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Stability of Regularized Hastings–Levitov Aggregation in the Subcritical Regime

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Abstract: We prove bulk scaling limits and fluctuation scaling limits for a two-parameter class ALE(α, η) of continuum planar aggregation models. The class includes regularized versions of the Hastings–Levitov family HL(α) and continuum versions of the family of dielectric-breakdown models, where the local attachment intensity for new particles is specified as a negative power $-\eta$ of the density of arc length with respect to harmonic measure. The limit dynamics follow solutions of a certain Loewner–Kufarev equation, where the driving measure is made to depend on the solution and on the parameter $\zeta = \alpha + \eta$. Our results are subject to a subcriticality condition $\zeta \leqslant 1$: this includes HL(α) for $\alpha \leqslant 1$ and also the case $\alpha = 2$, $\eta = -1$ corresponding to a continuum Eden model. Hastings and Levitov predicted a change in behaviour for HL(α) at $\alpha = 1$, consistent with our results. In the regularized regime considered, the fluctuations around the scaling limit are shown to be Gaussian, with independent Ornstein–Uhlenbeck processes driving each Fourier mode, which are seen to be stable if and only if $\zeta \leqslant 1$.

Contents

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1. Introduction

1.1. Hastings–Levitov aggregation. In many physical contexts there appear clusters whose shape is complex, formed apparently by some mechanism of random growth. It has long been a challenge to account for the observed variety of complex cluster shapes, starting from plausible physical principles governing the aggregation of individual microscopic particles. For clusters which are essentially two-dimensional, there is an approach introduced by Carleson and Makarov [4] and Hastings and Levitov [10], in which clusters are encoded as a composition of conformal maps, one for each particle. In this approach, a growing cluster is modelled by an increasing sequence of compact sets $K_n \subseteq \mathbb{C}$ which are assumed to be simply connected. We will take the initial set K_0 to be the closed unit disk $\{|z| \leqslant 1\}$. The increments $K_n \setminus K_{n-1}$ are then thought of as a sequence of particles added to the cluster. The idea is to study the clusters K_n via the conformal isomorphisms

$$\Phi_n : D_0 \to D_n$$

where D_n is the complementary domain $\mathbb{C} \setminus K_n$ and Φ_n is normalized by $\Phi_n(\infty) = \infty$ and $\Phi_n'(\infty) > 0$. Then $\Phi_0(z) = z$ for all z and K_n has logarithmic capacity $\Phi_n'(\infty) > 1$ for all $n \geqslant 1$. This formulation is convenient because the harmonic measure from ∞ on the boundary ∂D_n, which provides a natural way to choose the location of the next particle, is then simply the image under Φ_n of the uniform distribution on $\partial D_0 = \{|z| = 1\}$. Having chosen a random angle Θ_{n+1} to locate the next particle, and a model particle P_{n+1} attached to K_0 at $e^{i\Theta_{n+1}}$, for example a small disk tangent to K_0, the cluster map is updated to

$$\Phi_{n+1} = \Phi_n \circ F_{n+1} \tag{1}$$

where F_{n+1} is the conformal isomorphism $D_0 \to D_0 \setminus P_{n+1}$, normalized similarly to Φ_n. Then Φ_{n+1} encodes the cluster

$$K_{n+1} = K_n \cup \Phi_n(P_{n+1}).$$

Thus, once we specify distributions for the angles Θ_n and model particles P_n, we have specified a mechanism to grow a random cluster.

We will write

$$\mathrm{cap}(K_n) = \log \Phi'(\infty), \quad c_n = \log F_n'(\infty)$$

and we will refer to $\mathrm{cap}(K_n)$ as the capacity[1] of K_n and c_n as the capacity of P_n. Then

$$\mathrm{cap}(K_n) = c_1 + \cdots + c_n.$$

We will be looking for scaling limits where the particle capacities c_n and the associated particles P_n become small, but where n is chosen sufficiently large that the cluster capacities $\mathrm{cap}(K_n)$ grow macroscopically.

A simple case is to choose Θ_{n+1} uniformly distributed on the unit circle and to take $P_{n+1} = e^{i\Theta_{n+1}}P$, where P is a small disk tangent to the unit disk at 1, of radius $r(c)$, chosen so that P has capacity c. Then in fact $r(c)/\sqrt{c}$ has a positive limit as $c \to 0$. The location of the new particle $\Phi_n(P_{n+1})$ is then distributed according to harmonic measure on ∂K_n. However, if we assume that ∂K_n is approximately linear on the scale of P, then we would have

$$\Phi_n(P_{n+1}) \approx \Phi_n(e^{i\Theta_{n+1}}) + \Phi_n'(e^{i\Theta_{n+1}})P \tag{2}$$

so we would add an approximate disk of diameter proportional to $\sqrt{c}|\Phi_n'(e^{i\Theta_{n+1}})|$.

In order to compensate for this distortion, Hastings and Levitov proposed the HL(α) family of models where, once Θ_{n+1} is chosen, we choose P_{n+1} to be a particle of capacity

$$c_{n+1} = |\Phi_n'(e^{i\Theta_{n+1}})|^{-\alpha}c.$$

Then, in the case $\alpha = 2$, the particles added to the cluster would be approximately of constant size. The approximation (2) is in fact misleading, at least on a microscopic level, because ∂K_n develops inhomogeneities on the scale of the particles. Nevertheless, HL(2) has been considered as a variant of diffusion-limited aggregation (DLA) [28], with some justification, see [10], derived from numerical experiments.

In general, the HL(α) model offers a convenient mechanism for such experiments, and moves away from the lattice formulation of [28] which has been shown to lead to unphysical effects on large scales (see for example [8]). Moreover, it might be hoped that an evolving family of conformal maps would present a more tractable framework for the analysis of scaling limits than other growth models, while potentially sharing the same bulk scaling limit and fluctuation universality class. That is the direction explored in this paper.

Besides the mechanism of diffusive aggregation, based on harmonic measure, there is another one-parameter family of models, conceived originally in the lattice case, called dielectric breakdown models [20], which interpolates between DLA and the Eden model [7]. In the Eden model, each boundary site is chosen with equal probability. In the continuum setting, for an Eden-type model we would choose an attachment point on the boundary according to normalized arc length, which has density proportional to $|\Phi_n'(e^{i\theta})|$ with respect to harmonic measure. We can widen our family of models to include a continuum analogue of dielectric breakdown models (DBM) by choosing

$$\mathbb{P}(\Theta_{n+1} \in d\theta | \Phi_n) \propto |\Phi_n'(e^{i\theta})|^{-\eta} d\theta.$$

[1] This is an abuse of terminology since it is then $e^{\mathrm{cap}(K_n)}$ which is the logarithmic capacity.

The case $\eta = -1$ then provides a continuum variant of the Eden model.

In a law-of-large-numbers regime, it might be guessed that bulk characteristics of the cluster for the model incorporating both the α and η modifications would depend only on their sum $\zeta = \alpha + \eta$ since, once this is fixed, up to a global time-scaling, the growth rate of capacity due to particles attached near $e^{i\theta}$ does not depend further on α or η. We will show, in the regime which we can address, that this is indeed true.

In this paper we investigate the two-parameter family of models just described, but modified by the introduction of a regularization parameter $\sigma > 0$, which controls the minimum length scale over which feedback occurs through c_{n+1} and Θ_{n+1}. Specifically, we require

$$\mathbb{P}(\Theta_{n+1} \in d\theta | \Phi_n) \propto |\Phi_n'(e^{\sigma+i\theta})|^{-\eta} d\theta, \quad c_{n+1} = |\Phi_n'(e^{\sigma+i\Theta_{n+1}})|^{-\alpha} c. \tag{3}$$

This model was introduced in [27] as the (discrete-time) aggregate Loewner evolution model. We will require throughout that $\sigma \gg \sqrt{c}$ (and sometimes more) and we will restrict attention to the subcritical regime $\zeta \leqslant 1$. This includes the Eden case ($\alpha = 2$, $\eta = -1$) but excludes continuum DLA ($\alpha = 2$, $\eta = 0$). In the regularized models, we will show fluctuation behaviour which is universal over all choices of particle family. Our first main result shows that, in this regime, in the limit $c \to 0$, disks are stable, that is, an initial disk cluster remains close to a disk as particles are added and its capacity becomes large. Our second main result is to prove convergence of the normalized fluctuations of the cluster around its deterministic limit, to an explicit Gaussian process. The constraint $\zeta \leqslant 1$ appears sharp for this behaviour: we see an explicit dependence of the fluctuations on α and η and, in particular, an exponential instability of rate $(\zeta - 1)k$ in the kth Fourier mode if we formally take $\zeta > 1$.

1.2. Statement of results. In this section, we define the continuous-time ALE(α, η) model, which is our object of study, and we specify our standing assumptions for individual particles. We then state our main results.

Our model is constructed as a composition of univalent functions on the exterior unit disk $D_0 = \{|z| > 1\}$. Each of these functions corresponds to a choice of attachment angle $\theta \in [0, 2\pi)$ and a basic particle P. Recall that $K_0 = \{|z| \leqslant 1\}$. By a basic particle P we mean a non-empty subset of D_0 such that $K_0 \cup P$ is compact and simply connected. Set $D = D_0 \setminus P$. By the Riemann mapping theorem, there is a $c \in (0, \infty)$ and a conformal isomorphism $F : D_0 \to D$ with Laurent expansion of the form

$$F(z) = e^c \left(z + \sum_{k=0}^{\infty} a_k z^{-k} \right). \tag{4}$$

Then F is uniquely determined by P, and P has capacity c. Our model depends on three parameters $\alpha, \eta \in \mathbb{R}$ and $\sigma \in (0, \infty)$, along with the choice of a family of basic particles $(P^{(c)} : c \in (0, \infty))$ with $P^{(c)}$ of capacity c. The associated maps $F_c : D_0 \to D^{(c)}$ then have the form (4) with $a_k = a_k(c)$ for all k. We assume throughout that F_c extends continuously to $\{|z| \geqslant 1\}$. We require that our particle family is nested

$$P^{(c_1)} \subseteq P^{(c_2)} \quad \text{for } c_1 < c_2 \tag{5}$$

and satisfies, for some $\Lambda \in [1, \infty)$,

$$\delta(c) \leqslant \Lambda r_0(c) \quad \text{for all } c \tag{6}$$

where

$$r_0(c) = \sup\{|z| - 1 : z \in P^{(c)}\}, \quad \delta(c) = \sup\{|z-1| : z \in P^{(c)}\}.$$

In our results, only small values of c are of interest. For such c, the last condition (6) forces our particles $P^{(c)}$ to concentrate near the point 1 while never becoming too flat against the unit circle.

The following are all examples of particle families satisfying both conditions (5) and (6):

$$P^{(c)}_{\text{slit}} = (1, 1+\delta(c)], \quad P^{(c)}_{\text{bump}} = \{z \in D_0 : |z-1| \leqslant \delta(c)\}$$

and

$$P^{(c)}_{\text{disk}} = \{z \in D_0 : |z - 1 - r(c)| \leqslant r(c)\}, \quad r(c) = \delta(c)/2$$

where in each case δ is a suitable increasing homeomorphism of $(0, \infty)$.

It will be convenient to place our aggregation models from the outset in continuous time. By a (continuous-time) aggregate Loewner evolution of parameters $\alpha, \eta \in \mathbb{R}$, or ALE$(\alpha, \eta)$, we mean a finite-rate, continuous-time Markov chain $(\Phi_t)_{t\geqslant 0}$ taking values in the set of univalent functions $D_0 \to D_0$, starting from $\Phi_0(z) = z$, which, when in state ϕ, jumps to $\phi \circ F_{c(\theta,\phi),\theta}$ at rate $\lambda(\theta, \phi)d\theta/(2\pi)$, where

$$F_{c,\theta}(z) = e^{i\theta} F_c(e^{-i\theta} z), \quad c(\theta, \phi) = c|\phi'(e^{\sigma+i\theta})|^{-\alpha}, \quad \lambda(\theta, \phi) = c^{-1}|\phi'(e^{\sigma+i\theta})|^{-\eta}. \tag{7}$$

Since $\sigma > 0$, the rate $\lambda(\theta, \phi)$ is continuous in θ, so the total jump rate is finite. The model may be thought of equivalently in term of the random process of compact sets $(K_t)_{t\geqslant 0}$ given by

$$K_0 = \{|z| \leqslant 1\}, \quad K_t = K_0 \cup \{z \in D_0 : z \notin \Phi_t(D_0)\}.$$

The effect of the jump just described is then to add to the current cluster the set $\phi(e^{i\theta} P^{(c(\theta,\phi))})$ thereby increasing its capacity by $c(\theta, \phi)$.

An explicit realisation of this Markov chain can be constructed as follows. Given a univalent function $\phi : D_0 \to D_0$, define the normalising constant

$$Z_\phi = \int_0^{2\pi} |\phi'(e^{\sigma+i\theta})|^{-\eta} d\theta.$$

Starting from $\Phi_0(z) = z$, suppose that a realisation of $(\Phi_s)_{0\leqslant s\leqslant t}$ has been constructed up to some $t \geqslant 0$, and that $\Phi_t = \phi$. Sample independently a random time $T \sim \text{Exp}(c^{-1} Z_\phi/(2\pi))$ and random angle Θ with density function $|\phi'(e^{\sigma+i\theta})|^{-\eta}/Z_\phi$. Then set $\Phi_s = \phi$ for $t < s < t + T$, and $\Phi_{t+T} = \phi \circ F_{c(\Theta,\phi),\Theta}$. It is straightforward to verify that this construction gives a Markov chain with distribution corresponding to the specification above.

Denote the jump times of the Markov chain by T_k, $k = 1, 2, \ldots$. By the explicit construction,

$$\Phi_{T_{n+1}} = \Phi_{T_n} \circ F_{n+1}$$

where $F_n = F_{C_n,\Theta_n}$, for capacity C_n and attachment angle Θ_n satisfying

$$\mathbb{P}(\Theta_{n+1} \in d\theta | \Phi_{T_n}) \propto |\Phi'_{T_n}(e^{\sigma+i\theta})|^{-\eta} d\theta, \quad C_{n+1} = |\Phi'_{T_n}(e^{\sigma+i\Theta_{n+1}})|^{-\alpha} c.$$

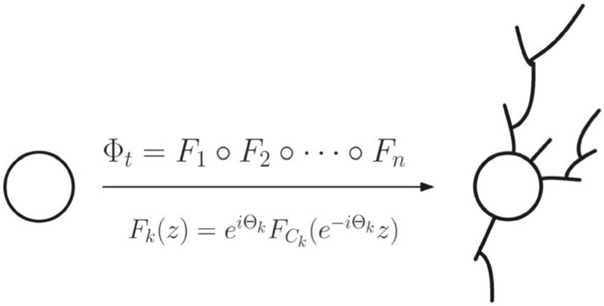

Fig. 1. Cluster map with n particles

Therefore, if $T_n \leqslant t < T_{n+1}$, we have

$$\Phi_t = F_1 \circ \cdots \circ F_n,$$

as in Fig. 1. Moreover, the capacity $\mathcal{T}_t$ of the cluster K_t is then given by

$$\mathcal{T}_t = \log \Phi_t'(\infty) = C_1 + \cdots + C_n.$$

For certain parameter values, the process $(\Phi_t)_{t \geqslant 0}$ may explode, that is, may take infinitely many jumps in a finite time interval. We will show in Proposition A.1 that explosion occurs if and only if both $\eta < 0$ and $\zeta = \alpha + \eta < 0$, and in this case we also have $\mathcal{T}_t \to \infty$ at the explosion time. This phenomenon is however irrelevant to our main results on scaling limits, since explosion is excluded by these results (with high probability) over the relevant time interval. Hence we will make no attempt to define Φ_t beyond explosion.

By reference to (1) and (3), it is immediate that the jump-chain $(\Phi_{T_n})_{n \geqslant 0}$ is exactly the discrete-time aggregate Loewner evolution process $(\Phi_n)_{n \geqslant 0}$ in the introductory discussion. In particular, in the case $\eta = \sigma = 0$, $(\Phi_t)_{t \geqslant 0}$ is the original Hastings–Levitov process embedded in continuous time as a Poisson process with jumps of rate c^{-1}. For clarity, from now on we denote the discrete-time process by $(\Phi_n^{\text{disc}})_{n \geqslant 0}$. Prior work on ALE models [21,27] was framed in terms of this discrete-time process. The continuous-time framework allows a more local specification of the dynamics, without the need to normalise the distribution of attachment angles. It further allows us to organise the computation of martingales in terms of a standard calculus for Poisson random measures.

We can now state our first main result. Define

$$t_\zeta = \begin{cases} \infty, & \text{if } \zeta \geqslant 0, \\ |\zeta|^{-1}, & \text{if } \zeta < 0. \end{cases}$$

and for $t < t_\zeta$ set

$$\tau_t = \begin{cases} t, & \text{if } \zeta = 0, \\ \zeta^{-1} \log(1 + \zeta t), & \text{otherwise.} \end{cases}$$

Note that $\tau_t \to \infty$ as $t \to t_\zeta$ for all ζ.

The result identifies the small-particle scaling limit of K_t in the case $\zeta \leqslant 1$ as a disk of radius e^{τ_t}, with quantified error estimates. It is proved in Proposition 5.7. The range

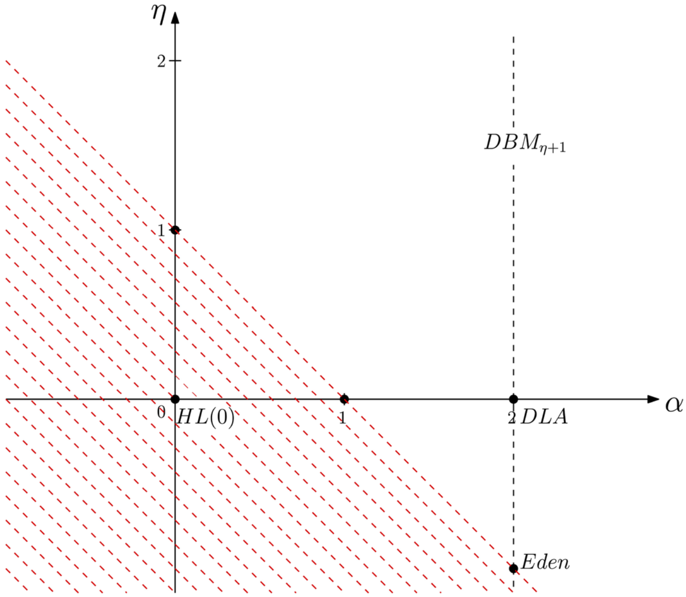

Fig. 2. Domain of stability for ALE(α, η)

of parameter values to which the result applies is indicated by the region shaded red in Fig. 2, with diagonal lines showing parameter pairs (α, η) sharing a common bulk scaling limit. Recall that $\mathcal{T}_t = \log \Phi_t'(\infty)$, which is the capacity of K_t, and set

$$\hat{\Phi}_t(z) = \Phi_t(z)/e^{\mathcal{T}_t}.$$

Theorem 1.1. *For all $\alpha, \eta \in \mathbb{R}$ with $\zeta = \alpha + \eta \leqslant 1$, for all $\varepsilon \in (0, 1/3]$ and $\nu \in (0, \varepsilon/4]$, for all $m \in \mathbb{N}$ and $T \in [0, t_\zeta)$, there is a constant $C = C(\alpha, \eta, \Lambda, \varepsilon, \nu, m, T) < \infty$ with the following property. In the case $\zeta < 1$, for all $c \leqslant 1/C$ and all $\sigma \geqslant c^{1/2-\varepsilon}$, with probability exceeding $1 - c^m$, for all $t \leqslant T$,*

$$|\mathcal{T}_t - \tau_t| \leqslant C \left(c^{1/2-\nu} + c^{1-4\nu} \left(\frac{e^\sigma}{e^\sigma - 1} \right)^2 \right)$$

and, for all $|z| = r \geqslant 1 + c^{1/2-\varepsilon}$,

$$|\hat{\Phi}_t(z) - z| \leqslant C \left(c^{1/2-\nu} + c^{1-4\nu} \left(\frac{e^\sigma}{e^\sigma - 1} \right)^2 \right).$$

Moreover, in the case $\zeta = 1$, for all $c \leqslant 1/C$ and all $\sigma \geqslant c^{1/3-\varepsilon}$, with probability exceeding $1 - c^m$, for all $t \leqslant T$,

$$|\mathcal{T}_t - \tau_t| \leqslant C \left(c^{1/2-\nu} + c^{1-4\nu} \left(\frac{e^\sigma}{e^\sigma - 1} \right)^2 \right)$$

and, for all $|z| = r \geqslant 1 + c^{1/2-\varepsilon}$,

$$|\hat{\Phi}_t(z) - z| \leqslant C \left(c^{1/2-\nu} \left(\frac{e^\sigma}{e^\sigma - 1} \right)^{1/2} + c^{1-4\nu} \left(\frac{e^\sigma}{e^\sigma - 1} \right)^{5/2} \right).$$

We will show a similar result for the discrete-time process $(\Phi_n^{\text{disc}})_{n\geqslant 0}$. Set

$$\mathcal{T}_n^{\text{disc}} = \log(\Phi_n^{\text{disc}})'(\infty), \quad \hat{\Phi}_n^{\text{disc}}(z) = \Phi_n^{\text{disc}}(z)/e^{\mathcal{T}_n^{\text{disc}}}.$$

Define

$$n_\alpha = \begin{cases} \infty, & \text{if } \alpha \geqslant 0, \\ |\alpha|^{-1}, & \text{if } \alpha < 0 \end{cases}$$

and for $n < n_\alpha/c$ set

$$\tau_n^{\text{disc}} = \begin{cases} cn, & \text{if } \alpha = 0, \\ \alpha^{-1}\log(1 + \alpha c n), & \text{otherwise.} \end{cases} \tag{8}$$

The following result is proved at the end of Sect. 5.2. The case $\alpha = 0$ is Theorem 1.1 in [21] but with an improvement to the constraints on r and σ, and the corresponding upper bound, in the $\eta = 1$ case.

Theorem 1.2. *For all* $\alpha, \eta \in \mathbb{R}$ *with* $\zeta = \alpha + \eta \leqslant 1$, *for all* $\varepsilon \in (0, 1/3]$ *and* $\nu \in (0, \varepsilon/4]$, *for all* $m \in \mathbb{N}$ *and* $N \in [0, n_\alpha)$, *not necessarily an integer, there is a constant* $C = C(\alpha, \eta, \Lambda, \varepsilon, \nu, m, N) < \infty$ *with the following property. In the case* $\zeta < 1$, *for all* $c \leqslant 1/C$ *and all* $\sigma \geqslant c^{1/2-\varepsilon}$, *with probability exceeding* $1 - c^m$, *for all* $n \leqslant N/c$,

$$|\mathcal{T}_n^{\text{disc}} - \tau_n^{\text{disc}}| \leqslant C c^{1-4\nu} \left(\frac{e^\sigma}{e^\sigma - 1} \right)^2$$

and, for all $|z| = r \geqslant 1 + c^{1/2-\varepsilon}$,

$$|\hat{\Phi}_n^{\text{disc}}(z) - z| \leqslant C \left(c^{1/2-\nu} + c^{1-4\nu} \left(\frac{e^\sigma}{e^\sigma - 1} \right)^2 \right).$$

Moreover, in the case $\zeta = 1$, *for all* $c \leqslant 1/C$ *and all* $\sigma \geqslant c^{1/3-\varepsilon}$, *with probability exceeding* $1 - c^m$, *for all* $n \leqslant N/c$,

$$|\mathcal{T}_n^{\text{disc}} - \tau_n^{\text{disc}}| \leqslant C c^{1-4\nu} \left(\frac{e^\sigma}{e^\sigma - 1} \right)^2$$

and, for all $|z| = r \geqslant 1 + c^{1/2-\varepsilon}$,

$$|\hat{\Phi}_n^{\text{disc}}(z) - z| \leqslant C \left(c^{1/2-\nu} \left(\frac{e^\sigma}{e^\sigma - 1} \right)^{1/2} + c^{1-4\nu} \left(\frac{e^\sigma}{e^\sigma - 1} \right)^{5/2} \right).$$

We turn to our second main result, which describes the limiting fluctuations of ALE(α, η). Denote by $\mathcal{H}$ the set of all holomorphic functions on $D_0 = \{|z| > 1\}$ which are bounded at ∞. We equip $\mathcal{H}$ with the metric

$$d(f, g) = \sum_{n=0}^{\infty} 2^{-n} \left(\sup_{|z| \geqslant 1+1/n} |f(z) - g(z)| \wedge 1 \right).$$

Then $\mathcal{H}$ is a complete separable metric space. Define for $t < t_\zeta$

$$\hat{\Psi}_t(z) = \hat{\Phi}_t(z) - z, \quad \Psi_t^{\mathrm{cap}} = \mathcal{T}_t - \tau_t.$$

Let $(B_t)_{t \geqslant 0}$ be a (real) Brownian motion. Let $(B_t(k))_{t \geqslant 0}$ for $k \geqslant 0$ be a sequence of independent complex Brownian motions, independent of $(B_t)_{t \geqslant 0}$. We can define continuous Gaussian processes $(\Gamma_t(k))_{t<t_\zeta}$ and $(\Gamma_t^{\mathrm{cap}})_{t<t_\zeta}$ by the following Ornstein–Uhlenbeck-type stochastic differential equations

$$d\Gamma_t(k) = e^{-\alpha\tau_t} \left(\sqrt{2} e^{-\eta\tau_t/2} dB_t(k) - (1 + (1-\zeta)k)\Gamma_t(k) e^{-\eta\tau_t} dt \right), \quad \Gamma_0(k) = 0,$$
$$d\Gamma_t^{\mathrm{cap}} = e^{-\alpha\tau_t} \left(e^{-\eta\tau_t/2} dB_t - \zeta \Gamma_t^{\mathrm{cap}} e^{-\eta\tau_t} dt \right), \quad \Gamma_0^{\mathrm{cap}} = 0.$$

We show in Sect. 6.2 that the series

$$\hat{\Gamma}_t(z) = \sum_{k=0}^{\infty} \Gamma_t(k) z^{-k}$$

converges in $\mathcal{H}$, uniformly on compacts in $[0, t_\zeta)$, almost surely. In fact $(\hat{\Gamma}_t)_{t<t_\zeta}$ satisfies the following stochastic differential equation in $\mathcal{H}$

$$d\hat{\Gamma}_t = e^{-\alpha\tau_t} \left(\sqrt{2} e^{-\eta\tau_t/2} d\hat{B}_t - (Q_0 + 1)\hat{\Gamma}_t e^{-\eta\tau_t} dt \right), \quad \hat{\Gamma}_0 = 0$$

where $Q_0 f(z) = -(1-\zeta) z f'(z)$ and

$$\hat{B}_t(z) = \sum_{k=0}^{\infty} B_t(k) z^{-k}.$$

The following two results are proved in Sect. 6.

Theorem 1.3. *Assume that* $\zeta = \alpha + \eta \in (-\infty, 1]$. *Fix* $T \in [0, t_\zeta)$ *and* $\varepsilon > 0$ *and consider the limit* $c \to 0$ *with* $\sigma \to 0$ *subject to the constraint*

$$\sigma \geqslant \begin{cases} c^{1/4-\varepsilon}, & \text{if } \zeta < 1, \\ c^{1/5-\varepsilon}, & \text{if } \zeta = 1. \end{cases}$$

Then

$$c^{-1/2}(\hat{\Psi}_t, \Psi_t^{\mathrm{cap}})_{t \leqslant T} \to (\hat{\Gamma}_t, \Gamma_t^{\mathrm{cap}})_{t \leqslant T}$$

weakly in the Skorokhod space $D([0, T], \mathcal{H} \times \mathbb{R})$.

As in the bulk scaling limit, we can deduce an analogous discrete-time fluctuation theorem. The case $\alpha = 0$ recovers Theorem 1.2 in [21]. Define for $t \geqslant 0$

$$\hat{\Psi}_t^{\text{disc}}(z) = \hat{\Phi}_{\lfloor t \rfloor}^{\text{disc}}(z) - z.$$

We have seen already in Theorem 1.2, for $N < n_\alpha$, that $(\mathcal{T}_n^{\text{disc}} - \tau_n^{\text{disc}})_{n \leqslant N/c}$ does not fluctuate at scale $\sqrt{c}$. We can define a continuous Gaussian process $(\hat{\Gamma}_t^{\text{disc}})_{t<n_\alpha}$ in $\mathcal{H}$ by

$$d\hat{\Gamma}_t^{\text{disc}} = \frac{\sqrt{2}d\hat{B}_t - (Q_0 + 1)\hat{\Gamma}_t^{\text{disc}}dt}{1 + \alpha t}, \quad \hat{\Gamma}_0^{\text{disc}} = 0.$$

Theorem 1.4. *Assume that $\zeta = \alpha + \eta \in (-\infty, 1]$. Fix $N \in [0, n_\alpha)$, not necessarily an integer, and fix $\varepsilon > 0$. In the limit $c \to 0$ with $\sigma \to 0$ considered in Theorem 1.3, we have*

$$c^{-1/2}(\hat{\Psi}_{t/c}^{\text{disc}})_{t \leqslant N} \to (\hat{\Gamma}_t^{\text{disc}})_{t \leqslant N}$$

weakly in $D([0, N], \mathcal{H})$.

1.3. Commentary and review of related work. Hastings and Levitov [10] introduced the family of planar aggregation models HL(α), which are the cases $\eta = \sigma = 0$ of our ALE(α, η) model. They discovered by numerical experiments that, for small particles, the models underwent a transition at $\alpha = 1$: for $\alpha \leqslant 1$ the cluster grows like a disk, while for $\alpha > 1$ it exhibits fractal properties. There are two natural scaling-limit regimes under which mathematical results have been established: capacity rescaling and the small-particle limit. Under capacity-rescaling, the particle capacity parameter c is kept fixed, and the cluster is rescaled to have logarithmic capacity 1, before the limit is taken as the number of particles goes to infinity. This corresponds to studying the limit of the map $\hat{\Phi}_n^{\text{disc}}(z)$ as $n \to \infty$. Under the small-particle limit, the parameter $c \to 0$, but the rate at which particles arrive is increased to ensure a non-trivial limit. This is the regime followed in the present paper, and corresponds to studying the limit of the process $(\Phi_{n(t)}^{\text{disc}}(z))_{t \geqslant 0}$ as $c \to 0$, where $n(t)$ is a suitable embedding of arrival times into continuous time. In most results to date, the embedding $n(t) = \lfloor t/c \rfloor$ has been used.

The HL(0) model is the most mathematically tractable model in the Hastings–Levitov family as in this case the particle maps, F_n, are i.i.d. It has been investigated rigorously in a series of works [24] (existence of a bulk scaling limit under capacity rescaling), [22] (bulk small-particle scaling limit), [26] (fluctuation small-particle scaling limit). Several variants exist, for example [1,2] (versions of HL(0) grown in the upper half-plane) and [14,17] (anisotropic versions of HL(0)). The σ-regularized variant of HL(α) was proposed in [15], where it was shown for slit maps that, if $\sigma \gg (\log(1/c))^{-1/2}$, there is disk-like behaviour in the small-particle limit for all $\alpha \geqslant 0$: it appeared that the observed fractal properties of HL(α) for $\alpha > 1$ were suppressed by strong regularization. In contrast, for the weaker regularization used in the present paper, the conjectured phase transition at $\alpha = 1$ (or $\zeta = 1$) becomes visible at the level of fluctuations. The method of [15] used a comparison with an HL(0)-type model which breaks down for smaller values of σ. Regularized versions of HL(α) under capacity-rescaling are considered in [24] (estimates for the dimension) and [16] (fluctuation limit when $0 < \alpha < 2$ and $\sigma = \infty$).

The regularized ALE(α, η) model first appeared in [27] where it was shown that, for slit maps, if $\alpha \geqslant 0$ and $\eta > 1$, σ-regularized ALE(α, η) converges to a growing slit in the small-particle limit, provided $\sigma \to 0$ sufficiently fast as $c \to 0$. This result is a consequence of the singularities of the derivative of the slit map on the cluster boundary, which causes the cluster growth to concentrate at the tips of particles. Similar degeneracies are exploited in two recent papers [12,13]. In [12] it was shown that, when $\eta < -2$, ALE$(0, \eta)$ converges to a SLE$_4$ curve. It is conjectured that, by making appropriate choices of particle shape, one can get convergence to SLE$_\kappa$ for any $\kappa \geqslant 4$. In [13], it is shown that ALE$(0, \eta)$, initiated from a needle-like configuration, converges to a Laplacian-path model [5]. Another model that fits into this framework is Quantum Loewner Evolution (QLE) [19]. The paper [27] contains a comprehensive discussion of connections between these and related models, so we do not repeat this here.

A new approach was begun in [21], treating regularized ALE$(0, \eta)$ as a Markov chain in univalent functions. By martingale arguments, a bulk small-particle scaling limit and fluctuation scaling limit were shown, subject to the constraint $\eta \leqslant 1$ and to restrictions on σ as a fractional power of c. These limits (in contrast to those above) turn out not to depend on the details of individual particle shapes. In this paper, we extend the analysis of [21] to ALE(α, η), subject now to the constraint $\zeta = \alpha + \eta \leqslant 1$. Thus we now include regularized HL(α) for $\alpha \leqslant 1$. Hastings and Levitov had argued that there should be a trade-off between α and η, with only ζ affecting the bulk scaling limit, and on this basis proposed HL(1), that is ALE$(1, 0)$, as a continuum variant of the Eden model. A more direct continuum analogue of the Eden model is ALE$(2, -1)$. Our results, in the regularized case, both justify the trade-off argument and show a disk scaling limit whenever $\zeta \leqslant 1$. On the other hand, we show that ALE$(1, 0)$ and ALE$(2, -1)$ have different fluctuation behaviour. As in [21], the behaviour of fluctuations as a function of ζ is consistent with the conjectured transition in behaviour at $\zeta = 1$. We emphasise that scaling limits for the conjectured supercritical regime $\zeta > 1$ lie outside the scope of the present paper.

