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INDUCED SUBGRAPHS AND TREE DECOMPOSITIONS I. EVEN-HOLE-FREE GRAPHS OF BOUNDED DEGREE

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Abstract. Treewidth is a parameter that emerged from the study of minor closed classes of graphs (i.e. classes closed under vertex and edge deletion, and edge contraction). It in some sense describes the global structure of a graph. Roughly, a graph has treewidth k if it can be decomposed by a sequence of noncrossing cutsets of size at most k into pieces of size at most $k+1$. The study of hereditary graph classes (i.e. those closed under vertex deletion only) reveals a different picture, where cutsets that are not necessarily bounded in size (such as star cutsets, 2-joins and their generalization) are required to decompose the graph into simpler pieces that are structured but not necessarily bounded in size. A number of such decomposition theorems are known for complex hereditary graph classes, including even-hole-free graphs, perfect graphs and others. These theorems do not describe the global structure in the sense that a tree decomposition does, since the cutsets guaranteed by them are far from being noncrossing. They are also of limited use in algorithmic applications.

We show that in the case of even-hole-free graphs of bounded degree the cutsets described in the previous paragraph can be partitioned into a bounded number of well-behaved collections. This allows us to prove that even-hole-free graphs with bounded degree have bounded treewidth, resolving a conjecture of Aboulker, Adler, Kim, Sintiari and Trotignon [arXiv:2008.05504]. As a consequence, it follows that many algorithmic problems can be solved in polynomial time for this class, and that even-hole-freeness is testable in the bounded degree graph model of property testing. In fact we prove our results for a larger class of graphs, namely the class of C_4-free odd-signable graphs with bounded degree.

1. Introduction

All graphs in this paper are finite and simple. A *hole* of a graph G is an induced cycle of G of length at least four. A graph is *even-hole-free* if it has no hole with an even number of vertices.

Even-hole-free graphs have been studied extensively; see [23] for a survey. The first polynomial time recognition algorithm for this class of graphs was obtained in [9]. This algorithm is based on a decomposition theorem from [8] that uses 2-joins and star, double star, and triple star cutsets to decompose the graph into simpler pieces. Later, a stronger decomposition theorem, using only star cutsets and 2-joins, was obtained in [12], leading to a faster recognition algorithm. Further improvements resulted in the best currently known algorithm with running time $\mathcal{O}(n^9)$ [6, 15]. This progress required deep insights into the behavior of even-hole-free graphs; however the global structure of graphs in this class is still not well understood. Moreover, there are several natural optimization problems whose complexity for this class remains unknown (among those are the vertex coloring problem and the maximum weight stable set problem). The key difficulty is to make use of star cutsets, and in particular to understand how several star cutsets in a given

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graph interact. In this paper we address this problem, by showing that star cutsets in an even-hole-free graph of bounded degree can be partitioned into a bounded number of well-behaved collections, which in turn allows us to bound the treewidth of such graphs.

Let $G = (V, E)$ be a graph. A *tree decomposition* (T, χ) of G is a tree T and a map $\chi : V(T) \to 2^{V(G)}$ such that the following hold:

(i) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(ii) For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
(iii) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

If (T, χ) is a tree decomposition of G and $V(T) = \{t_1, \ldots, t_n\}$, the sets $\chi(t_1), \ldots, \chi(t_n)$ are called the *bags of* (T, χ). The *width* of a tree decomposition (T, χ) is $\max_{t \in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted $\mathrm{tw}(G)$, is the minimum width of a tree decomposition of G.

Many NP-hard algorithmic problems can be solved in polynomial time in graphs with bounded treewidth. For a full discussion, see [5]. While tree decomposition s, and classes of graphs of bounded treewidth, play an important role in the study of graphs with forbidden minors, the problem of connecting tree decompositions with forbidden induced subgraphs has so far remained open. Clearly, in order to get a class of bounded treewidth, one needs to forbid, for example, large cliques, large complete bipartite graphs, large walls, and the line graphs of large walls. However, all of these obstructions , except for large cliques, contain even holes. Further, in [20], a bound on the treewidth of planar even-hole-free graphs was proven. On the other hand, [21] contains a construction of a family of even-hole-free graphs with no K_4, and with unbounded treewidth. The graphs in this construction have both unbounded degree and contain large clique minors. In [1] it was examined whether both of these are necessary. They show that any graph that excludes a fixed graph as a minor either has small treewidth or contains (as an induced subgraph) a large wall or the line graph of a large wall. This implies that even-hole-free graphs that exclude a fixed graph as a minor have bounded treewidth (generalizing the result of [20]). Furthermore, the following conjecture was made (and proved for subcubic graphs) in [1]:

Conjecture 1.1. *For every $\delta \geq 0$ there exists k such that even-hole-free graphs with maximum degree at most δ have treewidth at most k.*

The main result of the present paper is the proof of Conjecture 1.1, in fact, the following slight strengthening of it. We *sign* a graph G by assigning $0, 1$ weights to its edges. A graph G is *odd-signable* if there exists a signing such that every triangle and every hole in G has odd weight. Thus even-hole-free graphs are a subclass of odd-signable graphs.

Theorem 1.2. *For every $\delta \geq 0$ there exists k such that C_4-free odd-signable graphs with maximum degree at most δ have treewidth at most k.*

It follows from Theorem 1.2 that vertex coloring, maximum stable set, and many other NP-hard algorithmic problems can be solved in polynomial time for even-hole-free graphs with bounded maximum degree. Another consequence of Theorem 1.2 is that even-hole-freeness is testable in the bounded degree graph model of property testing, since even-hole-freeness is expressible in monadic second-order logic with modulo counting (CMSO) and CMSO is testable on bounded treewidth [4]. See [1] for an excellent survey that motivates the study of Conjecture 1.1 and surrounding problems, and in particular contains a detailed discussion of property testing algorithms.

1.1. **Outline of the proof of Theorem 1.2.** A graph G has bounded treewidth if and only if every connected component of G has bounded treewidth. Therefore, we prove that connected C_4-free odd-signable graphs with bounded degree have bounded treewidth.

In [14], a number of parameters tied to treewidth are discussed. Let G be a graph, let $c \in [\frac{1}{2}, 1)$, and let k be a nonnegative integer. For $S \subseteq V(G)$, a $(k, S, c)^*$-*separator* is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every component of $G \setminus X$ contains at most $c|S|$ vertices of S. The *separation number* $\text{sep}^*_c(G)$ is the minimum k such that there exists a $(k, S, c)^*$-separator for every $S \subseteq V(G)$. The separation number is related to treewidth through the following lemma.

Lemma 1.3 ([14]). *For every graph G and for all $c \in [\frac{1}{2}, 1)$, the following holds:*

$$sep^*_c(G) \leq tw(G) + 1 \leq \frac{1}{1-c} sep^*_c(G).$$

A set $S \subseteq V(G)$ is *d-bounded* if there exist $v_1, \ldots, v_{d'}$, with $d' \leq d$, such that $S \subseteq N^d[v_1] \cup \ldots \cup N^d[v_{d'}]$. For a graph G and weight function w on its vertices, if X is a subgraph of G or a subset of $V(G)$, then $w(X)$ is the sum of the weights of vertices in X. Let G be a graph and let $w : V(G) \to [0, 1]$ be a weight function of G such that $w(G) = 1$. By $w^{\max}$ we denote the maximum weight of a vertex in G. A set $Y \subseteq V(G)$ is a *(w, c, d)-balanced separator* of G if Y is d-bounded and $w(Z) \leq c$ for every component Z of $G \setminus Y$. The following lemma shows that if G is a graph with maximum degree δ and G has a (w, c, d)-balanced separator for every weight function $w : V(G) \to [0, 1]$ with $w(G) = 1$, then G has bounded treewidth.

Lemma 1.4. *Let δ, d be positive integers with $\delta \leq d$, let $c \in [\frac{1}{2}, 1)$, and let $\Delta(d) = d + d\delta + d\delta^2 + \ldots + d\delta^d$. Let G be a graph with maximum degree δ. Suppose that for every $w : V(G) \to [0, 1]$ with $w(G) = 1$ and $w^{\max} < \frac{1}{\Delta(d)}$, G has a (w, c, d)-balanced separator. Then, $tw(G) \leq \frac{1}{1-c}\Delta(d)$.*

Proof. Note that $\Delta(d)$ is an upper bound for the size of a d-bounded set in G. Let $S \subseteq V(G)$. If $|S| \leq \Delta(d)$, then S is a $(\Delta(d), S, c)^*$-separator of G. Now, assume $|S| > \Delta(d)$. Let $w_S : V(G) \to [0, 1]$ be such that $w_S(v) = \frac{1}{|S|}$ for $v \in S$ and $w_S(v) = 0$ for $v \in V(G) \setminus S$. Then, $w_S(G) = 1$ and $w_S^{\max} < \frac{1}{\Delta(d)}$, so G has a (w_S, c, d)-balanced separator. Specifically, for all $S \subseteq V(G)$ such that $|S| > \Delta(d)$, there exists a set X such that $|X| \leq \Delta(d)$, and $w_S(Z) \leq c$ for all components Z of $G \setminus X$. It follows that X is a $(\Delta(d), S, c)^*$-separator of G. Therefore, G has a $(\Delta(d), S, c)^*$-separator for every $S \subseteq V(G)$. It follows that $\text{sep}^*_c(G) \leq \Delta(d)$, and by Lemma 1.3, $\text{tw}(G) \leq \frac{1}{1-c}\Delta(d)$. ∎

In this paper, we prove that connected C_4-free odd-signable graphs with bounded degree have bounded treewidth. Specifically, we prove the following theorem:

Theorem 1.5. *Let δ, d be positive integers. Let G be a connected C_4-free odd-signable graph with maximum degree δ and let $w : V(G) \to [0, 1]$ be a weight function such that $w(G) = 1$. Let $f(2, \delta) = 2(\delta + 1)^2 + 1$, and let $c \in [\frac{1}{2}, 1)$. Assume that $d \geq 49\delta + 4f(2, \delta)\delta - 4$ and $(1 - c) + [w^{\max} + 3f(2, \delta)\delta 2^\delta(1 - c) + 2(\delta - 1)2^\delta(1 - c)](\delta + \delta^2) < \frac{1}{2}$. Then, G has a (w, c, d)-balanced separator.*

We can then prove our main result:

Theorem 1.6. *Let δ be a positive integer and let G be a connected C_4-free odd-signable graph with maximum degree δ. Then, there exists $c \in [\frac{1}{2}, 1)$ and positive integer $d \geq \delta$ such that $tw(G) \leq \frac{1}{1-c}(d + d\delta + d\delta^2 + \ldots + d\delta^d)$.*

Proof. Let $f(2, \delta) = 2(\delta + 1)^2 + 1$. Let d be an integer such that $d \geq 49\delta + 4f(2, \delta)\delta - 4$, and let $\Delta(d) = d + d\delta + d\delta^2 + \ldots + d\delta^d$. Note that there exists $c \in [\frac{1}{2}, 1)$ such that $(1 - c) + [\frac{1}{\Delta(d)} + 3f(2, \delta)\delta 2^\delta(1 - c) + 2(\delta - 1)2^\delta(1 - c)](\delta + \delta^2) < \frac{1}{2}$. Let $w : V(G) \to [0, 1]$ be a weight function of G such that $w(G) = 1$ and $w^{\max} < \frac{1}{\Delta(d)}$. Then by Theorem 1.5, G has a (w, c, d)-balanced separator. The result now follows from Lemma 1.4. ∎

Let us now discuss the main ideas of the proof of Theorem 1.5. We will give precise definitions of the concepts used below later in the paper; the goal of the next few paragraphs is just to give the reader a road map of where we are going. A *separation* (or *decomposition*) of a graph G is a triple of disjoint vertex sets (A, C, B) such that $A \cup C \cup B = V(G)$ and there are no edges from A to B. To "decompose along (A, C, B)" means to delete A. Usually, to prove a result that a certain graph family has bounded treewidth, one attempts to construct a collection of "non-crossing separations", which roughly means that the separations "cooperate" with each other, and the pieces that are obtained when the graph is simultaneously decomposed by all the separtions in the collection "line up" to form a tree structure. Such collections of separations are called "laminar."

In the case of C_4-free odd-signable graphs, there is a natural family of separations to turn to, given by Lemmas 4.4, 4.5, and 4.6. A key point here is that all the decompositions above are forced by the presence of certain induced subgraphs that we call "forcers." In essence it is shown that the corresponding decomposition of the forcer extends to the whole graph, and when the graph is decomposed along the decomposition, part of the forcer is removed.

