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From whole numbers to fractions to word problems: Hierarchical relations in mathematics knowledge for Chinese Grade 6 students

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ABSTRACT

It is well established in the literature that fraction knowledge is important for learning more advanced mathematics, but the hierarchical relations among whole number arithmetic, fraction knowledge, and mathematics word problem-solving are not well understood. In the current study, Chinese Grade 6 students (N = 1160; 465 girls; M_{age} = 12.1 years, SD = 0.6) completed whole number arithmetic (addition, subtraction, multiplication, and division), fraction (mapping, equivalence, comparison, and arithmetic), and mathematics word problem-solving assessments. They also completed two control measures: number writing speed and nonverbal intelligence. Structural equation modeling was used to investigate the hierarchical relations among these assessments. Among the four fraction tasks, the correlations were low to moderate, suggesting that each task may tap into a unique aspect of fraction understanding. In the model, whole number arithmetic was directly related to all four fraction tasks, but was only indirectly related to mathematics word problem-solving, through fraction arithmetic. Only fraction arithmetic, the most advanced fraction skill, directly predicted mathematics word problem-solving. These findings are consistent with the view that students need to build these associations into their mathematics hierarchy to advance their mathematical competence.

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Introduction

Several decades ago, Werner (1957) introduced the orthogenetic principle of development wherein development emerges through a continuous process of integration and differentiation. Since then, this theory has been applied to cognitive change, including the development of mathematical competence (Siegler & Chen, 2008; Xu & LeFevre, 2021; Xu et al., 2021). Consistent with the views of educators, mathematics learning is more effective when the interconnection of closely related concepts is facilitated (Ireland & Mouthaan, 2020; Li & Huang, 2013; Ma, 2010; Schmidt et al., 2002; Snider, 2004). Mathematical development involves the integration of various sets of number associations wherein integration reflects the construction of a higher-level understanding of number through the process of combining subsets of associations into a single mental representation (Clements et al., 2023; Hiebert, 1988; Siegler & Chen, 2008; Xu et al., 2019). Within this mental representation, the development of whole number skills, such as learning to represent and manipulate whole numbers to characterize cardinal (e.g., 4 is smaller than 5 and bigger than 3), ordinal (e.g., 4 comes after 3 and before 5), and arithmetic (e.g., 2 + 2, 5 − 1, 2 × 2, 8 ÷ 2) associations (Lyons et al., 2016; Merkley & Ansari, 2016; Xu et al., 2021), is fundamental. When solving mathematical problems, these associations need to be differentially activated, depending on the context (Ashcraft, 1982; Campbell, 1994; Siegler & Chen, 2008; Verguts & Fias, 2005; Xu et al., 2019).

The development of mathematical skills is a hierarchical process wherein complex mathematical skills are dependent on strong foundational skills (Hiebert, 1988; Merkley & Ansari, 2016; Siegler & Lortie-Forgues, 2014). Associations among numbers within a single mental representation have been explored in studies of arithmetic (Ashcraft, 1982; Campbell, 1994, 1995; Rickard, 2005; Siegler, 1988; Verguts & Fias, 2005). Expanding on the models in these studies, the *hierarchical symbol integration* (HSI) model posits that integration of mathematical skills occurs when newly acquired skills become interconnected with previously acquired skills, such that solvers can efficiently and flexibly access number knowledge to solve mathematical problems (Xu et al., 2019, 2023). Consistent with previous developmental studies, which have demonstrated a pattern if evolving relations among fundamental cardinal, ordinal, and arithmetic skills (Lyons et al., 2014; Sasanguie & Vos, 2018; Xu & LeFevre, 2021), the HSI model posits that less-skilled individuals are more likely to rely on procedural solutions that access basic number associations when solving mathematical problems. Thus, the use of such solutions may have stronger relations with more basic associations than with more advanced ones. For example, when determining the cost to purchase five shirts, each priced at £8, less-skilled individuals may rely on repeated addition to add 8 five times. In contrast, high-skilled individuals are likely to use multiplicative associations (8 × 5) to solve such problems, suggesting that strong performance, often indexed through speed and accuracy, may reflect the ability to access more advanced number associations. Students who have established a strong foundation of whole number knowledge will be equipped to integrate more advanced skills into their hierarchy of knowledge (Hiebert, 1988; Siegler & Lortie-Forgues, 2014).

One of the more advanced skills that students are introduced to relatively early in their formal schooling is fraction knowledge, which serves as the "gatekeeper" for learning more advanced mathematics (English & Halford, 1995; Mack, 1995; Smith et al., 2005). Fractions represent a relation between two magnitudes expressed in units of each other (Thompson & Saldanha, 2003). To advance their mathematical competence, students need to update their number representations and associations to incorporate rational numbers into their hierarchy (Barbieri et al., 2021; Booth & Newton, 2012; Booth et al., 2014; Siegler et al., 2011; Xu et al., 2023). For example, prior to fraction learning, students' mental representations are based on whole number rules (Ni & Zhou, 2005). Once fractions are introduced, students need to incorporate new associations that compete with their current representations, such as recognizing that 2 and 3 are smaller in magnitude than 4 and 9 but that $\frac{2}{3}$ is greater than $\frac{4}{9}$. In the current study, we investigated the relations in this hierarchy by considering performance on whole number arithmetic, fraction tasks, and problem solving for Chinese Grade 6 students.

The development of different types of fraction knowledge

There are many separate skills that students need to develop to gain a comprehensive understanding of fractions. Students must first understand that a fraction represents a relation between the part (s) of an equally partitioned unit and the total number of parts regardless of the size, shape, or arrangement of the parts (Charalambous & Pitta-Pantazi, 2007). Fraction mapping tasks, in which students convert pictorial representations of magnitude to fraction symbols, have been frequently used to measure students' understanding of these part–whole relations (Hecht et al., 2003; Hecht & Vagi, 2010; Jiang et al., 2020; Jordan et al., 2006; Mazzocco et al., 2013). Although this is arguably the most basic fraction skill, students can still have difficulty in moving from whole number to fraction notation. For instance, Di Lonardo Burr et al. (2022) found that one fifth of Chinese Grade 4 students consistently made errors when mapping fractions.

Contrary to whole numbers in which each number maps onto a single quantity, different fractions can represent the same magnitude (e.g., $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$, etc.; Pedersen & Bjerre, 2021). Ni (1999) showed that Grades 5 and 6 are a critical time for Chinese students to develop an understanding of fraction equivalence. In particular, students were able to map pictorial representations of non-reduced fractions (e.g., a circle with 2 of 4 parts shaded) to a symbolic representation (e.g., $\frac{1}{2}$) but continued to have difficulty in comparing equivalent fractions presented in symbolic form. Identifying and constructing equivalent fractions and understanding that two fractions can represent the same magnitude is a difficult concept for many students (Behr et al., 1984; Wong, 2010).