Hastings and Levitov [10] identify a Loewner–Kufarev-type equation, which they propose as governing the small-particle limit of HL(α), citing a discussion of Shraiman and Bensimon [25] for the Hele–Shaw flow, where α is taken to be 2. This is the LK(α) equation, which is the subject of the next section. As noted by Sola in a contribution to [18], there is a lack of mathematical theory for the LK(α) equation, except in the case $\alpha = 2$ when some special techniques become available (we refer the reader to [11] and to the monograph [9] which contains an extensive list of references). In this paper, since our focus is on clusters initiated as a disk, we are able to use an explicit solution of the equation, along with its linearization around that solution, so we do not rely on a general theory. However, the particle interpretation established here offers some evidence that for $\alpha \leqslant 1$, the LK(α) equation may have a suitable existence, uniqueness and stability theory, and that it may be possible to derive the equation as a limit of particle models.

Our results depend on constraints on the regularization parameter σ, though substantially weaker ones than those used in [15]. These constraints limit the interactions of individual particles and place us in the simplest case of Gaussian fluctuations. At a technical level, for Theorem 1.1, these constraints come from the need to have $\bar{\delta}(e^\sigma) \leqslant c^\varepsilon$ in Proposition 5.3, while for Theorem 1.3 they are needed to show that the Poisson integral process $(\Pi_t)_{t \geqslant 0}$ is a good approximation to the fluctuations in Proposition 5.7. In the case $\zeta = 1$, the regularizing operator Q obtained by linearization of the LK(ζ) equation collapses from a fixed multiple of the Cauchy operator to σ times the second derivative. In general, for scaling regimes where $\sigma \to 0$ faster than our fluctuation results allow, it remains possible that ALE(α, η) has different universal fluctuation behaviour, such as KPZ, as has been conjectured for the lattice Eden model.

1.4. Structure of the paper. In the next section, we discuss the Loewner–Kufarev equation for the limit dynamics. Then, in Sect. 3, we derive an interpolation formula between ALE(α, η) and solutions of the limit equation. The terms in this formula are estimated in Sect. 4. Equipped with these estimates, we show the bulk scaling limit in Sect. 5 and the fluctuation scaling limit in Sect. 6. We collect in Appendix A some further estimates needed in the course of the paper, including estimates on the conformal maps which encode single particles and particle families.

2. Loewner–Kufarev Equation

Let $\mathcal{S}$ denote the set of univalent holomorphic functions ϕ on $\{|z| > 1\}$ with $\phi(\infty) = \infty$ and $\phi'(\infty) \in (0, \infty)$. Then each $\phi \in \mathcal{S}$ has the form

$$\phi(z) = e^c \left(z + \sum_{k=0}^{\infty} a_k z^{-k} \right)$$

for some $c \in \mathbb{R}$ and some sequence $(a_k : k \geqslant 0)$ in $\mathbb{C}$. Fix parameters $\zeta \in \mathbb{R}$ and $\sigma \geqslant 0$. Given $\phi_0 \in \mathcal{S}$, consider the following Cauchy problem for $(\phi_t)_{t \geqslant 0}$ in $\mathcal{S}$

$$\dot{\phi}_t = a(\phi_t) \tag{9}$$

where

$$a(\phi)(z) = z\phi'(z) \fint_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} \left| \phi'(e^{\sigma + i\theta}) \right|^{-\zeta} d\theta.$$

The case $\sigma = 0$ of this equation is the equation proposed by Hastings and Levitov as scaling limit for HL(ζ), which we will call the LK(ζ) equation. When $\zeta = 0$, the value of σ is immaterial and there is a unique solution given by

$$\phi_t(z) = \phi_0(e^t z).$$

When $\zeta = 2$ and $\sigma = 0$, (9) is the Loewner–Kufarev equation associated to the Hele–Shaw flow. For $\sigma > 0$, we will refer to (9) as the σ-regularized LK(ζ) equation. We will be interested in the subcritical case $\zeta \in (-\infty, 1]$.

The general form of the Loewner–Kufarev equation is given by

$$\dot{\phi}_t(z) = z\phi_t'(z) \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} \mu_t(d\theta)$$

with $(\mu_t : t \geqslant 0)$ a given family of measures on $[0, 2\pi)$. Thus the σ-regularized LK(ζ) equation is obtained by requiring that the driving measures are given by

$$\mu_t(d\theta) = \left| \phi_t'(e^{\sigma + i\theta}) \right|^{-\zeta} d\theta / (2\pi).$$

Note that, when $\zeta = \alpha + \eta$, the density of these driving measures is the product of the density of the local attachment rate and the local particle capacity (7) for ALE(α, η). By the Loewner–Kufarev theory, for any solution $(\phi_t)_{t\geqslant 0}$ of (9), the sets

$$K_t = \mathbb{C} \setminus \{\phi_t(z) : |z| > 1\}$$

form an increasing family of simply-connected compacts, with capacities given by

$$\tau_t = \operatorname{cap}(K_t) = \log \phi_t'(\infty) = \log \phi_0'(\infty) + \int_0^t \mu_s([0, 2\pi))ds.$$

2.1. Linearization. We compute the linearization of (9) around a solution $(\phi_t)_{t\geqslant 0}$. For ψ holomorphic in $\{|z| > 1\}$, we have

$$(\nabla a(\phi)\psi)(z) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} a(\phi + \varepsilon\psi)(z) = z\psi'(z)h(z) - \zeta z\phi'(z)g(z)$$

where

$$h(z) = ⨍_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} |\phi'(e^{\sigma+i\theta})|^{-\zeta} d\theta$$

and, setting $\rho = \psi'/\phi'$,

$$g(z) = ⨍_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} |\phi'(e^{\sigma+i\theta})|^{-\zeta} \operatorname{Re} \rho(e^{\sigma+i\theta})d\theta. \tag{10}$$

Note that first-order variations in $\mathcal{S}$ have the form

$$\psi(z) = \delta z + \sum_{k=0}^{\infty} \psi_k z^{-k}, \quad \delta \in \mathbb{R}, \quad \psi_k \in \mathbb{C}.$$

The process of first-order variations $(\psi_t)_{t\geqslant 0}$ around a solution $(\phi_t)_{t\geqslant 0}$ can be expected to satisfy the linearized equation

$$\dot{\psi}_t = \nabla a(\phi_t)\psi_t.$$

2.2. Linear stability of disk solutions in the subcritical case. Fix $\tau_0 \in (0, \infty)$. A trial solution $\phi_t(z) = e^{\tau_t} z$ for (9) leads to the equation

$$\dot{\tau}_t = e^{-\zeta\tau_t}.$$

We solve to obtain $(\tau_t)_{t<t_\zeta}$, with $\tau_t \to \infty$ as $t \to t_\zeta$, given by

$$\tau_t = \begin{cases} \tau_0 + t, & \text{if } \zeta = 0, \\ \zeta^{-1} \log(e^{\zeta\tau_0} + \zeta t), & \text{otherwise} \end{cases}$$

where

$$t_\zeta = \begin{cases} \infty, & \text{if } \zeta \geqslant 0, \\ e^{\zeta\tau_0}/|\zeta|, & \text{if } \zeta < 0. \end{cases}$$

For the associated solutions $(\phi_t)_{t<t_\zeta}$, the the sets K_t form a growing family of disks. We call such a $(\phi_t)_{t<\tau_\zeta}$ a disk solution.

For disk solutions, we have $\phi_t'(z) = e^{\tau_t}$ for all z, so we can evaluate the integral (10) to obtain

$$(\nabla a(\phi_t)\psi)(z) = -Q\psi(z)\dot{\tau}_t$$

where

$$Q\psi(z) = -z\psi'(z) + \zeta z\psi'(e^\sigma z) = -D\psi(z) + \zeta e^{-\sigma} D\psi(e^\sigma z). \tag{11}$$

Here and below, we write $D\psi(z)$ for the radial derivative $z\psi'(z)$. Consider the action of Q on the set of holomorphic functions on $\{|z| > 1\}$ which are bounded at infinity. Then Q is a multiplier operator

$$Q\psi(z) = \sum_{k=0}^{\infty} q(k)\psi_k z^{-k}, \quad \psi(z) = \sum_{k=0}^{\infty} \psi_k z^{-k}$$

where

$$q(k) = k(1 - \zeta e^{-\sigma(k+1)}).$$

It is straightforward to obtain the following lower bounds. We have

$$q(k) \geqslant \begin{cases} k, & \text{if } \zeta \leqslant 0, \\ k/(1-\zeta), & \text{if } \zeta \in (0,1), \\ (1-1/e)((\sigma k^2) \wedge k), & \text{if } \zeta = 1. \end{cases}$$

Define for $\tau \geqslant 0$

$$P(\tau) = e^{-\tau Q}. \tag{12}$$

At least formally, at a disk solution, the linearized equation $\dot{\psi}_t = \nabla a(\phi_t)\psi_t$ has solution given by

$$\psi_t(z) = P(\tau_t - \tau_0)\psi_0(z). \tag{13}$$

In the case $\sigma = 0$, we have $q(k) = (1-\zeta)k$ so $P(\tau)\psi(z) = \psi(e^{(1-\zeta)\tau}z)$ for suitable ψ. Thus, if $\zeta > 1$, as for example in the Hele–Shaw case when $\zeta = 2$, we see that ψ_t is holomorphic in $\{|z| > 1\}$ only if ψ_0 extends to a holomorphic function in the larger domain $\{|z| > e^{(1-\zeta)(\tau_t-\tau_0)}\}$. On the other hand, if $\zeta \leqslant 1$, then $P(\tau_t - \tau_0)$ preserves the set of holomorphic first-order variations, so the variation ψ_t as given by (13) remains holomorphic for all t. We will show that this stability property in fact also holds whenever $\sigma \geqslant 0$ and $\zeta \leqslant 1$.

Define for $r > 1$

$$\|\psi\|_{p,r} = \left(⨍_0^{2\pi} |\psi(re^{i\theta})|^p d\theta \right)^{1/p}.$$

For a multiplier operator M, given by

$$M\psi(z) = \sum_{k=0}^{\infty} m(k)\psi_k z^{-k}, \quad \psi(z) = \sum_{k=0}^{\infty} \psi_k z^{-k}$$

let us write $A = A(M)$ for the smallest constant such that

$$|m(0)| \leqslant A, \quad \sum_{k=0}^{\infty} |m(k+1) - m(k)| \leqslant A.$$

The Marcinkiewicz multiplier theorem is recalled in Sect. A.3. This implies in particular that, for all $p \in (0, \infty)$, there is a constant $C = C(p) < \infty$ such that, for all $r > 1$, we have

$$\|M\psi\|_{p,r} \leqslant CA(M)\|\psi\|_{p,r}.$$

We use this criterion to obtain some estimates on the operators $P(\tau)$ and $DP(\tau)$ for $\tau \geqslant 0$. Note that, if $0 \leqslant m(k) \leqslant m^*$, then $A(M) \leqslant N(M)m^*$ where $N(M)$ is the number of maximal intervals of constant sign in the sequence of increments $(m(k+1) - m(k) : k \geqslant 0)$.

Lemma 2.1. *For all $n \geqslant 0$ and all $p \in (1, \infty)$, there is a constant $C = C(n, p) < \infty$ such that, for all $\sigma \geqslant 0$, for all holomorphic functions ψ on $\{|z| > 1\}$ bounded at ∞, all $\tau \geqslant 0$ and all $r > 1$, we have, for $\zeta < 1$,*

$$\|D^n P(\tau)\psi\|_{p,r} \leqslant \frac{C\|\psi\|_{p,r}}{(1-\zeta^+)^n \tau^n}$$

and for $\zeta = 1$

$$\|D^n P(\tau)\psi\|_{p,r} \leqslant \frac{Ce^{\sigma n}\|\psi\|_{p,r}}{\tau^n \wedge (\sigma\tau)^{n/2}}.$$

Proof. Consider first the case where $\zeta \leqslant 0$. We split

$$q(k) = q_1(k) + q_2(k), \quad q_1(k) = k, \quad q_2(k) = |\zeta| k e^{-\sigma(k+1)}.$$

Then, with obvious notation, $P(\tau) = P_1(\tau)P_2(\tau)$ so

$$\|D^n P(\tau)\|_{p,r} \leqslant \|D^n P_1(\tau)\|_{p,r}\|P_2(\tau)\|_{p,r}.$$

The sequence of multipliers $k^n e^{-\tau k}$ for $(-D)^n P_1(\tau)$ is bounded by $(n/\tau)^n$ and its increments change sign at most once, so $A(D^n P_1(\tau)) \leqslant 2(n/\tau)^n$. The sequence of multipliers $e^{-\tau q_1(k)}$ for $P_2(\tau)$ is bounded by 1 and its increments change sign at most once, so $A(P_2(\tau)) \leqslant 2$. Hence $\|D^n P(\tau)\|_{p,r} \leqslant C/\tau^n$ as claimed.

Consider next the case $\zeta \in (0, 1)$. We make another split

$$q(k) = q_1(k) + q_2(k), \quad q_1(k) = (1-\zeta)k, \quad q_2(k) = \zeta k(1 - e^{-\sigma(k+1)}).$$

Then $P(\tau) = P_1(\tau)P_2(\tau)$ again, where the notation now corresponds to the new split. We have $A(D^n P_1(\tau)) \leqslant 2(n/((1-\zeta)\tau))^n$ by the argument used for $D^n P_1$ in the case $\zeta \leqslant 0$. The sequence of multipliers $e^{-\tau q_1(k)}$ for $P_2(\tau)$ is bounded by 1 and is decreasing, so $A(P_2(\tau)) \leqslant 1$. Hence $\|D^n P(\tau)\|_{p,r} \leqslant C/((1-\zeta)\tau)^n$ as claimed.

Consider finally the case $\zeta = 1$. We now write

$$q(k) = \hat{q}(k) + q_3(k), \quad \hat{q}(k) = e^{-(1+\sigma)}(q_1(k) \wedge q_2(k)), \quad q_1(k) = k, \quad q_2(k) = \sigma k^2$$

and write $\hat{P}(\tau)$ for the operator with multipliers $e^{-\tau\hat{q}(k)}$ and so on. As already observed, the sequence of multipliers $k^n e^{-\tau k}$ for $(-D)^n P_1(\tau)$ is bounded by $(n/\tau)^n$ and its increments change sign at most once, so $A(D^n P_1(\tau)) \leqslant 2(n/\tau)^n$. The sequence of multipliers $k^n e^{-\tau\sigma k^2}$ for $(-D)^n P_2(\tau)$ is bounded by $(n/(\sigma\tau))^{n/2}$ and its increments also change sign at most once, so $A(DP_2(\tau)) \leqslant 2$. We use the inequality

$$|a_1 \vee b_1 - a_2 \vee b_2| \leqslant |a_1 - a_2| \vee |b_1 - b_2|$$

to deduce that $A(D\hat{P}(\tau)) \leqslant Ce^{\sigma n}/(\tau^n \wedge (\sigma\tau)^{n/2})$. Finally, it is straightforward to check that the sequence of multipliers $e^{-\tau q_3(k)}$ for $P_3(\tau)$ is bounded by 1 and decreasing, so $A(P_3(\tau)) \leqslant 1$. Hence $\|D^n P(\tau)\|_{p,r} \leqslant \|D^n \hat{P}(\tau)\|_{p,r} \|P_3(\tau)\|_{p,r} \leqslant Ce^{\sigma n}/(\tau^n \wedge (\sigma\tau)^{n/2})$ as claimed. □

2.3. Transformation to (Schlicht function, capacity) coordinates. Write $\mathcal{S}_1$ for the set of 'Schlicht functions at ∞' on $\{|z| > 1\}$, given by

$$\mathcal{S}_1 = \{\phi \in \mathcal{S} : \phi'(\infty) = 1\}.$$

It will be convenient to use coordinates $(\hat{\phi}, \tau)$ on $\mathcal{S}$, given by

$$\hat{\phi}(z) = e^{-\tau}\phi(z), \quad \tau = \log \phi'(\infty).$$

Then $\hat{\phi} \in \mathcal{S}_1$ and $\tau \in \mathbb{R}$. It is straightforward to show that, for a solution $(\phi_t)_{t\geqslant 0}$ to (9), the transformed variables $(\hat{\phi}_t, \tau_t)_{t\geqslant 0}$ satisfy

$$(\dot{\hat{\phi}}_t, \dot{\tau}_t) = b(\hat{\phi}_t, \tau_t) = (\hat{b}, b^{\text{cap}})(\hat{\phi}_t, \tau_t) \tag{14}$$

where

$$\begin{aligned} \hat{b}(\hat{\phi}, \tau)(z) &= e^{-\zeta\tau} z\hat{\phi}'(z) ⨍_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} |\hat{\phi}'(e^{\sigma+i\theta})|^{-\zeta} d\theta \\ &\quad - e^{-\zeta\tau}\hat{\phi}(z) ⨍_0^{2\pi} |\hat{\phi}'(e^{\sigma+i\theta})|^{-\zeta} d\theta, \\ b^{\text{cap}}(\hat{\phi}, \tau) &= e^{-\zeta\tau} ⨍_0^{2\pi} |\hat{\phi}'(e^{\sigma+i\theta})|^{-\zeta} d\theta. \end{aligned}$$

On linearizing (14) around a solution $(\hat{\phi}_t, \tau_t)_{t\geqslant 0}$, we obtain equations for first-order variations $(\hat{\psi}_t, \psi_t^{\text{cap}})_{t\geqslant 0}$ in the new coordinates, where now $\hat{\psi}_t$ is bounded at ∞ for all t, reflecting the normalization of $\hat{\phi}_t$. These are then related to the first-order variations $(\psi_t)_{t\geqslant 0}$ in the old coordinates by

$$\psi_t(z) = e^{\tau_t}(\hat{\psi}_t(z) + \psi_t^{\text{cap}}\hat{\phi}_t(z)).$$

For a disk solution $(\phi_t)_{t<t_\zeta}$, we have $\hat{\phi}_t(z) = z$ and $b(\hat{\phi}_t, \tau) = (0, e^{-\zeta\tau})$. The equations for first-order variations are then given by

$$\dot{\hat{\psi}}_t(z) = -(Q+1)\hat{\psi}_t(z)\dot{\tau}_t, \quad \dot{\psi}_t^{\text{cap}} = -\zeta\psi_t^{\text{cap}}\dot{\tau}_t$$

with solutions

$$\hat{\psi}_t(z) = e^{-(\tau_t - \tau_0)} P(\tau_t - \tau_0)\hat{\psi}_0(z), \quad \psi_t^{\text{cap}} = e^{-\zeta(\tau_t - \tau_0)}\psi_0^{\text{cap}}.$$

3. Interpolation Formula for Markov Chain Fluid Limits

We use an interpolation formula between continuous-time Markov chains and differential equations, which we first review briefly in a general setting. This formula is then applied to an ALE(α, η) aggregation process $(\Phi_t)_{t\geqslant 0}$ with capacity parameter c, regularization parameter σ and particle family $(P^{(c)} : c \in (0, \infty))$, taking as limit equation the σ-regularized LK(ζ) equation with $\zeta = \alpha + \eta$. We use (Schlicht function, capacity) coordinates for both the process and the limit equation.

3.1. General form of the interpolation formula. Let $(X_t)_{t\geqslant 0}$ be a continuous-time Markov chain with state-space E and transition rate kernel q, starting from x_0 say. Suppose for this general discussion that $E = \mathbb{R}^d$. Let b be a vector field on E with continuous bounded derivative ∇b. Write $(\xi_t(x) : t \geqslant 0, x \in E)$ for the flow of b. The compensated jump measure of $(X_t)_{t\geqslant 0}$ is the signed measure $\tilde{\mu}^X$ on $E \times (0, \infty)$ given by

$$\tilde{\mu}^X(dy, dt) = \mu^X(dy, dt) - q(X_{t-}, dy)dt, \quad \mu^X = \sum_{t:X_t \neq X_{t-}} \delta_{(X_t,t)}.$$

Set $x_t = \xi_t(x_0)$ and define, for $s \in [0, t]$,

$$Z_s = x_t + \nabla\xi_{t-s}(x_s)(X_s - x_s).$$

Then $Z_0 = x_t$ and $Z_t = X_t$ and, on computing the martingale decomposition of $(Z_s)_{s\leqslant t}$, we obtain the interpolation formula

$$X_t - x_t = M_t + A_t \tag{15}$$

where

$$M_t = \int_{E\times(0,t]} \nabla\xi_{t-s}(x_s)(y - X_{s-})\tilde{\mu}^X(dy, ds)$$

and

$$A_t = \int_0^t \nabla\xi_{t-s}(x_s)(\beta(X_s) - b(x_s) - \nabla b(x_s)(X_s - x_s))ds$$

where β is the drift of $(X_t)_{t\geqslant 0}$, given by

$$\beta(x) = \int_E (y - x)q(x, dy).$$

The identification of martingales associated with finite-rate continuous-time Markov chains is standard. The particular pathwise formulation in terms of the jump measure used here is developed in detail in [6]. We will use this formula in a case where the state-space E is infinite-dimensional. Rather than justify its validity generally in such a context, in the next section, we will prove directly the special case of the formula which we require. Note that the integrands in M_t and A_t depend on t. Nevertheless, we will call M_t the martingale term and A_t the drift term.

3.2. Proof of the formula for ALE(α, η). Let $(\Phi_t)_{t\geqslant 0}$ be an ALE(α, η) aggregation process with capacity parameter c, regularization parameter σ and particle family $(P^{(c)} : c \in (0,\infty))$. See Sect. 1.2 and (7) for the specification of this process. We use (Schlicht function, capacity) coordinates, as in Sect. 2.3, to obtain a continuous-time Markov chain $(X_t)_{t\geqslant 0} = (\hat{\Phi}_t, \mathcal{T}_t)_{t\geqslant 0}$ in $\mathcal{S}_1 \times [0,\infty)$. When in state $x = (\hat{\phi}, \tau)$, for all $\theta \in [0, 2\pi)$, this process makes a jump of size $(\Delta(\theta, z, c(\theta), \hat{\phi}), c(\theta))$ at rate $\lambda(\theta) d\theta/(2\pi)$, where

$$\Delta(\theta, z, c, \hat{\phi}) = e^{-c}\hat{\phi}(F_c(\theta, z)) - \hat{\phi}(z)$$

and

$$\begin{aligned} c(\theta) &= c(\theta, \hat{\phi}, \tau) = ce^{-\alpha\tau}|\hat{\phi}'(e^{\sigma+i\theta})|^{-\alpha}, \\ \lambda(\theta) &= \lambda(\theta, \hat{\phi}, \tau) = c^{-1}e^{-\eta\tau}|\hat{\phi}'(e^{\sigma+i\theta})|^{-\eta}. \end{aligned}$$

We can and do assume that the process is constructed from a Poisson random measure μ on $[0, 2\pi) \times [0, \infty) \times (0, \infty)$ of intensity $(2\pi)^{-1} d\theta dv dt$ by the following stochastic differential equation:

$$\begin{aligned} \hat{\Phi}_t(z) &= \int_{E(t)} H_s(\theta, z) 1_{\{v \leqslant \Lambda_s(\theta)\}} \mu(d\theta, dv, ds), \\ \mathcal{T}_t &= \int_{E(t)} C_s(\theta) 1_{\{v \leqslant \Lambda_s(\theta)\}} \mu(d\theta, dv, ds) \end{aligned}$$

where

$$E(t) = [0, 2\pi) \times [0, \infty) \times (0, t]$$

and

$$\begin{aligned} H_s(\theta, z) &= \Delta(\theta, z, C_s(\theta), \hat{\Phi}_{s-}), \quad C_s(\theta) = c(\theta, \hat{\Phi}_{s-}, \mathcal{T}_{s-}), \\ \Lambda_s(\theta) &= \lambda(\theta, \hat{\Phi}_{s-}, \mathcal{T}_{s-}). \end{aligned}$$

We use the vector field $b = (\hat{b}, b^{\text{cap}})$ of the σ-regularized LK(ζ) equation (14), written in (Schlicht function, capacity) coordinates. Consider the disk solution $(x_t)_{t\geqslant 0} = (\hat{\phi}_t, \tau_t)_{t<t_\zeta}$ with initial capacity $\tau_0 = 0$, which is given by

$$\hat{\phi}_t(z) = z, \quad \tau_t = \begin{cases} t, & \text{if } \zeta = 0, \\ \zeta^{-1}\log(1+\zeta t), & \text{if } \zeta \neq 0, \end{cases} \quad t_\zeta = \begin{cases} \infty, & \text{if } \zeta \geqslant 0, \\ |\zeta|^{-1}, & \text{if } \zeta < 0. \end{cases} \tag{16}$$

We will compute the form of the interpolation formula in this case and then prove directly that it holds. Note that

$$b(x_t) = (\hat{b}, b^{\text{cap}})(\hat{\phi}_t, \tau_t) = (0, e^{-\zeta\tau_t})$$

and, for $y = (\hat{\psi}, \psi^{\text{cap}})$,

$$\nabla b(x_t) y = -e^{-\zeta\tau_t}((Q+1)\hat{\psi}, \zeta\psi^{\text{cap}})$$

and the first-order variation at time t due to a variation y at time $s \leqslant t$ is given by

$$\nabla \xi_{t-s}(x_s) y = (e^{-(\tau_t - \tau_s)} P(\tau_t - \tau_s)\hat{\psi}, e^{-\zeta(\tau_t - \tau_s)}\psi^{\text{cap}}).$$

Write $\tilde{\mu}$ for the compensated Poisson random measure

$$\tilde{\mu}(d\theta, dv, ds) = \mu(d\theta, dv, ds) - (d\theta/2\pi)dvds.$$

Fix $t \geqslant 0$ and set $\bar{\tau}_s = \tau_t - \tau_s$. We alert the reader to the concealed dependence of $\bar{\tau}_s$ on t. The martingale term $M_t = (\hat{M}_t, M_t^{\text{cap}})$ in the interpolation formula may then be written

$$\hat{M}_t(z) = \int_{E(t)} e^{-\bar{\tau}_s} P(\bar{\tau}_s) H_s(\theta, z) 1_{\{v \leqslant \Lambda_s(\theta)\}} \tilde{\mu}(d\theta, dv, ds),$$
$$M_t^{\text{cap}} = \int_{E(t)} e^{-\zeta\bar{\tau}_s} C_s(\theta) 1_{\{v \leqslant \Lambda_s(\theta)\}} \tilde{\mu}(d\theta, dv, ds).$$

The drift $\beta = (\hat{\beta}, \beta^{\text{cap}})$ for $(\hat{\Phi}, \mathcal{T})$ is given by

$$\hat{\beta}(\hat{\phi}, \tau)(z) = ⨍_0^{2\pi} \Delta(\theta, z, c(\theta, \hat{\phi}, \tau), \hat{\phi})\lambda(\theta, \hat{\phi}, \tau)d\theta,$$
$$\beta^{\text{cap}}(\hat{\phi}, \tau) = ⨍_0^{2\pi} c(\theta, \hat{\phi}, \tau)\lambda(\theta, \hat{\phi}, \tau)d\theta.$$

Write $\hat{\Psi}_s(z) = \hat{\Phi}_s(z) - \hat{\phi}_s(z) = \hat{\Phi}_s(z) - z$ and $\Psi_s^{\text{cap}} = \mathcal{T}_s - \tau_s$. Then we have formally

$$\nabla b(x_s)(X_s - x_s) = -e^{-\zeta\tau_s}((Q+1)\hat{\Psi}_s, \zeta\Psi_s^{\text{cap}})$$

and so

$$\nabla\xi_{t-s}(x_s)\nabla b(x_s)(X_s - x_s) = -e^{-\zeta\tau_s}(e^{-\bar{\tau}_s} P(\bar{\tau}_s)(Q+1)\hat{\Psi}_s, e^{-\zeta\bar{\tau}_s}\zeta\Psi_s^{\text{cap}}).$$

The following interpolation identities may then be obtained formally by splitting equation (15) into its Schlicht function and capacity components.

Proposition 3.1. *For all $t < t_\zeta$ and all $|z| > 1$, we have*

$$\hat{\Psi}_t(z) = \hat{M}_t(z) + \hat{A}_t(z), \quad \Psi_t^{\text{cap}} = M_t^{\text{cap}} + A_t^{\text{cap}} \tag{17}$$

where

$$\hat{A}_t(z) = \int_0^t e^{-\bar{\tau}_s} P(\bar{\tau}_s) \left(\hat{\beta}(\hat{\Phi}_s, \mathcal{T}_s) + e^{-\zeta\tau_s}(Q+1)\hat{\Psi}_s\right)(z)ds,$$
$$A_t^{\text{cap}} = \int_0^t e^{-\zeta\bar{\tau}_s} \left(\beta^{\text{cap}}(\hat{\Phi}_s, \mathcal{T}_s) - e^{-\zeta\tau_s} + \zeta e^{-\zeta\tau_s}\Psi_s^{\text{cap}}\right) ds.$$

Proof. Fix $t < t_\zeta$. For $x \in [0, t]$, recall that $\bar{\tau}_x = \tau_t - \tau_x$ and define for $|z| > 1$

$$\hat{\Psi}_{x,t}(z) = e^{-\bar{\tau}_x} P(\bar{\tau}_x)(\hat{\Phi}_x - \hat{\phi}_x)(z), \quad \Psi_{x,t}^{\text{cap}} = e^{-\zeta\bar{\tau}_x}(\mathcal{T}_x - \tau_x).$$

Set

$$\hat{M}_{x,t}(z) = \int_{E(x)} e^{-\bar{\tau}_s} P(\bar{\tau}_s) H_s(\theta, z) 1_{\{v \leqslant \Lambda_s(\theta)\}} \tilde{\mu}(d\theta, dv, ds),$$
$$M_{x,t}^{\text{cap}} = \int_{E(x)} e^{-\zeta\bar{\tau}_s} C_s(\theta) 1_{\{v \leqslant \Lambda_s(\theta)\}} \tilde{\mu}(d\theta, dv, ds)$$

and

$$\hat{A}_{x,t}(z) = \int_0^x e^{-\bar{\tau}_s} P(\bar{\tau}_s) \left(\hat{\beta}(\hat{\Phi}_s, \mathcal{T}_s) + e^{-\zeta \tau_s} (Q+1) \hat{\Psi}_s \right)(z) ds,$$

$$A^{\text{cap}}_{x,t} = \int_0^x e^{-\zeta \bar{\tau}_s} \left(\beta^{\text{cap}}(\hat{\Phi}_s, \mathcal{T}_s) - e^{-\zeta \tau_s} + \zeta e^{-\zeta \tau_s} \Psi^{\text{cap}}_s \right) ds.$$

We will show that, for all $x \in [0, t]$ and all $|z| > 1$,

$$\hat{\Psi}_{x,t}(z) = \hat{M}_{x,t}(z) + \hat{A}_{x,t}(z), \quad \Psi^{\text{cap}}_{x,t} = M^{\text{cap}}_{x,t} + A^{\text{cap}}_{x,t}.$$

The case $x = t$ gives the claimed identities. In the case $x = 0$, all terms are 0. The left-hand and right-hand sides are piecewise continuously differentiable in x, except for finitely many jumps, at the jump times of $(\Phi_x)_{0 \leqslant x \leqslant t}$, which occur when μ has an atom at (θ, v, x) with $v \leqslant \Lambda_x(\theta)$. It will suffice to check that the jumps and derivatives agree. Now $\hat{A}_{x,t}(z)$ and $A^{\text{cap}}_{x,t}$ are continuous in x and, at the jump times of Φ_x, the jumps in $\hat{\Psi}_{x,t}(z)$ and $\Psi^{\text{cap}}_{x,t}$ are given by

$$\begin{aligned}
\Delta \hat{\Psi}_{x,t}(z) &= e^{-\bar{\tau}_x} P(\bar{\tau}_x) \Delta \hat{\Phi}_x(z) \\
&= e^{-\bar{\tau}_x} P(\bar{\tau}_x)(e^{-C_x(\theta)} \hat{\Phi}_{x-} \circ F_{C_x(\theta)}(\theta, .) - \hat{\Phi}_{x-})(z) \\
&= e^{-\bar{\tau}_x} P(\bar{\tau}_x) H_x(\theta, z) = \Delta \hat{M}_{x,t}(z)
\end{aligned}$$

and

$$\Delta \Psi^{\text{cap}}_{x,t} = e^{-\zeta \bar{\tau}_x} \Delta \mathcal{T}_x = e^{-\zeta \bar{\tau}_x} C_x(\theta) = \Delta M^{\text{cap}}_{x,t}.$$

So it remains to check the derivatives. We will use a spectral calculation for the semigroup of multiplier operators $P(\tau) = e^{-\tau Q}$, whose justification is straightforward. Recall that $\dot{\tau}_t = e^{-\zeta \tau_t}$. We have

$$\frac{d}{dx} \bar{\tau}_x = -e^{-\zeta \tau_x}, \quad \frac{d}{dx} e^{-\bar{\tau}_x} = e^{-\bar{\tau}_x} e^{-\zeta \tau_x}, \quad \frac{d}{dx} e^{-\zeta \bar{\tau}_x} = \zeta e^{-\zeta \bar{\tau}_x} e^{-\zeta \tau_x}$$

and

$$\frac{d}{dx} P(\bar{\tau}_x) = e^{-\zeta \tau_x} Q P(\bar{\tau}_x).$$

So, between the jump times, we have

$$\frac{d}{dx} \hat{\Psi}_{x,t}(z) = e^{-\zeta \tau_x} e^{-\bar{\tau}_x} P(\bar{\tau}_x)(Q+1) \hat{\Psi}_x(z),$$

$$\frac{d}{dx} \Psi^{\text{cap}}_{x,t}(z) = -e^{-\zeta \bar{\tau}_x} e^{-\zeta \tau_x} (1 - \zeta \Psi^{\text{cap}}_x)$$

and

$$\frac{d}{dx} \hat{M}_{x,t}(z) = -\int_0^{2\pi} e^{-\bar{\tau}_x} P(\bar{\tau}_x) H_x(\theta, z) \Lambda_x(\theta) d\theta = -e^{-\delta_x} P(\bar{\tau}_x) \hat{\beta}(\hat{\Phi}_x, \mathcal{T}_x)(z),$$

$$\frac{d}{dx} M^{\text{cap}}_{x,t} = -\int_0^{2\pi} e^{-\zeta \bar{\tau}_x} C_x(\theta) \Lambda_x(\theta) d\theta = -e^{-\zeta \bar{\tau}_x} \beta^{\text{cap}}(\hat{\Phi}_x, \mathcal{T}_x)$$

and

$$\frac{d}{dx}\hat{A}_{x,t}(z) = e^{-\bar{\tau}_x} P(\bar{\tau}_x)\left(\hat{\beta}(\hat{\Phi}_x, \mathcal{T}_x) + e^{-\zeta\tau_x}(Q+1)\hat{\Psi}_x\right)(z),$$
$$\frac{d}{dx}A^{\text{cap}}_{x,t} = e^{-\zeta\bar{\tau}_x}\left(\beta^{\text{cap}}(\hat{\Phi}_x, \mathcal{T}_x) - e^{-\zeta\tau_x} + \zeta e^{-\zeta\tau_x}\Psi^{\text{cap}}_x\right).$$

Hence, between the jump times,

$$\frac{d}{dx}\hat{\Psi}_{x,t}(z) = \frac{d}{dx}(\hat{M}_{x,t}(z) + \hat{A}_{x,t}(z)), \quad \frac{d}{dx}\Psi^{\text{cap}}_{x,t} = \frac{d}{dx}(M^{\text{cap}}_{x,t} + A^{\text{cap}}_{x,t})$$

as required. □

4. Estimation of Terms in the Interpolation Formula

We obtain some estimates on the terms in the interpolation formula (17) for ALE(α, η) when it is close to the disk solution (16) of the LK(ζ) equation, with $\zeta = \alpha + \eta$. For $\delta_0 \in (0, 1/2]$, define

$$T_0 = T_0(\delta_0) = \inf\left\{t \in [0, t_\zeta) : \sup_{\theta\in[0,2\pi)} |\hat{\Psi}'_t(e^{\sigma+i\theta})| > \delta_0 \text{ or } |\Psi^{\text{cap}}_t| > \delta_0\right\}.$$

We estimate first the martingale term and then the drift term.