Unfortunately, the decompositions above are very far from being non-crossing, and therefore we cannot use them in traditional ways to get tree decompositions. What turns out to be true, however, is that, due to the bound on the maximum degree of the graph, this collection of decompositions can be partitioned into a bounded number of laminar collections $X_1, \dots, X_p$ (where p depends on the maximum degree). We can then proceed as follows. Let G be a connected C_4-free odd-signable graph with maximum degree δ and let $w : V(G) \to [0, 1]$ be such that $w(G) = 1$. In view of Lemma 1.3, to prove Theorem 1.5, we would like to show that for certain c and d, G has a (w, c, d)-balanced separator; we may assume that no such separator exists. We first decompose G, simultaneously, by all the decompositions in X_1. Since X_1 is a laminar collection, by Lemma 2.1 this gives a tree decomposition of G, and we identify one of the bags of this decomposition as the "central bag" for X_1; denote it by β_1. Then, β_1 corresponds to an induced subgraph of G, and we can show that β_1 has no (w_1, c, d_1)-balanced separator for certain w_1 and d_1 that depend on w and d. We next focus on β_1, and decompose it using X_2, and so on. At step i, having decomposed by $X_1, \dots, X_i$, we focus on a central bag β_i that does not have a (w_i, c, d_i)-separator for suitably chosen w_i, d_i.

The fact that all the separations at play come from forcers ensures that at step i, after decomposing by $X_1, \dots, X_i$, none of the forcers that were "responsible" for the decompositions in $X_1, \dots, X_i$ is present in the central bag β_i (as part of each such forcer was removed in the decomposition process). It then follows that when we reach β_p, all we are left with is a "much simpler" graph (one that contains no forcers), where we can find a (w_p, c, d_p)-balanced separator directly, thus obtaining a contradiction.

The remainder of the paper is devoted to proving Theorem 1.5. In Section 1.2, we review key definitions and preliminaries. In Section 2, we define laminar collections of separations, and describe a tree decomposition corresponding to a laminar collection of separations. In Section 3, we prove results about clique cutsets and balanced separators. In Sections 4 and 5, we define forcers and prove results about forcers, star cutsets, and balanced separators. In Section 6, we prove a bound on separation number in graphs with no star cutset. Finally, in Section 7, we prove Theorem 1.5.

1.2. **Terminology and notation.** Let G and H be graphs. We say that G *contains* H if G has an induced subgraph isomorphic to H. We say that G is *H-free* if G does not contain H. If $\mathcal{H}$ is a set of graphs, we say that G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. For $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by X, and $G \setminus X = G[V(G) \setminus X]$. In this paper, we use induced subgraphs and their vertex sets interchangeably. Let $v \in V(G)$. The *open neighborhood of* v, denoted $N(v)$, is the set of all vertices in $V(G)$ adjacent to v. The *closed neighborhood of* v, denoted $N[v]$, is $N(v) \cup \{v\}$. Let $X \subseteq V(G)$. The *open neighborhood of* X, denoted $N(X)$, is the set of all vertices in $V(G) \setminus X$ with a neighbor in X. The *closed neighborhood of* X, denoted $N[X]$, is $N(X) \cup X$. If H is an induced subgraph of G and $X \subseteq H$, then $N_H(X)$ $(N_H[X])$ denotes the open (closed) neighborhood of X in H. Let $Y \subseteq V(G)$ be disjoint from X. Then, X is *anticomplete* to Y if there are no edges between X and Y. We use $X \cup v$ to mean $X \cup \{v\}$.

Given a graph G, a *path in* G is an induced subgraph of G that is a path. If P is a path in G, we write $P = p_1\text{-}\dots\text{-}p_k$ to mean that p_i is adjacent to p_j if and only if $|i-j| = 1$. We call the vertices p_1 and p_k the *ends of* P, and say that P is *from* p_1 *to* p_k. The *interior of* P, denoted by P^*, is the set $V(P) \setminus \{p_1, p_k\}$. The *length* of a path P is the number of edges in P. A *cycle* C is a sequence of vertices $p_1p_2\dots p_kp_1$, $k \geq 3$, such that $p_1 \dots p_k$ is a path, p_1p_k is an edge, and there are no other edges in C. The *length* of C is the number of edges in C. We denote a cycle of length four by C_4.

If $v \in V(G)$ and $X \subseteq V(G)$, a *shortest path from* v *to* X is the shortest path with one end v and the other end in X. If $v \in V(G)$, then $N^d_G(v)$ (or $N^d(v)$ when there is no danger of confusion) is the set of all vertices in $V(G)$ at distance exactly d from v, and $N^d_G[v]$ (or $N^d[v]$) is the set of vertices at distance at most d from v. Similarly, if $X \subseteq V(G)$, $N^d_G(X)$ (or $N^d(X)$) is the set of all vertices in $V(G)$ at distance exactly d from X, and $N^d[X]$ (or $N^d[X]$) is the set of all vertices in $V(G)$ at distance at most d from X.

Next we describe a few types of graphs that we will need. They are illustrated in Figure 1. A *theta* is a graph consisting of three internally vertex-disjoint paths $P_1 = a\text{-}\dots\text{-}b$, $P_2 = a\text{-}\dots\text{-}b$, and $P_3 = a\text{-}\dots\text{-}b$ of length at least 2, such that no edges exist between the paths except the three edges incident with a and the three edges incident with b. A *prism* is a graph consisting of three vertex-disjoint paths $P_1 = a_1\text{-}\dots\text{-}b_1$, $P_2 = a_2\text{-}\dots\text{-}b_2$, and $P_3 = a_3\text{-}\dots\text{-}b_3$ of length at least 1, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles and no edges exist between the paths except those of the two triangles. A *pyramid* is a graph consisting of three paths $P_1 = a\text{-}\dots\text{-}b_1$, $P_2 = a\text{-}\dots\text{-}b_2$, and $P_3 = a\text{-}\dots\text{-}b_3$ of length at least 1, two of which have length at least 2, vertex-disjoint except at a, and such that $b_1b_2b_3$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident with a.

A *wheel* (H, x) is a hole H and a vertex x such that x has at least three neighbors in H. A wheel (H, x) is *even* if x has an even number of neighbors on H. The following lemma characterizes odd-signable graphs in terms of forbidden induced subgraphs.

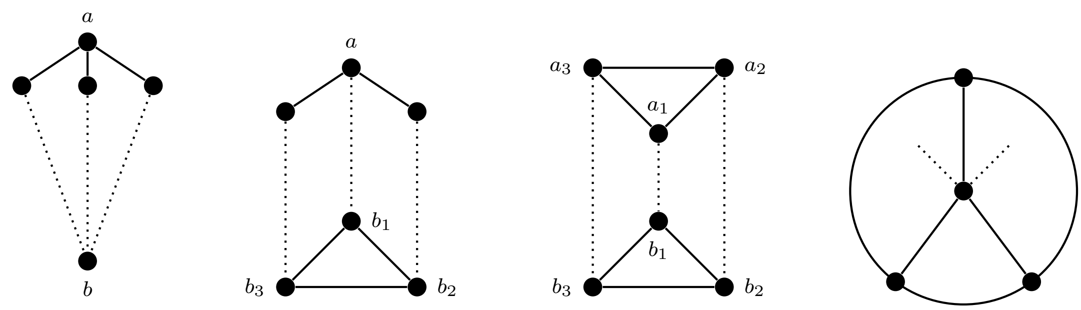

FIGURE 1. Theta, pyramid, prism, and wheel

Theorem 1.7. ([7]) *A graph is odd-signable if and only if it is (even wheel, theta, prism)-free.*

A *cutset* $C \subseteq V(G)$ of G is a set of vertices such that $G \setminus C$ is disconnected. A *star cutset* in a graph G is a cutset $S \subseteq V(G)$ such that either $S = \emptyset$ or for some $x \in S$, $S \subseteq N[x]$. A *clique* is a set $K \subseteq V(G)$ such that every pair of vertices in K are adjacent. A *clique cutset* is a cutset $C \subseteq V(G)$ such that C is a clique.

2. BALANCED SEPARATORS AND LAMINAR COLLECTIONS

The goal of this section is to develop the notion of a "central bag" for a laminar collection of separations, and to study the properties of the central bag. The main result is Lemma 2.6, that connects the existence of a balanced separator in the whole graph with the existence of one in the central bag of a laminar collection of separations. Note that in a later paper by the authors and their coauthors [2], a simpler way to define central bags is given.

For the remainder of the paper, unless otherwise specified, we assume that if G is a graph, then $w : V(G) \to [0,1]$ is a weight function of G with $w(G) = 1$, and $w^{\max} = \max_{v \in V(G)} w(v)$. A *separation* of a graph G is a triple of disjoint vertex sets (A, C, B) such that $A \cup C \cup B = V(G)$ and A is anticomplete to B. A separation (A, C, B) is *proper* if A and B are nonempty. A set $X \subseteq V(G)$ is a *clique star* if there exists a nonempty clique K in G such that $K \subseteq X \subseteq N[K]$. The clique K is called the *center* of X. A separation $S = (A, C, B)$ is a *star separation* if C is a clique star, and the *center* of S is the center of C. For $\varepsilon \in [0,1]$, a separation $S = (A, C, B)$ is *ε-skewed* if $w(A) < \varepsilon$ or $w(B) < \varepsilon$. For the remainder of the paper, if $S = (A, C, B)$ is ε-skewed, we assume that $w(A) < \varepsilon$. Let $S_1 = (A_1, C_1, B_1)$ and $S_2 = (A_2, C_2, B_2)$ be two separations. For $i = 1, 2$, let $X_i = A_i \cup C_i$ and $Y_i = C_i \cup B_i$. We say S_1 and S_2 are *non-crossing* if for some $i \in \{1, 2\}$, either $X_i \subseteq X_{3-1}$ and $Y_{3-i} \subseteq Y_i$, or $X_i \subseteq Y_{3-i}$ and $X_{3-i} \subseteq Y_i$. If S_1 and S_2 are not non-crossing, then S_1 and S_2 *cross*.

Let $\mathcal{C}$ be a collection of separations of G. The collection $\mathcal{C}$ is *laminar* if the separations of $\mathcal{C}$ are pairwise non-crossing. The *separation dimension* of $\mathcal{C}$, denoted $\dim(\mathcal{C})$, is the minimum number of laminar collections of separations with union $\mathcal{C}$.

Let G be a graph and let (T, χ) be a tree decomposition of G. Suppose that $e = t_1t_2$ is an edge of T and let T_1 and T_2 be the connected components of $T \setminus e$, where for $i = 1, 2$, t_i is a vertex of T_i. Up to symmetry between t_1 and t_2, the separation of G corresponding to e, denoted S_e, is defined as follows: $S_e = (D_e^{t_1}, C_e, D_e^{t_2})$, where $C_e = \chi(t_1) \cap \chi(t_2)$, $D_e^{t_1} = \left(\bigcup_{t \in T_1} \chi(t)\right) \setminus C_e$, and $D_e^{t_2} = \left(\bigcup_{t \in T_2} \chi(t)\right) \setminus C_e$. The following lemma shows that given a laminar collection of separations $\mathcal{C}$ of G, there exists a tree decomposition $(T_{\mathcal{C}}, \chi_{\mathcal{C}})$ of G such that there is a bijection between $\mathcal{C}$ and the separations corresponding to edges of $(T_{\mathcal{C}}, \chi_{\mathcal{C}})$.

Lemma 2.1 ([19])**.** *Let G be a graph and let $\mathcal{C}$ be a laminar collection of separations of G. Then there is a tree decomposition $(T_{\mathcal{C}}, \chi_{\mathcal{C}})$ of G such that*

(i) for all $S \in \mathcal{C}$, there exists $e \in E(T_{\mathcal{C}})$ such that $S = S_e$, and
(ii) for all $e \in E(T_{\mathcal{C}})$, $S_e \in \mathcal{C}$.

We call $(T_{\mathcal{C}}, \chi_{\mathcal{C}})$ a *tree decomposition corresponding to $\mathcal{C}$*. Suppose $\mathcal{C}$ is a laminar collection of ε-skewed separations of G, and let $(T_{\mathcal{C}}, \chi_{\mathcal{C}})$ be a tree decomposition corresponding to $\mathcal{C}$. For $e \in E(T_{\mathcal{C}})$, $S_e = (A_e, C_e, B_e)$, where $w(A_e) < \varepsilon$. We define the directed tree $T'_{\mathcal{C}}$ to be the orientation of $T_{\mathcal{C}}$ given by directing edge $e = t_1t_2$ of $T_{\mathcal{C}}$ from t_1 to t_2 if $A_e = D_e^{t_1}$ (so $e = (t_1, t_2)$ in $T'_{\mathcal{C}}$), and from t_2 to t_1 if $A_e = D_e^{t_2}$ (so $e = (t_2, t_1)$ in $T'_{\mathcal{C}}$). If $w(A_e) < \varepsilon$ and $w(B_e) < \varepsilon$, then edge e is directed arbitrarily.