To further comprehend magnitude of fractions, students need to expand their knowledge from a discrete system (i.e., whole numbers) to a continuous system (i.e., rational numbers), processing fractions as a single magnitude as opposed to treating the numerator and denominator as two independent numbers (Stafylidou & Vosniadou, 2004). For many, acquiring this understanding is challenging (Chi et al., 1994; Meert et al., 2010; Ni & Zhou, 2005; Rinne et al., 2017). For example, when comparing the magnitudes of two fractions, most Chinese students in Grade 4 erroneously processed the whole number components of fractions discretely (Di Lonardo Burr et al., 2022). However, by Grade 6 most Chinese students had a full understanding of fraction magnitude, as evidenced by their high accuracy (> 80%) on a fraction comparison task (Bailey et al., 2015).

Beyond magnitude understanding, students need to learn how to perform arithmetic operations with fractions. Knowledge of fraction magnitude and fraction arithmetic co-develop gradually from Grade 4 to Grade 6 (Bailey et al., 2017). Unfortunately, an immature understanding of fraction magnitude can impede the learning of fraction arithmetic (Lortie-Forgues et al., 2015; Siegler et al., 2011). For example, when solving addition and subtraction problems with uncommon denominators, some students erroneously apply whole number arithmetic rules, adding or subtracting both numerators and denominators (Gabriel et al., 2013; Schumacher & Malone, 2017). In China, fraction arithmetic is one of the later-learned fraction skills, with fraction addition and subtraction first introduced in the second term of Grade 5 and fraction multiplication and division subsequently introduced in the first term of Grade 6 (Ministry of Education, 2011). Chinese students have shown mastery in fraction arithmetic (i.e., > 90% accuracy) in all four operations by the end of Grade 6 (Bailey et al., 2015; Torbeyns et al., 2004).

Overall, learning fractions is challenging, and new misconceptions can present themselves at various stages of fraction learning, including on foundational tasks such as mapping pictorial magnitudes to fraction symbols and to more advanced tasks involving comparisons and arithmetic operations (see a review in Gabriel et al., 2013). These tasks presumably tap into different aspects of fraction knowledge, whether conceptual, procedural, or both. Accordingly, the correlations among different types of fraction skills are typically low to moderate in strength (Bailey et al., 2015; Di Lonardo Burr et al., 2022; Hecht & Vagi, 2010; Siegler & Pyke, 2013), suggesting that students can excel in certain areas while still struggling with others.

How is whole number arithmetic related to fraction knowledge?

Fraction knowledge is an expansion of a network of number symbol associations built on from prior knowledge of whole numbers (Ni & Zhou, 2005; Siegler et al., 2011). For many students, the familiarity of whole number knowledge initially impedes the acquisition of fraction knowledge (Chi et al., 1994; Ni & Zhou, 2005). In particular, students erroneously apply their whole number understanding when solving tasks that involve mapping (Di Lonardo Burr et al., 2022), comparing (Meert et al., 2010), ordering (Malone & Fuchs, 2017), or performing arithmetic operations with fractions (Braithwaite & Siegler, 2023). To overcome this erroneous way of thinking about fractions, students may need to reorganize their existing knowledge of whole numbers to facilitate the acquisition of fraction understanding (Vosniadou, 1994).

Although whole number knowledge may initially impede the acquisition of fraction knowledge, aspects of whole number knowledge play a role in the reorganization of broader number system knowledge (Siegler et al., 2011; Steffe & Olive, 2010). In particular, multiplicative skills (whole number multiplication and division) are the foundation for which students develop fraction knowledge because both concepts require an understanding that quantities can be flexibly measured in units relative to each other (Harel & Confrey, 1994; Mack, 1993; Steffe, 1992; Steffe & Olive, 2010; Thompson & Saldanha, 2003). Indeed, in early stages of fraction learning, students' multiplicative skills have been linked to learning and performance on various fraction tasks, including fraction mapping (Xu et al., 2022), fraction equivalency (Ni, 2001), and composite measures of fraction knowledge that include magnitude understanding (Hansen et al., 2015; Namkung et al., 2018; Stelzer et al., 2019, 2021). For fraction arithmetic, in particular, the procedures directly involve the use of whole number arithmetic. Overall, findings from a large-scale meta-analysis found that whole number arithmetic consistently predicted various fraction skills, independent of students' age, intelligence, and working memory (Lin & Powell, 2021), supporting the view that whole number arithmetic skills are important for fraction learning.

Whole number arithmetic and fraction skills predicting mathematics word problem-solving

There is limited existing literature examining the relations among whole number arithmetic, fraction knowledge, and word problem-solving. Word problems vary in difficulty, require different types of mathematical knowledge (e.g., whole number arithmetic, fraction arithmetic), and require correct interpretation of the presented situation to determine the appropriate equation and solution. The existing literature is predominantly focused on word problem-solving within the context of fractions and algebra. For example, Hecht et al. (2003) found that among American students in Grade 5, a composite measure of conceptual understanding of fractions (including mapping pictorial magnitudes to fraction symbols and magnitude comparison) was a unique predictor of fraction word problem-solving after controlling for working memory and classroom behavior. In contrast, whole number arithmetic (i.e., addition and multiplication) was not a unique predictor of fraction word problem-solving. Thus, a conceptual understanding of fractions is important for solving fraction word problems because this understanding likely assists students in accurately translating problems into accurate mental representations and applying the appropriate computations (Byrnes & Wasik, 1991; Dark & Benbow, 1990; Hecht, 1998; Hecht et al., 2003; Hegarty et al., 1995). For older students in Grades 6 to 8, Barbieri et al. (2021) examined profile membership based on algebraic knowledge, as assessed through tasks including algebraic word problem-solving. They found that among a range of fraction skills, including magnitude comparison, part–whole knowledge, and number line estimation, fraction arithmetic emerged as the best profile predictor, suggesting that for experienced fraction learners proficiency in fraction arithmetic may be the best predictor of their word problem-solving performance. Based on these previous findings and the complex demands associated with word problem-solving, in the current study we included word problems as an outcome variable and placed it at the top of the hierarchy.

Although there is some evidence for a relation between fraction arithmetic and word problem-solving, to our knowledge few studies have considered the hierarchical relations among whole number arithmetic, fraction skills, and mathematics word problem-solving. In the current study, we operationalized mathematical word problems as real-life scenarios that require the strategic use of arithmetic operations on the numerical data provided within the problem statement (Verschaffel et al., 2000). The process of solving word problems relies on accurate translation of the numerical relations embedded within these scenarios into mathematical equations (Dark & Benbow, 1990). Contrary to pure arithmetic tasks, with word problems students not only need to perform a correct calculation but also must identify the correct procedure by interpreting the scenario they are presented with (Di Lonardo Burr et al., 2021). Students' representations of number knowledge, which include an understanding of whole and rational numbers, may support them in selecting an appropriate and efficient procedure.