4.1. Estimates for the martingale terms. Recall that the martingale term $(\hat{M}_t, M^{\text{cap}}_t)$ in the interpolation formula is given by

$$\hat{M}_t(z) = \int_{E(t)} e^{-\bar{\tau}_{t,s}} P(\bar{\tau}_{t,s}) H_s(\theta, z) 1_{\{v\leqslant\Lambda_s(\theta)\}}\tilde{\mu}(d\theta, dv, ds),$$
$$M^{\text{cap}}_t = \int_{E(t)} e^{-\zeta\bar{\tau}_{t,s}} C_s(\theta) 1_{\{v\leqslant\Lambda_s(\theta)\}}\tilde{\mu}(d\theta, dv, ds)$$

where $E(t) = [0, 2\pi) \times [0, \infty) \times (0, t]$ and

$$\bar{\tau}_{t,s} = \tau_t - \tau_s, \quad C_s(\theta) = c(\theta, \hat{\Phi}_{s-}, \mathcal{T}_{s-}), \quad \Lambda_s(\theta) = \lambda(\theta, \hat{\Phi}_{s-}, \mathcal{T}_{s-}) \tag{18}$$

with

$$c(\theta, \hat{\phi}, \tau) = ce^{-\alpha\tau}|\hat{\phi}'(e^{\sigma+i\theta})|^{-\alpha}, \quad \lambda(\theta, \hat{\phi}, \tau) = c^{-1}e^{-\eta\tau}|\hat{\phi}'(e^{\sigma+i\theta})|^{-\eta}$$

and

$$H_s(\theta, z) = \Delta(\theta, z, C_s(\theta), \hat{\Phi}_{s-}), \quad \Delta(\theta, z, c, \hat{\phi}) = e^{-c}\hat{\phi}(F_c(\theta, z)) - \hat{\phi}(z).$$

Consider the following approximations to $\hat{M}_t(z)$ and M^{cap}_t, which are obtained by replacing $\hat{\Phi}_{s-}$ by $\hat{\phi}_s$, $\mathcal{T}_{s-}$ by τ_s and $e^{-c}F_c(\theta, z) - z$ by $2cz/(e^{-i\theta}z - 1)$. (Under our assumptions on the particle family, the last approximation becomes good in the limit $c \to 0$. See Sect. A.2 and in particular equation (110).) Define

$$\hat{\Pi}_t(z) = \int_{E(t)} e^{-\bar{\tau}_{t,s}} P(\bar{\tau}_{t,s}) H(\theta, z) 2c_s 1_{\{v \leqslant \lambda_s\}} \tilde{\mu}(d\theta, dv, ds), \tag{19}$$

$$\Pi_t^{\text{cap}} = \int_{E(t)} e^{-\zeta \bar{\tau}_{t,s}} c_s 1_{\{v \leqslant \lambda_s\}} \tilde{\mu}(d\theta, dv, ds) \tag{20}$$

where

$$c_s = ce^{-\alpha \tau_s}, \quad \lambda_s = c^{-1} e^{-\eta \tau_s}$$

and

$$H(\theta, z) = \frac{z}{e^{-i\theta} z - 1} = \sum_{k=0}^{\infty} e^{i(k+1)\theta} z^{-k}. \tag{21}$$

Lemma 4.1. *For all $\alpha, \eta \in \mathbb{R}$, all $p \geqslant 2$ and all $T < t_\zeta$, there is a constant $C(\alpha, \eta, p, T) < \infty$, such that, for all $c \in (0, 1]$, all $\sigma \geqslant 0$ and all $\delta_0 \in (0, 1/2]$,*

$$\Big\| \sup_{t \leqslant T_0(\delta_0) \wedge T} |M_t^{\text{cap}}| \Big\|_p \leqslant C\sqrt{c}$$

and

$$\Big\| \sup_{t \leqslant T_0(\delta_0) \wedge T} |M_t^{\text{cap}} - \Pi_t^{\text{cap}}| \Big\|_p \leqslant C(c + \sqrt{c\delta_0}).$$

Proof. We write T_0 for $T_0(\delta_0)$ in the proofs. Consider the martingale $(M_t)_{t < t_\zeta}$ given by

$$M_t = \int_{E(t)} e^{\zeta \tau_s} C_s(\theta) 1_{\{v \leqslant \Lambda_s(\theta),\, s \leqslant T_0\}} \tilde{\mu}(d\theta, dv, ds).$$

By an inequality of Burkholder, for all $p \geqslant 2$, there is a constant $C(p) < \infty$ such that, for all $t \geqslant 0$,

$$\|M_t^*\|_p \leqslant C(p) \left(\|\langle M \rangle_t\|_{p/2}^{1/2} + \|(\Delta M)_t^*\|_p \right). \tag{22}$$

We write here M_t^* for $\sup_{s \leqslant t} |M_s|$ and similarly for other processes. See [3, Theorem 21.1] for the discrete-time case. The continuous-time case follows by a standard limit argument. Now

$$\langle M \rangle_t = \int_0^{T_0 \wedge t} ⨍_0^{2\pi} e^{2\zeta \tau_s} C_s(\theta)^2 \Lambda_s(\theta) d\theta ds$$

and

$$\Delta M_t = |M_t - M_{t-}| \leqslant e^{\zeta \tau_t} \sup_{\theta \in [0, 2\pi)} C_t(\theta).$$

For all $t \leqslant T_0 \wedge T$ and all $\theta \in [0, 2\pi)$, we have

$$e^{\zeta \tau_t} \leqslant C, \quad C_t(\theta) \leqslant Cc, \quad \Lambda_t(\theta) \leqslant C/c \tag{23}$$

so $\langle M\rangle_t \leqslant Cc$ and $(\Delta M)^*_t \leqslant Cc$. Here and below, we write C for a finite constant of the dependence allowed in the statement. The value of C may vary from one instance to the next. We remind the reader that $C_t(\theta)$ and $\Lambda_t(\theta)$ are defined at (18). Hence

$$\|M^*_t\|_p \leqslant C\sqrt{c}.$$

Since $M^{\text{cap}}_t = e^{-\zeta\tau_t}M_t$ for all $t \leqslant T_0$, the first claimed estimate follows.

For the second estimate, we use instead the martingale $(M_t)_{t\geqslant 0}$ given by

$$M_t = \int_{E(t)} e^{\zeta\tau_s}\left(C_s(\theta)1_{\{v\leqslant\Lambda_s(\theta)\}} - c_s 1_{\{v\leqslant\lambda_s\}}\right)1_{\{s\leqslant T_0\}}\tilde{\mu}(d\theta, dv, ds).$$

Then

$$\langle M\rangle_t = \int_0^{T_0\wedge t}\int_0^\infty \fint_0^{2\pi} e^{2\zeta\tau_s}\left(C_s(\theta)1_{\{v\leqslant\Lambda_s(\theta)\}} - c_s 1_{\{v\leqslant\lambda_s\}}\right)^2 d\theta dv ds.$$

For $t \leqslant T_0 \wedge T$ and $\theta \in [0, 2\pi)$, we have

$$|C_t(\theta) - c_t| \leqslant Cc\delta_0, \quad |\Lambda_t(\theta) - \lambda_t| \leqslant C\delta_0/c \tag{24}$$

so

$$\int_0^\infty \left(C_t(\theta)1_{\{v\leqslant\Lambda_t(\theta)\}} - c_t 1_{\{v\leqslant\lambda_t\}}\right)^2 dv \leqslant Cc\delta_0. \tag{25}$$

Then $\langle M\rangle_t \leqslant Cc\delta_0$ and $(\Delta M)_t \leqslant Cc$. Hence, by Burkholder's inequality,

$$\|M^*_t\|_p \leqslant C(c + \sqrt{c\delta_0}).$$

Since $M^{\text{cap}}_t - \Pi^{\text{cap}}_t = e^{-\zeta\tau_t}M_t$ for all $t \leqslant T_0$, the second claimed estimate follows. □

Note that, since $\hat{\Phi}_t$ takes values in $\mathcal{S}_1$, the holomorphic function $\hat{\Psi}_t(z) = \hat{\Phi}_t(z) - z$ is bounded at ∞ and hence has a limiting value $\hat{\Psi}_t(\infty)$. The same is true for the terms $\hat{M}_t$ and $\hat{A}_t$ in the interpolation formula. Instead of estimating these terms directly, we estimate first their values at ∞ and then their radial derivatives $D\hat{M}_t$ and $D\hat{A}_t$, since this gives the best control of the derivative of $\hat{\Phi}_t$ near the unit circle, which drives the dynamics of the process.

Lemma 4.2. *For all $\alpha, \eta \in \mathbb{R}$ with $\zeta = \alpha + \eta \leqslant 1$, all $p \geqslant 2$ and all $T < t_\zeta$, there is a constant $C(\alpha, \eta, \Lambda, p, T) < \infty$, such that, for all $c \in (0, 1]$, all $\sigma \geqslant 0$, all $\delta_0 \in (0, 1/2]$ and all $t \leqslant T$,*

$$\Big\|\sup_{s\leqslant T_0(\delta_0)\wedge t}|\hat{M}_s(\infty)|\Big\|_p^p \leqslant Cc^{p/2}\left(1 + \int_0^t \|\hat{\Psi}_{s-}(\infty)1_{\{s\leqslant T_0(\delta_0)\}}\|_p^p ds\right)$$

and

$$\Big\|\sup_{s\leqslant T_0(\delta_0)\wedge t}|\hat{M}_s(\infty) - \hat{\Pi}_s(\infty)|\Big\|_p^p \leqslant C\left(\left(c + \sqrt{c\delta_0}\right)^p + c^{p/2}\int_0^t \|\hat{\Psi}_{s-}(\infty)1_{\{s\leqslant T_0(\delta_0)\}}\|_p^p ds\right).$$

Proof. By considering the Laurent expansions of F_c and $\hat{\phi}$, we have

$$\Delta(\theta, \infty, c, \hat{\phi}) = a_0(c)e^{i\theta} + (e^{-c} - 1)\hat{\psi}(\infty), \quad \hat{\psi}(z) = \hat{\phi}(z) - z. \tag{26}$$

Consider the martingale $(M_t)_{t<t_\zeta}$ given by

$$M_t = \int_{E(t)} e^{\tau_s} \left(a_0(C_s(\theta))e^{i\theta} + (e^{-C_s(\theta)} - 1)\hat{\Psi}_{s-}(\infty) \right) 1_{\{v \leqslant \Lambda_s(\theta),\, s \leqslant T_0\}} \tilde{\mu}(d\theta, dv, ds).$$

Then $\hat{M}_t(\infty) = e^{-\tau_t} M_t$ for all $t \leqslant T_0$. By Proposition A.5, $|a_0(c)| \leqslant Cc$ for all c. Hence

$$\begin{aligned} \langle M \rangle_t &= \int_0^{T_0 \wedge t} ⨍_0^{2\pi} e^{2\tau_s} \left| a_0(C_s(\theta))e^{i\theta} + (e^{-C_s(\theta)} - 1)\hat{\Psi}_{s-}(\infty) \right|^2 \Lambda_s(\theta) d\theta ds \\ &\leqslant Cce^{2\tau_t} \int_0^{T_0 \wedge t} (1 + |\hat{\Psi}_{s-}(\infty)|^2) ds \end{aligned}$$

and, for $p \geqslant 2$, since

$$|(\Delta M)_t^*|^p \leqslant \int_{E(t)} e^{p\tau_s} \left| a_0(C_s(\theta))e^{i\theta} + (e^{-C_s(\theta)} - 1)\hat{\Psi}_{s-}(\infty) \right|^p 1_{\{v \leqslant \Lambda_s(\theta), s \leqslant T_0\}} \mu(d\theta, dv, ds)$$

we have

$$\begin{aligned} \|(\Delta M)_t^*\|_p^p &\leqslant \mathbb{E} \int_0^{T_0 \wedge t} ⨍_0^{2\pi} e^{p\tau_s} \left| a_0(C_s(\theta))e^{i\theta} + (e^{-C_s(\theta)} - 1)\hat{\Psi}_{s-}(\infty) \right|^p \Lambda_s(\theta) d\theta ds \\ &\leqslant Cc^{p-1} e^{p\tau_t} \mathbb{E} \int_0^{T_0 \wedge t} (1 + |\hat{\Psi}_{s-}(\infty)|^p) ds. \end{aligned}$$

The first claimed estimate then follows from Burkholder's inequality (22).

For the second estimate, we consider instead the martingale $(M_t)_{t<t_\zeta}$ given by

$$\begin{aligned} M_t = \int_{E(t)} e^{\tau_s} \Big(& \left(a_0(C_s(\theta)) 1_{\{v \leqslant \Lambda_s(\theta)\}} - 2c_s 1_{\{v \leqslant \lambda_s\}} \right) e^{i\theta} \\ & + (e^{-C_s(\theta)} - 1)\hat{\Psi}_{s-}(\infty) 1_{\{v \leqslant \Lambda_s(\theta)\}} \Big) 1_{\{s \leqslant T_0\}} \tilde{\mu}(d\theta, dv, ds). \end{aligned}$$

Then $\hat{M}_t(\infty) - \hat{\Pi}_t(\infty) = e^{-\tau_t} M_t$ for all $t \leqslant T_0$. By Proposition A.5, we have $|a_0(c) - 2c| \leqslant Cc^{3/2}$. We combine this with (23) and (24) to see that

$$\int_0^\infty \left| a_0(C_t(\theta)) 1_{\{v \leqslant \Lambda_t(\theta)\}} - 2c_t 1_{\{v \leqslant \lambda_t\}} \right|^p dv \leqslant C(c^{3p/2-1} + c^{p-1}\delta_0) \leqslant C(c^p + c^{p-1}\delta_0).$$

The second estimate then follows by Burkholder's inequality as above. □

Recall that, for $p \in [1, \infty)$ and $r > 1$, we set

$$\|\psi\|_{p,r} = \left(⨍_0^{2\pi} |\psi(re^{i\theta})|^p d\theta \right)^{1/p}.$$

For a measurable function Ψ on $\Omega \times \{|z| > 1\}$, we set

$$|||\Psi|||_{p,r} = \left(\mathbb{E} ⨍_0^{2\pi} |\Psi(re^{i\theta})|^p d\theta \right)^{1/p}.$$

Lemma 4.3. *For all $\alpha, \eta \in \mathbb{R}$ with $\zeta = \alpha + \eta \leqslant 1$, all $\varepsilon \in (0, 1/2)$, all $p \geqslant 2$ and all $T < t_\zeta$, there is a constant $C(\alpha, \eta, \varepsilon, \Lambda, p, T) < \infty$ such that, for all $c \in (0, 1]$, all $\sigma \geqslant 0$, all $\delta_0 \in (0, 1/2]$ and all $t \leqslant T$, for all $r \geqslant 1 + c^{1/2-\varepsilon}$, for $\rho = (1+r)/2$, we have, in the case $\zeta < 1$,*

$$\||| D\hat{M}_t 1_{\{t \leqslant T_0(\delta_0)\}} |||_{p,r} \leqslant \frac{C\sqrt{c}}{r} \left(1 + r \sup_{s \leqslant t} |||D\hat{\Psi}_{s-} 1_{\{s \leqslant T_0(\delta_0)\}}|||_{p,\rho}\right) \left(\frac{r}{r-1}\right) \tag{27}$$

and

$$\begin{aligned} &||| D(\hat{M}_t - \hat{\Pi}_t) 1_{\{t \leqslant T_0(\delta_0)\}} |||_{p,r} \\ &\leqslant \frac{C\sqrt{c}}{r} \left(\sqrt{\delta_0} + r \sup_{s \leqslant t} |||D\hat{\Psi}_{s-} 1_{\{s \leqslant T_0(\delta_0)\}}|||_{p,\rho}\right) \left(\frac{r}{r-1}\right) + \frac{Cc}{r}\left(\frac{r}{r-1}\right)^2 \end{aligned} \tag{28}$$

while in the case $\zeta = 1$ the same bounds hold with $\left(\frac{r}{r-1}\right)$ replaced in the first term on the right-hand side by $\left(\frac{r}{r-1}\right) + \frac{1}{\sqrt{\sigma}} \left(\frac{r}{r-1}\right)^{1/2}$.

Proof. Recall that we write T_0 for $T_0(\delta_0)$. Fix $t \leqslant T < t_\zeta$. For $s \in [0, t]$, we will write $\bar{\tau}_s$ for $\bar{\tau}_{t,s} = \tau_t - \tau_s$. Consider for $|z| > 1$, the martingale $(M_x(z))_{0 \leqslant x \leqslant t}$ given by

$$\begin{aligned} M_x(z) &= \int_{E(x)} \tilde{H}_s(\theta, z) 1_{\{v \leqslant \Lambda_s(\theta),\, s \leqslant T_0\}} \tilde{\mu}(d\theta, dv, ds), \\ \tilde{H}_s(\theta, z) &= e^{-\bar{\tau}_s} DP(\bar{\tau}_s) H_s(\theta, z). \end{aligned}$$

By Burkholder's inequality, for $p \geqslant 2$ and all $|z| > 1$,

$$\|M_t(z)\|_p \leqslant C(p) \left(\|\langle M(z)\rangle_t\|_{p/2}^{1/2} + \|(\Delta M(z))_t^*\|_p \right). \tag{29}$$

On the event $\{t \leqslant T_0\}$, we have $D\hat{M}_t(z) = M_t(z)$ so, on taking the $\|.\|_{p,r}$-norm in (29), we obtain

$$||| D\hat{M}_t 1_{\{t \leqslant T_0\}} |||_{p,r} \leqslant |||M_t|||_{p,r} \leqslant C(p) \left(|||\langle M(.)\rangle_t\, |||_{p/2,r}^{1/2} + ||| \,(\Delta M(.))_t^* |||_{p,r} \right). \tag{30}$$

Now

$$\langle M(z)\rangle_t = \int_0^{T_0 \wedge t} \fint_0^{2\pi} |\tilde{H}_s(\theta, z)|^2 \Lambda_s(\theta) d\theta ds$$

and

$$(\Delta M(z))_t^* \leqslant \sup_{s \leqslant T_0 \wedge t, \theta \in [0, 2\pi)} |\tilde{H}_s(\theta, z)|. \tag{31}$$

Also

$$|(\Delta M(z))_t^*|^p \leqslant \int_{E(t)} |\tilde{H}_s(\theta, z)|^p 1_{\{v \leqslant \Lambda_s(\theta), s \leqslant T_0\}} \mu(d\theta, dv, ds)$$

so

$$\|(\Delta M(z))_t^*\|_p^p \leqslant \mathbb{E}\int_0^{T_0\wedge t} ⨍_0^{2\pi} |\tilde{H}_s(\theta, z)|^p \Lambda_s(\theta) d\theta ds.$$

We have $\Lambda_s(\theta) \leqslant C/c$ for all $s \leqslant T_0$ and $\theta \in [0, 2\pi)$. Hence

$$\langle M(z)\rangle_t \leqslant \frac{C}{c}\int_0^{T_0\wedge t} ⨍_0^{2\pi} |\tilde{H}_s(\theta, z)|^2 d\theta ds \tag{32}$$

and

$$\|\langle M(.)\rangle_t\|_{p/2,r} \leqslant \frac{C}{c}\int_0^{T_0\wedge t} ⨍_0^{2\pi} \|\tilde{H}_s(\theta, .)\|_{p,r}^2 d\theta ds. \tag{33}$$

Similarly,

$$|||(\Delta M(.))_t^*|||_{p,r}^p \leqslant \frac{C}{c}\mathbb{E}\int_0^{T_0\wedge t} ⨍_0^{2\pi} \|\tilde{H}_s(\theta, .)\|_{p,r}^p d\theta ds. \tag{34}$$

We will split the jump $\Delta(\theta, z, c, \hat{\phi})$ as the sum of several terms, and thereby split $H_s(\theta, z)$ and hence M_t also as a sum of terms. For each of these terms, we will use one of the inequalities (32), (33) and one of (31), (34) to obtain a suitable upper bound for the right-side of (30). These bounds will combine to prove the first claimed estimate.

Recall that $\hat{\phi}(z) = z + \hat{\psi}(z)$, so

$$\Delta(\theta, z, c, \hat{\phi}) = \Delta_0(\theta, z, c) + \left(e^{-c}\hat{\psi}(F_c(\theta, z)) - \hat{\psi}(z)\right) \tag{35}$$

where

$$\Delta_0(\theta, z, c) = e^{-c}F_c(\theta, z) - z.$$

We further split the second term by expanding in Taylor series, using an interpolation from z to $F_c(\theta, z)$. For $u \in [0, 1]$, define

$$F_{c,u}(\theta, z) = e^{uf_c(\theta,z)}z, \quad f_c(\theta, z) = \log(F_c(\theta, z)/z).$$

Then $F_{c,0}(\theta, z) = z$ and $F_{c,1}(\theta, z) = F_c(\theta, z)$. Fix c, θ and z and set

$$g(u) = e^{-cu}\hat{\psi}(F_{c,u}(\theta, z))$$

then

$$g^{(k)}(u) = e^{-cu}\sum_{j=0}^{k}\binom{k}{j}(-c)^{k-j} f_c(\theta, z)^j D^j\hat{\psi}(F_{c,u}(\theta, z)).$$

Set $m = \lceil 1/(8\varepsilon)\rceil$ and recall that our constants C are allowed to depend on ε. Then

$$\begin{aligned} e^{-c}\hat{\psi}(F_c(\theta, z)) - \hat{\psi}(z) &= g(1) - g(0) \\ &= \sum_{k=1}^{m}\frac{g^{(k)}(0)}{k!} + \int_0^1 \frac{(1-u)^m}{m!}g^{(m+1)}(u)du \\ &= \sum_{k=1}^{m+1}\Delta_k(\theta, z, c, \hat{\psi}) \end{aligned} \tag{36}$$

where, for $k = 1, \dots, m$,

$$\Delta_k(\theta, z, c, \hat{\psi}) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-c)^{k-j} f_c(\theta, z)^j D^j \hat{\psi}(z)$$

and

$$\Delta_{m+1}(\theta, z, c, \hat{\psi}) = \frac{1}{m!} \int_0^1 (1-u)^m e^{-cu} \sum_{j=0}^{m+1} \binom{m+1}{j} (-c)^{m+1-j} f_c(\theta, z)^j D^j \hat{\psi}(F_{c,u}(\theta, z)) du.$$

Let us write

$$H_s^0(\theta, z) = \Delta_0(\theta, z, C_s(\theta)),$$
$$H_s^k(\theta, z) = \Delta_k(\theta, z, C_s(\theta), \hat{\Phi}_{s-}), \quad k = 1, \dots, m+1$$

and

$$\tilde{H}_s^k(\theta, z) = e^{-\bar{\tau}_s} DP(\bar{\tau}_s) H_s^k(\theta, z)$$

and

$$M_x^k(z) = \int_{E(x)} \tilde{H}_s^k(\theta, z) 1_{\{v \leqslant \Lambda_s(\theta), s \leqslant T_0\}} \tilde{\mu}(d\theta, dv, ds).$$

We consider first the contribution of

$$\Delta_0(\theta, z, c) = e^{-c} F_c(\theta, z) - z.$$

We make the further split $\Delta_0 = \Delta_{0,0} + \Delta_{0,1}$, where

$$\Delta_{0,0}(\theta, z, c) = \frac{a_0(c) z}{e^{-i\theta} z - 1} = a_0(c) \sum_{k=0}^{\infty} e^{i(k+1)\theta} z^{-k}$$

and

$$\Delta_{0,1}(\theta, z, c) = e^{-c} F_c(\theta, z) - z - \frac{a_0(c) z}{e^{-i\theta} z - 1}.$$

We will exploit the more explicit form of $\Delta_{0,0}$, which is the main term as $c \to 0$ under our particle assumptions (4), (5) and (6), to obtain better estimates. We have, with obvious notation,

$$H_s^{0,0}(\theta, z) = a_0(C_s(\theta)) \sum_{k=0}^{\infty} e^{i(k+1)\theta} z^{-k}$$

so, for $\tau \geqslant 0$,

$$DP(\tau) H_s^{0,0}(\theta, z) = a_0(C_s(\theta)) \sum_{k=1}^{\infty} e^{i(k+1)\theta} (-k) e^{-\tau q(k)} z^{-k}.$$

By Proposition A.5, $|a_0(c)| \leqslant Cc$ for all c. So, for $|z| = r$ and $\tau \geqslant 0$,

$$|DP(\tau)H_s^{0,0}(\theta, z)| \leqslant Cc \sum_{k=1}^{\infty} kr^{-k} \leqslant \frac{Cc}{r}\left(\frac{r}{r-1}\right)^2 \tag{37}$$

and

$$\begin{aligned}\fint_0^{2\pi} |DP(\tau)H_s^{0,0}(\theta, z)|^2 d\theta &\leqslant Cc^2 \fint_0^{2\pi} \left|\sum_{k=1}^{\infty} e^{i(k+1)\theta}(-k)e^{-\tau q(k)} z^{-k}\right|^2 d\theta \\ &\leqslant Cc^2 \sum_{k=1}^{\infty} k^2 e^{-2\tau q(k)} r^{-2k}.\end{aligned}$$

Hence we have

$$\begin{aligned}\langle M^{0,0}(z)\rangle_t &\leqslant \frac{C}{c}\int_0^{T_0\wedge t} \fint_0^{2\pi} |DP(\bar{\tau}_s)H_s^{0,0}(\theta, z)|^2 d\theta ds \\ &\leqslant Cc \sum_{k=1}^{\infty} k^2 r^{-2k} \int_0^t e^{-2\bar{\tau}_s q(k)} ds \leqslant Cc \sum_{k=1}^{\infty} \frac{k^2 r^{-2k}}{q(k)}.\end{aligned}$$

We used the facts that $(d/ds)\bar{\tau}_s = -\dot{\tau}_s$ and $\dot{\tau}_s = e^{-\zeta\tau_s}$ and $e^{\zeta\tau_s} \leqslant C$ to see that, for all $\lambda > 0$,

$$\int_0^t \lambda e^{-\lambda\bar{\tau}_s} ds \leqslant C \int_0^t \lambda e^{-\lambda\bar{\tau}_s} \dot{\tau}_s ds \leqslant C. \tag{38}$$

We will use similar estimates for other integrals of $(\bar{\tau}_s)_{s\leqslant t}$ without further explanation. Now $q(k) \geqslant (1-\zeta^+)k$ so we obtain, for $\zeta < 1$,

$$\langle M^{0,0}(z)\rangle_t \leqslant \frac{Cc}{r^2}\left(\frac{r}{r-1}\right)^2.$$

On the other hand, for $\zeta = 1$, we have $q(k) \geqslant ((\sigma k^2) \wedge k)/C$ so we obtain

$$\langle M^{0,0}(z)\rangle_t \leqslant \frac{Cc}{r^2}\left(\left(\frac{r}{r-1}\right)^2 + \frac{1}{\sigma}\left(\frac{r}{r-1}\right)\right).$$

We use (31) and (37) to obtain, for $|z| = r > 1$,

$$|(\Delta M^{0,0}(z))_t^*| \leqslant \sup_{s\leqslant T_0\wedge t, \theta\in[0,2\pi)} |DP(\bar{\tau}_s)H_s^{0,0}(\theta, z)| \leqslant \frac{Cc}{r}\left(\frac{r}{r-1}\right)^2.$$

On substituting the estimates for $\langle M^{0,0}(z)\rangle_t$ and $(\Delta M^{0,0}(z))_t^*$ into (30), we obtain for $r \geqslant 1 + \sqrt{c}$ and $p \geqslant 2$, for $\zeta < 1$,

$$|||\, M_t^{0,0} |||_{p,r} \leqslant \frac{C\sqrt{c}}{r}\left(\frac{r}{r-1}\right) \tag{39}$$

while, for $\zeta = 1$,

$$\||| M_t^{0,0} \|||_{p,r} \leqslant \frac{C\sqrt{c}}{r}\left(\left(\frac{r}{r-1}\right) + \frac{1}{\sqrt{\sigma}} \left(\frac{r}{r-1}\right)^{1/2} \right). \tag{40}$$

We turn to the contribution of $\Delta_{0,1}$. For $s \leqslant T_0$ and all $\theta \in [0, 2\pi)$, we have

$$C_s(\theta) \leqslant Cc. \tag{41}$$

By Proposition A.6, there is a family of functions $(Q_u : u \in [0, 1])$, each holomorphic on $\{|z| > 1\}$, such that

$$|Q_u(z)| \leqslant \frac{C\sqrt{u}|z|}{|z-1|^2} \tag{42}$$

and such that

$$H_s^{0,1}(\theta, z) = \Delta_{0,1}(\theta, z, C_s(\theta)) = e^{-C_s(\theta)} \int_0^{C_s(\theta)} Q_u(\theta, z) du \tag{43}$$

where $Q_u(\theta, z) = e^{i\theta} Q_u(e^{-i\theta} z)$. We use the Laurent series

$$Q_u(z) = \sum_{k=1}^{\infty} a_u(k) z^{-k}$$

to write

$$DP(\tau) H_s^{0,1}(\theta, z) = e^{-C_s(\theta)} \sum_{k=1}^{\infty} (-k) e^{-\tau q(k)} e^{i(k+1)\theta} z^{-k} \int_0^{C_s(\theta)} a_u(k) du.$$

Hence we obtain, for $|z| = r > 1$,

$$|DP(\tau) H_s^{0,1}(\theta, z)| \leqslant \int_0^{Cc} \sum_{k=1}^{\infty} k r^{-k} |a_u(k)| du \leqslant \frac{Cc^{3/2}}{r} \left(\frac{r}{r-1}\right)^3$$

where we used

$$\sum_{k=1}^{\infty} k r^{-k} |a_u(k)| du \leqslant \left(\sum_{k=1}^{\infty} k^2 (r/\rho)^{-2k} \right)^{1/2} \left(\sum_{k=1}^{\infty} |a_u(k)|^2 \rho^{-2k} \right)^{1/2}$$

and

$$\sum_{k=1}^{\infty} |a_u(k)|^2 \rho^{-2k} = \|Q_u\|_{2,\rho}^2 \leqslant \frac{Cu}{r^2} \left(\frac{r}{r-1}\right)^3.$$

Now

$$⨍_0^{2\pi} |DP(\tau) Q_u(\theta, z)|^2 d\theta = \sum_{k=1}^{\infty} k^2 e^{-2\tau q(k)} |a_k(u)|^2 r^{-2k}$$

so, using again (38),

$$\int_0^t \fint_0^{2\pi} |DP(\bar\tau_s)Q_u(\theta,z)|^2 d\theta ds \leqslant C\sum_{k=1}^\infty \frac{k^2|a_k(u)|^2 r^{-2k}}{q(k)}.$$

Hence, using the same lower bounds for $q(k)$ as above, we obtain, for $\zeta < 1$,

$$\int_0^t \fint_0^{2\pi} |DP(\bar\tau_s)Q_u(\theta,z)|^2 d\theta ds \leqslant C\sum_{k=1}^\infty k|a_k(u)|^2 r^{-2k} \leqslant \frac{Cu}{r^2}\left(\frac{r}{r-1}\right)^4$$

and, for $\zeta = 1$,

$$\int_0^t \fint_0^{2\pi} |DP(\bar\tau_s)Q_u(\theta,z)|^2 d\theta ds \leqslant \frac{Cu}{r^2}\left(\left(\frac{r}{r-1}\right)^4 + \frac{1}{\sigma}\left(\frac{r}{r-1}\right)^3\right).$$

