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Rigorous Higher Order Poincaré Optical Vortex modes

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The state of polarisation of a general form of an optical vortex mode is represented by the vector $\hat{\boldsymbol{\epsilon}}_m$ which is associated with a vector light mode of order $m > 0$. It is formed as a linear combination of two product terms involving the phase functions $e^{\pm im\phi}$ times the optical spin unit vectors $\sigma^{\mp}$. Any such state of polarisation corresponds to a unique point (Θ_P, Φ_P) on the surface of the order m unit Poincaré sphere. However, albeit a key property, the general form of the vector potential in the Lorenz gauge $\mathbf{A} = \hat{\boldsymbol{\epsilon}}_m \Psi_m$ from which the fields are derived, including the longitudinal fields, has neither been considered, nor has had its consequences explored. Here we show that the spatial dependence of Ψ_m can be found rigorously by demanding that the product $\hat{\boldsymbol{\epsilon}}_m \Psi_m$ satisfies the vector paraxial equation. For a given order m this leads to a unique Ψ_m which has no azimuthal phase of the kind $e^{i\ell\phi}$ and it is a solution of a scalar partial differential equation with ρ and z as the only variables. The theory is employed to evaluate the angular momentum for a general Poincaré mode of order m yielding the angular momentum for right- and left- circularly polarised, elliptically-polarised, linearly-polarised and radially- and azimuthally-polarised higher order modes. We find that in applications involving Laguerre-Gaussian modes, only the modes of order $m \geq 2$ have non-zero angular momentum. All modes have zero angular momentum for points on the equatorial circle for which $\cos\Theta_P = 0$.

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1. INTRODUCTION

Higher order twisted vector modes of light have recently been highlighted as important for useful applications, most notably in quantum communication and optical manipulation of matter [1–10]. They are expected to lead to enhanced optical properties enabling higher order encoding protocols for increased bandwidth in optical communications [11] and it is also expected that their angular momentum, spin and chirality are enhanced. In particular, greater optical chirality would be desirable for chiroptical processes [12] leading to stronger interaction with chiral molecules and to enhanced enantioselectivity [13].

Padgett and Courtial [14] considered the first order case in which they generalised the standard Poincaré sphere representation of zero order polarised light to include first order. The Poincaré higher order representation has been focused on in a 2011 report by Milione et al [15]. Only those vector modes for which the polarisation state corresponds to a point on the surface of the unit Poincaré sphere can be classed as Poincaré modes and it is this type of vector modes that is the subject of this paper. As we shall emphasise, the radially- and azimuthally-polarised optical vortex modes are true first order Poincaré modes.

In the Poincaré modes of order m the optical polarisation $\hat{\boldsymbol{\epsilon}}_m$ arises as a linear superposition of tensor product states involving circular polarisation states $\sigma^{\pm} = (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})/\sqrt{2}$ and spatial phase $e^{\pm im\phi}$. In every higher order Poincare mode the state of polarisation is identified uniquely by a point of angular coordinates (Θ_P, Φ_P) on the surface of the order m unit Poincaré sphere. The superposition of two terms creates a vector polarisation which is spatially inhomogeneous, as opposed to the scalar case of pure elliptical polarisation, including circular and linear polarisation [16, 17]. The focus here is on the case of vector polarisations, specifically those associated with higher order Poincaré modes.

Note that the integer $m \geq 0$ includes the lowest order, namely $m = 1$, to be referred to as the first order light mode. Each order m is distinguished uniquely by its own Poincaré sphere. The Poincaré sphere for $m = 0$ incorporates all elliptically-polarised (including circularly- and linearly- polarised) Gaussian vortex modes. As we shall show, the first order $m = 1$ has a different Poincaré sphere and different modes and includes, in addition, the radially and azimuthally-polarised optical vortex modes [18] as well as quasi- circular and quasi-elliptical modes, in which the polarisation vector has a phase factor $e^{\pm i\phi}$. The only permitted amplitude function for the first order, for example of the Laguerre-Gaussian (LG) form, is $\mathrm{LG}_p^{m=1}$. Note, however, that the radially-polarised modes of concern here are those that can be described as pure radially-polarised modes. These have no helical wave fronts which means they do not possess a phase function of the kind $e^{i\ell\phi}$ and so have no angular momentum. As highlighted in a recent article [19], it is now possible to create radially-polarised modes with helical wave fronts with an arbitrary phase function $e^{i\ell\phi}$. Koksal et al [19] proceeded to evaluate the optical properties of these new forms of radially-polarised optical modes and pointed out that such modes have the usual advantage of being easy to focus to small waists and now have, in addition, the vortex property of helical wave fronts. It should, however, be borne in mind that the kind of radially-polarised modes discussed here are the pure kind which have no helical wave fronts.

