323 where $\widetilde{\mathbf{L}}^+(n) := \mathbb{E} \left[\widetilde{\mathbf{L}}(0, \tau_n) \right]$, denotes the expected number of times the process visits 0
324 before hitting level $n \in \mathbb{N}^+$.

325 *Proof.* Following the same line of logic as in [15], we note that the events $\{\tau_a^+ < \tau_{-b}^-\}$ and
326 $\{\tau_a < \tau_{-b}\}$ are equivalent due to the upward skip-free property of $\{X_n\}_{n \geq 0}$. This follows
327 from the fact that in order to drop below $-b$ and then hit a , the process must visit $-b$
328 on the way. Thus, conditioning on possible events and employing the Markov additive
329 property, we obtain

$$330 \quad \mathbb{P}(\tau_a < \infty, J_{\tau_a}) = \mathbb{P}(\tau_a < \tau_{-b}, J_{\tau_a}) + \mathbb{P}(\tau_a > \tau_{-b}, J_{\tau_{-b}}) \mathbb{P}(\tau_{a+b} < \infty, J_{\tau_{a+b}})$$

331 and

$$332 \quad \mathbb{P}(\tau_{-b} < \infty, J_{\tau_{-b}}) = \mathbb{P}(\tau_a > \tau_{-b}, J_{\tau_{-b}}) + \mathbb{P}(\tau_a < \tau_{-b}, J_{\tau_a}) \mathbb{P}(\tau_{-(a+b)} < \infty, J_{\tau_{-(a+b)}}).$$

333 Now, by recalling that $\mathbb{P}(\tau_a < \infty, J_{\tau_a}) = \widetilde{\mathbf{G}}^a$, solving the second equation w.r.t. $\mathbb{P}(\tau_a >$
334 $\tau_{-b}, J_{\tau_{-b}})$ and substituting the resulting equation into the first, yields

$$335 \quad \mathbb{P}(\tau_a < \tau_{-b}, J_{\tau_a}) \left[\mathbb{P}(\tau_{-(a+b)} < \infty, J_{\tau_{-(a+b)}}) \widetilde{\mathbf{G}}^{a+b} - \mathbf{I} \right] = \mathbb{P}(\tau_{-b} < \infty, J_{\tau_{-b}}) \widetilde{\mathbf{G}}^{a+b} - \widetilde{\mathbf{G}}^a. \quad (3.9)$$

336 Finally, by multiplying through by $-\widetilde{\mathbf{G}}^{-(a+b)}$ on the right, we have

$$337 \quad \mathbb{P}(\tau_a < \tau_{-b}, J_{\tau_a}) \left[\widetilde{\mathbf{G}}^{-(a+b)} - \mathbb{P}(\tau_{-(a+b)} < \infty, J_{\tau_{-(a+b)}}) \right] = \widetilde{\mathbf{G}}^{-b} - \mathbb{P}(\tau_{-b} < \infty, J_{\tau_{-b}}),$$

338 or equivalently

$$339 \quad \mathbb{P}(\tau_a < \tau_{-b}, J_{\tau_a}) = \widetilde{\mathbf{W}}(b) \widetilde{\mathbf{W}}(a+b)^{-1},$$

340 given that $\widetilde{\mathbf{W}}(\cdot)^{-1}$ exists (see Remark 7). Note that the above result is derived in the
341 absence of the occupation mass matrix, $\widetilde{\mathbf{L}}$, within the definition of $\widetilde{\mathbf{W}}(n)$, reinforcing the
342 point that the scale matrix is uniquely defined up to a (matrix) multiplicative constant.
343 The choice for including $\widetilde{\mathbf{L}}$ in the definition of $\widetilde{\mathbf{W}}(n)$, which is only well defined as long as
344 $\widetilde{\mathbf{L}}$ has finite entries (see Remark 3 for conditions), will become apparent in the following.

345 To prove Eq. (3.7), let us take the transform of the scale matrix and recall the definition
 346 given in Eq. (3.6), to obtain

$$347 \quad \sum_{n=0}^{\infty} z^n \widetilde{\mathbf{W}}(n) = \sum_{n=0}^{\infty} z^n \widetilde{\mathbf{G}}^{-n} \widetilde{\mathbf{L}} - \sum_{n=0}^{\infty} z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \widetilde{\mathbf{L}}, \quad (3.10)$$

348 where the first term on the r.h.s. satisfies

$$349 \quad \sum_{n=0}^{\infty} z^n \widetilde{\mathbf{G}}^{-n} \widetilde{\mathbf{L}} = \sum_{n=0}^{\infty} \left(z \widetilde{\mathbf{G}}^{-1} \right)^n \widetilde{\mathbf{L}} = \left(\mathbf{I} - z \widetilde{\mathbf{G}}^{-1} \right)^{-1} \widetilde{\mathbf{L}}, \quad (3.11)$$

350 for all $z \in (0, \gamma)$, where $\gamma := \min\{|\lambda_i| : \lambda_i \in \Lambda(\widetilde{\mathbf{G}})\}$.

351 For the second term of Eq. (3.10), under the conditions of Theorem 1, we have

$$352 \quad \begin{aligned} (\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1} &= \sum_{n=0}^{\infty} z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \widetilde{\mathbf{L}} + \sum_{n=1}^{\infty} z^{-n} \mathbb{P}(\tau_n < \infty, J_{\tau_n}) \widetilde{\mathbf{L}} \\ 353 \quad &= \sum_{n=0}^{\infty} z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \widetilde{\mathbf{L}} + \sum_{n=1}^{\infty} z^{-n} \widetilde{\mathbf{G}}^n \widetilde{\mathbf{L}} \\ 354 \end{aligned}$$

355 for all $z \in (0, 1]$ such that $(\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1}$ exists. Moreover, for $z \in (\rho(\widetilde{\mathbf{G}}), 1]$ ($\rho(\widetilde{\mathbf{G}}) < 1$ is
 356 true as long as $\widetilde{\mathbf{G}}$ is invertible and this follows from the assumption that the matrix $\widetilde{\mathbf{A}}_1$
 357 is non-singular, see also Remark 7), the geometric series on the r.h.s. converges and the
 358 above equation can be re-written as

$$359 \quad \begin{aligned} (\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1} &= \sum_{n=0}^{\infty} z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \widetilde{\mathbf{L}} + \left((\mathbf{I} - z^{-1} \widetilde{\mathbf{G}})^{-1} - \mathbf{I} \right) \widetilde{\mathbf{L}} \\ 360 \quad &= \sum_{n=0}^{\infty} z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \widetilde{\mathbf{L}} - (\mathbf{I} - z \widetilde{\mathbf{G}}^{-1})^{-1} \widetilde{\mathbf{L}}, \end{aligned} \quad (3.12)$$

361 once we prove a common domain of convergence, i.e., $(\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1}$ exists for some $z \in$
 362 $(\rho(\widetilde{\mathbf{G}}), 1]$. In fact, for $\rho(\widetilde{\mathbf{G}}) < 1$, see Lemma 4 in [8], it can be shown that the zeros of
 363 $\det[\mathbf{I} - \widetilde{\mathbf{F}}(z)]$ coincide with the eigenvalues of $\widetilde{\mathbf{G}}$ for $z \in (0, 1]$ and thus, the above holds.

364 Now, note that if we multiply Eq. (3.12) from the left by $\mathbf{I} - z \widetilde{\mathbf{G}}^{-1}$ and from the right
 365 by $\mathbf{I} - \widetilde{\mathbf{F}}(z)$, then both sides of the resulting equation are analytic for $z \in (0, 1]$. Hence,
 366 since the matrices $(\mathbf{I} - z \widetilde{\mathbf{G}}^{-1})$ and $(\mathbf{I} - \widetilde{\mathbf{F}}(z))$ are invertible as long as $z \notin \Lambda(\widetilde{\mathbf{G}})$ and thus
 367 for $z \in (0, \gamma)$, the aforementioned multiplication can be reversed and Eq. (3.12) holds for
 368 $z \in (0, \gamma)$ by analytic continuation. The result follows by substituting the above equation,
 369 along with Eq. (3.11), into Eq. (3.10) and using analytic continuation to extend the domain
 370 from $z \in (0, \gamma)$ to $z \in (0, 1]$ such that $z \notin \Lambda(\widetilde{\mathbf{G}})$.