Hence, for $|z| = r > 1$ and $\zeta < 1$, we have

$$\begin{aligned}\langle M^{0,1}(z)\rangle_t &\leqslant \frac{C}{c}\int_0^{T_0\wedge t} e^{-\bar\tau_s} \fint_0^{2\pi} |DP(\bar\tau_s)H_s^{0,1}(\theta,z)|^2 d\theta ds\\ &\leqslant \frac{C}{c}\int_0^{T_0\wedge t}\fint_0^{2\pi}\left(\int_0^{Cc}|DP(\bar\tau_s)Q_u(\theta,z)|du\right)^2 d\theta ds\\ &\leqslant C\int_0^{Cc}\int_0^{T_0\wedge t}\fint_0^{2\pi}|DP(\bar\tau_s)Q_u(\theta,z)|^2 d\theta ds du\\ &\leqslant \frac{C}{r^2}\left(\frac{r}{r-1}\right)^4\int_0^{Cc} u du = \frac{Cc^2}{r^2}\left(\frac{r}{r-1}\right)^4\end{aligned}$$

while, for $\zeta = 1$, similarly,

$$\langle M^{0,1}(z)\rangle_t \leqslant \frac{Cc^2}{r^2}\left(\left(\frac{r}{r-1}\right)^4 + \frac{1}{\sigma}\left(\frac{r}{r-1}\right)^3\right).$$

Also, for all $s \leqslant T_0$ and $|z| = r > 1$, we have

$$|\Delta M_s^{0,1}(z)| \leqslant \sup_{s\leqslant T_0} |DP(\bar\tau_s)H_s^{0,1}(\theta,z)| \leqslant \frac{Cc^{3/2}}{r}\left(\frac{r}{r-1}\right)^3.$$

Hence we obtain, for $p \geqslant 2$ and $r \geqslant 1+\sqrt{c}$, for $\zeta < 1$,

$$|||\, M_t^{0,1} |||_{p,r} \leqslant \frac{Cc}{r}\left(\frac{r}{r-1}\right)^2 \tag{44}$$

and, for $\zeta = 1$, similarly,

$$|||\, M_t^{0,1} |||_{p,r} \leqslant \frac{Cc}{r}\left(\frac{r}{r-1}\right)\left(\left(\frac{r}{r-1}\right) + \frac{1}{\sqrt{\sigma}}\left(\frac{r}{r-1}\right)^{1/2}\right). \tag{45}$$

We consider next, for $k = 1, \dots, m$, the contribution of

$$\Delta_k(\theta, z, c, \hat\psi) = \frac{1}{k!}\sum_{j=0}^k \binom{k}{j}(-c)^{k-j} f_c(\theta,z)^j D^j\hat\psi(z).$$

In order to avoid the appearance of a spurious log term in the case $\zeta = 1$ we treat this contribution a little differently. We take an additional derivative, estimate the derivative and finally integrate that estimate. We have

$$f_c(\theta, z) = \int_0^c L_u(\theta, z)du$$

where $L_u(\theta, z) = e^{i\theta} L_u(e^{-i\theta} z)$ and $L_u(z)$ is given by (117). Then

$$\begin{aligned} H_s^k(\theta, z) &= \Delta_k(\theta, z, C_s(\theta), \hat{\Psi}_{s-}) \\ &= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-C_s(\theta))^{k-j} \left(\int_0^{C_s(\theta)} L_u(\theta, z)du \right)^j D^j \hat{\Psi}_{s-}(z) \end{aligned} \tag{46}$$

so

$$\begin{aligned} &D^2 P(\tau) H_s^k(\theta, z) \\ &\quad = \frac{1}{k!}(-C_s(\theta))^k D^2 P(\tau)\hat{\Psi}_{s-}(z) + \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} (-C_s(\theta))^{k-j} \\ &\quad \int_0^{C_s(\theta)} \dots \int_0^{C_s(\theta)} D^2 P(\tau)(L_{u_1,\dots,u_j}(\theta, .) D^j \hat{\Psi}_{s-})(z) du_1 \dots du_j \end{aligned}$$

where

$$L_{u_1,\dots,u_j}(\theta, z) = \prod_{i=1}^{j} L_{u_i}(\theta, z).$$

Hence, for $s \leqslant T_0$,

$$\begin{aligned} |D^2 P(\tau) H_s^k(\theta, z)| &\leqslant C c^k |D^2 P(\tau)\hat{\Psi}_{s-}(z)| \\ &+ C \sum_{j=1}^{k} c^{k-j} \int_0^{Cc} \dots \int_0^{Cc} |D^2 P(\tau)(L_{u_1,\dots,u_j}(\theta, .) D^j \hat{\Psi}_{s-})(z)| du_1 \dots du_j \end{aligned}$$

so

$$\begin{aligned} &\left(\fint_0^{2\pi} |D^2 P(\bar{\tau}_s) H_s^k(\theta, z)|^2 d\theta \right)^{1/2} \\ &\quad \leqslant C c^k h_s(z) + C \sum_{j=1}^{k} c^{k-j} \int_0^{Cc} \dots \int_0^{Cc} h_{s,u_1,\dots,u_j}(z) du_1 \dots du_j \end{aligned}$$

where

$$\begin{aligned} h_s(z) &= |D^2 P(\bar{\tau}_s)\hat{\Psi}_{s-}(z)|, \\ h_{s,t_1,\dots,t_j}(z) &= \left(\fint_0^{2\pi} |D^2 P(\bar{\tau}_s)(L_{u_1,\dots,u_j}(\theta, .) D^j \hat{\Psi}_{s-})(z)|^2 d\theta \right)^{1/2}. \end{aligned}$$

By Proposition A.6, for $|z| = r \geqslant 1 + \sqrt{c}$ and $u \leqslant Cc$,

$$|L_u(z)| \leqslant \frac{C|z|}{|z-1|}$$

so

$$\|L_{u_1,\dots,u_j}\|_{p,r} \leqslant C\left(\frac{r}{r-1}\right)^{j-1/p}$$

and so by Proposition A.7, for $j = 1, \dots, k$ and $\rho = (r+1)/2$ and $\rho' = (3r+1)/4$, for $\zeta < 1$,

$$\begin{aligned}\|h_{s,u_1,\dots,u_j}\|_{p,r} &\leqslant \|D^2 P(\bar{\tau}_s)\|_{p,\rho'\to r}\|L_{u_1,\dots,u_j}\|_{2,\rho'}\|D^j\hat{\Psi}_{s-}\|_{p,\rho'} \\ &\leqslant C\left(\left(\frac{r}{r-1}\right)^2 \wedge \frac{1}{\bar{\tau}_s^2}\right)\left(\frac{r}{r-1}\right)^{j-1/2}\left(\frac{r}{r-1}\right)^{j-1}\|D\hat{\Psi}_{s-}\|_{p,\rho}.\end{aligned}$$

In estimating $\|D^2 P(\tau)\|_{p,\rho'\to r}$, we used the better of two estimates – either the case $n = 0$ of Lemma 2.1 in conjunction with (119) or the case $n = 2$ of Lemma 2.1. A similar but easier estimate holds for $\|h_s\|_{p,r}$. Now

$$\begin{aligned}\langle DM^k(z)\rangle_t &\leqslant \frac{C}{c}\int_0^{T_0\wedge t} \fint_0^{2\pi} |D^2 P(\bar{\tau}_s)H_s^k(\theta,z)|^2 d\theta ds \\ &\leqslant \frac{C}{c}\int_0^{T_0\wedge t}\left(c^k h_s(z) + \sum_{j=1}^k c^{k-j}\int_0^{Cc}\dots\int_0^{Cc} h_{s,u_1,\dots,u_j}(z)du_1\dots du_j\right)^2 ds\end{aligned}$$

so, for $r \geqslant 1 + \sqrt{c}$ and $\zeta < 1$,

$$\begin{aligned}&\|\langle DM^k(.)\rangle_t\|_{p/2,r} \\ &\leqslant \frac{C}{c}\int_0^{T_0\wedge t}\left(c^k\|h_s\|_{p,r} + \sum_{j=1}^k c^{k-j}\int_0^{Cc}\dots\int_0^{Cc}\|h_{s,u_1,\dots,u_j}\|_{p,r}du_1\dots du_j\right)^2 ds \\ &\leqslant Cc^{2k-1}\left(\frac{r}{r-1}\right)^{4k-3}\int_0^{T_0\wedge t}\left(\left(\frac{r}{r-1}\right)^4 \wedge \frac{1}{\bar{\tau}_s^4}\right)\|D\hat{\Psi}_{s-}\|_{p,\rho}^2 ds \\ &\leqslant Cc\left(\frac{r}{r-1}\right)\int_0^{T_0\wedge t}\left(\left(\frac{r}{r-1}\right)^4 \wedge \frac{1}{\bar{\tau}_s^4}\right)\|D\hat{\Psi}_{s-}\|_{p,\rho}^2 ds\end{aligned}$$

and so

$$|||\langle DM^k(.)\rangle_t|||_{p/2,r} \leqslant Cc\left(\frac{r}{r-1}\right)^4 \sup_{s\leqslant t}|||D\hat{\Psi}_{s-}1_{\{s\leqslant T_0\}}|||_{p,\rho}^p.$$

We used here the inequality

$$\int_0^\infty a^p \wedge s^{-p} ds \leqslant \left(\frac{p}{p-1}\right)a^{p-1} \tag{47}$$

which holds for all $a > 0$ and all $p > 1$, to see that

$$\int_0^t \left(\left(\frac{r}{r-1} \right)^4 \wedge \frac{1}{\bar{\tau}_s^4} \right) ds \leqslant C \int_0^t \left(\left(\frac{r}{r-1} \right)^4 \wedge \frac{1}{\bar{\tau}_s^4} \right) \dot{\tau}_s ds \leqslant C \left(\frac{r}{r-1} \right)^3 .$$

For $p \geqslant 2$ and $r > 1$, we have

$$|||(\Delta DM^k(.))_t^*|||_{p,r}^p \leqslant \frac{C}{c} \mathbb{E} \int_0^{T_0 \wedge t} ⨍_0^{2\pi} \|D^2 P(\bar{\tau}_s) H_s^k(\theta, .)\|_{p,r}^p d\theta ds$$

and, from (46), for $r \geqslant 1 + \sqrt{c}$, estimating as above but now using the second estimate of Proposition A.7, we get

$$\begin{aligned} \|D^2 P(\bar{\tau}_s) H_s^k(\theta, .)\|_{p,r} &\leqslant Cc^k \left(\left(\frac{r}{r-1} \right)^2 \wedge \frac{1}{\bar{\tau}_s^2} \right) \left(\frac{r}{r-1} \right)^{2k-1-1/p} \|D\hat{\Psi}_{s-}\|_{p,\rho} \\ &\leqslant Cc \left(\left(\frac{r}{r-1} \right)^2 \wedge \frac{1}{\bar{\tau}_s^2} \right) \left(\frac{r}{r-1} \right)^{1-1/p} \|D\hat{\Psi}_{s-}\|_{p,\rho} \end{aligned}$$

so, for $r \geqslant 1 + \sqrt{c}$,

$$\begin{aligned} &||| \, (\Delta DM^k(.))_t^* |||_{p,r}^p \\ &\leqslant Cc^{p-1} \left(\frac{r}{r-1} \right)^{p-1} \mathbb{E} \int_0^{T_0 \wedge t} \left(\left(\frac{r}{r-1} \right)^{2p} \wedge \frac{1}{\bar{\tau}_s^{2p}} \right) \|D\hat{\Psi}_{s-}\|_{p,\rho}^p ds \\ &\leqslant Cc^{p-1} \left(\frac{r}{r-1} \right)^{3p-2} \sup_{s \leqslant t} |||D\hat{\Psi}_{s-} 1_{\{s \leqslant T_0\}} \, |||_{p,\rho}^p . \end{aligned} \tag{48}$$

On substituting the estimates for $\langle DM^k(z) \rangle_t$ and $(\Delta DM^k(.))_t^*$ into (30), we obtain for $r \geqslant 1 + \sqrt{c}$ and $p \geqslant 2$, for $\zeta < 1$,

$$||| \, DM_t^k |||_{p,r} \leqslant C\sqrt{c} \left(\frac{r}{r-1} \right)^2 \sup_{s \leqslant t} |||D\hat{\Psi}_{s-} 1_{\{s \leqslant T_0\}} \, |||_{p,\rho} . \tag{49}$$

In the case $\zeta = 1$, we have to modify the above estimation in using

$$\|D^2 P(\tau)\psi\|_{p,r} \leqslant C \left(\left(\frac{r}{r-1} \right)^2 \wedge \left(\frac{1}{\tau^2} \vee \frac{1}{\sigma\tau} \right) \right) \|\psi\|_{p,\rho}.$$

We obtain in this case

$$||| \, DM_t^k |||_{p,r} \leqslant C\sqrt{c} \left(\left(\frac{r}{r-1} \right)^2 + \frac{1}{\sqrt{\sigma}} \left(\frac{r}{r-1} \right)^{3/2} \right) \sup_{s \leqslant t} |||D\hat{\Psi}_{s-} 1_{\{s \leqslant T_0\}} \, |||_{p,\rho} . \tag{50}$$

Now, since the holomorphic functions M_t^k and $\hat{\Psi}_{s-}$ vanish at ∞, we have

$$M_t^k(z) = -\int_1^\infty \frac{DM_t^k(az)}{a} da$$

and, for $a \geqslant 1$,

$$\|\hat{\Psi}_{s-}\|_{p,a\rho} \leqslant \frac{C}{a}\|\hat{\Psi}_{s-}\|_{p,\rho}$$

so, on integrating (49) and (50) we obtain, for $\zeta < 1$,

$$|||\, M_t^k |||_{p,r} \leqslant C\sqrt{c}\left(\frac{r}{r-1}\right)\sup_{s\leqslant t}|||D\hat{\Psi}_{s-}1_{\{s\leqslant T_0\}}\,|||_{p,\rho}\,. \tag{51}$$

while, for $\zeta = 1$,

$$|||\, M_t^k |||_{p,r} \leqslant C\sqrt{c}\left(\left(\frac{r}{r-1}\right)+\frac{1}{\sqrt{\sigma}}\left(\frac{r}{r-1}\right)^{1/2}\right)\sup_{s\leqslant t}|||D\hat{\Psi}_{s-}1_{\{s\leqslant T_0\}}\,|||_{p,\rho}\,. \tag{52}$$

(We remark that, if a similar argument is used to estimate $|||M_t^k|||_{p,r}$ directly, then one obtains the same estimate (51) for $\zeta < 1$ but one faces in the case $\zeta = 1$ the integral

$$\int_0^t \left(\frac{r}{r-1}\right)\wedge\frac{1}{\sigma\bar{\tau}_s}ds.$$

The $p = 1$ case of (47) then generates a log term, which our method avoids.)

We consider finally the contribution of

$$\Delta_{m+1}(\theta, z, c, \hat{\psi}) = \frac{1}{m!}\int_0^1 (1-u)^m e^{-cu}\sum_{j=0}^{m+1}\binom{m+1}{j} (-c)^{m+1-j} f_c(\theta, z)^j D^j\hat{\psi}(F_{c,u}(\theta, z))du.$$

Then

$$\begin{aligned} H_s^{m+1}(\theta, z) &= \Delta_{m+1}(\theta, z, C_s(\theta), \hat{\Psi}_{s-}) \\ &= \frac{1}{m!}\int_0^1 (1-u)^m e^{-C_s(\theta)u}\sum_{j=0}^{m+1}\binom{m+1}{j} \\ &\quad (-C_s(\theta))^{m+1-j} f_{C_s(\theta)}(\theta, z)^j D^j\hat{\Psi}_{s-}(F_{C_s(\theta),u}(\theta, z))du. \end{aligned}$$

By Proposition A.5, we have

$$|f_c(\theta, z)| \leqslant \frac{Cc|z|}{|e^{-i\theta}z-1|}.$$

Hence, for $s \leqslant T_0$ and $\tau \geqslant 0$,

$$\begin{aligned} &\|DP(\tau)H_s^{m+1}(\theta, .)\|_{p,r} \\ &\leqslant Cc^{m+1}\|DP(\tau)\hat{\Psi}_{s-}\|_{p,r} + C\|DP(\tau)\|_{p,\rho'\to r} \\ &\quad \sum_{j=1}^{m+1} c^{m+1-j}\|f_{C_s(\theta)}(\theta, .)^j\|_{p,\rho'}\|D^j\hat{\Psi}_{s-}(F_{C_s(\theta),u}(\theta, .))\|_{\infty,\rho'}. \end{aligned}$$

By Lemma 2.1, for $\zeta < 1$,

$$\|DP(\tau)\|_{p,\rho'\to r} \leqslant C\left(\frac{r}{r-1}\right)\wedge\frac{1}{\tau}.$$

We have

$$\|f_{C_s(\theta)}(\theta,.)^j\|_{p,\rho'} \leqslant Cc^j\left(\frac{r}{r-1}\right)^{j-1/p}$$

and, since $|F_{c,u}(\theta,z)| \geqslant |z|$, we have

$$\|D^j\hat{\Psi}_{s-}(F_{C_s(\theta),u}(\theta,.))\|_{\infty,\rho'} \leqslant \|D^j\hat{\Psi}_{s-}\|_{\infty,\rho'} \leqslant C\left(\frac{r}{r-1}\right)^{j-1+1/p}\|D\hat{\Psi}_{s-}\|_{p,\rho}.$$

Hence, for $\zeta < 1$, we have

$$\|DP(\tau)H_s^{m+1}(\theta,.)\|_{p,r} \leqslant Cc^{m+1}\left(\left(\frac{r}{r-1}\right)\wedge\frac{1}{\tau}\right)\left(\frac{r}{r-1}\right)^{2m+1}\|D\hat{\Psi}_{s-}\|_{p,\rho}$$

so, using (33),

$$\|\langle M^{m+1}(.)\rangle_t\|_{p/2,r} \leqslant Cc^{2m+1}\left(\frac{r}{r-1}\right)^{4m+2}\int_0^{T_0\wedge t}\left(\left(\frac{r}{r-1}\right)^2\wedge\frac{1}{\bar{\tau}_s^2}\right)\|D\hat{\Psi}_s\|_{p,\rho}^2 ds$$

and so

$$\begin{aligned}|||\langle M^{m+1}(.)\rangle_t|||_{p/2,r} &\leqslant Cc^{2m+1}\left(\frac{r}{r-1}\right)^{4m+3}\sup_{s\leqslant t}|||D\hat{\Psi}_s 1_{\{s\leqslant T_0\}}|||_{p,\rho}^2\\ &\leqslant Cc\left(\frac{r}{r-1}\right)^2\sup_{s\leqslant t}|||D\hat{\Psi}_s 1_{\{s\leqslant T_0\}}|||_{p,\rho}^2 .\end{aligned}$$

Here we have used our choice of $m \geqslant 1/(8\varepsilon)$ and the assumption $r \geqslant 1+c^{1/2-\varepsilon}$ to see that

$$c^{2m}\left(\frac{r}{r-1}\right)^{4m+1} \leqslant C.$$

The bound (48) remains valid with M^{m+1} in place of M^k. Hence for $\zeta < 1$

$$|||M_t^{m+1}|||_{p,r} \leqslant C\sqrt{c}\left(\frac{r}{r-1}\right)\sup_{s\leqslant t}|||D\hat{\Psi}_s 1_{\{s\leqslant T_0\}}|||_{p,\rho} . \tag{53}$$

For $\zeta = 1$, given the weaker bound for $\|DP(\tau)\|_{p,r}$ in Lemma 2.1, we adapt the argument as above to obtain

$$\begin{aligned}&|||\langle M^{m+1}(.)\rangle_t|||_{p/2,r}\\ &\leqslant Cc\left(\left(\frac{r}{r-1}\right)^2+\frac{1}{\sigma}\left(\frac{r}{r-1}\right)\right)\sup_{s\leqslant t}|||D\hat{\Psi}_s 1_{\{s\leqslant T_0\}}|||_{p,\rho}^2\end{aligned}$$

where log term has been absorbed using our choice of m, and then

$$|||\, M_t^{m+1} |||_{p,r} \leqslant C\sqrt{c}\Bigg(\left(\frac{r}{r-1}\right) + \frac{1}{\sqrt{\sigma}} \left(\frac{r}{r-1}\right)^{1/2} \Bigg) \sup_{s\leqslant t} |||\, D\hat{\Psi}_s 1_{\{s\leqslant T_0\}} \,|||_{p,\rho} \,. \tag{54}$$

Now

$$M_t = M_t^{0,0} + M_t^{0,1} + \sum_{k=1}^{m+1} M_t^k$$

and we have shown that all terms on the right-hand side can be bounded by the right-hand side in (27), so this first estimate is now proved.

It remains to show the second estimate. Fix $t \geqslant 0$ and consider, for $|z| > 1$, the martingale $(\Pi_x(z))_{x\geqslant 0}$ given by

$$\Pi_x(z) = \int_{E(x)} e^{-\bar{\tau}_s} P(\bar{\tau}_s) DH(\theta, z) 2c_s 1_{\{v\leqslant \lambda_s,\, s\leqslant T_0\}} \tilde{\mu}(d\theta, dv, ds).$$

Set $\tilde{M}_x(z) = M_x^{0,0}(z) - \Pi_x(z)$. Then

$$\tilde{M}_x(z) = \int_{E(x)} e^{-\bar{\tau}_s} \left(a_0(C_s(\theta)) 1_{\{v\leqslant \Lambda_s(\theta)\}} - 2c_s 1_{\{v\leqslant \lambda_s\}} \right) DP(\bar{\tau}_s) H(\theta, z) 1_{\{s\leqslant T_0\}} \tilde{\mu}(d\theta, dv, ds)$$

and

$$D(\hat{M}_t - \hat{\Pi}_t) = M_t - \Pi_t = \tilde{M}_t + M_t^{0,1} + \sum_{k=1}^{m+1} M_t^k .$$

For all but the first term on the right, the bounds (44), (45), (51), (52), (53), (54), are sufficient for (28). It remains to show a suitable bound on $\tilde{M}_t$. We use the estimate (25) to see that, for $\zeta < 1$,

$$\begin{aligned} \langle \tilde{M}(z) \rangle_t &= \int_0^{T_0\wedge t} \int_0^\infty \int_0^{2\pi} e^{-2\bar{\tau}_s} |a_0(C_s(\theta)) 1_{\{v\leqslant \Lambda_s(\theta)\}} \\ &\quad - 2c_s 1_{\{v\leqslant\lambda_s\}}|^2 |DP(\bar{\tau}_s) H(\theta, z)|^2 d\theta dv ds \\ &\leqslant \frac{Cc\delta_0}{r^2} \left(\frac{r}{r-1}\right)^2 \end{aligned}$$

while for $\zeta = 1$ we obtain similarly

$$\langle \tilde{M}(z) \rangle_t \leqslant \frac{Cc\delta_0}{r^2} \Bigg(\left(\frac{r}{r-1}\right)^2 + \frac{1}{\sigma} \left(\frac{r}{r-1}\right) \Bigg).$$

Otherwise we can proceed as for $M^{0,0}$ to arrive as the following estimates, which suffice for (28). For $\zeta < 1$, we have

$$|||\tilde{M}_t|||_{p,r} \leqslant \frac{C\sqrt{c\delta_0}}{r} \left(\frac{r}{r-1}\right) + \frac{Cc}{r} \left(\frac{r}{r-1}\right)^2$$

while for $\zeta = 1$

$$|||\tilde{M}_t|||_{p,r} \leqslant \frac{C\sqrt{c\delta_0}}{r}\left(\left(\frac{r}{r-1}\right) + \frac{1}{\sqrt{\sigma}}\left(\frac{r}{r-1}\right)^{1/2}\right) + \frac{Cc}{r}\left(\frac{r}{r-1}\right)^2.$$

□

4.2. Estimates for the drift terms. We turn to the drift terms, beginning with estimates for the drift $(\hat{\beta}, \beta^{\text{cap}})$ of the ALE(α, η) process. Recall that $(\mathcal{T}_t)_{t\geqslant 0}$ has drift given by

$$\beta^{\text{cap}}(\hat{\phi}, \tau) = ⨍_0^{2\pi} c(\theta, \hat{\phi}, \tau)\lambda(\theta, \hat{\phi}, \tau)d\theta$$

where

$$c(\theta, \hat{\phi}, \tau) = ce^{-\alpha\tau}|\hat{\phi}'(e^{\sigma+i\theta})|^{-\alpha}, \quad \lambda(\theta, \hat{\phi}, \tau) = c^{-1}e^{-\eta\tau}|\hat{\phi}'(e^{\sigma+i\theta})|^{-\eta}.$$

Lemma 4.4. *For all $\zeta \in \mathbb{R}$ and all $T < t_\zeta$, there is a constant $C(\zeta, T) < \infty$ such that, for all $\delta_0 \in (0, 1/2]$, all $t \leqslant T$, all $\hat{\phi} \in \mathcal{S}_1$ and all $\tau \geqslant 0$, we have*

$$|\beta^{\text{cap}}(\hat{\phi}, \tau) - e^{-\zeta\tau_t} + \zeta e^{-\zeta\tau_t}\psi_t^{\text{cap}}| \leqslant C\delta_0^2$$

whenever $|\psi_t^{\text{cap}}| \leqslant \delta_0$ and $|\hat{\psi}'(e^{\sigma+i\theta})| \leqslant \delta_0$ for all θ, where $\psi_t^{\text{cap}} = \tau - \tau_t$ and $\hat{\psi}(z) = \hat{\phi}(z) - z$.

Proof. We have

$$c(\theta, \hat{\phi}, \tau)\lambda(\theta, \hat{\phi}, \tau) = e^{-\zeta\tau}|\hat{\phi}'(e^{\sigma+i\theta})|^{-\zeta} = e^{-\zeta\tau_t}e^{-\zeta\psi_t^{\text{cap}}}|1 + \hat{\psi}'(e^{\sigma+i\theta})|^{-\zeta}$$

and, for $|w| \leqslant 1/2$,

$$|1 + w|^{-\zeta} = 1 - \zeta \operatorname{Re} w + \varepsilon(w), \quad |\varepsilon(w)| \leqslant C|w|^2$$

so

$$c(\theta, \hat{\phi}, \tau)\lambda(\theta, \hat{\phi}, \tau) = e^{-\zeta\tau_t}\left(1 - \zeta\psi_t^{\text{cap}} - \zeta \operatorname{Re} \hat{\psi}'(e^{\sigma+i\theta}) + \gamma_t(\theta, \hat{\phi}, \tau)\right) \tag{55}$$

where

$$|\gamma_t(\theta, \hat{\phi}, \tau)| \leqslant C\delta_0^2 \tag{56}$$

whenever $|\psi_t^{\text{cap}}| \leqslant \delta_0$ and $|\hat{\psi}'(e^{\sigma+i\theta})| \leqslant \delta_0$ for all θ. For $\hat{\phi} \in \mathcal{S}_1$, $\hat{\psi}$ is holomorphic in $\{|z| > 1\}$ and bounded at ∞, so

$$⨍_0^{2\pi} \operatorname{Re} \hat{\psi}'(e^{\sigma+i\theta})d\theta = 0. \tag{57}$$

The claimed estimate follows on integrating (55) in θ. □

Recall that the drift of $(\hat{\Phi}_t)_{t\geqslant 0}$ is given by

$$\hat{\beta}(\hat{\phi},\tau)(z)=\int_0^{2\pi}\Delta(\theta,z,c(\theta,\hat{\phi},\tau),\hat{\phi})\lambda(\theta,\hat{\phi},\tau)d\theta$$

where

$$\Delta(\theta,z,c,\hat{\phi})=e^{-c}\hat{\phi}(F_c(\theta,z))-\hat{\phi}(z),\quad F_c(\theta,z)=e^{i\theta}F_c(e^{-i\theta}z).$$

It is convenient in the following statement to use the notation

$$\|\phi\|_{p,r,0}=\|\phi-\phi(\infty)\|_{p,r}$$

for functions ϕ holomorphic in $\{|z|>1\}$ and bounded at ∞.

Lemma 4.5. *For all $\alpha,\eta\in\mathbb{R}$ and all $T<t_\zeta$, there is a constant $C(\alpha,\eta,\Lambda,T)<\infty$ with the following property. For all $c\in(0,1/C]$, all $\sigma>0$, all $\delta_0\in(0,1/2]$, all $t\leqslant T$, all $\hat{\phi}\in\mathcal{S}_1$ and all $\tau\geqslant 0$, we have*

$$|\hat{\beta}(\hat{\phi},\tau)(\infty)+e^{-\zeta\tau_t}(Q+1)\hat{\psi}(\infty)|\leqslant C(\delta_0\sqrt{c}+\delta_0^2)+C(c+\delta_0)|\hat{\psi}(\infty)| \tag{58}$$

whenever $|\psi_t^{\text{cap}}|\leqslant\delta_0$ and $|\hat{\psi}'(e^{\sigma+i\theta})|\leqslant\delta_0$ for all θ, where $\psi_t^{\text{cap}}=\tau-\tau_t$ and $\hat{\psi}(z)=\hat{\phi}(z)-z$.

Moreover, for all $\alpha,\eta\in\mathbb{R}$, all $\varepsilon\in(0,1/2]$, all $p\geqslant 2$ and all $T<\tau_\zeta$, there is a constant $C(\alpha,\eta,\Lambda,\varepsilon,p,T)<\infty$ with the following property. For all $c\in(0,1/C]$, all $\sigma>0$, all $\delta_0\in(0,1/2]$, all $t\leqslant T$, all $\hat{\phi}\in\mathcal{S}_1$ and all $\tau\geqslant 0$, for all $r\geqslant 1+c^{1/2-\varepsilon}$ and $\rho=(3r+1)/4$, we have

$$\begin{aligned}&\|\hat{\beta}(\hat{\phi},\tau)+e^{-\zeta\tau_t}(Q+1)\hat{\psi}\|_{p,r,0}\\&\quad\leqslant\frac{C\delta_0^2}{r}\left(1+\log\left(\frac{r}{r-1}\right)\right)+\frac{C\delta_0}{r}\left(1+\log\left(\frac{r}{r-1}\right)\right)r\|D\hat{\psi}\|_{p,\rho}\\&\qquad+\frac{C\delta_0\sqrt{c}}{r}\left(\frac{r}{r-1}\right)+\frac{Cc}{r}\left(\frac{r}{r-1}\right)r\|D\hat{\psi}\|_{p,\rho}\end{aligned} \tag{59}$$

whenever $|\psi_t^{\text{cap}}|\leqslant\delta_0$ and $|\hat{\psi}'(e^{\sigma+i\theta})|\leqslant\delta_0$ for all θ.

Proof. We use the split (35) and the Taylor expansion (36) to write

$$\Delta(\theta,z,c,\hat{\phi})=\Delta_0(\theta,z,c)+\sum_{k=1}^{m+1}\Delta_k(\theta,z,c,\hat{\psi})$$

where $m=\lceil 1/(8\varepsilon)\rceil$. We further split

$$\Delta_0(\theta,z,c)=\frac{2cz}{e^{-i\theta}z-1}+\tilde{\Delta}_0(\theta,z,c) \tag{60}$$

and

$$\Delta_1(\theta,z,c,\hat{\psi})=c(D\hat{\psi}(z)-\hat{\psi}(z))+\tilde{\Delta}_1(\theta,z,c,\hat{\psi}).$$

Note that we now split Δ_0 slightly differently to the split $\Delta_0 = \Delta_{0,0} + \Delta_{0,1}$ used for the martingale term: where before we had $a_0(c)$ we now approximate by $2c$, putting an additional error into the remainder term $\tilde{\Delta}_0$. Set

$$\tilde{\Delta}(\theta, z, c, \hat{\phi}) = \tilde{\Delta}_0(\theta, z, c) + \tilde{\Delta}_1(\theta, z, c, \hat{\psi}) + \sum_{k=2}^{m+1} \Delta_k(\theta, z, c, \hat{\psi})$$

and note that

$$e^{-c}\hat{\phi}(F_c(\theta, z)) - \hat{\phi}(z) = c\left(\frac{2z}{e^{-i\theta}z - 1} + D\hat{\psi}(z) - \hat{\psi}(z)\right) + \tilde{\Delta}(\theta, z, c, \hat{\psi}). \quad (61)$$

We use equation (55) to write

$$\begin{aligned}
\hat{\beta}(\hat{\phi}, \tau)(z) &= \fint_0^{2\pi} \left(\frac{2z}{e^{-i\theta}z - 1} + D\hat{\psi}(z) - \hat{\psi}(z)\right) c(\theta, \hat{\phi}, \tau)\lambda(\theta, \hat{\phi}, \tau)d\theta \\
&\quad + \fint_0^{2\pi} \tilde{\Delta}(\theta, z, c(\theta, \hat{\phi}, \tau), \hat{\psi})\lambda(\theta, \hat{\phi}, \tau)d\theta \\
&= e^{-\zeta\tau_t} \fint_0^{2\pi} \left(D\hat{\psi}(z) - \hat{\psi}(z) + \frac{2z}{e^{-i\theta}z - 1}\left(1 - \zeta\psi_t^{\text{cap}} - \zeta \operatorname{Re}\hat{\psi}'(e^{\sigma+i\theta})\right)\right) d\theta \\
&\quad + e^{-\zeta\tau_t} \fint_0^{2\pi} \frac{2z}{e^{-i\theta}z - 1}\gamma_t(\theta, \hat{\phi}, \tau)d\theta \\
&\quad + (D\hat{\psi}(z) - \hat{\psi}(z)) \fint_0^{2\pi} (c(\theta, \hat{\phi}, \tau)\lambda(\theta, \hat{\phi}, \tau) - e^{-\zeta\tau_t})d\theta \\
&\quad + \fint_0^{2\pi} \tilde{\Delta}(\theta, z, c(\theta, \hat{\phi}, \tau), \hat{\psi})\lambda(\theta, \hat{\phi}, \tau)d\theta.
\end{aligned}$$

Now $\hat{\psi}'(z) \to 0$ as $z \to \infty$, so

$$\begin{aligned}
&e^{-\zeta\tau_t} \fint_0^{2\pi} \left(D\hat{\psi}(z) - \hat{\psi}(z) + \frac{2z}{e^{-i\theta}z - 1}\left(1 - \zeta\psi_t^{\text{cap}} - \zeta \operatorname{Re}\hat{\psi}'(e^{\sigma+i\theta})\right)\right) d\theta \\
&\quad = e^{-\zeta\tau_t}\left(D\hat{\psi}(z) - \hat{\psi}(z) - \zeta e^{-\sigma} D\hat{\psi}(e^{\sigma}z)\right) = -e^{-\zeta\tau_t}(Q+1)\hat{\psi}(z).
\end{aligned}$$

Hence

$$\begin{aligned}
&\hat{\beta}(\hat{\phi}, \tau)(\infty) + e^{-\zeta\tau_t}(Q+1)\hat{\psi}(\infty) \\
&\quad = 2e^{-\zeta\tau_t} \fint_0^{2\pi} e^{i\theta}\gamma_t(\theta, \hat{\phi}, \tau)d\theta - \hat{\psi}(\infty) \fint_0^{2\pi} (c(\theta, \hat{\phi}, \tau)\lambda(\theta, \hat{\phi}, \tau) - e^{-\zeta\tau_t})d\theta \\
&\qquad + \fint_0^{2\pi} \tilde{\Delta}(\theta, \infty, c(\theta, \hat{\phi}, \tau), \hat{\psi})\lambda(\theta, \hat{\phi}, \tau)d\theta \qquad (62)
\end{aligned}$$

and

$$\begin{aligned}
&\hat{\beta}(\hat{\phi}, \tau)(z) + e^{-\zeta \tau_t}(Q+1)\hat{\psi}(z) \\
&\quad = e^{-\zeta \tau_t} \fint_0^{2\pi} \frac{2z}{e^{-i\theta}z - 1} \gamma_t(\theta, \hat{\phi}, \tau) d\theta \\
&\qquad + (D\hat{\psi}(z) - \hat{\psi}(z)) \fint_0^{2\pi} (c(\theta, \hat{\phi}, \tau)\lambda(\theta, \hat{\phi}, \tau) - e^{-\zeta \tau_t}) d\theta \\
&\qquad + \fint_0^{2\pi} \tilde{\Delta}(\theta, z, c(\theta, \hat{\phi}, \tau), \hat{\psi})\lambda(\theta, \hat{\phi}, \tau) d\theta. \qquad (63)
\end{aligned}$$

We will estimate the terms on the right-hand sides of (62) and (63), assuming from now on that t, $\hat{\phi}$ and τ are chosen so that $|\psi_t^{\text{cap}}| \leqslant \delta_0$ and $|\hat{\psi}'(e^{\sigma+i\theta})| \leqslant \delta_0$ for all θ.

