Article:

Valmorbida, G. orcid.org/0000-0002-1710-8590, Drummond, R. orcid.org/0000-0002-2586-1718 and Duncan, S.R. orcid.org/0000-0002-9525-7305 (2019) Regional analysis of slope-restricted Lurie systems. IEEE Transactions on Automatic Control, 64 (3). pp. 1201-1208. ISSN 0018-9286

https://doi.org/10.1109/tac.2018.2849618

Regional Analysis of Slope Restricted Lurie Systems

G. Valmorbida, R. Drummond and S. R. Duncan

***Abstract*—This paper considers the stability analysis of nonlinear Lurie type systems where the nonlinearity is both (locally) sector and slope restricted. Convex conditions for verifying stability, computing outer estimates of reachable sets and upper bounds on the induced $\mathcal{L}_2$ gain in a local or global domain are proposed. The conditions use a Lyapunov function that is quadratic on both the states and the nonlinearity and has an integral term on the nonlinearity. Numerical examples outline the benefits of the proposed approach.**

I. Introduction

The stability analysis of feedback loops consisting of linear time-invariant systems and sector bounded nonlinearities can be studied via the passivity properties of the elements in the interconnection. This is known as the absolute stability problem and can be analysed using the celebrated Circle and Popov criteria [17], where the key assumption is that the nonlinearity exists in a sector. The assumption that the nonlinearity is sector bounded might be overly conservative whenever the nonlinearities are known or their slopes can be bounded. The study of the class of slope-restricted nonlinear systems using the framework of absolute stability theory was first proposed in two papers; a frequency domain condition given in [6] and a geometrical condition based upon the construction of a Lyapunov function (LF) in [29]. It is noted that several positivity conditions on the LF were relaxed in [29].

In addition to the Lyapunov functions associated with the Circle and Popov criteria, different LF's have been proposed for studying Lurie systems: composite LFs [14]; LFs with quadratic components on both the nonlinearities and the states and Lurie-Postnikov terms were studied in [29, 24, 20, 25]. For quadratic LFs associated with the Circle criterion, the positivity of the LF is enforced with a positive-definite Lyapunov matrix [17]. In the case of LF's with a Lurie-Postnikov type term, which are associated with the Popov criterion, the positivity of the LF requires the positivity of the Lyapunov matrix, but does not necessarily impose the positivity of the Lurie-Postnikov integral terms' coefficients [12, 19]. An IQC formulation of this result that also does not require the positivity of the coefficients is presented in [15].

For the case of nonlinearities that are sector and slope bounded in a set containing the origin, we are interested in obtaining *local* certificates for gains, reachable sets and estimates of the basin of attraction. This provides tighter results and allows unbounded nonlinearities to be studied. Examples of systems modeled with unbounded nonlinearities include the driven Stirling engine [11] and electrical energy storage devices known as supercapacitors [7]. Estimates of region of attraction for sector bound nonlinear systems obtained with the Popov criterion have been considered in [28, 27, 23], and more recently in [13] using Semi-Definite Programming (SDP).

For global stability analysis, frequency domain methods that include a multiplier into the feedback loop, have been shown to improve results at the expense of computational complexity. The most famous example of such frequency-based methods are the multipliers of Zames and Falb [31] and their computational implementation [21]. A recent review on the contribution of the works of O'Shea [18] and Zames & Falb [31] is given in [3]. Even more recently, a local multiplier result was developed in [8] using dissipation inequalities. For systems with multiple slope restricted nonlinearities, a frequency domain criterion generalizing previous results for SISO systems and the associated multipliers has been presented in [10, 5, 22].

A. Contribution

Our results focus on the local analysis of Lurie type systems with slope-restricted nonlinearities. We develop LFs that are quadratic in both the state and the nonlinear terms and contain a Lurie-Postnikov integral term. We present conditions for the positivity of the LF that *do not impose* the positivity of the Lurie-Postnikov terms coefficients nor require that the quadratic terms on the nonlinearities are positive definite. We also present connections between our results and recent results in the literature that use similar LF structures.

The conditions verify dissipation inequalities that rely on inequalities associated with the sector and the slope bounds. In cases where the sector inequalities hold only locally, we discuss how to guarantee the inclusion of level sets in the region where the sector inequalities hold. These inclusion conditions allow us to estimate the region of attraction using contractive and invariant sets defined by the level sets of the computed LF. This allows us to analyse the effect of additive exogenous inputs and outputs to derive conditions for the computation of reachable sets and local induced gains. We also highlight the constraints of the convex optimization formulation used to illustrate the results with numerical examples. The results presented here extend the results in [26], where only the stability analysis was studied.

Notation The set of real valued matrices of dimensions $n \times m$ is denoted $\mathbb{R}^{n\times m}$, the set of symmetric matrices of dimension n is denoted $\mathbb{S}^n$, the set of diagonal matrices is denoted $\mathbb{D}$, the set of positive semi-definite diagonal matrices is denoted $\mathbb{D}_{\geq 0}$, and $He(A) := A + A^T$. The interior of a set $\mathcal{D} \subset \mathbb{R}^m$ is denoted $\mathcal{D}^\circ$, I_m denotes the identity matrix of dimension m. For $\rho \geq 0$, we use $\mathcal{E}(V, \rho) = \{x \in \mathbb{R}^n \mid V(x) \leq \rho\}$ i.e. the ρ sublevel set of V. We drop the arguments of some functions when it is clear from the context but include them when a statement contains both a signal and its evaluation at the argument. We denote the time-derivative of a function of time, x by $\dot{x}$ and we use ∂ to denote the sub-differential operator.

G. Valmorbida is with Laboratoire des Signaux et Systèmes, CentraleSupélec, CNRS, Univ. Paris-Sud, Université Paris-Saclay, 3 Rue Joliot-Curie, Gif–sur–Yvette 91192, France, Email: {giorgio.valmorbida}@l2s.centralesupelec.fr. R. Drummond and S. R. Duncan are with Department of Engineering Science, University of Oxford, 17 Parks Road, OX1 3PJ Oxford, United Kingdom, Email: {ross.drummond,stephen.duncan}@eng.ox.ac.uk. Work supported by the EPSRC project J010537/1.

II. Problem statement

Consider the linear time-invariant (LTI) system with input nonlinearities

$$\left\{\begin{array}{rcl}\dot{x} & = & Ax + B\phi(y) + B_w w \\ y & = & Cx + D\phi(y) + D_w w \\ z & = & C_z x + D_z\phi(y) + D_{zw} w\end{array}\right. \qquad (1)$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $w \in \mathbb{R}^{m_w}$.

The nonlinearity $\phi : \mathcal{Y} \to \mathbb{R}^m$, $\mathcal{Y} \subseteq \mathbb{R}^m$, is assumed to be *time-invariant, memoryless, Lipschitz on* $\mathcal{Y}^\circ$, *decentralized*

$$\phi(y) = [\phi_1(y_1) \quad \phi_2(y_2) \quad \ldots \phi_m(y_m)]^T, \qquad (2a)$$

sector bounded

$$\frac{\phi_i(y_i)}{y_i} \in [\underline{\delta}_i, \overline{\delta}_i] \quad \forall y \in \mathcal{Y}_0 \subseteq \mathcal{Y} \qquad (2b)$$

which implies $\phi(0) = 0$, and *slope restricted*

$$\partial\phi_i(y_i) \in [\underline{\gamma}_i, \overline{\gamma}_i] \quad \forall y \in \mathcal{Y}_0 \subseteq \mathcal{Y}, \qquad (2c)$$

where $\underline{\gamma}_i \leq \underline{\delta}_i$ and $\overline{\delta}_i \leq \overline{\gamma}_i$. We also introduce the matrices $\underline{\Delta} := diag(\underline{\delta}_1, \ldots, \underline{\delta}_m)$, $\overline{\Delta} := diag(\overline{\delta}_1, \ldots, \overline{\delta}_m)$ $\underline{\Gamma} := diag(\underline{\gamma}_1, \ldots, \underline{\gamma}_m)$, $\overline{\Gamma} := diag(\overline{\gamma}_1, \ldots, \overline{\gamma}_m)$ to compactly express the sector and slope bounds. The Lipschitz assumption on ϕ implies that $\partial\phi_i(y_i) = \frac{d\phi_i}{dy_i}$ almost everywhere, relaxing the requirement for the nonlinearity to be continuously differentiable [25, Section 2].