Thus although the state of polarisation $\hat{\boldsymbol{\epsilon}}_m$ is well known, the general form of the vector potential $\mathbf{A} = \hat{\boldsymbol{\epsilon}}_m \Psi_m$ associated with this polarisation has neither been considered, nor its consequences explored for arbitrary order m. Here we show that the spatial dependence of Ψ_m can be arrived at rigorously by demanding that $\mathbf{A}$ satisfies the vector paraxial equation. This leads to the unique realisation that all higher order modes $\Psi_m(\rho, z)$ have no azimuthal phase and, strictly, all admissible Ψ_m must be solutions of a specific scalar equation.

This paper is organised as follows. In section 2 we define the vector potential $\mathbf{A}$ for a general paraxial optical vortex endowed with the higher order polarisation state $\hat{\boldsymbol{\epsilon}}_m$ which is defined in terms of the higher order Poincaré sphere. In section 3 we outline the derivation of the vector paraxial equation and emphasise the main requirement that $\mathbf{A} = \hat{\boldsymbol{\epsilon}}\Psi_m$ must satisfy this equation. Section 4 considers the polarisation in cylindrical polar coordinates $\mathbf{r} = (\rho, \phi, z)$ and moves on to Section 5 by substituting the form of $\mathbf{A}$ into the vector paraxial equation. Here we outline the steps leading to conditions on the function Ψ_m for modes of order m, namely (a) a clear statement on the azimuthal dependence of Ψ_m and (b) the outlines of the derivation of the second order differential equation to be satisfied by Ψ_m involving only ρ and z-dependence. Having derived the equations for Ψ_m we proceed in section 6 to derive the electric and magnetic fields for for a Poincaré mode of order m. In section 7 we evaluate the axial component of the angular momentum of the order m mode and highlight in tabular form its values for various points on the order m unit Poincaré sphere. Section 8 contains our conclusions further comments.

2. POINCARÉ HIGHER ORDER MODES

The paraxial vector potential associated with a monochromatic optical vortex in a higher order state of polarisation is, as stated above,

$$\mathbf{A} = \hat{\boldsymbol{\epsilon}}_m \Psi(\mathbf{r}) e^{ik_z z} = \tilde{\mathbf{A}} e^{ik_z z} \tag{1}$$

where k_z is the axial vector and we do not show the factor $e^{i\omega t}$ associated with frequency ω. We follow Allen et al [20] and Haus [21], adopting the Lorenz gauge as the basis for the derivation of the electromagnetic fields, including the longitudinal components [22]. The fields can be derived entirely from the vector potential $\mathbf{A}$. The order m polarisation vector $\hat{\boldsymbol{\epsilon}}_m$ of any specific higher order mode coincides with a specific point on the surface of the order $m \geq 0$ unit Poincare sphere. It can be written as follows

$$\hat{\boldsymbol{\epsilon}}_m = e^{im\phi} \frac{(\hat{\mathbf{x}} - i\hat{\mathbf{y}})}{\sqrt{2}} \sin\left(\frac{\Theta_P}{2}\right) e^{i\Phi_P/2} + e^{-im\phi} \frac{(\hat{\mathbf{x}} + i\hat{\mathbf{y}})}{\sqrt{2}} \cos\left(\frac{\Theta_P}{2}\right) e^{-i\Phi_P/2} \tag{2}$$

where Θ_P and Φ_P are Poincaré sphere angles such that the north and south pole points are $(\Theta_P = 0, \Phi_P)$ and $(\Theta_P = \pi, \Phi_P)$ and the equatorial circle has $\Theta_P = \pi/2$ and in all case we have $0 \leq \Phi_P \leq 2\pi$. The vector $\hat{\boldsymbol{\epsilon}}_m$ in Eq.(2) is the most general polarisation state vector, and stated here, using our convention which differs slightly from that adopted by Milione et al [15]. The validity of the polarisation states, which were first introduced in [14] has already been confirmed experimentally [4, 5, 9].

