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# Reconstruction of the time-dependent source in thermal grooving by surface diffusion

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ABSTRACT

In hot polycrystalline materials, when a vertical flat grain boundary meets a horizontal surface the grain boundary forms a groove in the surface. Mathematically modelling features of such thermal grooving mechanism is therefore very important in characterizing polycrystalline materials composed of tiny grains intersecting an external free surface. With this aim in mind, we formulate and investigate a novel inverse problem of reconstructing the unknown time-dependent source term entering the fourth-order parabolic equation of thermal grooving by surface diffusion from a given integral observation. We formulate and prove in Theorems 2.3–2.7 that this linear inverse problem is well-posed. However, in practice, the ideal regularity of data under which the inverse source problem is stable is never satisfied due to the inherent non-smoothness of the measurement. Consequently, this leads to the inverse problem with raw data becoming ill-posed. In order to obtain accurate and stable solutions, we develop and compare two numerical methods, namely, a time-discrete method and an optimization method. We obtain error estimates and convergence rates for the time-discrete method. For the optimization method, an objective functional, which is proved to be Fréchet differentiable, is introduced and the conjugate gradient method (CGM), regularized by the discrepancy principle, is developed to compute the minimizer yielding the source term. The results of two numerical tests illustrate the performance of the two methods for both exact and noisy measured data.

## 1. Introduction

Fourth-order parabolic partial differential equations are utilized to explain the quantitative theory of thermal grooving through surface diffusion mechanism [1], the free vibration in beams and shafts [2], the epitaxial thin film growth [3], the long range effect of insects dispersal [4], etc. They are also applied in image processing to balance the trade-off between noise removal and edge preservation [5].

Higher $2m$-order parabolic inverse source problems given by

$$\begin{cases} u_t + (-1)^m a(x,t)\partial_x^{2m} u = g(x,t)f(x,t) + h(x,t), & (x,t) \in (0,1)\times(0,T) =: Q_T, \\ \partial_x^j u|_{x\in\{0,1\}} = 0, \quad j = \overline{0, m-1}, & t \in (0,T), \\ u(x,0) = \phi(x), & x \in (0,1), \end{cases} \tag{1.1}$$

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where $m \in \mathbb{N}^*$, were considered to determine the unknown time-dependent source term $f(x,t) = f(t)$ in [6] or space-dependent source term $f(x,t) = f(x)$ in [7], from a given integral observation. Under certain assumptions, the authors proved the well-posedness of the weak solutions to both these inverse problems even in the degenerate case when the coefficient $a(x,t)$ may vanish on a zero-measure set. The determination of the unknown source term in fourth-order hyperbolic equations has also been investigated. For instance, the time- or space-dependent load in a vibrating Euler–Bernoulli beam was reconstructed from boundary measurements in [8,9] and from final time overdetermination in [10,11], respectively. Also, the shear force in an Euler–Bernoulli cantilever beam was obtained from measured boundary deflection or bending moment in [12,13], respectively. We finally mention that the identification of the both space- and time-dependent force $f(x,t)$ in the Euler–Bernoulli equation from its space boundary values investigated in [14] is too much to hope for since the inverse problem in this general case is seriously underdetermined.

As a practical application related to characterizing the strength and stability of polycrystalline materials, we consider the study of a groove that forms when a vertical grain boundary meets a horizontal surface, which occurs, e.g., in the thermal treatment and metallization of electronic components of power modules, [15]. Recently, the time-dependent Mullins' coefficient $a(t)$ in such a problem modelled by:

$$\begin{cases} u_t + a(t)u_{xxxx}(x,t) = h(x,t), & (x,t) \in Q_T, \\ u_x(0,t) = u_x(1,t) = u_{xxx}(0,t) = u_{xxx}(1,t) = 0, & t \in (0,T), \\ u(x,0) = \phi(x), & x \in (0,1), \end{cases} \tag{1.2}$$

was reconstructed along with $u(x,t)$ from the measurement of the profile $u(x_0,t) = E(t)$ for $t \in [0,T]$ at a selected fixed point $x_0 \in [0,1]$, see [16], or from the total mass integral condition $\int_0^1 u(x,t)dx = \mathrm{E}(t)$ for $t \in [0,T]$, see [17]. Also, the identification of the pair $(p(t), u(x,t))$ satisfying the inverse problem

$$\begin{cases} u_t + a(t)u_{xxxx}(x,t) = p(t)u + h(x,t), & (x,t) \in Q_T, \\ u_x(0,t) = u(1,t) = u_{xxx}(0,t) = u_{xx}(1,t) = 0, & t \in (0,T), \\ u(x,0) + \int_0^T u(x,t)dt = \phi(x), & x \in (0,1), \\ u(0,t) = \nu(t), & t \in [0,T], \end{cases} \tag{1.3}$$

was investigated theoretically and numerically in [18,19], respectively.

In this paper, the thermal grooving coefficient $a = D_s\gamma_s\omega_V/(k_B\mathrm{T})$, where $D_s$ is the surface diffusivity, $\gamma_s$ is the surface energy, $\omega_V$ is the atomic volume, T is the absolute temperature and $k_B$ is the Boltzmann constant, is assumed to be known and constant, and, for simplicity, taken to be equal to unity. Instead, the right-hand side of the governing equation is assumed to contain a heat source whose time-dependent intensity is unknown and has to be determined. We consider therefore the mathematical model for thermal grooving given by

$$\begin{cases} u_t + u_{xxxx} = f(t)g(x,t) + h(x,t) =: \mathrm{F}(x,t) & (x,t) \in Q_T, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, & t \in (0,T), \\ u(x,0) = \phi(x), & x \in (0,1), \end{cases} \tag{1.4}$$

where $f$, $g$ and $h$ are the source term components, and $\phi$ is the initial profile from which the groove grows. The boundary conditions in (1.4) are those associated to simply-supported beams, but clamped-beam boundary conditions may also be considered [20].

As mentioned before, the fourth-order partial differential equation in (1.4) models the grain boundary small grooving by surface diffusion which occurs at moderate temperature when a vertical flat grain boundary meets a horizontal flat surface [1,21,22]. Surface diffusion is also the principal mechanism of mass transport at certain metal (e.g. gold) surfaces [23]. The simpler mased transport based on evaporation-condensation at the surface of other metals (e.g. magnesium) is not considered herein. More details on the physical background of thermal grooving can be found in [24].

The direct problem (1.4) consists of obtaining the thermal grooving profile $u(x,t)$ when the source terms and initial status are specified. In contrast, the inverse source problem consists of obtaining both $u(x,t)$ and the time-dependent source term $f(t)$ in (1.4) from the integral observation of the mass/energy of the system, given by

$$\int_0^1 \omega(x)u(x,t)dx = \psi(t), \quad t \in (0,T), \tag{1.5}$$

where $\omega(x)$ is a given weight function. Such non-local mass/energy specification was previously considered in many studies in heat transfer for the parabolic heat equation [25–29]. Of course, one cannot identify the most general source $F(x,t)$ in (1.4) from the time-dependent measurement (1.5) since we can always add the function $\tilde{u}(x,t) = tx^3(1-x)^3\cos(\pi x)/\omega(x)$, (assuming $C^4[0,1] \ni \omega(x) \neq 0$ for all $x \in [0,1]$) to $u(x,t)$ and obtain a new solution $(u(x,t) + \tilde{u}(x,t), F(x,t) + \tilde{u}_t(x,t) + \tilde{u}_{xxxx}(x,t))$ to (1.4) and (1.5).

In this paper, the significant contribution to the literature is that we fully solve in terms of both theory and numerics the newly formulated inverse source problem given by Eqs. (1.4) and (1.5). This problem needs to be solved in order to identify the unknown internal forces acting on a heated polycrystal and understand better the thermal grooving formation. Based on the arguments in [6,30], we investigate the existence and uniqueness of the solution using the contraction mapping theorem. Afterwards, the continuous dependence of $f(t)$ on the measurement and input data (1.5) is proved and a stability estimate is established. Some alternative arguments are also possible based on the semi-group theory [30] for the fourth-order linear partial differential operator $\partial_t + \partial_{xxxx}$ defined in (1.4).

We next consider the numerical determination of the unknown source term $f(t)$ from the integral observation (1.5). The time-discrete method [31] is introduced to obtain the unknown source term. We first obtain error estimates for this method for exact measured data. Then, using the cubic spline function method [32], error estimates for noisy data are also obtained. As a second method, the conjugate gradient method (CGM) is developed for minimizing the Tikhonov regularization functional in order to obtain a stable solution to the linear but ill-posed inverse source problem. We finally mention that although, due to physical considerations, the models given in Eqs. (1.2)–(1.4) are one-dimensional in space, mathematically they also make sense in higher dimensions and some of the techniques developed in this paper are extendable to these situations; furthermore the unknown coefficients $a(t)$, $p(t)$ or $f(t)$ in (1.2)–(1.4) are time-dependent only and therefore their identifications are, in principle, not affected by the space multi-dimensionality.

The paper is organized as follows: the well-posedness of the inverse problem (1.4) and (1.5) is investigated in Section 2. The time-discrete method is presented in Section 3 with its error estimate. The minimizer of the objective functional is utilized to approximate its solution in Section 4, and the CGM is established based on the Fréchet derivative. Then two numerical examples are considered in Section 5. Finally, Section 6 highlights the conclusions of this paper.

## 2. Mathematical analysis

Denote $V := H^2(0,1) \cap H_0^1(0,1)$ and $W := \{w \in H^4(0,1) \mid w(0) = w(1) = w''(0) = w''(1) = 0\}$.

### 2.1. Direct problem

Using [33, Proposition 2.1] we obtain the following theorem giving the well-posedness of the direct problem (1.4).

**Theorem 2.1.** *Let $f \in L^2(0,T)$, $g \in L^\infty(0,T;L^2(0,1))$ and $h \in L^2(Q_T)$.*

*(i) If $\phi \in L^2(0,1)$, then the direct problem (1.4) has a unique solution $u \in C([0,T];L^2(0,1)) \cap L^2(0,T;V) =: \mathcal{V}$. Moreover, there exists $C = C(T) > 0$ such that*

$$\|u\|_{L^\infty(0,T;L^2(0,1))\cap L^2(0,T;V)} \le C\left(\|fg+h\|_{L^2(Q_T)} + \|\phi\|_{L^2(0,1)}\right). \tag{2.1}$$

*(ii) If $\phi \in V$, then the direct problem (1.4) has a unique solution $u \in C([0,T];V) \cap L^2(0,T;W)$. Moreover, there exists $C = C(T) > 0$ such that*

$$\|u\|_{L^\infty(0,T;V)\cap L^2(0,T;W)} \le C\left(\|fg+h\|_{L^2(Q_T)} + \|\phi\|_V\right). \tag{2.2}$$

**Remark 2.1.** Theorem 2.1 for the problem (1.4) with homogeneous boundary conditions can be extended to the problem with inhomogeneous boundary conditions given by

$$\begin{cases} u_t + u_{xxxx} = f(t)g(x,t) + h(x,t) =: \mathrm{F}(x,t), & (x,t) \in Q_T, \\ u(0,t) = \mu_1(t), \quad u(1,t) = \mu_2(t), & t \in (0,T), \\ u_{xx}(0,t) = \mu_3(t), \quad u_{xx}(1,t) = \mu_4(t), & t \in (0,T), \\ u(x,0) = \phi(x), & x \in (0,1), \end{cases} \tag{2.3}$$

where $(\mu_i)_{i=\overline{1,4}}$ are the boundary data.

**Theorem 2.2.** *Letting the assumptions of Theorem 2.1 be satisfied along with $\phi \in L^2(0,1)$, and supposing that $\mu_i \in H^1(0,T)$ for $i = \overline{1,4}$, then there exists a unique solution $u \in C([0,T];L^2(0,1)) \cap L^2(0,T;H^2(0,1)) =: \tilde{\mathcal{V}}$ to the direct problem (2.3), which moreover satisfies the following estimate:*

$$\|u\|_{L^\infty(0,T;L^2(0,1))\cap L^2(0,T;H^2(0,1))} \le C\left(\|fg+h\|_{L^2(Q_T)} + \|\phi\|_{L^2(0,1)} + \|(\mu_1,\mu_2,\mu_3,\mu_4)\|_{[H^1(0,T)]^4}\right). \tag{2.4}$$

**Proof.** We extend the proof of [33, Proposition 2.4]. Consider the auxiliary (or lifting, or satisfier) function

$$\rho(x,t) := (1-x)\mu_1(t) + x\mu_2(t) + \left(-\frac{x^3}{6} + \frac{x^2}{2} - \frac{x}{3}\right)\mu_3(t) + \left(\frac{x^3}{6} - \frac{x}{6}\right)\mu_4(t), \tag{2.5}$$

which satisfies the boundary conditions of the problem (2.3). Defining

$$v := u - \rho, \quad L^2(0,1) \ni \varphi := \phi - \rho(x,0), \quad L^2(Q_T) \ni \mathrm{F}_1 := \mathrm{F} - \psi_t, \tag{2.6}$$

problem (2.3) recasts as

$$\begin{cases} v_t + v_{xxxx} = \mathrm{F}_1(x,t), & (x,t) \in Q_T, \\ u(0,t) = 0, \quad u(1,t) = 0, & t \in (0,T), \\ u_{xx}(0,t) = 0, \quad u_{xx}(1,t) = 0, & t \in (0,T), \\ u(x,0) = \varphi(x), & x \in (0,1). \end{cases} \tag{2.7}$$

From part (i) of Theorem 2.1 it follows that the direct problem (2.7) has a unique solution $v \in C([0,T]; L^2(0,1)) \cap L^2(0,T;V)$, which satisfies

$$\|v\|_{L^\infty(0,T;L^2(0,1))\cap L^2(0,T;V)} \leq C \left( \|\mathrm{F}_1\|_{L^2(Q_T)} + \|\varphi\|_{L^2(0,1)} \right). \tag{2.8}$$

Since $\psi$ satisfies the inhomogeneous boundary conditions in (2.3), it follows that $v + \rho = u \in C([0,T]; L^2(0,1)) \cap L^2(0,T;H^2(0,1))$ is a solution to problem (2.3). Now, because of the continuous injection $H^1(0,T) \hookrightarrow L^\infty(0,T)$, it can be obtained that

$$\|\mathrm{F}_1\|_{L^2(Q_T)} \leq C \left( \|\mathrm{F}\|_{L^2(Q_T)} + \|(\mu_1,\mu_2,\mu_3,\mu_4)\|_{[H^1(0,T)]^4} \right),$$
$$\|\varphi\|_{L^2(0,1)} \leq C \left( \|\phi\|_{L^2(0,1)} + \|(\mu_1,\mu_2,\mu_3,\mu_4)\|_{[H^1(0,T)]^4} \right),$$

which combined with (2.8) yield (2.4). Finally, inequality (2.1) and the linearity of the problem yields the uniqueness of solution. □

### 2.2. Inverse problem

The existence and uniqueness of the solution to the inverse problem given by Eqs. (1.4) and (1.5) can be obtained under the following conditions:

(a) $\omega \in W$;
(b) $\psi \in H^1(0,T)$ satisfies the compatibility condition $\int_0^1 \omega(x)\phi(x)dx = \psi(0)$;
(c) there exists a positive constant $g_0$ satisfying $\left|\int_0^1 \omega(x)g(x,t)dx\right| \geq g_0 > 0$ for all $t \in (0,T)$.