The well posedness of the algebraic loop in (1) is guaranteed if there exists a unique solution to the implicit equation $F(\mu) := \mu - D\phi(\mu) = \nu$, that is, a mapping $\mu(\nu)$ satisfying $F(\mu(\nu)) = \nu$. Following [30, Claim 1], for functions ϕ that are differentiable almost everywhere, the well-posedness of the loop is obtained if $JF(\mu)$, the Jacobian of F, belongs to a compact and convex set of invertible matrices for almost all values of μ (see [30, Proposition 2]). In the Appendix we show that the above conditions on the Jacobian hold true provided the inequality in the assumption below is verified.

Assumption 1 (Well-posedness): There exists a matrix $W \in \mathbb{D}^m_{\geq 0}$ such that

$$2W - He(W(I - D\underline{\Gamma})^{-1}D(\overline{\Gamma} - \underline{\Gamma})) > 0. \qquad (3)$$

Provided Assumption 1 holds, we can define the following set

$$\mathcal{X}_0 := \{x \in \mathbb{R}^n \mid y \in \mathcal{Y}_0, F(y) = Cx\}, \qquad (4)$$

where $\mathcal{Y}_0 \subseteq \mathcal{Y} \subseteq \mathbb{R}^m$ corresponds to the set where the sector and the slope restrictions hold, as defined in (2). We also define the following set

$$\mathcal{XW}_0 := \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^{m_w} \mid y \in \mathcal{Y}_0, x \in \mathcal{X}_0, F(y) = Cx + D_w w\}. \qquad (5)$$

Under Assumption 1, this paper provides a solution to the following problem:

Problem 1: For system (1) with ϕ satisfying (2):

a) For $w \equiv 0$, certify the stability of the origin with an estimate of the region of attraction (ERA) contained in $\mathcal{X}_0$;
b) Compute reachable sets contained in $\mathcal{X}_0$ for disturbances satisfying $w \in \{w \in \mathcal{L}_2 | \|w\|_2 \leq \rho^{\frac{1}{2}}\}$, and $(x(t), w(t)) \in \mathcal{XW}_0$;
c) Compute the (local) induced $\mathcal{L}_2$ gains between w and z, with $w \in \{w \in \mathcal{L}_2 | \|w\|_2 \leq \rho^{\frac{1}{2}}\}$, and $(x(t), w(t)) \in \mathcal{XW}_0$.

In case the sector and slope bounds (2b) and (2c) hold globally, i.e. $\mathcal{Y}_0 = \mathbb{R}^m$, global properties will be obtained by setting $\mathcal{X}_0 = \mathbb{R}^n$ and $\mathcal{XW}_0 \in \mathbb{R}^n \times \mathbb{R}^{m_w}$.

III. Sector inequalities

In this section we present inequalities related to the sector and slope bounds of the nonlinearities in system (1). These inequalities are required for assessing the positivity of quadratic-like expressions.

Define $s_1 : \mathbb{R}^{m\times m} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $s_2 : \mathbb{R}^{m\times m} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $s_3 : \mathbb{R}^{m\times m} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ as

$$\begin{array}{rcl}s_1(T, \phi, \theta) & := & (\phi - \underline{\Delta}\theta)\, T\left(\overline{\Delta}\theta - \phi\right) \\ s_2(T, \phi, \theta) & := & (\phi - \underline{\Gamma}\theta)\, T\left(\overline{\Gamma}\theta - \phi\right) \\ s_3(T, \phi_1, \phi_2, \theta_1, \theta_2) & := & ((\phi_1 - \phi_2) - \underline{\Gamma}(\theta_1 - \theta_2)) \\ & & \times T\left(\overline{\Gamma}(\theta_1 - \theta_2) - (\phi_1 - \phi_2)\right).\end{array}$$

The following lemma is associated with the sector boundedness of the functions ϕ_i.

Lemma 1: If $T_1 \in D^m_{\geq 0}$ and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ satisfies (2), then

$$s_1(T_1, \phi(\theta), \theta) \geq 0 \qquad (6)$$

for all $\theta \in \mathcal{Y}_0$.

In the following two lemmas, we consider $\theta : [0, \infty) \to \mathcal{Y}_0$, $\theta(t) \in \mathcal{C}^1(t)$ to obtain inequalities for the slope restrictions of ϕ.

Lemma 2: If $T_2 \in \mathbb{D}^m_{\geq 0}$ and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ satisfies (2), then

$$s_2(T_2, \dot{\phi}(\theta), \dot{\theta}) \geq 0 \qquad (7)$$

almost everywhere for $\theta \in \mathcal{Y}_0$.

From (2c) we have $(\partial\phi_i(\theta_i) - \underline{\gamma}_i)(\overline{\gamma}_i - \partial\phi_i(\theta_i)) \geq 0$.

Lemma 3: If $T_3 \in \mathbb{D}^m_{\geq 0}$ and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ satisfies (2c), then

$$s_3(T_3, \phi(\theta_1), \phi(\theta_2), \theta_1, \theta_2) \geq 0 \qquad (8)$$

for all $\theta_1, \theta_2 \in \mathcal{Y}_0$.

The above lemma shows that the slope restriction with non-negative bounds satisfies the *incremental sector boundedness property* [32, Definition 1].

IV. Main results

This section is concerned with Lyapunov functions of the form

$$V(x) = V_0(x) + \sum_{i=1}^{m} \lambda_i \int_0^{\tilde{y}_i(x)} (\phi_i(s) - \underline{\delta}_i s)\; ds, \qquad (9a)$$

where

$$V_0(x) = \left[\begin{array}{c} x \\ \phi(\tilde{y}(x)) \end{array}\right]^T \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{array}\right] \left[\begin{array}{c} x \\ \phi(\tilde{y}(x)) \end{array}\right], \qquad (9b)$$

and $\tilde{y}$ is the the solution of

$$\tilde{y}(x) = Cx + D\phi(\tilde{y}(x)). \qquad (10)$$

The use of function $V_0(x)$, with $\phi(\tilde{y})$ was proposed in [4] in the context of the analysis of saturating systems, where the positivity of $V_0(x)$ was enforced by imposing $P := \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{smallmatrix}\right] > 0$ [4]. We refer to the integral terms in (9a) as the Lurie-Postnikov terms. For the sake of compactness of notation we use $\tilde{\phi}$ to denote $\phi(\tilde{y}(x))$. One way to enforce the positivity of $V(x)$ is to impose $P > 0$ and $\lambda_i \geq 0$. When $P_{12} = 0$, $P_{22} = 0$, the relaxation of the non-negativity of the coefficients λ_i was considered in [19, 12, 1]. Define $\Lambda := diag(\lambda_1, \ldots, \lambda_m)$ and $\tilde{\Lambda} := diag(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m)$ which are used in the following lemma that gives conditions for the positivity of V without imposing positive-definiteness of P, nor the non-negativity of the coefficients λ_i.

Lemma 4: Consider V in (9) where ϕ satisfies (2a)-(2b) and Assumption 1 holds. If there exists a matrix $\tilde{\Lambda} \in \mathbb{D}^m_{\geq 0}$ such that

$$\Lambda \geq -\tilde{\Lambda}, \tag{11a}$$

$$V_0(x) - \frac{1}{2}\tilde{y}^T(x)(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}\tilde{y}(x) > 0, \ \forall x \in \mathcal{X}_0, \tag{11b}$$

then $V(x) > 0$, $\forall x \in \mathcal{X}_0 \subset \mathbb{R}^n$.