From (55) and (56), we have $|\gamma_t(\theta, \hat{\phi}, \tau)| \leqslant C\delta_0^2$ and

$$\left| \fint_0^{2\pi} (c(\theta, \hat{\phi}, \tau)\lambda(\theta, \hat{\phi}, \tau) - e^{-\zeta \tau_t}) d\theta \right| \leqslant C\delta_0.$$

We use (26) to see that

$$\begin{aligned}
\tilde{\Delta}(\theta, \infty, c, \hat{\psi}) &= \Delta(\theta, \infty, c, \hat{\psi}) - 2ce^{i\theta} + c\hat{\psi}(\infty) \\
&= (a_0(c) - 2c)e^{i\theta} + (e^{-c} - 1 + c)\hat{\psi}(\infty).
\end{aligned}$$

Write $c(\theta)$ for $c(\theta, \hat{\phi}, \tau)$ and $\lambda(\theta)$ for $\lambda(\theta, \hat{\phi}, \tau)$. Then

$$|c(\theta) - c_t| \leqslant C\delta_0 c, \quad |\lambda(\theta) - \lambda_t| \leqslant C\delta_0 c^{-1}$$

and, by Proposition A.5, we have

$$|a_0(c(\theta)) - 2c(\theta)| \leqslant Cc^{3/2}, \quad |(a_0(c(\theta)) - 2c(\theta)) - (a_0(c_t) - 2c_t)| \leqslant Cc^{3/2}\delta_0.$$

We can now estimate in (62) to obtain (58).

It remains to prove (59). For $|z| = r > 1$, we have

$$\begin{aligned}
\left| \fint_0^{2\pi} \frac{2e^{i\theta}}{e^{-i\theta}z - 1} \gamma_t(\theta, \hat{\phi}, \tau) d\theta \right| &\leqslant C\delta_0^2 \fint_0^{2\pi} \frac{1}{|e^{-i\theta}z - 1|} d\theta \\
&\leqslant \frac{C\delta_0^2}{r} \left(1 + \log\left(\frac{r}{r-1}\right)\right). \qquad (64)
\end{aligned}$$

Since $\hat{\psi}$ is bounded at ∞, by Marcinkiewicz's multiplier theorem, $\|\hat{\psi}\|_{p,r,0} \leqslant C\|D\hat{\psi}\|_{p,r}$ for all $p > 1$ and $r > 1$. Hence

$$\begin{aligned}
&\left\| (D\hat{\psi}(z) - \hat{\psi}(z)) \fint_0^{2\pi} (c(\theta, \hat{\phi}, \tau)\lambda(\theta, \hat{\phi}, \tau) - e^{-\zeta \tau_t}) d\theta \right\|_{p,r,0} \\
&\quad \leqslant C\delta_0 \|D\hat{\psi}\|_{p,r}. \qquad (65)
\end{aligned}$$

It remains to deal with the final term in (63). We first estimate the function obtained on replacing $c(\theta, \hat{\phi}, \tau)$ and $\lambda(\theta, \hat{\phi}, \tau)$ in that term by $c_t = ce^{-\alpha\tau_t}$ and $\lambda_t = c^{-1}e^{-\eta\tau_t}$. Note that, in the case $F_c(z) = e^c z$ and $m = 1$, the Taylor expansion (36) has the form

$$e^{-c}\hat{\phi}(e^c z) - \hat{\phi}(z) = c(D\hat{\psi}(z) - \hat{\psi}(z)) + c^2 \int_0^1 (1-u)e^{-cu}(D^2\hat{\psi}(e^{cu}z) - 2D\hat{\psi}(e^{cu}z) + \hat{\psi}(e^{cu}z))du.$$

On the other hand, by Cauchy's theorem,

$$⨍_0^{2\pi} \hat{\phi}(F_c(\theta, z))d\theta = \hat{\phi}(e^c z).$$

Hence, on integrating in θ in (61), we see that

$$⨍_0^{2\pi} \tilde{\Delta}(\theta, z, c, \hat{\psi})d\theta = c^2 \int_0^1 (1-u)e^{-cu}(D^2\hat{\psi}(e^{cu}z) - 2D\hat{\psi}(e^{cu}z) + \hat{\psi}(e^{cu}z))du$$

so, for $r > 1$ and $\rho = (3r+1)/4$,

$$\left\| ⨍_0^{2\pi} \tilde{\Delta}(\theta, ., c_t, \hat{\psi})\lambda_t d\theta \right\|_{p,r,0} \leqslant Cc\left(\frac{r}{r-1}\right)\|D\hat{\psi}\|_{p,\rho}. \tag{66}$$

It remains to deal with the error made in replacing $c(\theta, \hat{\phi}, \tau)$ and $\lambda(\theta, \hat{\phi}, \tau)$ by c_t and λ_t. We make a further split

$$\tilde{\Delta}_0(\theta, z, c) = \tilde{\Delta}_{0,0}(\theta, z, c) + \tilde{\Delta}_{0,1}(\theta, z, c), \quad \tilde{\Delta}(\theta, z, c) = \bar{\Delta}(\theta, z, c) + \tilde{\Delta}_{0,1}(\theta, z, c)$$

where

$$\tilde{\Delta}_{0,0}(\theta, z, c) = e^{-c}F_c(\theta, z) - z - \frac{a_0(c)z}{e^{-i\theta}z - 1}, \quad \tilde{\Delta}_{0,1}(\theta, z, c) = \frac{(a_0(c) - 2c)z}{e^{-i\theta}z - 1}.$$

Thus $\tilde{\Delta}_{0,0} = \Delta_{0,1}$, as considered in estimating the martingale terms, and $\tilde{\Delta}_{0,1}$ is the additional error introduced by the new split (60). We first estimate the $\tilde{\Delta}_{0,1}$ term. Since $|\psi_t^{\text{cap}}| \leqslant \delta_0$ and $|\hat{\psi}'(e^{\sigma+i\theta})| \leqslant \delta_0$ for all θ, we have

$$|c(\theta, \hat{\phi}, \tau) - c_t| \leqslant Cc\delta_0, \quad |\lambda(\theta, \hat{\phi}, \tau) - \lambda_t| \leqslant C\delta_0/c.$$

Hence, by Proposition A.5, for $c \leqslant 1/C$,

$$|(a_0(c(\theta, \hat{\phi}, \tau)) - 2c(\theta, \hat{\phi}, \tau))\lambda(\theta, \hat{\phi}, \tau) - (a_0(c_t) - 2c_t)\lambda_t| \leqslant C\delta_0\sqrt{c}$$

and, estimating as for (64), we obtain

$$\|\tilde{\Delta}_{0,1}(\theta, ., c(\theta, \hat{\phi}, \tau))\lambda(\theta, \hat{\phi}, \tau) - \tilde{\Delta}_{0,1}(\theta, ., c_t)\lambda_t\|_{p,r,0} \leqslant \frac{C\delta_0\sqrt{c}}{r}\left(1 + \log\left(\frac{r}{r-1}\right)\right). \tag{67}$$

By Proposition A.5, for $c \leqslant 1/C$,

$$|\tilde{\Delta}_{0,0}(\theta, z, c)| \leqslant \frac{Cc^{3/2}|z|}{|e^{-i\theta}z - 1|^2}$$

and, for $c_1, c_2 \in (0, c]$ and $|z| \geqslant 1 + \sqrt{c}$,

$$|\tilde{\Delta}_{0,0}(\theta, z, c_1) - \tilde{\Delta}_{0,0}(\theta, z, c_2)| \leqslant \frac{C\sqrt{c}|c_1 - c_2||z|}{|e^{-i\theta}z - 1|^2}$$

so

$$|\tilde{\Delta}_{0,0}(\theta, z, c(\theta, \hat{\phi}, \tau))\lambda(\theta, \hat{\phi}, \tau) - \tilde{\Delta}_{0,0}(\theta, z, c_t)\lambda_t| \leqslant \frac{C\delta_0\sqrt{c}|z|}{|e^{-i\theta}z - 1|^2}$$

so, for $|z| = r \geqslant 1 + \sqrt{c}$,

$$⨍_0^{2\pi} |\tilde{\Delta}_{0,0}(\theta, z, c(\theta, \hat{\phi}, \tau))\lambda(\theta, \hat{\phi}, \tau) - \tilde{\Delta}_{0,0}(\theta, z, c_t)\lambda_t|d\theta \leqslant \frac{C\delta_0\sqrt{c}}{r - 1}. \tag{68}$$

We have

$$\tilde{\Delta}_1(\theta, z, c, \hat{\psi}) = \left(\log\left(\frac{F_c(\theta, z)}{z}\right) - c\right) D\hat{\psi}(z)$$

so, by Proposition A.5, for $c \leqslant 1/C$,

$$|\tilde{\Delta}_1(\theta, z, c, \hat{\psi})| \leqslant \frac{Cc}{|e^{-i\theta}z - 1|}|D\hat{\psi}(z)|$$

and, for $c_1, c_2 \in (0, c]$ and $|z| \geqslant 1 + \sqrt{c}$,

$$|\tilde{\Delta}_1(\theta, z, c_1, \hat{\psi}) - \tilde{\Delta}_1(\theta, z, c_2, \hat{\psi})| \leqslant \frac{C|c_1 - c_2|}{|e^{-i\theta}z - 1|}|D\hat{\psi}(z)|$$

so

$$|\tilde{\Delta}_1(\theta, z, c(\theta, \hat{\phi}, \tau))\lambda(\theta, \hat{\phi}, \tau) - \tilde{\Delta}_1(\theta, z, c_t)\lambda_t| \leqslant \frac{C\delta_0}{|e^{-i\theta}z - 1|}|D\hat{\psi}(z)|$$

so, for $|z| = r \geqslant 1 + \sqrt{c}$,

$$\begin{aligned} &⨍_0^{2\pi} |\tilde{\Delta}_1(\theta, z, c(\theta, \hat{\phi}, \tau))\lambda(\theta, \hat{\phi}, \tau) - \tilde{\Delta}_1(\theta, z, c_t)\lambda_t|d\theta \\ &\quad \leqslant C\delta_0\left(1 + \log\left(\frac{r}{r-1}\right)\right)|D\hat{\psi}(z)|. \end{aligned} \tag{69}$$

For $k = 2, \ldots, m$, we have

$$\Delta_k(\theta, z, c, \hat{\psi}) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-c)^{k-j} f_c(\theta, z)^j D^j\hat{\psi}(z)$$

where $f_c(\theta, z) = \log(F_c(\theta, z)/z)$. By Proposition A.5, for $c \leqslant 1/C$ and $|z| = r > 1$,

$$|f_c(\theta, z)| \leqslant \frac{Ccr}{|e^{-i\theta}z - 1|}$$

and, for $c_1, c_2 \in (0, c]$ and $|z| = r \geqslant 1 + \sqrt{c}$,

$$|f_{c_1}(\theta, z) - f_{c_2}(\theta, z)| \leqslant \frac{C|c_1 - c_2|r}{|e^{-i\theta}z - 1|}$$

so, for $j = 0, 1, \ldots, k$,

$$|c_1^{k-j} f_{c_1}(\theta, z)^j - c_2^{k-j} f_{c_2}(\theta, z)^j| \leqslant \frac{Cc^{k-1}|c_1 - c_2|r^j}{|e^{-i\theta}z - 1|^j}$$

so

$$|\Delta_k(\theta, z, c, \hat{\psi})| \leqslant Cc^k \sum_{j=0}^{k} \frac{r^j}{|e^{-i\theta}z - 1|^j} |D^j \hat{\psi}(z)|$$

and

$$\begin{aligned} &|\Delta_k(\theta, z, c_1, \hat{\psi}) - \Delta_k(\theta, z, c_2, \hat{\psi})| \\ &\leqslant Cc^{k-1}|c_1 - c_2| \sum_{j=0}^{k} \frac{r^j}{|e^{-i\theta}z - 1|^j} |D^j \hat{\psi}(z)| \end{aligned}$$

so

$$\begin{aligned} &|\Delta_k(\theta, z, c(\theta, \hat{\phi}, \tau))\lambda(\theta, \hat{\phi}, \tau) - \Delta_k(\theta, z, c_t)\lambda_t| \\ &\leqslant Cc^{k-1}\delta_0 \sum_{j=0}^{k} \frac{r^j}{|e^{-i\theta}z - 1|^j} |D^j \hat{\psi}(z)| \end{aligned}$$

so

$$\begin{aligned} &\fint_0^{2\pi} |\Delta_k(\theta, z, c(\theta, \hat{\phi}, \tau))\lambda(\theta, \hat{\phi}, \tau) - \Delta_k(\theta, z, c_t)\lambda_t| d\theta \\ &\leqslant Cc^{k-1}\delta_0 \left(\frac{r}{r-1}\right)^{k-1} \sum_{j=0}^{k} |D^j \hat{\psi}(z)| \end{aligned}$$

and so, for $r \geqslant 1 + 2\sqrt{c}$,

$$\begin{aligned} &\left\| \fint_0^{2\pi} (\Delta_k(\theta, ., c(\theta, \hat{\phi}, \tau))\lambda(\theta, \hat{\phi}, \tau) - \Delta_k(\theta, ., c_t)\lambda_t) d\theta \right\|_{p,r,0} \\ &\leqslant Cc^{k-1}\delta_0 \left(\frac{r}{r-1}\right)^{2k-2} \|D\hat{\psi}\|_{p,\rho} \leqslant Cc\delta_0 \left(\frac{r}{r-1}\right)^2 \|D\hat{\psi}\|_{p,\rho} \end{aligned} \tag{70}$$

where we used the inequality $\|\hat{\psi}\|_{p,r,0} \leqslant C\|D\hat{\psi}\|_{p,r}$ in the $j = 0$ term.

In the final step, we use our assumption that $r \geqslant 1 + c^{1/2-\varepsilon}$ and our choice of $m = \lceil 1/(8\varepsilon) \rceil$ to see that

$$c^m \left(\frac{r}{r-1} \right)^{2m+1+1/p} \leqslant Cc \left(\frac{r}{r-1} \right)^2.$$

Recall that

$$\Delta_{m+1}(\theta, z, c, \hat{\psi}) = \frac{1}{m!} \int_0^1 (1-u)^m e^{-cu} \sum_{j=0}^{m+1} \binom{m+1}{j} (-c)^{m+1-j} f_c(\theta, z)^j D^j \hat{\psi}(F_{c,u}(\theta, z)) du$$

and, for $|z| = r > 1$, since $|F_{c,u}(\theta, z)| \geqslant r$, by (118), we find, for $\rho' = (7r+1)/8$,

$$|D^j \hat{\psi}(F_{c,u}(\theta, z))| \leqslant C \left(\frac{r}{r-1} \right)^{1/p} \|D^j \hat{\psi}\|_{p,\rho'}.$$

So, for $|z| = r > 1$,

$$|\Delta_{m+1}(\theta, z, c, \hat{\psi})| \leqslant Cc^{m+1} \left(\frac{r}{r-1} \right)^{1/p} \left(\frac{r}{|e^{-i\theta} z - 1|} \right)^{m+1} \|D^{m+1} \hat{\psi}\|_{p,\rho'}$$

so

$$\begin{aligned} &⨍_0^{2\pi} |\Delta_{m+1}(\theta, z, c(\theta, \hat{\phi}, \tau)) \lambda(\theta, \hat{\phi}, \tau) - \Delta_{m+1}(\theta, z, c_t) \lambda_t| d\theta \\ &\leqslant Cc^m \left(\frac{r}{r-1} \right)^{m+1/p} \|D^{m+1} \hat{\psi}\|_{p,\rho'} \end{aligned}$$

and so

$$\begin{aligned} &\left\| ⨍_0^{2\pi} (\Delta_{m+1}(\theta, ., c(\theta, \hat{\phi}, \tau)) \lambda(\theta, \hat{\phi}, \tau) - \Delta_{m+1}(\theta, ., c_t) \lambda_t) d\theta \right\|_{p,r,0} \\ &\leqslant Cc^m \left(\frac{r}{r-1} \right)^{2m+1/p} \|D\hat{\psi}\|_{p,\rho} \leqslant Cc \left(\frac{r}{r-1} \right) \|D\hat{\psi}\|_{p,\rho}. \end{aligned} \tag{71}$$

The claimed estimate is obtained by combining (64), (65), (66), (67), (68), (69), (70) and (71). □

Recall that the drift term $(\hat{A}_t, A_t^{\text{cap}})$ in the interpolation formula (17) is given by

$$\begin{aligned} \hat{A}_t(z) &= \int_0^t e^{-(\tau_t - \tau_s)} P(\tau_t - \tau_s) \left(\hat{\beta}(\hat{\Phi}_s, \mathcal{T}_s) + e^{-\zeta \tau_s} (Q+1) \hat{\Psi}_s \right) (z) ds, \\ A_t^{\text{cap}} &= \int_0^t e^{-\zeta(\tau_t - \tau_s)} \left(\beta^{\text{cap}}(\hat{\Phi}_s, \mathcal{T}_s) - e^{-\zeta \tau_s} + \zeta e^{-\zeta \tau_s} \Psi_s^{\text{cap}} \right) ds \end{aligned}$$

where $\Psi_s^{\text{cap}} = \mathcal{T}_s - \tau_s$ and $\hat{\Psi}_s(z) = \hat{\Phi}_s(z) - z$. Recall also that

$$T_0(\delta_0) = \inf \left\{ t \in [0, t_\zeta) : \sup_{\theta \in [0, 2\pi)} |\hat{\Psi}_t'(e^{\sigma + i\theta})| > \delta_0 \text{ or } |\Psi_t^{\text{cap}}| > \delta_0 \right\}.$$

Lemma 4.6. *For all $\zeta \in \mathbb{R}$ and all $T < t_\zeta$, there is a constant $C(\zeta, T) < \infty$ such that, for all $\sigma > 0$, all $\delta_0 \in (0, 1/2]$ and all $t \leqslant T_0(\delta_0) \wedge T$, we have*

$$|A_t^{\mathrm{cap}}| \leqslant C\delta_0^2.$$

Proof. For all $t \leqslant T_0(\delta_0) \wedge t_\zeta$ and all θ, we have $|\Psi_t^{\mathrm{cap}}| \leqslant \delta_0$ and $|\hat{\Psi}_t'(e^{\sigma+i\theta})| \leqslant \delta_0$ for all θ. Hence, by Lemma 4.4, for $t \leqslant T_0(\delta_0) \wedge T$,

$$|A_t^{\mathrm{cap}}| \leqslant e^{-\zeta\tau_t} \int_0^t e^{\zeta\tau_s} |\beta^{\mathrm{cap}}(\hat{\Phi}_s, \mathcal{T}_s) - e^{-\zeta\tau_s} + \zeta e^{-\zeta\tau_s}\Psi_s^{\mathrm{cap}}| ds \leqslant C\delta_0^2.$$

□

Lemma 4.7. *For all $\alpha, \eta \in \mathbb{R}$ with $\zeta = \alpha + \eta \leqslant 1$ and all $T < t_\zeta$, there is a constant $C(\alpha, \eta, \Lambda, T) < \infty$ with the following property. For all $c \in (0, 1/C]$, all $\sigma > 0$, all $\delta_0 \in (0, 1/2]$ and all $t \leqslant T$,*

$$\sup_{s \leqslant t \wedge T_0(\delta_0)} |\hat{A}_s(\infty)| \leqslant C(\delta_0\sqrt{c} + \delta_0^2) + C(c + \delta_0) \int_0^{t \wedge T_0(\delta_0)} |\hat{\Psi}_s(\infty)| ds. \tag{72}$$

Moreover, for all such α, η and T, for all $\varepsilon \in (0, 1/2]$ and all $p \geqslant 2$, there is a constant $C(\alpha, \eta, \Lambda, \varepsilon, p, T) < \infty$ with the following property. For all $c \in (0, 1/C]$, all $\sigma \geqslant 0$, all $\delta_0 \in (0, 1/2]$ and all $t \leqslant T$, for all $r \geqslant 1 + c^{1/2-\varepsilon}$, for $\rho = (1+r)/2$, we have in the case $\zeta < 1$

$$\begin{aligned} &||| D\hat{A}_t 1_{\{t \leqslant T_0(\delta_0)\}} |||_{p,r} \\ &\leqslant \frac{C}{r}\left(1 + \log\left(\frac{r}{r-1}\right)\right)\left(\delta_0^2\left(1 + \log\left(\frac{r}{r-1}\right)\right) + \delta_0\sqrt{c}\left(\frac{r}{r-1}\right)\right) \\ &\quad + C\left(1 + \log\left(\frac{r}{r-1}\right)\right)\left(\delta_0\left(1 + \log\left(\frac{r}{r-1}\right)\right) + c\left(\frac{r}{r-1}\right)\right) \sup_{s \leqslant t} ||| D\hat{\Psi}_s 1_{\{s \leqslant T_0(\delta_0)\}} |||_{p,\rho} \end{aligned} \tag{73}$$

while for $\zeta = 1$ the estimate (73) holds with the first factor of $1 + \log(\frac{r}{r-1})$ replaced by $1 + \log(\frac{r}{r-1}) + \frac{1}{\sqrt{\sigma}}$ in each term on the right.

We remark that some of the log terms in (73) can be avoided when $\zeta < 1$ by the same strategy used for (51). However, this does not work in the case $\zeta = 1$ because that strategy also replaces the term $\frac{1}{\sqrt{\sigma}}$ by $\frac{1}{\sigma}$ which, for our main results, leads to a weaker conclusion. The $\frac{1}{\sqrt{\sigma}}$ in (51) arises in a different way. Since spurious log terms for $\zeta < 1$ do not affect the main results, and to economise the argument, we will not present the slightly stronger estimates than (73) that are available for $\zeta < 1$.

Proof. The estimate (72) follows immediately from (58). Set $\rho' = (3r+1)/4$. For $\zeta < 1$, by Lemma 2.1, we have

$$\begin{aligned}
&||| D\hat{A}_t 1_{\{t\leqslant T_0\}}|||_{p,r} \\
&\leqslant ||| \int_0^{T_0\wedge t} e^{-\bar{\tau}_s} DP(\bar{\tau}_s)(\hat{\beta}(\hat{\Phi}_s,\mathcal{T}_s) + e^{-\zeta\tau_s}(Q+1)\hat{\Psi}_s)ds|||_{p,r} \\
&\leqslant \int_0^t |||DP(\bar{\tau}_s)(\hat{\beta}(\hat{\Phi}_s,\mathcal{T}_s) + e^{-\zeta\tau_s}(Q+1)\hat{\Psi}_s)1_{\{s\leqslant T_0\}}\,|||_{p,r}\, ds. \\
&\leqslant C\sup_{s\leqslant t} |||(\hat{\beta}(\hat{\Phi}_s,\mathcal{T}_s) + e^{-\zeta\tau_s}(Q+1)\hat{\Psi}_s)1_{\{s\leqslant T_0\}}\,|||_{p,\rho',0} \int_0^t \left(\frac{r}{r-1}\right)\wedge\frac{1}{\bar{\tau}_s}ds.
\end{aligned} \tag{74}$$

By Lemma 4.5, for $s\leqslant T_0$,

$$\begin{aligned}
&\|\hat{\beta}(\hat{\Phi}_s,\mathcal{T}_s) + e^{-\zeta\tau_s}(Q+1)\hat{\Psi}_s\|_{p,\rho',0} \\
&\quad\leqslant \frac{C\delta_0^2}{r}\left(1+\log\left(\frac{r}{r-1}\right)\right) + \frac{C\delta_0}{r}\left(1+\log\left(\frac{r}{r-1}\right)\right) r\|D\hat{\Psi}_s\|_{p,\rho} \\
&\qquad + \frac{C\delta_0\sqrt{c}}{r}\left(\frac{r}{r-1}\right) + \frac{Cc}{r}\left(\frac{r}{r-1}\right) r\|D\hat{\Psi}_s\|_{p,\rho}
\end{aligned}$$

so, for $s\leqslant t$,

$$\begin{aligned}
&||| (\hat{\beta}(\hat{\Phi}_s,\mathcal{T}_s) + e^{-\zeta\tau_s}(Q+1)\hat{\Psi}_s)1_{\{s\leqslant T_0\}}|||_{p,\rho',0} \\
&\leqslant \frac{C\delta_0^2}{r}\left(1+\log\left(\frac{r}{r-1}\right)\right) + \frac{C\delta_0}{r}\left(1+\log\left(\frac{r}{r-1}\right)\right) r\sup_{s\leqslant t}|||D\hat{\Psi}_s 1_{\{s\leqslant T_0\}}|||_{p,\rho} \\
&\quad + \frac{C\delta_0\sqrt{c}}{r}\left(\frac{r}{r-1}\right) + \frac{Cc}{r}\left(\frac{r}{r-1}\right) r\sup_{s\leqslant t}|||D\hat{\Psi}_s 1_{\{s\leqslant T_0\}}\,|||_{p,\rho}\ .
\end{aligned}$$

Since

$$\int_0^t \left(\frac{r}{r-1}\right)\wedge\frac{1}{\bar{\tau}_s}ds \leqslant C\left(1+\log\left(\frac{r}{r-1}\right)\right)$$

these estimates combine to prove (73). In the case $\zeta = 1$, the estimate of Lemma 2.1 leads to a different integral on the right in (74), for which we have the following bound

$$\int_0^t \left(\frac{r}{r-1}\right)\wedge\left(\frac{1}{\bar{\tau}_s}\vee\frac{1}{\sqrt{\sigma\bar{\tau}_s}}\right)ds \leqslant C\left(1+\log\left(\frac{r}{r-1}\right)+\frac{1}{\sqrt{\sigma}}\right).$$

Hence we obtain the modified form of (73) claimed for $\zeta = 1$. □

5. Bulk Scaling Limit for ALE(α, η)

Recall that we write our ALE(α, η) process $(\Phi_t)_{t\geqslant 0}$ in (Schlicht function, capacity) coordinates $(\hat{\Phi}_t, \mathcal{T}_t)$, and that we set

$$\hat{\Psi}_t(z) = \hat{\Phi}_t(z) - \hat{\phi}_t(z), \quad \Psi_t^{\text{cap}} = \mathcal{T}_t - \tau_t$$

where $(\hat{\phi}_t, \tau_t)_{t<t_\zeta}$ is the disk solution to the LK(ζ) equation with initial capacity $\tau_0 = 0$. We obtained the following interpolation formula (17)

$$\hat{\Psi}_t(z) = \hat{M}_t(z) + \hat{A}_t(z), \quad \Psi_t^{\text{cap}} = M_t^{\text{cap}} + A_t^{\text{cap}}$$

and have estimated the terms on the right-hand sides in the preceding section. We now put these estimates together to obtain first L^p-estimates and then pointwise high-probability estimates which allow us to prove Theorems 1.1 and 1.2.

5.1. L^p-estimates. Recall that

$$T_0(\delta_0) = \inf\left\{t \in [0, t_\zeta) : \sup_{\theta\in[0,2\pi)} |\hat{\Psi}_t'(e^{\sigma+i\theta})| > \delta_0 \text{ or } |\Psi_t^{\text{cap}}| > \delta_0\right\}.$$

Proposition 5.1. *For all $\alpha, \eta \in \mathbb{R}$, all $p \geqslant 2$ and all $T < t_\zeta$, there is a constant $C(\alpha, \eta, p, T) < \infty$ such that, for all $c \in (0, 1]$ and all $\delta_0 \in (0, 1/2]$,*

$$\Big\| \sup_{t\leqslant T\wedge T_0(\delta_0)} |\Psi_t^{\text{cap}}| \Big\|_p \leqslant C(\sqrt{c} + \delta_0^2)$$

and

$$\Big\| \sup_{t\leqslant T\wedge T_0(\delta_0)} |\Psi_t^{\text{cap}} - \Pi_t^{\text{cap}}| \Big\|_p \leqslant C(c + \sqrt{c\delta_0} + \delta_0^2)$$

and

$$\Big\| \sup_{t\leqslant T\wedge T_0(\delta_0)} |\hat{\Psi}_t(\infty)| \Big\|_p \leqslant C(\sqrt{c} + \delta_0^2)$$

and

$$\Big\| \sup_{t\leqslant T\wedge T_0(\delta_0)} |\hat{\Psi}_t(\infty) - \hat{\Pi}_t(\infty)| \Big\|_p \leqslant C(c + \sqrt{c\delta_0} + \delta_0^2).$$

Proof. The first two estimates follow immediately from Lemmas 4.1 and 4.6. From Lemmas 4.2 and 4.7, we obtain, for all $t \leqslant T$,

$$\Big\| \sup_{s\leqslant t\wedge T_0(\delta_0)} |\hat{\Psi}_t(\infty)| \Big\|_p^p \leqslant C(\sqrt{c} + \delta_0^2)^p + C\int_0^t \|\hat{\Psi}_s(\infty) 1_{\{s\leqslant T_0(\delta_0)\}}\|_p^p ds$$

from which the third estimate follows by Gronwall's lemma. The fourth estimate follows from the third, together with Lemmas 4.2 and 4.7. ◻

Fix $\sigma > 0$ and set

$$R = \frac{r}{r-1}, \quad L = 1 + \log R, \quad R_1 = R + \frac{\sqrt{R}}{\sqrt{\sigma}}, \quad L_1 = L + \frac{1}{\sqrt{\sigma}}.$$

Define

$$\delta(r) = (\sqrt{c}R + \delta_0^2 L^2 + \delta_0\sqrt{c}LR)/r, \quad \bar{\delta}(r) = \sqrt{c}R + \delta_0 L^2, \tag{75}$$

$$\delta_1(r) = (\sqrt{c}R_1 + \delta_0^2 LL_1 + \delta_0\sqrt{c}L_1R)/r, \quad \bar{\delta}_1(r) = \sqrt{c}R_1 + \delta_0 LL_1. \tag{76}$$

The next estimates follow immediately from Lemmas 4.3 and 4.7.

Proposition 5.2. *For all $\alpha, \eta \in \mathbb{R}$ with $\zeta = \alpha+\eta \leqslant 1$, all $\varepsilon \in (0, 1/2]$, all $p \geqslant 2$ and all $T < t_\zeta$, there is a constant $C(\alpha, \eta, \Lambda, \varepsilon, p, T) < \infty$ with the following property. For all $c \in (0, 1]$, all $\delta_0 \in (0, 1/2]$, all $r, e^\sigma \geqslant 1 + c^{1/2-\varepsilon}$ and all $t \leqslant T$, setting $\rho = (1+r)/2$, we have, for $\zeta < 1$,*

$$||| D\hat{\Psi}_t 1_{\{t \leqslant T_0(\delta_0)\}} |||_{p,r} \leqslant C\delta(r) + C\bar{\delta}(r) \sup_{s \leqslant t} ||| D\hat{\Psi}_{s-} 1_{\{s \leqslant T_0(\delta_0)\}} |||_{p,\rho} \tag{77}$$

while, for $\zeta = 1$,

$$||| D\hat{\Psi}_t 1_{\{t \leqslant T_0(\delta_0)\}} |||_{p,r} \leqslant C\delta_1(r) + C\bar{\delta}_1(r) \sup_{s \leqslant t} ||| D\hat{\Psi}_{s-} 1_{\{s \leqslant T_0(\delta_0)\}} |||_{p,\rho}\,. \tag{78}$$

The preceding estimate may be improved by an iterative argument to obtain the following result.