In view of the degeneracy arising from the variations with the Poincare Φ_P, it is convenient to focus on the case where $\Phi_P = 0$, so that we are retaining only the Θ_P variations on the Poincaré sphere. This means we will be considering the polarisation states at all points lying on the $\Phi = 0$ longitude joining the north pole at $\Theta_P = 0$ to the equator at $\Theta_P = \pi/2$ and finally to the south pole at $\Theta_P = \pi$. The polarisation at any given point on this longitude also depends on the order m.

3. VECTOR PARAXIAL EQUATION

The optical modes in question are assumed to be propagating along the z-axis, so we can write $\mathbf{A}$ as follows

$$\mathbf{A} = \tilde{A} e^{ik_z z}; \quad \tilde{A} = \hat{\boldsymbol{\epsilon}}_m \Psi \tag{3}$$

where $k_z \hat{\mathbf{z}}$ is the axial wavevector. This vector potential must satisfy the Helmholtz equation. We have

$$\nabla_\perp^2 \left(\tilde{A} e^{ik_z z}\right) + \frac{\partial^2}{\partial z^2} \left(\tilde{A} e^{ik_z z}\right) + k_z^2 \left(\tilde{A} e^{ik_z z}\right) = 0 \tag{4}$$

where we have set $\nabla^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2}$. It is easy to show that on evaluating the derivatives in the middle term gives

$$\frac{\partial^2}{\partial z^2} \left(\tilde{A} e^{ik_z z}\right) = \left\{ \left(\frac{\partial^2 \tilde{A}}{\partial z^2}\right) - k_z^2 \tilde{A} + 2ik_z \left(\frac{\partial \tilde{A}}{\partial z}\right) \right\} e^{ik_z z} \tag{5}$$

Substituting back in Eq.(4) we have

$$\nabla_\perp^2 \left(\tilde{A} e^{ik_z z}\right) + \left\{ \left(\frac{\partial^2 \tilde{A}}{\partial z^2}\right) - k_z^2 \tilde{A} + 2ik_z \left(\frac{\partial \tilde{A}}{\partial z}\right) \right\} e^{ik_z z} + k_z^2 \left(\tilde{A} e^{ik_z z}\right) = 0 \tag{6}$$

The k_z^2 terms cancel and we drop the $\frac{\partial^2 \tilde{\boldsymbol{A}}}{\partial z^2}$ term as required in the paraxial regime and it is convenient not to show the common phase factor $e^{ik_z z}$. We have the vector paraxial equation for the vector potntial $\tilde{\boldsymbol{A}}$

$$\nabla_\perp^2 \tilde{\boldsymbol{A}} + 2ik_z \frac{\partial \tilde{\boldsymbol{A}}}{\partial z} = 0 \tag{7}$$

Finally, we express $\nabla_\perp^2 \tilde{\boldsymbol{A}}$ in terms of cylindrical polar coordinates and obtain

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \tilde{\boldsymbol{A}}}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \tilde{\boldsymbol{A}}}{\partial \phi^2} + +2ik_z\left(\frac{\partial \tilde{\boldsymbol{A}}}{\partial z}\right) = 0 \tag{8}$$

This is the vector paraxial equation that must be satisfied by the vector potential $\tilde{\boldsymbol{A}}$. Recalling that $\tilde{\boldsymbol{A}}$ is given by the vector $\hat{\boldsymbol{\epsilon}}_{\boldsymbol{m}}\Psi$, in which the polarisation $\hat{\boldsymbol{\epsilon}}_{\boldsymbol{m}}$ is well-defined, our main task is to derive the conditions which determine the form of Ψ.

4. POLARISATION IN CYLINDRICAL COORDINATES

The general polarisation vector of the order m Poincaré sphere can be written as follows

$$\hat{\boldsymbol{\epsilon}}_m = e^{im\phi}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})\mathcal{U}_P + e^{-im\phi}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})\mathcal{V}_P \tag{9}$$

where $\mathcal{U}_P$ and $\mathcal{V}_P$ are Poincaré functions

$$\mathcal{U}_P = \frac{1}{\sqrt{2}}\sin\left(\frac{\Theta_P}{2}\right)e^{i\Phi_P/2}; \quad \mathcal{V}_P = \frac{1}{\sqrt{2}}\cos\left(\frac{\Theta_P}{2}\right)e^{-i\Phi_P/2} \tag{10}$$

It is convenient to express $\hat{\boldsymbol{\epsilon}}_m$ entirely in terms of plane polar coordinates (ρ, ϕ) and the associated unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ as follows