A *sink* of a directed graph G is a vertex v such that each edge incident with v is oriented toward v. Every directed tree has at least one sink. A directed tree T is an *in-arborescence* if there exists a root $v \in V(T)$ such that for every $u \in V(T)$, there is exactly one directed path from u to v in T. The following lemma shows that when $\mathcal{C}$ is a laminar collection of ε-skewed separations satisfying an additional property, $T'_{\mathcal{C}}$ is an in-arborescence.

Lemma 2.2. *Let $\varepsilon, \varepsilon_0 > 0$ be such that $\varepsilon + \varepsilon_0 < \frac{1}{2}$. Let G be a graph and let $\mathcal{C}$ be a laminar collection of ε-skewed separations of G such that $w(C) \leq \varepsilon_0$ for all (A, C, B) in $\mathcal{C}$. Let $(T_{\mathcal{C}}, \chi_{\mathcal{C}})$ be a tree decomposition corresponding to $\mathcal{C}$. Then, the directed tree $T'_{\mathcal{C}}$ is an in-arborescence.*

Proof. Let $x \in V(T'_{\mathcal{C}})$ be a sink of $T'_{\mathcal{C}}$. We prove by induction on the distance from x in $T_{\mathcal{C}}$ that for every vertex $u \in V(T'_{\mathcal{C}})$, the path from u to x in $T_{\mathcal{C}}$ is a directed path from u to x in $T'_{\mathcal{C}}$. Since x is a sink, the base case follows immediately. Suppose that there is a directed path from v to x in $T'_{\mathcal{C}}$ for all vertices v of distance i from x, and consider vertex u of distance $i + 1$ from x. Let $P = u\text{-}v\text{-}v'\text{-}\dots\text{-}x$ be the path from u to x in $T_{\mathcal{C}}$. By induction, the path $v\text{-}v'\text{-}\dots\text{-}x$ is a directed path from v to x in $T'_{\mathcal{C}}$. Suppose that $(v, u) \in E(T'_{\mathcal{C}})$. Let T_1 be the component of

$T'_{\mathcal{C}} \setminus (v, u)$ containing v, and let T_2 be the component of $T'_{\mathcal{C}} \setminus (v, v')$ containing v. Because S_{vu} and $S_{vv'}$ are ε-skewed separations of G, we have that

$$w\left(\left(\bigcup_{t\in T_1} \chi_{\mathcal{C}}(t)\right) \setminus (\chi_{\mathcal{C}}(v) \cap \chi_{\mathcal{C}}(u))\right) < \varepsilon \tag{1}$$

and

$$w\left(\left(\bigcup_{t\in T_2} \chi_{\mathcal{C}}(t)\right) \setminus \big(\chi_{\mathcal{C}}(v) \cap \chi_{\mathcal{C}}(v')\big)\right) < \varepsilon. \tag{2}$$

Together, (1) and (2) imply that $w(G) < 2\varepsilon + 2\varepsilon_0 < 1$, a contradiction. Therefore, the directed tree $T'_{\mathcal{C}}$ is an in-arborescence. ■

Lemma 2.3. *Let $c \in [\frac{1}{2}, 1)$ and let d be a positive integer. Let G be a graph, let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$, and suppose G has no (w, c, d)-balanced separator. Let $S = (A, C, B)$ be a separation of G such that C is d-bounded. Then, S is $(1 - c)$-skewed.*

Proof. Since C is d-bounded and G has no (w, c, d)-balanced separator, we may assume $w(B) > c$. Since $1 = w(G) \geq w(A) + w(B)$ and $w(B) > c$, it follows that $w(A) < 1 - c$, and so S is $(1 - c)$-skewed. ■

Let G be a graph with maximum degree δ. Note that $\delta + \delta^2$ is an upper bound for the maximum size of a clique star in G. Let $\beta \subseteq V(G)$. For a laminar collection X of ε-skewed star separations of G, β is *perpendicular* to X if $\beta \cap A = \emptyset$ for all $(A, C, B) \in X$.

Lemma 2.4. *Let δ be a positive integer, let $c \in [\frac{1}{2}, 1)$, and let $m \in [0, 1]$, with $(1-c)+m(\delta+\delta^2) < \frac{1}{2}$. Let G be a connected graph with maximum degree δ and let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$ and $w^{\max} \leq m$. Let X be a laminar collection of $(1 - c)$-skewed star separations of G. Let (T_X, χ_X) be a tree decomposition corresponding to X. (Note that since $(1 - c) + w^{\max}(\delta + \delta^2) < \frac{1}{2}$, it follows from Lemma 2.2 that T'_X is an in-arborescence.) Let v be the root of T'_X and let $\beta = \chi_X(v)$. Then β is connected and perpendicular to X.*

Proof. Suppose $(A, C, B) \in X$. Then, C is a clique star, so $|C| \leq \delta+\delta^2$ and $w(C) \leq w^{\max}(\delta+\delta^2)$. First, we show that β is connected. Let $e_1, \ldots, e_m$ be the edges of T_X incident with v and let $S_{e_1}, \ldots, S_{e_m}$ be the corresponding separations, where $S_{e_i} = (A_{e_i}, C_{e_i}, B_{e_i})$ and $w(A_{e_i}) < 1 - c$. Then, $V(G) \setminus \beta = \bigcup_{i=1}^m A_{e_i}$. Since $A_{e_1}, \ldots, A_{e_m}$ are pairwise disjoint and anticomplete, for every connected component D of $G \setminus \beta$ there exists $1 \leq i \leq m$ such that $D \subseteq A_{e_i}$. Since $N(A_{e_i}) \cap \beta \subseteq C_{e_i}$ and $C_{e_i} \subseteq N[K_{e_i}]$ for some clique $K_{e_i} \subseteq C_{e_i}$, it follows that the neighborhood in β of every connected component of $G \setminus \beta$ is contained in a unique connected component of β. Therefore, since G is connected, β is connected.

Now we show that β is perpendicular to X. Let $(A, C, B) \in X$, let $e = t_1t_2$ be the edge of T_X such that $S_e = (A, C, B)$, and let T_1 and T_2 be the components of $T_X \setminus e$ containing t_1 and t_2, respectively. Up to symmetry between T_1 and T_2, assume that $A = (\cup_{t\in T_1}\chi_X(t))\setminus\chi_X(t_2)$. Then, $e = (t_1, t_2)$ in T'_X. Since v is the root of T'_X, it follows that $v \in V(T_2)$, and thus $\beta \subseteq \cup_{t\in T_2}\chi_X(t)$. Therefore, $\beta \cap A = \emptyset$, so β is perpendicular to X. ■

Let G be a connected graph with maximum degree δ and let X be a laminar collection of ε-skewed star separations of G, where $\varepsilon + w^{\max}(\delta + \delta^2) < \frac{1}{2}$. Let (T_X, χ_X) be a tree decomposition corresponding to X. Let $v \in V(T_X)$ and $\beta = \chi_X(v)$ be as in Lemma 2.4; then β is connected and perpendicular to X. We call β the *central bag* for T_X. Let $e_1, \ldots, e_m$ be the edges of T_X incident with v where $e_i = v_iv$, and let $S_{e_1}, \ldots, S_{e_m}$ be the corresponding separations of G, where $S_{e_i} = (A_{e_i}, C_{e_i}, B_{e_i})$. Since $C_{e_i} = \chi_X(v) \cap \chi_X(v_i)$, it follows that $C_{e_i} \subseteq \chi_X(v) = \beta$ for every $i \in \{1, \ldots, m\}$.

For every C_{e_i}, let K_{e_i} be a center of C_{e_i}. We let $v_{e_i} \in K_{e_i}$ chosen arbitrarily be the *anchor* of C_{e_i}. For $v \in V(G)$, let $I_v \subseteq \{1, \dots, m\}$ be the set of indices i such that v is the anchor of C_{e_i}. Then, the *weight function* w_X *on* β *with respect to* T_X is a function $w_X : \beta \to [0, 1]$ such that $w_X(v) = w(v) + \sum_{i \in I_v} w(A_{e_i})$ for all $v \in \beta$.

Lemma 2.5. *Let δ be a positive integer and let $\varepsilon, m \in [0, 1]$, with $\varepsilon + m(\delta + \delta^2) < \frac{1}{2}$. Let G be a connected graph with maximum degree δ and let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$ and $w^{\max} \leq m$. Let X be a laminar collection of ε-skewed star separations of G. Let (T_X, χ_X) be a tree decomposition corresponding to X, let β be the central bag for T_X, and let w_X be the weight function on β with respect to T_X. Then, $w_X(\beta) = w(G) = 1$. Furthermore, if every clique K of G is the center of at most one star separation in X, then $w_X^{\max} \leq w^{\max} + 2^\delta \varepsilon$.*

Proof. By the definition of w_X, we have $w_X(\beta) = \sum_{v \in \beta} w_X(v) = \sum_{v \in V(G) \setminus \bigcup_{i=1}^m A_{e_i}} w(v) + \sum_{i=1}^m w(A_{e_i}) = w(G) = 1$.

Suppose every clique K of G is the center of at most one star separation in X. Because the maximum degree of G is δ, every vertex $v \in V(G)$ is in at most 2^δ cliques of G. It follows that every vertex $v \in V(G)$ is the anchor of at most 2^δ separations of X, so $|I_v| \leq 2^\delta$. Since X is a collection of ε-skewed separations, $w(A_{e_i}) < \varepsilon$ for all $i \in I_v$. Therefore, $w_X^{\max} \leq w^{\max} + 2^\delta \varepsilon$. ∎

The following lemma shows that if G does not have a (w, c, d)-balanced separator and X is a laminar collection of star separations of G, then the central bag for T_X does not have a $(w_X, c, d-2)$-balanced separator.

Lemma 2.6. *Let δ, d be positive integers with $d > 2$, let $c \in [\frac{1}{2}, 1)$, and let $m \in [0, 1]$, with $(1-c) + m(\delta + \delta^2) < \frac{1}{2}$. Let G be a connected graph with maximum degree δ, let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$ and $w^{\max} \leq m$, and suppose that G does not have a (w, c, d)-balanced separator. Let X be a laminar collection of star separations of G. Then, the central bag β for X exists (in particular, β is perpendicular to X), $w_X(\beta) = 1$, and β does not have a $(w_X, c, d-2)$-balanced separator.*

Proof. Since X is a collection of star separations, it follows that C is 2-bounded for every $(A, C, B) \in X$. Since G does not have a $(w, c, 2)$-balanced separator Lemma 2.3 implies that every separation in X is $(1-c)$-skewed. Let (T_X, χ_X) be a tree decomposition corresponding to X. Then, by Lemma 2.4, the central bag β for X exists, and by Lemma 2.5, $w_X(\beta) = 1$.

Suppose that Y is a $(w_X, c, d-2)$-balanced separator of β. We claim that $N^2_\beta[Y]$ is a (w, c, d)-balanced separator of G. Since Y is $(d-2)$-bounded, it follows that $N^2_\beta[Y]$ is d-bounded. Let $Q_1, \dots, Q_\ell$ be the components of $\beta \setminus Y$. Let $t \in V(T_X)$ be such that $\beta = \chi_X(t)$. Let $e_1, \dots, e_m$ be the edges of T_X incident with t, let $S_{e_1}, \dots, S_{e_m}$ be the corresponding separations, where $S_{e_i} = (A_{e_i}, C_{e_i}, B_{e_i})$ and $w(A_{e_i}) < 1 - c$, and let c_{e_i} be the anchor of C_{e_i} for $i = 1, \dots, m$. Then, $V(G) \setminus \beta = \bigcup_{i=1}^m A_{e_i}$ and A_{e_i} is anticomplete to A_{e_j} for $i \neq j$. For $v \in V(G)$, let $I_v \subseteq \{1, \dots, m\}$ be the set of all i such that v is the anchor of C_{e_i}. For $i = 1, \dots, \ell$, let $A_i = \bigcup_{v \in Q_i} \left(\bigcup_{j \in I_v} A_{e_j} \right)$, let $Q'_i = (Q_i \setminus N^2_\beta[Y])$, and let $Z_i = Q'_i \cup A_i$.

(1) Z_i is anticomplete to Z_j for $i \neq j$.

Suppose there is an edge e from Z_i to Z_j. Since Q'_i is anticomplete to Q'_j and A_i is anticomplete to A_j, we may assume that e is from $A_{e_{i'}}$ to Q'_j, where $A_{e_{i'}} \subseteq A_i$. Since $N(A_{e_{i'}}) \cap \beta \subseteq C_{e_{i'}}$, it follows that $C_{e_{i'}} \cap Q'_j \neq \emptyset$. Let $v \in C_{e_{i'}} \cap Q'_j$ and let P be a shortest path from $c_{e_{i'}}$ to v through β. Since $c_{e_{i'}}, v \in C_{e_{i'}}$ and $C_{e_{i'}}$ is a clique star, it follows that P is of length at most 2. Since $c_{e_{i'}} \in Q_i$ and $v \in Q_j$, it follows that P goes through Y and thus P is of length exactly 2. Let $P = c_{e_{i'}}$-y-v, where $y \in Y$. Then, $v \in N^2_\beta[y] \subseteq N^2_\beta[Y]$, a contradiction (since $v \in Q'_j$). This proves (1).