Proposition 5.3. *For all $\alpha, \eta \in \mathbb{R}$ with $\zeta = \alpha+\eta \leqslant 1$, all $\varepsilon \in (0, 1/2]$, all $p \geqslant 2$ and all $T < t_\zeta$, there is a constant $C(\alpha, \eta, \Lambda, \varepsilon, p, T) < \infty$ with the following property. In the case $\zeta < 1$, for all $c \in (0, 1]$, all $r, e^\sigma \geqslant 1 + c^{1/2-\varepsilon}$ and all $t \leqslant T$, for all $\nu \in (0, \varepsilon/2]$, setting $\delta_0 = c^{1/2-\nu}e^\sigma/(e^\sigma - 1)$, we have*

$$||| D\hat{\Psi}_t 1_{\{t \leqslant T_0(\delta_0)\}} |||_{p,r} \leqslant \frac{C\sqrt{c}}{r}\left(\frac{r}{r-1}\right) + \frac{Cc^{1-2\nu}}{r}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^2 \left(1 + \log\left(\frac{r}{r-1}\right)\right)^2.$$

Moreover, in the case $\zeta = 1$ and $\varepsilon \leqslant 1/3$, for all $c \in (0, 1]$, all $r \geqslant 1 + c^{1/2-\varepsilon}$, all $e^\sigma \geqslant 1 + c^{1/3-\varepsilon}$ and all $t \leqslant T$, for $\nu \in (0, \varepsilon/2]$, setting $\delta_0 = c^{1/2-\nu}e^\sigma/(e^\sigma - 1)$, we have

$$\begin{aligned} &||| D\hat{\Psi}_t 1_{\{t \leqslant T_0(\delta_0)\}} |||_{p,r} \\ &\leqslant \frac{C\sqrt{c}}{r}\left(\left(\frac{r}{r-1}\right) + \frac{1}{\sqrt{\sigma}}\left(\frac{r}{r-1}\right)^{1/2}\right) \\ &\quad + \frac{Cc^{1-2\nu}}{r}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^2 \left(1 + \log\left(\frac{r}{r-1}\right) + \frac{1}{\sqrt{\sigma}}\right)\left(1 + \log\left(\frac{r}{r-1}\right)\right). \end{aligned} \tag{79}$$

Proof. We begin with a crude estimate which allows us to restrict further consideration to small values of c. The function $\hat{\Phi}_t(z)$ is univalent on $\{|z| > 1\}$, with $\hat{\Phi}_t(z) \sim z$ as $z \to \infty$. So, by a standard distortion estimate, for all $|z| = r > 1$,

$$|\hat{\Phi}_t'(z) - 1| \leqslant \frac{1}{r^2 - 1}$$

and so

$$\|D\hat{\Psi}_t\|_{p,r} = r\|\hat{\Phi}_t' - 1\|_{p,r} \leqslant \frac{1}{r-1}. \tag{80}$$

It is straightforward to check that this implies the claimed estimates in the case where $c > 1/C$, for any given constant C of the allowed dependence. Hence it will suffice to consider the case where $c \leqslant 1/C$.

Consider first the case $\zeta < 1$. On substituting the chosen value of δ_0 in (75), we obtain

$$\delta(r) = \frac{1}{r}\left(\sqrt{c}R + c^{1-2\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^2 L^2 + c^{1-\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)LR\right),$$
$$\bar{\delta}(r) = \sqrt{c}R + c^{1/2-\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)L^2.$$

Note that, for $\rho = (1+r)/2$, we have $R(\rho) \leqslant 2R(r)$ and $L(\rho) \leqslant 2L(r)$, so $\bar{\delta}(\rho) \leqslant 4\bar{\delta}(r)$ and $\bar{\delta}(\rho) \leqslant 4\bar{\delta}(r)$. Note also that, for $r \geqslant 1 + c^{1/2-\varepsilon/2}$ and $e^\sigma \geqslant 1 + c^{1/2-\varepsilon}$, for all sufficiently small c,

$$C^*\bar{\delta}(r) \leqslant 2C^*(c^{\varepsilon/2} + c^{\varepsilon/2}(1 + \log(1/c))^2) \leqslant c^{\varepsilon/3} \leqslant 1$$

where C^* is the constant in Proposition 5.2. We restrict to such c. Set $C_0 = 1$ and for $k \geqslant 0$ define recursively $C_{k+1} = 2^{k+1}C_k + 1$. We will show that, for all $k \geqslant 0$, all $r \geqslant 1 + 2^k c^{1/2-\varepsilon/2}$ and all $t \leqslant T$,

$$|||\, D\hat{\Psi}_t 1_{\{t\leqslant T_0(\delta_0)\}}|||_{p,r} \leqslant C_k\left(\frac{(C^*\bar{\delta}(r))^k}{r-1} + C^*\delta(r)\right). \tag{81}$$

The case $k = 0$ is implied by (80). Suppose inductively that (81) holds for k, for all $r \geqslant 1 + 2^k c^{1/2-\varepsilon/2}$ and all $t \leqslant T$. Take $r \geqslant 1 + 2^{k+1}c^{1/2-\varepsilon/2}$ and $t \leqslant T$. Then $\rho = (r+1)/2 \geqslant 1 + 2^k c^{1/2-\varepsilon/2}$ so, for all $s \leqslant t$,

$$|||D\hat{\Psi}_s 1_{\{s\leqslant T_0\}}|||_{p,\rho} \leqslant C_k\left(\frac{(C^*\bar{\delta}(\rho))^k}{\rho-1} + C^*\delta(\rho)\right) \leqslant 2^{k+1}C_k\left(\frac{(C^*\bar{\delta}(r))^k}{r-1} + C^*\delta(r)\right).$$

Since $r \geqslant 1 + c^{1/2-\varepsilon/2}$, we can use Proposition 5.2 with ε replaced by $\varepsilon/2$ and substitute the last inequality into (77) to obtain

$$\begin{aligned}|||D\hat{\Psi}_t 1_{\{t\leqslant T_0(\delta_0)\}}|||_{p,r} &\leqslant 2^{k+1}C_k\left(\frac{(C^*\bar{\delta}(r))^{k+1}}{r-1} + C^*\bar{\delta}(r)\delta(r)\right) + C^*\delta(r)\\ &\leqslant C_{k+1}\left(\frac{(C^*\bar{\delta}(r))^{k+1}}{r-1} + C^*\delta(r)\right).\end{aligned}$$

Hence (81) holds for $k+1$ and the induction proceeds. Choose now $k = \lceil 3/\varepsilon\rceil$. Then

$$\frac{(C^*\bar{\delta}(r))^k}{r-1} \leqslant \frac{c^{\varepsilon k/3}}{r-1} \leqslant \frac{c}{r-1} \leqslant \delta(r).$$

For c sufficiently small, we have $c^{\varepsilon/2} \leqslant 2^{-k/2}$. Then, for all $r \geqslant 1 + c^{1/2-\varepsilon}$, we have $r \geqslant 1 + 2^k c^{1/2-\varepsilon/2}$, so we obtain

$$|||D\hat{\Psi}_t 1_{\{t \leqslant T_0(\delta_0)\}}|||_{p,r} \leqslant C_k \left(\frac{(C^* \bar{\delta}(r))^k}{r-1} + C^* \delta(r) \right) \leqslant 2C_k C^* \delta(r).$$

For c sufficiently small, we have

$$c^{1-\nu} \left(\frac{e^\sigma}{e^\sigma - 1} \right) LR \leqslant \sqrt{c} R$$

so this is a bound of the claimed form.

We turn to the case $\zeta = 1$. On substituting the chosen value of δ_0 in (76), we obtain

$$\delta_1(r) = \frac{1}{r} \bigg(\sqrt{c} R_1 + c^{1-2\nu} R (e^\sigma)^2 L L_1 + c^{1-\nu} R(e^\sigma) L_1 R \bigg),$$
$$\bar{\delta}_1(r) = \sqrt{c} R_1 + c^{1/2-\nu} R(e^\sigma) L L_1.$$

Note that, for $\rho = (1+r)/2$, we have $R_1(\rho) \leqslant 2R_1(r)$ and $L_1(\rho) \leqslant 2L_1(r)$, so $\bar{\delta}_1(\rho) \leqslant 4\bar{\delta}_1(r)$ and $\delta_1(\rho) \leqslant 4\delta_1(r)$. Note also that, for $r \geqslant 1 + c^{1/2-\varepsilon/2}$ and $e^\sigma \geqslant 1 + c^{1/3-\varepsilon}$, for all sufficiently small c,

$$C^* \bar{\delta}(r) \leqslant 4C^*(c^{\varepsilon/2} + c^{\varepsilon/2}(1 + \log(1/c))^2) \leqslant c^{\varepsilon/3} \leqslant 1$$

where C^* is the constant in Proposition 5.2. We restrict to such c. Set $C_0 = 1$ and for $k \geqslant 0$ define recursively $C_{k+1} = 2^{2k+1} C_k + 1$. Then, by an analogous inductive argument, we obtain, for all $k \geqslant 0$, all $t \leqslant T$ and all $r \leqslant 1 + 2^k c^{1/2}$,

$$|||\ D\hat{\Psi}_t 1_{\{t \leqslant T_0(\delta_0)\}}|||_{p,r} \leqslant C_k \left(\frac{(C^* \bar{\delta}_1(r))^k}{r-1} + C^* \delta_1(r) \right). \tag{82}$$

Choose now $k = \lceil 1/\varepsilon \rceil$ and assume that $r \geqslant 1 + c^{1/2-\varepsilon}$. Then

$$\frac{\bar{\delta}_1(r)^k}{r-1} \leqslant \frac{c^{\varepsilon k}}{r-1} \leqslant \frac{c}{r-1} \leqslant \delta_1(r).$$

and, for c sufficiently small, we have $c^\varepsilon \leqslant 2^{-k}$, so $r \geqslant 1 + 2^k c^{1/2}$ and so

$$|||D\hat{\Psi}_t 1_{\{t \leqslant T_0(\delta_0)\}}|||_{p,r} \leqslant 2C^* C_k \delta_1(r).$$

□

We note also the following estimates, which are deduced from (28) and (73) using the estimates of Proposition 5.3

Proposition 5.4. *For all* $\alpha, \eta \in \mathbb{R}$ *with* $\zeta = \alpha + \eta \leqslant 1$, *all* $\varepsilon \in (0, 1/2]$, *all* $p \geqslant 2$ *and all* $T < t_\zeta$, *there is a constant* $C(\alpha, \eta, \Lambda, \varepsilon, p, T) < \infty$ *with the following property. In the case* $\zeta < 1$, *for all* $c \in (0, 1]$, *all* $r, e^\sigma \geqslant 1 + c^{1/2-\varepsilon}$ *and all* $t \leqslant T$, *for all* $\nu \in (0, \varepsilon/2]$, *setting* $\delta_0 = c^{1/2-\nu} e^\sigma/(e^\sigma - 1)$, *we have*

$$|||D(\hat{\Psi}_t - \hat{\Pi}_t) 1_{\{t \leqslant T_0(\delta_0)\}}|||_{p,r} \leqslant \frac{C}{r} \Bigg(c \left(\frac{r}{r-1} \right)^2 + \sqrt{c\delta_0} \left(\frac{r}{r-1} \right) + \delta_0^2 \left(1 + \log \left(\frac{r}{r-1} \right) \right)^2 \Bigg).$$

Moreover, in the case $\zeta = 1$ and $\varepsilon \leqslant 1/3$, for all $c \in (0, 1]$, all $r \geqslant 1 + c^{1/2-\varepsilon}$, all $e^\sigma \geqslant 1 + c^{1/3-\varepsilon}$ and all $t \leqslant T$, for $\nu \in (0, \varepsilon]$, setting $\delta_0 = c^{1/2-\nu} e^\sigma/(e^\sigma - 1)$, we have

$$\begin{aligned}&|||\, D(\hat{\Psi}_t - \hat{\Pi}_t) 1_{\{t \leqslant T_0(\delta_0)\}} |||_{p,r} \\ &\leqslant \frac{C}{r}\Bigg(c\Bigg(\left(\frac{r}{r-1}\right) + \frac{1}{\sqrt{\sigma}}\left(\frac{r}{r-1}\right)^{1/2}\Bigg)^2 + \sqrt{c\delta_0}\Bigg(\left(\frac{r}{r-1}\right) + \frac{1}{\sqrt{\sigma}}\left(\frac{r}{r-1}\right)^{1/2}\Bigg) \\ &+ \delta_0^2\left(1 + \log\left(\frac{r}{r-1}\right)\right)\left(1 + \log\left(\frac{r}{r-1}\right) + \frac{1}{\sqrt{\sigma}}\right)\Bigg).\end{aligned}$$

We turn now to some estimates needed for the discrete-time results Theorems 1.2 and 1.4. Write $\mathcal{V}_t$ for the number of particles added by time t and define for $t < t_\zeta$

$$\nu_t = \alpha^{-1}((1 + \zeta t)^{\alpha/\zeta} - 1).$$

It is straightforward to see that, for all $\alpha, \eta \in \mathbb{R}$, we have $\nu_t \to n_\alpha$ as $t \to t_\zeta$. Also

$$c\mathcal{V}_t = \int_{E(t)} c 1_{\{v \leqslant \Lambda_s(\theta)\}} \mu(d\theta, dv, ds), \quad \nu_t = \int_0^t e^{-\eta \tau_s} ds. \tag{83}$$

Proposition 5.5. *For all $\alpha, \eta \in \mathbb{R}$, all $p \geqslant 2$ and all $T < t_\zeta$, there is a constant $C(\alpha, \eta, p, T) < \infty$ such that, for all $c \in (0, 1]$ and all $\delta_0 \in (0, 1/2]$,*

$$\Big\| \sup_{t \leqslant T \wedge T_0(\delta_0)} |c\mathcal{V}_t - \nu_t| \Big\|_p \leqslant C(\sqrt{c} + \delta_0^2).$$

Proof. Recall from Sect. 3.2 that

$$\mathcal{T}_t = \int_{E(t)} C_s(\theta) 1_{\{v \leqslant \Lambda_s(\theta)\}} \mu(d\theta, dv, ds), \quad \tau_t = \int_0^t e^{-\zeta \tau_s} ds$$

where

$$C_s(\theta) = c|\Phi'_{s-}(e^{\sigma + i\theta})|^{-\alpha}, \quad \zeta = \alpha + \eta.$$

If we substitute the explicit appearances of α in the preceding line by 0, then $C_s(\theta)$ becomes c and $e^{-\zeta\tau_s}$ becomes $e^{-\eta\tau_s}$. Then, applying these substitutions in the line above, we recover the integral representations (83) of $c\mathcal{V}_t$ and ν_t. The claimed estimate results from following through this modification in the calculations leading to Proposition 5.1. The details are left to the reader. □

We can also improve on the estimate of $\mathcal{T}_t$ by τ_t in Proposition 5.1. Define, for $c\mathcal{V}_t < n_\alpha$,

$$\tilde{\mathcal{T}}_t = \tau^{\text{disc}}_{\mathcal{V}_t}$$

where $\tau^{\text{disc}}_n = \alpha^{-1} \log(1 + \alpha c n)$ as at (8). We leave any modifications needed for the case $\alpha = 0$ to the reader. By allowing $\tilde{\mathcal{T}}_t$ to depend on the random time-scale of particle arrivals, we remove the main source of error when estimating $\mathcal{T}_t$ by τ_t.

Proposition 5.6. *For all $\alpha, \eta \in \mathbb{R}$, all $p \geqslant 2$, all $T < t_\zeta$ and all $N < n_\alpha$, there is a constant $C(\alpha, \eta, p, T, N) < \infty$ such that, for all $c \leqslant 1/C$ and all $\delta_0 \in (0, 1/2]$,*

$$\Big\| \sup_{t \leqslant T \wedge T(\delta_0),\, c\mathcal{V}_t \leqslant N} |\mathcal{T}_t - \tilde{\mathcal{T}}_t| \Big\|_p \leqslant C(c + \delta_0^2).$$

Proof. Set

$$\tilde{C}_t = \tau^{\text{disc}}_{\mathcal{V}_{t-}+1} - \tau^{\text{disc}}_{\mathcal{V}_{t-}} = \alpha^{-1} \log\left(1 + \frac{\alpha c}{1 + \alpha c \mathcal{V}_{t-}}\right).$$

Then

$$\tilde{\mathcal{T}}_t = \int_{E(t)} \tilde{C}_s 1_{\{v \leqslant \Lambda_s(\theta)\}} \mu(d\theta, dv, ds)$$

so

$$\begin{aligned} \mathcal{T}_t - \tilde{\mathcal{T}}_t &= \int_{E(t)} (C_s(\theta) - \tilde{C}_s) 1_{\{v \leqslant \Lambda_s(\theta)\}} \mu(d\theta, dv, ds) \\ &= \int_{E(t)} (C_s(\theta) - \tilde{C}_s) 1_{\{v \leqslant \Lambda_s(\theta)\}} \tilde{\mu}(d\theta, dv, ds) \\ &\quad + \int_0^t ⨍_0^{2\pi} (C_s(\theta) - \tilde{C}_s) \Lambda_s(\theta) d\theta ds. \end{aligned}$$

We have, for $c\mathcal{V}_t \leqslant N$,

$$|\tilde{C}_t - c e^{-\alpha \tilde{\mathcal{T}}_{t-}}| \leqslant C c^2$$

and, for $t \leqslant T_0(\delta_0)$, as in the proof of Lemma 4.4,

$$\begin{aligned} |C_t(\theta) - c e^{-\alpha \mathcal{T}_{t-}} (1 + \alpha \operatorname{Re} \hat{\Psi}'_{t-}(e^{\sigma + i\theta}))| &\leqslant C c \delta_0^2, \\ |c \Lambda_t(\theta) - e^{-\eta \mathcal{T}_{t-}} (1 + \eta \operatorname{Re} \hat{\Psi}'_{t-}(e^{\sigma + i\theta}))| &\leqslant C \delta_0^2, \\ |C_t(\theta) \Lambda_t(\theta) - e^{-\zeta \mathcal{T}_{t-}} (1 + \zeta \operatorname{Re} \hat{\Psi}'_{t-}(e^{\sigma + i\theta}))| &\leqslant C \delta_0^2 \end{aligned}$$

so

$$|C_t(\theta) - \tilde{C}_t| \leqslant C c |\mathcal{T}_{t-} - \tilde{\mathcal{T}}_{t-}| + C c \delta_0$$

and, using (57),

$$\left| ⨍_0^{2\pi} C_t(\theta) \Lambda_t(\theta) d\theta - e^{-\zeta \mathcal{T}_{t-}} \right| \leqslant C \delta_0^2$$

and

$$\left| ⨍_0^{2\pi} \tilde{C}_t \Lambda_t(\theta) d\theta - e^{-\alpha \tilde{\mathcal{T}}_{t-}} e^{-\eta \mathcal{T}_{t-}} \right| \leqslant C(c + \delta_0^2)$$

and so

$$\left| ⨍_0^{2\pi} (C_t(\theta) - \tilde{C}_t) \Lambda_t(\theta) d\theta \right| \leqslant C |\mathcal{T}_{t-} - \tilde{\mathcal{T}}_{t-}| + C(c + \delta_0^2).$$

Set

$$f(t)=\mathbb{E}\left(\sup_{s\leqslant t\wedge T_0(\delta_0),\, c\mathcal{V}_s\leqslant N}|\mathcal{T}_s-\tilde{\mathcal{T}}_s|^p\right)$$

Then, by Burkholder's and Jensen's inequalities, for $p\geqslant 2$, and all $t\leqslant T$,

$$f(t)\leqslant C(c^p+\delta_0^{2p})+C\int_0^t f(s)ds$$

and the claimed estimate follows by Gronwall's lemma. □

5.2. Spatially-uniform high-probability estimates. We now pass from the L^p-estimates of the preceding section to pointwise estimates which hold with high probability on the function $\hat{\Psi}_t(z)=\hat{\Phi}_t(z)-z$, uniformly in $t\in[0,T]$ and $|z|\geqslant r(c)$ as $c\to 0$, for a suitable function $r(c)$, which is specified in the next result, and tends to 1 as $c\to 0$. In order to show the desired uniformity, we combine the usual L^p-tail estimate with suitable dissections of $[0,T]$ and $\{|z|\geqslant r(c)\}$, choosing p large to deal with an increasing number of terms as $c\to 0$. We see at the same time that the event $\{T_0(\delta_0)>T\}$, to which our previous estimates were restricted, is in fact an event of high probability as $c\to 0$, thus closing the argument for convergence to a disk. The following result contains Theorem 1.1.

Proposition 5.7. *For all $\alpha,\eta\in\mathbb{R}$ with $\zeta=\alpha+\eta\leqslant 1$, all $\varepsilon\in(0,1/2]$ and all $\nu\in(0,\varepsilon/4]$, all $m\in\mathbb{N}$ and all $T<t_\zeta$, there is a constant $C(\alpha,\eta,\Lambda,\varepsilon,\nu,m,T)<\infty$ with the following property. In the case $\zeta<1$, for all $c\leqslant 1/C$, for $e^\sigma\geqslant 1+c^{1/2-\varepsilon}$ and $\delta_0=c^{1/2-\nu}e^\sigma/(e^\sigma-1)$, there is an event $\Omega_0\subseteq\{T_0(\delta_0)>T\}$ of probability exceeding $1-c^m$ on which, for all $t\leqslant T$ and all $|z|=r\geqslant 1+c^{1/2-\varepsilon}$,*

$$|\Psi_t^{\mathrm{cap}}|\leqslant C\left(c^{1/2-\nu}+c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma-1}\right)^2\right) \tag{84}$$

and

$$|\hat{\Psi}_t(z)|\leqslant C\left(c^{1/2-\nu}+c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma-1}\right)^2\right) \tag{85}$$

and

$$|D\hat{\Psi}_t(z)|\leqslant \frac{C}{r}\left(c^{1/2-\nu}\left(\frac{r}{r-1}\right)+c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma-1}\right)^2\right) \tag{86}$$

and

$$|\Psi_t^{\mathrm{cap}}-\Pi_t^{\mathrm{cap}}|\leqslant C\left(c^{3/4-2\nu}\left(\frac{e^\sigma}{e^\sigma-1}\right)^{1/2}+c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma-1}\right)^2\right) \tag{87}$$

and

$$|\hat{\Psi}_t(z)-\hat{\Pi}_t(z)|\leqslant Cc^{3/4-2\nu}\left(\frac{e^\sigma}{e^\sigma-1}\right)^{1/2}+Cc^{1-4\nu}\left(\left(\frac{r}{r-1}\right)+\left(\frac{e^\sigma}{e^\sigma-1}\right)^2\right). \tag{88}$$

Moreover, in the case $\zeta = 1$ *with* $\varepsilon \in (0, 1/3]$, *for all* $c \leqslant 1/C$, *for* $e^\sigma \geqslant 1 + c^{1/3-\varepsilon}$ *and* $\delta_0 = c^{1/2-\nu} e^\sigma/(e^\sigma - 1)$ *there is an event* $\Omega_0 \subseteq \{T_0(\delta_0) > T\}$ *of probability exceeding* $1 - c^m$ *on which, for all* $t \leqslant T$, *the estimates (84) and (87) hold and, for all* $|z| = r \geqslant 1 + c^{1/2-\varepsilon}$,

$$|\hat{\Psi}_t(z)| \leqslant C\left(c^{1/2-\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^{1/2} + c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^{5/2}\right)$$

and

$$|D\hat{\Psi}_t(z)| \leqslant \frac{C}{r}\left(c^{1/2-\nu}\left(\left(\frac{r}{r-1}\right) + \frac{1}{\sqrt{\sigma}}\left(\frac{r}{r-1}\right)^{1/2}\right) + c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^{5/2}\right)$$

and

$$|\hat{\Psi}_t(z) - \hat{\Pi}_t(z)| \leqslant Cc^{3/4-\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right) + Cc^{1-4\nu}\left(\left(\frac{r}{r-1}\right) + \left(\frac{e^\sigma}{e^\sigma - 1}\right)^{5/2}\right). \quad (89)$$

Proof. We will give details for the case $\zeta \in [0, 1)$. Some minor modifications are needed for the case $\zeta = 1$ because of the weaker L^p-estimate (79) which applies in that case, and also for the case $\zeta < 0$. These are left to the reader.

Fix $\alpha, \eta, \varepsilon, \nu, m$ and T as in the statement. By adjusting the value of ε, it will suffice to consider the case where $e^\sigma \geqslant 1 + 2c^{1/2-\varepsilon}$, and to find an event $\Omega_0 \subseteq \{T_0(\delta_0) > T\}$, of probability exceeding $1 - c^m$, on which the claimed estimates holds whenever $r \geqslant 1 + 2c^{1/2-\varepsilon}$ and $t \leqslant T$. There is a constant $C < \infty$ of the desired dependence, such that $\delta_0 \leqslant 1/2$ whenever $c \leqslant 1/C$. We restrict to such c. Set

$$\delta = c^{m+3}, \quad t(n) = \delta n, \quad N = \lfloor T/\delta \rfloor, \quad N_0 = \lfloor (T_0(\delta_0) \wedge T)/\delta \rfloor.$$

Recall that $\mathcal{V}_t$ denotes the number of particles added to the cluster by time t. Consider the event

$$\Omega_1 = \{\mathcal{V}_{t(n)} - \mathcal{V}_{t(n-1)} \leqslant 1 \text{ for all } n \leqslant N_0 \text{ and } \mathcal{V}_{t(N_0)} = \mathcal{V}_{T_0(\delta_0)\wedge T}\}.$$

Note that, on Ω_1, for all $t \leqslant T_0(\delta_0) \wedge T$, there exists $n \in \{1, \dots, N_0\}$ such that $\hat{\Psi}_t = \hat{\Psi}_{t(n)}$. Since $\delta_0 \leqslant 1/2$, there is a constant $C < \infty$ of the desired dependence such that the process $(\mathcal{V}_t)_{t \leqslant T_0(\delta_0)}$ is a thinning of a Poisson process of rate C/c. Hence

$$\mathbb{P}(\Omega_1^c) \leqslant N(C/c)^2\delta^2 + (C/c)/\delta \leqslant C\delta/c^2 = Cc^{m+1}$$

and hence $\mathbb{P}(\Omega_1^c) \leqslant c^m/3$ for all $c \leqslant 1/(3C)$. We restrict to such c.

Fix an integer $p \geqslant 2$, to be chosen later, depending on m and ν. By Proposition 5.1, there is a constant $C < \infty$ of the desired dependence such that, for $\mu_0 = C\left(\sqrt{c} + \delta_0^2\right)$, we have

$$\Big\| \sup_{t \leqslant T_0(\delta_0)\wedge T} |\Psi_t^{\text{cap}}| \Big\|_p \leqslant \mu_0, \quad \Big\| \sup_{t \leqslant T_0(\delta_0)\wedge T} |\hat{\Psi}_t(\infty)| \Big\|_p \leqslant \mu_0.$$

Set $\lambda_0 = (6c^{-m})^{1/p}$ and consider the event

$$\Omega_2 = \{|\Psi_t^{\text{cap}}| \leqslant \lambda_0\mu_0 \text{ and } |\hat{\Psi}_t(\infty)| \leqslant \lambda_0\mu_0 \text{ for all } t \leqslant T_0(\delta_0) \wedge T\}.$$

Then $\mathbb{P}(\Omega_2^c) \leqslant 2\lambda_0^{-p} = c^m/3$. We choose $p \geqslant m/\nu$. Then, since $e^\sigma \geqslant 1 + 2c^{1/2-\varepsilon}$ and $\nu \leqslant \varepsilon$, there is a constant $C < \infty$ of the desired dependence such that, for $c \leqslant 1/C$, on the event Ω_2, for all $t \leqslant T_0(\delta_0) \wedge T$,

$$|\Psi_t^{\mathrm{cap}}| \leqslant \lambda_0\mu_0 \leqslant Cc^{-\nu}(\sqrt{c} + \delta_0^2) = C\left(c^{1/2-\nu} + c^{1-3\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^2\right) \leqslant \delta_0. \quad (90)$$

We restrict to such c. Set

$$K = \min\{k \geqslant 1 : 2^k c^{1/2-\varepsilon} \geqslant 1\}.$$

Then $K \leqslant \lfloor \log(1/c) \rfloor + 1$. For $k = 1, \ldots, K$, set

$$r(k) = 1 + 2^k c^{1/2-\varepsilon}, \quad \rho(k) = \frac{r(k) + 1}{2}.$$

Then $\rho(k) \geqslant \rho(1) = 1 + c^{1/2-\varepsilon}$ for all k and $r(K) \in [2, 4]$. By Proposition 5.3, there is a constant $C < \infty$ of the desired dependence such that, for $k = 1, \ldots, K$ and all $t \leqslant T$,

$$|||D\hat{\Psi}_t 1_{\{t \leqslant T_0(\delta_0)\}}|||_{p,\rho(k)} \leqslant \mu(r(k))$$

where

$$\mu(r) = \frac{C}{r}\left(\sqrt{c}\left(\frac{r}{r-1}\right) + c^{1-3\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^2\right).$$

Set $\lambda = \left(3KTc^{-2m-3}\right)^{1/p}$ and consider the event

$$\Omega_3 = \bigcap_{n=1}^{N}\bigcap_{k=1}^{K}\{\|D\hat{\Psi}_{t(n)}\|_{p,\rho(k)} 1_{\{t(n) \leqslant T_0(\delta_0)\}} \leqslant \lambda\mu(r(k))\}.$$

Then

$$\mathbb{P}(\|D\hat{\Psi}_{t(n)}\|_{p,\rho(k)} 1_{\{t(n) \leqslant T_0(\delta_0)\}} > \lambda\mu(r(k))) \leqslant \lambda^{-p}$$

so

$$\mathbb{P}(\Omega_3^c) \leqslant KN\lambda^{-p} \leqslant KT\lambda^{-p}/\delta = c^m/3.$$

Fix $r \geqslant 1 + 2c^{1/2-\varepsilon}$. Then $r(k) \leqslant r < r(k+1)$ for some $k \in \{1, \ldots, K\}$, where we set $r(K+1) = \infty$. Note that $zD\hat{\Psi}_t(z)$ is a bounded holomorphic function on $\{|z| > \rho(1)\}$. We use the inequality (118) to see that, on the event Ω_3, for $n \leqslant N_0$,

$$\begin{aligned} r\|D\hat{\Psi}_{t(n)}\|_{\infty,r} &\leqslant r(k)\|D\hat{\Psi}_{t(n)}\|_{\infty,r(k)} \\ &\leqslant \left(\frac{r(k)+1}{r(k)-1}\right)^{1/p} r(k)\|D\hat{\Psi}_{t(n)}\|_{p,\rho(k)} \leqslant (2c^{-1/2})^{1/p} r(k)\lambda\mu(k) \end{aligned}$$

so

$$\|D\hat{\Psi}_{t(n)}\|_{\infty,r} \leqslant (2c^{-1/2})^{1/p}\lambda\mu(r(k)) \leqslant 2(2c^{-1/2})^{1/p}\lambda\mu(r).$$

We choose $p \geqslant (2m+4)/\nu$. Then there is a constant $C < \infty$ of the desired dependence such that, for $c \leqslant 1/C$, on Ω_3, for $n = 1, \dots, N_0$ and all $r \geqslant 1 + 2c^{1/2-\varepsilon}$, we have

$$\|D\hat{\Psi}_{t(n)}\|_{\infty,r} \leqslant \frac{C}{r}\left(c^{1/2-\nu}\left(\frac{r}{r-1}\right) + c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^2\right)$$

and

$$\|\hat{\Psi}'_{t(n)}\|_{\infty,e^\sigma} \leqslant c^{1/2-\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right) = \delta_0.$$

We restrict to such c. Set

$$\Omega_0 = \Omega_1 \cap \Omega_2 \cap \Omega_3.$$

Then $\mathbb{P}(\Omega_0^c) \leqslant c^m$ and, on the event Ω_0, for all $t \leqslant T_0(\delta_0) \wedge T$ and all $r \geqslant 1 + 2c^{1/2-\varepsilon}$,

$$\|D\hat{\Psi}_t\|_{\infty,r} \leqslant \frac{C}{r}\left(c^{1/2-\nu}\left(\frac{r}{r-1}\right) + c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^2\right)$$

and

$$\|\hat{\Psi}'_t\|_{\infty,e^\sigma} \leqslant \delta_0.$$

In conjunction with (90), this forces $T_0(\delta_0) > T$ on Ω_0 and so concludes the proof of (84) and (86).

We deduce (85) using the identity

$$\psi(z) = \psi(\infty) - \int_1^\infty D\psi(sz)s^{-1}ds.$$

On the event Ω_2, for all $t \leqslant T_0(\delta_0) \wedge T$,

$$|\hat{\Psi}_t(\infty)| \leqslant C\left(c^{1/2-\nu} + c^{1-3\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^2\right).$$

On the other hand, $\Omega_0 \subseteq \Omega_2$ and on Ω_0 we have $T_0(\delta_0) > T$ and, using (86), for $t \leqslant T$ and $|z| = r \geqslant 1 + c^{1/2-\varepsilon}$,

$$\begin{aligned}\int_1^\infty |D\hat{\Psi}_t(sz)|s^{-1}ds &\leqslant \int_1^\infty \frac{C}{rs}\left(c^{1/2-\nu}\left(\frac{sr}{sr-1}\right) + c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^2\right)s^{-1}ds \\ &\leqslant \frac{C}{r}\left(c^{1/2-\nu}\left(1 + \log\left(\frac{r}{r-1}\right)\right) + c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma - 1}\right)^2\right).\end{aligned}$$

Since $r \geqslant 1 + c^{1/2}$, the log factor can be absorbed in $c^{1/2-\nu}$ by adjustment of ν. Then, on combining the last two estimates, we obtain (85).

The estimate (87) may now be deduced from Proposition 5.1 using standard L^p tail estimates. The details are left to the reader.