371 To prove Eq. (3.8), we use similar arguments to those used for the result of Proposition
 372 1, to show that for $n \geq 0$

$$373 \quad \tilde{\mathbf{L}} = \tilde{\mathbf{L}}^+(n) + \mathbb{P}(\tau_n < \infty, J_{\tau_n}) \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \tilde{\mathbf{L}}, \quad (3.13)$$

374 where $\tilde{\mathbf{L}}^+(n) := \mathbb{E}(\tilde{\mathbf{L}}(0, \tau_n))$, from which it follows that

$$375 \quad \tilde{\mathbf{L}}^+(n) = \left[\mathbf{I} - \mathbb{P}(\tau_n < \infty, J_{\tau_n}) \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \right] \tilde{\mathbf{L}} = \left[\mathbf{I} - \tilde{\mathbf{G}}^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \right] \tilde{\mathbf{L}}.$$

376 Multiplying this expression through by $\tilde{\mathbf{G}}^{-n}$ (on the left) and recalling the form of $\tilde{\mathbf{W}}(n)$
 377 given in Eq. (3.6), the result follows immediately. So far we assume only that $\rho(\tilde{\mathbf{G}}) < 1$,
 378 hence by Remark 4 that either $v < 1$ or that $v = 1$ and $\kappa'(0) > 0$. To handle the remaining
 379 (limiting) case of $v = 1$ and $\kappa'(0) \leq 0$ we can follow the proof of Theorem 1 in [15].
 380 Namely we can use the representation (3.8) of the scale function, take $v \rightarrow 1$ and observe
 381 that matrices $\tilde{\mathbf{G}}$, $\tilde{\mathbf{L}}^+(n)$ and $\tilde{\mathbf{F}}(z)$ properly converge. □

382
 383 **Remark 6.** *In [16] the authors derive an equivalent result to Theorem 2 for a continuous-*
 384 *time MAP in the lattice and non-lattice case. Although their study focuses purely on the*
 385 *continuous-time case, they do point out the connection for the discrete-time model (Remark*
 386 *6 in [16]) but do not provide any proof or further details.*

387 **Remark 7** (Invertibility of $\tilde{\mathbf{L}}^+(n)$, $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{W}}(n)$). *Throughout the proof of the previous*
 388 *theorem and results earlier in this paper, we required invertibility of the fundamental matrix*
 389 *$\tilde{\mathbf{G}}$ and the scale matrix $\tilde{\mathbf{W}}(n)$. We will now look at under what conditions such invertibility*
 390 *holds:*

391 (i) *Following similar arguments as in [16], since the level process starts at $X_0 = 0$,*
 392 *the expected number of visits at 0 before the process reaches level $n \geq 0$, namely*
 393 $\tilde{\mathbf{L}}^+(n) = \mathbb{E}[\tilde{\mathbf{L}}(0, \tau_n)]$, *satisfies*

$$394 \quad \tilde{\mathbf{L}}^+(n) = \mathbf{I} + \mathbf{\Pi}_n \tilde{\mathbf{L}}^+(n),$$

395 where $\mathbf{\Pi}_n$ is a probability matrix with i, j -th element containing the probability of a
 396 second visit to level 0 before reaching level n and doing so in phase j , conditioned
 397 on the starting point $(0, i)$. Note that $\mathbf{\Pi}_n$ is clearly a sub-stochastic, non-negative
 398 matrix, which implies $\rho(\mathbf{\Pi}_n) < 1$ and thus $\mathbf{I} - \mathbf{\Pi}_n$ is invertible. Hence, $\tilde{\mathbf{L}}^+(n)$ is also
 399 invertible, since from the above expression it follows that $(\mathbf{I} - \mathbf{\Pi}_n) \tilde{\mathbf{L}}^+(n) = \mathbf{I}$.

400 (ii) *In order to show that $\tilde{\mathbf{G}}$ is invertible, recall that*

$$401 \quad \tilde{\mathbf{G}} = \sum_{m=-1}^{\infty} \tilde{\mathbf{A}}_{-m} \tilde{\mathbf{G}}^{m+1} = \tilde{\mathbf{A}}_1 + \sum_{m=0}^{\infty} \tilde{\mathbf{A}}_{-m} \tilde{\mathbf{G}}^{m+1},$$

402 from which it follows that

$$403 \quad \tilde{\mathbf{A}}_1 = \tilde{\mathbf{G}} - \sum_{m=0}^{\infty} \tilde{\mathbf{A}}_{-m} \tilde{\mathbf{G}}^{m+1} = \left(\mathbf{I} - \sum_{m=0}^{\infty} \tilde{\mathbf{A}}_{-m} \tilde{\mathbf{G}}^m \right) \mathbf{G} = \left(\mathbf{I} - \mathbf{\Pi}_1 \right) \mathbf{G}$$

404

405 Therefore, since $\mathbf{I} - \mathbf{\Pi}_1$ is invertible, $\tilde{\mathbf{G}}$ is invertible provided that $\tilde{\mathbf{A}}_1$ is invertible.
 406 Finally, since $\tilde{\mathbf{L}}^+(n)$ is invertible and given $\tilde{\mathbf{G}}$ is invertible, then by Eq. (3.8) it is
 407 clear that $\tilde{\mathbf{W}}(n)$ is also invertible.

408 Although Theorem 2 provides a number of representations for $\tilde{\mathbf{W}}$, in the discrete case the
 409 scale matrix also satisfies a recursive relation. The recursion below generalises the recursion
 410 for the scale function derived in [3] and has also been discussed in [16].

411 **Corollary 1.** For $b \geq 1$, the scale matrix $\tilde{\mathbf{W}}(\cdot)$, defined in Theorem 2, satisfies the follow-
 412 ing recursive equation

$$\tilde{\mathbf{W}}(b+1) = \tilde{\mathbf{A}}_1^{-1} \left(\tilde{\mathbf{W}}(b) - \sum_{m=0}^{b-1} \tilde{\mathbf{A}}_{-m} \tilde{\mathbf{W}}(b-m) \right), \quad (3.14)$$

413 with $\tilde{\mathbf{W}}(1) = \tilde{\mathbf{A}}_1^{-1}$.

Proof. To prove the recursive relation, consider the two-sided hitting probability $\mathbb{P}(\tau_a^+ <$
 $\tau_{-b}^-; J_{\tau_a^+})$ and condition on the first time step. Then, for $a, b \geq 1$, we have

$$\begin{aligned} \mathbb{P}(\tau_a^+ < \tau_{-b}^-; J_{\tau_a^+}) &= \sum_{m=-(b-1)}^1 \tilde{\mathbf{A}}_m \mathbb{P}_m(\tau_a^+ < \tau_{-b}^-; J_{\tau_a^+}) \\ &= \sum_{m=-(b-1)}^1 \tilde{\mathbf{A}}_m \mathbb{P}(\tau_{a-m}^+ < \tau_{-(b+m)}^-; J_{\tau_{a-m}^+}), \end{aligned}$$

where the last equality follows from the Markov additive property. Further, using Theorem
 2 and multiplying on the right by $\tilde{\mathbf{W}}(a+b)$, the above expression can be re-written as

$$\tilde{\mathbf{W}}(b) = \sum_{m=-(b-1)}^1 \tilde{\mathbf{A}}_m \tilde{\mathbf{W}}(b+m).$$

414 and the recursive expression given in Eq. (3.14) follows directly after some basic algebraic
 415 manipulations. For $\tilde{\mathbf{W}}(1)$, recall Remark 7 that $\tilde{\mathbf{L}}^+(1) = (\mathbf{I} - \mathbf{\Pi}_1)^{-1}$ and also that $\tilde{\mathbf{A}}_1^{-1} =$
 416 $\tilde{\mathbf{G}}^{-1}(\mathbf{I} - \mathbf{\Pi}_1)^{-1} = \tilde{\mathbf{G}}^{-1} \tilde{\mathbf{L}}^+(1) = \tilde{\mathbf{W}}(1)$, from Theorem 2. \square

417 **Remark 8.** Under the same line of logic as Remark 5, we recall that the above results
 418 are more general than explicitly stated. For example, by superimposing killing Eq. (3.5) is
 419 equivalent to

$$420 \quad \mathbb{E}\left(v^{\tau_a^+}; \tau_a^+ < \tau_{-b}^-, J_{\tau_a^+}\right) = \widetilde{\mathbf{W}}_v(b)\widetilde{\mathbf{W}}_v(a+b)^{-1}, \quad (3.15)$$

421 for $v \in (0, 1]$, where $\widetilde{\mathbf{W}}_v(\cdot)$ is defined in Eq. (3.4) with the rest of the results amended
 422 accordingly.

