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Matrix factorizations of the discriminant of S_n [☆]

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ABSTRACT

Consider the symmetric group S_n acting as a reflection group on the polynomial ring $k[x_1, \dots, x_n]$ where k is a field, such that $\mathrm{Char}(k)$ does not divide $n!$. We use Higher Specht polynomials to construct matrix factorizations of the discriminant of this group action: these matrix factorizations are indexed by partitions of n and respect the decomposition of the coinvariant algebra into isotypical components. The maximal Cohen–Macaulay modules associated to these matrix factorizations give rise to a noncommutative resolution of the discriminant and they correspond to the nontrivial irreducible representations of S_n. All our constructions are implemented in Macaulay2 and we provide several examples. We also discuss extensions of these results to Young subgroups of S_n and indicate how to generalize them to the reflection groups $G(m, 1, n)$.

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1. Introduction

The classical discriminant $D(f)$ of a polynomial f in one variable over a field k detects whether f has a multiple root. If f is of degree d, then its discriminant can be expressed as an irreducible quasi-homogeneous polynomial in the coefficients of f, and $D(f)$ vanishes exactly when f has a multiple root. In general, an explicit formula for $D(f)$ consists of many monomial terms (e.g., for $d = 6$ the discriminant has 246 terms), and several compact *determinantal formulae* are known, that is, $D(f)$ can be written as determinant of a matrix with entries polynomials in the coefficients of f: the most famous determinantal formula is due to Sylvester, and there are other determinantal representations due to Bezout and Cayley, see (Gelfand et al., 2008, Chapter 12, 1). One can show that these matrices are equivalent in the sense that they have isomorphic cokernels, see (Hovinen, 2009, Thm. 2.2.6). From a more homological point of view, making use of *matrix factorizations*, these cokernels yield maximal Cohen–Macaulay (= CM)-modules of rank 1 over the hypersurface ring defined by $D(f)$.

Now a guiding question for our investigations is: *can one find other non-equivalent determinantal formulae for $D(f)$, and more generally, find other matrix factorizations of $D(f)$, and even classify them?*

In this paper we will explicitly determine several matrix factorizations of $D(f)$ that are coming from an interpretation of $D(f)$ as discriminant of the reflection group action of S_n on k^n, in particular, our matrix factorizations will correspond to isotypical components of the coinvariant algebra.

Before commenting on the contents of the present paper, we review some results that lead to our work. When f has degree four, Hovinen studied matrix factorizations for the classical discriminant in his thesis (Hovinen, 2009), where he describes several non-equivalent determinantal formulae (in particular, the open swallowtail) using deformation theory and also gives a complete classification of the homogeneous rank 1 modules of $D(f)$ (Hovinen, 2009, Thm. 3.2.1, Thm. 4.4.7). In singularity theory, discriminants occur in various guises, often as so-called free divisors. Free divisors were first studied by Saito (1980) and are hypersurfaces, whose singular locus is a CM-module over the coordinate ring. Discriminants of reflection groups have been studied from this point of view in Saito (1993); Orlik and Terao (1992), and other discriminants include discriminants of versal deformations of several types of singularities, see Buchweitz et al. (2009) for an overview and further references.

Here we interpret the classical discriminant as the discriminant of the reflection group S_n acting on k^n: let G be any finite reflection group $G \subseteq \mathrm{GL}(n,k)$ acting on the vector space k^n. Then G also acts on $S := \mathrm{Sym}_k(k^n)$. Denote by $R := S^G$ the invariant ring under the group action and further by $\mathcal{A}(G)$ the reflection arrangement in k^n, and by $V(\Delta)$ the discriminant in the (smooth) quotient space k^n/G. Note that the hypersurface $V(\Delta)$ is given by the reduced polynomial $\Delta \in R$ and is simply the projection of $\mathcal{A}(G)$ onto the quotient. Moreover, in the case of $G = S_n$ and $k = \mathbb{C}$, it is well-known that $V(\Delta)$ is isomorphic to the classical discriminant $V(D(f))$, where f is a polynomial of degree n, see Section 2.4.

This interpretation allows us to use representation theory, in particular the McKay correspondence (see e.g. Buchweitz, 2012 for more background information and references). In Buchweitz et al. (2020) a McKay correspondence was established for the discriminants $V(\Delta)$ of true reflection groups, also see Buchweitz et al. (2018) for a more leisurely account:

Theorem *(Cor. 4.20 in Buchweitz et al., 2020). Let $G \subseteq \mathrm{GL}(n,k)$ be a true reflection group acting on S and let $R = S^G$ be the invariant ring, $\Delta \in R$ be the discriminant polynomial, and $z \in S$ be the polynomial defining the reflection arrangement $\mathcal{A}(G)$. Then the nontrivial irreducible G-representations are in bijection with the isomorphism classes of (graded) $R/(\Delta)$-direct summands of the CM module $S/(z)$ over $R/(\Delta)$.*

In particular, these direct summands are unique up to isomorphism of $R/(\Delta)$-modules, and hence so are their matrix factorizations.

Now we can state a refined version of our guiding question: *Can we write down matrix factorizations for the direct summands of $S/(z)$ explicitly and also find a geometric interpretation of them?*

So far, all matrix factorizations for isotypical components have been determined for the case when G is a true reflection group of rank 2, see Buchweitz et al. (2020), and for the case of the family of rank 2 complex reflection groups $G(m,p,2)$, see May (2023). For higher rank reflection groups, a complete answer is only known for the special case S_4 (Buchweitz et al., 2020, Section 6). There has been progress on determining the direct summands of $S/(z)$ that correspond to logarithmic (co-)residues (Buchweitz et al., 2020, Thm. 5.9). The other isotypical components have yet to be determined in general. However, in this paper we determine explicit matrix factorizations for $S/(z)$ for $G = S_n$, which answers the first part of the question above for these G. In order to find a geometric interpretation, a starting point would be to analyze the matrix factorizations yielding CM modules of rank 1, and thus determinantal formulae for the discriminant.

The main problem in writing down the matrix factorizations is to find a suitable R-basis of $S/(z)$ that respects the decomposition in isotypical components. To this end we will use modifications of *higher Specht polynomials*. Higher Specht polynomials themselves are a generalization of the classical Specht polynomials and were introduced by Ariki et al. (1997) for the groups $G(m, p, n)$, also see Terasoma and Yamada (1993) for the case S_n. These polynomials were further studied for generalizations of coinvariant rings (Gillespie and Rhoades, 2021), such as Garsia–Procesi modules (introduced in Garsia and Procesi, 1992). They form a basis of the coinvariant algebra and are indexed by standard Young tableaux T, P of shape λ, where λ is a partition of n. Note that partitions of n are in bijection with the irreducible representations of S_n.

Our main result is the following:

Theorem 1.1 *(cf. Theorem 3.14). Let $\Delta \in R$ be the discriminant polynomial of $G = S_n$ under the action on S, let λ be a partition of n and denote by $T \in ST(\lambda)$ a standard Young tableau of shape λ. Then multiplication by z on S defines a matrix factorization (z, z) of Δ, which decomposes as*

$$(z, z) = \bigoplus_{\lambda \vdash n} \bigoplus_{T \in ST(\lambda)} (z|_{M_T}, z|_{N_{T'}}) , \tag{1}$$

where M_T is the R-module generated by the modified higher Specht polynomials, and $N_{T'}$ is the R-module generated by the higher Specht polynomials for the conjugate tableau T'. In terms of CM-modules, this decomposition can be written as

$$S/(z) = \bigoplus_{\lambda \vdash n} \bigoplus_{T \in ST(\lambda)} M_T \cong \bigoplus_{\lambda \vdash n} \bigoplus_{T' \in ST(\lambda)} N_{T'} .$$

We will define *modified Higher Specht polynomials* H_T^P so that matrix given by multiplication by z in terms of the bases F_T^P and H_T^P of S as an R-module, is block diagonal as written in equation (1).

For our computations we used the computer algebra system Macaulay2 (Grayson and Stillman). In Section 3.1 the code is described in more detail and also a link to a GitHub repository is provided.

Furthermore, we follow Ariki et al. (1997) and also determine a decomposition of $S/(z)$ into isotypical components corresponding to Young subgroups $S_{n_1} \times \cdots \times S_{n_m} \leqslant S_n$, where $\sum_{i=1}^m n_i = n$, using our modified higher Specht polynomials, see Theorem 4.14.

In Subsection 4.1 we indicate how to generalize our results to the family of complex reflection groups $G(m, p, n)$, with the explicit examples of $G(2, 1, n)$, also known as hyperoctahedral groups and denoted by B_n. These groups are of interest in many combinatorial applications, see e.g. Debus et al. (2023) for more detail. Writing down higher Specht polynomials and subsequently our matrix factorizations for more general groups $G(m, p, n)$ become notationally quite challenging and we will not pursue this topic further in this paper.

In order to get a complete answer for the question above, one also needs to consider the exceptional reflection groups of rank $\geqslant 2$ (for $k = \mathbb{C}$ these are the 15 groups $G_{23}, \ldots, G_{37}$ in the Shephard–Todd classification). Beyond the case of the family $G(m, p, n)$, it is not quite clear how to find an equivalent of a “Specht basis” so we pose the

Question 1.2. Can one find analogues for Higher Specht polynomials for all pseudo-reflection groups G, that is, find a natural basis of the coinvariant algebra S/R_+ which is compatible with the decomposition of S/R_+ into G-irreducible modules?

The paper is structured as follows: in Section 2 we recall basics of matrix factorizations, Young diagrams and introduce Δ as the discriminant of S_n acting on k^n. In Section 3 we prove our main result (Theorem 3.14) about the decomposition $S/(z)$ into isotypical components, using (modified) Higher Specht polynomials. We also give an explicit description of the matrix factorization for an isotypical component in Theorem 3.20 and close the section with examples and a description of our code in Section 3.1. Finally, in Section 4 we generalize this decomposition to Young subgroups of S_n and treat the groups $B_n = G(2, 1, n)$.

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2. Preliminaries

2.1. Matrix factorizations

Matrix factorizations were introduced by Eisenbud (1980) to study homological properties of hypersurface rings. Here we recall the main results that will be needed later, following the expositions in Leuschke and Wiegand (2012); Yoshino (1990).

Definition 2.1. Let B be a commutative ring and let $f \in B$. A *matrix factorization* of f is a pair (φ, ψ) of homomorphisms between free B-modules of the same rank n, with $\varphi : F \to G$ and $\psi : G \to F$, such that

$$\psi\varphi = f \cdot 1_F \qquad \text{and} \qquad \varphi\psi = f \cdot 1_G \,.$$

We may choose bases for F, G, and then, equivalently, φ, ψ are square matrices of size $n \times n$ over B, such that

$$\psi \cdot \varphi = f \cdot 1_{B^n} \qquad \text{and} \qquad \varphi \cdot \psi = f \cdot 1_{B^n} \,.$$

In the following, we will always assume that B is either a regular local ring or that B is a graded polynomial ring.

Recall that a *morphism of matrix factorizations* (φ_1, ψ_1) and (φ_2, ψ_2) of f is a pair of matrices (α, β) such that the following diagram commutes:

$$\begin{tikzcd} B^{n_1} \arrow[r, "\psi_1"] \arrow[d, "\alpha"] & B^{n_1} \arrow[r, "\varphi_1"] \arrow[d, "\beta"] & B^{n_1} \arrow[d, "\alpha"] \\ B^{n_2} \arrow[r, "\psi_2"] & B^{n_2} \arrow[r, "\varphi_2"] & B^{n_2} \end{tikzcd}$$

We say that two matrix factorizations are *equivalent* if there is a morphism (α, β) in which α, β are isomorphisms. Furthermore, for two matrix factorizations (φ_1, ψ_1) and (φ_2, ψ_2) of f, their *sum* is defined as

$$(\varphi_1, \psi_1) \oplus (\varphi_2, \psi_2) = \left(\begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix}, \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} \right) .$$

With these notions, matrix factorizations of f form an additive category, denoted by $\mathrm{MF}_B(f)$.