For the estimate (88), define

$$\tilde{\mu}_0 = C(c + \sqrt{c\delta_0} + \delta_0^2)$$

where C is the constant in Proposition 5.1, and define

$$\tilde{\mu}(r) = \frac{C}{r}\left(c\left(\frac{r}{r-1}\right)^2 + \sqrt{c\delta_0}\left(\frac{r}{r-1}\right) + \delta_0^2\left(1+\log\left(\frac{r}{r-1}\right)\right)\right).$$

where C is the constant of Proposition 5.4. Set $\tilde{\Omega}_0 = \Omega_1 \cap \tilde{\Omega}_2 \cap \tilde{\Omega}_3$, where

$$\tilde{\Omega}_2 = \Omega_2 \cap \{|\hat{\Psi}_t(\infty) - \hat{\Pi}_t(\infty)| \leqslant \lambda_0\tilde{\mu}_0 \text{ for all } t \leqslant T_0(\delta_0) \wedge T\}$$

and

$$\tilde{\Omega}_3 = \Omega_3 \cap \bigcap_{n=1}^{N}\bigcap_{k=1}^{K}\{\|D(\hat{\Psi}_{t(n)} - \hat{\Pi}_{t(n)})\|_{p,\rho(k)} 1_{\{t(n)\leqslant T_0(\delta_0)\}} \leqslant \lambda\tilde{\mu}(r(k))\}.$$

We follow a similar argument to above to see that $\mathbb{P}(\tilde{\Omega}_0^c) \leqslant 2c^m$ and on $\tilde{\Omega}_0$ we have $T_0(\delta_0) > T$ and for $t \leqslant T$

$$|\hat{\Psi}_t(\infty) - \hat{\Pi}_t(\infty)| \leqslant C\left(c^{3/4-3\nu/2}\left(\frac{e^\sigma}{e^\sigma-1}\right)^{1/2} + c^{1-3\nu}\left(\frac{e^\sigma}{e^\sigma-1}\right)^2\right)$$

and for $|z| = r \geqslant 1 + 2c^{1/2-\varepsilon}$,

$$\|D(\hat{\Psi}_t - \hat{\Pi}_t)\|_{\infty,r} \leqslant \frac{C}{r}\left(c^{1-\nu}\left(\frac{r}{r-1}\right)^2 + c^{3/4-3\nu/2}\left(\frac{e^\sigma}{e^\sigma-1}\right)^{1/2}\right.$$
$$\left.\left(\frac{r}{r-1}\right) + c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma-1}\right)^2\right).$$

Finally we can integrate as above to deduce (88). □

Proof of Theorem 1.2. We will write the argument for the case $\zeta < 1$, omitting the modifications needed for $\zeta = 1$, which are left to the reader. Since $N < n_\alpha$, we can choose $\delta = \delta(\alpha, \eta, N) > 0$ and $T < t_\zeta$ such that $\nu_T = N + \delta$. Choose δ_0 and Ω_0 as in Proposition 5.7, with the choice of T just made. Write C for the constant appearing in Proposition 5.7 and set

$$\Delta = C\left(c^{1/2-\nu} + c^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma-1}\right)^2\right).$$

Then, for all $|z| \geqslant 1 + c^{1/2-\varepsilon}$ and all $t \leqslant T$, on the event Ω_0, we have $|\hat{\Phi}_t(z) - z| \leqslant \Delta$. Then, by Propositions 5.5 and 5.6, choosing δ_0 as in Proposition 5.7 and using an L^p-tail estimate for suitably large p, there is an event $\Omega_1 \subseteq \Omega_0$, of probability exceeding $1 - 2c^m$, on which, for all $t \leqslant T$, both $|c\mathcal{V}_t - \nu_t| \leqslant \Delta$ and, provided $c\mathcal{V}_t \leqslant N$, also

$$|\mathcal{T}_t - \tilde{\mathcal{T}}_t| \leqslant Cc^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma-1}\right)^2.$$

We can choose C so that, for $c \leqslant 1/C$, we have $\Delta \leqslant \delta$, so $c\mathcal{V}_T \geqslant N + \delta - \Delta \geqslant N$ always on Ω_1. Now, for all $n \leqslant N/c$, we have $\mathcal{V}_t = n$ for some $t \leqslant T$ with $c\mathcal{V}_t \leqslant N$, so on Ω_1, for all $|z| \geqslant 1 + c^{1/2-\varepsilon}$, we have

$$|\hat{\Phi}_n^{\text{disc}}(z) - z| \leqslant \Delta, \quad |\mathcal{T}_n^{\text{disc}} - \tau_n^{\text{disc}}| \leqslant Cc^{1-4\nu}\left(\frac{e^\sigma}{e^\sigma-1}\right)^2.$$

□

6. Fluctuation Scaling Limit for ALE(α, η)

Given an ALE(α, η) process $(\Phi_t)_{t\geqslant 0}$, recall that

$$\hat{\Phi}_t(z) = \Phi_t(z)/e^{\mathcal{T}_t}, \quad \mathcal{T}_t = \log \Phi_t'(\infty).$$

The fluctuations in these coordinates are given by

$$\hat{\Psi}_t(z) = \hat{\Phi}_t(z) - z, \quad \Psi_t^{\text{cap}} = \mathcal{T}_t - \tau_t, \quad \tau_t = \zeta^{-1}\log(1+\zeta t).$$

Recall that we write $\mathcal{H}$ for the set of holomorphic functions on $\{|z| > 1\}$ which are bounded at ∞, and we use on $\mathcal{H}$ the topology of uniform convergence on $\{|z| \geqslant r\}$ for all $r > 1$. In this section we prove Theorem 1.3 and then, at the end, we deduce Theorem 1.4.

6.1. Reduction to Poisson integrals. Our starting point is the interpolation formula (17)

$$\hat{\Psi}_t(z) = \hat{M}_t(z) + \hat{A}_t(z), \quad \Psi_t^{\text{cap}} = M_t^{\text{cap}} + A_t^{\text{cap}}.$$

As a first step, we study the approximations $\hat{\Pi}_t(z)$ and Π_t^{cap} to $\hat{M}_t(z)$ and M_t^{cap} which have a simple form and which prove to be the dominant terms in the considered limit. Set

$$H(\theta, z) = \frac{z}{e^{-i\theta}z - 1} = \sum_{k=0}^{\infty} e^{i(k+1)\theta} z^{-k}.$$

Recall the multiplier operator $P(\tau)$ defined at (12). Then

$$P(\tau)H(\theta, z) = \sum_{k=0}^{\infty} e^{-q(k)\tau} e^{i(k+1)\theta} z^{-k}.$$

Recall that $c_t = ce^{-\alpha\tau_t}$ and $\lambda_t = c^{-1}e^{-\eta\tau_t}$, and that we define for $|z| > 1$

$$\hat{\Pi}_t(z) = \int_{E(t)} e^{-(\tau_t-\tau_s)} P(\tau_t - \tau_s) H(\theta, z) 2c_s 1_{\{v\leqslant\lambda_s\}} \tilde{\mu}(d\theta, dv, ds), \tag{91}$$

$$\Pi_t^{\text{cap}} = \int_{E(t)} e^{-\zeta(\tau_t-\tau_s)} c_s 1_{\{v\leqslant\lambda_s\}} \tilde{\mu}(d\theta, dv, ds). \tag{92}$$

The following result allows us to deduce the weak limit of the normalized fluctuations from that of the Poisson integrals $(\hat{\Pi}_t, \Pi_t^{\text{cap}})_{t\geqslant 0}$.

Proposition 6.1. *For all $\alpha, \eta \in \mathbb{R}$ with $\zeta = \alpha + \eta \leqslant 1$,*

$$c^{-1/2}(\hat{\Psi}_t - \hat{\Pi}_t, \Psi_t^{\text{cap}} - \Pi_t^{\text{cap}}) \to 0$$

in $\mathcal{H} \times \mathbb{R}$ uniformly on compacts in $[0, t_\zeta)$, in probability, in the limit $c \to 0$ and $\sigma \to 0$ considered in Theorem 1.3.

Proof. In Theorem 1.3, for $\zeta < 1$, we restrict to $\sigma \geqslant c^{1/4-\varepsilon}$ and take $\delta_0 = c^{1/2-\nu}e^{\sigma}/(e^{\sigma}-1)$ with $\nu \leqslant \varepsilon/4$. On the other hand, for $\zeta = 1$, we restrict to $\sigma \geqslant c^{1/5-\varepsilon}$ and take $\delta_0 = c^{1/2-\nu}e^{\sigma}/(e^{\sigma}-1))$. In both cases, the right-hand sides in (87), (88) and (89) are therefore small compared to $\sqrt{c}$ in the considered limit. The claim thus follows from Proposition 5.7. □

Since the integral (91) converges absolutely for all ω, we can exchange limits to see that

$$\hat{\Pi}_t(z) = \sum_{k=0}^{\infty} \Pi_t(k) z^{-k}$$

where

$$\Pi_t(k) = 2\int_{E(t)} e^{-(1+q(k))(\tau_t-\tau_s)} e^{i(k+1)\theta} c_s 1_{\{v\leqslant\lambda_s\}} \tilde{\mu}(d\theta, dv, ds).$$

Set $q_0(k) = (1-\zeta)k$ and define, for all $\zeta \in (-\infty, 1]$ and $t < t_\zeta$,

$$\Pi_t^0(k) = 2\int_{E(t)} e^{-(1+q_0(k))(\tau_t-\tau_s)} e^{i(k+1)\theta} c_s 1_{\{v\leqslant\lambda_s\}} \tilde{\mu}(d\theta, dv, ds).$$

Proposition 6.2. *For all $\alpha, \eta \in \mathbb{R}$ with $\alpha + \eta = \zeta \leqslant 1$, and all $t < t_\zeta$, there is a constant $C(\alpha, \eta, t) < \infty$ such that, for all $k \geqslant 0$,*

$$\left\| \sup_{s\leqslant t} |\Pi_s(k)| \right\|_2 \leqslant C\sqrt{c}, \quad \left\| \sup_{s\leqslant t} |\Pi_s^{\text{cap}}| \right\|_2 \leqslant C\sqrt{c}$$

and

$$\left\| \sup_{s\leqslant t} |\Pi_s(k) - \Pi_s^0(k)| \right\|_2 \leqslant Ck^2\sigma\sqrt{c}.$$

Moreover, C may be chosen so that, for all $h \in [0, 1]$ and all stopping times $T \leqslant t - h$,

$$\|\Pi_{T+h}^0(k) - \Pi_T^0(k)\|_2 \leqslant C\sqrt{c}(\sqrt{h} + kh), \quad \|\Pi_{T+h}^{\text{cap}} - \Pi_T^{\text{cap}}\|_2 \leqslant C\sqrt{ch}.$$

Proof. The estimates for $(\Pi_t^{\text{cap}})_{t<t_\zeta}$ are standard and are left to the reader. For $(\Pi_t(k))_{t<t_\zeta}$, we use time-dissection to obtain estimates with good dependence on k. Set $\kappa = 1 + q(k)$ and define

$$M_t(k) = e^{\kappa\tau_t}\Pi_t(k) = \int_{E(t)} e^{\kappa\tau_s} e^{i(k+1)\theta} 2c_s 1_{\{v\leqslant\lambda_s\}} \tilde{\mu}(d\theta, dv, ds).$$

Set $n = \lceil \kappa\tau_t \rceil$ and $t(n) = t$. Set $t(i) = i/\kappa$ for $i = 0, 1, \dots, n-1$. Then $t(i+1) - t(i) \leqslant 1/\kappa$ for all i. We have

$$\mathbb{E}(|M_t(k)|^2) = 4\int_0^t e^{2\kappa\tau_s} c_s^2 \lambda_s ds \leqslant Cc\int_0^t e^{2\kappa\tau_s}\dot{\tau}_s ds \leqslant Cce^{2\kappa\tau_t}/\kappa$$

so, by Doob's L^2-inequality,

$$\left\| \sup_{s\leqslant t} |M_s(k)| \right\|_2 \leqslant Ce^{\kappa\tau_t}\sqrt{c/\kappa}.$$

Now, for $t(i) \leqslant s \leqslant t(i+1)$,

$$|\Pi_s(k)| \leqslant e^{-\kappa\tau_{t(i)}}|M_s(k)|$$

so

$$\Big\| \sup_{t(i)\leqslant s\leqslant t(i+1)} |\Pi_s(k)| \Big\|_2 \leqslant Ce^{-\kappa\tau_{t(i)}} \Big\| \sup_{s\leqslant t(i+1)} |M_s(k)| \Big\|_2 \leqslant C\sqrt{c/\kappa}$$

and so

$$\Big\| \sup_{s\leqslant t} |\Pi_s(k)| \Big\|_2 \leqslant C\sqrt{c}. \tag{93}$$

For the second estimate, set $\kappa_0 = 1 + (1-\zeta)k$ and note that

$$0 \leqslant |\kappa - \kappa_0| = |\zeta|k(1 - e^{-\sigma(k+1)}) \leqslant |\zeta|k(k+1)\sigma.$$

Restrict for now to the case $\zeta \geqslant 0$, when $\kappa \geqslant \kappa_0$, and define

$$M_t^0(k) = \int_{E(t)} e^{\kappa_0\tau_s} e^{i(k+1)\theta} 2c_s 1_{\{v\leqslant\lambda_s\}} \tilde{\mu}(d\theta, dv, ds)$$

and

$$\tilde{M}_t(k) = M_t(k) - M_t^0(k) = \int_{E(t)} (e^{\kappa\tau_s} - e^{\kappa_0\tau_s}) e^{i(k+1)\theta} 2c_s 1_{\{v\leqslant\lambda_s\}} \tilde{\mu}(d\theta, dv, ds).$$

Note that

$$0 \leqslant e^{\kappa\tau_s} - e^{\kappa_0\tau_s} \leqslant (\kappa - \kappa_0)\tau_s e^{\kappa\tau_s}$$

so, by a similar argument,

$$\Big\| \sup_{s\leqslant t} |e^{-\kappa\tau_s}\tilde{M}_s(k)| \Big\|_2 \leqslant C(\kappa - \kappa_0)\sqrt{c}.$$

Now

$$\Pi_s(k) - \Pi_s^0(k) = e^{-\kappa\tau_s}\tilde{M}_s(k) + (e^{-\kappa\tau_s} - e^{-\kappa_0\tau_s})M_s^0(k)$$

so

$$|\Pi_s(k) - \Pi_s^0(k)| \leqslant e^{-\kappa\tau_s}|\tilde{M}_s(k)| + (\kappa - \kappa_0)\tau_s|\Pi_s^0(k)|$$

and so

$$\Big\| \sup_{s\leqslant t} |\Pi_s(k) - \Pi_s^0(k)| \Big\|_2 \leqslant C(\kappa - \kappa_0)\sqrt{c} \leqslant Ck^2\sigma\sqrt{c}.$$

For $\Pi_s^0(k)$, we used the estimate (93) with κ replaced by κ_0, which is the special case $\sigma = 0$. A similar argument holds in the case $\zeta < 0$, with the roles of κ and κ_0 interchanged, which leads to the same estimate. It remains to show the third estimate, which we will do for general $\sigma \geqslant 0$. We have

$$\begin{aligned}\Pi_{T+h}(k) - \Pi_T(k) &= e^{-\kappa\tau_{T+h}}M_{T+h}(k) - e^{-\kappa\tau_T}M_T(k) \\ &= e^{-\kappa\tau_{T+h}}\tilde{M}_h(k) - (e^{-\kappa(\tau_{T+h}-\tau_T)} - 1)\Pi_T(k)\end{aligned}$$

where we redefine

$$\tilde{M}_h(k) = M_{T+h}(k) - M_T(k) = \int_{E(T+h)\setminus E(T)} e^{\kappa\tau_s} e^{i(k+1)\theta} 2c_s 1_{\{v \leqslant \lambda_s\}} \tilde{\mu}(d\theta, dv, ds).$$

Now

$$\mathbb{E}(|\tilde{M}_h(k)|^2 | T) = 4 \int_T^{T+h} e^{2\kappa\tau_s} c_s^2 \lambda_s ds$$

so

$$\mathbb{E}(|e^{-\kappa\tau_{T+h}} \tilde{M}_h(k)|^2) \leqslant Cch.$$

On the other hand, since $T \leqslant t$,

$$\|(e^{-\kappa(\tau_{T+h}-\tau_T)} - 1)\Pi_T(k)\|_2 \leqslant C\kappa h \left\| \sup_{s\leqslant t} |\Pi_s(k)| \right\|_2 \leqslant C(k+1)h\sqrt{c}.$$

The claimed estimate follows. □

6.2. Gaussian limit process. By Proposition 6.1, in order to compute the weak limit of $c^{-1/2}(\hat{\Psi}_t, \Psi_t^{\text{cap}})_{t<t_\zeta}$, it suffices to compute the weak limit of $c^{-1/2}(\hat{\Pi}_t, \Pi_t^{\text{cap}})_{t<t_\zeta}$. This process is a deterministic linear function of the compensated Poisson random measure $\tilde{\mu}$. We are guided to find the weak limit process by replacing $\tilde{\mu}$ in (91) and (92) by a Gaussian white noise on $[0, 2\pi) \times [0, \infty) \times (0, \infty)$ of the same intensity. At the same time, we set $\sigma = 0$ in the limit,[2] replacing the multiplier operator $P(\tau)$ by the corresponding operator $P_0(\tau)$ when $\sigma = 0$. Then, using the scaling properties of white noise, we arrive at candidate limit processes $(\hat{\Gamma}_t(z))_{t<t_\zeta}$ and $(\Gamma_t^{\text{cap}})_{t<t_\zeta}$ which are defined as follows. Let W be a Gaussian white noise on $[0, 2\pi) \times (0, \infty)$ of intensity $(2\pi)^{-1} d\theta dt$. Define for each $|z| > 1$ and $t \in [0, t_\zeta)$

$$\hat{\Gamma}_t(z) = 2 \int_0^t \int_0^{2\pi} e^{-(\tau_t - \tau_s)} P_0(\tau_t - \tau_s) H(\theta, z) e^{-(\alpha+\eta/2)\tau_s} W(d\theta, ds),$$

$$\Gamma_t^{\text{cap}} = \int_0^t \int_0^{2\pi} e^{-\zeta(\tau_t - \tau_s)} e^{-(\alpha+\eta/2)\tau_s} W(d\theta, ds)$$

where these Gaussian integrals are understood by the usual L^2 isometry. Define for $t \geqslant 0$ and $k \geqslant 0$

$$B_t(k) = \sqrt{2} \int_0^t \int_0^{2\pi} e^{i(k+1)\theta} W(d\theta, ds), \quad B_t = \int_0^t \int_0^{2\pi} W(d\theta, ds).$$

[2] It is not necessary to pass to the limit $\sigma \to 0$. Indeed, the best Gaussian approximation for given $\sigma > 0$ would be obtained using P instead of P_0. The limit $c \to 0$ with σ fixed then holds uniformly in σ, subject to the restrictions stated in Theorem 1.3, and the limit processes for σ fixed converge weakly to the case $\sigma = 0$. We have stated only the joint limit, since this seems to us of main interest, and since the limit fluctuations have in this case a slightly simpler form.

We can and do choose versions of $(B_t(k))_{t\geqslant 0}$ and $(B_t)_{t\geqslant 0}$ which are continuous in t. Then $(B_t(k))_{t\geqslant 0}$ is a complex Brownian motion for all k, $(B_t)_{t\geqslant 0}$ is a real Brownian motion, and all these processes are independent. Note that, almost surely, for all $t < t_\zeta$,

$$\Gamma_t^{\text{cap}} = \int_0^t e^{-\zeta(\tau_t-\tau_s)} e^{-(\alpha+\eta/2)\tau_s} dB_s.$$

Define for $t \in [0, t_\zeta)$ and $k \geqslant 0$

$$\Gamma_t(k) = \sqrt{2}\int_0^t e^{-(1-\zeta)k(\tau_t-\tau_s)} e^{-(\alpha+\eta/2)\tau_s} dB_s(k).$$

The following estimate may be obtained by (a simpler version of) the argument used for Proposition 6.2.

Proposition 6.3. *For all $\alpha, \eta \in \mathbb{R}$ with $\alpha+\eta = \zeta \leqslant 1$, and all $t < t_\zeta$, there is a constant $C(\alpha, \eta, t) < \infty$ such that, for all $k \geqslant 0$,*

$$\left\| \sup_{s\leqslant t} |\Gamma_s(k)| \right\|_2 \leqslant C.$$

The following identity holds in L^2 for all $|z| > 1$ and $t < t_\zeta$

$$\hat{\Gamma}_t(z) = \sum_{k=0}^{\infty} \Gamma_t(k) z^{-k}. \tag{94}$$

By Proposition 6.3, almost surely, the right-hand side in (94) converges uniformly on compacts in $[0, t_\zeta)$, uniformly on $\{|z| \geqslant r\}$, for all $r > 1$. So we can and do use (94) to choose a version of $\hat{\Gamma}_t(z)$ for each $t < t_\zeta$ and $|z| > 1$ such that $(\hat{\Gamma}_t)_{t<t_\zeta}$ is a continuous process in $\mathcal{H}$ and (94) holds for all ω.

The processes $(\Gamma_t(k))_{t<t_\zeta}$ and $(\Gamma_t^{\text{cap}})_{t<t_\zeta}$ are also characterized by the following Ornstein–Uhlenbeck-type stochastic differential equations

$$d\Gamma_t(k) = e^{-\alpha\tau_t}\left(\sqrt{2}e^{-\eta\tau_t/2} dB_t(k) - (1+(1-\zeta)k)\Gamma_t(k) e^{-\eta\tau_t} dt\right), \quad \Gamma_0(k) = 0,$$
$$d\Gamma_t^{\text{cap}} = e^{-\alpha\tau_t}\left(e^{-\eta\tau_t/2} dB_t - \zeta\Gamma_t^{\text{cap}} e^{-\eta\tau_t} dt\right), \quad \Gamma_0^{\text{cap}} = 0.$$

These equations can be put in a simpler form by switching to the time-scale

$$\nu_t = \int_0^t e^{-\eta\tau_s} ds$$

which arises as the limit as $c \to 0$ of a time-scale where particles arrive at a constant rate. Write $\nu \mapsto t(\nu) : [0, n_\alpha) \to [0, t_\zeta)$ for the inverse map and set

$$\tilde{\Gamma}_\nu(k) = \Gamma_{t(\nu)}(k), \quad \tilde{\Gamma}_\nu^{\text{cap}} = \Gamma_{t(\nu)}^{\text{cap}}$$

and

$$\tilde{\tau}_\nu = \tau_{t(\nu)}, \quad \tilde{B}_\nu(k) = \int_0^{t(\nu)} e^{-\eta\tau_s/2} dB_s(k), \quad \tilde{B}_\nu = \int_0^{t(\nu)} e^{-\eta\tau_s/2} dB_s.$$

Then $e^{-\alpha\tilde{\tau}_\nu} = (1+\alpha\nu)^{-1}$. Also $(\tilde{B}_\nu(k))_{\nu<n_\alpha}$ is a complex Brownian motion for all k, $(\tilde{B}_\nu)_{\nu<n_\alpha}$ is a real Brownian motion, and these processes are independent. Then we have

$$d\tilde{\Gamma}_\nu(k) = (1+\alpha\nu)^{-1}\left(\sqrt{2}d\tilde{B}_\nu(k) - (1+(1-\zeta)k)\tilde{\Gamma}_\nu(k)d\nu\right), \quad \tilde{\Gamma}_0(k) = 0,$$
$$d\tilde{\Gamma}_\nu^{\mathrm{cap}} = (1+\alpha\nu)^{-1}\left(d\tilde{B}_\nu - \zeta\tilde{\Gamma}_\nu^{\mathrm{cap}}d\nu\right), \quad \tilde{\Gamma}_0^{\mathrm{cap}} = 0. \tag{95}$$

We can define a Brownian motion $(\tilde{B}_\nu)_{\nu<n_\alpha}$ in $\mathcal{H}$ by

$$\tilde{B}_\nu(z) = \sum_{k=0}^{\infty}\tilde{B}_\nu(k)z^{-k}.$$

Set

$$\tilde{\Gamma}_\nu(z) = \sum_{k=0}^{\infty}\tilde{\Gamma}_\nu(k)z^{-k} = \hat{\Gamma}_{t(\nu)}(z).$$

On summing the equations (95), we see that $(\tilde{\Gamma}_\nu)_{\nu<n_\alpha}$ satisfies the following stochastic integral equation in $\mathcal{H}$

$$\tilde{\Gamma}_\nu(z) = \int_0^\nu \frac{\sqrt{2}d\tilde{B}_s(z) - \tilde{\Gamma}_s(z)ds + (1-\zeta)D\tilde{\Gamma}_s(z)ds}{1+\alpha s}.$$

6.3. Convergence. Given Proposition 6.1, the following result will complete the proof of Theorem 1.3.

Proposition 6.4. *For all $\alpha, \eta \in \mathbb{R}$ with $\alpha + \eta = \zeta \leqslant 1$ and all $T < t_\zeta$, we have*

$$c^{-1/2}(\hat{\Pi}_t, \Pi_t^{\mathrm{cap}})_{t\geqslant 0} \to (\hat{\Gamma}_t, \Gamma_t^{\mathrm{cap}})_{t\leqslant T}$$

weakly in $D([0,T], \mathcal{H}\times\mathbb{R})$ as $c \to 0$ and $\sigma \to 0$ as in Theorem 1.3.

Proof. By Proposition 6.2, it will suffice to show the claimed limit with $(\hat{\Pi}_t)_{t\leqslant T}$ replaced by $(\hat{\Pi}_t^0)_{t\leqslant T}$. We first show that

$$c^{-1/2}((\Pi_t^0(k) : k \geqslant 0), \Pi_t^{\mathrm{cap}})_{t\leqslant T} \to ((\Gamma_t(k) : k \geqslant 0), \Gamma_t^{\mathrm{cap}})_{t\leqslant T}$$

in the sense of finite-dimensional distributions. For all $n \geqslant 1$, all $k_1, \dots, k_n \geqslant 0$ and all $t_1, \dots, t_n \leqslant T$, any real-linear function of $c^{-1/2}(\Pi_{t_j}^0(k_j), \Pi_{t_j}^{\mathrm{cap}} : j = 1, \dots, n)$ can be written in the form

$$F = \int_{E(T)} \tilde{f}_t(\theta)1_{\{v\leqslant\lambda_t\}}\tilde{\mu}(d\theta, dv, dt)$$

where

$$\tilde{f}_t(\theta) = c^{-1/2}f_t(\theta)c_t$$

and $(\theta, t) \mapsto f_t(\theta) : [0, 2\pi) \times (0, T] \to \mathbb{R}$ is bounded, measurable and independent of c. Set

$$\sigma_t^2 = ⨍_0^{2\pi} f_t(\theta)^2 d\theta.$$

The same linear function applied to $(\Gamma_{t_j}(k_j), \Gamma_{t_j}^{\mathrm{cap}} : j = 1, \dots, n)$ gives the random variable

$$G = \int_0^T \int_0^{2\pi} \tilde{f}_t(\theta)\lambda_t^{1/2} W(d\theta, dt) = \int_0^T \int_0^{2\pi} f_t(\theta) e^{-(\alpha+\eta/2)\tau_t} W(d\theta, dt).$$

Then

$$\mathbb{E}(F^2) = \mathbb{E}(G^2) = \int_0^T ⨍_0^{2\pi} \tilde{f}_t(\theta)^2 \lambda_t d\theta dt = \int_0^T e^{-(2\alpha+\eta)\tau_t} \sigma_t^2 dt$$

and, using the Campbell–Hardy formula, as $c \to 0$,

$$\begin{aligned} \mathbb{E}(e^{iuF}) &= \exp\left(\int_0^T ⨍_0^{2\pi} (e^{iu\tilde{f}_t(\theta)} - 1 - iu\tilde{f}_t(\theta))\lambda_t d\theta dt \right) \\ &\to \exp\left(-\frac{u^2}{2} \int_0^T e^{-(2\alpha+\eta)\tau_t} \sigma_t^2 dt \right) = \mathbb{E}(e^{iuG}). \end{aligned}$$

The claimed convergence of finite-dimensional distributions follows, by convergence of characteristic functions.

Now, Proposition 6.2 shows that the processes $(\Pi_t^0(k))_{t\leqslant T}$ and $(\Pi_t^{\mathrm{cap}})_{t\leqslant T}$ all satisfy Aldous's tightness criterion in $D([0, T], \mathbb{C})$. Hence

$$c^{-1/2}((\Pi_t^0(k) : k \geqslant 0), \Pi_t^{\mathrm{cap}})_{t\leqslant T} \to ((\Gamma_t(k) : k \geqslant 0), \Gamma_t^{\mathrm{cap}})_{t\leqslant T}$$

weakly in $D([0, T], \mathbb{C}^{\mathbb{Z}^+} \times \mathbb{R})$ as $c \to 0$. Hence, for all $K \geqslant 0$,

$$c^{-1/2}(p_K(\hat{\Pi}_t^0), \Pi_t^{\mathrm{cap}})_{t\leqslant T} \to (p_K(\hat{\Gamma}_t), \Gamma_t^{\mathrm{cap}})_{t\leqslant T}$$

weakly in $D([0, T], \mathcal{H} \times \mathbb{R})$ as $c \to 0$, where, for $f(z) = \sum_{k=0}^\infty a_k z^{-k}$,

$$p_K(f)(z) = \sum_{k=0}^K a_k z^{-k}.$$

For $|z| = r$, we have

$$|(f - p_K(f))(z)| \leqslant \sum_{k=K+1}^\infty |a_k| r^{-k}.$$

Hence, it will suffice to show, for $r > 1$ and all $\varepsilon > 0$, that

$$\lim_{K\to\infty} \limsup_{c\to 0} \mathbb{P}\left(c^{-1/2} \sup_{t\leqslant T} \sum_{k=K+1}^\infty |\Pi_t^0(k)| r^{-k} > \varepsilon \right) = 0.$$

But, since $\alpha + \eta = \zeta \leqslant 1$, by Proposition 6.2, there is a constant $C(\alpha, \eta, T) < \infty$ such that, for all $c > 0$ and all $r > 1$,

$$\left\| c^{-1/2} \sup_{t \leqslant T} \sum_{k=K}^{\infty} |\Pi_t^0(k)| r^{-k} \right\|_2 \leqslant \frac{C r^{-K}}{r-1}.$$

The desired limit follows. □

Proof of Theorem 1.4. We will argue via the Skorokhod representation theorem. It will suffice to show the claimed convergence for all sequences $c_k \to 0$ and $\sigma_k \to 0$ subject to the constraint assumed in Theorem 1.3. Given $N < n_\alpha$, choose $\delta > 0$ and $T < t_\zeta$ such that $\nu_T = N + \delta$, as in the proof of Theorem 1.2. By Theorem 1.3 and Propositions 5.5 and 5.7, and since $D([0, T], \mathcal{H})$ is a complete separable metric space, there is a probability space on which are defined a sequence of ALE(α, η) processes $(\Phi_t^{(k)})_{t \geqslant 0}$, with common particle family $(P^{(c)} : c \in (0, \infty))$, and a Gaussian process $(\hat{\Gamma}_t)_{t < t_\zeta}$ with the following properties:

(a) $(\Phi_t^{(k)})_{t \geqslant 0}$ has capacity parameter c_k and regularization parameter σ_k,
(b) $(\hat{\Gamma}_t)_{t < t_\zeta}$ has the distribution of the limit Gaussian process in Theorem 1.3,
(c) almost surely, as $k \to \infty$,

$$\sup_{t \leqslant T} |c \mathcal{V}_t^{(k)} - \nu_t| \to 0$$

and, for all $r > 1$,

$$\sup_{t \leqslant T} \sup_{|z| \geqslant r} \left| c^{-1/2} \hat{\Psi}_t^{(k)}(z) - \hat{\Gamma}_t(z) \right| \to 0.$$

Here, $\mathcal{V}_t^{(k)}$ denotes the number of particles added in $(\Phi_t^{(k)})_{t \geqslant 0}$ by time t. Define for $n \geqslant 0$ and $\nu < n_\alpha$

$$J_n^{(k)} = \inf\{t \geqslant 0 : \mathcal{V}_t^{(k)} = n\}, \quad t(\nu) = \zeta^{-1}((1 + \alpha\nu)^{\zeta/\alpha} - 1).$$

From (c), we deduce that, almost surely, as $k \to \infty$,

$$\sup_{n \leqslant N/c} |J_n^{(k)} - t(cn)| \to 0$$

and, for $\nu \in [0, N]$ and $n = \lfloor \nu/c \rfloor$, the following limit holds in $\mathcal{H}$

$$c^{-1/2} \hat{\Psi}_{\nu/c}^{(k),\text{disc}} = c^{-1/2} \hat{\Psi}_n^{(k),\text{disc}} = c^{-1/2} \hat{\Psi}_{J_n^{(k)}}^{(k)} \to \hat{\Gamma}_{t(\nu)}.$$

But $(\hat{\Gamma}_{t(\nu)})_{\nu < n_\alpha}$ has the same distribution as $(\hat{\Gamma}_\nu^{\text{disc}})_{\nu < n_\alpha}$. Hence

$$c^{-1/2} (\hat{\Psi}_\nu^{(k),\text{disc}})_{\nu \leqslant N} \to (\hat{\Gamma}_\nu^{\text{disc}})_{\nu \leqslant N}$$

weakly in $D([0, N], \mathcal{H})$. □

Data Availability Statement Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

A. Miscellaneous Estimates

A.1. Explosion in continuous-time ALE(α, η). In this paper we adopted a continuous-time formulation of ALE. When the cluster is in state ϕ, we add a particle with harmonic measure coordinate $\theta \in [0, 2\pi)$ at rate $c^{-1}|\phi(e^{\sigma+i\theta})|^{-\eta} d\theta/(2\pi)$. In previous works, this process had been considered in discrete time, that is, jump by jump. Besides being mathematically convenient, the continuous-time formulation has physical meaning since it considers the process in the natural physical time-scale. We now determine exactly for which parameter values α and η there is pathwise explosion for ALE. The definition and running assumptions (5) and (6) for ALE(α, η) are given in Sect. 1.2.