Proof: Use (11a) to obtain a positive-definite lower bound for (9a) as follows. If Assumption 1 holds, the mapping $\tilde{y} : \mathcal{X}_0 \to \mathcal{Y}_0$ is well defined. We can then prove that $V(x)$ is positive-definite in $\mathcal{X}_0$ by obtaining a positive-definite lower bound as follows

$$\begin{array}{rcl} V(x) & = & V_0(x) + \sum_{i=1}^m \lambda_i \int_0^{\tilde{y}_i(x)} (\phi_i(s) - \underline{\delta}_i s) ds \\ & \geq & V_0(x) - \sum_{i=1}^m \tilde{\lambda}_i \int_0^{\tilde{y}_i(x)} (\phi_i(s) - \underline{\delta}_i s) ds \\ & = & V_0(x) - \frac{1}{2}\tilde{y}^T(x)(\overline{\Delta} - \overline{\Delta})\tilde{\Lambda}\tilde{y}(x) \\ & & \quad - \sum_{i=1}^m \tilde{\lambda}_i \int_0^{\tilde{y}_i(x)} (\phi_i(s) - \underline{\delta}_i s) ds \\ & = & V_0(x) - \frac{1}{2}\tilde{y}^T(x)(\overline{\Delta} - \overline{\Delta} - \underline{\Delta})\tilde{\Lambda}\tilde{y}(x) \\ & & \quad - \sum_{i=1}^m \tilde{\lambda}_i \int_0^{\tilde{y}_i(x)} \phi_i(s) ds \\ & = & V_0(x) - \frac{1}{2}\tilde{y}^T(x)(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}\tilde{y}(x) \\ & & \quad + \sum_{i=1}^m \tilde{\lambda}_i \int_0^{\tilde{y}_i(x)} ((\overline{\delta}_i s - \phi_i(s)) ds \\ & = & \underbrace{V_0(x) - \frac{1}{2}\tilde{y}^T(x)(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}\tilde{y}(x)}_{>0 \text{ from (11b)}} \\ & & \quad \underbrace{+ \sum_{i=1}^m \tilde{\lambda}_i \int_0^{\tilde{y}_i(x)} (\overline{\delta}_i s - \phi_i(s)) ds}_{\geq 0 \text{ from } \tilde{\Lambda} \geq 0 \text{ and (2b)}} . \end{array} \tag{12}$$

■

The following theorem presents conditions for the stability of the origin of Lurie system (1) with slope-restricted nonlinearities:

Theorem 1: For nonlinearities ϕ satisfying (2) if there exists a matrix $P \in \mathbb{R}^{(n+m)\times(n+m)}$, matrices $\Lambda \in \mathbb{D}^m$, $\tilde{\Lambda}$, $T_j \in \mathbb{D}^m_{\geq 0}$, $j \in \{0, \ldots, 4\}$, and a scalar $\rho > 0$ such that (11a) holds,

$$V_0(x) - \frac{1}{2}\tilde{y}^T(x)(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}\tilde{y}(x) - s_1(T_0, \tilde{\phi}, \tilde{y}(x)) > 0 \tag{13a}$$

$\forall x \in \mathbb{R}^n, \ \tilde{\phi} \in \mathbb{R}^m$,

$$\begin{aligned} - \left\langle \begin{bmatrix} \nabla_x V \\ \nabla_{\tilde{\phi}} V \end{bmatrix}, \begin{bmatrix} \dot{x} \\ \dot{\tilde{\phi}} \end{bmatrix} \right\rangle - \Psi(z, w) - s_1(T_1, \tilde{\phi}, \tilde{y}(x)) \\ - s_1(T_2, \phi, y(x, w)) - s_2(T_3, \dot{\tilde{\phi}}, \dot{\tilde{y}}(\dot{\tilde{\phi}}, x, \phi, w)) \\ - s_3(T_4, \tilde{\phi}, \phi, \tilde{y}(x), y(x, w)) > 0 \end{aligned} \tag{13b}$$

$\forall x \in \mathbb{R}^n, \ \phi \in \mathbb{R}^m, \ \tilde{\phi} \in \mathbb{R}^m, \ \dot{\tilde{\phi}} \in \mathbb{R}^m, \ w \in \mathbb{R}^{m_w}$ and

$$\mathcal{E}(V, \rho) \subseteq \mathcal{X}_0 \tag{13c}$$

hold with

a) $\Psi \equiv 0$ and $w \equiv 0$ (which gives $\tilde{\phi} = \phi$ so that $s_3 \equiv 0$ and allows us to set $T_2 = 0$);
b) $\Psi(z, w) = w^T w$;
c) $\Psi(z, w) = w^T w - \eta^{-2} z^T z$;

then

a) (stability) the origin of (1) is locally asymptotically stable and $\mathcal{E}(V, \rho)$ is an estimate of its region of attraction. In the case $\mathcal{X}_0 = \mathbb{R}^n$, the origin is globally asymptotically stable.
b) (reachable set) $x(0) = 0$ and $\|w\|_2 \leq \rho^{\frac{1}{2}}$, $(x(t), w(t)) \in \mathcal{XW}_0$, so that $x(t) \in \mathcal{E}^\circ(V, \rho)$ for all $t \geq 0$;
c) (local finite $\mathcal{L}_2$-gain) $x(0) = 0$ and $\|w\|_2 \leq \rho^{\frac{1}{2}}$, $(x(t), w(t)) \in \mathcal{XW}_0$, imply $\|z\|_2 < \eta\|w\|_2$, that is, the induced $\mathcal{L}_2$ gain from w to z is bounded by η for every input satisfying $\|w\|_2 \leq \rho^{\frac{1}{2}}$.

Proof: If (13a) holds,

$$V_0(x) - \frac{1}{2}\tilde{y}^T(x)(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}\tilde{y}(x) > s_1(T_0, \tilde{\phi}, \tilde{y}(x))$$

from Lemma 1 and $s_1(T_0, \phi(\tilde{y}), \tilde{y}) \geq 0$ holds for all $x \in \mathcal{X}_0$, thus (11b) holds. Following Lemma 4 if (11a) also holds, then $V(x) \geq 0$, $\forall x \in \mathcal{X}_0$.

We use $\dot{V}(x, \tilde{\phi}, \dot{\tilde{\phi}}, \phi, w)$ to express the time-derivative of $V(x)$ along the trajectories of (1)

$$\dot{V}(x, \tilde{\phi}, \dot{\tilde{\phi}}, \phi, w) = \left\langle \begin{bmatrix} \nabla_x V \\ \nabla_{\tilde{\phi}} V \end{bmatrix}, \begin{bmatrix} Ax + B\phi + B_w w \\ \dot{\tilde{\phi}} \end{bmatrix} \right\rangle.$$

From (13b) we have

$$\begin{aligned} - \dot{V}(x, \tilde{\phi}, \dot{\tilde{\phi}}, \phi, w) - \Psi(z, w) > s_1(T_1, \tilde{\phi}, \tilde{y}(x, \tilde{\phi})) \\ + s_1(T_2, \phi, y(x, \phi, w)) + s_2(T_3, \dot{\tilde{\phi}}, \dot{\tilde{y}}(x, \dot{\tilde{\phi}}, \phi, w)) \\ + s_3(T_4, \tilde{\phi}, \phi, \tilde{y}(x, \tilde{\phi}), y(x, \phi, w)). \end{aligned}$$

If (2) holds, the relations in Lemmas 1-3 give

$$-\dot{V}(x, \tilde{\phi}, \dot{\tilde{\phi}}, \phi, w) - \Psi(z, w) > 0, \ \forall x \in \mathcal{X}_0. \tag{14}$$