The main reason to consider matrix factorizations is that they correspond to maximal Cohen–Macaulay modules over a hypersurface ring: For any non-unit $f \neq 0$ in B we denote by $A = B/(f)$ the hypersurface ring defining $V(f) \subseteq \mathrm{Spec}(B)$. Let further $\mathfrak{C}(A)$ be the *category of maximal Cohen-Macaulay modules* over the ring A.

Theorem 2.2 *(Eisenbud's matrix factorization theorem, see Eisenbud, 1980, 6.1, 6.3). Assume that B is a regular local ring. Let $A = B/(f)$ be as above and let (φ, ψ) be a matrix factorization of f. Then the functor* $\mathrm{Coker}(\varphi, \psi) = \mathrm{Coker}(\varphi)$ *induces an equivalence of categories*

$$\underline{MF}_B(f) := MF_B(f)/\{(1, f)\} \simeq \mathfrak{C}(A)\ .$$

This shows that instead of directly calculating the maximal Cohen-Macaulay modules over A, we can instead construct matrix factorizations of f.

Remark 2.3. This theorem also holds in the *graded case*, that is, when B is a graded polynomial ring and f is a homogeneous element. Then one considers the categories of graded matrix factorizations and of graded CM-modules, see e.g., Yoshino (1990). In this paper we implicitly work in the graded situation, although we will not care too much about the actual degrees.

Remark 2.4. Note that when our ring B is an integral domain, then given one component of a matrix factorization φ, the other is uniquely determined by the formula

$$\psi = f\varphi^{-1}.$$

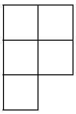
In fact, φ is a component of a matrix factorization precisely when φ is invertible in the fraction field of B and ψ defined above has entries in B rather than in its fraction field. In addition, when we consider minimal generators of the kernel of the map $\varphi : (B/(f))^{n_1} \to (B/(f))^{n_1}$, it is the image of the map $\psi : (B/(f))^{n_1} \to (B/(f))^{n_1}$. Hence, ψ gives the reduced first reduced syzygy of φ and we write $\psi = \mathrm{syz}^1_B \varphi$.

2.2. Young diagrams

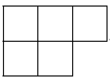
Here we recall basic facts about Young diagrams and representations of S_n, for more detail see Fulton (1996).

Consider $n \in \mathbb{N} \setminus \{0\}$. Let λ be a partition of n, denoted $\lambda \vdash n$, i.e. $\lambda = (\lambda_0, \dots, \lambda_{k-1})$, such that $1 \leqslant k \leqslant n$, $0 < \lambda_{i+1} \leqslant \lambda_i$ and $\sum_i \lambda_i = n$. A partition can also be represented as a *Young diagram*, which is constructed in the following way: Given a partition λ of n, the Young diagram associated to λ is a collection of left justified rows of squares called cells. Enumerate the rows from 0 to $k-1$, top to bottom, the number of cells in row i is λ_i. The partitions uniquely determine the Young diagram so we use the same notation λ for the partition and the Young diagram. We call a Young diagram associated to a partition of n, a Young diagram of size n. For the partition λ we write λ' for the *conjugate* partition, which is defined by taking the transpose of its Young diagram.

Example 2.5. Let $n = 5$ and $\lambda = (2, 2, 1)$ then the Young diagram is:



and $\lambda' = (3, 2)$ with Young diagram

.

Definition 2.6. A *Young tableau* is a Young diagram of size n, where each cell contains a number from 1 to n such that each number 1 to n appears only once. A Young tableau with an underlying Young diagram λ is said to be of *shape* λ. A Young tableau is called *standard* if the sequence of entries in the rows and columns are strictly increasing, and the set of standard Young tableaux of shape λ is denoted by $\mathrm{ST}(\lambda)$.

For a Young tableau $T \in \mathrm{ST}(\lambda)$ we write T' for its *conjugate tableau*, that is, T' is obtained by transposing T and its entries. Note that $T' \in \mathrm{ST}(\lambda')$, where λ' is the conjugate partition of λ.

Example 2.7. Let $n = 5$ and consider the Young diagram λ from the previous example. Then the following are Young tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

are also both standard tableaux.

The partition $\lambda = (2, 2, 1)$ has the conjugate $\lambda' = (3, 1)$, and the conjugates of the above tableaux are

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}\,.$$

Definition 2.8. Let $\lambda \vdash n$ and let T be a standard tableau of shape λ. We define the *word* $w(T)$ to be the sequence obtained by reading each column from bottom to top starting from the left. We write $w(T)^j$ for the j-th term in this sequence, where $j = 0, \dots, n-1$. The *index* $i(T) = i(w(T))$ is inductively defined as follows: write subscripts on the elements of the sequence $w(T)$ by first writing 1_0 in $w(T)$, then if you have k_p, add a subscript $(k+1)_p$ if $k+1$ is to the right of k in $w(T)$ or $(k+1)_{p+1}$ if $k+1$ is to the left of k in $w(T)$. We write the subscripts in the cells of a tableau $i(T)$, matching the entries of T. Further we define $\hat{i}(T)$ to be $i(T)$ written in non increasing order and $|i(T)|$ to be the sum of the indexes. This notion will be needed in Section 3, in particular Lemma 3.19.

Example 2.9.

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad\qquad i(T) = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$w(T) = (5, 3, 1, 4, 2) \qquad\qquad (5_2, 3_1, 1_0, 4_1, 2_0)$$

$$\hat{i}(T) = (2, 1, 1, 0, 0) \qquad\qquad |i(T)| = 4.$$

It is widely known that Young diagrams of size n, and thus partitions of n, are in bijection with the irreducible representations of S_n over a field k, where $\mathrm{Char}(k)$ does not divide $|S_n| = n!$, see (Fulton and Harris, 1991, Section 4) for $\mathrm{Char}(k) = 0$ and (James, 1978, Section 10, 11) for $\mathrm{Char}(k) \nmid n!$. We will sometimes denote the irreducible representations V_λ of S_n simply by their corresponding partitions λ.

2.3. The action of S_n on k^n

S_n naturally acts on a finite dimensional vector space V of dimension n over a field k where $\mathrm{Char}(k)$ does not divide $|S_n| = n!$. The quotient variety V/S_n is smooth by the theorem of Chevalley (1955). By fixing a basis $\{x_1, \dots, x_n\}$ of V, we form the symmetric algebra of V, $\mathrm{Sym}_k(V) \cong k[x_1, \dots, x_n]$. The action of S_n on V can be naturally extended to $S = k[x_1, \dots, x_n]$ via $\pi \cdot f(x) = f(\pi(x))$ for $\pi \in S_n$. We denote by R the invariant ring $R = S^{S_n}$. Note that we have $\mathrm{Spec}(S) = V$ and $\mathrm{Spec}(R) = V/S_n$. The theorem of Chevalley–Shepard–Todd also shows that $R \cong k[e_1, \dots, e_n]$, where e_i are the elementary symmetric polynomials in $x_1, \dots, x_n$. Note that R is a graded polynomial ring with $\deg e_i = i$.

Let $\mathcal{A}(S_n)$ be the set of reflecting hyperplanes of the action of S_n, the so-called reflection arrangement of S_n. Let $H \in \mathcal{A}(S_n)$ be such a hyperplane, and let α_H be a linear form defining H in S. Then

$$z = \prod_{H \in \mathcal{A}(S_n)} \alpha_H = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$$

is a polynomial in S defining the reflection arrangement of S_n, that is $V(z) = \bigcup_{H \in \mathcal{A}(S_n)} H$.

Definition 2.10. The discriminant polynomial of the S_n action on V is defined by:

$$\Delta = z^2 = \prod_{H \in \mathcal{A}(S_n)} \alpha_H^2 = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)^2 .$$

This is defined as an element of S but Δ is also invariant under the group action, see (Orlik and Terao, 1992, Lemma 6.44), and so can be expressed as an element of $R = k[e_1, \ldots, e_n]$. We note that $R/(\Delta)$ is a hypersurface ring and from Chevalley's theorem, S is a free R-module of rank $|S_n| = n!$.

Let (R_+) be the ideal generated by $e_1, \ldots, e_n$ in S and let $S/(R_+)$ the *coinvariant algebra*. The structure of S as a graded free R-module is given by Chevalley's theorem (Chevalley, 1955), Chevalley assumes k is a field of characteristic 0 but the result holds more generally for fields k with characteristic not dividing $|S_n|$, see (Bourbaki, 1981, Chapter 5, Section 2, Theorem 2). As a graded R-module S can be decomposed as:

$$S \cong S/(R_+) \otimes_k R.$$

Denote the set of irreducible representations V_λ of S_n by $\mathrm{irrep}(S_n)$. The R-module $S/(R_+)$ carries the regular representation, in particular:

$$S/(R_+) \cong \bigoplus_{V_\lambda \in \mathrm{irrep}(S_n)} V_\lambda^{\dim V_\lambda} .$$

We thus denote the λ-direct summand (the *λ-isotypical component*) of S by $S_\lambda = V_\lambda^{\dim V_\lambda} \otimes_k R$. The polynomial z is the relative invariant for the determinantal representation, see (Orlik and Terao, 1992, Theorem 6.37) (This was proved in Stanley, 1977, Theorem 3.1 for characteristic 0.). That is, z generates the direct summand of $S/(R_+)$ corresponding to $V_{\det} = V_\lambda$ where λ is given by the Young diagram

$$T = \begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \\ \hline \end{array}$$

Note that if $a \in S_\lambda$, then $za \in S_{\lambda \otimes \det}$. In the following we always denote $\lambda \otimes \det$ by λ' and note that λ' corresponds to the conjugate representation $V_{\lambda'}$ of V_λ.

Recalling Definition 2.1, we have that multiplication by z induces the matrix factorization (z, z) over R of Δ:

$$\begin{tikzcd} S \arrow[r, "z"] \arrow[d, "\cong"] & S \arrow[r, "z"] \arrow[d, "\cong"] & S \arrow[d, "\cong"] \\ R^{n!} \arrow[r, "z"] & R^{n!} \arrow[r, "z"] & R^{n!} \end{tikzcd}$$

This matrix factorization of $\Delta \in R$ corresponds to the maximal Cohen–Macaulay module $\mathrm{Coker}(z) = S/(z)$. It was shown in Buchweitz et al. (2020) that $\mathrm{End}_{R/(\Delta)}(S/(z))$ has global dimension n and $S/(z)$ is a faithful $R/(\Delta)$-module and so is a *noncommutative resolution* of the discriminant and that the direct summands of $S/(z)$ correspond to the nontrivial irreducible representations of S_n. More precisely, $\mathrm{End}_{R/(\Delta)}(S/(z)) \cong A/AeA$ as rings, where $A = S_n * S$ is the skew group ring and e is the idempotent for the trivial representation, (Buchweitz et al., 2020, Theorem 4.17). Thus the indecomposable projective modules of these two rings are in bijection and hence it follows that the maximal Cohen–Macaulay modules over $R/(\Delta)$ corresponding to the direct summands of $S/(z)$ are pairwise non-isomorphic.