The current research

The HSI model offers insights into differences among students' whole number and fraction skills. The model was originally tested with adults separated into two groups based on skill level (i.e., adults who had completed elementary school education in Canada were less skilled than those educated in China). For more-skilled adults (i.e., those who, on average, responded to problems with efficiency and accuracy), arithmetic fluency, but not basic number knowledge (number comparison and ordinal judgments), directly related to more advanced mathematics outcomes, including fraction knowledge and mathematics word problem-solving. Xu et al. interpreted this finding as evidence for full integration. In contrast, for the less-skilled adults (i.e., those who, on average, responded to problems slowly and/or inaccurately), both ordinal judgments and arithmetic fluency directly related to more advanced mathematics outcomes (Xu et al., 2019). Moreover, in a second sample of Canadian-educated adults who had varying levels of mathematical skills, both whole number arithmetic (specifically multiplication and division) and fraction arithmetic uniquely predicted algebraic knowledge (Xu et al., 2023). Notably, fraction arithmetic predicted substantially more unique variance in algebra than multiplicative skills, which Xu et al. interpreted as evidence that these individuals had not fully integrated their whole and rational number knowledge. Within the context of the HSI model, the findings from these studies suggest that, progressing along the hierarchy of skills, when full integration has occurred, allowing for efficient and flexible access to number knowledge (as evidenced by both accurate and quick responses), only the higher-order skill uniquely predicts the next level in the hierarchy.

In the current study, for the first time, these predictions were tested with a population wherein whole number arithmetic skills are expected to be integrated with fraction skills. Based on curriculum expectations and the findings of previous studies with Chinese students, by Grade 6, proficiency—defined as obtaining solutions to problems both quickly and accurately—is expected for whole number arithmetic as well as numerous fraction tasks, including mapping, fraction magnitude, and fraction addition and subtraction[1] (Bailey et al., 2015; Ministry of Education, 2011; Torbeyns et al., 2015; Xu et al., 2022).

We first explored the strength of the correlations among four fraction tasks (i.e., mapping, equivalence, comparison, and fraction arithmetic) to determine whether there was a strong overlap in knowledge of different fraction concepts. Next, we investigated the hierarchy of mathematical skills, considering whole number arithmetic, fraction skills, and word problem-solving. To our knowledge, this is the first study to simultaneously investigate multiple fraction skills and their relation to problem-solving within this hierarchy. We hypothesized that whole number arithmetic skills would predict performance on all four fraction tasks. Because Lin and Powell (2021) found that the predictive influence of whole number arithmetic on fraction skills decreased after controlling for domain-general cognitive skills, such as intelligence, we included the Raven's progressive matrices test as a control variable of intelligence. We further hypothesized that fraction arithmetic, the most advanced fraction skill, would be the key predictor of mathematical word problem-solving, with variability in fraction arithmetic superseding the influence of whole number arithmetic and other types of fraction skills (mapping, equivalence, and comparison). We considered fraction arithmetic to be the most advanced fraction skill because, for these Chinese Grade 6 students, it is the most recently acquired and therefore the least practiced.

[1] Multiplication and division with fractions are not introduced until Grade 6 and thus were not tested in the current study.

Method

Participants

The current study was approved by the institutional review board of Shandong Normal Univeristy. Participants included 1160 monolingual Chinese students in Grade 6 (695 boys, 465 girls; M_{age} = 12.1-years, *SD* = 0.6) recruited from 22 classrooms (*n* = 47–58 students per classroom) in two public elementary schools in the northern part of China. The median education level for both mothers and fathers was a high school diploma, representative of low–middle socioeconomic status in China.

Procedure

Group testing, administered by two experimenters who either had completed or were working toward a bachelor's degree in education, was carried out in each classroom during school hours. The experimenters and teachers followed the test protocols and explicit instructions constructed by the researchers.

Measures

The data presented here come from a large longitudinal project focusing on the effect of individual–environmental interactions in changes in mathematics anxiety (Li et al., 2021, 2023). The current study addressed a unique set of theoretical questions that have not been reported elsewhere. All measures described here were completed in the fall of Grade 6 with the exception of nonverbal intelligence, which was completed in the fall of Grade 3. The fraction assessments and anonymized data for the measures analyzed in the current study are available for download at the Open Science Framework (https://osf.io/m72ph/).

Nonverbal intelligence

Students completed a paper-and-pencil version of the Raven's progressive matrices test to assess their ability to recognize patterns and relationships among geometric figures (Raven, 1938). Specifically, students were asked to find the missing element in a pattern among six geometric figures in a multiple-choice format. In this task, students were given 40 min to complete five sets of problems (12 problems in each set) of increasing difficulty. Scoring was the total number of correct responses. Internal reliability (Cronbach's α) based on accuracy on individual trials obtained in the current sample was .86.

Number writing speed

Considering that the primary tasks of focus in our study were timed paper-and-pencil tasks, we included a number writing speed task as a control variable. A total of 60 single-digit Arabic digits were presented in three columns. Students were given 1 min to copy each of the numbers as quickly as possible. Scoring was the total number of digits students copied.

Whole number arithmetic

Students completed four subsets from the Chinese adapted version of the arithmetical ability subscale from the Heidelberg Rechen: addition, subtraction, multiplication, and division (Haffner et al., 2005; adapted from Wu & Li, 2006). For each operation, students were given 1 min to complete a maximum of 40 arithmetic problems, presented in two columns, in order. Problems increased in difficulty, ranging from single-digit to three-digit problems. For each operation, scoring was the total number of correct responses. The reported reliability (Cronbach's α) for the Chinese adapted version of the arithmetical ability subscale was .88 (Wu & Li, 2005). This reliability was established through a large national assessment in China involving 14,693 students from Grades 1 to 6.

Addition. In the first column, problems had single- and double-digit addends (e.g., 1 + 6 =__, 12 + 3 =__), with no sums greater than 20. In the second column, problems had single-, double-, and triple-digit addends (e.g., 6 + 16 =__, 16 + 27 =__, 234 + 567 =__).

Subtraction. In the first column, problems had single- and double-digit minuends and subtrahends (e.g., 5–3 =__, 10–5 =__), with no minuends greater than 20. In the second column, problems had double- and triple-digit minuends and single-, double-, and triple-digit subtrahends (e.g., 15–8 =__, 39–22 =__, 765–432 =__).