Thus if

a) $\Psi(z, w) \equiv 0$, we have that $\dot{V}$ is negative for all $x \in \mathcal{X}_0$. Since from (13c) the time-derivative of V is negative along the trajectories of system (1) provided the sector inequalities hold, that is, provided the trajectories belong to the set $\mathcal{X}_0$ which, from (13c) contains the set $\mathcal{E}(V, \rho)$. Following [17, Theorem 4.1], with (13a) and (13b) that hold in the sublevel set, $\mathcal{E}(V, \rho)$ is an invariant and contractive set and hence provides an estimate of the region of attraction of (1).
b) $\Psi(z, w) = -w^T w$, $x_0 = 0$, integrate (14) from 0 to t^* to obtain $\int_0^{t^*} w^T(\tau)w(\tau)\mathrm{d}\tau > V(t^*)$ since $V(0) = 0$. Hence, provided $\|w\|_2^2 = \int_0^{t^*} w^T(\tau)w(\tau)\mathrm{d}\tau \leq \rho$ we have that $x(t^*) \in \mathcal{E}^\circ(V(x), \rho)$. From (13c) the sector inequalities hold so (13a) and (13b) hold.
c) $\Psi(z, w) = -w^T w + \eta^{-2} z^T z$ and $x_0 = 0$, integrate from 0 to t^* to obtain $\int_0^{t^*} w^T(\tau)w(\tau)\mathrm{d}\tau > \int_0^{t^*} \eta^{-2} z^T(\tau)z(\tau)\mathrm{d}\tau + V(x(t^*))$. Since $V(x(t^*)) \geq 0$, then $\|w\|_2^2 > \eta^{-2}\|z\|_2^2$ for any $t^* \in [0, \infty)$. From $\|w\|_2 \leq \rho^{\frac{1}{2}}$ and $\int_0^{t^*} \eta z^T(\tau)z(\tau)\mathrm{d}\tau \geq 0$ the above inequality implies $V(x(t^*)) < \rho$, thus from (13c) we have $x(t^*) \in \mathcal{X}_0$ for any $t^* \in [0, \infty)$, hence (13a) and (13b) hold for $\|w\|_2 \leq \rho^{\frac{1}{2}}$.

■

Remark 1: The use of Lemma 2 in the proof of Theorem 1, requires $\tilde{y}$ to be differentiable. From (10) we have $\frac{d\tilde{y}}{dt} = C\frac{dx}{dt} + D\partial\phi(\tilde{y})\frac{d\tilde{y}}{dt}$, which can be written as $(I - D\partial\phi(\tilde{y}))\frac{d\tilde{y}}{dt} = C\frac{dx}{dt}$. Thus if $(I - D\partial\phi(\tilde{y}))$, is non-singular for all $\tilde{y} \in \mathcal{Y}_0$, $\frac{d\tilde{y}}{dt}$ exists and is given by $\frac{d\tilde{y}}{dt} = (I - D\partial\phi(\tilde{y}))^{-1}C\frac{dx}{dt}$. From Proposition 1 in the Appendix we have that Assumption 1 guarantees the invertibility of $(I - D\partial\phi(\tilde{y}))$ thus, the existence of $\frac{d\tilde{y}}{dt}$. ⋆

Note that the set inclusion (13c) is required to guarantee that the sector inequalities in Lemmas 1-3 hold so that (13b) implies (14). Moreover, from Assumption 1 and the fact that $(x(t), w(t)) \in \mathcal{XW}_0$ we have $y(t) \in \mathcal{Y}_0$ $\forall t \geq 0$. The condition on the disturbance $(x(t), w(t)) \in \mathcal{XW}_0$ can be dropped in two cases: 1) for $D_w = 0$, we have $\tilde{y} \equiv y$ and (13c) implies that

$y(t) \in \mathcal{Y}_0$, for all $t \geq 0$; 2) for the case $\mathcal{Y}_0 = \mathbb{R}^m$, the inequalities from Lemmas 1-3 hold globally so (13c) is trivially satisfied.

A convenient property of the quadratic inequalities (13a)-(13b) is that their representation is affine on P, Λ, $\tilde{\Lambda}$, T_i, $i = \{0, \ldots, 4\}$. Whenever the inclusion (13c) is also formulated in terms of affine inequalities on these variables and the system matrices (A, B, C, D) and the sector bounds $\underline{\Delta}$, $\overline{\Delta}$, $\underline{\Gamma}$, $\overline{\Gamma}$ are given, we can set the problem of computing these variables as a convex semi-definite program. Numerical examples illustrate the solution to these convex semi-definite programs in Section V and the corresponding linear matrix inequalities (LMIs) are detailed in the Appendix.

A. Inclusion conditions

To satisfy local properties of (1) with Theorem 1 we have to guarantee the inclusion (13c). For sets of the form

$$\mathcal{X}_0 = \left\{ x \in \mathbb{R}^n \mid (\tilde{y}_j(x) - \underline{\tilde{y}}_j)(\tilde{y}_j(x) - \overline{\tilde{y}}_j) \leq 0, j = 1 \ldots m \right\}, \tag{15}$$

a condition for the set inclusion is provided by the following lemma.

Lemma 5: If there exist scalars $\alpha_j > 0$ such that

$$-\alpha_j(\tilde{y}_j(x) - \underline{\tilde{y}}_j)(\tilde{y}_j(x) - \overline{\tilde{y}}_j) \geq (\rho - V(x)) \tag{16}$$

$j = 1, \ldots, m$ then (13c) holds.

Proof: If the above inequality holds, then for all x satisfying $(\rho - V(x)) \geq 0$ the inequality $-(\tilde{y}_j(x) - \underline{\tilde{y}}_j)(\tilde{y}_j(x) - \overline{\tilde{y}}_j) \geq 0$ holds and $x \in \mathcal{X}_0$ $\forall x \in \mathcal{E}(V, \rho)$, hence the set inclusion. ■

For the function $V(x)$ in (9), the inequalities (16) become

$$-\alpha_j \underline{\tilde{y}}_j \overline{\tilde{y}}_j - \rho + \alpha_j(\underline{\tilde{y}}_j + \overline{\tilde{y}}_j)\tilde{y}_j(x) - \alpha_j \tilde{y}_j^2(x) + V_0(x) + \sum_{i=1}^{m} \lambda_i \int_0^{\tilde{y}_i} \phi_i(s) - \underline{\delta}_i s \, \mathrm{d}s \geq 0, \tag{17}$$

$j = 1, \ldots, m$. The reason for expressing nonlinearities in quadratic-like forms is to frame the inclusion condition of Theorem 1 as a set of affine matrix inequalities on the unknown coefficients λ_i. Whenever only its bounds are given, as in (2a), consider $\tilde{\lambda}_i$ satisfying $\lambda_i \geq -\tilde{\lambda}_i$ to obtain the following lower bound for the Lurie-Postnikov terms in (17) (see (12))

$$\sum_{i=1}^{m} \lambda_i \int_0^{\tilde{y}_i} \phi_i(s) - \underline{\delta}_i s \, \mathrm{d}s \geq -\frac{1}{2}\tilde{y}^T(x)(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}\tilde{y}(x) \geq 0. \tag{18}$$

Provided the inequalities

$$-\alpha_j \underline{\tilde{y}}_j \overline{\tilde{y}}_j - \rho + \alpha_j(\underline{\tilde{y}}_j + \overline{\tilde{y}}_j)\tilde{y}_j(x) - \alpha_j \tilde{y}_j^2(x) + V_0(x) - \frac{1}{2}\tilde{y}^T(x)(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}\tilde{y}(x) \geq 0, \tag{19}$$

$j = 1, \ldots, m$, hold, we have that (17) holds and hence guarantees set inclusion (13c). A lower bound on the Lurie-Postnikov terms that guarantee inclusion conditions for sector nonlinearities similar to (18), was proposed in [13].