423 3.2.2 Two-sided exit theory for arbitrary $\widetilde{\mathbf{A}}_1$

424 In Theorem 2, we rely on the fact that $\widetilde{\mathbf{A}}_1$ is non-singular, which in turn ensures $\widetilde{\mathbf{G}}$ is
 425 non-singular by Remark 7. However, it turns out that a similar result can also be derived
 426 for arbitrary $\widetilde{\mathbf{A}}_1$ in terms of matrices closely related to the $\widetilde{\mathbf{W}}$ scale matrix.

427 To see this, let us define $\widetilde{\mathbf{L}}^-(n) := \mathbb{E}(\widetilde{\mathbf{L}}(0, \tau_{-n}))$ for $n \geq 0$, $\widetilde{\mathbf{M}}(n) := \mathbb{E}(\widetilde{\mathbf{L}}(-n, \tau_{-(n+1)}))$
 428 and recall $\widetilde{\mathbf{R}}$ is related to the ‘time-reversed’ counterpart of $\widetilde{\mathbf{G}}$ (see Remark 5). Then, we
 429 have the following theorem.

430 **Theorem 3.** Assume the matrix $\widetilde{\mathbf{A}}_1$ is singular. Then, there exists a matrix $\widetilde{\mathbf{V}} : \mathbb{N} \rightarrow$
 431 $\mathbb{R}^{N \times N}$ with $\widetilde{\mathbf{V}}(0) = \mathbf{I}$, which is invertible and satisfies

$$432 \quad \mathbb{P}(\tau_a^+ < \tau_{-b}^-, J_{\tau_a^+}) = \widetilde{\mathbf{V}}(b)\widetilde{\mathbf{R}}^a\widetilde{\mathbf{V}}(a+b)^{-1}, \quad (3.16)$$

433 where

$$434 \quad \widetilde{\mathbf{V}}(n) = \widetilde{\mathbf{L}}^-(n) = \left[\mathbf{I} - \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \widetilde{\mathbf{G}}^n \right] \widetilde{\mathbf{L}}.$$

435 Furthermore, it holds that

$$436 \quad \widetilde{\mathbf{L}}^-(n) = \sum_{k=-1}^{n-1} \widetilde{\mathbf{M}}(k)\widetilde{\mathbf{R}}^k \quad (3.17)$$

437 and for $z \in (0, 1]$ such that $z \notin \Lambda(\widetilde{\mathbf{G}})$, also

$$438 \quad \sum_{n=0}^{\infty} z^n \widetilde{\mathbf{M}}(n) = \left(\mathbf{I} - \widetilde{\mathbf{F}}(z) \right)^{-1} \left(\mathbf{I} - z^{-1} \widetilde{\mathbf{R}} \right). \quad (3.18)$$

439 *Proof.* Assume now that the matrix $\widetilde{\mathbf{G}}$ is singular (which, by Remark 7, is equivalent to
 440 the requirement that the matrix $\widetilde{\mathbf{A}}_1$ is singular). Then, from equation (3.9) we can obtain
 441 an alternative representation for the two-sided exit probability of the form

$$442 \quad \mathbb{P}(\tau_a^+ < \tau_{-b}^-, J_{\tau_a^+}) = \widetilde{\mathbf{H}}(b)\widetilde{\mathbf{G}}^a\widetilde{\mathbf{H}}(a+b)^{-1},$$

443 where

$$444 \quad \widetilde{\mathbf{H}}(n) = \mathbf{I} - \mathbb{P}(\tau_{-n}^- < \infty, J_{\tau_{-n}^-}) \widetilde{\mathbf{G}}^n,$$

445 for $n \geq 0$, as long as this matrix is invertible (see below). Moreover, by similar arguments
 446 as in Eq. (3.13), it follows that

$$447 \quad \tilde{\mathbf{L}} = \tilde{\mathbf{L}}^-(n) + \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \mathbb{P}(\tau_n < \infty, J_{\tau_n}) \tilde{\mathbf{L}},$$

448 or equivalently

$$449 \quad \tilde{\mathbf{L}}^-(n) = \left[\mathbf{I} - \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \tilde{\mathbf{G}}^n \right] \tilde{\mathbf{L}}.$$

450 Now, although we do not discuss in much details here the definition and probabilistic
 451 interpretation of the matrix $\tilde{\mathbf{R}}$, [16] explain that the matrix $\tilde{\mathbf{R}}^n$ comprises of i, j -th elements
 452 representing the expected number of visits to level $n \geq 0$ in phase j before the first return
 453 to the level 0, given $X_0 = 0$ and $J_0 = i$. Hence, using this interpretation, we observe that

$$454 \quad \tilde{\mathbf{G}}^n \tilde{\mathbf{L}} = \mathbb{E} \left(\tilde{\mathbf{L}}(n, \infty) \right) = \tilde{\mathbf{L}} \tilde{\mathbf{R}}^n$$

455 and therefore, straightforward calculations show that Eq. (3.16) holds for $\tilde{\mathbf{V}}(n) = \tilde{\mathbf{L}}^-(n)$
 456 as long as this matrix is invertible for all $n \geq 0$. Note that this can easily be verified by
 457 employing the same argument as in (i) of Remark 7 for $n \leq 0$ and considering $\mathbf{\Pi}_{-n}$ instead
 458 of $\mathbf{\Pi}_n$.

459 To prove Eq. (3.17), we use similar arguments as [16] and employ the Markov property
 460 to obtain

$$461 \quad \tilde{\mathbf{L}}^-(n+1) = \tilde{\mathbf{L}}^-(n) + \tilde{\mathbf{M}}(n) \tilde{\mathbf{R}}^n,$$

462 and, in particular, $\tilde{\mathbf{L}}^-(1) = \tilde{\mathbf{M}}(0)$, from which the result follows directly.

463 Finally, to prove the transform in Eq. (3.18), we again follow the methodology of [16]
 464 and first note that by conditioning on the first time period, for $n \geq 1$, we have

$$465 \quad \tilde{\mathbf{M}}(n) = \sum_{m=-1}^n \tilde{\mathbf{A}}_{-m} \tilde{\mathbf{M}}(n-m), \quad (3.19)$$

466 whilst, for $n = 0$, it follows that

$$467 \quad \tilde{\mathbf{M}}(0) = \mathbf{I} + \tilde{\mathbf{A}}_1 \tilde{\mathbf{M}}(1) + \tilde{\mathbf{A}}_0 \tilde{\mathbf{M}}(0). \quad (3.20)$$

468 Taking transforms on both sides of Eq. (3.19) and noting the above expression for $\tilde{\mathbf{M}}(0)$,
 469 after some algebraic manipulations (see Appendix), we obtain

$$470 \quad \sum_{n=0}^{\infty} z^n \tilde{\mathbf{M}}(n) = \mathbf{I} - z^{-1} \tilde{\mathbf{A}}_1 \tilde{\mathbf{M}}(0) + \tilde{\mathbf{F}}(z) \sum_{k=0}^{\infty} z^k \tilde{\mathbf{M}}(k)$$

$$471 \quad = \mathbf{I} - z^{-1} \tilde{\mathbf{R}} + \tilde{\mathbf{F}}(z) \sum_{k=0}^{\infty} z^k \tilde{\mathbf{M}}(k), \quad (3.21)$$

472

473 where in the last equality we have use the probabilistic interpretation of $\tilde{\mathbf{R}}$ to note that
 474 $\tilde{\mathbf{R}} = \tilde{\mathbf{A}}_1 \tilde{\mathbf{L}}^-(1) = \tilde{\mathbf{A}}_1 \tilde{\mathbf{M}}(0)$. The result follows directly by solving the above expression for
 475 the transform and holds as long as $\mathbf{I} - \tilde{\mathbf{F}}(z)$ is invertible. \square

476 Although the result of Theorem 3 is clearly more general than that of Theorem 2, as it
 477 does not require invertibility of $\tilde{\mathbf{A}}_1$, it deviates from the well known form and methodology
 478 of scale matrices (functions) seen throughout the literature. As such, since the purpose of
 479 this paper is to demonstrate and derive the fully discrete analogue of the well known ‘scale
 480 theory’ for MACs, we will assume the invertibility of $\tilde{\mathbf{A}}_1$ throughout the rest of this paper
 481 but point out that all the following results could also be generalised to the arbitrary case
 482 (see [16] for more details of such results in the continuous-time setting).