Multiplication. In the first column, problems had single-digit multiplicands and multipliers (e.g., 2 $\times$ 2 =__, 9 $\times$ 6 =__). In the second column, problems had single- and double-digit multiplicands and multipliers, all less than 20 (e.g., 16 $\times$ 12 =__, 11 $\times$ 14 =__).

Division. In the first column, problems had single- and double-digit dividends and single-digit divisors (e.g., 6 $\div$ 2 =__, 28 $\div$ 4 =__). These problems were complementary to the problems found on a 9 $\times$ 9 multiplication table. In the second column, problems had double- and triple-digit dividends and single-digit divisors (e.g., 49 $\div$ 7 =__, 119 $\div$ 7 =__).

Fraction skills

Fraction mapping. Students were given 1 min to complete a maximum of 20 fraction mapping problems, presented in two columns, in order. Each trial included a picture with shaded and unshaded areas. Students were asked to write down the fraction that described the shaded portion of the picture (e.g., a rectangle showing 1 of 3 portions shaded corresponds to the fraction $\frac{1}{3}$). The problems progressively became more difficult, starting with fractions like $\frac{1}{2}$ and $\frac{1}{3}$ and advancing to $\frac{7}{10}$ and $\frac{5}{8}$. Scoring was the total number of correct responses. Split-half (odd/even) reliability based on accuracy on individual trials obtained in the current sample was .84.

Fraction equivalence. Students were given 1 min to complete a maximum of 20 fraction equivalence problems, presented in two columns, in order. For each trial, students were presented with a fraction (e.g., $\frac{1}{3}$) and an accompanying picture of a shape segmented into equal-sized pieces, with some pieces shaded. In the first column, 10 problems consisted of pictures with non-reducible fractions (e.g., a circle with 4 of its 5 pieces shaded representing $\frac{4}{5}$). In the second column, 10 problems consisted of pictures with reducible fractions (e.g., a circle with 2 of its 6 pieces shaded representing $\frac{1}{3}$). For both types of problems, half of the trials had fractions that accurately matched their pictorial representations, and the other half had fractions that mismatched with their pictorial representations. If both the fraction and the shape represented the same magnitude, students were asked to put a "✓"; otherwise, they were asked to put an "$\times$". Scoring was the total number of correct responses. Split-half (odd/even) reliability based on accuracy on individual trials obtained in the current sample was .78.

Fraction comparison. Students were given 1 min to complete a maximum of 20 fraction comparison problems, presented in two columns, in order. Each trial consisted of two fractions. Students were asked to circle the fraction with the larger magnitude. There were 11 congruent and 9 incongruent trials. For congruent trials, the relative magnitude of components (numerator and/or denominator) corresponded to the relative magnitude of the whole fractions (e.g., $\frac{7}{9} < \frac{8}{9}$); for incongruent trials, the relative magnitude of components did not correspond to the relative magnitude of the whole fractions (e.g., $\frac{2}{3} > \frac{4}{7}$). Scoring was the total number of correct responses. Split-half (odd/even) reliability based on accuracy on individual trials obtained in the current sample was .87.

Fraction arithmetic. Students completed two fraction arithmetic tasks: addition and subtraction. For each operation, students were given 2 min to complete a maximum of 20 problems, presented in two columns, in order. For both operations, the first 10 problems had either common denominators (Problems 1–5; e.g., $\frac{1}{4} + \frac{2}{4} =$ __, $\frac{2}{9} - \frac{1}{9} =$ __) or common numerators (Problems 6–10; e.g., $\frac{1}{5} + \frac{1}{6} =$ __, $\frac{3}{4} - \frac{3}{5} =$ __), followed by 10 problems that had neither common denominators nor common numerators (Problems 11–20; e.g., $\frac{1}{2} + \frac{3}{4} =$ __, $\frac{5}{9} - \frac{1}{3} =$ __). Students were instructed to use the margin of the testing sheet for rough work. Scoring was the total number of correct responses for each operation. Split-half (odd/even) reliabilities based on accuracy on individual trials obtained in the current sample were .95 and .91 for addition and subtraction, respectively.

Mathematics word problem-solving

Students completed the revised version of the arithmetical subscale of the Wechsler Intelligence Scale for Chinese Children (WISC; Zhang, 2009). Students solved mathematics problems of increasing difficulty (e.g., "What is the cost to purchase three pens, each priced at 80 pennies?"; "You spent 18 pounds on a desk that was on sale for $\frac{1}{3}$ of the original price. What was the original price of the desk?") In accordance with the Chinese WISC manual, students were presented with a maximum of 10 problems from the testing booklets (Problems 10–19). All students started with Problem 10. If students answered Problems 10 and 11 correctly, they were given 9 points for the previous problems and proceeded to Problem 12. If students failed to answer either question correctly, they were given 0 points for the previous problems. If students made errors on three consecutive problems, they were still allowed to continue because the testing was done in a group setting; however, any additional problems answered were not scored.

For Problems 10 to 14, students earned 1 point for providing an accurate response. For Problems 15 to 19, which were more challenging, bonus points were awarded for solving the problems correctly within specific time limits. The number of bonus points for each problem varied based on its difficulty. For example, for Problem 19, 3 bonus points were awarded if solved within 40 s, 2 bonus points if solved within 60 s, and 1 bonus point if solved within 90 s. For these problems, the experimenter announced when each of the time limit benchmarks (e.g., 40 s, 60 s, 90 s) had been reached; students were instructed to place a check mark on the answer sheet in the appropriate column (e.g., 0–40 s, 41–60 s, 61–90 s) if they completed the problem within the corresponding time limit. Scoring was the total number of correct responses plus any bonus points earned (max = 30). The internal reliability obtained from the WISC technical report was excellent, with Cronbach's α = .86 (Zhang, 2009).

Data analysis

To account for potential classroom effects, we conducted multilevel structural equation modeling with class entered as a random effect using Mplus 7.0 (Muthén & Muthén, 1998). First, we anticipated high correlations among the whole number arithmetic operations (addition, subtraction, multiplication, and division) and fraction arithmetic operations (fraction addition and subtraction); thus, we specified two latent factors in the model: *whole number arithmetic* and *fraction arithmetic*. Next, we constructed a model with direct paths from whole number arithmetic to fraction skills and mathematics word problem-solving, and from fraction skills to mathematics word problem-solving. Within the model, we controlled for nonverbal intelligence, number writing speed, and gender. Given the large sample size, the comparative fit index (CFI > .90), root mean square error of approximation (RMSEA < .06), and standardized root mean square residual (SRMR < .08) were used to examine model fit (Hu & Bentler, 1999; Kline, 2011).