When the nonlinearity that satisfies the sector condition is known, in some cases it is possible to explicitly write the Lurie-Postnikov term in a quadratic-like form. As an example, consider the nonlinearities $\ln(1 + \tilde{y}_i)$ and $\frac{\tilde{y}_i}{1+\tilde{y}_i}$

$$\begin{aligned} \int_0^{\tilde{y}_i} \ln(1+s) - \underline{\delta}_i s \, \mathrm{d}s &= \ln(1+\tilde{y}_i)(1+\tilde{y}_i) - \tilde{y}_i - \tfrac{1}{2}\underline{\delta}_i \tilde{y}_i^2 \\ \int_0^{\tilde{y}_i} \tfrac{s}{1+s} - \underline{\delta}_i s \, \mathrm{d}s &= -\ln(1+\tilde{y}_i) + \tilde{y}_i - \tfrac{1}{2}\underline{\delta}_i \tilde{y}_i^2, \end{aligned} \tag{20}$$

which can be expressed as quadratic-like forms in the vector $[1 \;\; \tilde{y}_i \;\; \ln(1+\tilde{y}_i)]^T$. These nonlinearities present sector and slope bounds that hold only in the interval $\left[\underline{\tilde{y}}_j, \overline{\tilde{y}}_j\right]$ as detailed in the table below: note that for both $\ln(1+\tilde{y}_j)$ and $\frac{\tilde{y}_j}{1+\tilde{y}_j}$, (2)

$\phi(\tilde{y}_j)$	$\underline{\delta}$	$\overline{\delta}$	$\underline{\gamma}$	$\overline{\gamma}$
$\ln(1+\tilde{y}_j)$	$\frac{\ln(1+\overline{\tilde{y}}_j)}{\overline{\tilde{y}}_j}$	$\frac{\ln(1+\underline{\tilde{y}}_j)}{\underline{\tilde{y}}_j}$	$\frac{1}{1+\overline{\tilde{y}}_j}$	$\frac{1}{1+\underline{\tilde{y}}_j}$
$\frac{\tilde{y}_j}{1+\tilde{y}_j}$	$\frac{1}{1+\overline{\tilde{y}}_j}$	$\frac{1}{1+\underline{\tilde{y}}_j}$	$\frac{\overline{\tilde{y}}_j}{(1+\overline{\tilde{y}}_j)^2}$	$\frac{\underline{y}_j}{(1+\underline{\tilde{y}}_j)^2}$

Table I

LOCAL SECTOR AND SLOPE BOUNDS FOR $\ln(1+\tilde{y}_j)$ AND $\frac{\tilde{y}_j}{1+\tilde{y}_j}$ FOR $\mathcal{X}_0$ AS IN (15) WITH $\underline{\tilde{y}}_j > -1$.

holds with $\mathcal{Y} = (-1, \infty)$ thus $\mathcal{Y}_0 = \left[\underline{\tilde{y}}_j, \overline{\tilde{y}}_j\right]$ is defined with $-1 < \underline{\tilde{y}}_j < 0$ and $0 < \overline{\tilde{y}}_j$.

B. Discussion on the proposed LF

The function (9) was introduced in [29] to study single-input single-output (SISO) systems with slope-restricted nonlinearities satisfying $\underline{\gamma} = -\infty$ or $\overline{\gamma} = \infty$, yielding a graphical criterion involving the frequency response of the linear part. A main feature of the result presented in this paper is that neither the Lurie-Postnikov coefficient λ nor the corresponding P_{22} block (scalar in the SISO case) are required to be positive definite. The same Lyapunov structure was used in [16] where the extension of the frequency domain criteria of [29] to the MIMO case was presented.

Convex optimization based approaches using the quadratic-like term in (9) have also been proposed [24, 20, 25], although none of these references addresses the positivity of the LF as proposed by Lemma 4. In [24], the positivity of (9) it is obtained by imposing $P > 0$ and $\Lambda > 0$ and the slope restriction is addressed by considering a norm-bounded inequality. In [20] and [25], the slope restriction is studied with the inequality of Lemma 2 and the proposed Lyapunov functions contain additional Lurie-Postnikov type terms with non-negative coefficients and impose $P \geq 0$ ($P > 0$ in [25]). The remark below shows that the additional terms on these papers can be recast in the form (9) where the block P_{22} is allowed to be negative definite.

Remark 2: (Additional Lurie-Postnikov terms for slope-restricted nonlinearities) In [20] and [25], Lyapunov function structures containing the term $V_0(x)$ as in (9b) were studied for the stability and induced $\mathcal{L}_2$ gain analysis for system (1) with additive disturbance terms. When compared to (9a) the structures in [20] and [25] use additional integral terms. It is shown in [25] that some of the additional Lurie-Postnikov terms in [20] were redundant. We now discuss how (9a) compares with the LF of [25], which can be written as

$$\bar{V}(x) = \begin{bmatrix} x \\ \tilde{\phi} \end{bmatrix}^T \bar{P} \begin{bmatrix} x \\ \tilde{\phi} \end{bmatrix} + \sum_{j=1}^{4}\sum_{i=1}^{m} \mu_{1,i} \int_0^{\tilde{y}_i(x)} \bar{g}_{j,i}(s)\mathrm{d}s \tag{21}$$

where $\bar{g}_{1,i}(s) = \phi_i(s)$, $\bar{g}_{2,i}(s) = \overline{\delta}_i s - \phi_i(s)$, $\bar{g}_{3,i} = (\overline{\gamma}_i - \partial\phi_i(s))\, s$, $\bar{g}_{4,i} = \partial\phi_i(s)\left(\overline{\delta}_i s - \phi_i(s)\right)$ and $\bar{P} > 0$ and $\mu_{j,i} \geq 0$, $i = 1, \ldots, m$, $j = 1, \ldots, 4$. For ϕ satisfying (2) with $\underline{\delta}_i = \underline{\gamma}_i = 0$, $i = 1, \ldots, m$ we clearly have $g_{j,i}(x) \geq 0$, $j = 1, \ldots, 4$, $i = 1, \ldots, m$.

By using the relations

$$\int_0^{\tilde{y}_i} \phi_i(s)\partial\phi_i(s)\mathrm{d}s = \frac{1}{2}\phi_i^2(\tilde{y}_i)$$
$$\int_0^{\tilde{y}_i} \partial\phi_i(s)s\mathrm{d}s = \phi_i(\tilde{y}_i)\tilde{y}_i + \int_0^{\tilde{y}_i} \phi_i(s)\mathrm{d}s,$$

it is straightforward to obtain

$$\sum_{j=1}^{4}\sum_{i=1}^{m} g_{j,i}(x) = \begin{bmatrix} x \\ \phi(\tilde{y}(x)) \end{bmatrix}^T M \begin{bmatrix} x \\ \phi(\tilde{y}(x)) \end{bmatrix} + \sum_{i=1}^{m}(\mu_{1,i} - \mu_{2,i} + \mu_{3,i} - \overline{\delta}_i\mu_{4,i}) \int_0^{\tilde{y}_i(x)} \phi_i(s)\mathrm{d}s$$

with $M = \begin{bmatrix} C^T & 0 \\ D^T & I \end{bmatrix} \begin{bmatrix} \overline{\Delta}M_2 + \overline{\Gamma}M_3 & \frac{1}{2}(\overline{\Delta}M_4 - M_3) \\ \frac{1}{2}(\overline{\Delta}M_4 - M_3) & -\frac{1}{2}M_4 \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}$ where $M_j = diag(\mu_{j,1}, \ldots, \mu_{j,m})$, $j = 1, \ldots, 4$. Thus (9a) is obtained from (21) by setting $P = \bar{P} + M$ and $\lambda_i = (\mu_{1,i} - \mu_{2,i} + \mu_{3,i} - \overline{\delta}_i\mu_{4,i})$. Note that the matrix $\bar{P} + M$ is not necessarily positive definite since its lower, right diagonal block $\bar{P}_{22} - \frac{1}{2}M_4$ may not be positive definite. Note also that the Lurie-Postnikov term coefficients $\bar{\mu}_i := (\mu_{1,i} - \mu_{2,i} + \mu_{3,i} - \overline{\delta}_i\mu_{4,i})$ can also be negative since $\mu_{j,i} \geq 0$ does not imply $\bar{\mu}_i \geq 0$. ⋆

For the specific case of saturation or deadzone nonlinearities, the integral terms can be incorporated to the quadratic-like term V_0. This fact has been observed in [4]. In [9], the slope restriction of the deadzone is accounted for (see [9, Fact 2]). In both [4] and [9], the positive definiteness of $V_0(x)$ is obtained by imposing $P > 0$.