(2) If $c_{e_i} \in Y$, then A_{e_i} is anticomplete to Z_j for $j \in \{1, \dots, \ell\}$.

Suppose $c_{e_i} \in Y$. Then, $C_{e_i} \subseteq N^2_\beta[Y]$. Since $N(A_{e_i}) \cap \beta \subseteq C_{e_i}$, it follows that A_{e_i} is anticomplete to Q'_j for all $j = 1, \dots, \ell$. Therefore, A_{e_i} is anticomplete to Z_j for all $j = 1, \dots, \ell$. This proves (2).

Let $I_Y \subseteq \{1, \dots, m\}$ be the set of all i such that $c_{e_i} \in Y$. Then, $V(G) \setminus N^2_\beta[Y] = \left(\bigcup_{i \in I_Y} A_{e_i}\right) \cup \left(\bigcup_{j=1}^{\ell} Z_j\right)$. Suppose Z is a component of $V(G) \setminus N^2_\beta[Y]$. It follows from (1) and (2) that either $Z \subseteq A_{e_i}$ for some $i \in I_Y$, or $Z \subseteq Z_j$ for some $j \in \{1, \dots, \ell\}$. Since $w_X(Q_i) \leq c$, it follows that $w(Z_i) \leq c$ for all $i = 1, \dots, \ell$. Further, since every separation in X is $(1-c)$-skewed and $c \in [\frac{1}{2}, 1)$, it follows that $w(A_{e_i}) < (1-c) \leq c$ for all $i \in I_Y$. Therefore, $w(Z) \leq c$, and $N^2_\beta[Y]$ is a (w, c, d)-balanced separator of G, a contradiction. ■

3. Balanced separators and clique separations

In this section, we show that if G is a connected graph with no balanced separator, then there exists a connected induced subgraph of G with no balanced separator and no clique cutset. The central bag from Lemma 2.6 is the primary tool for finding such an induced subgraph.

A separation (A, C, B) of a graph G is a *clique separation* if C is a clique. A clique cutset C is *minimal* if every $c \in C$ has a neighbor in every component of $G \setminus C$. Note that in a connected graph G, $|C| \geq 1$ for every minimal clique cutset C of G.

Lemma 3.1. *Let G be a connected graph and let $\mathcal{C}$ be a collection of clique separations of G such that C is a minimal clique cutset for all $(A, C, B) \in \mathcal{C}$ and for every two distinct separations (A_1, C_1, B_1), $(A_2, C_2, B_2) \in \mathcal{C}$, $C_1 \neq C_2$. Then, $\dim(\mathcal{C}) = 1$. In particular, $\mathcal{C}$ is laminar.*

Proof. Let $S_1 = (A_1, C_1, B_1)$ and $S_2 = (A_2, C_2, B_2)$ be clique separations of G such that C_1 and C_2 are minimal clique cutsets of G. Since C_1 is a clique and A_2 is anticomplete to B_2, either $C_1 \cap A_2 = \emptyset$ or $C_1 \cap B_2 = \emptyset$. We may assume that $C_1 \cap A_2 = \emptyset$. Similarly, we may assume that $C_2 \cap A_1 = \emptyset$. If $A_1 \cap A_2 = \emptyset$, then $A_2 \subseteq B_1$ and $A_1 \subseteq B_2$, so S_1 and S_2 are non-crossing (since $A_2 \cup C_2 \subseteq B_1 \cup C_1$ and $A_1 \cup C_1 \subseteq B_2 \cup C_2$). Therefore, we may assume that $A_1 \cap A_2 \neq \emptyset$. Since $C_1 \neq C_2$, either $C_1 \cap B_2 \neq \emptyset$ or $C_2 \cap B_1 \neq \emptyset$. Assume up to symmetry that $C_1 \cap B_2 \neq \emptyset$. Since $A_1 \subseteq A_2 \cup B_2$ and A_2 is anticomplete to B_2, every component of A_1 is either a subset of A_2 or a subset of B_2. Since $A_1 \cap A_2 \neq \emptyset$, there exists a connected component A of A_1 such that $A \subseteq A_2$. Let $c \in C_1 \cap B_2$. Then, c is anticomplete to A, contradicting that C_1 is a minimal clique cutset. It follows that S_1 and S_2 are non-crossing. Therefore, $\dim(\mathcal{C}) = 1$. ■

Let G be a graph and let C be a minimal clique cutset of G. The *minimal clique separation S for C* is defined as follows: $S = (A, C, B)$, where B is a largest weight connected component of $G \setminus C$ and $A = V(G) \setminus (B \cup C)$.

Lemma 3.2. *Let $c \in [\frac{1}{2}, 1)$. Let G be a graph, let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$, and suppose G has no $(w, c, 1)$-balanced separator. Let C be a minimal clique cutset of G. Then, the minimal clique separation S for C is unique and S is $(1-c)$-skewed.*

Proof. Since G has no $(w, c, 1)$-balanced separator, C is not a $(w, c, 1)$-balanced separator. It follows that if B is a largest weight connected component of $G \setminus C$, then $w(B) > c$. Since $c \in [\frac{1}{2}, 1)$ and $w(G) = 1$, the largest weight connected component of $G \setminus C$ is unique, and thus S is unique. Since C is a 1-bounded set and G has no $(w, c, 1)$-balanced separator, it follows from Lemma 2.3 that S is $(1-c)$-skewed. ■

In the following lemma, we prove that if k is the minimum size of a clique cutset in G and $\mathcal{C}$ is the collection of all minimal clique separations of G for clique cutsets of size k, then the central bag β for $\mathcal{C}$ does not contain a clique cutset of size less than or equal to k. Note that a minimum size clique cutset is a minimal clique cutset.

Lemma 3.3. *Let δ be a positive integer, let k be a nonnegative integer, let $c \in [\frac{1}{2}, 1)$, and let $m \in [0,1]$, with $(1-c) + m(\delta + \delta^2) < \frac{1}{2}$. Let G be a connected graph with maximum degree δ and let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$ and $w^{\max} \leq m$. Suppose G does not have a $(w, c, 1)$-balanced separator, and suppose the smallest clique cutset in G has size k. Let $\mathcal{C}$ be the collection of all minimal clique separations of G such that $|C| = k$ for every $(A, C, B) \in \mathcal{C}$. Then, $\mathcal{C}$ is laminar, and if $(T_{\mathcal{C}}, \chi_{\mathcal{C}})$ is the tree decomposition of G corresponding to $\mathcal{C}$ and β is the central bag for $T_{\mathcal{C}}$, then β does not have a clique cutset of size less than or equal to k.*

Proof. Since G is connected, $k \geq 1$. Since G does not have a $(w, c, 1)$-balanced separator and $c \in [\frac{1}{2}, 1)$, it follows that every minimal clique cutset of size k in G corresponds to exactly one minimal clique separation in $\mathcal{C}$. Therefore, by Lemma 3.1, $\mathcal{C}$ is laminar, and by Lemma 3.2, every separation in $\mathcal{C}$ is $(1-c)$-skewed. Let $v \in V(T_{\mathcal{C}})$ be such that $\beta = \chi_{\mathcal{C}}(v)$ is the central bag for $T_{\mathcal{C}}$, and suppose β has a clique cutset of size less than or equal to k. Let (A_v, C_v, B_v) be a minimal clique separation of β such that $|C_v| \leq k$. Let $v_1, \ldots, v_m$ be the vertices of $T_{\mathcal{C}}$ adjacent to v, let $e_i = vv_i$ be the edge from v to v_i for $i = 1, \ldots, m$, and let $S_{e_1}, \ldots, S_{e_m}$ be the clique separations corresponding to $e_1, \ldots, e_m$, where $S_{e_i} = (D^v_{e_i}, C_{e_i}, D^{v_i}_{e_i})$ as in Section 2. Since $\beta \cap \chi_{\mathcal{C}}(v_i) = C_{e_i}$ and C_{e_i} is a clique, it follows that $C_{e_i} \cap A_v = \emptyset$ or $C_{e_i} \cap B_v = \emptyset$ for all $i = 1, \ldots, m$. Let A be the union of A_v and all $D^{v_i}_{e_i}$ for i such that $C_{e_i} \cap B_v = \emptyset$, and let B be the union of B_v and all $D^{v_i}_{e_i}$ for i such that $D^{v_i}_{e_i} \not\subseteq A$. For $i \neq j$, $D^{v_i}_{e_i}$ and $D^{v_j}_{e_j}$ are disjoint and anticomplete to each other. By properties of the tree decomposition, $\beta \cup \bigcup_{i=1}^m D^{v_i}_{e_i} = V(G)$. Therefore, it follows that (A, C_v, B) is a clique separation of G with $|C_v| \leq k$.

Since the smallest clique cutset in G has size k, it follows that $|C_v| = k$. Let $S = (X, C_v, Y)$ be the minimal clique separation for C_v in G. It follows that $S \in \mathcal{C}$, so by Lemma 2.4, $\beta \subseteq C_v \cup Y$. But since (A, C_v, B) is a clique separation of G, it follows that two components of $G \setminus C_v$ intersect β, a contradiction. ■

In the following theorem, we use Lemmas 2.6 and 3.3 to find an induced subgraph of G that has no clique cutset and no balanced separator.

Theorem 3.4. *Let δ, d be positive integers, with $d > 2\delta - 2$. Let $c \in [\frac{1}{2}, 1)$ and let $m \in [0,1]$, with $(1-c) + [m + (\delta-1)2^\delta(1-c)](\delta + \delta^2) < \frac{1}{2}$. Let G be a connected graph with maximum degree δ, let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$ and $w^{\max} \leq m$, and suppose G has no (w, c, d)-balanced separator. Then, there exists a sequence $(\alpha_0, w_0), (\alpha_1, w_1), \ldots, (\alpha_{\delta'}, w_{\delta'})$ such that $\delta' < \delta$, $(\alpha_0, w_0) = (G, w)$ and for $i \in \{0, \ldots, \delta'\}$, the following hold:*

- *α_i is a connected induced subgraph of G and w_i is a weight function on α_i such that $w_i(\alpha_i) = 1$ and $w_i^{\max} \leq w^{\max} + i2^\delta(1-c)$.*
- *α_i has no $(w_i, c, d-2i)$-balanced separator.*
- *If $i > 0$ then α_i is the central bag for a tree decomposition corresponding to a collection of minimal clique separations of α_{i-1}.*
- *$\alpha_{\delta'}$ does not have a clique cutset.*

Proof. We may assume that G has a clique cutset, otherwise the result holds with $\delta' = 0$. If $\delta = 1$, then G consists of a single edge, contradicting the assumption that G has a clique cutset. Therefore, $\delta \geq 2$ and so $d > 2$. Since the maximum degree of G is δ and every vertex in a minimal clique cutset C has a neighbor in every component of $G \setminus C$, it follows that every minimal clique cutset of G has size at most $\delta - 1$. Let j_0 be the size of the smallest clique cutset of G. Note that since G is connected, $j_0 \geq 1$. Since G has no (w, c, d)-balanced separator and $d \geq 1$, G has no $(w, c, 1)$-balanced separator. Let $\mathcal{C}_1$ be the collection of all minimal clique separations of G that correspond to clique cutsets of size j_0. By Lemma 3.2, every separation in $\mathcal{C}_1$ is $(1-c)$-skewed and for every two distinct separations $(A_1, C_1, B_1), (A_2, C_2, B_2) \in \mathcal{C}_1$, $C_1 \neq C_2$. Therefore, by Lemma 3.1, $\mathcal{C}_1$ is laminar. Let $(T_{\mathcal{C}_1}, \chi_{\mathcal{C}_1})$ be the tree decomposition of G corresponding to $\mathcal{C}_1$. By Lemma 2.6, the central bag for $T_{\mathcal{C}_1}$ exists and does not have a $(w_{\mathcal{C}_1}, c, d-2)$-balanced separator. Let α_1 be the central bag for $T_{\mathcal{C}_1}$ and let $w_1 = w_{\mathcal{C}_1}$. By Lemma 2.5, $w_1(\alpha_1) = 1$ and $w_1^{\max} \leq w^{\max} + 2^\delta(1-c)$. Since $(1-c) + w^{\max}(\delta + \delta^2) \leq (1-c) + [w^{\max} + (\delta-1)2^\delta(1-c)](\delta+\delta^2) < \frac{1}{2}$, by Lemma 2.4, α_1 is connected. It follows from Lemma 3.3 that α_1 does not have a clique cutset of size less than or equal to j_0. If α_1 does not have a clique cutset, then $\delta' = 1$ and the sequence ends. Otherwise, for $i \in \{2, \dots, \delta-1\}$, we define (α_i, w_i) inductively. For $i \in \{2, \dots, \delta-1\}$, suppose (α_{i-1}, w_{i-1}) are such that α_{i-1} is the central bag for a tree decomposition corresponding to a collection of minimal clique separations of α_{i-2} and w_{i-1} is the corresponding weight function on α_{i-1}, α_{i-1} is a connected induced subgraph of G with no (w_{i-1}, c, d_{i-1})-balanced separator for $d_{i-1} = d - 2(i-1)$, $w_{i-1}(\alpha_{i-1}) = 1$, and $w_{i-1}^{\max} \leq w^{\max} + (i-1)2^\delta(1-c)$. Further, suppose the smallest clique cutset in α_{i-1} has size j_{i-1}, where $\delta > j_{i-1} \geq i$.