2.4. Discriminants of reflection groups and discriminants of deformations (over $k = \mathbb{C}$)

Here we briefly comment on the connection between the classical discriminant of a polynomial (as discussed in the introduction) and discriminants of reflection groups: let $k = \mathbb{C}$ and let $G \subseteq \mathrm{GL}(n, k)$ be a finite complexified Coxeter group. That is, G is of type $A_k, B_k, D_k, E_6, E_7, E_8, I_2(p), F_4, H_3$, or H_4, see e.g. Humphreys (1990) for the classification. Then Arnol'd has shown that the discriminant of the reflection group G in $\mathbb{C}^n/G$ is isomorphic to the discriminant of a semi-universal deformation of the singularity of the same type, see Arnol'd (1972) for type ADE, and Arnol'd et al. (1985) for more details. Since S_n in its reflection representation corresponds to the Coxeter group A_{n-1}, our discriminant $V(\Delta)$ is isomorphic to the discriminant of the semi-universal deformation of an A_{n-1}-singularity. A semi-universal deformation of the singularity $k[x]/(x^n)$ is given by

$$F = x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 .$$

The discriminant of F is the classical discriminant of a polynomial of degree n.

For a concrete example, look at the correspondence for $n = 3$: Consider a cubic monic polynomial $f(x) = x^3 + ax^2 + bx + c$ with $a, b, c \in k$. Using Sylvester's formula, one calculates that the discriminant $D(f)$ is given as

$$D(f)(a,b,c) = -4a^3c + a^2b^2 + 18abc - 4b^3 - 27c^2 .$$

$D(f)$ is a quasi-homogeneous polynomial in $k[a,b,c]$ with $\deg(a) = 1, \deg(b) = 2, \deg(c) = 3$. Moreover, one can always achieve $a = 0$, and then the discriminant is of the well-known form

$$D(f)(b,c) = 4b^3 + 27c^2 .$$

On the other hand, we calculate the discriminant Δ of the action of S_3 on k^3 resp. $k[x_1, x_2, x_3]$ from its Saito matrix (see Saito, 1993 for Coxeter groups and Orlik and Terao, 1992 for complex reflection groups): the Saito matrix is given as JJ^T, where J is the Jacobian matrix of the basic invariants. In the case of S_3, we can take the power sums $s_i = \sum_{j=1}^{3} x_j^i$, $i = 1, 2, 3$ for the basic invariants and then $J = (\frac{\partial s_i}{\partial x_j})_{i,j=1,\ldots,3}$ (up to multiplication with a constant)

$$JJ^T = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} = \begin{pmatrix} 3 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{pmatrix} .$$

One calculates $s_4 = \frac{1}{6}(s_1^4 - 6s_2s_1^2 + 3s_2^2 + 8s_3s_1)$ and further

$$\Delta = \det(JJ^T) = 3s_2s_1^4 - 7s_1^2s_2^2 + 12s_1s_2s_3 + s_2^3 - 6s_3^2 - 1/3s_1^6 - 8/3s_3s_1^3 .$$

A coordinate change (and restricting to the invariant hyperplane $s_1 = x_1 + x_2 + x_3 = 0$) shows that this defines the same curve as $D(f)$, namely the cusp $\Delta = 4s_2^3 + 27s_3^2$ in $R = k[x_1, x_2, x_3]^{S_3} \cong k[s_2, s_3]$.

3. Decomposition of (z, z) for S_n

During this section fix $n \geqslant 3$. We consider the decomposition of the coinvariant algebra $S/(R_+)$ and the multiplication of z restricted to each isotypical component S_λ, where each λ corresponds to a Young tableau. Basis elements for the isotypical components S_λ are then given by Higher Specht polynomials (Ariki et al., 1997), we follow the definitions as in Ariki et al. (1997). However, for our purposes we will define a modification of these polynomials, see Definition 3.4.

Definition 3.1. Let $\lambda \vdash n$ and $T_1, T_2 \in \mathrm{ST}(\lambda)$. We define the *Last Letter Ordering (LL)* in the following way. Let $1 \leqslant k \leqslant n$ be the largest integer that is written in a different position for both tableaux T_1 and T_2. If the row in which k appears in T_2 is above the row it appears in T_1, then we say $T_1 < T_2$.

Example 3.2. Let $n = 5$. Consider the following two tableaux T_1 and T_2 on the partition $(3, 2)$.

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \cline{1-2}\end{array} = T_2$$

The set of elements that are in different positions is $\{2, 3\}$ thus the maximal element which has a different position is 3. Note that in T_1 the element 3 is written in the second row, which is below the first row where 3 is written in T_2.

Definition 3.3. Given a Young tableau T of shape λ, we define two subgroups of S_n, the *Row Stabilizer* $R(T)$ which are all elements of the group ring kS_n that permute elements within the same row, and similarly the *Column Stabilizer* $C(T)$ which permutes elements within the same columns of T. With these subgroups we define the following

$$r_T = \sum_{\pi \in R(T)} \pi \qquad c_T = \sum_{\rho \in C(T)} \mathrm{sgn}(\rho)\rho \ .$$

Lastly we define the *Young Symmetrizers*

$$\varepsilon_T = \frac{f^\lambda}{n!} c_T r_T \qquad \text{and} \qquad \sigma_T = \frac{f^\lambda}{n!} r_T c_T$$

where f^λ is the number of standard tableaux of shape λ. These are both idempotents of kS_n.

In the following we use multi-index notation $x_{w(T)}^{i(P)} = x_{w(T)^0}^{i(P)^0} \cdots x_{w(T)^{n-1}}^{i(P)^{n-1}}$. In addition, we further simplify notation by writing

$$x_T^P = x_{w(T)}^{i(P)}.$$

Definition 3.4. Let T, P be two standard Young tableaux of shape λ. The *higher Specht polynomials* are defined as

$$F_T^P = \varepsilon_T . x_T^P = \varepsilon_T . x_{w(T)}^{i(P)} \ .$$

We further define the *modified higher Specht polynomials* as

$$H_T^P = \sigma_T . x_T^P \ .$$

Definition 3.5. For a tableau $P \in \mathrm{ST}(\lambda)$ let M^P the R-submodule of S generated by the *modified* higher Specht polynomials $\{H_T^P \mid T \in \mathrm{ST}(\lambda)\}$ and let N^P the R-submodule of S generated by *normal* higher Specht polynomials $\{F_T^P \mid T \in \mathrm{ST}(\lambda)\}$.

Theorem 3.6. *(Ariki et al., 1997, Theorem 1, (2)) Let $P \in ST(\lambda)$, then the image of N^P in $S/(R_+)$ is a S_n-subrepresentation of S isomorphic to the irreducible representation corresponding to shape of P.*

Theorem 3.7. *Let $P \in ST(\lambda)$, then N^P is a S_n-subrepresentation of S isomorphic to M^P.*

Proof. Recall that $kS_n\varepsilon_T$ and $kS_n\sigma_T$ are both isomorphic to the irreducible representation V_λ (Fulton and Harris, 1991, Exercise 4.4). Therefore if we consider an ordering $\{T_1, \ldots, T_k\}$ of the standard tableaux of shape λ according to the last letter ordering, we can fix $\pi_i \in S_n$ such that $\pi_i(T_1) = T_i$.

Note that (Ariki et al., 1997, Lemma 5) shows that $\pi_i\varepsilon_{T_i}$ and $\pi_i\sigma_{T_i}$ are a basis for V_λ in kS_n. Let φ be an isomorphism $\varphi : kS_n\varepsilon_T \to kS_n\sigma_T$. Then any element $f \in N^P = \langle F_T^P \mid T \in \mathrm{ST}(\lambda)\rangle$ can be written as $(c_1\pi_1\varepsilon_{T_1} + \cdots + c_k\pi_k\varepsilon_{T_k}).x_T^P$. Consider the map between N^P and M^P defined by

$$(c_1\pi_1\varepsilon_{T_1} + \cdots + c_k\pi_k\varepsilon_{T_k}).x_T^P \longmapsto (c_1\varphi(\pi_1\varepsilon_{T_1}) + \cdots + c_k\varphi(\pi_k\varepsilon_{T_k})).x_T^P.$$

The above map is an isomorphism since φ is an isomorphism. □

Definition 3.8. For a tableau $T \in \mathrm{ST}(\lambda)$ let M_T the R-submodule of S generated by $\{H_T^P \mid P \in \mathrm{ST}(\lambda)\}$ and N_T the R-submodule generated by $\{F_T^P \mid P \in \mathrm{ST}(\lambda)\}$.

Remark 3.9. The modules M_T and N_T are not irreducible representations of S_n and are free R-modules.

Theorem 3.10. *(Terasoma and Yamada, 1993, Theorem 1) The collection*

$$\bigcup_{\lambda\vdash n}\{F_T^P \mid T \in ST(\lambda),\, P \in ST(\lambda)\}$$

forms a k-basis for $S/(R_+)$. And thus they also form a R-basis for the free R-module S.

Example 3.11. Let T be the following standard tableau:

$$T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\\ \vdots \\ \begin{array}{|c|}\hline n-1 \\ \hline n \\ \hline \end{array}$$

Note that T gives rise to the determinantal representation $V_{\det}$ of S_n. The Young Symmetrizer ε_T is given by:

$$\varepsilon_T = \frac{1}{n!}c_T r_T = \frac{1}{n!}\left(\sum_{\pi\in C(T)} \mathrm{sgn}(\pi)\pi\right)\mathrm{id} = \frac{1}{n!}\sum_{\pi\in S_n}\mathrm{sgn}(\pi)\pi.$$

The index $i(T) = (n-1, n-2, \ldots, 1, 0)$ and so $F_T^T = \varepsilon_T(x_1^0 x_2^1 \cdots x_n^{n-1})$. The higher Specht polynomial F_T^T is the polynomial $\frac{1}{n!}z$. Moreover, in this case we also have

$$\sigma_T = \frac{1}{n!}r_T c_T = \frac{1}{n!}\mathrm{id}\left(\sum_{\pi\in C(T)}\mathrm{sgn}(\pi)\pi\right) = \frac{1}{n!}\varepsilon_T,$$

and so $H_T^T = F_T^T = \frac{1}{n!}z$.

Lemma 3.12. *If $T_1 < T_2$ then $\varepsilon_{T_1}\varepsilon_{T_2} = \sigma_{T_2}\sigma_{T_1} = 0$.*

Proof. The proof for $\varepsilon_{T_1}\varepsilon_{T_2} = 0$ is widely known, see (Ariki et al., 1997, Lemma 4) or (Stembridge, 2011, Proposition 1). The equality $\sigma_{T_2}\sigma_{T_1} = 0$ can be seen by applying the anti-automorphism of kS_n given by $w \mapsto w^{-1}$. □

Lemma 3.13. *Let T be a Young tableau of shape λ and T' its conjugate, then we have*

$$\varepsilon_T(zf) = z\sigma_{T'}(f)$$

for any polynomial $f \in S$.

Proof. We first observe that for a Young tableau T of shape λ, $R(T) = C(T')$, $C(T) = R(T')$ and so

$$\varepsilon_T = \sum_{r\in R(T),c\in C(T)} \operatorname{sgn}(c)rc = \sum_{c\in C(T'),r\in R(T')} \operatorname{sgn}(r)cr.$$

For any $\pi \in S_n$, we have that $\pi(z) = \operatorname{sgn}(\pi)z$, and so for any polynomial f;

$$\begin{aligned}\varepsilon_T(zf) &= \sum_{r\in R(T),c\in C(T)} \operatorname{sgn}(c)rc(zf) = \sum_{c\in C(T'),r\in R(T')} \operatorname{sgn}(r)cr(zf)\\ &= z\left(\sum_{c\in C(T'),r\in R(T')} \operatorname{sgn}(c)cr(f))\right) = z(\sigma_{T'}(f)) \quad \square\end{aligned}$$

Theorem 3.14. *For the discriminant Δ of S_n, the matrix factorization defined by the reduced hyperplane arrangement, (z, z), can be decomposed in the following way:*

$$(z, z) = \bigoplus_{\lambda\vdash n} \bigoplus_{T\in ST(\lambda)} (z|_{M_T}, z|_{N_{T'}}).$$

Where $(z|_{M_T}, z|_{N_{T'}})$ are the matrix factorizations:

$$M_T \xrightarrow{z|_{M_T}} N_{T'} \xrightarrow{z|_{N_{T'}}} M_T$$

and M_T the R-submodule of S generated by $\{H^P_T \mid P \in ST(\lambda)\}$ and $N_{T'}$ the R-submodule generated by $\{F^{P'}_{T'} \mid P' \in ST(\lambda')\}$.