V. Numerical Formulation and Examples

In this section we present numerical solutions for the inequalities presented in Theorem 1. The computation of the stability certificates, reachable sets and local induced $\mathcal{L}_2$-gains are based on the solution to the SDPs obtained from the inequalities of Theorem 1. The associated constraints to the SDP we solve are detailed in the Appendix. For nonlinearities that yield sector and slope bounds that hold only locally, we guarantee the set inclusion (13c) by solving the inequalities (17) for the case where the nonlinearity is known and has an explicit quadratic-like representation, or, if it is only known to satisfy sector bounds we use a lower bound to the integral term and solve (19) otherwise.

In the following example we optimize sector and slope bounds using different structures of the Lyapunov function (9).

Example 1: This example computes the maximum sector and slope restriction for the SISO system described by $G_1(s) = \frac{0.2s^2}{s^4+0.4s^3+6s^2+0.1s+1}$. The sector and slope conditions are defined by a parameter ϵ, as $\underline{\delta} = 0$, $\overline{\delta} = \epsilon$, $\underline{\gamma} = -0.5\epsilon$, $\overline{\gamma} = 1.5\epsilon$. Via a bissection algorithm, we obtain bounds for the parameter ϵ such that the global stability of system (1) is guaranteed. Table II gives the results comparing the bounds of $V(x)$ to the bounds obtained with $V_0(x)$, together with the special cases of V given by $V_Q := x^TP_{11}x$ and $V_{LP} := x^TP_{11}x + \sum_{i=1}^{m}\lambda_i\int_0^{y_i}\phi(s)\mathrm{d}s$ ∘

	V_Q	V_{LP}	V_0	V
ϵ	0.730	1.272	0.730	2.422

Table II
Maximum bound on ϵ for global stability of system $G_1(s)$.

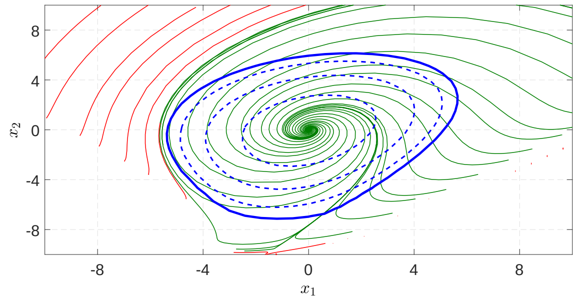

Figure 1. Estimate of the region of attraction with the Lyapunov function (9) (dark blue). Trajectories asymptotically converging to the origin are shown in green, while diverging trajectories are depicted in red .

Example 2: Consider the system

$$\begin{cases} \dot{x}_1 &= -x_2 + \ln(1+y_1) + 2\frac{y_2}{1+y_2} \\ \dot{x}_2 &= x_1 - 0.65x_2 + \ln(1+y_1) + \frac{y_2}{1+y_2} \\ y_1 &= 0.1(x_1 + x_2) - 0.2\frac{y_2}{1+y_2} \\ y_2 &= 0.1(x_2 - x_1). \end{cases}$$

This can be readily put in the form (1) with $\phi_1(y_1) = \ln(1+y_1)$, $\phi_2(y_2) = \frac{y_2}{1+y_2}$. In order to compute a region of attraction of its origin, we fix the interval of interest $y_1 \in [-.4, 50]$, $y_2 \in [-.5, 50]$ thus defining the slope and sector bounds for the nonlinearities according to Table I. We obtain the inclusion inequality (17) by explicitly computing the Lurie-Postnikov terms as in (20) and fixing $\rho = 1$. We then obtain an ERA by solving the convex optimization problem that minimizes $Trace(P_{11})$ subject to (22a)-(22c), (17) (see the Appendix). The level sets obtained are depicted in Figure 1. Inner level sets of the LF are also depicted and show that incorporating the Lurie-Postnikov terms and the nonlinearities in V_0 may yield an asymmetric ERA with respect to the origin. Note also that the innermost level set is indistinguishable from an ellipsoid, showing that close to the equilibrium point, the term $x^TP_{11}x$ dominates the non-quadratic terms of the LF. ∘

Example 3: This example computes upper bounds for the local induced $\mathcal{L}_2$ gain η of an idealised Stirling engine. The dynamic equations are obtained from (3) of [11] with damping factor $c = 50$ and nonlinearity $\phi(y) = y/(1+y)$

$$\dot{x}_1 = x_2 - cx_1 - cw$$
$$\dot{x}_2 = -\frac{x_1}{1+x_1}$$
$$y = x_1$$
$$z = x_1.$$

The gain depends upon both the local domain and the magnitude of the disturbance whose norm is upper bounded by $\|w\|_2 \leq \rho^{\frac{1}{2}}$. For this example, the upper bound on the domain is set as $\overline{\tilde{y}} = 0.5$ and η is computed for each $\{\underline{\tilde{y}},\ \rho\} = \{1,\ 2,\ 5,\ 6,\ 8\} \times 10^{\{-2,\ -1\}}$. Figure 2 shows minimal upper bounds for η searched over the values of $\underline{\tilde{y}}$ for fixed ρ. The bounds were computed using $V(x)$ subject to (22a)-(22c), (17) and a local Popov criterion obtained using $V_{LP}(x)$ and the substitution of a lower bound for the LF given by V_C into (16), a similar method to [13]. Tighter bounds were obtained using $V(x)$ for all values. ∘

As pointed out in Remark 2, a single Lurie-Postnikov term may replace the four non-negative Lurie-Postnikov terms associated to each input in the Lyapunov function studied in [25,

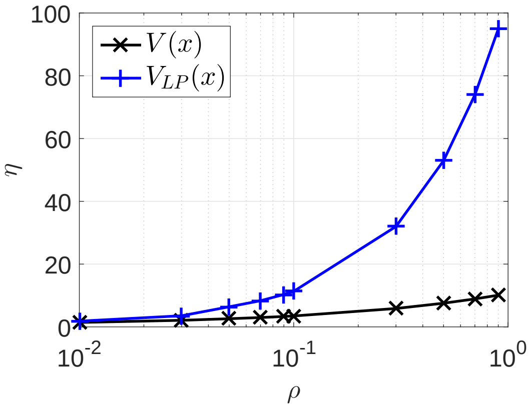

Figure 2. Induced $\mathcal{L}_2$ gain bounds for an idealised Stirling engine.

Theorem 5]. However, in this paper, these terms and the matrix P_{22} are not necessarily non-negative. We have performed the global stability and gain computations for the examples in [2] and [25] to illustrate the fact that the global analysis using the presented results yield the same results as the ones obtained with a more complex Lyapunov function. Indeed, the conditions of Theorem 1 matched the stability bounds obtained with the results of [20] for the balanced realization of all transfer functions in [2, Table 3]. Similarly, the solution to the inequalities of Theorem 1 give the same $\mathcal{L}_2$ gain bounds as the ones in [25, Theorem 5] for the systems defined in of [25, Table 2].

VI. Conclusions

In this paper, stability analysis of Lurie type systems with slope-restricted nonlinearities was carried out for LFs that have a quadratic-like term on the state and the nonlinearity and Lurie-Postnikov type terms. We have proposed relaxed conditions for the positivity of the LF (cf. Lemma 4) and have used sector inequalities to propose conditions for the global and local properties of solutions to Lurie systems. Importantly, the LF structure allows for negative coefficients in the Lurie-Postnikov term.

Numerical solutions to the dissipation inequalities of the main result (cf. Theorem 1) can be obtained with the solutions to SDPs. The proposed numerical formulation is a convex optimisation problem since the SDP constraints are affine both on the Lyapunov/storage function coefficients and the multipliers associated to sector inequalities. The local stability analysis with the computation of ERAs and local gain analysis are illustrated with numerical examples.