Since $\delta > i$ and $d > 2\delta - 2$, it follows that $d - 2(i-1) \geq 1$. Since α_{i-1} has no $(w_{i-1}, c, d-2(i-1))$-balanced separator, it follows that α_{i-1} has no $(w_{i-1}, c, 1)$-balanced separator. Let $\mathcal{C}_i$ be the collection of all minimal clique separations of α_{i-1} that correspond to clique cutsets of size j_{i-1}. By Lemmas 3.2 and 3.1, $\mathcal{C}_i$ is laminar. Since $w_{i-1}^{\max} \leq w^{\max} + (i-1)2^\delta(1-c)$ and $i < \delta$, it follows that $(1-c) + w_{i-1}^{\max}(\delta+\delta^2) < (1-c) + [w^{\max} + (\delta-1)2^\delta(1-c)](\delta+\delta^2) < \frac{1}{2}$. Since $d > 2\delta - 2$, $i < \delta$, and $\delta \geq 2$, it follows that $d_{i-1} = d - 2(i-1) \geq d - 2(\delta-2) > 2$. Since α_{i-1} has no $(w_{i-1}, c, d-2(i-1))$-balanced separator, $d_{i-1} > 2$ and $(1-c) + w_{i-1}^{\max}(\delta+\delta^2) < \frac{1}{2}$, it follows from Lemma 2.6 that the central bag for $\mathcal{C}_i$ exists and does not have a $(w_{\mathcal{C}_i}, c, d_i)$-balanced separator, where $d_i = d_{i-1} - 2 = d - 2i \geq 1$. Let $T_{\mathcal{C}_i}$ be the tree decomposition of α_{i-1} corresponding to $\mathcal{C}_i$. Let α_i be the central bag for $T_{\mathcal{C}_i}$ and let $w_i = w_{\mathcal{C}_i}$ be the weight function on α_i with respect to $T_{\mathcal{C}_i}$. By Lemma 2.5, $w_i(\alpha_i) = 1$ and $w_i^{\max} \leq w_{i-1}^{\max} + 2^\delta(1-c) \leq w^{\max} + i2^\delta(1-c)$. Since $(1-c) + w_{i-1}^{\max}(\delta+\delta^2) < \frac{1}{2}$, by Lemma 2.4, α_i is connected. If α_i has no clique cutset, then $\delta' = i$ and the sequence ends. Otherwise, let j_i be the size of the smallest clique cutset in α_i. By Lemma 3.3, it follows that $j_i > j_{i-1}$, so $j_i \geq i+1$. Since the maximum size of a minimal clique cutset in G, and thus in α_i, is $\delta - 1$, $j_i < \delta$. Thus, minimal clique cutsets used in this proof are of sizes in $\{1, \dots, \delta-1\}$, so $\delta' < \delta$. Therefore, the sequence $(\alpha_1, w_1), \dots, (\alpha_{\delta'}, w_{\delta'})$ is well-defined and satisfies the theorem. Further, by construction, $\alpha_{\delta'}$ does not have a clique cutset. ■

We call $\alpha_{\delta'}$ the *clique-free bag* for G.

4. Star cutsets and forcers

Let G be a graph. A cutset C of G is a *clique star cutset* of G if C is a clique star. Recall that a star separation $S = (A, C, B)$ is proper if C is a clique star cutset. In this section we study properties of separations associated with clique star cutsets. In particular, we establish the notion of a canonical separation that corresponds to a given clique, and show how to partition a set of canonical clique separations into a bounded number of laminar collections; this is done in Lemma 4.2. Then we list several lemmas showing that certains subgraphs are clique star cutset forcers (Lemmas 4.4, 4.5, and 4.6, summarized in Lemma 4.7). Finally we show that repeatedly taking central bags leads to a forcer-free subgraph (this is done in Theorem 4.11).

In the following lemma, we show that if two proper star separations cross, then their centers are not anticomplete to each other.

Lemma 4.1. *Let G be a theta-free graph with no clique cutset, let K_1 and K_2 be cliques of G, and let $\mathcal{S}_1 = (A_1, C_1, B_1)$ and $\mathcal{S}_2 = (A_2, C_2, B_2)$ be proper star separations such that $C_1 \subseteq N[K_1]$ and $C_2 \subseteq N[K_2]$. Suppose S_1 and S_2 cross. Then, K_1 and K_2 are not anticomplete to each other.*

Proof. Suppose K_1 is anticomplete to K_2. Then, $K_1 \cap N[K_2] = \emptyset$, so K_1 is contained in a connected component of $G \setminus C_2$. Similarly, K_2 is contained in a connected component of $G \setminus C_1$. Up to symmetry between A and B, assume that $K_1 \subseteq B_2$ and $K_2 \subseteq B_1$. Then, $C_1 \cap A_2 = \emptyset$ and $C_2 \cap A_1 = \emptyset$. Since S_1 and S_2 cross, it follows that $A_1 \cap A_2 \neq \emptyset$. Let $A = A_1 \cap A_2$. Suppose $C_1 \subseteq B_2$. Then, C_1 is anticomplete to A. Because $A \subseteq A_1$ and A_1 is anticomplete to B_1, it follows that B_1 is anticomplete to A. Finally, since $A_1 \cap C_2 = \emptyset$, it follows that $A_1 \setminus A \subseteq B_2$, so A is anticomplete to $A_1 \setminus A$. Therefore, A is anticomplete to $G \setminus A$, a contradiction, so $C_1 \cap C_2 \neq \emptyset$.

Let $C = C_1 \cap C_2$, let A' be a connected component of A, and let $C' = N_C(A')$. Suppose there exists $c_1, c_2 \in C'$ such that $c_1c_2 \notin E(G)$. Then, G contains a theta between c_1 and c_2 through A', K_1, and K_2, a contradiction. Therefore, C' is a clique. Since $A_1 \cap A_2$ is anticomplete to B_1 and B_2, it follows that $N(A) \subseteq C$, so $N(A') = C'$. Then, A' is a connected component of $G \setminus C'$, so C' is a clique cutset of G, a contradiction. ■

The next lemma shows that if Y is a set of cliques of size at most k, then there exists a partition of Y into $(k + \delta k) \sum_{j=0}^{k-1} \binom{\delta}{j} + 1$ parts such that every two cliques in the same part are anticomplete to each other.

Lemma 4.2. *Let δ, k be positive integers with $k \leq \delta$ and let $f(k, \delta) = (k + \delta k) \sum_{j=0}^{k-1} \binom{\delta}{j} + 1$. Let G be a graph with maximum degree δ and let $Y = \{K_1, \dots, K_t\}$ be a set of cliques of G of size at most k. Then, there exists a partition $(Y_1, \dots, Y_{f(k,\delta)})$ of Y such that for every $\ell \in \{1, \dots, f(k, \delta)\}$ and $K_i, K_j \in Y_\ell$, K_i is anticomplete to K_j.*

Proof. Let H be a graph with vertex set $V(H) = \{x_1, \dots, x_t\}$, and for $x_i, x_j \in V(H)$, $i \neq j$, let $x_ix_j \in E(H)$ if and only if K_i is not anticomplete to K_j in G. Let $x_i \in V(H)$ and let $x_j \in N_H(x_i)$.Then, K_i is not anticomplete to K_j, so $K_j \cap N[K_i] \neq \emptyset$. Let $v \in K_j \cap N[K_i]$. Then, $K_j \subseteq N[v]$. Since $|N[K_i]| \leq k + \delta k$ and $|N[u]| \leq \delta$ for all $u \in V(G)$, it follows that K_i is not anticomplete to at most $(k + \delta k) \sum_{j=0}^{k-1} \binom{\delta}{j}$ cliques of size at most k. Therefore, the maximum degree of H is at most $(k + \delta k) \sum_{j=0}^{k-1} \binom{\delta}{j}$.

Since the maximum degree of H is at most $(k + \delta k) \sum_{j=0}^{k-1} \binom{\delta}{j}$, it follows that the chromatic number of H is at most $(k + \delta k) \sum_{j=0}^{k-1} \binom{\delta}{j} + 1 = f(k, \delta)$. Let $C : V(H) \to \{1, \dots, f(k, \delta)\}$ be a coloring of H and let $Y_1, \dots, Y_{f(k,\delta)}$ be the color classes of C. Then, $(Y_1, \dots, Y_{f(k,\delta)})$ is a partition of Y such that if $\ell \in \{1, \dots, f(k, \delta)\}$ and $K_i, K_j \in Y_\ell$, then K_i is anticomplete to K_j. ■

Let G be a graph with weight function w and let K be a nonempty clique of G. A *canonical star separation for* K, denoted S_K, is defined as follows: $S_K = (A_K, C_K, B_K)$, where B_K is a largest weight connected component of $G \setminus N[K]$ if $G \setminus N[K] \neq \emptyset$ and $B_K = \emptyset$ otherwise, C_K is the union of K and the set of all vertices $v \in N[K]$ such that v has a neighbor in B_K, and $A_K = V(G) \setminus (B_K \cup C_K)$. The following lemma shows that if G has no balanced separator, then the canonical star separation is unique.

Lemma 4.3. *Let $c \in [\frac{1}{2}, 1)$. Let G be a graph with no $(w, c, 2)$-balanced separator and let K be a nonempty clique of G. Then, the canonical star separation S_K for K is unique and S_K is $(1 - c)$-skewed.*

Proof. Since G has no $(w, c, 2)$-balanced separator, $N[K]$ is not a $(w, c, 2)$-balanced separator. It follows that if B_K is a largest weight connected component of $G \setminus N[K]$, then $w(B_K) > c$. Since $c \in [\frac{1}{2}, 1)$ and $w(G) = 1$, the largest weight connected component of $G \setminus N[K]$ is unique, and thus S_K is unique. Since C_K is a 2-bounded set and G has no $(w, c, 2)$-balanced separator, it follows from Lemma 2.3 that S_K is $(1 - c)$-skewed. ■

Let G be a graph. Let X, Y, Z be disjoint subsets of $V(G)$. We say that X *separates* Y *from* Z if there exist distinct components C_Y, C_Z of $G \setminus X$ such that $Y \subseteq C_Y$ and $Z \subseteq C_Z$. Recall that a *wheel* (H, x) of G consists of a hole H and a vertex x that has at least three neighbors in H. A *sector* of (H, x) is a path P of H whose ends are adjacent to x, and such that x is anticomplete to P^* (recall that P^* is the set of interior vertices of P). A sector P is a *long sector* if P^* is nonempty. We now define several types of wheels that we will need. They are illustrated in Figure 2.

A wheel (H, x) is a *universal wheel* if x is complete to H. A wheel (H, x) is a *twin wheel* if $N(x) \cap H$ induces a path of length 2. If (H, x) is a twin wheel and x_1-x_2-x_3 is the path of length 2 induced by $N(x) \cap H$, we say x_2 is the *clone of x in H*. Note that if (H, x) is a twin wheel and x_2 is the clone of x in H, then $((H \setminus \{x_2\}) \cup \{x\}, x_2)$ is also a twin wheel. Suppose (H, x) is a twin wheel contained in a graph G and x_2 is the clone of x in H. We say (H, x) is *x-rich* if there is a path in G from x to $V(H) \setminus N[x]$ containing no neighbors of x_2 other than x, and *x_2-rich* if there is a path in G from x_2 to $V(H) \setminus N[x]$ containing no neighbors of x other than x_2. We say (H, x) is *x-poor* if it is not x-rich, and *x_2-poor* if it is not x_2-rich. We say that (H, x, x_2) is a *terminal twin wheel* if (H, x) is a twin wheel and x_2 is the clone of x in H, and (H, x) is either x-poor or x_2-poor. A wheel (H, x) is a *short pyramid* if $|N(x) \cap H| = 3$ and x has exactly one pair of adjacent neighbors in H. A wheel is *proper* if it is not a twin wheel or a short pyramid. If (H, x) is a short pyramid (resp. proper wheel), then x is said to be the *center* of a short pyramid (resp. proper wheel) in H.