Proposition A.1. *Let $(\Phi_t)_{t \geqslant 0}$ be an ALE(α, η) process. Denote by $(\mathcal{T}_t)_{t \geqslant 0}$ the associated process of capacities and by Z the explosion time of $(\Phi_t)_{t \geqslant 0}$. Then, on the event $\{Z < \infty\}$, we have $\mathcal{T}_t \to \infty$ as $t \to Z$. Moreover, if $\eta \geqslant 0$ or $\zeta = \alpha + \eta \geqslant 0$, then $Z = \infty$ almost surely, while if $\eta < 0$ and $\zeta < 0$ then $Z < \infty$ almost surely.*

Proof. The total jump rate $\lambda(\phi)$ at a state ϕ is given by

$$\lambda(\phi) = c^{-1} \fint_0^{2\pi} |\phi'(e^{\sigma+i\theta})|^{-\eta} d\theta$$

so, by distortion estimates, there is a constant $C(\eta, \sigma) < \infty$ such that

$$c^{-1} e^{-\eta\tau}/C \leqslant \lambda(\phi) \leqslant C c^{-1} e^{-\eta\tau}$$

where $\tau = \phi'(\infty)$. Similarly, there is a constant $C(\alpha, \sigma) < \infty$ such that the next jump in capacity $\Delta\tau$ satisfies

$$c e^{-\alpha\tau}/C \leqslant \Delta\tau \leqslant C c e^{-\alpha\tau}.$$

The upper bound in the first estimate implies that the jump rate is bounded if $\eta \geqslant 0$, and is bounded on compacts in τ if $\eta < 0$. Hence $Z = \infty$ almost surely if $\eta \geqslant 0$, and if $\eta < 0$ then $Z < \infty$ only if $\mathcal{T}_t \to \infty$ as $t \to Z$. Moreover, using also the upper bound of the second estimate, we see that

$$\mathcal{T}_t - C^2 \int_0^t e^{-\zeta \mathcal{T}_s} ds$$

is a local supermartingale up to Z. Hence $Z = \infty$ almost surely if $\zeta \geqslant 0$.

It remains to show that $Z < \infty$ almost surely in the case when $\eta < 0$ and $\zeta < 0$. For this we use the lower bounds in the estimates above. Set

$$\delta(\tau) = (c/C)e^{-\alpha\tau}, \quad \lambda(\tau) = (c^{-1}/C)e^{-\eta\tau}, \quad F(\tau) = \tau + \delta(\tau).$$

It will be convenient to choose $C \geqslant \alpha c$ so that F is increasing on $[0, \infty)$. We know that $\mathcal{T}_t$ jumps up by at least $\delta(\mathcal{T}_{t-})$ at rate at least $\lambda(\mathcal{T}_{t-})$. Consider the Markov chain $(X_t)_{t\geqslant 0}$ starting from 0 which jumps up by $\delta(X_{t-})$ at rate $\lambda(X_{t-})$. Since $\eta < 0$, for as long as $X_{t-} \leqslant \mathcal{T}_{t-}$, we may couple these processes so that $(X_t)_{t\geqslant 0}$ jumps whenever $(\mathcal{T}_t)_{t\geqslant 0}$ does. But $X_0 = \mathcal{T}_0 = 0$ and at each jump of X_t we have $X_t = F(X_{t-})$ and $\mathcal{T}_t \geqslant F(\mathcal{T}_{t-})$. Since F is increasing, the inequality $X_t \leqslant \mathcal{T}_t$ extends to all $t < Z$. It will therefore suffice to show that $(X_t)_{t\geqslant 0}$ explodes. Now the sequence of values $(x(n) : n \geqslant 0)$ taken by $(X_t)_{t\geqslant 0}$ is given by

$$x(n+1) = x(n) + \delta(x(n)), \quad x(0) = 0$$

and the holding times of $(X_t)_{t\geqslant 0}$ are independent exponential random variables of parameters $(\lambda(x(n)) : n \geqslant 0)$. Hence $(X_t)_{t\geqslant 0}$ explodes if and only if

$$\sum_{n=0}^{\infty} \lambda(x(n))^{-1} < \infty.$$

Note that, if $\alpha \leqslant 0$, then $x(n) \geqslant cn/C$ for all n so

$$\sum_{n=0}^{\infty} \lambda(x(n))^{-1} \leqslant Cc \sum_{n=0}^{\infty} e^{\eta cn/C} < \infty.$$

Assume then that $\alpha > 0$. For $x \geqslant 0$, define $(\psi_t(x) : t \geqslant 0)$ by

$$\dot{\psi}_t(x) = \delta(\psi_t(x)), \quad \psi_0(x) = x.$$

We can solve to obtain

$$\psi_t(x) = \frac{1}{\alpha} \log(e^{\alpha x} + \alpha ct/C).$$

Since $\alpha > 0$ and $\psi_t(x)$ is increasing in t,

$$\psi_1(x(n)) = x(n) + (c/C) \int_0^1 e^{-\alpha\psi_s(x(n))} ds \leqslant x(n) + \delta(x(n)) = x(n+1).$$

Since $\psi_t(x)$ is increasing in x, it follows by induction that, for all n,

$$x(n) \geqslant \psi_n(0) = \frac{1}{\alpha} \log(1 + \alpha cn/C).$$

Hence

$$e^{\eta x(n)} \leqslant (1 + \alpha cn/C)^{\eta/\alpha}.$$

But $\alpha + \eta = \zeta < 0$ so $\eta/\alpha < -1$ and so

$$\sum_{n=0}^{\infty} \lambda(x(n))^{-1} \leqslant Cc \sum_{n=0}^{\infty} (1 + \alpha cn/C)^{\eta/\alpha} < \infty$$

as required. □

A.2. Estimates for single-particle maps. Let P be a basic particle and let

$$F(z) = e^c \left(z + \sum_{k=0}^{\infty} a_k z^{-k} \right)$$

be the associated conformal map $D_0 \to D_0 \setminus P$. We assume that F extends continuously to $\{|z| \geqslant 1\}$. Set

$$r_0 = r_0(P) = \sup\{|z| - 1 : z \in P\},$$
$$\delta = \delta(P) = \inf\{r \geqslant 0 : |z - 1| \leqslant r \text{ for all } z \in P\}.$$

We assume throughout that $\delta \leqslant 1$. We use the following well known estimates on the capacity c. There is an absolute constant $C < \infty$ such that

$$r_0^2/C \leqslant c \leqslant C\delta^2. \tag{96}$$

The lower bound relies on Beurling's projection theorem and a comparison with the case of a slit particle. The upper bound follows from a comparison with the case $P_\delta = S_\delta \cap D_0$, where S_δ is the closed disk whose boundary intersects the unit circle orthogonally at $e^{\pm i\theta_\delta}$ with $\theta_\delta \in [0, \pi]$ is determined by $|e^{i\theta_\delta} - 1| = \delta$. See Pommerenke [23].
Write

$$\log \left(\frac{F(z)}{z} \right) = u(z) + iv(z)$$

where we understand the argument to be determined for each $z \in D_0$ so that the left-hand side is holomorphic in D_0 and such that $v(z) \to 0$ as $z \to \infty$. Then u and v are bounded and harmonic in D_0, with continuous extensions to $\{|z| \leqslant 1\}$, and $u(z) \to c$ as $z \to \infty$. Note also that

$$0 \leqslant u(e^{i\theta}) \leqslant \log(1 + r_0) \leqslant r_0 \quad \text{for all } \theta. \tag{97}$$

Lemma A.2. *Assume that* $16\delta \leqslant \pi$. *Then*

$$u(e^{i\theta}) = 0 \quad \textit{whenever } |\theta| \in [16\delta, \pi] \tag{98}$$

and

$$|v(e^{i\theta})| \leqslant 16\delta \quad \textit{for all } \theta. \tag{99}$$

Proof. Set

$$p_\delta = \mathbb{P}_\infty(B \text{ hits } S_\delta \text{ before leaving } D_0)$$

where B is a complex Brownian motion. Consider the conformal map f of D_0 to the upper half-plane $\mathbb{H}$ given by

$$f(z) = i\frac{z - 1}{z + 1}.$$

Set $b = f(e^{-i\theta_\delta}) = \sin\theta_\delta/(1 + \cos\theta_\delta)$. Since $\delta \leqslant 1$, we have

$$\theta_\delta \leqslant \delta\pi/3 \tag{100}$$

and then $b \leqslant 2\pi\delta/9$. By conformal invariance,

$$p_\delta = \mathbb{P}_i(B \text{ hits } f(S_\delta) \text{before leaving} \mathbb{H}) = 2\int_0^{2b/(1-b^2)} \frac{dx}{\pi(1+x^2)}.$$

Hence

$$p_\delta \leqslant 4b/\pi \leqslant 8\delta/9. \tag{101}$$

Now $e^{i\pi}$ is not a limit point of P so $e^{i\pi} = F(e^{i(\pi+\alpha)})$ for some $\alpha \in \mathbb{R}$. Then $u(e^{i(\pi+\alpha)}) = 0$ and we can and do choose α so that $\alpha + v(e^{i(\pi+\alpha)}) = 0$. Set

$$\theta^+ = \sup\{\theta \leqslant \pi + \alpha : u(e^{i\theta}) > 0\}, \quad \theta^- = \inf\{\theta \geqslant \pi + \alpha : u(e^{i\theta}) > 0\} - 2\pi.$$

Then $\theta^- \leqslant \theta^+$. We will show that $|\theta^\pm| \leqslant 16\delta$, which then implies (98). For $\theta \in [\theta^-, \theta^+]$, we have $F(e^{i\theta}) \in S_\delta$ so $|\theta + v(e^{i\theta})| \leqslant \theta_\delta$. Set $P^* = \{F(e^{i\theta}) : \theta \in [\theta^-, \theta^+]\}$. Then $P^* \subseteq S_\delta$ so, by conformal invariance,

$$\frac{\theta^+ - \theta^-}{2\pi} = \mathbb{P}_\infty(B \text{ hits } P^* \text{ on leaving} D_0 \setminus P) \leqslant p_\delta.$$

On the other hand, for $\theta, \theta' \in [\theta^+, \theta^- + 2\pi]$ with $\theta \leqslant \theta'$, by conformal invariance,

$$\begin{aligned}\frac{\theta' - \theta}{2\pi} &= \mathbb{P}_\infty\big(B \text{ hits } \big[e^{i(\theta + v(e^{i\theta}))}, e^{i(\theta' + v(e^{i\theta'}))}\big] \text{ on leaving } D_0 \setminus P\big) \\ &\leqslant \frac{\theta' + v(e^{i\theta'}) - \theta - v(e^{i\theta})}{2\pi}\end{aligned}$$

so v is non-decreasing on $[\theta^+, \theta^- + 2\pi]$, and so

$$\alpha + v(e^{i\theta^+}) \leqslant \alpha + v(e^{i(\pi+\alpha)}) = 0 \leqslant \alpha + v(e^{i\theta^-}).$$

Hence

$$\theta^+ - \alpha \leqslant 2\pi p_\delta + \theta^- - \alpha \leqslant 2\pi p_\delta + \theta_\delta - v(e^{i\theta^-}) - \alpha \leqslant 2\pi p_\delta + \theta_\delta \tag{102}$$

and similarly

$$\theta^- - \alpha \geqslant -2\pi p_\delta - \theta_\delta. \tag{103}$$

So we obtain, for all $\theta \in [\theta^-, \theta^+]$,

$$|\alpha + v(e^{i\theta})| \leqslant 2\theta_\delta + 2\pi p_\delta. \tag{104}$$

Since v is continuous and is non-decreasing on the complementary interval, this inequality then holds for all θ. Now v is bounded and harmonic in D_0 with limit 0 at ∞, so

$$\int_0^{2\pi} v(e^{i\theta}) d\theta = 0.$$

Hence

$$|\alpha| = \left| ⨍_0^{2\pi} (\alpha + v(e^{i\theta})) d\theta \right| \leqslant 2\theta_\delta + 2\pi p_\delta.$$

On combining this with (102), (103) and (104), we see that

$$|\theta^{\pm}| \leqslant 4\theta_\delta + 4\pi p_\delta, \quad |v(e^{i\theta})| \leqslant 4\theta_\delta + 4\pi p_\delta \quad \text{for all } \theta.$$

But $4\theta_\delta + 4\pi p_\delta \leqslant 44\pi\delta/9 \leqslant 16\delta$ by (100) and (101), so we have shown the claimed inequalities. □

Proposition A.3. *There is an absolute constant $C < \infty$ with the following properties. In the case where $\delta = \delta(P) \leqslant 1/C$, for all $|z| > 1$,*

$$\left|\log\left(\frac{F(z)}{z}\right) - c\right| \leqslant \frac{C\delta}{|z|} \tag{105}$$

and, for all $|z| > 1$ with $|z-1| \geqslant C\delta$,

$$\left|\log\left(\frac{F(z)}{z}\right) - c - \frac{2c}{z-1}\right| \leqslant \frac{C\delta c|z|}{|z-1|^2} \tag{106}$$

and

$$|a_0 - 2c| \leqslant C\delta c \tag{107}$$

and

$$\left|\log\left(\frac{F(z)}{z}\right) - c - \frac{a_0}{z-1}\right| \leqslant \frac{C\delta c}{|z-1|^2}. \tag{108}$$

Proof. Since $z\log(F(z)/z)$ is bounded and holomorphic in $\{|z| > 1\}$, (105) follows from (97) and (99) by the maximum principle. The inequality (107) follows from (106) on letting $z \to \infty$, since $z(\log(F(z)/z) - c) \to a_0$. Moreover, since $(z-1)^2(\log(F(z)/z) - c) - a_0 z$ is bounded and holomorphic on $\{|z| > 1\}$, (108) follows from (106) by the maximum principle, at the cost of replacing C by $6C$, say. We will show (106) holds whenever $|z-1| \geqslant 3a$, where $a = 16\delta$.
Since u is bounded and harmonic with $u(z) \to c$ as $z \to \infty$, we have

$$⨍_0^{2\pi} u(e^{i\theta})d\theta = c$$

and, for all $|z| > 1$,

$$u(z) = ⨍_0^{2\pi} u(e^{i\theta}) \operatorname{Re}\left(\frac{z+e^{i\theta}}{z-e^{i\theta}}\right) d\theta = c + ⨍_0^{2\pi} u(e^{i\theta}) \operatorname{Re}\left(\frac{2e^{i\theta}}{z-e^{i\theta}}\right) d\theta.$$

Let $\alpha \in (-\pi, \pi]$ and $\rho > 0$ be defined by

$$⨍_0^{2\pi} u(e^{i\theta})e^{i\theta}d\theta = c\rho e^{i\alpha}.$$

We use (98) to see that $|\alpha| \leqslant a$ and $\rho \in [\cos a, 1)$. Now

$$u(z) - c - \operatorname{Re}\left(\frac{2\rho c e^{i\alpha}}{z-e^{i\alpha}}\right) = ⨍_0^{2\pi} u(e^{i\theta}) \operatorname{Re}\left(\frac{2e^{i\theta}}{z-e^{i\theta}} - \frac{2e^{i\theta}}{z-e^{i\alpha}}\right) d\theta.$$

For $|z-1| \geqslant 2a$ and any θ such that $u(e^{i\theta}) > 0$, we have

$$|z-e^{i\alpha}| \geqslant |z-1|/2, \quad |z-e^{i\theta}| \geqslant |z-1|/2, \quad |e^{i\theta}-e^{i\alpha}| \leqslant 2a.$$

Hence, for $|z| > 1$ with $|z-1| \geqslant 2a$,

$$\left| u(z) - c - \operatorname{Re}\left(\frac{2\rho c e^{i\alpha}}{z-e^{i\alpha}}\right)\right| \leqslant 2 \fint_0^{2\pi} u(e^{i\theta}) \frac{|e^{i\theta}-e^{i\alpha}|}{|z-e^{i\theta}||z-e^{i\alpha}|} d\theta \leqslant \frac{16ac}{|z-1|^2}$$

and

$$\left| \frac{2}{z-1} - \frac{2\rho e^{i\alpha}}{z-e^{i\alpha}} \right| \leqslant \frac{2(1-\rho+|\rho e^{i\alpha}-1||z|)}{|z-1||z-e^{i\alpha}|} \leqslant \frac{12a|z|}{|z-1|^2}$$

and hence

$$\left| u(z) - c - \operatorname{Re}\left(\frac{2c}{z-1}\right)\right| \leqslant \frac{Cac|z|}{|z-1|^2}. \tag{109}$$

We can extend F to a holomorphic function in $\{|z-1| > a\}$ by setting $F(\bar{z}^{-1}) = \overline{F(z)}^{-1}$. Then u and v also extend and it is straightforward to check that the estimate (109) remains valid for all $|z-1| \geqslant 2a$. Since $v(z) \to 0$ as $z \to \infty$, a standard argument allows us to deduce from (109) that, for $|z-1| \geqslant 3a$,

$$\left| v(z) - \operatorname{Im}\left(\frac{2c}{z-1}\right)\right| \leqslant \frac{Cac|z|}{|z-1|^2}$$

and hence

$$\left| \log\left(\frac{F(z)}{z}\right) - c - \frac{2c}{z-1}\right| \leqslant \frac{Cac|z|}{|z-1|^2}.$$

□

We sometimes use exponentiated versions of the inequalities just proved, which are straightforward to deduce and are noted here for easy reference. There is an absolute constant $C < \infty$ with the following properties. Suppose that $\delta \leqslant 1/C$. Then, for all $|z| > 1$,

$$|e^{-c}F(z) - z| \leqslant C\delta$$

and, in the case $|z-1| \geqslant C\delta$,

$$\left| e^{-c}F(z) - z - \frac{2cz}{z-1}\right| \leqslant \frac{C\delta c|z|^2}{|z-1|^2} \tag{110}$$

and

$$\left| e^{-c}F(z) - z - \frac{a_0 z}{z-1}\right| \leqslant \frac{C\delta c|z|}{|z-1|^2}.$$

Proposition A.4. *There is an absolute constant $C < \infty$ with the following properties. Let P_1, P_2 be basic particles with $P_1 \subseteq P_2$. For $i = 1, 2$, write F_i for the associated conformal map $D_0 \to D_0 \setminus P_i$ and write c_i for the capacity of P_i. Set $\delta_i = \delta(P_i)$ and $a_{0,i} = a_0(P_i)$ and set*

$$\varepsilon_i(z) = \log\left(\frac{F_i(z)}{z}\right) - c_i - \frac{2c_i}{z-1}, \quad \varepsilon_{0,i}(z) = \log\left(\frac{F_i(z)}{z}\right) - c_i - \frac{a_{0,i}}{z-1}.$$

Assume that $\delta_2 \leqslant 1/C$. Then

$$|a_{0,2} - a_{0,1} - 2(c_2 - c_1)| \leqslant C\delta_2(c_2 - c_1) \tag{111}$$

and, for all $|z| > 1$ with $|z - 1| \geqslant C\delta_2$,

$$|\varepsilon_1(z) - \varepsilon_2(z)| \leqslant \frac{C\delta_2(c_2 - c_1)|z|}{|z-1|^2} \tag{112}$$

and

$$\left|\varepsilon_{0,1}(z) - \varepsilon_{0,2}(z)\right| \leqslant \frac{C\delta_2(c_2 - c_1)}{|z-1|^2}. \tag{113}$$

Proof. The inequalities (112) and (113) follow from (111) by the same argument used to deduce (107) and (108) from (106). Set $\tilde{P} = F_1^{-1}(P_2 \setminus P_1)$. Write $\tilde{F}$ for the associated conformal map $D_0 \to D_0 \setminus \tilde{P}$ and write $\tilde{c}$ for the capacity of $\tilde{P}$. Then

$$F_2 = F_1 \circ \tilde{F}, \quad c_2 = c_1 + \tilde{c}.$$

Note that, for $z \in \tilde{P}$, we have $F_1(z) \in P_2$, so $|F_1(z) - 1| \leqslant \delta_2$. But $|e^{-c_1}F_1(z) - z| \leqslant C\delta_1$ for all $|z| > 1$ and $c_1 \leqslant C\delta_1^2$. Hence $|z - 1| \leqslant C\delta_2$ for all $z \in \tilde{P}$ and so

$$\tilde{\delta} = \delta(\tilde{P}) \leqslant C\delta_2.$$

Hence, for C sufficiently large and $\delta_2 \leqslant 1/C$, for all $|z| > 1$ with $|z - 1| \geqslant C\delta_2$,

$$\left|\log\left(\frac{\tilde{F}(z)}{z}\right) - \tilde{c} - \frac{2\tilde{c}}{z-1}\right| \leqslant \frac{C\delta_2\tilde{c}|z|}{|z-1|^2} \tag{114}$$

and in particular

$$\left|\log\left(\frac{\tilde{F}(z)}{z}\right)\right| \leqslant \frac{C\tilde{c}|z|}{|z-1|}. \tag{115}$$

Set $z_t = z\exp(t\log(\tilde{F}(z)/z))$ and $f(t) = \log(F_1(z_t)/F_1(z))$. Then

$$\log\left(\frac{F_2(z)}{F_1(z)}\right) = f(1) - f(0) = \int_0^1 \dot{f}(t)dt = \log\left(\frac{\tilde{F}(z)}{z}\right)\int_0^1 F_1'(z_t)\left(\frac{F_1(z_t)}{z_t}\right)^{-1} dt$$

so

$$\begin{aligned}
&\varepsilon_2(z) - \varepsilon_1(z) \\
&\quad = \log\left(\frac{F_2(z)}{F_1(z)}\right) - \tilde{c} - \frac{2\tilde{c}}{z-1} \\
&\quad = \log\left(\frac{\tilde{F}(z)}{z}\right)\int_0^1 F_1'(z_t)\left(\frac{F_1(z_t)}{z_t}\right)^{-1} dt - \tilde{c} - \frac{2\tilde{c}}{z-1} \\
&\quad = \log\left(\frac{\tilde{F}(z)}{z}\right) - \tilde{c} - \frac{2\tilde{c}}{z-1} + \log\left(\frac{\tilde{F}(z)}{z}\right)\int_0^1 \left(F_1'(z_t)\left(\frac{F_1(z_t)}{z_t}\right)^{-1} - 1\right) dt.
\end{aligned}$$

Now $|\log(\tilde{F}(z)/z)| \leqslant C\delta_2$, so $|z_t - z| \leqslant C\delta_2$ for all t. Hence, for C sufficiently large and $|z-1| \geqslant C\delta_2$, we have $|z_t - 1| \geqslant C_0\delta_1$ for all t, where C_0 is the constant from Proposition A.3. Then

$$|e^{-c_1}F_1(z_t) - z_t| \leqslant C\delta_1, \quad |e^{-c_1}F_1'(z_t) - 1| \leqslant \frac{C\delta_1}{|z-1|}$$

where we used Cauchy's integral formula for the second inequality, adjusting the value of C if necessary. On combining these estimates with (114) and (115), we see that

$$|\varepsilon_2(z) - \varepsilon_1(z)| \leqslant \frac{C\delta_2(c_2 - c_1)|z|}{|z-1|^2}$$

as claimed. □

The following is a straightforward consequence of (96) and Propositions A.3 and A.4.

Proposition A.5. *Let $(P^{(c)} : c \in (0,1])$ be a family of basic particles and suppose that the associated conformal maps F_c are given by*

$$F_c(z) = e^c\left(z + \sum_{k=0}^{\infty} a_k(c) z^{-k}\right).$$

Fix $\Lambda \in [1,\infty)$ and assume that $\delta(c) \leqslant \Lambda r_0(c)$ for all c. Then there is a constant $C(\Lambda) < \infty$ such that, for all $c \leqslant 1/C$,

$$|a_0(c) - 2c| \leqslant Cc^{3/2}$$

and, for all $|z| > 1$,

$$\left|\log\left(\frac{F_c(z)}{z}\right) - c\right| \leqslant \frac{Cc}{|z-1|}$$

and

$$\left|\log\left(\frac{F_c(z)}{z}\right) - c - \frac{a_0(c)}{z-1}\right| \leqslant \frac{Cc^{3/2}}{|z-1|^2}$$

and

$$\left|e^{-c}F_c(z) - z - \frac{a_0(c)z}{z-1}\right| \leqslant \frac{Cc^{3/2}|z|}{|z-1|^2}.$$

Moreover, if $(P^{(c)} : c \in (0,1])$ *is nested, then* C *may be chosen so that, for all* $c_1, c_2 \in (0, c]$,

$$|(a_0(c_1) - 2c_1) - (a_0(c_2) - 2c_2)| \leqslant Cc^{1/2}|c_1 - c_2|$$

and, for all $|z-1| \geqslant C\sqrt{c}$,

$$\left|\left(\log\left(\frac{F_{c_1}(z)}{z}\right) - c_1\right) - \left(\log\left(\frac{F_{c_2}(z)}{z}\right) - c_2\right)\right| \leqslant \frac{C|c_1 - c_2|}{|z-1|}$$

and

$$\left|\left(\log\left(\frac{F_{c_1}(z)}{z}\right) - c_1 - \frac{a_0(c_1)}{z-1}\right) - \left(\log\left(\frac{F_{c_2}(z)}{z}\right) - c_2 - \frac{a_0(c_2)}{z-1}\right)\right| \leqslant \frac{C\sqrt{c}|c_1 - c_2|}{|z-1|^2}$$

and

$$\left|\left(e^{-c_1}F_{c_1}(z) - z - \frac{a_0(c_1)z}{z-1}\right) - \left(e^{-c_2}F_{c_2}(z) - z - \frac{a_0(c_2)z}{z-1}\right)\right| \leqslant \frac{C\sqrt{c}|c_1 - c_2||z|}{|z-1|^2}.$$

For our final particle estimates, we use the following integral representation for the family of particle maps

$$F_c(z) = z + \int_0^c DF_t(z) \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} \mu_t(d\theta) dt$$

for some measurable family of probability measures $(\mu_t : t \in (0, \infty))$, with μ_t supported on $\{\theta : |e^{i\theta} - 1| \leqslant \delta(t)\}$ for all t. This follows from our requirements that the particles $P^{(c)}$ have capacity c, are contained in $\{|z-1| \leqslant \delta(c)\}$ and are nested, by the Loewner–Kufarev theory. Our condition (6) and the inequality (96) then give a constant $C(\Lambda) < \infty$ such that

$$\operatorname{supp} \mu_t \subseteq \{\theta : |e^{i\theta} - 1| \leqslant C\sqrt{t}\}. \tag{116}$$

Define holomorphic functions L_t and Q_t on $\{|z| > 1\}$ by

$$L_t(z) = \int_0^{2\pi} l_t(\theta, z)\mu_t(d\theta), \quad Q_t(z) = \int_0^{2\pi} q_t(\theta, z)\mu_t(d\theta) \tag{117}$$

where

$$l_t(\theta, z) = \left(D\log\left(\frac{F_t(z)}{z}\right) + 1\right)\frac{z + e^{i\theta}}{z - e^{i\theta}}, \quad q_t(\theta, z) = DF_t(z)\frac{z + e^{i\theta}}{z - e^{i\theta}} - e^t z - \frac{2e^t e^{i\theta} z}{z-1}.$$

Note that $l_t(\theta, z) \to 1$ and $q_t(\theta, z) \to 0$ as $z \to \infty$, uniformly in θ. It is then straightforward to show the integral representations

$$\log\left(\frac{F_c(z)}{z}\right) = \int_0^c L_t(z)dt, \quad e^c\left(e^{-c}F_c(z) - z - \frac{a_0(c)z}{z-1}\right) = \int_0^c Q_t(z)dt.$$

Proposition A.6. *There is a constant $C(\Lambda) < \infty$ with the following property. For all $t \leqslant 1/C$ and all $|z| > 1$,*

$$|L_t(z)| \leqslant \frac{C|z|}{|z-1|} + \frac{C\sqrt{t}|z|}{|z-1|^2}, \quad |Q_t(z)| \leqslant \frac{C\sqrt{t}|z|}{|z-1|^2}.$$

Proof. We give the details for the second estimate, leaving the first which is similar but simpler to the reader. We split $q_t(\theta, z) = g_t(\theta, z) + h_t(\theta, z)$, where

$$g_t(\theta, z) = (DF_t(z) - e^t z)\frac{z + e^{i\theta}}{z - e^{i\theta}}, \quad h_t(\theta, z) = e^t z\left(\frac{z + e^{i\theta}}{z - e^{i\theta}} - 1 - \frac{2e^{i\theta}}{z-1}\right).$$

Now

$$\frac{z + e^{i\theta}}{z - e^{i\theta}} - 1 - \frac{2e^{i\theta}}{z-1} = \frac{2e^{i\theta}(e^{i\theta} - 1)}{(z - e^{i\theta})(z-1)}$$

so, on the support of μ_t, we have, for $|z-1| \geqslant 2C\sqrt{t}$,

$$|h_t(\theta, z)| \leqslant \frac{2Ce^t\sqrt{t}|z|}{|z-1|^2}$$

where C is the constant in (116). On the other hand, we showed above that, for all $|z| > 1$,

$$|F_t(z) - e^t z| \leqslant C\sqrt{t}$$

and F_t extends by reflection to a holomorphic function on $\{|z-1| > C\sqrt{t}\}$ satisfying the same inequality. Hence, by Cauchy's integral formula, for $|z-1| \geqslant 2C\sqrt{t}$,

$$|DF_t(z) - e^t z| \leqslant \frac{C\sqrt{t}|z|}{|z-1|}$$

and so, for θ in the support of μ_t,

$$|g_t(\theta, z)| \leqslant \frac{C\sqrt{t}|z|}{|z-1|^2}.$$

We have shown that, for all $|z-1| \geqslant C\sqrt{t}$,

$$|Q_t(z)| \leqslant \frac{C\sqrt{t}|z|}{|z-1|^2}.$$

□

A.3. Operator inequalities. Recall that, for a measurable function f on $\{|z| > 1\}$, for $p \in [1, \infty)$ and $r > 1$, we set

$$\|f\|_{p,r} = \left(⨍_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \quad \|f\|_{\infty,r} = \sup_{\theta \in [0,2\pi)} |f(re^{i\theta})|.$$

Suppose that f is holomorphic and is bounded at ∞. It is standard that, for $\rho \in (1, r)$,

$$\|f\|_{p,r} \leqslant \|f\|_{p,\rho}, \quad \|f\|_{\infty,r} \leqslant \left(\frac{\rho}{r-\rho} \right)^{1/p} \|f\|_{p,\rho}. \tag{118}$$

Moreover, there is an absolute constant $C < \infty$ such that

$$\|Df\|_{p,r} \leqslant \frac{C\rho}{r-\rho} \|f\|_{p,\rho} \tag{119}$$

where $Df(z) = zf'(z)$. The function f has a Laurent expansion

$$f(z) = \sum_{k=0}^{\infty} f_k z^{-k}.$$

Let M be an operator which acts as multiplication by m_k on the the kth Laurent coefficient. Thus

$$Mf(z) = \sum_{k=0}^{\infty} m_k f_k z^{-k}.$$

Assume that there exists a finite constant $A > 0$ such that, for all $k \geqslant 0$,

$$|m_k| \leqslant A$$

and, for all integers $K \geqslant 0$,

$$\sum_{k=2^K}^{2^{K+1}-1} |m_{k+1} - m_k| \leqslant A.$$

Then, by the Marcinkiewicz multiplier theorem [29, Vol. II, Theorem 4.14], for all $p \in (1, \infty)$, there is a constant $C = C(p) < \infty$ such that, for all $r > 1$,

$$\|Mf\|_{p,r} \leqslant CA\|f\|_{p,r}. \tag{120}$$

We will use also the following estimate.

Proposition A.7. *Let f and g be holomorphic in $\{|z| > 1\}$ and bounded at ∞. Let M be a multiplier operator and let $p \geqslant 2$. Set*

$$f_\theta(z) = f(e^{-i\theta} z).$$

and

$$h_p(z) = \left(⨍_0^{2\pi} |M(f_\theta . g)|^p (z) d\theta \right)^{1/p}.$$

Then, for all $r, \rho > 1$, *we have*

$$\|h_2\|_{p,r} \leqslant \|M\|_{p,\rho\to r}\|g\|_{p,\rho}\|f\|_{2,\rho}$$

and

$$\|h_p\|_{p,r} \leqslant \|M\|_{p,\rho\to r}\|g\|_{p,\rho}\|f\|_{p,\rho}$$

where

$$\|M\|_{p,\rho\to r} = \sup\{\|Mf\|_{p,r} : \|f\|_{p,\rho} \leqslant 1\}.$$

Proof. The second estimate is straightforward and is left to the reader. For the first, we can write

$$f(z) = \sum_{k=0}^{\infty} f_k z^{-k}, \quad g(z) = \sum_{k=0}^{\infty} g_k z^{-k}, \quad Mf(z) = \sum_{k=0}^{\infty} m_k f_k z^{-k}.$$

Then

$$M(f_\theta . g)(z) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} m_{j+k} f_k g_j e^{i\theta k} z^{-(k+j)}$$

so

$$h_2(z)^2 = \sum_{k=0}^{\infty} |f_k|^2 |M(\tau_k g)(z)|^2$$

where $\tau_k g(z) = z^{-k} g(z)$. Hence

$$\begin{aligned}\|h_2\|_{p,r}^2 = \|h_2^2\|_{p/2,r} &\leqslant \sum_{k=0}^{\infty} |f_k|^2 \|M(\tau_k g)\|_{p,r}^2 \leqslant \sum_{k=0}^{\infty} |f_k|^2 \|M\|_{p,\rho\to r}^2 \|\tau_k g\|_{p,\rho}^2 \\ &= \sum_{k=0}^{\infty} |f_k|^2 \rho^{-2k} \|M\|_{p,\rho\to r}^2 \|g\|_{p,\rho}^2 = \|M\|_{p,\rho\to r}^2 \|f\|_{2,\rho}^2 \|g\|_{p,\rho}^2.\end{aligned}$$

□

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