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{\rho}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi; \quad \hat{\boldsymbol{y}} = \hat{\boldsymbol{\rho}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi \tag{11}$$

We thus find for $\hat{\boldsymbol{\epsilon}}_m$

$$\hat{\boldsymbol{\epsilon}}_m = \mathcal{U}_P e^{i(m-1)\phi}(\hat{\boldsymbol{\rho}} - i\hat{\boldsymbol{\phi}}) + \mathcal{V}_P e^{-i(m-1)\phi}(\hat{\boldsymbol{\rho}} + i\hat{\boldsymbol{\phi}}) \tag{12}$$

which can be written as

$$\hat{\boldsymbol{\epsilon}}_m = \hat{\boldsymbol{\rho}}F - i\hat{\boldsymbol{\phi}}G \tag{13}$$

where F and G are as follows

$$F = \mathcal{U}_P e^{i(m-1)\phi} + \mathcal{V}_P e^{-i(m-1)\phi}; \quad G = \mathcal{U}_P e^{i(m-1)\phi} - \mathcal{V}_P e^{-i(m-1)\phi} \tag{14}$$

so that

$$F + G = 2\mathcal{U}_P e^{i(m-1)\phi}; \qquad F - G = 2\mathcal{V}_P e^{-i(m-1)\phi} \tag{15}$$

With $\hat{\boldsymbol{\epsilon}}_m$ given by Eq.(13), the explicit for of the paraxial equation is now as follows

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial[\hat{\boldsymbol{\epsilon}}_{\boldsymbol{m}}\Psi]}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2[\hat{\boldsymbol{\epsilon}}_{\boldsymbol{m}}\Psi]}{\partial \phi^2} + 2ik_z\frac{\partial}{\partial z}[\hat{\boldsymbol{\epsilon}}_{\boldsymbol{m}}\Psi] = 0 \tag{16}$$

and we recall that our task is to determine the conditions to be satified by Ψ.

5. GENERAL CONDITIONS ON Ψ

We consider the three terms in the vector paraxial equation in turn. Since $\hat{\boldsymbol{\epsilon}}_{\boldsymbol{m}}$ does not depend on the radial coordinate ρ, the first term of Eq.(16) follows immediately as

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial[\hat{\boldsymbol{\epsilon}}_{\boldsymbol{m}}\Psi]}{\partial \rho}\right) = (\hat{\boldsymbol{\rho}}F - i\hat{\boldsymbol{\phi}}G)\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Psi}{\partial \rho}\right) \tag{17}$$

For the second term, we obtain after some algebra

$$\frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\epsilon_m\Psi = \frac{1}{\rho^2}(\hat{\boldsymbol{\rho}}\mathcal{S} - i\hat{\boldsymbol{\phi}}\mathcal{T}) \tag{18}$$

where $\mathcal{S}$ and $\mathcal{T}$ can be written in terms of F and G as follows

$$\mathcal{S} = \left[\frac{\partial^2\Psi}{\partial \phi^2} - m^2\Psi\right]F + 2im\frac{\partial \Psi}{\partial \phi}G; \quad \mathcal{T} = \left[\frac{\partial^2\Psi}{\partial \phi^2} - m^2\Psi\right]G + 2im\frac{\partial \Psi}{\partial \phi}F \tag{19}$$

Since $\hat{\boldsymbol{\epsilon}}_{\boldsymbol{m}}$ does not depend on z, the third term of Eq.(16) can be treated as for the first term. We then have

$$(\hat{\boldsymbol{\rho}}F - i\hat{\boldsymbol{\phi}}G)\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial[\Psi]}{\partial \rho}\right) + \frac{1}{\rho^2}(\hat{\boldsymbol{\rho}}\mathcal{S} - i\hat{\boldsymbol{\phi}}\mathcal{T}) + 2ik_z\left(\hat{\boldsymbol{\rho}}F - i\hat{\boldsymbol{\phi}}G\right)\frac{\partial}{\partial z}[\Psi] = 0 \tag{20}$$