Proof. For an irreducible representation λ of G, recall that the map z takes elements from the isotypical component S_λ of S to the isotypical component $S_{\lambda'}$ of S where $\lambda' = \lambda \otimes \det$. Thus the matrix factorization decomposes immediately as:

$$(z, z) = \bigoplus_{\lambda\vdash n}(z|_{S_\lambda}, z|_{S_{\lambda'}}).$$

Let $T, P \in \mathrm{ST}(\lambda)$, and consider $H^P_T \in S_\lambda$ from above. Hence $zH^P_T \in S_{\lambda'}$. We can write zH^P_T as the following;

$$zH^P_T = \sum_{U,W\in \mathrm{ST}(\lambda')} g^{P,W}_{U,T} F^W_U \ , \tag{2}$$

where $g^{P,W}_{U,T}$ are in R, since the F^W_U form an R-basis of $S_{\lambda'}$ by Theorem 3.10. Recall that $F^P_T = \varepsilon_T.x^P_T$, and $H^P_T = \sigma_T.x^P_T$. If $T_1 < T_2$, then by Lemma 3.12 we have that $\varepsilon_{T_1}\varepsilon_{T_2} = \sigma_{T_2}\sigma_{T_1} = 0$ and hence $\varepsilon_{T_1} F^P_{T_2} = \varepsilon_{T_1}\varepsilon_{T_2}.x^P_{T_2} = 0$ and $\sigma_{T_2} H^P_{T_1} = \sigma_{T_2}\sigma_{T_1}.x^P_{T_1} = 0$. Order $\mathrm{ST}(\lambda') = (T'_1, \ldots, T'_k)$ in such a way that if $i < j$ then $T'_i < T'_j$. We want to calculate $zH^P_{T_1}$. Applying $\varepsilon_{T'_1}$ to both sides of equation (2) yields

$$\begin{aligned}\varepsilon_{T'_1} \sum_{U,W\in \mathrm{ST}(\lambda')} g^{P,W}_{U,T_1} F^W_U &= \sum_{U,W\in \mathrm{ST}(\lambda')} g^{P,W}_{U,T_1}(\varepsilon_{T'_1} F^W_U)\\ &= \sum_{W\in \mathrm{ST}(\lambda')} g^{P,W}_{T'_1,T_1} F^W_{T'_1} \ ,\end{aligned}$$

since T_1' is the least element in $\mathrm{ST}(\lambda')$. Further compute

$$\begin{aligned}\varepsilon_{T_1'}(zH_{T_1}^P) &= z(\sigma_{T_1} H_{T_1}^P)\\ &= zH_{T_1}^P\,.\end{aligned}$$

Thus we have that

$$zH_{T_1}^P = \sum_{W\in\mathrm{ST}(\lambda')} g_{T_1',T_1}^{P,W} F_{T_1'}^W\,.$$

With the argument above and the fact that for any $T_i \in \mathrm{ST}(\lambda)$ there exists a permutation π that permutes T_i and T_1 we have the following equation for any $T_i \in \mathrm{ST}(\lambda)$.

$$zH_{T_i}^P = \sum_{W\in\mathrm{ST}(\lambda)} (\mathrm{sgn}(\pi) g_{T_i',T_1}^{P,W}) F_{T_i'}^W\,.$$

This shows that for any $H_T^P \in S_\lambda$ we have that $zH_T^P \in \langle F_{T'}^P \mid P \in \mathrm{ST}(\lambda')\rangle$. Following a similar argument above we have that when we restrict z to $\langle F_{T'}^P \mid P \in \mathrm{ST}(\lambda')\rangle$ we have an element in $\langle H_T^P \mid P \in \mathrm{ST}(\lambda)\rangle$. Hence

$$zF_{T_i'}^P = \sum_{W\in\mathrm{ST}(\lambda')} (\mathrm{sgn}(\pi) h_{T_i,T_k'}^{P,W}) H_{T_i}^W\,.$$

We can see that for a $T \in \mathrm{ST}(\lambda)$ and using the notation above we can write the matrices of the maps as

$$\left[g_{T,T'}^{P,W}\right]\left[h_{T',T}^{P,W}\right]_{P,W'\in\mathrm{ST}(\lambda)} = \Delta\mathrm{Id}_{|\mathrm{ST}(\lambda)|\times|\mathrm{ST}(\lambda)|}. \quad \square$$

Theorem 3.15. *If $T_1, T_2 \in ST(\lambda)$ then there is a matrix factorization equivalence between $(z|_{N_{T_1}}, z|_{M_{T_1'}})$ and $(z|_{N_{T_2}}, z|_{M_{T_2'}})$.*

Proof. Consider $\pi \in S_n$ such that $\pi(T_1) = T_2$, thus for any $P \in \mathrm{ST}(\lambda)$ we have that

$$zH_{T_1}^P = \mathrm{sgn}(\pi) zH_{T_2}^P \text{ and } zF_{T_1'}^{P'} = \mathrm{sgn}(\pi) zF_{T_2'}^{P'}.$$

This means that $z|_{N_{T_1}} = \mathrm{sgn}(\pi) z|_{N_{T_2}}$ and $z|_{M_{T_1'}} = \mathrm{sgn}(\pi) z|_{M_{T_2'}}$. Therefore the matrices are the same up to multiplication by a scalar matrix, and so there is a matrix factorization equivalence between them. $\square$

Remark 3.16. Let λ be a Young diagram of size n, then $|\mathrm{ST}(\lambda)| = \dim S_\lambda$, so we get $\dim S_\lambda$ copies of the maximal Cohen–Macaulay-module $\mathrm{Coker}(z|_{N_{T_1}}, z|_{M_{T_1'}})$ in the decomposition.

Definition 3.17. Define a R-bilinear form $\langle -, - \rangle : S \times S \to R$ where for any $f, g \in S$ we have that $\langle f, g\rangle = \frac{1}{z}\sum_{\pi\in S_n} \mathrm{sgn}(\pi)\pi(fg)$.

Note that this bilinear form is similar to the one used in Ariki et al. (1997), except we do not set the variables to 0. Recall from Definition 2.8 that for a tableau T, $\hat{i}(T)$ is $i(T)$ written in non increasing order and $|i(T)|$ is the sum of the indexes. Consider the ordering on $\mathrm{ST}(\lambda)$, defined in the following way. If $|\hat{i}(P_1)| < |\hat{i}(P_2)|$, then $P_1 < P_2$; if $|\hat{i}(P_1)| = |\hat{i}(P_2)|$ and $\hat{i}(P_1) < \hat{i}(P_2)$ with respect to the reverse lexicographical ordering, then $P_1 < P_2$; lastly if $\hat{i}(P_1) = \hat{i}(P_2)$ and P_1 is smaller than P_2 with respect to the last letter order, then we set $P_1 < P_2$.

Definition 3.18. We define the co-charge $j(T)$ of a tableau T by $j(T) = i(T')$.

Lemma 3.19. *Let $P_1 < P_2$ with respect to the ordering above, then $\langle F_T^{P_1}, F_{T'}^{P_2'} \rangle = 0$*

Proof. The main idea here is that if $\deg(fg) < \deg(z) = \frac{n(n-1)}{2}$ then $\langle f, g \rangle$ is either 0 and if $\deg(fg) = \frac{n(n-1)}{2}$ then it is a constant. In these cases the result (Ariki et al., 1997, Proposition 1) for $\langle -, - \rangle$ hold, thus it is sufficient to show that if $P_1 < P_2$ then $\deg(F_T^{P_1} F_T'^{P_2'}) < \frac{n(n-1)}{2}$. The Lemma then follows from the case distinction:

(1) If $|\hat{i}(P_1)| < |\hat{i}(P_2)|$, then $|\hat{i}(P_1)| + |\hat{j}(P_2)| < \frac{n(n-1)}{2}$ by (Ariki et al., 1997, Lemma 1).
(2) If $|\hat{i}(P_1)| = |\hat{i}(P_2)|$ and $\hat{i}(P_1) < \hat{i}(P_2)$ in the reverse lexicographical ordering, then $|\hat{i}(P_1)| = \frac{n(n-1)}{2} - |\hat{j}(P_2)|$ and $|\hat{i}(P_1)| + |\hat{j}(P_2)| = \frac{n(n-1)}{2}$ and thus from the proof of (Ariki et al., 1997, Theorem 1) the results hold.
(3) If $|\hat{i}(P_1)| = |\hat{i}(P_2)|$ and $\hat{i}(P_1) = \hat{i}(P_2)$, then if $P_1 < P_2$ with respect to the last letter order the results hold using the same argument as in (2). □

Theorem 3.20. *Let λ be a Young diagram, where m is the dimension of the corresponding irreducible representation and $T \in ST(\lambda)$. The matrix factorization $(z|_{N_T}, z|_{M_{T'}})$ can be written explicitly as (A, B), where*

$$A = \begin{bmatrix} g_1^1 & \cdots & g_m^1 \\ \vdots & \ddots & \vdots \\ g_1^m & \cdots & g_m^m \end{bmatrix}$$

and B is a $m \times m$ matrix obtained by taking the first reduced syzygy of A, so $B = \mathrm{syz}_R^1 A$ and g_i^j are defined iteratively as:

$$g_i^j = \frac{\langle F_T^{T_j}, zH_T^{T_i} - g_i^1 F_{T'}^{T_1'} - \cdots - g_i^{j-1} F_{T'}^{T_{j-1}'} \rangle}{\langle F_T^{T_j}, F_{T'}^{T_j'} \rangle}$$

for $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant m$.

Proof. Note that $\langle F_T^P, F_{T'}^{P'} \rangle$ is a non-zero constant in k, and a formula is given in (Ariki et al., 1997, Proposition 1). Order $\mathrm{ST}(\lambda) = (P_1, \ldots, P_m)$ where if $i < j$ then $P_i < P_j$, so if $i < j$ then $\langle F_T^{P_i}, F_{T'}^{P_j'} \rangle = 0$. Consider the matrix describing $z|_{M_T} : M_T \to N_{T'}$ to have the entries $[g_i^j]_{i,j}$ where i, j index the rows and columns. We calculate

$$zH_T^{P_i} = g_i^1 F_{T'}^{P_1'} + \cdots + g_i^m F_{T'}^{P_m'} .$$

Plugging this into the bilinear form with $F_T^{P_j}$ yields

$$\langle F_T^{P_j}, zH_T^{P_i} \rangle = g_i^1 \langle F_T^{P_j}, F_{T'}^{P_1'} \rangle + \ldots + g_i^m \langle F_T^{P_j}, F_{T'}^{P_m'} \rangle . \tag{3}$$

For $1 < j$ the term $g_i^j \langle F_T^{P_1}, F_{T'}^{P_j'} \rangle = 0$, therefore $\langle F_T^{P_1}, zH_T^{P_i} \rangle = g_i^1 \langle F_T^{P_1}, F_{T'}^{P_1'} \rangle$. Thus we can then recursively write a formula for each entry.

$$g_i^1 = \frac{\langle F_T^{P_1}, zH_T^{P_i} \rangle}{\langle F_T^{P_1}, F_{T'}^{P_1'} \rangle}$$

$$\vdots$$

$$g_i^j = \frac{\langle F_T^{P_j}, zH_T^{P_i} - g_i^1 F_{T'}^{P_1'} - \cdots - g_i^{j-1} F_{T'}^{P_{j-1}} \rangle}{\langle F_T^{P_j}, F_{T'}^{P_j'} \rangle}$$

These are the entries in row i, and the matrix A can be computed by considering all i. $\square$

Remark 3.21. This gives a quicker computational way to calculate the matrix factorization corresponding to a irreducible representation of S_n, and thus a maximal Cohen–Macaulay module over R, for a specific irreducible representation λ of S_n.