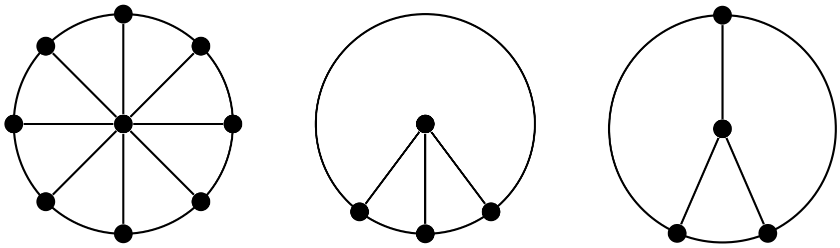

FIGURE 2. Universal wheel, twin wheel, and short pyramid

The following three lemmas show that proper wheels and short pyramids generate clique star cutsets.

Lemma 4.4 ([3], [12])**.** *Let G be a C_4-free odd-signable graph that contains a proper wheel (H, x) that is not a universal wheel. Let x_1 and x_2 be the endpoints of a long sector Q of (H, x). Let W be the set of all vertices h in $H \cap N(x)$ such that the subpath of $H \setminus \{x_1\}$ from x_2 to h contains an even number of neighbors of x, and let $Z = H \setminus (Q \cup N(x))$. Let $N' = N(x) \setminus W$. Then, $N' \cup \{x\}$ is a cutset of G that separates Q^* from $W \cup Z$.*

Lemma 4.5 ([11])**.** *Let G be a C_4-free odd-signable graph that contains a universal wheel (H, x). If $G = N[x]$ then for every two non-adjacent vertices a and b of H, $N[x] \setminus \{a, b\}$ is a cutset of G that separates a and b. If $G \setminus N[x] \neq \emptyset$ then for every connected component C of $G \setminus N[x]$, there exists $a \in H$ such that a has no neighbor in H, i.e. $N[x] \setminus \{a\}$ is a cutset of G that separates a from C.*

Lemma 4.6. ([8]) *Let G be a C_4-free odd-signable graph that contains a wheel (H, x) that is a short pyramid. Let x_1, x_2 and y be the neighbors of x in H such that x_1x_2 is an edge. For $i \in \{1, 2\}$ let H_i be the sector of (H, x) with ends y, x_i. Then, H_1 and H_2 are long sectors of (H, x), and $S = N(x) \cup N(y)$ is a cutset of G that separates $H_1 \setminus S$ from $H_2 \setminus S$.*

Let G be a graph. A *forcer* $F = (H, K)$ in G consists of a hole H and a clique K such that one of the following holds:

- (H, x) is a proper wheel of G and $K = \{x\}$.

- (H, x) is a short pyramid of G, $N(x) \cap H = \{x_1, x_2, y\}$ where x_1x_2 is an edge, and $K = \{x, y\}$.
- (H, x, x_2) is a terminal twin wheel of G, (H, x) is x_2-poor, and $K = \{x\}$.

If $F = (H, K)$ is a forcer, we say that K is the *center* of F. The forcer described in the first bullet is referred to as a *proper wheel forcer*, the one in the second bullet as a *short pyramid forcer*, and the one in the third bullet as a *twin wheel forcer*. A forcer $F = (H, K)$ is *strong* if it is not a twin wheel forcer. The following lemma shows that forcers generate clique star cutsets.

Lemma 4.7. *Let G be a C_4-free odd-signable graph and let $F = (H, K)$ be a forcer in G. Then, K is the center of a clique star cutset in G.*

Proof. If (H, x) is a proper wheel that is not a universal wheel, then by Lemma 4.4, x together with some of its neighbors is a clique star cutset in G. If (H, x) is a universal wheel, then by Lemma 4.5, x together with some of its neighbors is a clique star cutset in G. If (H, x) is a short pyramid and y is the common node of the two long sectors of (H, x), then by Lemma 4.6, x, y and its neighbors form a clique star cutset in G. It follows that if $F = (H, K)$ is a strong forcer, then the result holds. Therefore, assume $F = (H, K)$ is a twin wheel forcer. It follows that there exist $x \in V(G), x_2 \in V(H)$ such that (H, x, x_2) is a terminal twin wheel, (H, x) is x_2-poor, and $K = \{x\}$. Then, it follows that $N[K] \setminus x_2$ is a clique star cutset that separates x_2 from $H \setminus N[K]$. ■

The following lemma shows that if $F = (H, K)$ is a forcer and $S_K = (A_K, C_K, B_K)$ is the canonical star separation for K, then $A_K \cap H \neq \emptyset$.

Lemma 4.8. *Let G be a C_4-free odd-signable graph. Let $F = (H, K)$ be a forcer in G and let $S_K = (A_K, C_K, B_K)$ be a canonical star separation for K. Then, $A_K \cap H \neq \emptyset$. Furthermore, if for $c \in [\frac{1}{2}, 1)$, G has no $(w, c, 2)$-balanced separator, then S_K is a proper star separation.*

Proof. Let (H, x) be the wheel such that $F = (H, K)$. Suppose first that (H, x) is a wheel such that there exist two long sectors S_1, S_2 of (H, x). Lemmas 4.4 and 4.6 imply that $N[K]$ separates $S_1 \setminus N[K]$ from $S_2 \setminus N[K]$. It follows that for some $i \in \{1, 2\}$, $S_i \cap A_K \neq \emptyset$, and so $H \cap A_K \neq \emptyset$.

Next, suppose that (H, x) is a proper wheel with exactly one long sector S. If $B_K \cap H = \emptyset$, then $S^* \cap A_K \neq \emptyset$, so we may assume that $S^* \subseteq B_K$. By Lemma 4.4, for some $a \in N(x) \cap H$, a has no neighbor in B_K. Therefore, $a \in A_K$ and $A_K \cap H \neq \emptyset$.

Now, suppose that (H, x) is a universal wheel. We may assume that $G \neq N[K]$ (since otherwise $B_K = \emptyset$ and $A_K = H$). Then, it follows from Lemma 4.5 that for every component C of $G \setminus N[K]$, there exists $a \in H$ such that a has no neighbor in C. In particular, there exists $a \in H$ such that a has no neighbor in B_K. Therefore, $a \notin C_K$ and $a \notin B_K$, so $a \in A_K$ and $H \cap A_K \neq \emptyset$.

Finally, suppose that (H, x) is a twin wheel, and let x_2 be the clone of x in H. Then, (H, x, x_2) is a terminal twin wheel, (H, x) is x_2-poor, and $K = \{x\}$. Consider $G \setminus N[K]$. If $(H \setminus \{x_1, x_2, x_3\}) \cap B_K = \emptyset$, then $A_K \cap H \neq \emptyset$, so assume $(H \setminus \{x_1, x_2, x_3\}) \subseteq B_K$. Since (H, x) is x_2-poor, it follows that x_2 does not have a neighbor in B_K. Therefore, $x_2 \in A_K$, and $A_K \cap H \neq \emptyset$.

Now, suppose that $c \in [\frac{1}{2}, 1)$ and G has no $(w, c, 2)$-balanced separator. Then, $G \setminus N[K] \neq \emptyset$, and thus $B_K \neq \emptyset$. Since $A_K \neq \emptyset$, it follows that S_K is proper. ■

Let G' be an induced subgraph of G. A forcer $F = (H, K)$ is *active for G'* if $H \subseteq G'$ and $K \subseteq G'$.

Lemma 4.9. *Let δ be a positive integer, $c \in [\frac{1}{2}, 1)$, and $m \in [0, 1]$, with $(1 - c) + m(\delta + \delta^2) < \frac{1}{2}$. Let G be a connected C_4-free odd-signable graph with maximum degree δ, let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$ and $w^{\max} \leq m$, and suppose G does not have a $(w, c, 2)$-balanced separator. Let $\mathcal{F}$ be a set of forcers, let $Y = \{K : (H, K) \in \mathcal{F}\}$ be the set of centers of $\mathcal{F}$, and let $\mathcal{C}$ be the collection of canonical star separations for centers in Y. Suppose $\mathcal{C}$ is* *laminar and let $(T_{\mathcal{C}}, \chi_{\mathcal{C}})$ be the tree decomposition of G corresponding to $\mathcal{C}$. Then, the central bag β for $\mathcal{C}$ exists and no forcer in $\mathcal{F}$ is active for β.*

Proof. By Lemma 4.3, every separation in $\mathcal{C}$ is $(1-c)$-skewed. By Lemma 2.4, the central bag β for $\mathcal{C}$ exists (in particular, β is perpendicular to $\mathcal{C}$). Suppose $F = (H, K)$ is a forcer in $\mathcal{F}$ and let $S_K = (A_K, C_K, B_K)$ be the canonical star separation for K. Then, since β is perpendicular to $\mathcal{C}$, $\beta \cap A_K = \emptyset$, and hence $\beta \subseteq C_K \cup B_K$. By Lemma 4.8, it follows that $H \cap A_K \neq \emptyset$, so $H \not\subseteq \beta$ and F is not active for β. ■

The following theorem generalizes the results of Lemma 4.9. Recall the definition of clique-free bag from the end of Section 3: the clique-free bag of a graph G is an induced subgraph α of G, formed by taking repeated central bags, such that α does not have a clique cutset. (See Theorem 3.4 for details).

Theorem 4.10. *Let δ, d be positive integers, let k be a nonnegative integer, let $f(2,\delta) = 2(\delta+1)^2+1$, let $c \in [\frac{1}{2}, 1)$, and let $m \in [0,1]$, with $d > 2f(2,\delta)\delta + 2\delta$, and $(1-c) + \left[m + f(2,\delta)\delta 2^{\delta}(1-c)\right](\delta+\delta^2) < \frac{1}{2}$. Let G be a connected C_4-free odd-signable graph with maximum degree δ, let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$ and $w^{\max} \leq m$, and suppose that G does not have a (w, c, d)-balanced separator. Let $\mathcal{F}$ be a set of forcers of G. Then, there exists a sequence $(\beta_1, w_1), \ldots, (\beta_{2k+1}, w_{2k+1})$, where $\beta_{2k+1} \subseteq \beta_{2k} \subseteq \ldots \subseteq \beta_1 \subseteq \beta_0 = G$, $k \leq f(2,\delta)$, and for $i \in \{1, \ldots, 2k+1\}$, w_i is a weight function on β_i, with $w_i(\beta_i) = 1$, such that:*

- *for $i \in \{0, \ldots, k\}$, β_{2i+1} is the clique-free bag for β_{2i},*
- *for $i \in \{0, \ldots, k-1\}$, β_{2i+2} is the central bag for a tree decomposition corresponding to a laminar collection of proper star separations of β_{2i+1} with clique centers of size 1 or 2 (of size 1 if $\mathcal{F}$ does not contain a short pyramid forcer),*
- *for $i \in \{0, \ldots, k\}$, β_{2i+1} is connected and does not have a (w_{2i+1}, c, d_{2i+1})-balanced separator, for $d_{2i+1} = d - 2i\delta - 2(\delta-1)$, and for $i \in \{0, \ldots, k-1\}$, β_{2i+2} is connected and does not have a (w_{2i+2}, c, d_{2i+2})-balanced separator for $d_{2i+2} = d - 2(i+1)\delta$,*
- *$w_{2k+1}^{\max} \leq w^{\max} + f(2,\delta)\delta 2^{\delta}(1-c) + (\delta-1)2^{\delta}(1-c)$,*
- *no forcer in $\mathcal{F}$ is active for β_{2k+1},*
- *β_{2k+1} has no clique cutset.*

Proof. Let $Y = \{K : (H,K) \in \mathcal{F}\}$ be the set of centers of forcers in $\mathcal{F}$. For all $K \in Y$, $|K| \in \{1,2\}$, and if (H,K) is not a short pyramid forcer, then $|K| = 1$. Let $(Y_1, \ldots, Y_{f(2,\delta)})$ be a partition of Y as in Lemma 4.2 and let $\mathcal{F}_1, \ldots, \mathcal{F}_{f(2,\delta)}$ be a partition of $\mathcal{F}$ such that $Y_i = \{K : (H,K) \in \mathcal{F}_i\}$, for $i \in \{1, \ldots, f(2,\delta)\}$. Let β_1 be the clique-free bag for G and let w_1 be the weight function on β_1 from Theorem 3.4. By Theorem 3.4, β_1 has no clique cutset and no $(w_1, c, d-2(\delta-1))$-balanced separator, where $w_1(\beta_1) = 1$ and $w_1^{\max} \leq w^{\max} + (\delta-1)2^{\delta}(1-c)$. If no forcer in $\mathcal{F}$ is active for β_1, then $k = 0$, and the sequence ends.