The percentage of missing cases was extremely small (i.e., 0.1% on each of whole number multiplication, whole number division, and fraction tasks; 1.6% on mathematics word problem-solving); missing data were unlikely to influence the interpretation of the results (Enders, 2010). Notably, the Raven's matrices test was administered in Grade 3; as the study progressed, students continued to join the project in each grade, and thus 23.7% of the Grade 6 sample did not complete the Raven's matrices test. In addition, extreme outliers (all negative; defined as values with $|z$ scores$| > 3.29$; Field, 2013) were found for the following tasks: nonverbal intelligence ($n = 10$), number writing speed ($n = 5$), addition ($n = 5$), subtraction ($n = 2$), multiplication ($n = 7$), division ($n = 10$), fraction mapping ($n = 10$), fraction equivalence ($n = 1$), fraction comparison ($n = 2$), fraction addition ($n = 23$), fraction subtraction ($n = 12$), and mathematics word problem-solving ($n = 9$). Sensitivity analyses revealed the same pattern of results with and without outliers and when either the whole sample was included or only students who had completed the Raven's matrices test were included. Thus, full information maximum likelihood was used to estimate the final model, using all available information (Enders, 2010).

Results

Descriptive statistics

Descriptive statistics and correlations are shown in Tables 1 and 2, respectively. Except for nonverbal intelligence and number writing speed (p = .609), all correlations were significant (ps < .001). Word problem-solving was correlated with whole number arithmetic and fraction skills, with stronger correlations with fraction arithmetic skills. There were gender differences on two measures. Compared with girls, boys had lower scores on fraction mapping (17.0 vs. 17.4), t(1123.76) = - −2.69, p = .007, d = .15, but had higher scores on fraction comparison (16.3 vs. 16.0), t(1020.45) = 2.36, p = .018, d = .14. Therefore, gender differences on fraction mapping and fraction comparison were considered in subsequent analyses.

Whole number arithmetic and fraction performance

The distributions of scores for each operation for whole number arithmetic and for the four fraction tasks are shown in Figs. 1 and 2, respectively. For whole number arithmetic, performance varied with operation, F(2.94, 3393.48) = 734.78, MSE = 6.88, p < .001, η_p^2 = .39. Specifically, students had the best performance on division, followed by multiplication, addition, and subtraction (using Bonferroni adjustment, all pairwise comparisons were significant, ps < .001). Correlations among the four operations were high, ranging from .65 to .70. Notably, the distribution of multiplication was narrower than that of the other operations, suggesting that these Chinese students have memorized the multiplication table. Similarly, we speculate that the superior performance on division problems reflects the inverse relationship between multiplication and division, with students retrieving and inverting multiplication facts to efficiently solve division problems.

For the fraction tasks, overall, students had high performance (mean accuracy > 75%) on all four fraction tasks (see Table 1). The distributions of the scores and the strength of the correlations among the fraction tasks varied. Consistent with previous research (Xu et al., 2022), the correlations among the fraction tasks were low to moderate, suggesting that different fraction tasks require different types of fraction knowledge. With respect to fraction arithmetic, there was a small, albeit significant, difference across the two operations, with slightly higher performance on subtraction problems (M = 16.3) than on addition problems (M = 15.7), t(1158) = −6.44, p < .001, Cohen's d = .19.

Multilevel structural equation modeling

We hypothesized that whole number arithmetic would predict all four types of fraction skills. Furthermore, among all predictors, we expected that only fraction arithmetic would uniquely predict mathematics word problem-solving. A multilevel structural equation model containing classroom as a random effect was fit to the data. First, an intercept-only model that contained a classroom variable was fit. We found that the intraclass correlation coefficients for the endogenous variables of the model ranged from .05 to .34, indicating variabilities on these variables among the classrooms.

Next, we added a confirmatory factor analysis to the intercept-only model to test factor loadings for whole number arithmetic (i.e., addition, subtraction, multiplication, and division) and fraction arithmetic (i.e., fraction addition and fraction subtraction). The model, shown in Fig. 3, had good fit to the data, $\chi^2(8)$ = 33.94, p < .001, SRMR = .02, CFI = .99, RMSEA = .05. Students had excellent performance on all four operations; the correlations among the operations were strong, and the factor loadings were excellent (ps < .001).

Lastly, we tested the multilevel structural equation model. The final model fit was acceptable, $\chi^2(43)$ = 122.50, p < .001, SRMR = .02, CFI = .98, RMSEA = .04. For readability, the control variables (i.e., nonverbal intelligence, number writing speed, and gender) are shown in the figure legend. In

Table 1
Descriptive statistics.

Variable	*N*	*M*	*SD*	Skew	Min	Max
Control measures						
Nonverbal intelligence	885	38.0	7.5	−0.9	8	58
Number writing speed	1160	53.0	10.1	−1.4	5	60
Whole number arithmetic						
Addition	1160	31.5	5.0	−0.4	8	40
Subtraction	1160	30.1	5.0	−0.4	13	40
Multiplication	1159	33.5	3.0	−0.8	11	40
Division	1158	34.8	4.8	−1.4	9	40
Fraction skills						
Mapping	1159	17.2	2.7	−1.7	0	20
Equivalence	1159	16.1	2.8	−0.4	5	20
Comparison	1159	16.2	2.8	−0.5	6	20
Addition	1159	15.7	4.4	−1.6	0	20
Subtraction	1159	16.3	3.9	−1.6	0	20
Mathematics outcome						
Word problem-solving	1141	22.4	3.4	−1.1	9	29

Table 2
Correlations among measures.

Variable	Control		Whole number arithmetic				Fraction skills				
	1	2	3	4	5	6	7	8	9	10	11
Nonverbal intelligence	–										
Number writing speed	.02	–									
Whole number addition	.17	.34	–								
Whole number subtraction	.17	.28	.65	–							
Whole number multiplication	.18	.29	.67	.68	–						
Whole number division	.21	.19	.68	.70	.68	–					
Fraction mapping	.17	.37	.46	.45	.44	.39	–				
Fraction equivalence	.19	.27	.50	.54	.44	.46	.45	–			
Fraction comparison	.24	.19	.41	.46	.41	.46	.33	.48	–		
Fraction addition	.21	.15	.45	.50	.46	.50	.35	.37	.42	–	
Fraction subtraction	.18	.15	.47	.51	.48	.54	.38	.36	.42	.71	–
Word problem-solving	.12	.16	.30	.33	.29	.32	.23	.27	.25	.34	.35

Note. All values are significant at $p < .001$ except the correlation between nonverbal intelligence and number writing speed.

support of our first hypothesis, whole number arithmetic predicted all four fraction skills (see Fig. 3). Moreover, only fraction arithmetic uniquely predicted mathematics word problem-solving. However, the indirect effect from whole number arithmetic to mathematics word problem-solving through fraction arithmetic ($\beta = .21$) was significant ($p < .001$). Thus, in support of our second hypothesis, fraction arithmetic superseded the influence of whole number arithmetic and other fraction skills in predicting students' performance in mathematical word problem-solving for students in Grade 6. No other indirect effects from whole number arithmetic to mathematics word problem-solving were significant ($ps < .05$).