Remark 3.22. Recall that if L is an lower triangular $k \times k$ matrix, we can write $L = D + N$ where D is diagonal and N is strictly lower triangular. If D is invertible, then $D^{-1}N$ is nilpotent and we have the well known formula

$$\begin{aligned} L^{-1} &= D^{-1} - D^{-1}ND^{-1} + D^{-1}ND^{-1}ND^{-1} - \cdots \\ &= D^{-1}\left(\sum_{i=0}^{n}(ND^{-1})^i\right). \end{aligned}$$

If we define matrices

$$\begin{aligned} U_{ij} &= \langle F_T^{P_j}, F_{T'}^{P'_i}\rangle \\ G_{ij} &= g_i^j \\ X_{ij} &= \langle F_T^{P_j}, zH_T^{P_i}\rangle \end{aligned}$$

then Equation (3) gives $X = GL$ and Lemma 3.19 shows that L is lower triangular. So we can solve the recursive formula in Theorem 3.20 as

$$\begin{aligned} G &= XL^{-1} \\ &= X(D^{-1} - D^{-1}ND^{-1} + D^{-1}ND^{-1}ND^{-1} - \cdots) \\ &= XD^{-1}\left(\sum_{i=0}^{n}(ND^{-1})^i\right). \end{aligned}$$

Example 3.23. Let S_5 act on $\mathbb{C}[x_1, x_2, x_3, x_4, x_5]$ with the basic invariants e_i, $i = 1, \ldots, 5$. If we quotient out by the hyperplane $e_1 = x_1 + \cdots + x_5 = 0$, we get a set of invariants $t_1, \ldots, t_4$ of the action of S_5 on $k[x_2, x_3, x_4, x_5]$, where

$$t_i = e_{i+1}(-x_2 - x_3 - x_4 - x_5, x_2, x_3, x_4, x_5).$$

The discriminant Δ of this group action is given by:

$$\begin{aligned} \Delta = &-\frac{1}{3\,600}t_1^3t_2^2t_3^2 + \frac{1}{900}t_1^4t_3^3 + \frac{1}{900}t_1^3t_2^3t_4 - \frac{1}{200}t_1^4t_2t_3t_4 + \frac{3}{400}t_1^5t_4^2 - \\ &\frac{3}{1\,600}t_2^4t_3^2 + \frac{1}{100}t_1t_2^2t_3^3 - \frac{2}{225}t_1^2t_3^4 + \frac{3}{400}t_2^5t_4 - \frac{7}{160}t_1t_2^3t_3t_4 + \\ &\frac{7}{180}t_1^2t_2t_3^2t_4 + \frac{11}{192}t_1^2t_2^2t_4^2 - \frac{1}{16}t_1^3t_3t_4^2 + \frac{4}{225}t_3^5 - \frac{1}{9}t_2t_3^3t_4 + \frac{5}{32}t_2^2t_3t_4^2 + \\ &\frac{5}{36}t_1t_3^2t_4^2 - \frac{25}{96}t_1t_2t_4^3 + \frac{125}{576}t_4^4. \end{aligned}$$

Let $\lambda = \ydiagram{4,1}$ be a partition of n corresponding to the standard representation of S_n, then the matrix factorization of $S/(z)$ corresponding to λ is (A, B) where

$$\left[\begin{matrix} -6{t_1}^2t_3+24{t_3}^2-180t_2t_4 & 3{t_2}^2t_3-23t_1t_2t_4-25{t_4}^2 & -2t_1t_2t_3+12{t_1}^2t_4-20t_3t_4 \\ -36t_2t_3+270t_1t_4 & -4t_1t_2t_3+36{t_1}^2t_4+40t_3t_4 & -32{t_3}^2+90t_2t_4 \\ 12t_1t_2+600t_4 & -16t_2t_3+120t_1t_4 & 12{t_2}^2-32t_1t_3 \\ 144{t_2}^2-384t_1t_3 & 16t_1{t_2}^2-48{t_1}^2t_3-64{t_3}^2+80t_2t_4 & 32t_2t_3-240t_1t_4 \\ 36{t_1}^2+240t_3 & -24{t_2}^2+64t_1t_3 & 4t_1t_2+200t_4 \end{matrix}\right.$$
$$\left.\begin{matrix} 6t_2{t_3}^2-18{t_2}^2t_4+3t_1t_3t_4 & 2{t_1}^2t_2t_4+12t_2t_3t_4+10t_1{t_4}^2 \\ -8t_1{t_3}^2+24t_1t_2t_4+75{t_4}^2 & 18{t_2}^2t_4-48t_1t_3t_4 \\ -32{t_3}^2+90t_2t_4 & -12{t_1}^2t_4-80t_3t_4 \\ 8t_1t_2t_3-72{t_1}^2t_4-80t_3t_4 & -18{t_2}^2t_3+64t_1{t_3}^2-46t_1t_2t_4-50{t_4}^2 \\ -12t_2t_3+90t_1t_4 & -4{t_1}^2t_3-48{t_3}^2+60t_2t_4 \end{matrix}\right]$$

Fig. 1. The matrix A for the partition $(3,2) \vdash 5$.

$$A = \begin{pmatrix} t_4 & -\frac{1}{50}\,t_1t_3 & -\frac{1}{50}\,t_2t_3 + \frac{1}{10}\,t_1t_4 & -\frac{1}{25}\,t_3^2 + \frac{1}{10}\,t_2t_4 \\ -\frac{8}{5}\,t_3 & \frac{2}{25}\,t_1t_2 - \frac{1}{2}\,t_4 & \frac{2}{25}\,t_2^2 - \frac{2}{15}\,t_1t_3 & \frac{3}{50}\,t_2t_3 - \frac{3}{10}\,t_1t_4 \\ \frac{6}{5}\,t_2 & -\frac{3}{25}\,t_1^2 + \frac{2}{5}\,t_3 & -\frac{4}{75}\,t_1t_2 + \frac{1}{3}\,t_4 & -\frac{1}{25}\,t_1t_3 \\ -\frac{4}{5}\,t_1 & -\frac{3}{10}\,t_2 & -\frac{4}{15}\,t_3 & -\frac{1}{2}\,t_4 \end{pmatrix}$$

and B is a 4×4 matrix with entries in R such that $\operatorname{Coker} B \cong \operatorname{syz}_R^1(A)$. Note that as the dimension of the representation corresponding to λ is 4, we get 4 copies of this matrix.

Example 3.24. Let $\lambda = \ydiagram{3,2}$ then the matrix factorization of $S/(z)$ corresponding to λ is (A, B) where A is the matrix given in Fig. 1 and $B = \Delta A^{-1}$ is a 5×5 matrix with entries in R.

Both of the matrices from Example 3.23 and 3.24 were obtained using the Macaulay2 package "PushForward" (Raicu et al., 2021).

3.1. Macaulay2 package walkthrough

In this section we will demonstrate how to obtain the matrix factorizations of Examples 3.23 and 3.24 from the Macaulay2 package available at Faber et al. (2022). Note that our package makes heavy use of the Macaulay2 packages "PushForward" and "SpechtModule" (Raicu et al., 2021; Spechtmodule, 2019). We present two methods of computing the matrix factorizations: the first method will be a general method involving Gröbner bases, which can be used for computing any of the matrix factorizations described in this paper, in particular the matrix factorizations for the product subgroups of Section 4. The second method will be a faster method, however, it can only be used for the matrix factorizations of the discriminant of S_n. Consider $m \geqslant 0$ and $(n_1, \ldots, n_m)$ to be a list of integers such that $n_i \geqslant 0$ and $\sum n_i = n$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be an m-tuple of Young diagrams of type $(n_1, \ldots, n_m)$. Such a λ will be called an *m-partition*. Note here that m-partitions are used extensively in Section 4. For both methods we will assume we are given the following data.

- An m-partition $\lambda \vdash n$ with the hook-length formula $h(\lambda) = d$.
- A polynomial ring $S = k[x_1, \ldots, x_n]$.
- $T \in \mathrm{ST}(\lambda)$ the standard tableau of shape λ obtained by filling each row with successive numbers.
- Two lists $L_1 = \{F_T^{V_1}, \ldots, F_T^{V_d}\}$ and $L_2 = \{H_{T'}^{V'_1}, \ldots, H_{T'}^{V'_d}\}$ of higher Specht polynomials such that $V'_i < V'_j$ (w.r.t. the last letter ordering Ariki et al., 1997) if $i < j$.
- A generating set for the subring of symmetric functions $\{E_1, \ldots, E_n\}$.

Considering the ring S as a module over $R = S^{S_n}$, let the submodules M_1 and M_2 be those generated by the Specht polynomials in L_1 and L_2 respectively. The goal is to obtain the matrix $G \in R^{d \times d}$ which corresponds to the multiplication by the polynomial z defined in Section 2.3. The entries of the matrix G can be denoted as seen below:

$$\begin{bmatrix} g_1^1 & \cdots & g_d^1 \\ \vdots & \ddots & \vdots \\ g_1^d & \cdots & g_d^d \end{bmatrix}$$

The entries $g_i^j \in R$ for all $1 \leqslant i, j \leqslant d$, and from the description of this matrix we know from Theorem 3.20 that for a given $1 \leqslant i \leqslant d$ we can describe $zF_T^{V_i}$ as below:

$$zF_T^{V_i} = \sum_{l=1}^{d} g_i^l H_{T'}^{V_l'} \tag{4}$$

We have created a Macaulay2 package to give exactly the data required given a m-partition λ and a polynomial ring S. First declare a polynomial ring of n variables, and a m-partition can be declared using the "makePar" function which accepts a list of lists in descending order. If we take the example of $((2,1),(1)) \vdash 4$ as a 2-list then we have the following code:

```
i1 : loadPackage "SpechtPolynomials"
o1 = SpechtPolynomials
o1 : Package
i2 : n = 4;
i3 : R = QQ[x_1..x_n];
i4 : par = makePar {{2,1},{1}};
```

Next a generating set for R is needed, this can be any generating set such as the elementary symmetric functions or the power sum functions. Let $E_i = x_1^i + \cdots + x_n^i$ be the first n power sum functions. To make this list in Macaulay2 run the code:

```
i5 : Sym = apply(1..n, i-> sum apply(gens R, x-> x^i));
```

In this case "Sym" is the list of polynomials that generate all symmetric functions. A list of higher Specht polynomials can be obtained from the partition and the function "HSP". The function will make a constructor for the higher Specht polynomials which takes in 3 arguments, a polynomial ring S, the number m of the m-partition, and a type $t \in \{1,2\}$. Note here that m-partitions appear for product submodules of S_n, see Section 4. Type 1 higher Specht polynomials are the polynomials F_T^P and type 2 are the modified polynomials H_T^P in Definition 3.4. To construct the list $L_1 = \{F_T^{V_1}, \ldots, F_T^{V_d}\}$ we start with:

```
i6 : F = HSP(R,#par,1);
```

All that is needed is the tableau T described in our data. This is obtained with the following function:

```
i7 : tab1 = first tabFromPar par;
```

The constructor F is used to generate the list L_1 by using a method to fix F_T^V with "tab1" on the bottom and letting V run through all standard tableaux of that size. This is done with the following code:

```
L_1 = (F_tab1)_1;
```

The tableaux T can be conjugated and the same steps repeated as above (with type $t = 2$) to obtain $L_2 = \{H_{T'}^{V_1'}, \ldots, H_{T'}^{V_d'}\}$.

```
i17 : tab2 = conjugate tab1;
i18 : H = HSP(R,#par,2);
i19 : L_2 = (H_tab2)_1;
```

All the data we need for computing the matrices has been initialized.