Separating the $\hat{\boldsymbol{\rho}}$ and the $-i\hat{\boldsymbol{\phi}}$ terms and putting them to zero, we have a set of two simultaneous equations with two solutions

$$\hat{\boldsymbol{\rho}}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial[\Psi]}{\partial\rho}\right)F+2ik_zF\frac{\partial}{\partial z}[\Psi]+\frac{1}{\rho^2}\left\{\left(\frac{\partial^2\Psi}{\partial\phi^2}-m^2\Psi\right)F+2im\frac{\partial\Psi}{\partial\phi}G\right\}\right]=0 \tag{21}$$

$$-i\hat{\boldsymbol{\phi}}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial[\Psi]}{\partial\rho}\right)G+2ik_zG\frac{\partial}{\partial z}[\Psi]+\frac{1}{\rho^2}\left\{\left(\frac{\partial^2\Psi}{\partial\phi^2}-m^2\Psi\right)G+2im\frac{\partial\Psi}{\partial\phi}F\right\}\right]=0 \tag{22}$$

These can be written as

$$aF+bG = 0 \tag{23}$$
$$bF+aG = 0 \tag{24}$$

where

$$a=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial[\Psi]}{\partial\rho}\right)+2ik_z\frac{\partial}{\partial z}[\Psi]+\frac{1}{\rho^2}\left(\frac{\partial^2\Psi}{\partial\phi^2}-m^2\Psi\right);\qquad b=\frac{2im}{\rho^2}\frac{\partial\Psi}{\partial\phi} \tag{25}$$

Setting the determinant of Eqs.(23) and (24) to zero, we have $(a^2-b^2)=0$. Explicitly this is $a=b$ and $a=-b$ for which the non-trivial solution is $a=b=0$. Setting $b=0$ yields

$$\frac{2im}{\rho^2}\frac{\partial\Psi}{\partial\phi}=0 \tag{26}$$

Thus Ψ has no $\phi-$ dependence and so can only be a function of ρ and z. Equation (26) also means the second derivative also vanishes $\frac{\partial^2\Psi}{\partial\phi^2}=0$, but only as long as $m\neq 0$. We are thus left with the equation that must be satisfied by the function Ψ which is $a=0$. It is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial[\Psi]}{\partial\rho}\right)+2ik_z\frac{\partial}{\partial z}[\mathcal{F}]-\frac{m^2}{\rho^2}[\Psi]=0;\quad (m\neq 0) \tag{27}$$

This scalar equation has been derived here by imposing the vector paraxial equation (8) on $\tilde{\mathbf{A}}=\hat{\boldsymbol{\epsilon}}_m\Psi_m$. The generic solution is $\Psi_m(\rho,z)$ and it is important to re-emphasise that it has no ϕ dependence for all Poincaré modes of order $m\neq 0$. The solutions are well-known and include the amplitude functions of Laguerre-Gaussian modes, Bessel modes and Bessel-Gaussian modes.

The vector potential can now be found by substituting for $\hat{\boldsymbol{\epsilon}}_m$ either in cylindrical coordinates using Eq.(12) or the original hybrid cylindrical/Cartesian form Eq.(9). We have for the all-cylindrical form

$$\tilde{\mathbf{A}}=\left[\mathcal{U}_Pe^{i(m-1)\phi}(\hat{\boldsymbol{\rho}}-i\hat{\boldsymbol{\phi}})+\mathcal{V}_Pe^{-i(m-1)\phi}(\hat{\boldsymbol{\rho}}+i\hat{\boldsymbol{\phi}})\right]\Psi_m(\rho,z)e^{ik_zz} \tag{28}$$

or the cylindrical/Cartesian form

$$\mathbf{A}=\left\{e^{im\phi}(\hat{\mathbf{x}}-i\hat{\mathbf{y}})\mathcal{U}_P+e^{-im\phi}(\hat{\mathbf{x}}+i\hat{\mathbf{y}})\mathcal{V}_P\right\}\Psi_m(\rho,z)e^{ik_zz} \tag{29}$$

Once the type of mode is decided by specifying Ψ_m the vector potential forms the basis for the evaluation of the characteristics of the Poincaré modes, notably, the energy-momentum , the optical spin as well as the orbital angular momentum. Note that Ψ_m is very general and is simply one of the solutions of the above paraxial equation, Eq.(27) without the phase $e^{im\phi}$ factor. Equation (12) or Eq.(29) supplies the variations of the vector potential at every point on the surface of the unit Poincaré sphere of any order m.