Discussion

Mathematical skills are ever evolving; the development of these skills is hierarchical, such that strong foundational skills are necessary for students to advance toward an understanding of more complex skills (Hiebert, 1988; Merkley & Ansari, 2016; Siegler & Lortie-Forgues, 2014). Among these skills, fractions receive a lot of attention in the mathematics learning literature; the required abstract

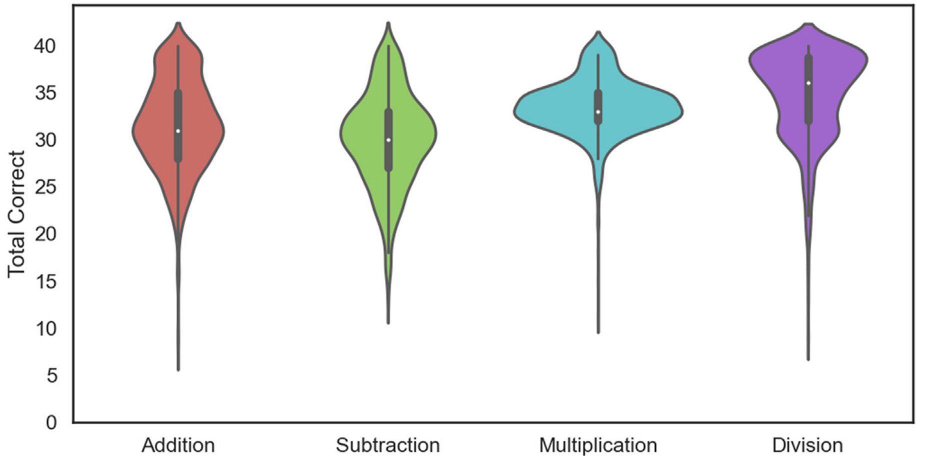

Fig. 1. Violin plots of whole number arithmetic tasks. Violin plots depict distributions of the data using density curves, with the width of each curve corresponding to the approximate frequency of data points within that region. The white dot is the median, the black bar in the center of the plot shows the interquartile range, and the thin black bar shows the range of scores.

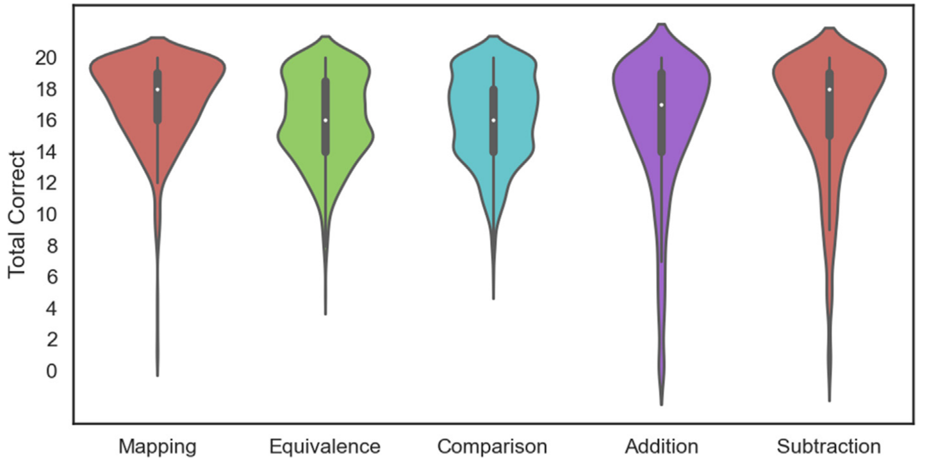

Fig. 2. Violin plots of fraction tasks. Violin plots depict distributions of the data using density curves, with the width of each curve corresponding to the approximate frequency of data points within that region. The white dot is the median, the black bar in the center of the plot shows the interquartile range, and the thin black bar shows the range of scores.

understanding that quantities can be measured flexibly in units of each other (Mack, 1993; Thompson & Saldanha, 2003) and the necessary inhibition of knowledge of the whole number system (Braithwaite & Siegler, 2018; Ni & Zhou, 2005; Siegler & Lortie-Forgues, 2017) is challenging for many students. The mathematical hierarchy, including fraction knowledge, has been investigated within the framework of the HSI model. With experienced learners, Xu et al. (2019) found that the strength of these interconnections varied by skill level. However, prior studies have included only single measures of fraction skills. In the current study, we expanded on these findings by including different types of fraction skills.