3.1.1. Gröbner basis method

This method is well known and works in the more general setting where we have an inclusion of polynomial rings $R \subseteq S$ where $S \cong R^n$ as an R-module, and we wish to compute multiplication by $z \in S$ as a matrix $\rho_z : R^n \to R^n$. We consider a auxiliary polynomial ring $\hat{S} = k[e_1, \dots, e_n, f_1, \dots, f_n, x_1, \dots, x_n]$ ordering the variables as presented. Thus we can consider the ideal I generated by the following sets $\{e_i - E_i \,|\, 1 \leqslant i \leqslant n\}$ and $\{f_i - H_{T'}^{V_i'} \,|\, 1 \leqslant i \leqslant d\}$. We can write the image of the elements $zF_T^{V_1}, \dots, zF_T^{V_d}$ in the ring $\hat{R}$ by I. Because of the ordering of these elements, the result will be written in terms of $e_1, \dots, e_n$ and $f_1, \dots, f_n$. Thus producing the columns of the matrix G when we consider that e_i represents the element E_i in R.

For our purposes this method is very slow. Essentially if $c_1, \dots, c_l$ are the size of the columns of the diagram of λ then the polynomial with highest degree d in the ideal I described above is given by the following

$$d = \max(n, \sum_{i=1}^{l} c_i - i)\,. \tag{5}$$

The sum is obtained by considering the highest charge of the standard Young tableaux of shape λ, given by the tableau obtained by filling the columns in successive order. This means however that we are computing the Gröbner basis of an ideal with highest degree polynomial being d in $2n + d$ variables, which is computationally slow. Thus we seek a better way of computing these ideals.

Remark 3.25. The effectivity of this method depends on the chosen monomial order for the Gröbner basis. It would be an interesting problem to find a monomial order tailored to our polynomials so that the computations will be faster.

3.1.2. Recursive method

Restricting the problem to the S_n case enables the use of the bilinear form described in Theorem 3.20 to compute these matrices using a recursive method.

This method can be implemented in Macaulay2, assuming we have worked with a partition $\lambda \vdash n$ and we obtain all the data described in the beginning of this section. We describe a basic example, consider the case that $n = 4$ and $\lambda = (2, 2)$. Then we begin with the following set up

```
n=4
par = makePar {{2,2}}
R = QQ[x_1..x_n]
Sym = apply(1..n, i-> sum apply(gens R, x-> x^i))
F = HSP(R,#par,1)
tab1 = first tabFromPar par
L1 = (F_tab1)_1
tab2 = conjugate tab1
H = HSP(R,#par,2)
L2 = (H_tab1)_1
L3 = reverse (H_tab2)_1
L4 = reverse (F_tab2)_1
```

This is very similar to the set up at the beginning of this section, except two more lists are needed in order to compute both matrices of the matrix factorization. There are four lists of higher Specht polynomials, “L1” and “L4” are of F type, likewise “L2” and “L3” are of H type. Now we make a function for the bilinear form described above and find our z-value.

```
z = product flatten apply(toList(1..n-1),
   i -> flatten apply(toList(i+1..n), j -> x_i-x_j))
Lz1 = apply(L1,f->z*f)
Lz2 = apply(L3,f->z*f)

bil = (f,g)->(
     return antiSymmetrize(f*g)//z;
     )
```

Lists “Lz1” and “Lz2” are the multiplication by z of the lists “L1” and “L3”. Then we can compute the matrices by using the recursive method written in Theorem 3.20.

```
G1 = {}
d = #L1
pol = 0_R;
for i from 0 to d-1 do (
     row = {};
     for j from 0 to d-1 do (
pol = bil(L2_(j),Lz1_i-sum apply(toList(0..#row-1),l->row_l*L3_l))//bil(L2_j,L3_(j));
print factor pol;
row = append(row,pol);
);
     G1 = append(G1,row);
     );

G2 = {};
pol = 0_R;
for i from 0 to d-1 do (
     row = {};
     for j from 0 to d-1 do (
pol = bil(L4_(j),Lz2_i-sum apply(toList(0..#row-1),l->row_l*L1_l))//bil(L4_j,L1_j);
print factor pol;
row = append(row,pol);
);
     G2 = append(G2,row);
     );

G1 = transpose matrix G1
G2 = transpose matrix G2
```

Let $e_i = \sum_j x_j^i$, which generate the subring R of symmetric functions in S. The entries in matrices G_1 and G_2 can be expressed in terms of the generators $\{e_1, \ldots, e_n\}$.

```
A = QQ[e_1..e_n]
B = QQ[e_1..e_n,x_1..x_n]
I = ideal apply(1..n, i->e_i-sum(apply(toList(x_1..x_n),f->f^i)))
H = B/I
phi = map(H,R)
rho = map(A,B)
G1 = rho lift(phi G1,B)
G2 = rho lift(phi G2,B)
```

This produces the following matrices

$$G1 = \begin{pmatrix} \frac{4}{3}e_2^2 - 4e_4 & -\frac{1}{3}e_2^3 - \frac{2}{3}e_3^2 \\ \frac{2}{3}e_2^3 + \frac{4}{3}e_3^2 & \frac{1}{3}e_2e_3^2 + \frac{1}{6}e_2^2e_4 + \frac{1}{6}e_4^2 \end{pmatrix}$$
$$G2 = \begin{pmatrix} -\frac{1}{2}e_2e_3^2 - \frac{1}{4}e_2^2e_4 - \frac{1}{4}e_4^2 & -\frac{1}{2}e_2^3 - e_3^2 \\ e_2^3 + 2e_3^2 & -2e_2^2 + 6e_4 \end{pmatrix}$$

One can check that they do indeed form a matrix factorization,

$$G1 \cdot G2 = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} .$$

This method offers a faster way to obtain the matrix factorizations. Given the tableau T we used to construct the lists of Specht polynomials, to compute each bilinear product to find the entries takes $|C(T)||R(T)|$ polynomial permutations and additions, with one polynomial division done. Thus if d is the number of standard tableaux of shape λ, each matrix takes $d|C(T)||R(T)|$ permutations, additions and divisions, which will be faster than the Gröbner basis method.

4. Decomposition for product submodules of S_n

In this section we generalize Theorem 3.14 to irreducible representations of the *Young Subgroups* of S_n. These subgroups are of the form $S_{n_1} \times \cdots \times S_{n_m} \leqslant S_n$ for any given m-tuple $(n_1, \dots, n_m)$ with $\sum_{i=1}^m n_i = n$. In particular the decomposition of (z, z) will correspond to the irreducible representations of $S_{n_1} \times \cdots \times S_{n_m}$. The irreducible representations of the Young subgroup $S_{n_1} \times \cdots \times S_{n_m}$ will be of the form $V_{n_1} \otimes \ldots \otimes V_{n_m}$ where each V_{n_i} is a irreducible representation of S_{n_i} thus we will discuss a basis for the coinvariant algebra $S/(R_+)$ indexed by these representations. While the decomposition will be more coarse than the one discussed in Section 3, the motivation for this section is that the construction can be used to describe the decomposition for the wreath product groups $G(m, 1, n)$ from the Shephard–Todd classification.

The definitions from Section 2.2 can be generalized to describe the representations of Young subgroups. Consider $m \geqslant 0$ and $(n_1, \dots, n_m)$ to be a list of integers such that $n_i \geqslant 0$ and $\sum n_i = n$. Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be an m-tuple of Young diagrams of type $(n_1, \dots, n_m)$. Such a λ will be called an *m-partition.* If $\lambda_i \vdash n_i$ for all $1 \leqslant i \leqslant m$, write $P(m, n)$ as the set of m-tuples of partitions with a total of n cells and define an operation called the shift $\mathrm{sh}^1(\lambda) = (\lambda_{m-1}, \lambda_0 \dots, \lambda_{m-2})$. An m-tuple of tableaux $T = (T_1, \dots, T_m)$ is of shape λ if each T_i is of shape λ_i, and is called an *m-tableau.* An m-tableau is standard if all of its tableaux are standard, with the set of all standard m-tableaux being $\mathrm{ST}(\lambda)$. We define $\mathrm{ST}(n_1, \dots, n_m)$ to be the set of standard m-tableaux of type $(n_1, \dots, n_m)$. If $T = (T^1, \dots, T^m)$ is a standard m-tableau of type $(n_1, \dots, n_m)$ then $T' = ((T^1)', \dots, (T^m)')$.

Example 4.1. Let $n = 7$, $m = 3$ then

$$T = \left(\begin{array}{|c|c|}\hline 1 & 7 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\, , - , \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 6 \\ \hline \end{array} \right)$$

is a standard m-tableau of type $(3, 0, 4)$. The conjugate tableau T' is given by

$$T' = \left(\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 7 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\, , - , \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 6 \\ \hline \end{array} \right)$$

Remark 4.2. We have defined T' differently to Ariki et al. (1997), where they also reverse the order of the tableau, this is so that, given a $T \in \mathrm{ST}(\lambda)$, the m-tableau T' is in $\mathrm{ST}(\lambda') = \mathrm{ST}(\lambda \otimes \det)$. The consequence of our definition is that we will not be able to use the same bilinear form reduction to get a similar result to Theorem 3.20 as before.

Similarly as in Section 3 we will define an ordering on $\mathrm{ST}(n_1, \dots, n_m)$ as an extension of the Last Letter ordering. First we will consider an ordering on $\mathrm{ST}(\lambda)$. Consider two standard m-tableaux $T_1 = (T_1^1, \dots, T_1^m)$ and $T_2 = (T_2^1, \dots, T_2^m)$ of the same shape. Let $1 \leqslant k \leqslant n$ be the greatest number that appears in different cells in both of the tableaux. We say $T_1 < T_2$ if either k is written in T_1^i and T_2^j with $i < j$, or it is written in a row in T_1^i below a row in T_2^i, for $1 \leqslant i \leqslant n$.