Otherwise, assume that there is a forcer in $\mathcal{F}_1$ active for β_1. Let $X_1 = \{S_K : K \in Y_1\}$ be the set of canonical star separations of β_1 for centers in Y_1. Since β_1 has no $(w_1, c, d-2(\delta-1))$-balanced separator and $d - 2(\delta-1) \geq 2$, by Lemma 4.3, every clique K appears as a center of at most one separation in X_1 and every separation in X_1 is $(1-c)$-skewed. Since β_1 has no clique cutset and cliques in Y_1 are pairwise anticomplete, and by Lemma 4.8 the separations in X_1 are all proper, it follows from Lemma 4.1 that X_1 is laminar. Since X_1 is a laminar collection of star separations of β_1 and $(1-c) + w_1^{\max}(\delta+\delta^2) \leq (1-c) + [w^{\max} + (\delta-1)2^{\delta}(1-c)](\delta+\delta^2) < \frac{1}{2}$, by Lemma 2.6, the central bag β_2 for X_1 exists and β_2 does not have a $(w_{X_1}, c, d-2\delta)$-balanced separator. Let $w_2 = w_{X_1}$ be the weight function on β_2 with respect to T_{X_1}, where T_{X_1} is the tree decomposition of β_1 corresponding to X_1. By Lemma 2.5, $w_2(\beta_2) = 1$ and $w_2^{\max} \leq w_1^{\max} + 2^{\delta}(1-c) \leq w^{\max} + \delta 2^{\delta}(1-c)$. By Lemma 4.9, it follows that no forcer in $\mathcal{F}_1$ is active for β_2. By Lemma 2.4, β_2 is connected.

For $i > 0$, we define (β_{2i+1}, w_{2i+1}) and (β_{2i+2}, w_{2i+2}) inductively. For $i \in \{1, \ldots, f(2,\delta)\}$, suppose (β_{2i}, w_{2i}) are such that β_{2i} is connected and has no (w_{2i}, c, d_{2i})-balanced separator for $d_{2i} = d - 2i\delta \geq 1$, $w_{2i}(\beta_{2i}) = 1$, and $w_{2i}^{\max} \leq w^{\max} + i\delta 2^{\delta}(1-c)$. Further, suppose there exists $I_i \subseteq \{1, \ldots, f(2,\delta)\}$ such that $i \leq |I_i| < f(2,\delta)$, no forcer in $\bigcup_{j \in I_i} \mathcal{F}_j$ is active for β_{2i}, and for all $j \in \{1, \ldots, f(2,\delta)\} \setminus I_i$, there is a forcer in $\mathcal{F}_j$ active for β_{2i}.

Since $d > 2f(2,\delta)\delta + 2\delta$ and $i < f(2,\delta)$, it follows that $d_{2i} = d - 2i\delta > 2\delta > 2\delta - 2$. Also, since β_{2i} has no (w_{2i}, c, d_{2i})-balanced separator and $(1-c) + [w_{2i}^{\max} + \delta 2^{\delta}(1-c)](\delta + \delta^2) \leq (1-c) + [w^{\max} + f(2,\delta)\delta 2^{\delta}(1-c)](\delta + \delta^2) < \frac{1}{2}$, the conditions of Theorem 3.4 for β_{2i} are satisfied. Let β_{2i+1} be the clique-free bag for β_{2i} and let w_{2i+1} be the weight function on β_{2i+1} from Theorem 3.4. By Theorem 3.4, β_{2i+1} does not have a $(w_{2i+1}, c, d_{2i} - 2(\delta-1))$-balanced separator, where $w_{2i+1}(\beta_{2i+1}) = 1$ and $w_{2i+1}^{\max} \leq w_{2i}^{\max} + (\delta-1)2^{\delta}(1-c) \leq w^{\max} + i\delta 2^{\delta}(1-c) + (\delta-1)2^{\delta}(1-c)$. Let $d_{2i+1} = d_{2i} - 2(\delta-1)$. If no forcer in $\mathcal{F}$ is active for β_{2i+1}, then $k = i$, and the sequence ends. Otherwise, let $\sigma_i \in \{1, \ldots, f(2,\delta)\} \setminus I_i$ be such that there is a forcer in $\mathcal{F}_{\sigma_i}$ that is active for β_{2i+1}. Let $X_{\sigma_i} = \{S_K : K \in Y_{\sigma_i}\}$ be the set of canonical star separations of β_{2i+1} for centers in Y_{σ_i}. Since β_{2i+1} has no (w_{2i+1}, c, d_{2i+1})-balanced separator, by Lemma 4.3, every clique K appears as the center of at most one separation in X_{σ_i} and every separation in X_{σ_i} is $(1-c)$-skewed. Since β_{2i+1} has no clique cutset and cliques in Y_{σ_i} are pairwise anticomplete and by Lemma 4.8 the separations in Y_{σ_i} are all proper, it follows from Lemma 4.1 that X_{σ_i} is laminar. Finally, $d_{2i+1} > 2$ and, since $i < f(2,\delta)$, $(1-c) + w_{2i+1}^{\max}(\delta + \delta^2) \leq (1-c) + \left[w^{\max} + f(2,\delta)\delta 2^{\delta}(1-c)\right](\delta + \delta^2) < \frac{1}{2}$, so by Lemma 2.6, the central bag β_{2i+2} for X_{σ_i} exists and β_{2i+2} does not have a $(w_{X_{\sigma_i}}, c, d_{2i+2})$-balanced separator, where $d_{2i+2} = d_{2i+1} - 2 = d - 2(i+1)\delta$. Let $w_{2i+2} = w_{X_{\sigma_i}}$ be the weight function on β_{2i+2} with respect to $T_{X_{\sigma_i}}$, where $T_{X_{\sigma_i}}$ is the tree decomposition of β_{2i+1} corresponding to X_{σ_i}. By Lemma 2.5, $w_{2i+2}(\beta_{2i+2}) = 1$ and $w_{2i+2}^{\max} \leq w_{2i+1}^{\max} + 2^{\delta}(1-c) \leq w^{\max} + (i+1)\delta 2^{\delta}(1-c)$. By Lemma 2.4, β_{2i+2} is connected. Let I_{i+1} be the set of all $j \in \{1, \ldots, f(2,\delta)\}$ such that no forcer in $\mathcal{F}_j$ is active for β_{2i+2}. Since $\beta_{2i+2} \subseteq \beta_{2i}$ and no forcer in $\bigcup_{j \in I_i} \mathcal{F}_j$ is active for β_{2i}, it follows that no forcer in $\bigcup_{j \in I_i} \mathcal{F}_j$ is active for β_{2i+2}. Further, since β_{2i+2} is the central bag for a tree decomposition corresponding to X_{σ_i}, it follows from Lemma 4.9 that no forcer in $\mathcal{F}_{\sigma_i}$ is active for β_{2i+2}. Therefore, $|I_{i+1}| \geq i+1$, and (β_{2i+2}, w_{2i+2}) satisfies the conditions of the induction. It follows that the sequence $(\beta_1, w_1), \ldots, (\beta_{2k+1}, w_{2k+1})$ is well-defined, $k \leq f(2,\delta)$, β_{2k+1} does not have a clique cutset, and no forcer in $\mathcal{F}$ is active for β_{2k+1}. ■

We call $(\beta_1, w_1), \ldots, (\beta_{2k+1}, w_{2k+1})$ as in Theorem 4.10 an *$\mathcal{F}$-decomposition* of G, and β_{2k+1} the *terminal bag* for $(\beta_1, w_1), \ldots, (\beta_{2k+1}, w_{2k+1})$. A graph G is *clean* if G does not contain a strong forcer. The following theorem shows that if $\mathcal{F}$ is the collection of all strong forcers of G and β_{2k+1} is the terminal bag for a $\mathcal{F}$-decomposition, then β_{2k+1} is clean.

Theorem 4.11. *Let δ, d be positive integers, let $f(2,\delta) = 2(\delta+1)^2 + 1$, let $c \in [\frac{1}{2}, 1)$, and let $m \in [0,1]$, with $d > 2f(2,\delta)\delta + 2\delta$, and $(1-c) + \left[m + f(2,\delta)\delta 2^{\delta}(1-c)\right](\delta + \delta^2) < \frac{1}{2}$. Let G be a connected C_4-free odd-signable graph with maximum degree δ, let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$ and $w^{\max} \leq m$, and suppose G does not have a (w, c, d)-balanced separator. Let $\mathcal{F}$ be the set of all strong forcers of G, and let $(\beta_1, w_1), \ldots, (\beta_{2k+1}, w_{2k+1})$ be an $\mathcal{F}$-decomposition. Then, the terminal bag β_{2k+1} is clean.*

Proof. Suppose β_{2k+1} contains a strong forcer $F = (H, K)$. Then, F is a strong forcer in G, so $F \in \mathcal{F}$. By Theorem 4.10, it follows that F is not active for β_{2k+1}, a contradiction. ■

5. Twin wheels in clean graphs

In this section we study twin wheels. It turns out that not all twin wheels are clique star cutset forcers, but some of them ("terminal" ones) are. The goal of this section is to show that the central bag for the collection of all twin wheel forcers of a clean graph G does not contain a terminal twin wheel.

Let G be a clean C_4-free odd-signable graph. The following two lemmas describe the behavior of twin wheels in G. Lemma 5.1 follows from the proof of Lemma 8.4 in [12] and Lemma 5.2 follows from the proof of Theorem 1.5 in [12]. For completeness we include their proofs.

Lemma 5.1. ([12]) *Let G be a clean C_4-free odd-signable graph. Let (H, x) be a twin wheel contained in G. Let x_1-x_2-x_3 be the subpath of H such that $N(x) \cap H = \{x_1, x_2, x_3\}$. Suppose there exists a vertex $u \in V(G)$ such that $N(u) \cap (H \cup x) = \{x, x_1, x_1'\}$, where x_1' is the neighbor of x_1 in $H \setminus x_2$. Then, (H, x) is x_2-poor.*

Proof. Let x_1-p_1-$\dots$-p_k-x_3 be the long sector of (H, x), and let $P = p_1$-$\dots$-p_k. Suppose that (H, x) is x_2-rich. Then there exists a path $Q = q_1$-$\dots$-q_l in $G \setminus (N[x] \setminus \{x_2\})$ from x_2 to P. We may assume that Q is chosen to be the minimal such path. Then, q_l has a neighbor in P, x_1 and x_3 are the only nodes of H that may have a neighbor in $Q \setminus q_l$, x_2 is adjacent to q_1, and x_2 does not have a neighbor in $Q \setminus q_1$. Let p_i (resp. $p_{i'}$) be the neighbor of q_l in P with lowest (resp. highest) index.

(1) Both u and x_1 have a neighbor in Q.

$N(u) \cap Q \neq \emptyset$, else $Q \cup \{p_1, \dots, p_i, x_1, x_2, u, x\}$ induces a proper wheel with center x_1, contradicting the assumption that G is clean. Now suppose that $N(x_1) \cap Q = \emptyset$. Let H' be the hole induced by $Q \cup \{p_1, \dots, p_i, x_1, x_2\}$. Since G is clean, (H', u) is a twin wheel, and hence $i = 1$ and $N(u) \cap Q = \{q_l\}$. Since $\{u, x, x_3, q_l\}$ cannot induce a C_4, x_3q_l is not an edge. Since $\{u, x, x_2, q_1\}$ cannot induce a C_4, $l > 1$. Suppose $i' = 1$. If $N(x_3) \cap Q = \emptyset$, then $Q \cup H$ induces a theta. So $N(x_3) \cap Q \neq \emptyset$. Let q_s be the node of $N(x_3) \cap Q$ with highest index. Then $\{q_s, \dots q_l, p_1, x_1, x, x_3, u\}$ induces a proper wheel with center u, a contradiction. So $i' > 1$. But then $\{q_l, p_{i'}, \dots, p_k, u, x_1, x_2, x_3, x\}$ induces a proper wheel with center x, a contradiction. This proves (1).

(2) $N(x_3) \cap Q = \emptyset$.

Suppose x_3 has a neighbor in Q. By (1), let q_s (resp. q_t) be the node of Q with the lowest index adjacent to x_1 (resp. u). If $s \leq t$, then $\{q_1, \dots, q_t, u, x, x_1, x_2\}$ induces a proper wheel with center x_1. So $s > t$. In particular, $t < l$ and $s > 1$. If x_3 has a neighbor in $Q \setminus q_l$, then $(Q \setminus q_l) \cup P \cup \{u, x, x_3\}$ contains a theta between u and x_3. So x_3 has no neighbor in $Q \setminus q_l$, and hence $N(x_3) \cap Q = \{q_l\}$. Let H' be the hole induced by $Q \cup \{x_2, x_3\}$. Since $H' \cup x_1$ cannot induce a theta, (H', x_1) is a wheel. Since $s > 1$, (H', x_1) is a proper wheel or a short pyramid, contradicting that G is clean. This proves (2).