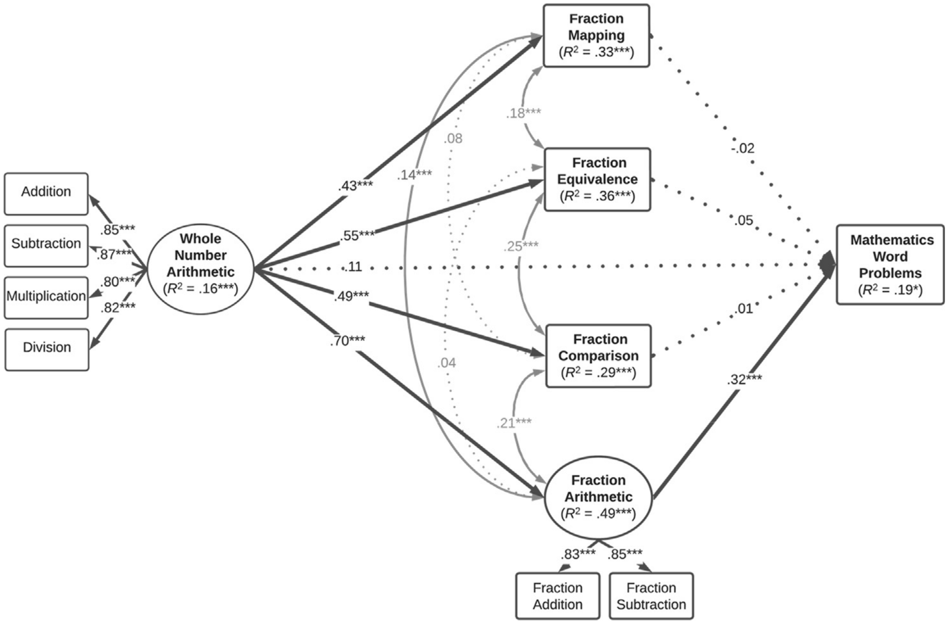

Fig. 3. Multilevel structural equation model (N = 1160). Numbers on the arrows are standardized coefficients. $^{*}p < .05$; $^{**}p < .01$; $^{***}p < .001$. Dashed lines represent nonsignificant paths. Gray lines represent covariance. Classroom was included as a random effect. Nonverbal intelligence and number writing speed were controlled for whole number arithmetic (β = .22 and β = .33, ps < .001, respectively), fraction mapping (β = .07, p = .023 and β = .22, p < .001, respectively), fraction equivalence (β = .05, p = .056 and β = .09, p = .002, respectively), fraction comparison (β = .13, p < .001 and β = .02, p = .558, respectively), fraction arithmetic (β = .08, p = .008 and $\beta = -.05$, p = .170, respectively), and mathematics word problem-solving (β = .02, p = .422 and β = .07, p = .130, respectively). Gender was controlled for fraction mapping (β = .07, p = .023) and fraction comparison ($\beta = -.05$, p = .019).

How strongly are different fraction skills related to one another?

It is not uncommon in the literature for fraction skills to be assessed using a composite measure (e.g., Hansen et al., 2015; Jordan et al., 2013; Namkung et al., 2018; Siegler et al., 2012; Stelzer et al., 2019, 2021). Among the few studies of American students (Grades 4–8) that have examined different types of fraction skills separately, mapping, comparison, and arithmetic skills had low to moderate correlations (Bailey et al., 2015; Hecht & Vagi, 2010; Siegler & Pyke, 2013). In the current study, we also found low to moderate correlations among the different types of fraction skills for Chinese students in Grade 6, a group of students who demonstrated excellent performance across all four fraction tasks. Notably, Di Lonardo Burr et al. (2022) also found a weak correlation between fraction mapping and comparison for Chinese students in Grade 4 who had just started to learn fractions, and errors were not consistent between tasks, suggesting that mastery of one task does not naturally extend to mastery of another task. Thus, although many skills fall under the umbrella of fraction knowledge, different types of knowledge are likely required for different tasks.

The existing research on equivalent fractions often takes a qualitative approach to show that fraction equivalence extends the part–whole understanding of fraction mapping by requiring some knowledge of the magnitude of fractions (Ni, 2001; Wong, 2010). In the current study, we included fraction equivalence as one of our fraction tasks. This task included both pictorial and symbolic representations of fractions, requiring both mapping and magnitude knowledge. Therefore, the finding that equivalence was more strongly correlated with both fraction mapping and fraction comparison tasks than with fraction arithmetic was expected. However, again these correlations were low to moderate in strength, highlighting the importance of mastering a variety of different types of fraction skills. In particular, students should master foundational, conceptual fraction skills before moving to more advanced concepts of fractions such as fraction arithmetic (Li & Huang, 2013; Sun, 2019).

Building a hierarchy of mathematical knowledge

Expanding the HSI model, we investigated the relations among whole number arithmetic and different types of fraction skills and how these skills related to word problem-solving. Whole number arithmetic skills are foundational in the mathematical hierarchy. These skills are important for later-learned mathematics, including fraction understanding. For example, in their meta-analysis, Lin and Powell (2021) found very large effect sizes between procedural whole number arithmetic skills and both conceptual and procedural fraction knowledge. Consistent with these findings, in the current study we found that whole number arithmetic skills were directly related to all four types of fraction skills for Chinese students in Grade 6, demonstrating *very large effect sizes* (i.e., > .40, according to the evaluation criteria in Funder & Ozer, 2019). Possibly, the consistent relations found between whole number arithmetic and fraction skills reflect a shared underlying understanding of magnitude representation (Fazio et al., 2014; Siegler, 2016; Siegler et al., 2011). This expanded understanding of magnitude (i.e., 2 and 3 are smaller in magnitude than 4 and 9, but $\frac{2}{3}$ is greater than $\frac{4}{9}$) reflect a stronger representation of integrated whole and rational number knowledge.

Whole number arithmetic is used to solve certain types of fraction tasks, such as comparison (e.g., converting to decimals through division; cross-multiplying and then comparing) and fraction arithmetic (e.g., finding common denominators; adding or subtracting numerators), as well as to solve mathematical word problems. When students have not fully integrated their whole and rational number knowledge, foundational skills, such as whole number arithmetic, continue to directly predict more advanced mathematics performance, including on fraction, algebra, and word problem-solving tasks (Xu et al., 2019, 2023). However, as the hierarchy of mathematics knowledge develops, for students who have integrated whole and rational number knowledge, these more basic whole number arithmetic skills no longer uniquely predict mathematics outcomes when more advanced skills, such as fraction skills, are considered (Xu et al., 2023). Consistent with the proficient whole number arithmetic and fraction skills noted in previous studies with Chinese students (Bailey et al., 2015; Torbeyns et al., 2015; Xu et al., 2022) and curriculum expectations (Ministry of Education, 2011), in the current study Chinese Grade 6 students had proficient performance on whole number arithmetic and fraction tasks. Notably, whole number arithmetic was not directly related to word problem-solving. Thus, in accord with the HSI model, we speculate that whole number arithmetic indirectly predicting word problem-solving through fraction skills reflects the integration of whole and rational number knowledge. Students' strong performance in both whole number and fraction tasks suggests that they can efficiently and effectively select appropriate strategies, depending on the context, to solve mathematical problems.

To solve mathematical word problems, students need to understand the problem situation, accurately translate the situation into a mathematical equation, and subsequently apply the correct computations. This process can be particularly difficult for word problems involving fractions, with Mostert and Hickendorff (2023) finding that, with the exception of division, Dutch students in Grades 6 to 8 had better fraction arithmetic performance when equations were presented symbolically compared with when they needed to be extracted from a scenario in a word problem. Because of the added complexity of correctly interpreting the situation, word problems can be thought of as a more advanced mathematical competency.

In the current study, we placed word problems at the top of our hierarchy. Our task consisted of problems of varying difficulty levels, with some requiring whole number arithmetic knowledge (e.g., determining the cost to purchase three pencils, each priced at 80 pennies) and others requiring fraction knowledge (e.g., determining the total time traveled if $\frac{1}{5}$ of the journey was traveled at one speed and the remainder of the journey was traveled at another speed). In a study with Chinese Grade 6 students, Jiang et al. (2014) found that to solve difficult problems similar to our example involving speed, most students transformed the problem into a mathematical statement involving one or more arithmetic operations (Jiang et al., 2014). More generally, Jiang et al. (2014) suggested that students rely on abstract strategies to solve these types of problems. Consistent with this view, we showed that variability in the more advanced mathematical skill, fraction arithmetic (Barbieri et al., 2021), superseded variability in the less advanced whole number arithmetic and magnitude-based fraction skills (i.e., mapping, equivalence, and comparison), becoming the only unique predictor of mathematics word problem-solving. This pattern is consistent with findings from a prior study with Chinese-educated learners where later-acquired advanced mathematical skills superseded variability in basic skills when predicting mathematical word problem-solving (Xu et al., 2019).