Example 4.3. Let $n = 4$, $m = 2$ and $\lambda = \left(\ydiagram{2,1}, \ydiagram{1}\right)$ then, with respect to the Last Letter ordering on $\mathrm{ST}(\lambda)$ we have

$$\left(\begin{ytableau} 1 & 2 \\ 4 \end{ytableau}, \begin{ytableau} 3 \end{ytableau}\right) < \left(\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}, \begin{ytableau} 4 \end{ytableau}\right).$$

Now we consider when the last number that appears in the different cells is on the same tableau for both m-tableaux. Let $n = 4, m = 2$ and $\lambda = \left(\ydiagram{2,2}, \ydiagram{2,1}\right)$ then with respect to the Last Letter ordering on $\mathrm{ST}(\lambda)$ we have

$$\left(\begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}, \begin{ytableau} 5 & 6 \\ 7 \end{ytableau}\right) < \left(\begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}, \begin{ytableau} 5 & 7 \\ 6 \end{ytableau}\right).$$

Theorem 4.4. *Let λ be a Young diagram of type $(n_1, \dots, n_m)$, then the last letter ordering is total in $ST(\lambda)$.*

Proof. Let $T_1, T_2 \in \mathrm{ST}(\lambda)$ then either $T_1 = T_2$ or there exists, at least 2 elements which appear in different boxes. Let k be the last number that changes, then k must appear in different rows otherwise one of T_1, T_2 would not be standard. Since the last number that changes appears in different rows then either $T_1 < T_2$ or $T_2 < T_1$. $\square$

If we consider the lexicographical ordering of the partitions of type $(n_1, \dots, n_m)$ we can fully order $\mathrm{ST}(n_1, \dots, n_m)$. Let us consider a ordering on partitions of n, if $\lambda_1 = (\alpha_1, \dots, \alpha_k)$ and $\lambda_2 = (\beta_1, \dots, \beta_l)$ are two partitions of n, and let $1 \leqslant i \leqslant \min(k, l)$ be the first integer that $\alpha_i - \beta_i \neq 0$, then if $\alpha_i - \beta_i > 0$ then $\lambda_1 < \lambda_2$. Using this ordering, it is easy to see that we can use the lexicographical ordering on a partition of type $(n_1, \dots, n_m)$ to have a total ordering. This way we can order tableaux of different partitions by comparing their shapes. This way if we have two tableaux T and V, if they are in the same partition we may order them using LL-order, and if they are in different partitions, give their order with lexicographical order on the partitions.

Example 4.5. Let us consider $m = 1$ and $n = 4$ then we can order the partitions the following way;

$$(4) < (3, 1) < (2, 2) < (2, 1, 1) < (1, 1, 1, 1)$$

If we consider $\lambda_1 = (3, 1)$ and $\lambda_2 = (2, 2)$ and $T \in \mathrm{ST}(\lambda_1)$ and $V \in \mathrm{ST}(\lambda_2)$ as below, by our ordering we have:

$$\begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau} < \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}$$

Now consider $m = 3$ and $n = 6$, and define $\lambda_1 = \left(\ydiagram{2,1}, \ydiagram{1}, \ydiagram{2}\right)$ and $\lambda_2 = \left(\ydiagram{2,1}, \ydiagram{1}, \ydiagram{1,1}\right)$ then since $\ydiagram{2} < \ydiagram{1,1}$ we have the following:

$$\left(\begin{ytableau} 2 & 3 \\ 5 \end{ytableau}, \begin{ytableau} 1 \end{ytableau}, \begin{ytableau} 4 & 6 \end{ytableau}\right) < \left(\begin{ytableau} 2 & 3 \\ 5 \end{ytableau}, \begin{ytableau} 1 \end{ytableau}, \begin{ytableau} 4 \\ 6 \end{ytableau}\right).$$

Definition 4.6. Let λ be an m-partition and consider an m-tableau $T = (T_1, \ldots, T_m)$ of shape λ. We call T *natural* if the numbers written in tableau T_i are contained in the set $\{\sum_{j=1}^{i-1} n_j + 1, \ldots, \sum_{j=1}^{i} n_j\}$. We denote the set of natural standard m-tableaux of shape λ by $\mathrm{NST}(\lambda)$ and $\mathrm{NST}(n_1, \ldots, n_m)$ is the set of all natural standard tableaux on all the partitions of type $(n_1, \ldots, n_m)$.

Example 4.7. Take $m = 3$ and $n = 5$, and consider the shape $\lambda = \left(\ydiagram{2,1}, -, \ydiagram{2}\right)$. To make a natural standard tableau with this shape we take the first three $\{1, 2, 3\}$ and assign them to the first tableau, we assign no numerals to the second tableau since it is empty, and the remaining numbers go into the last tableau. Then we have to sort the numerals in the tableau in order to make them standard. Thus

$$\left(\ytableaushort{13,2}, -, \ytableaushort{45}\right)$$

is an element of $\mathrm{NST}(\lambda)$.

We can define higher Specht polynomials via Young symmetrizers in a similar fashion to Definition 3.4, for a given m-tableau $T = (T^1, \ldots, T^m)$ we define $\varepsilon_T = \varepsilon_{T^1} \cdots \varepsilon_{T^m}$ and similarly $\sigma_T = \sigma_{T^1} \cdots \sigma_{T^m}$. The following theorem shows how $S/(R_+)$ decomposes into irreducible representations of a Young subgroup.

Theorem 4.8. *(Ariki et al., 1997, Theorem 1) Fix n and let $(n_1, \ldots, n_m)$ be a sequence such that $\sum_{i=1}^m n_i = n$. Then the collection:*

$$\bigcup_{\lambda \vdash (n_1, \ldots, n_m)} \{F_T^S \mid T \in NST(\lambda), S \in ST(\lambda)\}.$$

Form a k-basis for $S/(R_+)$. For $\lambda \vdash (n_1, \ldots, n_m)$, let $S \in ST(\lambda)$. Then the collection

$$\{F_T^S \mid T \in NST(\lambda)\}$$

forms a basis of the $S_{n_1} \times \cdots \times S_{n_m}$-submodule of $S/(R_+)$ which is isomorphic to irreducible representation V_λ.

Example 4.9. Let us consider the Young subgroup $S_1 \times S_2$ inside S_3. There are two 2-tuples of Young diagrams that partition (1,2) namely:

$$\lambda_1 = \left(\ydiagram{1}, \ydiagram{2}\right) \qquad \lambda_2 = \left(\ydiagram{1}, \ydiagram{1,1}\right)$$

$$\mathrm{NST}(\lambda_1) = \left\{\left(\ytableaushort{1}, \ytableaushort{23}\right)\right\} \qquad \mathrm{NST}(\lambda_2) = \left\{\left(\ytableaushort{1}, \ytableaushort{2,3}\right)\right\}$$

and

$$\mathrm{ST}(\lambda_1) = \left\{\left(\ytableaushort{1}, \ytableaushort{23}\right), \left(\ytableaushort{2}, \ytableaushort{13}\right), \left(\ytableaushort{3}, \ytableaushort{12}\right)\right\}$$

$$\mathrm{ST}(\lambda_2) = \left\{\left(\ytableaushort{1}, \ytableaushort{2,3}\right), \left(\ytableaushort{2}, \ytableaushort{1,3}\right), \left(\ytableaushort{3}, \ytableaushort{1,2}\right)\right\}.$$

Thus we get 3 copies of the irreducible representation corresponding to λ_1 and 3 copies of the irreducible representation corresponding to λ_2

Lemma 4.10. *Let $T_1, T_2 \in NST(n_1, \dots, n_m)$. If $T_1 < T_2$, it follows that $\varepsilon_{T_1}\varepsilon_{T_2} = 0$.*

Proof. Let $T_1, T_2 \in \mathrm{NST}(n_1, \dots, n_m)$, then we can commute terms of the Young symmetrizer such that $\varepsilon_{T_1}\varepsilon_{T_2} = \varepsilon_{T_1^1} \cdots \varepsilon_{T_1^m}\varepsilon_{T_2^1} \cdots \varepsilon_{T_2^m} = \varepsilon_{T_1^1}\varepsilon_{T_2^1} \cdots \varepsilon_{T_1^m}\varepsilon_{T_2^m}$ Let $T_1 < T_2$ and suppose that the last number that appears in different cells is contained in T_1^i. If T_1 and T_2 are of the same shape then $\varepsilon_{T_1^i}\varepsilon_{T_2^j} = \varepsilon_{T_2^j}\varepsilon_{T_1^i}$ for $i \neq j$. Now $T_1^i < T_2^i$, then the fact that $\varepsilon_{T_1}\varepsilon_{T_2} = 0$ follows from Lemma 3.12. If T_1^i and T_2^i are of different shapes, $\varepsilon_{T_1^i}\varepsilon_{T_2^i} = 0$ and thus $\varepsilon_{T_1}\varepsilon_{T_2} = 0$. □

Lemma 4.11. *Let T be a standard m-tuple of tableaux, then $z\varepsilon_T(f) = \sigma_{T'}(f)$ for any $f \in S$.*

Proof. Let $f \in S$ then $z\varepsilon_T(f) = z\varepsilon_{T^1} \dots \varepsilon_{T^m}(f) = \varepsilon_{T^1} \dots \varepsilon_{T^{m-1}} z\sigma_{(T^m)'}(f) = \sigma_{(T^1)'} \dots \sigma_{(T^m)'}(zf)$. □

Definition 4.12. Let λ be an m-tuple of Young diagrams of size $(n_1, \dots, n_m)$ and let $T \in \mathrm{NST}(\lambda)$. Let $M_T = \langle H_T^S \mid S \in \mathrm{ST}(\lambda)\rangle$ and $N_T = \langle F_T^S \mid S \in \mathrm{ST}(\lambda')\rangle$ be R-modules.

Remark 4.13. These are analogous modules to the ones defined in Definition 3.8 and are free R-submodules of S, but are not irreducible representations of S_n.

Theorem 4.14. *For the discriminant Δ of S_n, the matrix factorization defined by the reduced hyperplane arrangement, (z, z), can be decomposed in the following way:*

$$(z, z) = \bigoplus_{\lambda \vdash (n_1, \dots, n_m)} \bigoplus_{T \in NST(\lambda)} (z|_{M_T}, z|_{N_{T'}}) ,$$

and $(z|_{M_T}, z|_{N_{T'}})$ are the matrix factorizations:

$$M_T \xrightarrow{z|_{M_T}} N_{T'} \xrightarrow{z|_{N_{T'}}} M_T \ .$$

Proof. Recall that we can order all of the standard tableaux with n cells $\mathrm{NST}(n_1, \dots, n_m)$ such that if $i < j$ then $T_i > T_j$, thus $\varepsilon_{T_j}\varepsilon_{T_i} = \sigma_{T_i}\sigma_{T_j} = 0$. Let d to be the size of $\mathrm{NST}(n_1, \dots, n_m)$, we can write $S = \oplus_{1 \leqslant i \leqslant d} M_{T_i} = \oplus_{1 \leqslant i \leqslant d} N_{T_i}$. It is clear that since for a m-tableau T ε_T and σ_T are idempotent if $i < j$ then $\varepsilon_{T_i} F_{T_j} = 0$ and $\sigma_{T_j} H_{T_i} = 0$. Therefore consider $1 \leqslant k \leqslant d$, let P be a standard tableau of the same shape as T_k. Then we can split $zH_{T_k'}^P$ into the different components of S, where each $f_{T_i'} \in N_{T_i'}$. Calculate

$$zH_{T_k}^P = f_{T_1'} + f_{T_2'} + \dots + f_{T_k'} + \dots + f_{T_d'} \ . \tag{6}$$

Claim: For each $1 \leqslant j < k$, each component $f_{T_j'} = 0$.