Appendix

LMIs from Theorem 1

The quadratic inequalities in Theorem 1 and the inequality (19), which is a sufficient condition for (13c), are equivalent to linear matrix inequalities presented in (22), where M_Ψ satisfies $\Psi(w,z) = \zeta^T M_\Psi \zeta$ with $\zeta = \begin{bmatrix} x^T & \tilde{\phi}^T & \dot{\tilde{\phi}}^T & \phi^T & w^T \end{bmatrix}^T$. Whenever the nonlinearity is known and the Lurie term is expressed as a quadratic form, *ad hoc* inequalities replace (22d).

Conditions for Well Posedness of the algebraic loop

In [30, Proposition 2], it is shown that a condition for the algebraic loop to be well posed is that the Jacobian of $F(\mu) = \mu - D\phi(\mu)$, where it is defined, belongs to a compact, convex set of non-singular matrices. In this appendix we show that such a condition holds provided (3) holds. The only difference to the reasoning presented in [30, Proposition 2] is given by conditions related to the non-singularity of the Jacobian of $F(\mu)$.

The Jacobian of $F(\mu)$ is given by $JF(\mu) = I - D\partial\phi(\mu)$ a.e.. Thanks to the slope restriction of $\phi(\mu)$ in (2c), for almost all μ, $JF(\mu) \in \mathcal{M} := \overline{co}(\{I - D\Gamma, \Gamma \in \mathcal{G}\})$, where

$$\mathcal{G} := \left\{\Gamma \in \mathbb{D} : \Gamma = diag(\gamma_1, \gamma_2, \ldots, \gamma_m), \gamma_i \in \left[\underline{\gamma}_i, \overline{\gamma}_i\right], \forall i\right\}$$

and $\overline{co}(\mathcal{A})$ denotes the closed convex hull of the set $\mathcal{A}$. From the above description we have that the set $\mathcal{M}$ is convex and compact, the proposition below sets conditions for the matrices in the set $\mathcal{M}$ to be nonsingular, thus guaranteeing that the solution to the algebraic loop exists and is unique.

Proposition 1: Given a matrix $D \in \mathbb{R}^{m\times m}$, if there exists a matrix $W \in \mathbb{D}^m_{\geq 0}$ such that $2W - W(I - D\underline{\Gamma})^{-1}D(\overline{\Gamma} - \underline{\Gamma}) - (\overline{\Gamma} - \underline{\Gamma})D^T W((I - D\underline{\Gamma})^{-1})^T > 0$ then $I - D\Gamma$ is nonsingular for all matrices Γ belonging to the set $\mathcal{G}$.

Proof: If $(I - D\Gamma)$ is singular then there exists $z \in \mathbb{R}^m$, $z \neq 0$ such that $0 = (I - D\Gamma)z = ((I - D\underline{\Gamma}) - D(\Gamma - \underline{\Gamma}))z = (I - D\underline{\Gamma})z - D(\overline{\Gamma} - \underline{\Gamma})(\overline{\Gamma} - \underline{\Gamma})^{-1}(\Gamma - \underline{\Gamma}))z$. Define $\bar{z} = (\overline{\Gamma} - \underline{\Gamma})^{-1}(\Gamma - \underline{\Gamma}))z$ to obtain

$$(I - D\underline{\Gamma})\left[z - (I - D\underline{\Gamma})^{-1}D(\overline{\Gamma} - \underline{\Gamma})\bar{z}\right] = 0.$$

Multiply the above expression on the left by $\bar{z}^T W(I - D\underline{\Gamma})^{-1}$, to obtain

$$\bar{z}^T W z - \bar{z}^T W(I - D\underline{\Gamma})^{-1}D(\overline{\Gamma} - \underline{\Gamma})\bar{z} = 0.$$

Since for $\underline{\gamma}_i \leq \gamma_i \leq \overline{\gamma}_i$, $1 \geq (\overline{\gamma}_i - \underline{\gamma}_i)^{-1}(\gamma_i - \underline{\gamma}_i) \geq 0$ we have $\bar{z}^T W z = z^T(\overline{\Gamma} - \underline{\Gamma})^{-1}(\Gamma - \underline{\Gamma})Wz \geq z^T(\overline{\Gamma} - \underline{\Gamma})^{-2}(\Gamma - \underline{\Gamma})^2 Wz = \bar{z}^T W\bar{z}$. Thus, if $(I - D\Gamma)$ is singular we must have

$$\bar{z}^T \left(W - \frac{1}{2}He(W(I - D\underline{\Gamma})^{-1}D(\overline{\Gamma} - \underline{\Gamma}))\right)\bar{z} \leq 0,$$

which contradicts the inequality of the claim. Hence if the inequality in the claim holds the matrix $(I - D\Gamma)$ is non-singular for any $\Gamma \in \mathcal{G}$. ■

References

[1] M. Aizerman and F. Gantmacher, *Absolute stability of regulator systems*. Holden-Day. Oakland, CA, 1964.

[2] J. Carrasco, M. Maya-Gonzalez, A. Lanzon, and W. Heath, “LMI searches for anticausal and noncausal rational Zames–Falb multipliers,” *Systems & Control Letters*, vol. 70, pp. 17–22, 2014.

[3] J. Carrasco, M. Turner, and W. Heath, “Zames-Falb multipliers for absolute stability: From O’Shea’s contribution to convex searches,” *European Journal of Control*, vol. 28, pp. 1 – 19, 2016.

[4] D. Dai, T. Hu, A. R. Teel, and L. Zaccarian, “Piecewise-quadratic Lyapunov functions for systems with deadzones or saturations,” *Systems & Control Letters*, vol. 58, no. 5, pp. 365–371, 2009.

[5] F. J. D’amato, M. A. Rotea, A. V. Megretski, and U. T. Jönsson, “New results for analysis of systems with repeated nonlinearities,” *Automatica*, vol. 37, no. 5, pp. 739–747, 2001.

[6] A. Dewey, “On the stability of feedback systems with one differentiable nonlinear element,” *IEEE Transactions on Automatic Control*, vol. 11, no. 3, pp. 485–491, 1966.

$$\tilde{\Lambda} \geq 0,\ \Lambda \geq -\tilde{\Lambda},\ T_i \geq 0,\ i = 0, \dots, 4, T_{c,j} \geq 0,\ j = 0, \dots, m, \tag{22a}$$

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} - \frac{1}{2}\begin{bmatrix} C^T \\ D^T \end{bmatrix}(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}\begin{bmatrix} C & D \end{bmatrix} + He\left(\frac{1}{2}\begin{bmatrix} (\underline{\Delta}C)^T \\ (\underline{\Delta}D^T - I_m)^T \end{bmatrix} T_0 \begin{bmatrix} \overline{\Delta}C & (\overline{\Delta}D^T - I_m) \end{bmatrix}\right) > 0, \tag{22b}$$

$$\begin{aligned} -He\Bigg(&\left(\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -(\underline{\Delta}C)^T \\ (I - \underline{\Delta}D)^T \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \Lambda C & \Lambda D \end{bmatrix}\right)\begin{bmatrix} A & 0 & 0 & B & B_w \\ 0 & 0 & I_m & 0 & 0 \end{bmatrix}\right. \\ &+ \frac{1}{2}\begin{bmatrix} (\underline{\Delta}C)^T \\ (\underline{\Delta}D - I_m)^T \\ 0 \\ 0 \\ 0 \end{bmatrix} T_1 \begin{bmatrix} \overline{\Delta}C & (\overline{\Delta}D - I_m) & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} (\underline{\Delta}C)^T \\ 0 \\ 0 \\ (\underline{\Delta}D - I_m)^T \\ (\underline{\Delta}D_w)^T \end{bmatrix} T_2 \begin{bmatrix} \overline{\Delta}C & 0 & 0 & (\overline{\Delta}D - I_m) & \overline{\Delta}D_w \end{bmatrix} \\ &+ \frac{1}{2}\begin{bmatrix} A^T & 0 \\ 0 & 0 \\ 0 & I_m \\ B^T & 0 \\ B_w^T & 0 \end{bmatrix}\begin{bmatrix} (\underline{\Gamma}C)^T \\ (\underline{\Gamma}D - I_m)^T \end{bmatrix} T_3 \begin{bmatrix} \overline{\Gamma}C & (\overline{\Gamma}D - I_m) \end{bmatrix}\begin{bmatrix} A & 0 & 0 & B & B_w \\ 0 & 0 & I_m & 0 & 0 \end{bmatrix} \\ &+ \frac{1}{2}\begin{bmatrix} 0 \\ (I - \underline{\Gamma}D) \\ 0 \\ -(I - \underline{\Gamma}D) \\ \underline{\Gamma}D_w \end{bmatrix} T_4 \begin{bmatrix} 0 & (I - \overline{\Gamma}D) & 0 & -(I - \overline{\Gamma}D) & \overline{\Gamma}D_w \end{bmatrix} + M_\Psi\Bigg) > 0, \end{aligned} \tag{22c}$$