**Theorem 2.3.** *Suppose conditions (a)–(c) and the assumptions of Theorem 2.1 hold, and that $\phi \in L^2(0,1)$. Then, there exists a unique solution $(u,f) \in \mathcal{V} \times L^2(0,T)$ to the inverse problem given by Eqs. (1.4) and (1.5). Moreover, $f$ depends continuously upon the observation $\psi(t)$ and the input data $h(x,t)$ and $\phi(x)$, and satisfies the estimate*

$$\|f\|_{L^2(0,T)} \leq \frac{K}{g_0} \sum_{n=0}^{\infty} \frac{\kappa^n T^{n/2}}{(n!)^{1/2}}, \tag{2.9}$$

*where $K$ and $\kappa$ are two positive constants explicitly given by*

$$K = (T\|\omega''''\|_{L^2(0,1)} + \|\omega\|_{L^2(0,1)})\|h\|_{L^2(Q_T)} + \sqrt{T}\|\omega''''\|_{L^2(0,1)}\|\phi\|_{L^2(0,1)} + \|\psi\|_{H^1(0,T)},$$
$$\kappa = \frac{\sqrt{T}}{g_0}\|g\|_{L^\infty(Q_T)}\|\omega''''\|_{L^2(0,1)}.$$

**Proof.** Multiplying the first equation in (1.4) by $\omega(x)$, integrating the result over $[0,1]$ and using the homogeneous boundary conditions and conditions (a) and (b), we obtain

$$f(t) = \frac{1}{G(t)} \left( \int_0^1 \omega''''(x)u(x,t)dx + \psi'(t) - H(t) \right) \tag{2.10}$$

where

$$G(t) = \int_0^1 \omega(x)g(x,t)dt, \quad H(t) = \int_0^1 \omega(x)h(x,t)dx. \tag{2.11}$$

The approach of transferring derivatives to the weight function $\omega$ in (1.5) originates back to [30] and it has been used elsewhere in [34,35].

Based on (2.11), let us introduce the operator $\mathcal{A} : L^2(0,T) \to L^2(0,T)$ by the formula

$$\mathcal{A}(f) = \frac{1}{G(t)} \left( \int_0^1 \omega'''' u(f)dx + \psi'(t) - H(t) \right), \tag{2.12}$$

where $u = u(f) \in \mathcal{V}$ is the unique solution of the direct problem (1.4) for a given $f \in L^2(0,T)$, which implies that (2.10) can be written as the following fixed point operator equation:

$$f = \mathcal{A}(f). \tag{2.13}$$

We now prove that there exists a positive integer $n_0$ for which the operator $\mathcal{A}^{n_0}$ is a contraction operator on $L^2(0,T)$. For this, let $f^{(1)}, f^{(2)} \in L^2(0,T)$, and let $u^{(1)}(x,t)$, $u^{(2)}(x,t)$ be the solutions to the direct problem (1.4) corresponding to $f^{(1)}$, $f^{(2)}$, respectively. Then, the differences $w(x,t) := u^{(1)}(x,t) - u^{(2)}(x,t)$ and $F(t) := f^{(1)}(t) - f^{(2)}(t)$ satisfy the following problem:

$$\begin{cases} w_t + w_{xxxx} = F(t)g(x,t), & (x,t) \in Q_T, \\ w(0,t) = w(1,t) = w_{xx}(0,t) = w_{xx}(1,t) = 0, & t \in (0,T), \\ w(x,0) = 0, & x \in (0,1). \end{cases} \tag{2.14}$$

By part (i) of Theorem 2.1, there exists a unique solution $w(x,t) \in \mathcal{V}$ to the problem (2.14). Multiplying the first equation in (2.14) by $w(x,t)$, and integrating the result over $(0,1)$, we have

$$\frac{1}{2}\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2(0,1)} + \|w_{xx}(\cdot,t)\|^2_{L^2(0,1)} = \int_0^1 F(t)g(x,t)w(x,t)dx \le |F(t)|\|g(\cdot,t)\|_{L^2(0,1)}\|w(\cdot,t)\|_{L^2(0,1)}, \quad t \in [0,T], \tag{2.15}$$

which implies that

$$\frac{d}{dt}\|w(\cdot,t)\|_{L^2(0,1)} \le |F(t)|\|g(\cdot,t)\|_{L^2(0,1)}, \quad t \in [0,T].$$

This inequality yields that

$$\|w(\cdot,t)\|_{L^2(0,1)} \le \|g\|_{L^\infty(0,T;L^2(0,1))}\int_0^t |F(\tau)|d\tau, \quad t \in [0,T]. \tag{2.16}$$

Consequently, for any $t \in [0,T]$, using (2.16), we obtain

$$\begin{aligned} \|\mathcal{A}(f^{(1)}) - \mathcal{A}(f^{(2)})\|_{L^2(0,t)} &= \left(\int_0^t \left|\frac{1}{G(\tau)}\int_0^1 \omega''''(x)w(x,\tau)dx\right|^2 d\tau\right)^{1/2} \\ \le \frac{1}{g_0}\|\omega''''\|_{L^2(0,1)}\left(\int_0^t \|w(\cdot,\tau)\|^2_{L^2(0,1)}d\tau\right)^{1/2} &\le \kappa\left(\int_0^t \|F\|^2_{L^2(0,\tau)}d\tau\right)^{1/2} \le \kappa\sqrt{t}\|f^{(1)} - f^{(2)}\|_{L^2(0,t)}. \end{aligned} \tag{2.17}$$

Then, for any $n \in \mathbb{N}^*$, using mathematical induction and (2.17), we can obtain that

$$\|\mathcal{A}^n(f^{(1)}) - \mathcal{A}^n(f^{(2)})\|_{L^2(0,T)} \le \left(\frac{\kappa^{2n}T^n}{n!}\right)^{1/2}\|f^{(1)} - f^{(2)}\|_{L^2(0,T)}, \tag{2.18}$$

where the origin of $n!$ in the denominator of (2.17) comes from integrating $t^{n-1}/(n-1)!$ from 0 to $t$, in the induction process. Since there exists a positive integer $n_0$ such that

$$\frac{\kappa^{2n_0}T^{n_0}}{n_0!} < 1, \tag{2.19}$$

this yields that the operator $\mathcal{A}^{n_0}$ is a contraction operator on $L^2(0,T)$. Then the operator $\mathcal{A}^{n_0}$ has a unique fixed point $f \in L^2(0,T)$. From this and using that $\mathcal{A}^{n_0}(\mathcal{A}(f)) = \mathcal{A}^{n_0+1}(f) = \mathcal{A}(\mathcal{A}^{n_0}(f)) = \mathcal{A}(f)$, it follows that $\mathcal{A}(f) = f$, hence has the fixed point $f$. Uniqueness of this fixed point also follows immediately.

For any initial guess $f^0 \in L^2(0,T)$, we can use the method of successive approximations given by $f^{n+1} = \mathcal{A}(f^n)$ for $n \in \mathbb{N}$. We rewrite $n$ as $n = mn_0 + n_1$ and $0 \le n_1 \le n_0 - 1$ is an integer, then $m \to \infty$ implies that $n \to \infty$. Using $\mathcal{A}^{n_0}(f) = f$, (2.18) and (2.19) we get

$$\begin{aligned} \|f^n - f\|_{L^2(0,T)} &= \|f^{mn_0+n_1} - f\|_{L^2(0,T)} = \|\mathcal{A}(f^{mn_0+n_1-1}) - \mathcal{A}^{n_0}(f)\|_{L^2(0,T)} \\ = \cdots = \|\mathcal{A}^{n_0}(f^{(m-1)n_0+n_1}) - \mathcal{A}^{n_0}(f)\|_{L^2(0,T)} &\le \left(\frac{\kappa^{2n_0}T^{n_0}}{n_0!}\right)^{1/2}\|f^{(m-1)n_0+n_1} - f\|_{L^2(0,T)} \\ &\le \cdots \le \left(\frac{\kappa^{2n_0}T^{n_0}}{n_0!}\right)^{m/2}\|f^{n_1} - f\|_{L^2(0,T)} \to 0, \quad m \to \infty, \end{aligned}$$

which means the sequence $\{f^n\}_{n\in\mathbb{N}}$ converges to $f$ in the $L^2$-norm, and $f \in L^2(0,T)$ is the solution to the inverse problem given by Eqs. (1.4) and (1.5). Choosing $f^0 = 0$ and using that $f^n = \mathcal{A}^n(f^0)$, we obtain

$$\begin{aligned} \|f\|_{L^2(0,T)} &= \lim_{n\to\infty}\|f^n - f^0\|_{L^2(0,T)} = \lim_{n\to\infty}\|\mathcal{A}(f^{n-1}) - f^0\|_{L^2(0,T)} \\ &= \lim_{n\to\infty}\|\mathcal{A}(f^{n-1}) - \mathcal{A}(f^{n-2}) + \mathcal{A}(f^{n-2}) - \mathcal{A}(f^{n-3}) + \cdots + \mathcal{A}(f^0) - f^0\|_{L^2(0,T)} \\ &\le \sum_{n=1}^{\infty}\|\mathcal{A}^{n-1}(\mathcal{A}(f^0)) - \mathcal{A}^{n-1}(f^0)\|_{L^2(0,T)} \le \sum_{n=0}^{\infty}\left(\frac{\kappa^{2n}T^n}{n!}\right)^{1/2}\|\mathcal{A}(f^0) - f^0\|_{L^2(0,T)} \\ &= \sum_{n=0}^{\infty}\left(\frac{\kappa^{2n}T^n}{n!}\right)^{1/2}\|\mathcal{A}(0)\|_{L^2(0,T)}. \end{aligned} \tag{2.20}$$

Multiplying the first equation of (1.4) by $u(0)$ (denoting the solution of (1.4) with $f = f^0 = 0$), and integrating over $[0,1]$, we have

$$\frac{1}{2}\frac{d}{dt}\|u(0)(\cdot,t)\|^2_{L^2(0,1)} + \|u_{xx}(0)(\cdot,t)\|^2_{L^2(0,1)} \le \|h(\cdot,t)\|_{L^2(0,1)}\|u(0)(\cdot,t)\|_{L^2(0,1)}, \quad t \in [0,T], \tag{2.21}$$

which yields that

$$\frac{d}{dt}\|u(0)(\cdot,t)\|_{L^2(0,1)} \le \|h(\cdot,t)\|_{L^2(0,1)}, \quad t \in [0,T].$$

By the initial data in (1.4), we have

$$\|u(0)(\cdot,t)\|_{L^2(0,1)} \le \sqrt{t}\|h\|_{L^2(0,t;L^2(0,1))} + \|\phi\|_{L^2(0,1)}, \quad t \in [0,T].$$

Thus, we obtain

$$\|u(0)\|_{L^2(Q_T)} \le T\|h\|_{L^2(Q_T)} + \sqrt{T}\|\phi\|_{L^2(0,1)}. \tag{2.22}$$

From (2.11), (2.12), (2.22) and condition (c), and using the triangle inequality, we have

$$\begin{aligned}\|\mathcal{A}(0)\|_{L^2(0,T)} \le& \frac{1}{g_0}\left(\left\|\int_0^1 \omega''''(x)u(0)dx\right\|_{L^2(0,T)} + \|\psi'\|_{L^2(0,T)} + \|H\|_{L^2(0,T)}\right)\\ \le& \frac{1}{g_0}\left(\|\omega''''\|_{L^2(0,1)}\|u(0)\|_{L^2(Q_T)} + \|\psi\|_{H^1(0,T)} + \|\omega\|_{L^2(0,1)}\|h\|_{L^2(Q_T)}\right) = \frac{K}{g_0}.\end{aligned} \tag{2.23}$$

Hence, (2.20) and (2.23) yield the estimate (2.9). It is easy to check using the ratio test that the series in the right-hand side of (2.9) is absolutely convergent.

Finally, we consider the continuous dependence of $f$ upon the integral observation $\psi(t)$, and the input data $h(x,t)$ and $\phi(x)$. Let $(u^{(i)}(x,t), f^{(i)}(t)) \in \mathcal{V} \times L^2(0,T)$ for $i = 1,2$ be the solutions of the following two inverse problems:

$$\begin{cases} u_t^{(i)} + u_{xxxx}^{(i)} = f^{(i)}(t)g(x,t) + h^{(i)}(x,t), & (x,t) \in Q_T,\\ u^{(i)}(0,t) = u^{(i)}(1,t) = u_{xx}^{(i)}(0,t) = u_{xx}^{(i)}(1,t) = 0, & t \in (0,T),\\ u^{(i)}(x,0) = \phi^{(i)}(x), & x \in (0,1),\\ \int_0^1 \omega(x)u^{(i)}(x,t)dx = \psi^{(i)}(t), & t \in (0,T). \end{cases} \tag{2.24}$$

Then, the difference $u(x,t) := u^{(1)}(x,t) - u^{(2)}(x,t)$ satisfies the problem

$$\begin{cases} u_t + u_{xxxx} = (f^{(1)}(t) - f^{(2)}(t))g(x,t) + h^{(1)}(x,t) - h^{(2)}(x,t), & (x,t) \in Q_T,\\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, & t \in (0,T),\\ u(x,0) = \phi^{(1)}(x) - \phi^{(2)}(x), & x \in (0,1),\\ \int_0^1 \omega(x)u(x,t)dx = \psi^{(1)}(t) - \psi^{(2)}(t), & t \in (0,T). \end{cases} \tag{2.25}$$

Using the estimate (2.9), we obtain

$$\|f^{(1)} - f^{(2)}\|_{L^2(0,T)} \le \frac{K_1}{g_0}\sum_{n=0}^{\infty}\frac{\kappa^n T^{n/2}}{(n!)^{1/2}},$$

where

$$K_1 := (T\|\omega''''\|_{L^2(0,1)} + \|\omega\|_{L^2(0,1)})\|h^{(1)} - h^{(2)}\|_{L^2(Q_T)} + \sqrt{T}\|\omega''''\|_{L^2(0,1)}\|\phi^{(1)} - \phi^{(2)}\|_{L^2(0,1)} + \|\psi^{(1)} - \psi^{(2)}\|_{H^1(0,T)}.$$

This inequality implies the continuous dependence of $f$ upon the measurement $\psi(t)$, and the input data $h(x,t)$ and $\phi(x)$. The proof of the theorem is complete. □

**Theorem 2.4.** *Let the conditions in Theorem 2.1 hold and $\phi \in L^2(0,1)$. Let also the conditions (b) and (c) hold and $\omega \in V$. Suppose finally that*

$$\tilde{\kappa} := \frac{\sqrt{T}}{g_0}\|\omega''\|_{L^2(0,1)}\|g\|_{L^\infty(0,T;L^2(0,1))} < 1. \tag{2.26}$$