483 At this point it is natural to consider the corresponding downward exit problems (one
 484 and two-sided). However, in order to do this we must first discuss some fluctuation problems
 485 for the associated ‘reflected’ MAC process which is discussed in the following section.

486 4 Exit Problems For Reflected MACs

487 In this section, we deviate from the basic MAC described above and consider the associated
 488 two-sided reflection of the process $\{X_n\}_{n \geq 0}$ with respect to a strip $[-d, 0]$ with $d > 0$. The
 489 choice of strip is purely for notational convenience and can easily be converted to the
 490 general strip $[-b, a]$ by shifting the process appropriately. The main result of this section
 491 is given in Theorem 4 which is interesting in its own right, but is also used to derive the
 492 aforementioned downward exit problems of the original (un-reflected) MAC.

493 Following the same line of logic as in [15], let us define the reflected process by

$$494 \quad H_n = X_n + R_n^- - R_n^+,$$

495 where R_n^- and R_n^+ are known as regulators for the reflected process at the barriers $-d$ and
 496 0, respectively, which ensure that the process $\{H_n\}_{n \geq 0}$ remains within the strip $[-d, 0]$ for
 497 all $n \in \mathbb{N}$. Note that in continuous-time and space, the reflected process $\{H_n\}_{n \geq 0}$ corre-
 498 sponds to the solution of the so-called Skorohod problem (see [23]). By the construction of
 499 $\{H_n\}_{n \geq 0}$, it is clear that $\{R_n^-\}_{n \geq 0}$ and $\{R_n^+\}_{n \geq 0}$ are both non-decreasing processes, with
 500 $R_0^- = R_0^+ = 0$, when X_0 in $[-d, 0]$, which only increase during periods when $H_n = -d$
 501 and $H_n = 0$, respectively. Moreover, since $\{X_n\}_{n \geq 0}$ is ‘spectrally negative’ the upward
 502 regulator $\{R_n^+\}_{n \geq 0}$ increases by at most one per unit time.

503 Now, let us denote by ρ_k , the right inverse of the regulator $\{R_n^+\}_{n \geq 0}$, defined by

$$504 \quad \rho_k = \inf\{n \geq 0 : R_n^+ > k\} = \inf\{n \geq 0 : R_n^+ = k + 1\}, \quad (4.1)$$

505 such that $R_{\rho_k}^+ = k + 1$. Then, since an increase in $\{R_n^+\}_{n \geq 0}$ only occurs whilst $H_n = 0$, it
 506 follows that $H_{\rho_k} = 0$ and thus, $R_{\rho_k}^- = (k + 1) - X_{\rho_k}$. Hence, by the strong Markov property
 507 of $\{X_n\}_{n \geq 0}$, we have that $\{(R_{\rho_k}^-, J_{\rho_k})\}_{k \geq 0}$ is itself a MAC with random initial position

508 $(R_{\rho_0}^-, J_{\rho_0})$ when $X_0 \in [-d, 0]$ and non-negative jumps within the level process $\{R_{\rho_k}^-\}_{k \geq 0}$.
 509 Thus, in a similar way as for the original MAC (X, J) , we can define its p.g.m., given
 510 $X_0 = 0$, by

$$511 \quad \mathbb{E}(z^{R_{\rho_k}^-}; J_{\rho_k}) = (\tilde{\mathbf{F}}^*(z))^{k+1}, \quad \tilde{\mathbf{F}}^*(z) := \mathbb{E}(z^{R_{\rho_0}^-}; J_{\rho_0}). \quad (4.2)$$

512 **Remark 9.** In the continuous case, $X_0 = 0$ is a regular point on $(0, \infty)$ and thus, it
 513 follows that $\rho_0 = 0$ a.s. and thus $\mathbb{E}(z^{R_{\rho_0}^-}; J_{\rho_0}) = \mathbf{I}$ (see [15] for details). However, in the
 514 fully discrete set-up, we have already mentioned that $R_{\rho_0}^-$ is random for $X_0 = 0$ and is due
 515 to the possibility of the process experiencing a negative jump in the first time period such
 516 that $\rho_0 \neq 0$. Moreover, the process may drop below the lower level $-d$ (resulting in a jump
 517 in $\{R_n^-\}_{n \geq 0}$) before the stopping time ρ_0 , and justifies the choice of the p.g.m. $\mathbb{E}(z^{R_{\rho_0}^-}; J_{\rho_0})$
 518 above, compared to $\mathbb{E}(z^{R_{\rho_1}^-}; J_{\rho_1})$ in the continuous case (see [15]). On the other hand, we
 519 note that if $X_0 = 1$, then $\mathbb{E}_1(z^{R_{\rho_0}^-}; J_{\rho_0}) = \mathbf{I}$, since $R_0^+ = 1$, and thus $\rho_0 = 0$. The latter
 520 observation will play a crucial role in analysing the distribution of $(R_{\rho_0}^-, J_{\rho_0})$, which is given
 521 in the following theorem in terms of the second v -scale matrix, denoted $\tilde{\mathbf{Z}}_v$, and defined for
 522 $z \in (0, 1]$ and $v \in (0, 1]$, by

$$523 \quad \tilde{\mathbf{Z}}_v(z, n) = z^{-n} \left[\mathbf{I} + \sum_{k=0}^n z^k \tilde{\mathbf{W}}_v(k) (\mathbf{I} - v \tilde{\mathbf{F}}(z)) \right], \quad (4.3)$$

524 with $\tilde{\mathbf{Z}}_v(z, 0) = \mathbf{I}$, for all $z \in (0, 1]$ and $v \in (0, 1]$ and $\tilde{\mathbf{Z}}_1(z, n) =: \tilde{\mathbf{Z}}(z, n)$.

525 **Theorem 4.** For $z \in (0, 1]$, such that $z \notin \Lambda(\tilde{\mathbf{G}})$, and $x \in [-d, 1]$ it holds that $\tilde{\mathbf{Z}}(z, d+1)$
 526 is invertible and

$$527 \quad \mathbb{E}_x(z^{R_{\rho_0}^-}; J_{\rho_0}) = \tilde{\mathbf{Z}}(z, d+x) \tilde{\mathbf{Z}}(z, d+1)^{-1}, \quad (4.4)$$

528 where $\tilde{\mathbf{Z}}(z, n)$ is defined by Eq. (4.3).

529 *Proof.* The proof of this theorem actually follows a similar line of logic as the proof of
 530 Theorem 1 however, due to the nature of the reflected process, the calculations require
 531 greater attention.

532 First note that since $H_{\rho_k} = 0$ for each $k \in \mathbb{N}$, we have $X_{\rho_k} = k + 1 - R_{\rho_k}^-$ and thus
 533 $\{(X_{\rho_k}, J_{\rho_k})\}_{k \geq 0}$ is a MAC having unit (upward) drift and downward jumps described by
 534 $\{R_{\rho_k}^-\}_{k \geq 0}$ with random ‘initial’ position $X_{\rho_0} = 1 - R_{\rho_0}^-$. Moreover, its occupation mass in
 535 the bivariate state $(y, j) \in \mathbb{Z} \times E$ is defined by $\tilde{L}^*(y, j, \infty) = \sum_{k=0}^{\infty} 1_{(X_{\rho_k}=y, J_{\rho_k}=j)}$ and thus,
 536 from the occupation mass formula in Eq. (2.5), we have

$$537 \quad \sum_{k=0}^{\infty} z^{-X_{\rho_k}} 1_{(J_{\rho_k}=j)} = \sum_{m \in \mathbb{Z}} z^{-m} \tilde{L}^*(m, j, \infty).$$

538 Taking expectations on both sides of this expression, conditioned on the initial state $X_0 =$
 539 $x \in [-d, 1]$, and writing in matrix form yields

$$540 \quad \sum_{k=0}^{\infty} \mathbb{E}_x(z^{-X_{\rho_k}}; J_{\rho_k}) = \sum_{m \in \mathbb{Z}} z^{-m} \tilde{\mathbf{L}}_x^*(m, \infty), \quad (4.5)$$

541 where $\tilde{\mathbf{L}}_x^*(m, \infty)$ is the infinite-time occupation matrix with i, j -th element given by
 542 $(\tilde{\mathbf{L}}_x^*(m, \infty))_{ij} = \mathbb{E}_x(\tilde{L}^*(m, j, \infty) \mid J_0 = i)$.