A special case of the general result strictly arises when we set $\mathcal{U}_P=\mathcal{V}_P$ and the solution of the differential equation (27) is denoted as Ψ_m, identified as the amplitude function of of the Laguerre-Gaussian mode of winding number m and radial number p. The form of Ψ_m for a Laguerre-Gaussian mode is written as LG_p^m which has the well-known form to be displayed explicitly in section 7. When multiplied by the phase functions $e^{\pm im\phi}$ we obtain the complete solutions for combination of LG modes with winding numbers m and $-m$ and the same radial number p multiplied by σ^- and σ^+, respectively. We have

$$\mathbf{A}=\left\{LG_p^{+m}\sigma^-+LG_p^{-m}\sigma^+\right\} \tag{30}$$

It should be emphasised that this form is only applicable under the conditions $\mathcal{U}_P=\mathcal{V}_P$, which means $\Theta_P=\pi/2$ when $\Phi_P=0$. Thus for any order m, the form in Eq.(30) (which is often put forward as a prototype vector mode [1, 8, 15, 23]) corresponds to a specific point on the order m Poincaré sphere. For $m\geq 1$, the modes are higher order radially-polarised modes. As we show below all such modes have zero angular momentum.

The polarisation in Eq.(29) is the general form for the case of Laguerre-Gaussian higher order modes and covers every point in the entire Poincaré sphere when the Poincaré sphere angles Θ_P,Φ_P are specified. We can then write

$$\begin{aligned}\mathbf{A} &= \hat{\boldsymbol{\epsilon}}_m\Psi_m(\rho,z)\\ &= LG_p^m\frac{(\hat{\mathbf{x}}-i\hat{\mathbf{y}})}{\sqrt{2}}\mathcal{U}_P+LG_p^{-m}\frac{(\hat{\mathbf{x}}+i\hat{\mathbf{y}})}{\sqrt{2}}\mathcal{V}_P\end{aligned} \tag{31}$$

Every vector mode of order m as given by Eq.(31) in the case of Laguerre-Gaussian modes corresponds to a point of coordinates (Θ_P,Φ_P) on the surface of the unit Poincare sphere of order m.

6. ELECTROMAGNETIC FIELDS

In order to evaluate the corresponding electric and magnetic fields, we first write the vector potential Eq.(29) as the sum of two parts as follows

$$\mathbf{A} = \mathbf{A_1} + \mathbf{A_2} \tag{32}$$

$$\mathbf{A_1} = (\hat{\mathbf{x}} - i\hat{\mathbf{y}})\mathcal{G}^{(1)}(\mathbf{r}); \quad \mathbf{A_2} = (\hat{\mathbf{x}} + i\hat{\mathbf{y}})\mathcal{G}^{(2)}(\mathbf{r}) \tag{33}$$

where

$$\mathcal{G}^{(1)}(\mathbf{r}) = \mathcal{U}_P e^{im\phi}\Psi_m(\rho, z)e^{ik_z z}; \quad \mathcal{G}^{(2)}(\mathbf{r}) = \mathcal{V}_P e^{-im\phi}\Psi_m(\rho, z)e^{ik_z z} \tag{34}$$

The electric and magnetic fields of our generally-polarised mode are similarly written as the sums $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$ and $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ where $\mathbf{B}_i = \boldsymbol{\nabla} \times \mathbf{A}_i; \quad i = 1, 2$. The sequence of steps involves dealing first with the two parts of the magnetic field and from those use Maxwell's curl B equation to derive the corresponding electric field parts. We have for $\mathbf{B}_1$ and $\mathbf{E}_1$

$$\mathbf{B}_1 = \left\{ik_z(\hat{\mathbf{y}} + i\hat{\mathbf{x}}) - \hat{\mathbf{z}}\left(i\partial_x + \partial_y\right)\right\}\mathcal{G}^{(1)}(\mathbf{r}); \quad \mathbf{E}_1 = c\left\{ik_z(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) - \hat{\mathbf{z}}\left(\partial_x - i\partial_y\right)\right\}\mathcal{G}^{(1)}(\mathbf{r}) \tag{35}$$

The fields $\mathbf{B}_2$ and $\mathbf{E}_2$ emerge following exactly the same procedure as followed for $\mathbf{B}_1$ and $\mathbf{E}_1$. We obtain

$$\mathbf{B}_2 = \left\{ik_z(\hat{\mathbf{y}} - i\hat{\mathbf{x}}) + \hat{\mathbf{z}}\left(i\partial_x - \partial_y\right)\right\}\mathcal{G}^{(2)}(\mathbf{r}); \quad \mathbf{E}_2 = \left\{ick_z(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) - \hat{\mathbf{z}}c(\partial_x + i\partial_y)\right\}\mathcal{G}^{(2)}(\mathbf{r}) \tag{36}$$

The forms of electromagnetic fields displayed above which include the longitudinal components apply to any optical vortex. Their derivation follows standard procedure based on the vector potential with details given in recent reports [17, 19].