*Then, there exists a unique solution $(u,f) \in \mathcal{V} \times L^2(0,T)$ to the inverse problem given by Eqs. (1.4) and (1.5). Moreover, $f$ depends continuously upon the observation $\psi(t)$, and the input data $h(x,t)$ and $\phi(x)$, and satisfies the estimate*

$$\|f\|_{L^2(0,T)} \le \frac{e^{T/2}\|\omega\|_{H^2(0,1)}\tilde{K} + \|\psi\|_{H^1(0,T)}}{(1-\tilde{\kappa})g_0}, \tag{2.27}$$

*where $\tilde{K} = \|h\|_{L^2(Q_T)} + \|\phi\|_{L^2(0,1)}$.*

**Proof.** Multiplying the first equation in (1.4) by $\omega(x)$, and integrating the result over $[0,1]$ using the homogeneous boundary conditions, $\omega(0)=\omega(1)=0$ and condition (b), we obtain

$$f(t)=\frac{1}{G(t)}\left(\int_0^1 \omega''(x)u_{xx}(x,t)dx+\psi'(t)-H(t)\right) \tag{2.28}$$

where $G(t)$ and $H(t)$ are given by (2.11). This implies the following operator equation:

$$f=\mathcal{B}(f), \tag{2.29}$$

where $\mathcal{B}: L^2(0,T)\to L^2(0,T)$ is the operator defined as

$$\mathcal{B}(f)=\frac{1}{G(t)}\left(\int_0^1 \omega''(x)u_{xx}(f)dx+\psi'(t)-H(t)\right). \tag{2.30}$$

By using similar arguments as in the proof of Theorem 2.3, let $u^{(1)}(x,t)$ and $u^{(2)}(x,t)\in\mathcal{V}$ be the solutions to the direct problem (1.4) corresponding to the sources $f^{(1)}$ and $f^{(2)}\in L^2(0,T)$, respectively. Then, $w(x,t)=u^{(1)}(x,t)-u^{(2)}(x,t)$ and $F(t)=f^{(1)}(t)-f^{(2)}(t)$ satisfy the problem (2.14). Applying (2.15), we have

$$\|w_{xx}\|_{L^2(Q_T)}\le\sqrt{T}\|F\|_{L^2(0,T)}\|g\|_{L^\infty(0,T;L^2(0,1))}.$$

Hence, the above estimate and (2.30) yield that

$$\|\mathcal{B}(f^{(1)})-\mathcal{B}(f^{(2)})\|_{L^2(0,T)}=\left(\int_0^T\left|\frac{1}{G(t)}\int_0^1\omega''(x)w_{xx}(x,t)dx\right|^2dt\right)^{1/2}\le\frac{1}{g_0}\|\omega''\|_{L^2(0,1)}\|w_{xx}\|_{L^2(Q_T)}\le\tilde{\kappa}\|f^{(1)}-f^{(2)}\|_{L^2(0,T)}. \tag{2.31}$$

Then, the condition (2.26) implies that $\mathcal{B}$ is a contraction operator on $L^2(0,T)$, which has a unique fixed point, i.e., the inverse problem given by Eqs. (1.4) and (1.5) has a unique solution $(u,f)\in\mathcal{V}\times L^2(0,T)$. Meanwhile, for any $f^0\in L^2(0,T)$, the solution $f$ can be approximated by the successive approximations given by $f^{n+1}=\mathcal{B}(f^n)$ for $n\in\mathbb{N}$, and the sequence $\{f^n\}_{n\in\mathbb{N}}$ converges to $f$ in $L^2(0,T)$. Using the condition (2.26), applying the method used in (2.20) and (2.31), and taking $f^0=0$, we can obtain that

$$\|f\|_{L^2(0,T)}\le\sum_{n=0}^{\infty}\tilde{\kappa}^n\|\mathcal{B}(0)\|_{L^2(0,T)}=\frac{1}{1-\tilde{\kappa}}\|\mathcal{B}(0)\|_{L^2(0,T)}. \tag{2.32}$$

Multiplying the first equation of (1.4) by $u(0)$ and integrating over $[0,1]$, we have

$$\frac{1}{2}\frac{d}{dt}\|u(0)(\cdot,t)\|^2_{L^2(0,1)}+\|u_{xx}(0)(\cdot,t)\|^2_{L^2(0,1)}\le\frac{1}{2}\|h(\cdot,t)\|^2_{L^2(0,1)}+\frac{1}{2}\|u(0)(\cdot,t)\|^2_{L^2(0,1)},\quad t\in[0,T]. \tag{2.33}$$

Applying the Gronwall's inequality and integrating over $[0,T]$ we obtain

$$\|u(0)\|^2_{L^2(Q_T)}\le e^T(\|h\|^2_{L^2(Q_T)}+\|\phi\|^2_{L^2(0,1)})\le e^T\tilde{K}^2. \tag{2.34}$$

Also, integrating (2.33) with respect to time and using (2.34) we obtain

$$\|u_{xx}(0)\|^2_{L^2(Q_T)}\le\frac{1}{2}\|h\|^2_{L^2(Q_T)}+\frac{1}{2}e^T\tilde{K}^2\le e^T\tilde{K}^2. \tag{2.35}$$

Consequently, we have

$$\begin{aligned}\|\mathcal{B}(0)\|_{L^2(0,T)}\le&\frac{1}{g_0}\left(\left\|\int_0^1\omega''(x)u_{xx}(0)dx\right\|_{L^2(0,T)}+\|\psi'\|_{L^2(0,T)}+\|H\|_{L^2(0,T)}\right)\\ \le&\frac{1}{g_0}\left(\|\omega''\|_{L^2(0,1)}\|u_{xx}(0)\|_{L^2(Q_T)}+\|\psi\|_{H^1(0,T)}+\|\omega\|_{L^2(0,1)}\|h\|_{L^2(Q_T)}\right)\\ \le&\frac{1}{g_0}\left(e^{T/2}\|\omega\|_{H^2(0,1)}\tilde{K}+\|\psi\|_{H^1(0,T)}\right).\end{aligned} \tag{2.36}$$

The estimate (2.27) can now be derived from (2.32) and (2.36). In addition, the continuous dependence of $f(t)$ upon $\psi(t)$, $h(x,t)$ and $\phi(x)$ can be verified by applying the analogous arguments utilized in Theorem 2.3, and the proof is complete. □

**Remark 2.2.** Theorem 2.3 gives that the inverse problem given by Eqs. (1.4) and (1.5) is well-posed globally under the hypotheses (a)–(c) and conditions of Theorem 2.1, whilst Theorem 2.4, due to the restrictive condition (2.26), ensures only the local well-posedness, i.e. for $0<T<T^*:=\frac{g_0^2}{\|\omega''\|^2_{L^2(0,1)}\|g\|^2_{L^\infty(0,T;L^2(0,1))}}$. However, the unique solution to Eqs. (1.4) and (1.5) actually holds globally under the conditions of Theorem 2.4 without the assumption (2.26), as given and proved in the following theorem.

**Theorem 2.5.** *Let the assumptions in Theorem 2.4 hold except for (2.26). Then, there exists a unique solution $(u,f)\in\mathcal{V}\times L^2(0,T)$ to the inverse problem given by Eqs. (1.4) and (1.5).*

**Proof.** From Remark 2.2 we only need to consider the case $T^* \le T$. Taking

$$t^* = \frac{T^*}{4} = \frac{g_0^2}{4\|\omega''\|^2_{L^2(0,1)}\|g\|^2_{L^\infty(0,T;L^2(0,1))}} \le \frac{g_0^2}{4\|\omega''\|^2_{L^2(0,1)}\|g\|^2_{L^\infty(0,t^*;L^2(0,1))}},$$

then by using the same method applied above in Theorem 2.4, we have

$$\|\mathcal{B}(f^{(1)}) - \mathcal{B}(f^{(2)})\|_{L^2(0,t^*)} = \left(\int_0^{t^*}\left|\frac{1}{G(t)}\int_0^1 \omega''(x)w_{xx}(F)(x,t)dx\right|^2 dt\right)^{1/2}$$

$$\le \frac{\sqrt{t^*}}{g_0}\|\omega''\|_{L^2(0,1)}\|g\|_{L^\infty(0,t^*;L^2(0,1))}\|f^{(1)} - f^{(2)}\|_{L^2(0,t^*)} \le \frac{1}{2}\|f^{(1)} - f^{(2)}\|_{L^2(0,t^*)},$$

and thus there exists a unique solution to the inverse problem given by Eqs. (1.4) and (1.5) for $0 \le t \le t^*$. Next, for $t \ge t^*$, we consider the inverse problem on the time interval $(t^*, 2t^*)$ with the initial data $\phi(x) = u(x,t^*)$, since $t^*$ is independent of $\phi(x)$. Hence, analogous arguments imply that the solution to the inverse problem given by Eqs. (1.4) and (1.5) exists uniquely for $t^* \le t \le 2t^*$. Repeating the above method a finite number of times $[T/t^*] + 1$, we conclude that the inverse problem has a unique solution in $(0,T)$. □

We now consider the inverse problem (2.3) with inhomogeneous boundary conditions to recover $f(t)$ from the given integral observation (1.5). Using (2.5), (2.6) and the notations

$$\tilde{h} := h - \rho_t - \rho_{xxxx}, \quad \Psi := \psi - \int_0^1 \omega(x)\rho(x,t)dx,$$

the inverse problem given by Eqs. (1.5) and (2.3) is transformed into the inverse problem of determining $(v, f)$ with homogeneous boundary data given by

$$\begin{cases} v_t + v_{xxxx} = f(t)g(x,t) + \tilde{h}(x,t), & (x,t) \in Q_T, \\ v(0,t) = v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = 0, & t \in (0,T), \\ v(x,0) = \varphi(x), & x \in (0,1), \end{cases} \tag{2.37}$$

and the integral observation

$$\int_0^1 \omega(x)v(x,t)dx = \Psi(t), \quad t \in (0,T). \tag{2.38}$$

**Remark 2.3.** For the inverse problem given by Eqs. (1.5) and (2.3), using the result for the direct problem in Theorem 2.2 and applying the approach used in the proof of Theorem 2.3, we can obtain the following well-posedness results.

**Theorem 2.6.** *Suppose conditions (a)–(c) and the assumptions of Theorem 2.2 hold. Then, the inverse problem given by Eqs. (1.5) and (2.3) has a unique solution $(u, f) \in \tilde{\mathcal{V}} \times L^2(0,T)$. Moreover, $f$ depends continuously upon the observation $\psi(t)$, and the input data $h(x,t)$, $\mu_i(t)$ for $i = \overline{1,4}$, and $\phi(x)$, and satisfies the estimate:*

$$\|f\|_{L^2(0,T)} \le \frac{K_\mu}{g_0}\sum_{n=0}^{\infty}\frac{\kappa^n T^{n/2}}{(n!)^{1/2}}, \tag{2.39}$$

*where $K_\mu := K + \|\omega''''\|_{L^2(0,1)}\|(\mu_1,\mu_2,\mu_3,\mu_4)\|_{[H^1(0,T)]^4}$.*

**Theorem 2.7.** *Let the assumptions of Theorem 2.2 hold. Let also the conditions (b) and (c) hold, $\omega \in V$ and (2.26) be satisfied. Then, there exists a unique solution $(u, f) \in \tilde{\mathcal{V}} \times L^2(0,T)$ to the inverse problem given by Eqs. (1.5) and (2.3). Moreover, $f$ depends continuously upon the observation $\psi(t)$, and the input data $h(x,t)$, $\mu_i(t)$ for $i = \overline{1,4}$, and $\phi(x)$, and satisfies the estimate*

$$\|f\|_{L^2(0,T)} \le \frac{e^{T/2}\|\omega\|_{H^2(0,1)}\tilde{K}_\mu + \|\psi\|_{H^1(0,T)}}{(1-\tilde{\kappa})g_0}, \tag{2.40}$$

*and $\tilde{K}_\mu = \tilde{K} + \|(\mu_1,\mu_2,\mu_3,\mu_4)\|_{[H^1(0,T)]^4}$. Furthermore, as in Theorem 2.5, the existence and uniqueness also hold globally without the restriction (2.26).*

From Theorems 2.3–2.7, we can see that the well-posed solution to the inverse problem given by Eqs. (1.4) and (1.5) can be obtained if $\psi \in H^1(0,1)$, and under such condition, the unknown source term $f(t)$ can be calculated directly by using the formulation (2.10) or (2.28). However, the integral measurement $\psi$ in (1.5) contains noise such that the noisy data $\psi^\epsilon \in L^2(0,T)$ satisfies

$$\|\psi - \psi^\epsilon\|_{L^2(0,T)} \le \epsilon, \tag{2.41}$$

where $\epsilon \ge 0$ represents the level of noise. The differentiation of the noisy measurement $\psi^\epsilon$ is mildly ill-posed, i.e. small perturbations in the measured data can produce large effects on its derivative, which indicates that the formulation (2.10) or (2.28) cannot be applied to numerically obtain $f(t)$ directly. Therefore, when $\psi$ is replaced by its noisy measurement $\psi^\epsilon$, the time-discrete approximation along with the cubic spline function method [32] (Section 3), and the well-known Tikhonov regularization method together with the conjugate gradient method (CGM) (Section 4), will be employed to numerically obtain a stable approximation of the unknown source term $f(t)$.

## 3. Time-discrete approximation in case $g(x,t) = g(x)$

Divide the time interval $[0,T]$ into a uniform grid $t_k = (k-1)\Delta t$ for $k = \overline{1,K}$, with a time-step $\Delta t = \frac{T}{K-1}$ and $2 \le K \in \mathbb{N}$ time steps. Under the assumption (a) on the weight function $\omega \in W$ and using (2.10), the unknown source term $f(t)$ in the inverse problem given by Eqs. (1.4) and (1.5) with the space-dependent source component $g(x,t) = g(x)$ can be approximated by the following time-discrete scheme:

$$f^1 = \frac{\int_0^1 \omega''''(x)\phi(x)dx + (\psi')^1 - \int_0^1 \omega(x)h^1(x)dx}{\int_0^1 \omega(x)g(x)dx},$$
$$f^k = \frac{\int_0^1 \omega''''(x)u^{k-1}(x)dx + (\psi')^k - \int_0^1 \omega(x)h^k(x)dx}{\int_0^1 \omega(x)g(x)dx}, \quad k = \overline{2,K}, \tag{3.1}$$

where $f^k := f(t_k)$, $u^k(x) := u(x,t_k)$, $u^1(x) = \phi(x)$, $(\psi')^k := \psi'(t_k)$, $h^k(x) := h(x,t_k)$ and the function $u^k(x)$ solves the problem:

$$\begin{cases} \delta_t u^k + u^k_{xxxx} = f^k g + h^k, & k = \overline{2,K}, \\ u^k(0) = u^k(1) = u^k_{xx}(0) = u^k_{xx}(1) = 0, & k = \overline{2,K}, \end{cases} \tag{3.2}$$

where $\delta_t u^k = \frac{u^k - u^{k-1}}{\Delta t}$.