543 Let us now treat the left-hand side and right-hand side of Eq. (4.5) separately. Firstly,
 544 using the fact that $X_{\rho_k} = k + 1 - R_{\rho_k}^-$, along with the strong Markov and Markov additive
 545 properties of $\{R_{\rho_k}^-\}_{k \geq 0}$, the l.h.s. of Eq. (4.5) can be re-written in the form

$$\begin{aligned} 546 \quad \sum_{k=0}^{\infty} \mathbb{E}_x(z^{-X_{\rho_k}}; J_{\rho_k}) &= \sum_{k=0}^{\infty} z^{-(k+1)} \mathbb{E}_x(z^{R_{\rho_k}^-}; J_{\rho_k}) \\ 547 &= \sum_{k=0}^{\infty} z^{-(k+1)} \mathbb{E}_x(z^{R_{\rho_0}^-}; J_{\rho_0}) \mathbb{E}(z^{R_{\rho_{k-1}}^-}; J_{\rho_{k-1}}) \\ 548 &= \mathbb{E}_x(z^{R_{\rho_0}^-}; J_{\rho_0}) \sum_{k=0}^{\infty} z^{-(k+1)} (\tilde{\mathbf{F}}^*(z))^k \\ 549 &= \mathbb{E}_x(z^{R_{\rho_0}^-}; J_{\rho_0}) z^{-1} (\mathbf{I} - z^{-1} \tilde{\mathbf{F}}^*(z))^{-1}, \quad (4.6) \end{aligned}$$

550 for all $z \in (0, 1]$ such that $z > (\rho(\tilde{\mathbf{F}}^*(z)))$. We note that since $\{(X_{\rho_k}, J_{\rho_k})\}_{k \geq 0}$ is a MAC, it
 551 holds that $\mathbb{E}(z^{-X_{\rho_k}}; J_{\rho_k}) = (\mathbb{E}(z^{-X_{\rho_0}}; J_{\rho_0}))^{k+1}$. Now, let us define $\bar{\tau}_1 = \inf\{\rho_k \geq 0 : X_{\rho_k} =$
 552 $1\}$ and $\bar{\mathbf{G}}$ to be the probability transition matrix such that $\mathbb{P}(\bar{\tau}_1 < \infty, J_{\bar{\tau}_1}) = \bar{\mathbf{G}}$, which is
 553 sub-stochastic, (implying $\rho(\bar{\mathbf{G}}) < 1$) in the case of killing or no killing and negative drift.
 554 Then, based on similar arguments as those discussed in the proof of Theorem 2, since the
 555 eigenvalues of $\bar{\mathbf{G}}$ coincide with the roots of $\mathbf{I} - \mathbb{E}(z^{-X_{\rho_0}}; J_{\rho_0}) = (\mathbf{I} - z^{-1} \tilde{\mathbf{F}}^*(z))$, then we
 556 conclude that $\mathbf{I} - z^{-1} \tilde{\mathbf{F}}^*(z)$ is invertible for $z \in (\rho(\bar{\mathbf{G}}), 1]$. In fact, since $\{X_n\}_{n \geq 0}$ is an
 557 upward skip-free process, it follows that $\bar{\tau}_1 = \tau_1$ for $X_0 \in [-d, 1]$, which implies $\bar{\mathbf{G}} = \tilde{\mathbf{G}}$,
 558 and thus $\mathbf{I} - z^{-1} \tilde{\mathbf{F}}^*(z)$ is invertible for $z \in (\rho(\tilde{\mathbf{G}}), 1]$. Hence, by applying the same analytic
 559 continuation argument as in Theorem 2, the above expression holds for $z \in (\rho(\tilde{\mathbf{G}}), 1)$.

560 Now, for the r.h.s. of Eq. (4.5), let us introduce the matrix quantity $\tilde{\mathbf{C}}_{-y}$ whose individ-
 561 ual i, j -th elements denote the probability of the process $\{X_n\}_{n \geq 0}$ first hitting some level
 562 $-y < 0$ from initial states $X_0 = 0$ and $J_0 = i$, and then hitting the upper level $(d + 1) - y$
 563 whilst $J_n = j$, such that

$$\begin{aligned} 564 \quad \tilde{\mathbf{C}}_{-y} &= \mathbb{P}(\tau_{-y} < \infty, J_{\tau_{-y}}) \mathbb{P}_{-y}(\tau_{d+1-y} < \infty, J_{\tau_{d+1-y}}) \\ 565 &= \mathbb{P}(\tau_{-y} < \infty, J_{\tau_{-y}}) \mathbb{P}(\tau_{d+1} < \infty, J_{\tau_{d+1}}) = \mathbb{P}(\tau_{-y} < \infty, J_{\tau_{-y}}) \tilde{\mathbf{G}}^{d+1}. \quad (4.7) \end{aligned}$$

567 Using this quantity, it is possible to show that for $X_0 = x \in [-d, 1]$

$$568 \quad \tilde{\mathbf{L}}_x^*(m, \infty) = \left[1_{(m>0)} \mathbb{P}_x(\tau_m < \infty, J_{\tau_m}) + 1_{(m \leq 0)} \tilde{\mathbf{C}}_{m-(d+1)-x} \right] \sum_{k=0}^{\infty} \left(\tilde{\mathbf{C}}_{-(d+1)} \right)^k.$$

569 To see this, note that $\tilde{L}^*(m, j, \infty)$ corresponds to the (local) time points ρ_k (increases in
570 $\{R_n^+\}_{n \geq 0}$) such that $X_{\rho_k} = m$ and $J_{\rho_k} = j$, or alternatively, time points $k \geq 0$ for which
571 $\{R_n^+\}_{n \geq 0}$ is increasing and $X_k = m$ and $J_k = j$. Then, for $m > 0$, the first increase of
572 $\tilde{L}^*(m, j, \infty)$ is at τ_m , otherwise, for $m \leq 0$, $\{X_n\}_{n \geq 0}$ has to first visit the state (level)
573 $m - (d + 1)$ to ensure that at the next time the process $\{X_n\}_{n \geq 0}$ visits the level $m < 0$,
574 the ‘reflected process’ $\{H_n\}_{n \geq 0}$ was at its upper boundary in the previous time period
575 ($H_{n-1} = 0$), resulting in an increase of $\{R_n^+\}_{n \geq 0}$. Every subsequent increase of $\tilde{L}^*(m, j, \infty)$
576 is obtained in a similar way. Thus, the above equation follows by application of the strong
577 Markov and Markov additive properties.

578 Taking transforms on both sides of the above equation, it yields

$$579 \quad \sum_{m \in \mathbb{Z}} z^{-m} \tilde{\mathbf{L}}_x^*(m, \infty) = \left(\sum_{m=1}^{\infty} z^{-m} \mathbb{P}_x(\tau_m < \infty, J_{\tau_m}) + \sum_{m=-\infty}^0 z^{-m} \tilde{\mathbf{C}}_{m-(d+1)-x} \right) \sum_{k=0}^{\infty} \left(\tilde{\mathbf{C}}_{-(d+1)} \right)^k$$

$$580 \quad = \left(\sum_{m=1}^{\infty} z^{-m} \tilde{\mathbf{G}}^{m-x} + \sum_{m=-\infty}^0 z^{-m} \mathbb{P}(\tau_{m-(d+1)-x} < \infty, J_{\tau_{m-(d+1)-x}}) \tilde{\mathbf{G}}^{d+1} \right)$$

$$581 \quad \times \left(\mathbf{I} - \tilde{\mathbf{C}}_{-(d+1)} \right)^{-1}, \quad (4.8)$$