We prove the claim by induction. Let $j = 1$, then since $1 < k$ then $T_k < T_1$ thus $\sigma_{T_1} H_{T_k}^P = 0$. Note that

$$\varepsilon_{T_1'} zH_{T_k}^P = \varepsilon_{T_1'}(f_{T_1'} + \dots + f_{T_d'}).$$

From an analogue of Lemma 4.11 where σ and ε are swapped, we have

$$\begin{aligned} z(\sigma_{T_1} H^P_{T_k}) &= \varepsilon_{T'_1} f_{T'_1} + \cdots + \varepsilon_{T'_1} f_{T'_d} \\ 0 &= f_{T'_1}. \end{aligned}$$

Assume that the claim is true for $j-1$. Since $j<k$ then $T_j > T_k$ thus $\sigma_{T_j} H^P_T = 0$. Therefore we have the following computation, using again Lemma 4.11

$$\begin{aligned} \varepsilon_{T'_j} z H^P_{T_k} &= \varepsilon_{T'_j}(f_{T'_1} + \cdots + f_{T'_j} + \cdots + f_{T'_d}) \\ z(\sigma_{T_j} H^P_{T_k}) &= \varepsilon_{T'_j} f_{T'_1} + \cdots + \varepsilon_{T'_j} f_{T'_j} + \cdots + \varepsilon_{T'_j} f_{T'_d} \\ 0 &= f_{T'_j}\ . \end{aligned}$$

Therefore equation (6) reduces to

$$zH^S_{T_k} = f_{T'_k} + f_{T'_{k+1}} + \cdots + f_{T'_d}\ . \tag{7}$$

After applying $\varepsilon_{T'_k}$ to (7), the left hand side becomes $\varepsilon_{T'_k}(zH^P_{T_k}) = z(\sigma_{T_k} H^P_{T_k}) = zH^P_{T_k}$. If $j>k$, then $\varepsilon_{T'_k} f_{T'_j} = 0$, thus $zH^K_{T_k} = f_{T'_k}$. In other words $z|_{M_T} : M_T \to N_{T'}$. A similar argument can be made about $zF^P_{T_i}$ thus proving the statement.

The argument above shows that for a $T \in \mathrm{NST}(n_1, \dots, n_m)$, $\mathrm{Im}(z|_{M_T}) = N_{T'}$, similarly one could show that $\mathrm{Im}(z|_{N_{T'}}) = M_T$. Therefore the matrix factorization splits as

$$(z, z) = \bigoplus_{T \in NST(n_1,\dots,n_m)} (z|_{H_T}, z|_{F_{T'}})\ . \quad \square$$

Example 4.15. Consider the Young subgroup $S_1 \times S_2$ inside S_3 and let $\sigma_1, \sigma_2, \sigma_3$ be the invariants of the S_3 action, then the matrix representing the multiplication by z is given by;

$$\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}$$

where A is the 3×3 matrix

$$A = \begin{bmatrix} -2\sigma_2 & -2\sigma_1\sigma_3 & -6\sigma_3 \\ 4\sigma_1 & \sigma_1\sigma_2 + 3\sigma_3 & 4\sigma_2 \\ -6 & -2\sigma_2 & -2\sigma_1 \end{bmatrix}$$

and B is the 3×3 matrix

$$B = \begin{bmatrix} -\frac{1}{2}\sigma_1^2\sigma_2 + 2\sigma_2^2 - \frac{3}{2}\sigma_1\sigma_3 & -\sigma_1^2\sigma_3 + 3\sigma_2\sigma_3 & -\frac{1}{2}\sigma_1\sigma_2\sigma_3 + \frac{9}{2}\sigma_3^2 \\ 2\sigma_1^2 - 6\sigma_2 & \sigma_1\sigma_2 - 9\sigma_3 & 2\sigma_2^2 - 6\sigma_1\sigma_3 \\ -\frac{1}{2}\sigma_1\sigma_2 + \frac{9}{2}\sigma_3 & -\sigma_2^2 + 3\sigma_1\sigma_3 & -\frac{1}{2}\sigma_1\sigma_2^2 + 2\sigma_1^2\sigma_3 - \frac{3}{2}\sigma_2\sigma_3 \end{bmatrix}.$$

A is the matrix of $(z|_{H_T}, z|_{F_{T'}})$ where

$$T = \left(\boxed{1}, \boxed{2\ |\ 3}\right) \qquad T' = \left(\boxed{1}, \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}\right)$$

of $\mathrm{NST}(1,2)$.

Remark 4.16. The matrix factorizations from Theorem 3.14 and 4.14 are equivalent as matrix factorizations, since they are matrices that describe the same map. We get from one to the other by a change of basis of S as an R-module.

4.1. Extended example: B_n

This section discusses the discriminants of the *complex* reflection groups $G(m,1,n)$, see the Shephard–Todd classification (Shephard and Todd, 1954), and how one would compute them using the methods of this paper. Here we present the example of the complex reflection group $B_n = G(2,1,n)$, giving the general definitions were natural. Similar to S_n the group B_n is a *true* reflection group, that is, its reflections are of order 2.

Let $\mathrm{Perm}(n) \simeq S_n$ be the group of permutation matrices of size n. Furthermore let $D_n(m) \simeq (\mu_m)^n$ be the group of diagonal $n \times n$ matrices whose entries are m-roots of unity.

Definition 4.17. The generalized permutation group $G(m,n) := G(m,1,n) = \mathbb{Z}_m \ltimes S_n$ can be described as a group of matrices in the following way:

$$G(m,n) = \{M \mid M = PD, \text{ where } P \in \mathrm{Perm}(n) \text{ and } D \in D_n(m) \text{ and } \det(D)^n = 1\}.$$

Thus if $M \in G(m,n)$ then M is a permutation matrix with roots of unity as entries. As in Section 2.3 with S_n, we obtain an action of $G(m,n)$ on a vector space V by choosing a basis $x_1, \dots, x_n$. This action extends to the polynomial ring $S = k[x_1, \dots, x_n] \cong \mathrm{Sym}_k(\mathrm{V})$. It is well known, cf. Morita and Yamada (1998), that the invariants of $G(m,n)$ on this action are given by $\{f(x_1^m, \dots, x_n^m) \mid f \in S^{S_n}\}$. Furthermore let $e_i^{(m)} = e_i(x_1^m, \dots, x_n^m)$ where e_i are the elementary symmetric functions. The ring of symmetric functions of $G(m,n)$ can be described as $S^{G(m,n)} = k[e_1^{(m)}, \dots, e_n^{(m)}]$.

Recall that $P(m,n)$ the set of m-partitions of n. The irreducible representations of $G(m,n)$ are in one-to-one correspondence with $P(m,n)$ (Ariki and Koike, 1994). For the case of $B_n = G(2,n)$, the representations correspond to 2-partitions with n cells.

In a similar way to Definition 2.10 we can define the polynomials

$$z = \prod_{H \in \mathcal{A}(B_n)} \alpha_H = x_1 \cdots x_n \prod_{1 \leqslant i < j \leqslant n} (x_i^2 - x_j^2)$$

and

$$\Delta = \prod_{H \in \mathcal{A}(B_n)} \alpha_H^2 = x_1^2 \cdots x_n^2 \prod_{1 \leqslant i < j \leqslant n} (x_i^2 - x_j^2)^2,$$

where $z, \Delta \in A$ and $\Delta \in S^{B_n}$.

Example 4.18. The 2-partition that corresponds to the representation whose isotypical component is generated by z is:

$$\lambda_{alt}^1 = (-, \lambda_{alt})$$

with one standard 2-tableau T:

$$\left(-, \begin{array}{c} \boxed{1} \\ \vdots \\ \boxed{n} \end{array} \right).$$

As before the higher Specht polynomials, with a modification, give a basis of the coinvariant algebra that respects the $G(m,n)$ action over S. Given a m-partition $\lambda \vdash n$ of type $(n_1, \dots, n_m)$ and a standard m-tableau T of shape λ we define a monomial

$$\mu_T = \prod_{i=1}^{m} \prod_{a \in T^i} x_a^i.$$

Definition 4.19. Let T and P be two standard m-tableaux of shape λ. The *higher Specht polynomials* for $G(m,n)$ are defined as

$$\hat{F}_T^P = \mu_T F_T^P(x_1^m, \dots, x_n^m) \quad \text{and} \quad \hat{H}_T^P = \mu_T H_T^P(x_1^m, \dots, x_n^m)$$

Where F_T^P and H_T^P are the Higher Specht polynomials for the product subgroups of S_n.

Example 4.20. Continuing Example 4.18: then $\mu_T = x_1 \dots x_n$ and the higher Specht polynomials are

$$\hat{F}_T^T = x_1 \dots x_n F_T^T(x_1^2, \dots, x_n^2) = x_1 \dots x_n \prod_{i<j}(x_i^2 - x_j^2)$$

and

$$\hat{H}_T^T = x_1 \dots x_n H_T^T(x_1^2, \dots, x_n^2) = x_1 \dots x_n \prod_{i<j}(x_i^2 - x_j^2).$$

As before, tensoring a representation $\lambda = (\lambda_0, \dots, \lambda_{m-1})$ with λ_{alt}^1, gives the target of the map given by z and is important in calculating the matrix factorizations for Δ. Recall that the shift operator $\mathrm{sh}^1(\lambda) = (\lambda_{m-1}, \lambda_0 \dots, \lambda_{m-2})$. Then we have

$$\lambda \otimes \lambda_{alt}^1 = \mathrm{sh}^1(\lambda')$$

Example 4.21. In the case of B_n, we have

$$\lambda_{alt}^1 \otimes \lambda_{alt}^1 = \lambda_{triv}^0$$

Let T be the standard young tableau of shape λ_{alt}^0. Then

$$\hat{F}_T^T = \hat{H}_T^T = F_T^T(x_1^2, x_2^2) = \prod_{i<j}(x_i^2 - x_j^2).$$

We calculate the map of $z = x_1 \dots x_n \prod_{i<j}(x_i^2 - x_j^2)$ by first multiplying by $x_1 \dots x_n$ and then $\prod_{i<j}(x_i^2 - x_1^2)$.

Consider the multiplication by $\prod_{i<j}(x_i^2 - x_j^2)$. The 2-partition that corresponds to the representation whose isotypical component is generated by $(x_i^2 - x_j^2)$ is:

$$\left(\begin{array}{c} \boxed{1} \\ \vdots \\ \boxed{n} \end{array}, - \right).$$

Let P, T be standard tableaux of shape λ, then $\lambda \otimes \lambda_{alt}^0 = \lambda'$.

The multiplication by $\prod_{i<j}(x_i^2 - x_j^2)$ is given by

$$\prod_{i<j}(x_i^2 - x_j^2)\hat{F}_P^T = \prod_{i<j}(x_i^2 - x_j^2)\mu_T F_P^T(x_1^2, \dots, x_n^2)$$

$$\mu_T \prod_{i<j}(x_i^2 - x_j^2) F_P^T(x_1^2, \dots, x_n^2) = \sum_{k=1}^{d} \mu_T f_{T_k'}(x_1^2, \dots, x_n^2) H_{P'}^{T_k'}(x_1^2, \dots, x_n^2)$$

where $f_{T_i'}$ are the invariants from the proof of Theorem 4.14. The 2-partition that corresponds to the representation whose isotypical component is generated by $x_1 \cdots x_n$ is

$$\left(-, \boxed{1} \cdots \boxed{n}\right).$$

The multiplication by $x_1 \cdots x_n$ is given by

$$\begin{aligned} x_1 \cdots x_n \hat{F}_P^T &= x_1 \cdots x_n \mu_P F_P^T(x_1^2, \ldots, x_n^2) \\ &= x_I^2 \mu_{sh_1(P)} F_P^T(x_1^2, \ldots, x_n^2) \end{aligned}$$

where I is a subset of $\{1, \ldots, n\}$ which are contained in the last position of λ and $x_I = \prod_{i \in I} x_i$.

Combining these calculations when all the cells are in the last position (when $x_{\mu_I} = (x_1 \cdots x_n)^2$ is an invariant) or none of the cells are in the last position (when $x_I = 1$) is straightforward. When this is not the case, the calculation becomes more complicated, see for the easiest case (May, 2023, Lemma 6.12).

CRediT authorship contribution statement

Eleonore Faber: Writing – review & editing, Writing – original draft, Supervision, Software, Methodology, Investigation. **Colin Ingalls:** Writing – review & editing, Writing – original draft, Supervision, Software, Methodology, Investigation. **Simon May:** Writing – review & editing, Writing – original draft, Software, Methodology, Investigation. **Marco Talarico:** Writing – review & editing, Writing – original draft, Software, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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