The well-posedness of the system (3.1) and (3.2) is given in the following two lemmas.

**Lemma 3.1.** *Suppose that $\phi \in L^2(0,1)$, $g \in L^2(0,1)$, $h \in C(0,T;L^2(0,1))$, $\psi \in C^1(0,T)$, $\omega \in W$ and $\int_0^1 \omega(x)g(x)dx \neq 0$. Then, there exists a unique function pair $(u^k, f^k) \in V \times \mathbb{R}$ satisfying (3.1) and (3.2).*

**Proof.** For the fourth-order elliptic problem (3.2), its variational formulation can be obtained by using Green's theorem,

$$a(u^k,\chi) := \frac{1}{\Delta t}\int_0^1 u^k \chi dx + \int_0^1 u^k_{xx}\chi_{xx} dx = \frac{1}{\Delta t}\int_0^1 u^{k-1}\chi dx + \int_0^1 (f^k g + h^k)\chi dx =: \mathcal{F}_k(\chi),$$

for any $\chi \in V$. Clearly, $a(\mathrm{u},\chi)$ is bilinear and

$$|a(\mathrm{u},\chi)| \le \frac{1}{\Delta t}\|\mathrm{u}\|_{L^2(0,1)}\|\chi\|_{L^2(0,1)} + \|\mathrm{u}_{xx}\|_{L^2(0,1)}\|\chi_{xx}\|_{L^2(0,1)} \le C\|\mathrm{u}\|_V\|\chi\|_V,$$
$$a(\mathrm{u},\mathrm{u}) = \frac{1}{\Delta t}\|\mathrm{u}\|^2_{L^2(0,1)} + \|\mathrm{u}_{xx}\|^2_{L^2(0,1)} \ge C\|\mathrm{u}\|^2_V.$$

Hence $a(\cdot,\cdot)$ is a bilinear, continuous and coercive functional on $V$. We also have

$$|\mathcal{F}_k(\chi)| \le C\left(\|u^{k-1}\|_{L^2(0,1)} + |f^k| + 1\right)\|\chi\|_V, \quad k = \overline{2,K},$$
$$|f^k| \le C\left(\|\psi\|_{C^1(0,T)} + \|u^{k-1}\|_{L^2(0,1)} + 1\right), \quad k = \overline{2,K},$$

and $\mathcal{F}_k(\cdot)$ is a linear functional on $V$.

For $k = 2$, it is easy to find that $|f^2|$ is bounded, which implies the unique existence of $u^2(x) \in V$ due to the Lax–Milgram theorem. Using the above arguments with recursion for $k = \overline{3,K}$, we can obtain the boundness of $f^k$, i.e. $f^k \in \mathbb{R}$, and the unique existence of $u^k(x) \in V$ to (3.1) and (3.2). □

As in [34], we obtain the following lemma.

**Lemma 3.2.** *Let the assumptions of Lemma 3.1 hold. Assume further that $\phi \in V$. Then there exist positive constants $C$ and $\Delta t_0$ such that for any $\Delta t \le \Delta t_0$ and $j = \overline{2,K}$,*

$$\max_{j=\overline{2,K}} \|u^j\|^2_{L^2(0,1)} + \sum_{k=2}^{j}\|u^k - u^{k-1}\|^2_{L^2(0,1)} + \Delta t\sum_{k=2}^{j}\|u^k_{xx}\|^2_{L^2(0,1)} \le C, \tag{3.3}$$

$$\Delta t\sum_{k=2}^{j}\|\delta_t u^k\|^2_{L^2(0,1)} + \max_{j=\overline{2,K}}\|u^j_{xx}\|^2_{L^2(0,1)} + \sum_{k=2}^{j}\|u^k_{xx} - u^{k-1}_{xx}\|^2_{L^2(0,1)} \le C, \tag{3.4}$$

$$\max_{j=\overline{2,K}} |f^j|^2 \le C. \tag{3.5}$$

**Proof.** Multiplying by $u^k\Delta t$ the first equation in (3.2) and integrating over $[0,1]$, we have

$$\int_0^1 (u^k - u^{k-1})u^k dx + \Delta t\|u^k_{xx}\|^2_{L^2(0,1)} = \int_0^1 (f^k g + h^k)u^k dx\Delta t, \quad k = \overline{2,K}.$$

For every $j = \overline{2,K}$, we sum the above result up for $k = \overline{2,j}$ and have

$$\sum_{k=2}^{j}\int_0^1 (u^k - u^{k-1})u^k dx + \Delta t\sum_{k=2}^{j}\|u^k_{xx}\|^2_{L^2(0,1)} = \Delta t\sum_{k=2}^{j}\int_0^1 (f^k g + h^k)u^k dx.$$

Using Abel's lemma, we have

$$\sum_{k=2}^{j}\int_0^1 (u^k - u^{k-1})u^k dx = \frac{1}{2}\left(\|u^j\|^2_{L^2(0,1)} - \|\phi\|^2_{L^2(0,1)} + \sum_{k=2}^{j}\|u^k - u^{k-1}\|^2_{L^2(0,1)}\right).$$

Young's inequality implies that

$$\left|\sum_{k=2}^{j}\int_0^1 (f^k g + h^k)u^k dx\right| \le \frac{1}{2}\sum_{k=2}^{j}\left(|f^k|^2\|g\|^2_{L^2(0,1)} + \|h\|^2_{C(0,T;L^2(0,1))}\right) + \frac{1}{2}\sum_{k=2}^{j}\|u^k\|^2_{L^2(0,1)}.$$

Thus, we have

$$(1 - C\Delta t)\|u^j\|^2_{L^2(0,1)} + \sum_{k=2}^{j}\|u^k - u^{k-1}\|^2_{L^2(0,1)} + \Delta t\sum_{k=2}^{j}\|u^k_{xx}\|^2_{L^2(0,1)} \le C\left(1 + \Delta t\sum_{k=2}^{j}|f^k|^2 + \Delta t\sum_{k=2}^{j-1}\|u^k\|^2_{L^2(0,1)}\right) \tag{3.6}$$

where $C$ is a positive constant dependent on $g$, $h$ and $\phi$. Choosing $\Delta t \le \Delta t_0 = \frac{1}{C+1} > 0$, then $1 - C\Delta t \ge 1 - C\Delta t_0 > 0$ and

$$\begin{aligned}&\|u^j\|^2_{L^2(0,1)} + \sum_{k=2}^{j}\|u^k - u^{k-1}\|^2_{L^2(0,1)} + \sum_{k=2}^{j}\|u^k_{xx}\|^2_{L^2(0,1)}\\ \le&\frac{C\Delta t}{1 - C\Delta t_0}\sum_{k=2}^{j-1}\left(\|u^k\|^2_{L^2(0,1)} + \sum_{i=2}^{k}\|u^i - u^{i-1}\|^2_{L^2(0,1)} + \Delta t\sum_{i=2}^{k}\|u^i_{xx}\|^2_{L^2(0,1)}\right) + \frac{C}{1 - C\Delta t_0}\left(1 + \Delta t\sum_{k=2}^{j}|f^k|^2\right).\end{aligned}$$

The discrete Gronwall lemma implies that

$$\begin{aligned}&\max_{j=\overline{2,K}}\|u^j\|^2_{L^2(0,1)} + \sum_{k=2}^{j}\|u^k - u^{k-1}\|^2_{L^2(0,1)} + \Delta t\sum_{k=2}^{j}\|u^k_{xx}\|^2_{L^2(0,1)}\\ \le&\frac{C\Delta t}{1 - C\Delta t_0}\sum_{k=2}^{j}|f^k|^2 + \frac{C^2\Delta t}{(1 - C\Delta t_0)^2}e^{\frac{CT}{1-C\Delta t_0}}\sum_{k=2}^{j-1}\left(1 + \Delta t\sum_{i=1}^{k}|f^k|^2\right) \le C_1 + C_1\Delta t\sum_{k=2}^{j}|f^k|^2,\end{aligned}$$

where $C_1 = \left(\frac{C}{1-C\Delta t_0} + \frac{C^2 T}{(1-C\Delta t_0)^2}e^{\frac{CT}{1-C\Delta t_0}}\right) > 0$ depends on $g$, $h$, $\phi$, $\psi$ and $T$. From (3.1), (3.3) and the above inequality, we get

$$|f^j|^2 \le C\left(1 + \|u^{j-1}\|^2_{L^2(0,1)}\right) \le C + C\Delta t\sum_{k=2}^{j-1}|f^k|^2,$$

which yields (3.5) by using the discrete Gronwall lemma. Consequently, (3.3) can be derived.

Multiplying (3.2) by $\delta_t u^k\Delta t$ and using integration by parts, we have

$$\Delta t\sum_{k=2}^{j}\|\delta_t u^k\|^2_{L^2(0,1)} + \sum_{k=2}^{j}\int_0^1 u^k_{xx}(u^k_{xx} - u^{k-1}_{xx})dx = \Delta t\sum_{k=2}^{j}\int_0^1 (f^k g + h^k)\delta_t u^k dx,$$

and, via similar arguments, we get

$$\sum_{k=2}^{j}\int_0^1 u^k_{xx}(u^k_{xx} - u^{k-1}_{xx})dx = \frac{1}{2}\left(\|u^j_{xx}\|^2_{L^2(0,1)} - \|\phi''\|^2_{L^2(0,1)} + \sum_{k=2}^{j}\|u^k_{xx} - u^{k-1}_{xx}\|^2_{L^2(0,1)}\right),$$

and

$$\left|\sum_{k=2}^{j}\int_0^1 (f^k g + h^k)\delta_t u^k dx\right| \le \frac{1}{2}\sum_{k=2}^{j}\left(|f^k|^2\|g\|^2_{L^2(0,1)} + \|h\|^2_{C(0,T;L^2(0,1))}\right) + \frac{1}{2}\sum_{k=2}^{j}\|\delta_t u^k\|^2_{L^2(0,1)}.$$

Hence, the estimate (3.4) can be derived by using the discrete Gronwall inequality. The proof is complete. □

Based on Lemma 3.2, we can obtain an approximation for the time-dependent source, as given by Theorem 3.1. For this, let us first define piecewisely the following functions:

$$u_K(x,t) = u^{k-1}(x) + (t - t_{k-1})\delta_t u^k(x), \quad t \in (t_{k-1}, t_k], \quad k = \overline{2, K},$$
$$\bar{u}_K(x,t) = u^k(x), \quad t \in (t_{k-1}, t_k], \quad k = \overline{2, K}$$

and

$$\bar{f}_K(t) = \frac{\int_0^1 \omega''''(x)\bar{u}_K(x, t - \Delta t)dx + \bar{\psi}'_K(t) - \int_0^1 \omega(x)\bar{h}_K(x,t)dx}{\int_0^1 \omega(x)g(x)dx}, \quad t \in (0, T], \tag{3.7}$$

where $\bar{\psi}_K(t)$ and $\bar{h}_K(x,t)$ are defined by

$$\bar{\psi}_K(0) = \psi(0), \quad \bar{\psi}_K(t) = \psi(t_k), \quad t \in (t_{k-1}, t_k], \quad k = \overline{2, K},$$
$$\bar{h}_K(x,0) = h(x,0), \quad \bar{h}_K(x,t) = h(x,t_k), \quad t \in (t_{k-1}, t_k], \quad k = \overline{2, K}.$$

Then, we have that

$$\begin{cases} \int_0^1 (u_K)_t(x,t)\chi(x)dx + \int_0^1 (\bar{u}_K)_{xx}(x,t)\chi''(x)dx = \int_0^1 (\bar{f}_K(t)g(x) + \bar{h}_K(x,t))\chi(x)dx, \quad t \in (0,T], \ \forall \chi \in V, \\ u_K(x,0) = \phi(x), \quad x \in (0,1). \end{cases} \tag{3.8}$$

**Theorem 3.1.** *Let the assumptions of Lemma 3.2 hold. Assume further that $h \in H^1(0,T;L^2(0,1))$ and $\psi \in H^2(0,T)$. Then there exist positive constants $C$ and $\Delta t_0$ such that for any $\Delta t \le \Delta t_0$*

$$\int_0^T |\bar{f}_K(t) - f(t)|^2 dt \le C\Delta t. \tag{3.9}$$

**Proof.** Subtracting (2.10) from (3.7), we have

$$\bar{f}_K(t) - f(t) = \frac{\int_0^1 \omega''''(x)(\bar{u}_K(x,t-\Delta t) - u(x,t))dx + (\bar{\psi}'_K(t) - \psi'(t)) - \int_0^1 \omega(x)(\bar{h}_K(x,t) - h(x,t))dx}{\int_0^1 \omega(x)g(x)dx}. \tag{3.10}$$

Since $\psi \in H^2(0,1)$ and $h \in H^1(0,T;L^2(0,1))$, we have

$$\int_0^t |\bar{\psi}'_K(\tau) - \psi'(\tau)|^2 d\tau \le \sum_{k=2}^{K}\int_{t_k}^{t_{k-1}}\left|\int_\tau^{t_k}\psi''(\zeta)d\zeta\right|^2 d\tau \le C\Delta t,$$

$$\int_0^t \left|\int_0^1 \omega(x)(\bar{h}_K(x,\tau) - h(x,\tau))dx\right|^2 d\tau \le C\int_0^t \|\bar{h}_K(\cdot,\tau) - h(\cdot,\tau)\|^2_{L^2(0,1)}d\tau \le C\sum_{k=2}^{K}\int_{t_k}^{t_{k-1}}\left\|\int_\tau^{t_k} h_\zeta d\zeta\right\|^2_{L^2(0,1)} d\tau \le C\Delta t.$$

For the first term of the right-hand side of (3.10), we obtain

$$\left|\int_0^1 \omega''''(x)(\bar{u}_K(x,t-\Delta t) - u(x,t))dx\right| \le \left|\int_0^1 \omega''''(x)(\bar{u}_K(x,t-\Delta t) - \bar{u}_K(x,t))dx\right| + \left|\int_0^1 \omega''''(x)(\bar{u}_K(x,t) - u(x,t))dx\right|,$$

and (3.4) yields

$$\int_0^t \left|\int_0^1 \omega''''(x)(\bar{u}_K(x,\tau-\Delta t) - \bar{u}_K(x,\tau))dx\right|^2 d\tau \le C\int_0^t \|\bar{u}_K(\cdot,\tau-\Delta t) - \bar{u}_K(\cdot,\tau)\|^2_{L^2(0,1)}d\tau \le C(\Delta t)^2\sum_{k=2}^{K}\|\delta_t u^k\|^2_{L^2(0,1)}\Delta t \le C(\Delta t)^2,$$

and

$$\int_0^t \left|\int_0^1 \omega''''(x)(\bar{u}_K(x,\tau) - u(x,t))dx\right|^2 d\tau \le C\int_0^t \|\bar{u}_K(\cdot,\tau) - u(\cdot,\tau)\|^2_{L^2(0,1)}d\tau \le C\left((\Delta t)^2 + \int_0^t \|u_K - u\|^2_{L^2(0,1)}d\tau\right).$$