583 where we have used the fact that $\sum_{k=0}^{\infty} (\tilde{\mathbf{C}}_{-(d+1)})^k = (\mathbf{I} - \tilde{\mathbf{C}}_{-(d+1)})^{-1}$ in the presence of
584 killing, since $\tilde{\mathbf{C}}_{-(d+1)}$ is a sub-stochastic matrix and thus, its Perron-Frobenius eigenvalue
585 is less than 1. Now, the first term inside the brackets of the last expression is clearly
586 equivalent to $-(\mathbf{I} - z \tilde{\mathbf{G}}^{-1})^{-1} \tilde{\mathbf{G}}^{-x}$ for all $z \in (\rho(\tilde{\mathbf{G}}), 1]$, whilst by the change of variable
587 $k = m - (d + 1) - x$, the second term within the brackets becomes HERE!!

$$588 \quad z^{-(d+1)-x} \sum_{k=-\infty}^{-(d+1+x)} z^{-k} \mathbb{P}(\tau_k < \infty, J_{\tau_k}) \tilde{\mathbf{G}}^{d+1} = z^{-(d+1)-x} \sum_{m=d+1+x}^{\infty} z^m \mathbb{P}(\tau_{-m} < \infty, J_{\tau_{-m}}) \tilde{\mathbf{G}}^{d+1},$$

$$589$$

590 and thus, after some algebraic manipulations (see Appendix), Eq.(4.8) can be re-written
591 as

$$592 \quad \sum_{m \in \mathbb{Z}} z^{-m} \tilde{\mathbf{L}}_x^*(m, \infty) = z^{-1} \tilde{\mathbf{Z}}(z, d+x) (\mathbf{I} - \tilde{\mathbf{F}}(z))^{-1} \tilde{\mathbf{W}}(d+1)^{-1}, \quad (4.9)$$

593 where $\tilde{\mathbf{Z}}(z, n)$ is defined in Eq. (4.3). Finally, by combining Eqs. (4.6) and (4.9), we obtain
594 for $z \in (\rho(\tilde{\mathbf{G}}), 1)$

$$595 \quad \mathbb{E}_x(z^{R_{\rho_0}^-}; J_{\rho_0}) (\mathbf{I} - z^{-1} \tilde{\mathbf{F}}^*(z))^{-1} = \tilde{\mathbf{Z}}(z, d+x) (\mathbf{I} - \tilde{\mathbf{F}}(z))^{-1} \tilde{\mathbf{W}}(d+1)^{-1}. \quad (4.10)$$

596 To complete the proof, it remains to determine the form of the matrix $\widetilde{\mathbf{F}}^*(z)$. To do this,
 597 let $x = 1$ into the above expression which, after using the fact that $\mathbb{E}_1(z^{R_{\rho_0}^-}; J_{\rho_0}) = \mathbf{I}$ since
 598 in this case $\rho_0 = 1$ and taking inverses on both sides, gives

$$599 \quad \mathbf{I} - z^{-1}\widetilde{\mathbf{F}}^*(z) = \widetilde{\mathbf{W}}(d+1)(\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1}\widetilde{\mathbf{Z}}(z, d+1)^{-1}.$$

600 Note that this expression shows that $\widetilde{\mathbf{Z}}(z, d+1)$ is an invertible matrix as long as $\widetilde{\mathbf{W}}(d+1)$
 601 is invertible and after solving w.r.t. $\widetilde{\mathbf{F}}^*(z)$ also gives

$$602 \quad \widetilde{\mathbf{F}}^*(z) = z \left[\mathbf{I} - \widetilde{\mathbf{W}}(d+1)(\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1}\widetilde{\mathbf{Z}}(z, d+1)^{-1} \right]. \quad (4.11)$$

603 The result follows by substituting the above expression for $\widetilde{\mathbf{F}}^*(z)$ back into Eq.(4.10),
 604 re-arranging and employing analytic continuation in a similar way as previous. \square

606 **Remark 10.** We point out that setting $X_0 = x = 0$ in the result of Theorem 4, gives an
 607 equivalent representation for $\widetilde{\mathbf{F}}^*(z)$ in terms of the $\widetilde{\mathbf{Z}}$ scale matrix only, i.e.,

$$608 \quad \widetilde{\mathbf{F}}^*(z) = \widetilde{\mathbf{Z}}(z, d)\widetilde{\mathbf{Z}}(z, d+1)^{-1}.$$

609 Moreover, we note that based on its definition, it is also possible to use the recursive relation
 610 of $\widetilde{\mathbf{W}}(\cdot)$, given in Corollary 1, to obtain explicit values of $\widetilde{\mathbf{Z}}(z, \cdot)$.

611 Although the result of Theorem 4 is interesting in its own right, its main importance in
 612 this paper is as a stepping stone for proving a similar result for the associated one-sided
 613 reflected process (see Section 4.1 below) and consequently, the two-sided and one-sided (as
 614 a limiting case) downward exit problems for the original (non-reflected) MAC.

615 4.1 One-Sided Reflection

616 As discussed in the previous section, the downward exit problems can be solved using an
 617 auxiliary result for the one-sided (lower) reflected process. As such, let us define

$$618 \quad Y_n = X_n + R_n^{-b},$$

619 where $R_n^{-b} = -b - (-b \wedge \underline{X}_n)$ with $\underline{X}_n = \inf_{k \leq n} \{X_k\}$, denotes a lower reflecting barrier at
 620 the level $-b \leq 0$. Note that this is equivalent to shifting the two-sided reflected process
 621 of the previous section and letting the upper reflecting barrier tend to infinity. Then, by
 622 direct application of Theorem 4 we get the following corollary.

623 **Corollary 2.** For $X_0 = 0$, $z \in (0, 1]$ such that $z \notin \Lambda(\widetilde{\mathbf{G}})$, $a > 0$ and $b \geq 0$, it holds that

$$624 \quad \mathbb{E}\left(z^{R_{\tau_a}^{-b}}; J_{\tau_a}\right) = \widetilde{\mathbf{Z}}(z, b)\widetilde{\mathbf{Z}}(z, a+b)^{-1}, \quad (4.12)$$

625 *Proof.* Note that if we set $d = (a-1) + b$ in Theorem 4, then $\{(H_n + (a-1), R_n^-)\}_{n \geq 0}$
 626 up to time ρ_0 coincides with $\{(Y_n, R_n^{-b})\}_{n \geq 0}$ up to time τ_a , given that $H_0 + (a-1) = Y_0$.
 627 Hence, the result follows directly from Theorem 4 with $x = -(a-1)$. \square

628 5 Downward Exit Problems

629 For the one and two-sided downward exit problems, we are interested in the events $\{\tau_{-b}^- <$
630 $\infty\}$ and $\{\tau_{-b}^- < \tau_a^+\}$, respectively. Unlike the upward exit, due to the possibility of down-
631 ward jumps in the MAC, the stopping time τ_{-b}^- is not necessarily equivalent to the first
632 hitting time of the level $-b < 0$, i.e., $\tau_{-b}^- \neq \tau_{-b}$. It is for this reason that we cannot employ
633 the Markov type structure seen for the upward exit identities and, instead, rely on the
634 results of the reflected processes of the previous section.

635 Although it would appear easier to derive in the first instance, it turns out that the
636 one-sided downward exit problem can easily be obtained as a limiting case of the related
637 two-sided case and as such, is considered in the following.

638 5.1 Two-Sided Exit Downward - $\{\tau_{-b}^- < \tau_a^+\}$

639 For the two-sided downward exit problem, we are interested in the time of exiting the fixed
640 ‘strip’, $[-b, a]$, such that $\{\tau_{-b}^- < \tau_a^+\}$. Using the result for the transform of the downward
641 regulator for the one-sided reflected process, we obtain the following corollary.