7. ANGULAR MOMENTUM

The electric and magnetic fields associated with the general Poincaré mode are given by Eqs.(35) and (36) and it is from these fields that all optical properties follow. In particular, we now proceed to work out the cycle-averaged angular momentum. The angular momentum density is the moment of the energy density and is formally defined as follows:

$$\bar{\boldsymbol{j}} = \mathbf{r} \times \frac{1}{2\mu_0 c^2}\Re[\mathbf{E}^* \times \mathbf{B}]; \quad \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2; \quad \mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 \tag{37}$$

We shall only consider the evaluation of longitudinal (z-component) of the angular momentum. The cycle-averaged angular momentum density $\bar{j}_z$ is defined as

$$\bar{j}_z = \frac{1}{2\mu_0 c^2}\Re\left\{x(\mathbf{E}^* \times \mathbf{B})_y - y(\mathbf{E}^* \times \mathbf{B})_x)\right\} \tag{38}$$

We obtain on substituting from Eqs.(35) and (36)

$$\bar{j}_z = \frac{k_z}{2\mu_0 c}\left\{m\Psi_m^2 + \rho\Psi_m\Psi_m'\right\}\cos\Theta_P \tag{39}$$

The integral of this density over a cross-section in the $x - y$ plane yields the total angular momentum per unit length, $\bar{J}_z$

$$\bar{J}_z = \frac{k_z}{2\mu_0 c}\int_0^\infty \rho d\rho \int_0^{2\pi} d\phi \left\{m\Psi_m^2 + \rho\Psi_m\Psi_m'\right\}\cos\Theta_P \tag{40}$$

where the prime on Ψ means differentiation with respect to the radial coordinate ρ. It can be checked that the right hand side of Eq.(40) has the dimensions of angular momentum per unit length. Recall that Ψ_m is a solution of the differential equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi_m}{\partial\rho}\right) + 2ik_z\frac{\partial}{\partial z}\Psi_m - \frac{m^2}{\rho^2}\Psi_m = 0 \tag{41}$$

Strictly speaking, this is not the scalar paraxial equation as such, but it is the equation that follows from the paraxial equation had there been a term which includes the angular derivatives $\partial^2/\partial\phi^2$. Formally the solution of Eq.(41) is the amplitude function for a vortex mode of order m. In particular, one of the solutions is the amplitude function of winding number m for a Laguerre-Gaussian mode, so we can write

$$\Psi_m(\rho, z) = \mathcal{E}_0\frac{C_{|m|,p}}{\sqrt{1 + z^2/z_R^2}}\left(\frac{\rho\sqrt{2}}{w_0\sqrt{1 + z^2/z_R^2}}\right)^{|m|}\exp\left[\frac{-\rho^2}{w_0^2(1 + z^2/z_R^2)}\right]L_p^{|m|}\left\{\frac{2\rho^2}{w_0^2(1 + z^2/z_R^2)}\right\}e^{i\chi(\rho,z)}, \tag{42}$$

and $\chi(\rho, z)$ includes the Gouy and the curvature phases

$$\chi(\rho, z) = -(2p + |m| + 1)\arctan\left(\frac{z}{z_R}\right) + \frac{kz\rho^2}{2(z^2 + z_R^2)}, \tag{43}$$

Here w_0 is the beam waist, $z_R = w_0^2 k/2$ is the Rayleigh range, $C_{|m|,p} = \sqrt{p!/(p + |m|)!}$ and $L_p^{|m|}$ the associated Laguerre polynomial.