Therefore, we obtain that

$$\int_0^t |\bar{f}_K(\tau) - f(\tau)|^2 d\tau \le C\left(\Delta t + \int_0^t \|u_K(\cdot,\tau) - u(\cdot,\tau)\|^2_{L^2(0,1)}d\tau\right). \tag{3.11}$$

For any $\chi \in V$, the problem (1.4) with $g(x,t) = g(x)$ satisfies the following identity:

$$\begin{cases} \int_0^1 u_t(x,t)\chi(x)dx + \int_0^1 u_{xx}(x,t)\chi''(x)dx = \int_0^1 (f(t)g(x) + h(x,t))\chi(x)dx, & t \in (0,T], \\ u(x,0) = \phi(x), & x \in (0,1). \end{cases} \tag{3.12}$$

For each frozen $t \in (0,T]$, we take $\chi(x) = u_K(x,t) - u(x,t)$ in (3.8) and (3.12), subtract, unfreeze $t$ and integrate with respect to $t$ over $(0,\theta)$ with $\theta \in (0,T]$, to obtain

$$\int_0^\theta \int_0^1 [u_K(x,t) - u(x,t)]_t [u_K(x,t) - u(x,t)] dx dt + \int_0^\theta \|[u_K(\cdot,t) - u(\cdot,t)]_{xx}\|^2_{L^2(0,1)} dt$$
$$= \int_0^\theta \int_0^1 [\bar{f}_K(t) - f(t)] g(x) [u_K(x,t) - u(x,t)] dx dt + \int_0^\theta \int_0^1 [\bar{h}_K(x,t) - h(x,t)][u_K(x,t) - u(x,t)] dx dt$$
$$- \int_0^\theta \int_0^1 [\bar{u}_K(x,t) - u_K(x,t)]_{xx} [u_K(x,t) - u(x,t)]_{xx} dx dt. \tag{3.13}$$

It is obvious that

$$\int_0^\theta \int_0^1 [u_K(x,t) - u(x,t)]_t [u_K(x,t) - u(x,t)] dx dt = \frac{1}{2} \|u_K(\cdot,\theta) - u(\cdot,\theta)\|^2_{L^2(0,1)}.$$

Using (3.11) and Young inequality, we have

$$\left| \int_0^\theta \int_0^1 [\bar{f}_K(t) - f(t)] g(x) [u_K(x,t) - u(x,t)] dx dt \right|$$
$$\le C \left( \int_0^\theta |\bar{f}_K(t) - f(t)|^2 dt + \int_0^\theta \|u_K(\cdot,t) - u(\cdot,t)\|^2_{L^2(0,1)} dt \right) \le C \left( \Delta t + \int_0^\theta \|u_K(\cdot,t) - u(\cdot,t)\|^2_{L^2(0,1)} dt \right), \tag{3.14}$$

and similarly,

$$\left| \int_0^\theta \int_0^1 [\bar{h}_K(x,t) - h(x,t)][u_K(x,t) - u(x,t)] dx dt \right| \le C \left( \Delta t + \int_0^\theta \|u_K(\cdot,t) - u(\cdot,t)\|^2_{L^2(0,1)} dt \right).$$

For the third term in the right-hand side of (3.13), the Young and Cauchy inequalities and the estimate (3.4) imply that

$$\left| \int_0^\theta \int_0^1 [\bar{u}_K(x,t) - u_K(x,t)]_{xx} [u_K(x,t) - u(x,t)]_{xx} dx dt \right|$$
$$\le \frac{C}{\varepsilon} \int_0^\theta \|[\bar{u}_K(\cdot,t) - u_K(\cdot,t)]_{xx}\|^2_{L^2(0,1)} dt + \varepsilon \int_0^\theta \|[u_K(\cdot,t) - u(\cdot,t)]_{xx}\|^2_{L^2(0,1)} dt \le \frac{C}{\varepsilon} \Delta t + \varepsilon \int_0^\theta \|[u_K(\cdot,t) - u(\cdot,t)]_{xx}\|^2_{L^2(0,1)} dt,$$

for any $\varepsilon > 0$. Hence, (3.13) becomes

$$\|u_K(\cdot,\theta) - u(\cdot,\theta)\|^2_{L^2(0,1)} + \int_0^\theta \|[u_K(\cdot,t) - u(\cdot,t)]_{xx}\|^2_{L^2(0,1)} dt$$
$$\le \frac{C}{\varepsilon} \left( \Delta t + \int_0^\theta \|u_K(\cdot,t) - u(\cdot,t)\|^2_{L^2(0,1)} dt \right) + \varepsilon \int_0^\theta \|[u_K(\cdot,t) - u(\cdot,t)]_{xx}\|^2_{L^2(0,1)} dt.$$

This yields that

$$\max_{\theta \in [0,T]} \|u_K(\cdot,\theta) - u(\cdot,\theta)\|^2_{L^2(0,1)} \le C \Delta t,$$

by applying the Gronwall inequality and taking $\varepsilon = 1$. Meanwhile, from (3.11) this implies (3.9) and the proof of the theorem is complete. □

### *3.1. The cubic spline method*

As previously discussed at the end of Section 2, we actually need to recover $f(t)$ from the noisy data $\psi^\epsilon$. The conventional finite-difference scheme to compute the derivative of $\psi^\epsilon$ can only be applied for exact data $\psi$ or when $\epsilon$ is very small, due to the ill-posed process of numerical differentiating the noisy $\psi^\epsilon$. In order to obtain a stable derivative of the measured data $\psi^\epsilon$, the cubic spline function method [32] is employed. The natural cubic spline function $s(t)$ is constructed as follows:

(i) $s(t)$ is a twice differentiable natural cubic spline of time mesh grid $t_k$:

$$s(t_k+) = s(t_k-), \quad s'(t_k+) = s'(t_k-), \quad s''(t_k+) = s''(t_k-), \quad k = \overline{2, K-1},$$

where $s(t_k+) = \lim_{t \to t_k+} s(t)$ and $s(t_k-) = \lim_{t \to t_k-} s(t)$;

(ii) $s''(0) = s''(T) = 0$;

(iii) The third-order derivative of $s(t)$ at the time instant $t = t_k$ satisfies the following conditions:

$$s'''(t_k+) - s'''(t_k-) = \frac{\Delta t}{\alpha} (\psi^\epsilon(t_k) - s(t_k)), \quad k = \overline{2, K-1},$$

where $\alpha > 0$ is a regularization parameter.

**Lemma 3.3** ([32, *Theorem 2.5*]). *Suppose that $\psi$ and $\psi^\epsilon \in L^\infty(0,T)$ satisfy $\|\psi^\epsilon - \psi\|_{L^\infty(0,T)} \le \epsilon$. Suppose further that $\psi \in H^2(0,T)$. Then, the function $s(t)$ obtained by the above process (i)–(iii) satisfies the following estimates:*

$$\|s' - \psi'\|_{L^2(0,T)} \le \left( 2\Delta t + 4\alpha^{1/4} + \frac{\Delta t}{\pi} \right) \|s''\|_{L^2(0,T)} + \Delta t \frac{\alpha^{1/2}}{\epsilon} + \frac{2\epsilon}{\alpha^{1/4}}. \tag{3.15}$$

*With the choice $\alpha = \epsilon^2$, the estimate (3.15) yields*

$$\|s' - \psi'\|_{L^2(0,T)} \le \left(2\Delta t + 4\sqrt{\epsilon} + \frac{\Delta t}{\pi}\right)\|s''\|_{L^2(0,T)} + \Delta t + 2\sqrt{\epsilon} = \mathcal{O}(\Delta t + \sqrt{\epsilon}). \tag{3.16}$$

Therefore, the unknown quantity $f(t)$ can be determined from the cubic spline $s(t)$ generated above from $\psi^\epsilon$. Also, the system (3.1) and (3.2) can be rewritten as:

$$\begin{aligned} f_s^1 &= \frac{\int_0^1 \omega''''(x)\phi(x)dx + (s')^1 - \int_0^1 \omega(x)h^1(x)dx}{\int_0^1 \omega(x)g(x)dx}, \\ f_s^k &= \frac{\int_0^1 \omega''''(x)u_s^{k-1}(x)dx + (s')^k - \int_0^1 \omega(x)h^k(x)dx}{\int_0^1 \omega(x)g(x)dx}, \quad k = \overline{2,K}, \end{aligned} \tag{3.17}$$

and the function $u_s^k(x)$ solves the problem

$$\begin{cases} \delta_t u_s^k + (u_s^k)_{xxxx} = f_s^k g(x) + h^k(x), & k = \overline{2,K}, \\ u_s^k(0) = u_s^k(1) = (u_s^k)_{xx}(0) = (u_s^k)_{xx}(1) = 0, & k = \overline{2,K}, \end{cases} \tag{3.18}$$

where $\delta_t u_s^k = \frac{u_s^k - u_s^{k-1}}{\Delta t}$, $u_s^1(x) = \phi(x)$ and $(s')^k = s'(t_k)$. Like the definitions of functions $\bar{f}_K$, $u_K$ and $\bar{u}_K$, we can define the functions $\bar{f}_{Ks}$, $u_{Ks}$ and $\bar{u}_{ks}$ in the same way, respectively. Similar to Theorem 3.1 we obtain the following theorem.

**Theorem 3.2.** *Let the assumptions of Theorem 3.1 be satisfied and assume further that $\psi \in L^\infty(0,T)$. Then choosing $\alpha = \epsilon^2$, there exist positive constants $C$ and $\Delta t_0$ such that for any $\Delta t \le \Delta t_0$,*

$$\int_0^T \left|\bar{f}_{Ks}(t) - f(t)\right|^2 dt \le C(\Delta t + \epsilon). \tag{3.19}$$

**Proof.** Denoting by $d^k := f_s^k - f^k$, $w^k(x) := u_s^k(x) - u^k(x)$ and $S^k := (s')^k - (\psi')^k$, from (3.1), (3.2) and (3.17), (3.18), we have

$$d^1 = \frac{S^1}{\int_0^1 \omega(x)g(x)dx}, \quad d^k = \frac{\int_0^1 \omega''''(x)w^{k-1}(x)dx + S^k}{\int_0^1 \omega(x)g(x)dx}, \quad k = \overline{2,K}, \tag{3.20}$$

and

$$\begin{cases} \delta_t w^k + w^k_{xxxx} = d^k g(x), & k = \overline{2,K}, \\ w^k(0) = w^k(1) = w^k_{xx}(0) = w^k_{xx}(1) = 0, & k = \overline{2,K}, \end{cases} \tag{3.21}$$

with $w^1(x) = 0$. Applying the approaches used to prove Lemma 3.2, for every $j = \overline{2,K}$, we can establish that

$$\max_{j=\overline{2,K}} \|w^j\|^2_{L^2(0,1)} + \sum_{k=2}^{j} \|w^k - w^{k-1}\|^2_{L^2(0,1)} + \Delta t \sum_{k=2}^{j} \|w^k_{xx}\|^2_{L^2(0,1)} \le C, \tag{3.22}$$

$$\delta_t \sum_{k=2}^{j} \|w^k\|^2_{L^2(0,1)} \Delta t + \max_{j=\overline{2,K}} \|w^j_{xx}\|^2_{L^2(0,1)} + \sum_{k=2}^{j} \|w^k_{xx} - w^{k-1}_{xx}\|^2_{L^2(0,1)} \le C, \tag{3.23}$$

$$\max_{j=\overline{2,K}} |d^j|^2 \le C. \tag{3.24}$$

We define functions $w_K(x,t)$ and $\bar{w}_K(x,t)$ similarly to the definition of $u_K(x,t)$ and $\bar{u}_K(x,t)$. Then, (3.20) and (3.21) become

$$\bar{d}_K(t) = \bar{f}_{Ks}(t) - \bar{f}_K(t) = \frac{\int_0^1 \omega''''(x)\bar{w}_K(x, t-\Delta t)dx + \bar{S}_K(t)}{\int_0^1 \omega(x)g(x)dx}, \tag{3.25}$$

where $\bar{S}_K(t)$ is defined by

$$\bar{S}_K(0) = S^1, \quad \bar{S}_K(t) = S(t_k), \quad t \in (t_{k-1}, t_k], \quad k = \overline{2,K},$$

and

$$\begin{cases} \int_0^1 (\bar{w}_K)_t(x,t)\chi(x)dx + \int_0^1 (\bar{w}_K)_{xx}(x,t)\chi''(x)dx = \int_0^1 \bar{d}_K(t)g(x)\chi(x)dx, & t \in (0,T], \\ \bar{w}_K(x,0) = 0, & x \in (0,1), \end{cases} \tag{3.26}$$

for all $\chi \in V$. For the right-hand side of (3.25), we have

$$\int_0^t \left|\int_0^1 \omega''''(x)\bar{w}_K(x,\tau-\Delta t)dx\right|^2 d\tau \le C\int_0^t \|\bar{w}_K(\cdot,\tau-\Delta t)\|^2_{L^2(0,1)} d\tau$$

$$\leq C\int_0^t \|\bar{w}_K(\cdot,\tau)-w_K(\cdot,\tau)\|^2_{L^2(0,1)}d\tau + C\int_0^t \|w_K(\cdot,\tau)\|^2_{L^2(0,1)}d\tau \leq C(\Delta t)^2 + C\int_0^t \|w_K(\cdot,\tau)\|^2_{L^2(0,1)}d\tau, \tag{3.27}$$

and the estimate (3.16) in Lemma 3.3 implies that

$$\begin{aligned}\int_0^t |\bar{S}_K(\tau)|^2 d\tau &\leq \int_0^t |s'(\tau)-\psi'(\tau)|^2 d\tau + \int_0^t \left|\bar{S}_K(\tau)-(s'(\tau)-\psi'(\tau))\right|^2 d\tau \\ &= \int_0^t |s'(\tau)-\psi'(\tau)|^2 d\tau + \sum_{k=2}^K \int_{t_{k-1}}^{t_k} \left|[(s')^k - s'(\tau)] - [(\psi')^k - \psi'(\tau)]\right|^2 d\tau \\ &\leq C((\Delta t)^2+\epsilon) + \sum_{k=2}^K \int_{t_{k-1}}^{t_k} \left( \left|\int_\tau^{t_k} s''(\zeta)d\zeta\right|^2 + \left|\int_\tau^{t_k} \psi''(\zeta)d\zeta\right|^2 \right) d\tau \leq C(\Delta t + \epsilon).\end{aligned}$$

Thus, we obtain

$$\int_0^t |\bar{d}_K(\tau)|^2 d\tau \leq C(\Delta t + \epsilon) + C\int_0^t \|w_K(\cdot,\tau)\|^2_{L^2(0,1)}d\tau. \tag{3.28}$$