642 **Corollary 3.** *For $z \in [0, 1]$ such that $z \notin \Lambda(\tilde{\mathbf{G}})$, it holds that for any $a, b > 0$, we have*

$$643 \mathbb{E}\left(z^{-X_{\tau_{-b}^-}^-}; \tau_{-b}^- < \tau_a^+, J_{\tau_{-b}^-}^-\right) = z^{b-1} \left[\tilde{\mathbf{Z}}(z, b-1) - \tilde{\mathbf{W}}(b) \tilde{\mathbf{W}}(a+b)^{-1} \tilde{\mathbf{Z}}(z, a+b-1) \right].$$

644 *Proof.* Consider the one-sided reflected process of Section 4.1. Then, by the strong Markov
645 and Markov additive properties, it follows that for $b > 0$, we have

$$646 \mathbb{E}\left(z^{R_{\tau_a^+}^{-(b-1)}}; J_{\tau_a^+}^-\right) = \mathbb{P}(\tau_a^+ < \tau_{-b}^-; J_{\tau_a^+}^-) + \mathbb{E}\left(z^{-(b-1)-X_{\tau_{-b}^-}^-}; \tau_{-b}^- < \tau_a^+, J_{\tau_{-b}^-}^-\right) \mathbb{E}\left(z^{R_{\tau_{a+b-1}^+}^0}; J_{\tau_{a+b-1}^+}^-\right).$$

647 Re-arranging this expression and using the identities of Theorem 2 and Corollary 2 the
648 result follows immediately. \square

649 5.2 One-Sided Exit Downward

650 For the one-sided exit problem, we are now interested in the event that of down-crossing
651 the level $-b < 0$, whilst the upward movement of the MAC is un-restricted, i.e., $\{\tau_{-b}^- < \infty\}$
652 which, as already mentioned, can be viewed as a limiting case of the corresponding two-
653 sided problem as $a \rightarrow \infty$. In fact, this is the argument used to obtain the following
654 one-sided downward exit identity.

655 **Corollary 4.** *Assume we are not in the case of no-killing and zero drift, i.e., it is not true*
656 *that both $v = 1$ and $\kappa'(1) = 0$. Then, $\tilde{\mathbf{L}}$ is invertible and, for $z \in (0, 1]$ such that $z \notin \Lambda(\tilde{\mathbf{G}})$*
657 *and $b > 0$, we have*

$$658 \mathbb{E}\left(z^{-X_{\tau_{-b}^-}^-}; J_{\tau_{-b}^-}^-\right) = z^{b-1} \left[\tilde{\mathbf{Z}}(z, b-1) - z \tilde{\mathbf{W}}(b) \tilde{\mathbf{L}}^{-1} (\mathbf{I} - z \tilde{\mathbf{G}}^{-1})^{-1} \tilde{\mathbf{L}} (\tilde{\mathbf{F}}(z) - \mathbf{I}) \right]. \quad (5.1)$$

659 *Proof.* Firstly, the invertibility of $\tilde{\mathbf{L}}$ follows from Remark 3, for which it cannot hold that
660 both $v = 1$ and $\kappa'(1) = 0$. On the other hand, Eq. (5.2) follows from taking the limit of
661 the two-sided case (see Corollary 3) as the upper barrier tends to infinity, i.e., $a \rightarrow \infty$. In
662 order to evaluate the value of the limit of $\tilde{\mathbf{W}}(b)\tilde{\mathbf{W}}(a+b)^{-1}\tilde{\mathbf{Z}}(z, a+b-1)$ as $a \rightarrow \infty$, note
663 that by the definition of the scale matrix $\tilde{\mathbf{Z}}(z, n)$, and using Eq. (3.7), it follows that

$$\begin{aligned}
664 \quad \tilde{\mathbf{Z}}(z, a+b-1) &= z^{-(a+b-1)} \left(\mathbf{I} + \sum_{k=0}^{a+b-1} z^k \tilde{\mathbf{W}}(k) (\mathbf{I} - \tilde{\mathbf{F}}(z)) \right) \\
665 &= z^{-(a+b-1)} \sum_{k=a+b}^{\infty} z^k \tilde{\mathbf{W}}(k) (\tilde{\mathbf{F}}(z) - \mathbf{I}) \\
666 &= \sum_{n=1}^{\infty} z^n \tilde{\mathbf{W}}(n+a+b-1) (\tilde{\mathbf{F}}(z) - \mathbf{I}).
\end{aligned}$$

667 Moreover, by using the fact that $\tilde{\mathbf{W}}(n) = \tilde{\mathbf{G}}^{-n} \tilde{\mathbf{L}}(n)$ (see Theorem 2), multiplication of the
668 above expression by $\tilde{\mathbf{W}}(a+b)^{-1}$ on the left yields

$$669 \quad \tilde{\mathbf{W}}(a+b)^{-1} \tilde{\mathbf{Z}}(z, a+b-1) = \tilde{\mathbf{L}}^{-1}(a+b) \sum_{n=1}^{\infty} z^n \tilde{\mathbf{G}}^{-(n-1)} \tilde{\mathbf{L}}(n+a+b-1) (\tilde{\mathbf{F}}(z) - \mathbf{I}),$$

670 which, after taking $a \rightarrow \infty$ and using dominated convergence theorem, gives

$$\begin{aligned}
671 \quad \lim_{a \rightarrow \infty} \tilde{\mathbf{W}}(a+b)^{-1} \tilde{\mathbf{Z}}(z, a+b-1) &= \tilde{\mathbf{L}}^{-1} z \sum_{n=0}^{\infty} (z \tilde{\mathbf{G}}^{-1})^n \tilde{\mathbf{L}} (\tilde{\mathbf{F}}(z) - \mathbf{I}) \\
672 &= \tilde{\mathbf{L}}^{-1} z (\mathbf{I} - z \tilde{\mathbf{G}}^{-1})^{-1} \tilde{\mathbf{L}} (\tilde{\mathbf{F}}(z) - \mathbf{I}),
\end{aligned}$$

673 for $z \in (0, \gamma)$, where $\tilde{\mathbf{L}}$ is the infinite time occupation mass matrix defined in Proposition
674 1. Finally, by analytic continuation, it can be shown that the above holds for all $z \in (0, 1]$
675 such that $z \notin \Lambda(\tilde{\mathbf{G}})$ and thus, by taking the limit as $a \rightarrow \infty$ in Corollary 3, using the above
676 expressions and re-arranging, we obtain the result. \square

677 **Remark 11.** We point out once again that by explicitly imposing killing, Corollary 3 and
678 consequently Corollary 4 equivalently yield the following joint transforms for $v \in (0, 1]$

$$679 \quad \mathbb{E} \left(v^{\tau_{-b}^-} z^{-X_{\tau_{-b}^-}}; \tau_{-b}^- < \tau_a^+, J_{\tau_{-b}^-} \right) = z^{b-1} \left[\tilde{\mathbf{Z}}_v(z, b-1) - \tilde{\mathbf{W}}_v(b) \tilde{\mathbf{W}}_v(a+b)^{-1} \tilde{\mathbf{Z}}_v(z, a+b-1) \right],$$

680 and

$$681 \quad \mathbb{E} \left(v^{\tau_{-b}^-} z^{-X_{\tau_{-b}^-}}; J_{\tau_{-b}^-} \right) = z^{b-1} \left[\tilde{\mathbf{Z}}_v(z, b-1) - z \tilde{\mathbf{W}}_v(b) \tilde{\mathbf{L}}_v^{-1} (\mathbf{I} - z \tilde{\mathbf{G}}_v^{-1})^{-1} \tilde{\mathbf{L}}_v (\tilde{\mathbf{F}}_v(z) - \mathbf{I}) \right]. \quad (5.2)$$

682 where $\tilde{\mathbf{W}}_v(\cdot)$ and $\tilde{\mathbf{Z}}_v(z, \cdot)$ are defined as in Eqs. (3.4) and (4.3), respectively.

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741 **Appendix**

742 **Proof of Eq. (3.21).** It follows from the results of Eq. (3.19) and (3.20), that

$$\begin{aligned}
743 \quad \sum_{n=0}^{\infty} z^n \widetilde{\mathbf{M}}(n) &= \widetilde{\mathbf{M}}(0) + \sum_{n=1}^{\infty} z^n \sum_{m=-1}^n \widetilde{\mathbf{A}}_{-m} \widetilde{\mathbf{M}}(n-m) \\
744 \quad &= \mathbf{I} + \sum_{n=0}^{\infty} z^n \sum_{m=-1}^n \widetilde{\mathbf{A}}_{-m} \widetilde{\mathbf{M}}(n-m) \\
745 \quad &= \mathbf{I} + \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n+1} \widetilde{\mathbf{A}}_{-(n-k)} \widetilde{\mathbf{M}}(k) \\
746 \quad &= \mathbf{I} + \sum_{n=0}^{\infty} z^n \widetilde{\mathbf{A}}_{-n} \widetilde{\mathbf{M}}(0) + \sum_{n=0}^{\infty} z^n \sum_{k=1}^{n+1} \widetilde{\mathbf{A}}_{-(n-k)} \widetilde{\mathbf{M}}(k) \\
747 \quad &= \mathbf{I} + \sum_{n=0}^{\infty} z^n \widetilde{\mathbf{A}}_{-n} \widetilde{\mathbf{M}}(0) + \sum_{k=1}^{\infty} z^k \sum_{n=k-1}^{\infty} z^{n-k} \widetilde{\mathbf{A}}_{-(n-k)} \widetilde{\mathbf{M}}(k) \\
748 \quad &= \mathbf{I} + \sum_{n=0}^{\infty} z^n \widetilde{\mathbf{A}}_{-n} \widetilde{\mathbf{M}}(0) + \sum_{i=-1}^{\infty} z^i \widetilde{\mathbf{A}}_{-i} \sum_{k=1}^{\infty} z^k \widetilde{\mathbf{M}}(k) \\
749 \quad &= \mathbf{I} - z^{-1} \widetilde{\mathbf{A}}_1 \widetilde{\mathbf{M}}(0) + \widetilde{\mathbf{F}}(z) \sum_{k=0}^{\infty} z^k \widetilde{\mathbf{M}}(k), \\
750 \quad &
\end{aligned}$$