In our study, fraction arithmetic was the most recently introduced fraction concept. As such, this later-learned mathematics skill is also the most advanced and challenging for these students. Thus, a potential interpretation of our findings could be that difficult mathematics tasks predict more difficult mathematics tasks. However, performance on the fraction arithmetic task and the other fraction tasks was similarly high, with median accuracy scores at 80% or above for all fraction tasks (see Fig. 2). Thus, we speculate that, for this group of students with highly proficient fundamental and fraction skills, our results pattern may be better explained by the integration between whole and rational number knowledge. Consistent with this view, not all prior HSI model studies found that later-acquired advanced mathematical skills superseded variability in basic skills when predicting mathematical outcomes, particularly when students are in the process of learning new skills and forming new associations. For example, in a study with Canadian Grade 2 and 3 students (mean ages 8–9 years), despite learning additive associations (addition and subtraction) prior to multiplicative associations (multiplication and division), both subtraction and multiplication predicted performance on applied mathematics (Xu et al., 2021). Xu et al. (2021) suggested that because these students were still in the process of learning multiplication, they had not yet integrated this newly acquired skill into their mental representation, and thus more foundational skills (i.e., subtraction) continued to uniquely predict more advanced mathematics. Returning to our example from earlier (i.e., purchasing five £8 shirts), even though these students knew the procedure for multiplication, without acquisition of fluent multiplicative skills, they might still use less efficient strategies, such as repeating addition, as opposed to the most efficient strategy (i.e., multiplication). We posit that with practice and experience, various skills at both the foundational and more advanced levels of the hierarchy can become integrated into students' mental representations, enabling students to fluently and selectively access the specific skills needed to solve mathematical problems.

Limitations and future research

The study focused on the concurrent relations among whole number arithmetic, fraction skills, and mathematics word problem-solving for students in Grade 6. Although concurrent studies can provide valuable insights into the relations among different levels of mathematics knowledge, longitudinal research is needed to capture the development of fraction knowledge. Moreover, with grouped data, we can only speculate on average patterns. Thus, future studies that follow students from Grade 3 to Grade 6 and investigate individual differences through a person-centered approach would provide additional insights into the integration of fraction skills into students' hierarchy of mathematics knowledge. Such longitudinal studies, particularly those that include strategy analyses and focus on individual differences, would also provide stronger evidence for (or against) the HSI model given that they would capture the development of mathematical skills and the formation of new associations.

In the current study, we included only the reasoning task as a control variable without considering other cognitive skills known to be related to mathematics problem-solving. Specifically, working memory is crucial for extracting key information from word problems wherein students need to manage and update intermediate results during arithmetic operations and inhibit irrelevant information throughout the problem-solving process (Allen et al., 2019; Friso-van den Bos et al., 2013; Peng et al., 2016; Raghubar et al., 2010). Furthermore, a meta-analysis by Peng et al. (2016) revealed that working memory moderated the relationship between mathematics word problem-solving and fraction tasks, highlighting the shared variance between these two types of tasks. Future research should include a wider range of domain-general skills to investigate the links between fraction skills and mathematics word problem-solving.

Similarly, future research should consider the timing of measuring cognitive skills. In the current study, reasoning skills were measured in Grade 3, but mathematics performance was measured in Grade 6. Although cognitive skills tend to be relatively stable, it is possible that this difference in assessment timing is what led to the weaker correlations seen between reasoning and mathematics performance than have been noted in the literature (see meta-analysis by Peng et al., 2019).

When selecting the tasks for this study, we chose word problems for the top of the hierarchy. Notably, the standardized word problem assessment included some items that required fraction arithmetic. Thus, it is possible that the unique relation between fraction arithmetic and word problems is partially driven by these items. Without strategy reports, we cannot investigate whether students used fraction arithmetic or whether they used alternative whole number strategies to solve the problems. For example, in the problem "You spent 18 pounds on a desk that was on sale for $\frac{1}{3}$ of the original price. What was the original price of the desk?", students could use fraction arithmetic (e.g., $18 \div \frac{1}{3}$) or they could solve it using whole number arithmetic (e.g., 18×3). Future studies should aim to replicate the current findings with word problems in which solutions do not require fraction arithmetic procedures to test the generalizability of our findings and gather strategy reports to investigate the efficiency of solution strategies.

Our sample consisted of students with superior mathematics performance, and thus we chose to use timed assessments to obtain more variability in performance. Notably, performance was high across many of the tasks, and many of the students rarely made errors; thus, we did not systematically examine individual errors on the tasks. In China, a critical component of mathematics instruction is ensuring that students understand related concepts, particularly for struggling students (Ministry of Education, 2011). In the future, examining this hierarchy in more diverse populations, including students with mathematics difficulties, will help to generalize the findings from our model. Moreover, including open-ended questions and strategy reports will provide further insights into the developmental progression of students' fraction learning.

Conclusion

We found support for the hierarchical relations among whole number arithmetic, different types of fraction skills, and mathematics word problem-solving. Whole number knowledge is the foundation on which students build their fraction knowledge. However, as students develop more advanced skills, such as fraction skills, whole number arithmetic no longer uniquely predicted mathematics word problem-solving. Instead, the most advanced fraction skill, fraction arithmetic, was the key predictor of mathematics word problem-solving, capturing the integration of whole and rational number knowledge for these students. Overall, Chinese Grade 6 students demonstrated an excellent understanding of different types of fraction knowledge, but performance across these tasks had low to moderate correlations, suggesting that they likely tap into different skills. Practice with various types of fraction assessments is needed for students to integrate fraction knowledge into their mental representation.

CRediT authorship contribution statement

Chang Xu: Conceptualization, Data curation, Formal analysis, Methodology, Visualization, Writing – original draft, Writing – review & editing. **Sabrina Di Lonardo Burr:** Conceptualization, Formal analysis, Methodology, Visualization, Writing – original draft, Writing – review & editing. **Hongxia Li:** Conceptualization, Data curation, Investigation, Methodology, Project administration, Writing – review & editing. **Chang Liu:** Writing – review & editing. **Jiwei Si:** Conceptualization, Funding acquisition, Methodology, Resources, Supervision, Writing – review & editing.

Data availability

The data that support the findings of this study are openly available on Open Science Framework.

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