For each frozen $t\in(0,T]$, we take $\chi(x)=w_K(x,t)$ in (3.26), unfreeze $t$ and integrate with respect to $t$ over $(0,\theta)$ with $\theta\in(0,T]$, to obtain

$$\begin{aligned}&\int_0^\theta\int_0^1 (w_K)_t(x,t)w_K(x,t)dxdt + \int_0^\theta \|(w_K)_{xx}(\cdot,t)\|^2_{L^2(0,1)}dt \\ &= \int_0^\theta \bar{d}_K(t)\left(\int_0^1 g(x)w_K(x,t)dx\right)dt - \int_0^\theta\int_0^1 (\bar{w}_K - w_K)_{xx}(x,t)(w_K)_{xx}(x,t)dxdt.\end{aligned}$$

Clearly,

$$\int_0^\theta\int_0^1 (w_K)_t(x,t)w_K(x,t)dxdt = \frac{1}{2}\|w_K(\cdot,\theta)\|^2_{L^2(0,1)},$$

$$\left|\int_0^\theta\int_0^1 (\bar{w}_K - w_K)_{xx}(x,t)(w_K)_{xx}(x,t)dxdt\right| \leq C\Delta t + \int_0^\theta \|(w_K)_{xx}(x,t)\|^2_{L^2(0,1)}dt.$$

and (3.28) yields that

$$\left|\int_0^\theta \bar{d}_K(t)\left(\int_0^1 g(x)w_K(x,t)dx\right)dt\right| \leq C\left(\int_0^\theta |\bar{d}_K(t)|^2dt + \int_0^\theta \|w_K(\cdot,t)\|^2_{L^2(0,1)}dt\right) \leq C\left(\Delta t + \epsilon + \int_0^\theta \|w_K(\cdot,t)\|^2_{L^2(0,1)}\right).$$

Hence, we obtain that

$$\|w_K(\cdot,\theta)\|^2_{L^2(0,1)} \leq C\left(\Delta t + \epsilon + \int_0^\theta \|w_K(\cdot,t)\|^2_{L^2(0,1)}dt\right),$$

and Gronwall inequality leads to

$$\max_{\theta\in[0,T]} \|w_K(\cdot,\theta)\|^2_{L^2(0,1)} \leq C(\Delta t + \epsilon).$$

Using (3.28), we immediately obtain (3.19), which concludes the proof of the theorem. □

The above theorem indicates that the time-discrete scheme (3.17) and (3.18) can be used to determine $f(t)$. However, there are some limitations of such method as we had to further assume that: (i) the data $\psi\in L^\infty(0,T)\cap H^2(0,T)$; (ii) the source term $g$ is time-independent; (iii) the time-step $\Delta t$ should be small enough.

In order to solve the general inverse problem given by Eqs. (1.4) and (1.5) under the more general conditions of Theorem 2.5, the optimization method based on the CGM is developed in the next section.

## 4. Optimization method

In the context of Theorem 2.1, assuming $\phi\in L^2(0,1)$, $g\in L^\infty(0,T;L^2(0,1))$ and $h\in L^2(Q_T)$, let $V\ni u(x,t;f)$ (or $u(f)$) denote the unique solution of the problem (1.4) for a particular function $f(t)\in L^2(0,T)$. Then the inverse problem of recovering $f(t)$ can be reformulated as an operator equation given by

$$l(f)(t) := \int_0^1 \omega(x)u(x,t;f)dx = \psi^\epsilon(t), \quad t\in[0,T], \tag{4.1}$$

here $l$ maps $L^2(0,T)$ into $L^2(0,T)$. Note that the measurement $\psi^\epsilon$ may simulate the measured data at the point $x=x_0\in(0,1)$, namely,

$$u(x_0,t)=\psi^\epsilon(t), \quad t\in[0,T]. \tag{4.2}$$

if we take $\omega(x)=\delta(x-x_0)$, where $\delta(\cdot)$ is the Dirac delta function. However, since in this section the solution $u$ to the direct problem (1.4) is defined in the weak sense, the pointwise value of $u(x_0,t)$ does not make sense. Therefore, we consider the determination of the unknown source term $f(t)$ from (4.1) rather than from (4.2). Assume further that $\omega \in V$ and that conditions (b) and (c) are satisfied. Then, according to Theorem 2.5, there exists a unique solution $(u, f) \in \mathcal{V} \times L^2(0,T)$ of the inverse problem given by Eqs. (1.4) and (1.5).

The Tikhonov regularization can be utilized to solve the operator Eq. (4.1). This is based on minimizing the objective functional $J : L^2(0,T) \mapsto \mathbb{R}_+$ defined as:

$$J(f) := \frac{1}{2}\|l(f) - \psi^\epsilon\|^2_{L^2(0,T)} + \frac{\beta}{2}\|f\|^2_{L^2(0,T)} = \frac{1}{2}\int_0^T \left|\int_0^1 \omega(x)u(x,t;f)dx - \psi^\epsilon(t)\right|^2 dt + \frac{\beta}{2}\|f\|^2_{L^2(0,T)}, \tag{4.3}$$

where $\beta \geq 0$ is the regularization parameter, and $u(x,t)$ is the weak solution to the problem (1.4) satisfying the identity

$$-\int_{Q_T} u\eta_t dxdt + \int_{Q_T} u_{xx}\eta_{xx} dxdt = \int_{Q_T} (fg + h)\eta dxdt + \int_0^1 \phi(x)\eta(x,0)dx, \tag{4.4}$$

for all $\eta \in H^1(0,T; H^2(0,1))$ with $\eta|_{x\in\{0,1\}} = 0$ and $\eta|_{t=T} = 0$.

According to Theorem 2.1 and the estimate (2.1), for any $f \in L^2(0,T)$, we have

$$\|l(f)\|_{L^2(0,T)} \leq \|\omega\|_{L^2(0,1)}\|u\|_{L^2(Q_T)} \leq C\|\omega\|_{L^2(0,1)}\left(\|fg + h\|_{L^2(Q_T)} + \|\phi\|_{L^2(0,1)}\right).$$

In summary, the operator $l : f \mapsto \int_0^1 \omega(x)u(x,t;f)dx$ is linear and bounded from $L^2(0,T)$ to $L^2(0,T)$.

We next prove that the objective functional $J(f)$ is Fréchet differentiable. For this, we introduce the adjoint problem given by:

$$\begin{cases} -\lambda_t + \lambda_{xxxx} = \omega(x)\left(\int_0^1 \omega(x)u(x,t)dx - \psi^\epsilon(t)\right), & (x,t) \in Q_T, \\ \lambda(0,t) = \lambda(1,t) = \lambda_{xx}(0,t) = \lambda_{xx}(1,t) = 0, & t \in (0,T), \\ \lambda|_{t=T} = 0, & x \in [0,1]. \end{cases} \tag{4.5}$$

The problem (4.5) has a solution $\lambda \in \mathcal{V}$ by Theorem 2.1, since $\omega \in L^2(0,1)$, $u \in L^2(0,T; H^2(0,1)) \hookrightarrow L^2(0,T; L^\infty(0,1))$ and $\psi^\epsilon \in L^2(0,T)$.

**Theorem 4.1.** *Let the assumptions of Theorem 2.5 hold. Then, the objective functional $J(f)$ given by (4.3) is Fréchet differentiable, and the Fréchet derivative at $f \in L^2(0,T)$ is given by*

$$J'(f) = \int_0^1 \lambda(x,t)g(x,t)dx + \beta f, \tag{4.6}$$

*where $\lambda$ satisfies the adjoint problem (4.5). Moreover, the gradient $J'(f)$ is Lipschitz continuous.*

**Proof.** For $f \in L^2(0,T)$ take any increment $\delta f \in L^2(0,T)$ and denote $\delta u := u(f + \delta f) - u(f)$. Then, $\delta u$ satisfies the following problem:

$$\begin{cases} (\delta u)_t + (\delta u)_{xxxx} = \delta f(t)g(x,t), & (x,t) \in Q_T, \\ \delta u(0,t) = \delta u(1,t) = (\delta u)_{xx}(0,t) = (\delta u)_{xx}(1,t) = 0, & t \in (0,T), \\ \delta u(x,0) = 0, & x \in (0,1). \end{cases} \tag{4.7}$$

Applying Theorem 2.1 it follows that the problem (4.7) has a unique solution $\delta u \in \mathcal{V}$. Multiplying (4.7) by $\delta u$ and integrating over $(0,1)$, we get

$$\frac{1}{2}\frac{d}{dt}\|\delta u(\cdot,t)\|^2_{L^2(0,1)} + \|(\delta u)_{xx}(\cdot,t)\|^2_{L^2(0,1)} \leq \frac{1}{2}\|\delta f(t)g(\cdot,t)\|^2_{L^2(0,1)} + \frac{1}{2}\|\delta u(\cdot,t)\|^2_{L^2(0,1)}, \quad t \in [0,T].$$

Then, Gronwall's inequality implies that

$$\|\delta u\|_{L^2(Q_T)} \leq e^{T/2}\|\delta f g\|_{L^2(Q_T)} \leq e^{T/2}\|\delta f\|_{L^2(0,T)}\|g\|_{L^\infty(0,T;L^2(0,1))}. \tag{4.8}$$

Denoting $\delta J := J(f + \delta f) - J(f)$ and using (4.3), we have

$$\begin{aligned} \delta J = &\int_0^T \int_0^1 \omega(x)\delta u(x,t)\left(\int_0^1 \omega(x)u(x,t)dx - \psi^\epsilon(t)\right)dxdt \\ &+ \beta\int_0^T f(t)\delta f(t)dt + \frac{1}{2}\left\|\int_0^1 \omega(x)\delta u(x,t)dx\right\|^2_{L^2(0,T)} + \frac{\beta}{2}\|\delta f\|^2_{L^2(0,T)}. \end{aligned}$$

Using integration by parts in the first term and the homogeneous boundary conditions to the problems (4.5) and (4.7), we obtain

$$\int_0^T \int_0^1 \omega(x)\delta u(x,t)\left(\int_0^1 \omega(x)u(x,t)dx - \psi^\epsilon(t)\right)dxdt = \int_0^T \int_0^1 \lambda((\delta u)_t + (\delta u)_{xxxx})dxdt = \int_0^T \int_0^1 \delta f \lambda g dxdt, \tag{4.9}$$

Then, using the inequality (4.8), the third term satisfies the estimate

$$\left\|\int_0^1 \omega(x)\delta u(x,t)dx\right\|^2_{L^2(0,T)} \le \|\omega\|^2_{L^2(0,1)}\|\delta u\|^2_{L^2(Q_T)} \le e^T\|\omega\|^2_{L^2(0,1)}\|\delta f\|^2_{L^2(0,T)}\|g\|^2_{L^\infty(0,T;L^2(0,1))}.$$

Consequently, we have

$$\delta J = \int_0^T \delta f\left(\int_0^1 \lambda g dx + \beta f\right)dt + \mathcal{O}(\|\delta f\|^2_{L^2(0,T)}), \tag{4.10}$$

which means that the Fréchet derivative $J'(f)$ is given by (4.6).

Using (4.6), we have

$$J'(f+\delta f) - J'(f) = \int_0^1 \bar{\lambda} g dx + \beta\delta f \le \|\bar{\lambda}(\cdot,t)\|_{L^2(0,1)}\|g(\cdot,t)\|_{L^2(0,1)} + \beta\delta f, \tag{4.11}$$

where $\bar{\lambda}$ satisfies the problem

$$\begin{cases} -\bar{\lambda}_t + \bar{\lambda}_{xxxx} = \omega(x)\int_0^1 \omega(x)\delta u(x,t)dx, & (x,t)\in Q_T,\\ \bar{\lambda}(0,t) = \bar{\lambda}(1,t) = \bar{\lambda}_{xx}(0,t) = \bar{\lambda}_{xx}(1,t) = 0, & t\in(0,T),\\ \bar{\lambda}|_{t=T} = 0, & x\in[0,1], \end{cases} \tag{4.12}$$

where $\delta u(x,t)$ is the solution of the problem (2.14). By (4.8), the solution $\bar{\lambda}$ to (4.12) satisfies the estimate

$$\|\bar{\lambda}\|_{L^2(Q_T)} \le e^{T/2}\|\omega\|^2_{L^2(0,1)}\|\delta u\|_{L^2(Q_T)} \le e^T\|\omega\|^2_{L^2(0,1)}\|\delta f\|_{L^2(0,T)}\|g\|_{L^\infty(0,T;L^2(0,1))}.$$

Applying the above inequality, then (4.11) satisfies the estimate

$$\|J'(f+\delta f) - J'(f)\|_{L^2(0,T)} \le \|g\|_{L^\infty(0,T;L^2(0,1))}\|\bar{\lambda}\|_{L^2(Q_T)} + \beta\|\delta f\|_{L^2(0,T)} \le L\|\delta f\|_{L^2(0,T)},$$

where $L = e^T\|\omega\|^2_{L^2(0,1)}\|g\|^2_{L^\infty(0,T;L^2(0,1))} + \beta > 0$ is a positive constant independent of $f$ and $\delta f$. Thus the gradient $J'(f)$ is Lipschitz continuous and the proof is complete. □

**Theorem 4.2.** *Suppose that $f,\ f+\delta f \in L^2(0,T)$, then the Fréchet gradient (4.6) of the objective functional (4.3) satisfies*

$$\left\langle J'(f+\delta f) - J'(f), \delta f\right\rangle_{L^2(0,T)} = \left\|\int_0^1 \omega(x)\delta u(x,t)dx\right\|^2_{L^2(0,T)} + \beta\|\delta f\|^2_{L^2(0,T)}, \tag{4.13}$$

*where $\delta u$ satisfies the problem (4.7), and $\langle\cdot,\cdot\rangle_{L^2(0,T)}$ indicates the inner product in $L^2(0,T)$.*

**Proof.** Using the Fréchet derivative (4.6) and the problem (4.12), for any $f,\ f+\delta f \in L^2(0,T)$, we have

$$\left\langle J'(f+\delta f) - J'(f), \delta f\right\rangle_{L^2(0,T)} = \int_{Q_T} \bar{\lambda} g\delta f dxdt + \beta\|\delta f\|^2_{L^2(0,T)}. \tag{4.14}$$

From (4.7), we get

$$\int_{Q_T} \bar{\lambda} g\delta f dxdt = \int_{Q_T} \bar{\lambda}((\delta u)_t + (\delta u)_{xxxx})dxdt.$$

By the initial and boundary conditions of problems (4.7) and (4.12), and using integration by parts, we have

$$\int_{Q_T} \bar{\lambda} g\delta f dxdt = \int_{Q_T} \delta u(-\bar{\lambda}_t + \bar{\lambda}_{xxxx})dxdt = \int_{Q_T} \delta u\left(\omega(x)\int_0^1 \omega(x)\delta u(x,t)dx\right)dxdt = \left\|\int_0^1 \omega(x)\delta u(x,t)dx\right\|^2_{L^2(0,T)}.$$

Therefore, (4.13) can be derived by combining the above result with (4.14). □

**Remark 4.1.** For $\beta > 0$, expression (4.13) implies that the functional (4.3) is strictly convex, hence it has a unique minimizer, which is given by [36, Theorem 2.12],

$$L^2(0,T) \ni f_\epsilon^\beta = (l^*l + \beta I)^{-1}l^*\psi^\epsilon, \tag{4.15}$$

where $l^* : L^2(0,T) \mapsto L^2(0,T)$ is the adjoint operator of $l$.