751 where, in the last equality, we have used the series definition of $\widetilde{\mathbf{F}}(z)$ given in Eq. (2.2). \square

752 **Proof of Eq. (4.9).** To prove Eq. (4.9), first note that

$$753 \quad \sum_{n=0}^k z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \widetilde{\mathbf{L}} = \left[\sum_{n=0}^{\infty} z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) - \sum_{n=k+1}^{\infty} z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \right] \widetilde{\mathbf{L}}.$$

754 Then, solving Eq. (3.12) w.r.t. $\sum_{n=0}^{\infty} z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \widetilde{\mathbf{L}}$ and substituting into the above
755 equation, we have

$$756 \quad \sum_{n=0}^k z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \widetilde{\mathbf{L}} = (\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1} + (\mathbf{I} - z^{-1} \widetilde{\mathbf{G}}^{-1})^{-1} \widetilde{\mathbf{L}} - \sum_{n=k+1}^{\infty} z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \widetilde{\mathbf{L}}. \tag{A.2}$$

757 Now, at this point, consider the definition of the scale matrix, $\widetilde{\mathbf{W}}(n)$, given in Eq. (3.6).
758 Multiplying this expression through by z^n and summing from 0 to k on both sides, gives

$$759 \quad \sum_{n=0}^k z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) \widetilde{\mathbf{L}} = \sum_{n=0}^k z^n \widetilde{\mathbf{G}}^{-n} \widetilde{\mathbf{L}} - \sum_{n=0}^k z^n \widetilde{\mathbf{W}}(n), \tag{A.3}$$

760 and thus by equating the r.h.s. of Eqs. (A.2) and (A.3) and re-arranging, we obtain

$$\begin{aligned}
761 \quad \sum_{n=k+1}^{\infty} z^n \mathbb{P}(\tau_{-n} < \infty, J_{\tau_{-n}}) &= \sum_{n=0}^k z^n \widetilde{\mathbf{W}}(n) \widetilde{\mathbf{L}}^{-1} + (\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1} \widetilde{\mathbf{L}}^{-1} - \sum_{n=0}^k z^n \widetilde{\mathbf{G}}^{-n} \\
762 \quad &\quad + (\mathbf{I} - z \widetilde{\mathbf{G}}^{-1})^{-1} \\
763 \quad &= \sum_{n=0}^k z^n \widetilde{\mathbf{W}}(n) \widetilde{\mathbf{L}}^{-1} + (\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1} \widetilde{\mathbf{L}}^{-1} \\
764 \quad &\quad - (\mathbf{I} - z \widetilde{\mathbf{G}}^{-1})^{-1} (\mathbf{I} - (z \widetilde{\mathbf{G}}^{-1})^{k+1}) + (\mathbf{I} - z \widetilde{\mathbf{G}}^{-1})^{-1} \\
765 \quad &= \sum_{n=0}^k z^n \widetilde{\mathbf{W}}(n) \widetilde{\mathbf{L}}^{-1} + (\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1} \widetilde{\mathbf{L}}^{-1} \\
766 \quad &\quad + (\mathbf{I} - z \widetilde{\mathbf{G}}^{-1})^{-1} (z \widetilde{\mathbf{G}}^{-1})^{k+1}, \tag{A.4} \\
767
\end{aligned}$$

768 which provides an expression for the second term of Eq. (4.8). Thus, letting $k = d + x$ in
769 the above expression and substituting into Eq. (4.8), we have that

$$\begin{aligned}
770 \quad \sum_{m \in \mathbb{Z}} z^{-m} \widetilde{\mathbf{L}}_x^*(m, \infty) &= \left[-(\mathbf{I} - z \widetilde{\mathbf{G}}^{-1})^{-1} \widetilde{\mathbf{G}}^{-x} + z^{-(d+1)-x} \left(\sum_{n=0}^{d+x} z^n \widetilde{\mathbf{W}}(n) \widetilde{\mathbf{L}}^{-1} + (\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1} \widetilde{\mathbf{L}}^{-1} \right. \right. \\
771 \quad &\quad \left. \left. + (\mathbf{I} - z \widetilde{\mathbf{G}}^{-1})^{-1} (z \widetilde{\mathbf{G}}^{-1})^{d+x+1} \right) \widetilde{\mathbf{G}}^{d+1} \right] (\mathbf{I} - \widetilde{\mathbf{C}}_{-(d+1)})^{-1} \\
772 \quad &= z^{-(d+1)-x} \left[\sum_{n=0}^{d+x} z^n \widetilde{\mathbf{W}}(n) + (\mathbf{I} - \widetilde{\mathbf{F}}(z))^{-1} \right] \widetilde{\mathbf{L}}^{-1} \widetilde{\mathbf{G}}^{d+1} (\mathbf{I} - \widetilde{\mathbf{C}}_{-(d+1)})^{-1} \\
773 \quad &\tag{A.5}
\end{aligned}$$

774 Now, setting $n = d + 1$ in Eq. (3.6) and multiplying from the right by $\widetilde{\mathbf{L}}^{-1} \widetilde{\mathbf{G}}^{d+1}$, yields

$$\begin{aligned}
775 \quad \widetilde{\mathbf{W}}(d+1) \widetilde{\mathbf{L}}^{-1} \widetilde{\mathbf{G}}^{d+1} &= \mathbf{I} - \mathbb{P}(J_{\tau_{-(d+1)}}) \widetilde{\mathbf{G}}^{d+1} \\
776 \quad &= \mathbf{I} - \widetilde{\mathbf{C}}_{-(d+1)}, \\
777
\end{aligned}$$

778 by the definition of $\widetilde{\mathbf{C}}_{-y}$ given in Eq. (4.7) and thus, it follows that

$$779 \quad (\mathbf{I} - \widetilde{\mathbf{C}}_{-(d+1)})^{-1} = \widetilde{\mathbf{G}}^{-(d+1)} \widetilde{\mathbf{L}} \widetilde{\mathbf{W}}(d+1)^{-1}.$$

780 Finally, substituting the above equation into Eq. (A.5), we get that

$$\begin{aligned}
781 \quad \sum_{m \in \mathbb{Z}} z^{-m} \tilde{\mathbf{L}}_x^*(m, \infty) &= z^{-(d+1)-x} \left[\sum_{n=0}^{d+x} z^n \tilde{\mathbf{W}}(n) + (\mathbf{I} - \tilde{\mathbf{F}}(z))^{-1} \right] \tilde{\mathbf{W}}(d+1)^{-1} \\
782 \quad &= z^{-1} z^{-(d+x)} \left[\sum_{n=0}^{d+x} z^n \tilde{\mathbf{W}}(n) (\mathbf{I} - \tilde{\mathbf{F}}(z)) + \mathbf{I} \right] (\mathbf{I} - \tilde{\mathbf{F}}(z))^{-1} \tilde{\mathbf{W}}(d+1)^{-1} \\
783 \quad &= z^{-1} \tilde{\mathbf{Z}}(z, d+x) (\mathbf{I} - \tilde{\mathbf{F}}(z))^{-1} \tilde{\mathbf{W}}(d+1)^{-1}, \\
784
\end{aligned}$$

785 where the last equation follows from the definition of the $\tilde{\mathbf{Z}}$ scale matrix given in Eq. (4.3). □

786