$$\begin{aligned} &\begin{bmatrix} -(\underline{\tilde{y}}_j\overline{\tilde{y}}_j + \rho) & \frac{\underline{\tilde{y}}_j + \overline{\tilde{y}}_j}{2}C_j & \frac{\underline{\tilde{y}}_j + \overline{\tilde{y}}_j}{2}D_j \\ \frac{\underline{\tilde{y}}_j + \overline{\tilde{y}}_j}{2}C_j^T & -C_j^T C_j + P_{11} - \frac{1}{2}C^T(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}C & -C_j^T D_j + P_{12} - \frac{1}{2}C^T(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}D \\ \frac{\underline{\tilde{y}}_j + \overline{\tilde{y}}_j}{2}D_j^T & -D_j^T C_j + P_{12}^T - \frac{1}{2}D^T(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}C & -D_j^T D_j + P_{22} - \frac{1}{2}D^T(\overline{\Delta} - \underline{\Delta})\tilde{\Lambda}D \end{bmatrix} \\ &\qquad + He\left(\frac{1}{2}\begin{bmatrix} 0 \\ (\underline{\Delta}C)^T \\ (\underline{\Delta}D^T - I_m)^T \end{bmatrix} T_{c,j} \begin{bmatrix} 0 & \overline{\Delta}C & (\overline{\Delta}D^T - I_m) \end{bmatrix}\right) \geq 0 \quad j = 1, \dots, m. \end{aligned} \tag{22d}$$

[7] R. Drummond, D. Howey, and S. Duncan, "Low-order mathematical modelling of electric double layer supercapacitors using spectral methods," *Journal of Power Sources*, vol. 277, pp. 317–328, 2015.

[8] M. Fetzer and C. W. Scherer, "Zames-Falb multipliers for invariance," *IEEE Control Systems Letters*, vol. 1, no. 2, pp. 412–417, 2017.

[9] A. Garulli, A. Masi, G. Valmorbida, and L. Zaccarian, "Global stability and finite $\mathcal{L}_{2m}$-gain of saturated uncertain systems via piecewise polynomial Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 58, no. 1, pp. 242–246, Jan 2013.

[10] A. Halanay and V. Rasvan, "Absolute stability of feedback systems with several differentiable non-linearities," *International Journal of Systems Science*, vol. 22, no. 10, pp. 1911–1927, 1991.

[11] J. Hauser and B. Bamieh, "Dynamics of a driven Stirling engine," in *Proceedings of the 54th IEEE Conference on Decision and Control*. Osaka, Japan: IEEE, 2015, pp. 3730–3735.

[12] W. Heath and G. Li, "Lyapunov functions for the multivariable Popov criterion with indefinite multipliers," *Automatica*, vol. 45, no. 12, pp. 2977–2981, 2009.

[13] H. Hindi and S. Boyd, "Analysis of linear systems with saturation using convex optimization," in *Proc. of the 37th IEEE Conference on Decision and Control*, vol. 1. Tampa, FL: IEEE, 1998, pp. 903–908.

[14] T. Hu, B. Huang, and Z. Lin, "Absolute stability with a generalized sector condition," *IEEE Transactions on Automatic Control*, vol. 49, no. 4, pp. 535–548, April 2004.

[15] U. Jönsson, "Stability analysis with Popov multipliers and integral quadratic constraints," *Systems & Control Letters*, vol. 31, no. 2, pp. 85 – 92, 1997.

[16] R. Josselson and G. V. S. Raju, "Absolute stability of control systems with many sector and slope-restricted nonlinearities," *International Journal of Control*, vol. 19, no. 3, pp. 609–614, 1974.

[17] H. K. Khalil, *Nonlinear systems.* Prentice-Hall. Upper Saddle River, NJ, 2002, vol. 3.

[18] R. O'Shea, "An improved frequency-time domain stability criterion for autonomous continuous systems," *IEEE Transactions on Automatic Control*, vol. 12, no. 6, pp. 725–731, 1967.

[19] P. Park, "A revisited Popov criterion for nonlinear Lur'e systems with sector-restrictions," *International Journal of Control*, vol. 68, no. 3, pp. 461–470, 1997.

[20] ——, "Stability criteria of sector- and slope-restricted Lur'e systems," *IEEE Transactions on Automatic Control*,

vol. 47, no. 2, pp. 308–313, Feb 2002.

[21] M. Safonov, “Computer-aided stability analysis renders Popov criterion obsolete,” *IEEE Transactions on Automatic Control*, vol. 32, no. 12, pp. 1128–1131, 1987.

[22] M. G. Safonov and V. V. Kulkarni, “Zames-Falb multipliers for MIMO nonlinearities,” *International Journal of Robust and Nonlinear Control*, vol. 10, no. 11-12, pp. 1025–1038, 2000.

[23] V. R. Sastry, “Finite regions of attraction for the problem of Lur'e,” *International Journal of Control*, vol. 14, no. 4, pp. 789–790, 1971.

[24] J. Suykens, J. Vandewalle, and B. De Moor, “An absolute stability criterion for the Lur'e problem with sector and slope restricted nonlinearities,” *IEEE Transactions on Circuits and Systems. Part I: Fundamental Theory and Applications*, vol. 45, pp. 1007–1009, 1998.

[25] M. C. Turner and M. Kerr, “Lyapunov functions and $\mathcal{L}_2$ gain bounds for systems with slope restricted nonlinearities,” *Systems & Control Letters*, vol. 69, pp. 1–6, 2014.

[26] G. Valmorbida, R. Drummond, and S. R. Duncan, “Positivity conditions of Lyapunov functions for systems with slope restricted nonlinearities,” in *Proc. of the 2016 IEEE American Control Conference*. Boston, MA: IEEE, 2016, pp. 258–263.

[27] J. A. Walker and N. H. McClamroch, “Finite regions of attraction for the problem of Lur'e,” *International Journal of Control*, vol. 6, no. 4, pp. 331–336, 1967.

[28] S. Weissenberger, “Application of results from the absolute stability problem to the computation of finite stability domains,” *IEEE Transactions on Automatic Control*, vol. 13, no. 1, pp. 124–125, Feb 1968.

[29] V. A. Yakubovich, “The method of matrix inequalities in the stability theory of nonlinear control systems: II. Absolute stability in a class of nonlinearities with a condition on the derivative,” *Avtom. i Telemech.*, vol. 26, pp. 577–592, 1965.

[30] L. Zaccarian and A. R. Teel, “A common framework for anti-windup, bumpless transfer and reliable designs,” *Automatica*, vol. 38, no. 10, pp. 1735 – 1744, 2002.

[31] G. Zames and P. Falb, “Stability conditions for systems with monotone and slope-restricted nonlinearities,” *SIAM Journal on Control*, vol. 6, no. 1, pp. 89–108, 1968.

[32] F. Zhang, H. L. Trentelman, and J. M. A. Scherpen, “Dynamic feedback synchronization of Lur'e networks via incremental sector boundedness,” *IEEE Transactions on Automatic Control*, vol. 61, no. 9, pp. 2579–2584, 2016.