**Remark 4.2.** For the problem (2.3) with inhomogeneous boundary conditions, we can also define the objective functional $J(f)$ given by (4.3) to transfer the inverse problem into an optimization problem, and by the same arguments illustrated in the above theorem, the functional is Fréchet differentiable and shares the same form of gradient (4.6).

Since the solution of the inverse problem given by Eqs. (1.4) and (1.5) can be approximated by the minimizer of the objective functional $J(f)$ defined by (4.3), then the following iteration process based on the CGM is used to recover the source term $f(t)$ by minimizing $J(f)$:

$$f^{n+1} = f^n + \alpha^n d^n, \quad n \in \mathbb{N}, \tag{4.16}$$

where the subscript $n$ denotes the number of iterations, $f^0(t)$ is the initial guess for the coefficient $f(t)$, $\alpha^n$ is the search step size, also known as the learning rate, in passing from iteration $n$ to $n+1$, and $d^n$ is the search direction given by:

$$d^0 = -J'(f^0), \quad d^n = -J'(f^n) + \gamma^n d^{n-1}, \quad n \in \mathbb{N}^*, \tag{4.17}$$

and $\gamma^n$ is the conjugate coefficient given by:

$$\gamma^n = \frac{\|J'(f^n)\|_{L^2(0,T)}}{\|J'(f^{n-1})\|_{L^2(0,T)}}, \quad n \in \mathbb{N}^*, \tag{4.18}$$

and the search step size $\alpha^n$ is determined by

$$\alpha^n = \arg\min_{\alpha \geq 0} J(f^n + \alpha d^n).$$

In order to obtain $\alpha^n$, we consider the objective functional (4.3) with $f^{n+1}$, i.e.

$$J(f^{n+1}) = \frac{1}{2}\int_0^T \left| \int_0^1 \omega(x) u(f^n + \alpha^n d^n)(x,t)dx - \psi^\epsilon(t) \right|^2 dt + \frac{\beta}{2}\|f^n + \alpha^n d^n\|^2_{L^2(0,T)}.$$

Then, we have

$$\begin{aligned}\frac{\partial J(f^{n+1})}{\partial \alpha^n} &= \lim_{\delta\alpha \to 0} \frac{J(f^n + (\alpha^n + \delta\alpha)d^n) - J(f^n + \alpha^n d^n)}{\delta\alpha} \\ &= \int_0^T \left( \int_0^1 \omega u^n dx - \psi^\epsilon \right) \int_0^1 \omega \delta u^n dx dt + \beta \int_0^T f^n d^n dt \alpha^n \left\| \int_0^1 \omega \delta u^n dx \right\|^2_{L^2(0,T)} + \alpha^n \beta \|d^n\|^2_{L^2(0,T)},\end{aligned}$$

where $u^n := u(f^n)$, and $\delta u^n$ satisfies the following sensitivity problem:

$$\begin{cases} (\delta u^n)_t + (\delta u^n)_{xxxx} = d^n g(x,t), & (x,t) \in Q_T, \\ \delta u^n(0,t) = \delta u^n(1,t) = (\delta u^n)_{xx}(0,t) = (\delta u^n)_{xx}(1,t) = 0, & t \in (0,T), \\ \delta u^n(x,0) = 0, & x \in (0,1). \end{cases} \tag{4.19}$$

Using (4.9) with $d^n = \delta f$, we have

$$\int_0^T \left( \int_0^1 \omega u(f^n) dx - \psi^\epsilon \right) \int_0^1 \omega \delta u^n dx dt = \int_0^T \int_0^1 d^n \lambda(f^n) g dx dt.$$

This identity and the gradient $J'(f)$ given by (4.6) imply that

$$\frac{\partial J(f^{n+1})}{\partial \alpha^n} = \int_0^T J'(f^n) d^n dt + \alpha^n \left\| \int_0^1 \omega \delta u^n dx \right\|^2_{L^2(0,T)} + \alpha^n \beta \|d^n\|^2_{L^2(0,T)}.$$

Setting $\frac{\partial J(f^{n+1})}{\partial \alpha^n} = 0$, we obtain

$$\alpha^n = -\frac{\langle J'(f^n), d^n \rangle_{L^2(0,T)}}{\left\| \int_0^1 \omega \delta u^n dx \right\|^2_{L^2(0,T)} + \beta \|d^n\|^2_{L^2(0,T)}}, \quad n \in \mathbb{N}.$$

Using (4.10) with $\delta f = \alpha^n d^n$, we have

$$\begin{aligned}\frac{\partial J(f)}{\partial \alpha^n} &= \lim_{\alpha^n \to 0} \frac{J(f + \alpha^n d^n) - J(f)}{\alpha^n} \\ &= \lim_{\alpha^n \to 0} \frac{1}{\alpha^n} \left( \int_0^T \alpha^n d^n \left( \int_0^1 \lambda(f) g dx + \beta f^n \right) dt + \mathcal{O}(\|\alpha^n d^n\|^2_{L^2(0,T)}) \right) = \left\langle J'(f), d^n \right\rangle_{L^2(0,T)},\end{aligned}$$

and thus

$$\frac{\partial J(f^{n+1})}{\partial \alpha^n} = \left\langle J'(f^{n+1}), d^n \right\rangle_{L^2(0,T)} = 0.$$

Then, using (4.17), we obtain that

$$\alpha^n = \frac{\|J'(f^n)\|^2_{L^2(0,T)}}{\left\| \int_0^1 \omega \delta u^n dx \right\|^2_{L^2(0,T)} + \beta \|d^n\|^2_{L^2(0,T)}}, \quad n \in \mathbb{N}. \tag{4.20}$$

In case $\beta = 0$, for achieving regularization, the iteration process is stopped when the following discrepancy criterion is satisfied:

$$J(f^n) \le \tau\epsilon^2/2, \tag{4.21}$$

where $\tau$ is a safeguarding constant greater than one.

In summary, the CGM iterative algorithm for numerically reconstructing of the unknown source term $f(t)$ is as follows:

**Step 1.** Set $n = 0$ and choose an arbitrary initial guess $f^0 \in L^2(0,T)$.

**Step 2.** Solve the direct problem (1.4) for $u(f^n)$ and calculate the objective functional $J(f^n)$ given by (4.3).

**Step 3.** Solve the adjoint problem (4.5) (changing also $t \mapsto T - t$) to obtain $\lambda(f^n)$ and the derivative $J'(f^n)$ by (4.6). Compute the conjugate coefficient $\gamma^n$ by (4.18) and the search direction $d^n$ in (4.17).

**Step 4.** Solve the sensitivity problem (4.19) to obtain $\delta u^n$ numerically with $d^n$, and calculate the search direction $\alpha^n$ by (4.20).

**Step 5.** Update $f^{n+1}$ by (4.16).

**Step 6.** In case $\beta = 0$, if the stopping criterion (4.21) is satisfied, then go to **Step 7**. Else set $n = n + 1$, and go to **Step 2**.

**Step 7.** End.

**Remark 4.3.** In case of no noise, i.e. $\epsilon = 0$ in (2.41), if the exact data $\psi$ belongs to the range of the operator defined in (4.1), then the CGM with $\beta = 0$, described above, converges to the unique solution of the inverse problem given by Eqs. (1.4) and (1.5), [37, Theorem 7.9].

**Remark 4.4.** If $\beta = 0$, it is obvious that $J'(f)(T) = 0$ from (4.5) and (4.6). This shows that if the terminal time value $f^0(T)$ is not specified as the true value of $f(T)$, the numerical results of $f(t)$ will deviate from the exact values near the final time $t = T$. In order to avoid such shortcoming, we shall record data (1.5) a little longer, say, up to $t = \bar{T} > T$.

## 5. Numerical results and discussions

In this section, we consider the numerical determination of $f(t)$ by utilizing the time-discrete method prescribed in Section 3 and the CGM prescribed in Section 4, combined with the finite-difference scheme. Thus, we first establish the finite-difference method (FDM) to obtain the numerical solution of the following direct initial–boundary value problem:

$$\begin{cases} u_t + u_{xxxx} = F(x,t), & (x,t) \in Q_T, \\ u(0,t) = \mu_1(t), \quad u(1,t) = \mu_2(t), & t \in (0,T), \\ u_{xx}(0,t) = \mu_3(t), \quad u_{xx}(1,t) = \mu_4(t), & t \in (0,T), \\ u(x,0) = \phi(x), & x \in (0,1). \end{cases} \tag{5.1}$$

The main dependent variable to be determined is the function $u(x,t)$. When the source term $F$ is given by $F(x,t) = f(t)g(x,t) + h(x,t)$, the problem (5.1) becomes the direct problem (1.4) when $\mu_1(t) = \mu_2(t) = \mu_3(t) = \mu_4(t) = 0$.

Divide the domain $[0,1] \times [0,T]$ into the uniform grid:

$$x_i = (i-1)\Delta x, \quad i = \overline{1,I}, \quad t_k = (k-1)\Delta t, \quad k = \overline{1,K},$$

where the space and time mesh step sizes are given by $\Delta x = \frac{1}{I-1}$ and $\Delta t = \frac{T}{K-1}$, and denote the values of $u(x,t)$, $F(x,t)$, $\mu_i(t)$ for $i = \overline{1,4}$, and $\phi(x)$ at the node $(i,k)$ by:

$$u_i^k = u(x_i,t_k), \quad F_i^k = F(x_i,t_k), \quad \phi_i = \phi(x_i), \quad \mu_1^k = \mu_1(t_k), \quad \mu_2^k = \mu_2(t_k), \quad \mu_3^k = \mu_3(t_k), \quad \mu_4^k = \mu_4(t_k).$$

Also, denote

$$F_i^{k-\frac{1}{2}} = \frac{F_i^k + F_i^{k-1}}{2}, \quad \delta_t u_i^{k-\frac{1}{2}} = \frac{u_i^k - u_i^{k-1}}{\Delta t}, \quad \delta_x^2 u_i^k = \frac{u_{i-1}^k - 2u_i^k + u_{i+1}^k}{(\Delta x)^2}, \quad \delta_x^4 u_i^k = \delta_x^2(\delta_x^2 u_i^k).$$

Then, the Crank–Nicolson scheme for the initial–boundary value problem (5.1) is given by

$$\begin{cases} \delta_t u_i^{k-\frac{1}{2}} + \frac{1}{2}(\delta_x^4 u_i^k + \delta_x^4 u_i^{k-1}) = F_i^{k-\frac{1}{2}}, & i = \overline{2,I-1}, \ k = \overline{2,K}, \\ u_1^k = \mu_1^k, \quad u_I^k = \mu_2^k, & k = \overline{2,K}, \\ u_i^1 = \phi_i, & i = \overline{1,I}, \end{cases} \tag{5.2}$$

where, for $i = \overline{3,I-2}$:

$$\delta_x^4 u_i^k = \delta_x^2(\delta_x^2 u_i^k) = \delta_x^2 \left( \frac{1}{(\Delta x)^2}(u_{i-1}^k - 2u_i^k + u_{i+1}^k) \right) \frac{1}{(\Delta x)^4}(u_{i-2}^k - 4u_{i-1}^k + 6u_i^k - 4u_{i+1}^k + u_{i+2}^k), \quad k = \overline{2,K}, \tag{5.3}$$

for $i = 2$:

$$\delta_x^4 u_2^k = \frac{1}{(\Delta x)^2}(\delta_x^2 u_1^k - 2\delta_x^2 u_2^k + \delta_x^2 u_3^k) = \frac{1}{(\Delta x)^2}(\mu_3^k - 2\delta_x^2 u_2^k + \delta_x^2 u_3^k) = \frac{1}{(\Delta x)^4}((\Delta x)^2 \mu_3^k - 2u_1^k + 5u_2^k - 4u_3^k + u_4^k), \quad k = \overline{2,K}, \tag{5.4}$$

and for $i = I-1$:

$$\delta_x^4 u_{I-1}^k = \frac{1}{(\Delta x)^2}(\delta_x^2 u_{I-2}^k - 2\delta_x^2 u_{I-1}^k + \delta_x^2 u_I^k) = \frac{1}{(\Delta x)^2}(\delta_x^2 u_{I-2}^k - 2\delta_x^2 u_{I-1}^k + \mu_4^k)$$
$$= \frac{1}{(\Delta x)^4}(u_{I-3}^k - 4u_{I-2}^k + 5u_{I-1}^k - 2u_I^k + (\Delta x)^2 \mu_4^k), \quad k = \overline{2, K}. \tag{5.5}$$

From (5.3)–(5.5), the difference system (5.2) can be reformulated as a $(I-2)\times(I-2)$ system of linear algebraic equations of the form:

$$\begin{cases} A\mathbf{u}^k = B\mathbf{u}^{k-1} + \mathbf{F}^{k-1}, & k = \overline{2, K}, \\ u_1^k = \mu_1^k, \quad u_I^k = \mu_2^k, & k = \overline{2, K}, \\ \mathbf{u}^1 = [\phi_1, \phi_2, \dots, \phi_I]^{\mathrm{T}}, \end{cases} \tag{5.6}$$

where $\mathbf{u}^k = [u_2^k, u_3^k, \dots, u_{I-1}^k]^{\mathrm{T}}$,

$$A = \begin{bmatrix} 1+5r & -4r & r & & & & \\ -4r & 1+6r & -4r & r & & & \\ r & -4r & 1+6r & -4r & r & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & r & -4r & 1+6r & -4r & r \\ & & & r & -4r & 1+6r & -4r \\ & & & & r & -4r & 1+5r \end{bmatrix}, \quad B = \begin{bmatrix} 1-5r & 4r & -r & & & & \\ 4r & 1-6r & 4r & -r & & & \\ -r & 4r & 1-6r & 4r & -r & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -r & 4r & 1-6r & 4r & -r \\ & & & -r & 4r & 1-6r & 4r \\ & & & & -r & 4r & 1-5r \end{bmatrix},$$

and

$$\mathbf{F}^{k-1} = \begin{bmatrix} \Delta t F_2^{k-\frac{1}{2}} + 2r(\mu_1^{k-1} + \mu_1^k) - r(\Delta x)^2(\mu_3^{k-1} + \mu_3^k) \\ \Delta t F_3^{k-\frac{1}{2}} - r(\mu_1^{k-1} + \mu_1^k) \\ \Delta t F_4^{k-\frac{1}{2}} \\ \cdots \\ \Delta t F_{I-3}^{k-\frac{1}{2}} \\ \Delta t F_{I-2}^{k-\frac{1}{2}} - r(\mu_2^{k-1} + \mu_2^k) \\ \Delta t F_{I-1}^{k-\frac{1}{2}} + 2r(\mu_2^{k-1} + \mu_2^k) - r(\Delta x)^2(\mu_4^{k-1} + \mu_4^k) \end{bmatrix},$$

where $r = \frac{\Delta t}{2(\Delta x)^4}$. The finite-difference scheme (5.6) to numerically solve the initial–boundary value problem (5.1) is consistent and convergent of second-order $\mathcal{O}((\Delta t)^2 + (\Delta x)^2)$ in both space and time. Higher-order convergence can be obtained by employing semi-discrete time-dependent compact schemes [38].