We focus on the case in which the Rayleigh range z_R is large so that the amplitude function reduces to

$$\Psi_m(\rho) = \mathcal{E}_0 \sqrt{\frac{p!}{(p+|m|)!}} e^{-\frac{\rho^2}{w_0^2}} \left(\frac{\sqrt{2}\rho}{w_0}\right)^{|m|} L_p^{|m|}\left(\frac{2\rho^2}{w_0^2}\right) \tag{44}$$

where $\mathcal{E}_0$ is a normalisation constant. It is straightforward to evaluate the integrals in Eq.(40). The result is

$$\bar{J}_z = (m-1)\mathcal{J}_0 \cos\Theta_P; \quad (m \geq 1) \tag{45}$$

where $\mathcal{J}_0$ is a constant given by

$$\mathcal{J}_0 = \frac{k_z w_0^2 \pi \mathcal{E}_0^2}{4\mu_0 c} \tag{46}$$

It is also straightforward to check that $\mathcal{J}_0$ has the dimensions of angular momentum per unit length. Equation (45) shows the variations of the angular momentum carried by a general Poincaré mode of order m at a point (Θ_P, Φ_P) on the unit sphere. The result, as we explained earlier shows no dependence on Φ_P which indicates that all points on the line of latitude at a fixed Θ_P, but varying Φ_P have the same value of angular momentum. The function $\cos\Theta_P$ takes continuous values spanning the range $+1 \to -1$ as Θ_P changes from 0 to π. The angular momentum also varies with the order m, but the equation (45) does not apply to the lowest order $m = 0$ since it is clear that for $m = 0$ the function $\Psi_{m=0}$ becomes a Gaussian function and so the $m = 0$ mode does not carry angular momentum. Furthermore, as Eq.(45) shows, all first order $m = 1$ modes have zero angular momentum. In particular the point for which $\Theta_P = \pi/2$ corresponds to pure radially-polarised modes which are known not to carry angular momentum. By contrast all higher order modes for which $m \geq 2$ have angular momentum, which, as Eq.(45) shows, increases with increasing order m. These features of the result are summarised in Table 1.

8. CONCLUSIONS

In conclusion, the aim of this article was primarily to clarify the general form of the electromagnetic fields of vector Poincaré modes. Such modes are normally defined in terms of the polarisation vector $\hat{\boldsymbol{\epsilon}}$ with no indication of the amplitude function Ψ which contains the spatial variation of the mode. We have shown that the form of the amplitude function is only determined by demanding that the vector potential, from which the electric and magnetic fields are derived, must satisfy the vector paraxial equation. This condition has led us to two equations to be satisfied by Ψ. The first stipulates that Ψ has no phase dependence on the azimuthal angle ϕ and the second that Ψ must satisfy a second order differential equation involving the variables ρ and z only. The solutions of this differential equation turn out to be the amplitude functions Ψ_m of a scalar equation that has no dependence on ϕ. We have indicated that the Laguerre-Gaussian amplitude function satisfies this equation and we proceeded to evaluate the angular momentum of a general Poincaré mode of order m. We have found that only the modes of order $m \geq 2$ have non-zero angular momentum. The lower orders $m = 0, 1$ both have zero angular momentum for all points on the order $m = 0, 1$ Poincaré spheres. The prototype of the case $m = 0$ is azimuthally-polarised Gaussian modes and $m = 1$ is exemplified by the radially-polarised modes, but the results indicate that all first order modes, including quasi-circularly-polarised and in general quasi-elliptical modes have zero angular momentum.The condition $\mathcal{U}_P = \mathcal{V}_P$ means $\Theta_P = \pi/2$ when $\Phi_P = 0$. We have also shown that for any order m, the form in Eq.(30) (regarded as the prototype vector modes [1, 8, 15, 23]) are higher order radially-polarised modes and we have shown that all such modes have zero angular momentum.

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DISCLOSURES

The authors declare no conflicts of interest.

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Order m	Θ_P	Polarisation	$\bar{J}_z/\mathcal{J}_0$
$m = 1$	$(0 \to \pi)$	Elliptical	0
$m = 1$	$0\ or\ \pi$	circular	0
$m = 1$	$\pi/2$	Radial	0
$m \geq 1$	$\pi/2$	Radial	0
$m = 2$	$(0 \to \pi)$	Elliptical	$\cos\Theta_P$
$m > 2$	$(0 \to \pi)$	Elliptical	$(m-1)\cos\Theta_P$

Table 1. Variations of the magnitude of the scaled angular momentum per unit length $\bar{J}_z/\mathcal{J}_0$ with both the Poincaré angle Θ_P and the order m for modes localised on the longitude $\Phi_P = 0$ of the Poincare e sphere. It is clear that only for $m \geq 2$ and $\Theta_P \neq \pi/2$ that the modes possess angular momentum. The magnitude of the angular momentum increases as m increases for all points on the order m Poincaré sphere for which $\cos\Theta_P > 0$.

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