The direct, sensitivity and adjoint problems (1.4), (4.5) and (4.19), respectively, involved in the CGM are solved by using the FDM (5.6) under the assumption that their classical strong solutions exist; otherwise, the finite element method becomes an applicable alternative [39]. The trapezoidal rule is employed to deal with all the integrals involved. In addition, the accuracy error, as a function of the iteration number $n$, is defined as

$$E(f^n) = \|f - f^n\|_{L^2(0,T)}, \tag{5.7}$$

where $f^n$ is the $n$th iterate and $f$ is the true source term, if available. The noisy integral observation $\psi^\epsilon$ is simulated by adding to the exact data $\psi^{\text{exact}}$ errors given by

$$\psi^\epsilon = \psi^{\text{exact}} + \sigma \times \text{random}(1), \tag{5.8}$$

where $\sigma = \frac{p}{100} \times \max_{t\in[0,T]} |\psi^{\text{exact}}(t)|$ is the standard deviation, $p\%$ represents the percentage of noise and random(1) generates random values from a Gaussian normal distribution with zero mean and standard deviation equal to unity. Next, two numerical examples are considered to reconstruct the unknown source term $f(t)$ from the integral observation (5.8).

**Example 1.** In this example, we consider the input data

$$g(x,t) = \pi^4 \sin(\pi x), \quad h(x,t) = e^t \sin(\pi x), \quad \phi(x) = \sin(\pi x), \quad \omega(x) = \sin(\pi x), \quad \mu_i(t) = 0, \ i = \overline{1,4}, \quad \psi(t) = e^t/2.$$

With this data the analytical solution to the inverse problem (1.4) and (1.5) is given by

$$f(t) = e^t, \quad u(x,t) = e^t \sin(\pi x). \tag{5.9}$$

It is obvious that $\omega$ satisfies condition (a). We take the terminal time $T = 1$, and the mesh sizes $\Delta x = 0.01$ and $\Delta t = 0.0001$.

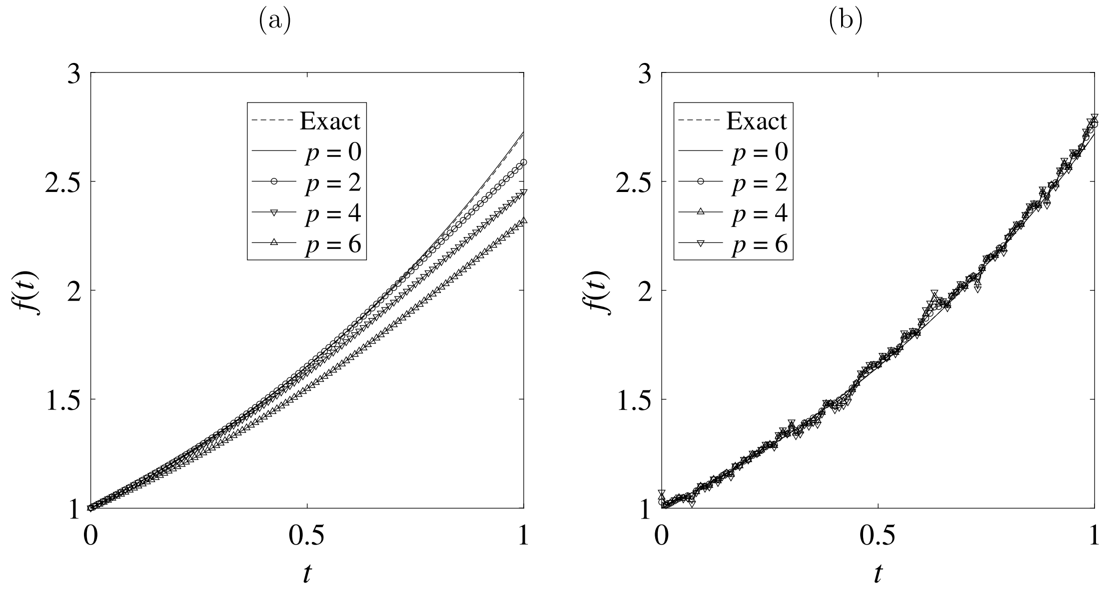


**Fig. 1.** The exact and numerical results for $f(t)$ obtained using (a) the time-discrete method and (b) the CGM, for $p \in \{0,2,4,6\}$ noise, for Example 1.

We first apply the time-discrete method described in Section 3 regularized by the cubic spline function method to reconstruct $f(t)$. The numerical results for $f(t)$ obtained with $\alpha = \epsilon^2$ are presented in Fig. 1(a). The accuracy errors are $E(f) \in \{0.0053, 0.0332, 0.0902, 0.1678\}$ for the percentages of noise $p\% \in \{0,2,4,6\}\%$, respectively.

Secondly, we apply the CGM. We take $\bar{T} = 1.2 > T = 1$ to avoid the stagnation at the final time. We also take $\beta = 0$ in (4.3) and employ the discrepancy principle criterion (4.21) (with $\tau \approx 1$) for stopping the iterations. The initial guess is chosen arbitrary, say $f^0(t) = 2$. Fig. 1(b) shows the numerical solutions for $f(t)$ at the stopping iteration numbers $n^* \in \{20,2,2,2\}$ for $p \in \{0,2,4,6\}$, respectively. In the case of no noise $p = 0$, the exact and numerical solutions overlap, which are graphically undistinguishable. The accuracy errors are $E(f^{n^*}) \in \{0.0005, 0.0171, 0.0257, 0.0355\}$ with $p \in \{0,2,4,6\}$ noise, respectively. From Fig. 1 it can be seen that stable and accurate solutions are obtained for the unknown source term $f(t)$.

Compared with the time-discrete method, the weight function $\omega(x)$ or the mesh size $\Delta t$ can be chosen arbitrary in the CGM. For instance, for $p = 0$, the accuracy error is $E(f^{20}) \in \{5.2347\times10^{-4}, 5.2109\times10^{-4}, 5.2300\times10^{-4}, 5.6716\times10^{-4}\}$ with the weights $\omega(x) \in \{1, x, x^2, e^x\}$, respectively. Also, for $p = 0$, for the mesh size $\Delta t \in \{0.01, 0.005, 0.001, 0.0005, 0.0001\}$ and $\Delta x = 0.01$, the corresponding accuracy errors $\{0.0013, 6.0157\times10^{-4}, 5.6810\times10^{-4}, 5.1657\times10^{-4}, 4.9899\times10^{-4}\}$ for the CGM obtained after 20 iterations are lower than the errors $\{0.4322, 0.2823, 0.0664, 0.0281, 0.0053\}$ of the time-discrete method. From Figs. 1(a) and 1(b), it can be seen that the numerical approximations for the time-discrete method are smooth since they are based on cubic splines, whilst for the CGM, the approximate solution given in (4.15) is only in $L^2(0,T)$ and hence it is expected to be non-smooth for random noisy data (5.8).

**Example 2.** The inverse problem given by Eqs. (1.5) and (2.3) with inhomogeneous boundary conditions is considered to reconstruct a piecewise continuous source term $f(t)$. We take

$$g(x,t) = e^{-t}(\pi^4(\sin(\pi x) + \cos(\pi x)) - 3) \times \begin{cases} 1, & t \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ -1, & \text{otherwise,} \end{cases}$$

$$h(x,t) = -e^{-t}(\sin(\pi x) + \cos(\pi x)), \quad \phi(x) = \sin(\pi x) + \cos(\pi x) + 3, \quad \omega(x) = 1,$$

$$\mu_1(t) = 4e^{-t}, \quad \mu_2(t) = 2e^{-t}, \quad \mu_3(t) = -\pi^2 e^{-t}, \quad \mu_4(t) = \pi^2 e^{-t}, \quad \psi(t) = \left(3 + \frac{2}{\pi}\right) e^{-t}.$$

The analytical solution to the inverse problem given by Eqs. (1.5) and (2.3) is given by

$$f(t) = \begin{cases} 1, & t \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ -1, & \text{otherwise,} \end{cases} \qquad u(x,t) = e^{-t}(\sin(\pi x) + \cos(\pi x) + 3). \tag{5.10}$$

Compared with the previous Example 1, the inverse problem considered in this example is more severe since the source term to be retrieved is a discontinuous function at the time instants $t = 1/4$ and $t = 3/4$. We only use the CGM based on minimizing (4.3) with $\beta = 0$, to reconstruct $f(t)$ with mesh sizes $\Delta x = \Delta t = 0.01$ and $\bar{T} = 1.2$. Here the initial guess for determining $f(t)$

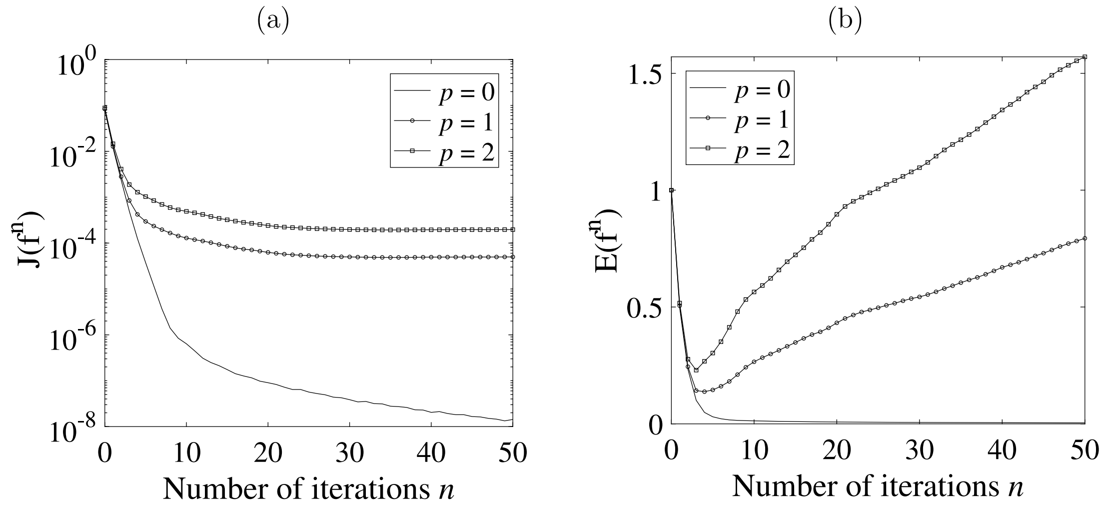


**Fig. 2.** (a) The objective functional $J(f^n)$ given by (4.3) with $\beta = 0$ and (b) the accuracy error $E(f^n)$ defined by (5.7), for $p \in \{0, 1, 2\}$ noise, for Example 2.

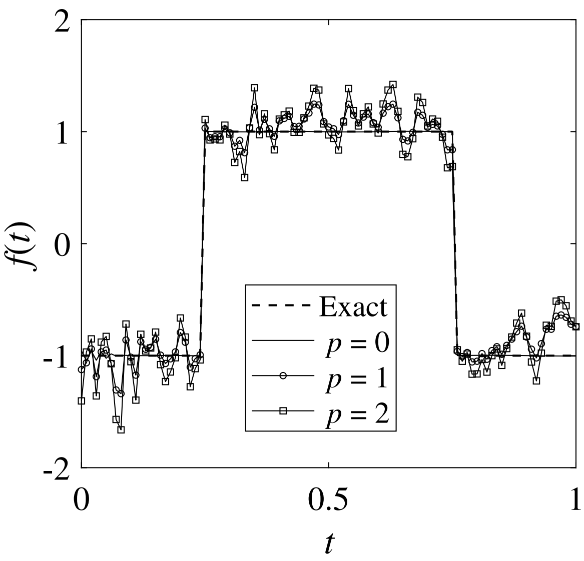


**Fig. 3.** The exact and numerical results for the source $f(t)$, for $p \in \{0, 1, 2\}$ noise, for Example 2.

is chosen as $f^0(t) = 0$. For the choices of the weight function $\omega(x) \in \{x, x^2, e^x, \sin(\pi x)\}$ and $p = 0$ noise, we calculate the error $E(f^{50}) \in \{0.0053, 0.0065, 0.0042, 0.0045\}$ after 50 iteration numbers.

The behaviour of the objective functional $J(f^n)$ defined by (4.3) with $\beta = 0$ to determine the unknown source $f(t)$, treated as a function of the iteration number $n$, is presented in Fig. 2(a) for the noise level $p \in \{0, 1, 2\}$. It is easy to observe that the objective functional is a monotonic decreasing function of the iteration number $n$, which converges to a small positive value rapidly. By utilizing the discrepancy principle (4.21) (with $\tau \approx 1$) and $\frac{\epsilon^2}{2} \in \{1.3 \times 10^{-8}, 8.9590 \times 10^{-4}, 0.0036\}$, we obtain the stopping iteration numbers $n^* \in \{50, 3, 3\}$ for $p \in \{0, 1, 2\}$, respectively. The accuracy error $E(f^n)$ defined by (5.7), as a function of $n$, is displayed in Fig. 2(b), with the values $\{0.0045, 0.1424, 0.2298\}$ for $p \in \{0, 1, 2\}$ noise, respectively. The numerical solutions for the unknown source term $f(t)$ are presented in Fig. 3 at the stopping iteration numbers $n^*$ with $p \in \{0, 1, 2\}$. From this figure it can be seen that the numerical solutions are stable and reasonably accurate bearing in mind the severely discontinuous source term.

## 6. Conclusion

In this paper, the determination of a time-dependent source term in a fourth-order parabolic problem related to thermal grooving by surface diffusion has been investigated from a given integral measurement. Based on the Fourier method of separating variables and the contraction mapping theorem, we obtain the well-posedness of the weak solution to the inverse problem for smooth data

(1.5). However, in practice the measured data is seldom smooth and therefore, the problem is still ill-posed in reality. To overcome the instability of numerically differentiating a noisy function, we first use the time-discrete method with cubic splines to obtain stably the unknown source term. The error estimate is also obtained under rigorous analysis. Another approach is the Tikhonov regularization method. Based on the Fréchet derivative of the objective functional, the CGM is applied to obtain the solution of the source term. Error estimates are possible based on the approach described in [40] for determining the source for the parabolic second-order heat equation.

Two numerical examples for continuous and discontinuous source term have been presented, and the discussion highlights that reasonably accurate and stable solutions of the time-dependent source term have been achieved by both methods. Error estimates for the numerical solutions when the data (1.5) is noisy are not established herein but we refer to the recent paper of Neubauer [41] where optimal convergence rates in the presence of discretizations and modelling errors have been obtained.

Future work will consider inverse problems for groove growth by surface subdiffusion modelled by fourth-order fractional equations [42].

## Data availability

No data was used for the research described in the article.

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