By (1), let q_s (resp. q_t) be the node of Q with lowest index adjacent to x_1 (resp. u). If $s = 1$ then $\{q_1, \dots, q_t, x, x_2, x_1, u\}$ induces a proper wheel with center x_1, a contradiction. So $s > 1$. By (2), $Q \cup \{p_{i'}, \dots, p_k, x_2, x_3\}$ induces a hole H'. But then, since $s > 1$, either $H' \cup x_1$ induces a theta, or (H', x_1) is a proper wheel or a short pyramid, a contradiction. ■

Lemma 5.2. ([12]) *Let G be a clean C_4-free odd-signable graph. Let (H, x) be a twin wheel contained in G, let $N(x) \cap H = \{x_1, x_2, x_3\}$, where x_2 is the clone of x in H, and suppose (H, x, x_2) is not a terminal twin wheel. Then, there exists a path $P = p_1$-$\dots$-p_k in $G \setminus (H \cup x)$ such that $N(p_1) \cap (H \cup x) = \{x\}$, $N(p_k) \cap (H \cup x)$ is an edge of $H \setminus \{x_1, x_2, x_3\}$, and P^* is anticomplete to $H \cup x$. Similarly, there exists a path $Q = q_1$-$\dots$-q_j in $G \setminus (H \cup x)$ such that $N(q_1) \cap (H \cup x) = \{x_2\}$, $N(q_j) \cap (H \cup x)$ is an edge of $H \setminus \{x_1, x_2, x_3\}$, and Q^* is anticomplete to $H \cup x$.*

Proof. Since (H,x) is not terminal, it follows that (H,x) is x-rich and x_2-rich. Let x_1-q_1-$\dots$-q_l-x_3 be the long sector of (H,x), and let Q be the path q_1-$\dots$-q_l. Then by Lemma 5.1, there does not exist a node u such that $N(u)\cap(H\cup x)=\{x,x_1,q_1\}$, and by symmetry, there does not exist a node u such that $N(u)\cap(H\cup x)=\{x,x_3,q_l\}$. Since (H,x) is x-rich, there exists a path $P=p_1$-$\dots$-p_k in $G\setminus(N[x_2]\setminus\{x\})$ from x to Q. We may assume that P is chosen to be the minimal such path. Then, p_k has a neighbor in Q, x_1 and x_3 are the only nodes of H that may have a neighbor in $P\setminus p_k$, and $N(p_1)\cap(H\cup x)=\{x\}$. Let q_i (resp. $q_{i'}$) be the neighbor of p_k in Q with lowest (resp. highest) index.

(1) $\{x_1,x_3\}$ is anticomplete to P.

Suppose that one of x_1,x_3 has a neighbor in P. Since $\{x_2,x_2,x_3,p_k\}$ does not induce a C_4, not both x_1,x_3 are adjacent to p_k. Since $H\cup(P\setminus p_k)$ does not contain a theta between x_1 and x_3, it follows that at least one of x_1,x_3 is anticomplete to $P\setminus p_k$. It follows (exchanging the roles of x_1,x_3 if necessary) that we may assume that x_3 has a neighbor in P, and x_1 is anticomplete to $P\setminus p_k$.

Since $\{x_1,p_1,x_3,x_2\}$ does not induce a C_4, it follows that if $k=1$, then x_1 is non-adjacent to p_k. Consequently, $P\cup\{x_1,x,q_1,\dots,q_i\}$ induces a hole H'. Since $H'\cup x_3$ does not induce a theta or a strong forcer, x_3 is adjacent to p_1 and $N(x_3)\cap H'\subseteq N(p_1)\cap H'$. If $N(x_3)\cap H'=\{x,p_1\}$, then $H'\cup\{x_2,x_3\}$ induces a proper wheel with center x. So $N(x_3)\cap H'=N(p_1)\cap H'$.

Let H'' be the hole induced by $(H'\setminus\{x,p_1\})\cup\{x_2,x_3\}$. Then (H'',x) is a twin wheel, and $N(p_1)\cap(H''\cup x)=\{x,x_3,x_3'\}$, where x_3' is the neighbor of x_3 in $H''\setminus x_2$. Since (H,x) is x_2-rich, there is a path R in $G\setminus(N[x]\setminus x_2)$ from x_2 to Q. It follows $R\cup\{q_{i'},\dots,q_l\}$ contains a path showing that (H'',x) is x_2-rich. But Lemma 5.1 (with p_1 playing the role of u) implies that (H'',x) is x_2-poor, a contradiction. This proves (1).

If $k=1$ then (since by (1) $\{x_1,x_3\}$ is anticomplete to P) $(H\setminus x_2)\cup P\cup x$ induces a theta or a strong forcer. So $k>1$. If $i=i'$ or $p_ip_{i'}$ is not an edge, then the graph induced by $(H\setminus x_2)\cup P\cup x$ contains a theta between x and either p_k (when $i\neq i'$) or p_i (when $i=i'$). So $p_ip_{i'}$ is an edge. By symmetry between x and x_2, the result follows.. ■

We now use 5.2 to show that twin wheel forcers can be used in a way similar to strong forcers.

Theorem 5.3. *Let δ,d be positive integers, let $f(2,\delta)=2(\delta+1)^2+1$, let $c\in[\frac{1}{2},1)$, and let $m\in[0,1]$, with $d>2f(2,\delta)\delta+2\delta$ and $(1-c)+[m+f(2,\delta)\delta 2^{\delta}(1-c)](\delta+\delta^2)<\frac{1}{2}$. Let G be a connected clean C_4-free odd-signable graph with maximum degree δ, let $w:V(G)\to[0,1]$ be a weight function on G with $w(G)=1$ and $w^{\max}\leq m$, and suppose G does not have a (w,c,d)-balanced separator. Let $\mathcal{T}$ be the set of all twin wheel forcers in G and let $(\beta_1,w_1),\dots,(\beta_{2k+1},w_{2k+1})$ be a $\mathcal{T}$-decomposition of G. Then, β_{2k+1} does not contain a terminal twin wheel.*

Proof. Let $\beta_0=G$.

(1) For $i\in\{1,\dots,2k+1\}$, if (H,x,x_2) is a terminal twin wheel in β_i, then (H,x,x_2) is a terminal twin wheel in β_{i-1}.

Let (H,x,x_2) be a terminal wheel in β_i, with $N(x)\cap H=\{x_1,x_2,x_3\}$, and suppose (H,x,x_2) is not a terminal wheel in β_{i-1}. Since (H,x,x_2) is not a terminal twin wheel in β_{i-1}, by Lemma 5.2 there exists a path $P=p_1$-$\dots$-p_m in β_{i-1} such that $N(p_1)\cap(H\cup x)=\{x_2\}$, $N(p_m)\cap(H\cup x)$ is an edge of $H\setminus\{x_1,x_2,x_3\}$, and P^* is anticomplete to $H\cup x$. Similarly, there exists a path $Q=q_1$-$\dots$-q_t in β_{i-1} such that $N(q_1)\cap(H\cup x)=\{x\}$, $N(q_t)\cap(H\cup x)$ is an edge of $H\setminus\{x_1,x_2,x_3\}$, and Q^* is anticomplete to $H\cup x$. Since (H,x,x_2) is a terminal twin wheel in β_i, we may assume that $V(P)\not\subseteq V(\beta_i)$. If i is odd, then by the definition of $\mathcal{T}$-decomposition, β_i is the clique-free bag of β_{i-1}. By the definition of the clique-free bag, it follows that β_i is an induced subgraph of β_{i-1} obtained by decomposing β_{i-1} with clique cutsets. Since $H \cup x \cup P$ does not have a clique cutset, it follows that $H \cup x \cup P$ is contained in β_i, a contradiction. Therefore, i is even, and so by the definition of $\mathcal{T}$-decomposition, β_i is the central bag for a tree decomposition corresponding to a laminar collection of proper star separations in β_{i-1}. Let $p_0 = x_2$ and let p_{m+1} be a neighbor of p_m in H. Let $\ell \in \{1, \dots, m\}$ and $j \in \{1, \dots, m+1\}$ be such that $\ell < j$, $p_{\ell-1}, p_j \in \beta_i$, and $p_s \notin \beta_i$ for $\ell \leq s < j$. It follows that $p_{\ell-1}$ and p_j have neighbors in a connected component of $\beta_{i-1} \setminus \beta_i$. Since β_i is the central bag for a tree decomposition corresponding to a collection of star separations in β_{i-1}, it follows that $p_{\ell-1}$ and p_j are in a star cutset of β_{i-1}. In particular, there exists $v \in \beta_i$ such that $p_{\ell-1}, p_j \in N[v]$. Since P^* is anticomplete to $H \cup x$, it follows that $v \notin H$.

Since there does not exist a path from x_2 to $H \setminus \{x_1, x_2, x_3\}$ in β_i not containing a neighbor of x, it follows that v is adjacent to x, and thus $p_{\ell-1}, p_j \neq v$. Let $N(p_m) \cap (H \cup x) = \{h_1, h_2\}$, where h_1 is on the path from x_1 to h_2 through $H \setminus x_2$. We may assume that if v is adjacent to one of h_1, h_2, then v is adjacent to h_1 and $h_1 = p_{m+1}$. Let R be the path from h_1 to x_1 not containing h_2 in H. Consider the hole H' given by x_1-x_2-p_1-P-p_m-h_1-R-x_1. Then, v has two non-adjacent neighbors $p_{\ell-1}$ and p_j in H'. Since G is clean and theta-free, it follows that (H', v) is a twin wheel. Since v is adjacent to both $p_{\ell-1}$ and p_j, and $p_{\ell-1}p_j$ is not an edge, and (H', v) is a twin wheel, either all the neighbors of v in H' are contained in $R \cup x_2$, or they are all contained in $P \cup \{p_0, p_{m+1}\}$. Since v has at least 2 neighbors in $P \cup \{p_0, p_{m+1}\}$, it follows that either $p_j = h_1 = p_{m+1}$, $p_{\ell-1} = p_0$, and $N(v) \cap (H \cup P) = \{x_1, x_2, h_1\}$, where h_1x_1 is an edge and v has no other neighbors in H because G is clean; or $j = \ell + 1$ and $N(v) \cap H' = \{p_{\ell-1}, p_\ell, p_{\ell+1}\}$. In the first case, $h_2 \in H \setminus N[v]$ and $p_m h_2$ is an edge, so P and $H \setminus N[v]$ are in the same connected component of $\beta_{i-1} \setminus N[v]$. Since $H \subseteq \beta_i$, it follows that $P \subseteq \beta_i$, a contradiction. Therefore, the second case holds. Now, consider the hole H'' given by x_1-x_2-p_1-P-$p_{\ell-1}$-v-p_j-P-p_m-h_1-R-x_1. Then, $N(x) \cap H'' = \{x_1, x_2, v\}$, and since G is clean, (H'', x) is not a short pyramid. Therefore, $p_{\ell-1} = x_2 = p_0$.

Let S be the path from h_2 to x_3 in $H \setminus \{h_1\}$. Since $N(v) \cap H' = \{p_0, p_1, p_2\}$, it follows that v has no neighbors in $P \setminus \{p_1, p_2\}$. Further, since v has three neighbors x_2, p_1, p_2 in the hole given by x_2-x_3-S-h_2-p_m-P-p_1-x_2, it follows that v has no neighbors in S. Therefore, let H''' be the hole given by x-v-p_2-P-p_m-h_2-S-x_3-x. Then, (H''', x_2) is a twin wheel, where x is the clone of x_2 in H'''. Furthermore, there is a path contained in $Q \cup (P \setminus p_1) \cup (H \setminus x_2)$ from x to $H''' \setminus \{v, x, x_3\}$ containing no neighbor of x_2 other than x, so (H''', x_2) is x-rich. But $N(p_1) \cap (H''' \cup x_2) = \{p_2, v, x_2\}$, contradicting Lemma 5.1. This proves (1).

Suppose that β_{2k+1} contains a terminal twin wheel (H, x, x_2). By (1), it follows that (H, x, x_2) is a terminal twin wheel in G, so we may assume that $F = (H, \{x\})$ is a twin wheel forcer in G. Then, by Theorem 4.10, F is not active for β_{2k+1}, a contradiction. Therefore, β_{2k+1} does not contain a terminal twin wheel. ■

The following lemma shows that if G is a graph with no balanced separator, no clique cutset, and no forcer, then G has no star cutset.

Lemma 5.4. *Let $c \in [\frac{1}{2}, 1)$. Let G be a theta-free graph, let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$ and $w^{\max} \leq m$, and suppose that G has no $(w, c, 1)$-balanced separator, G has no clique cutset, and G has no forcer. Then G has no star cutset.*

Proof. Suppose G has a star cutset C' centered at v and let (A', C', B') be a star separation such that $A', B' \neq \emptyset$. Let (A, C, B) be the canonical star separation for $\{v\}$. Since G has no $(w, c, 1)$-balanced separator, $G \setminus N[v] \neq \emptyset$, and therefore $B \neq \emptyset$. Without loss of generality let $B \subseteq B'$. Then, $A' \subseteq A$, and therefore $A \neq \emptyset$.

Let A^* be a component of A. Since G does not have a clique cutset, it follows that there exist $u_1, u_2 \in N(A^*)$ such that $u_1u_2 \notin E(G)$. Let P be a path from u_1 to u_2 through B and let Q be a shortest path from u_1 to u_2 through A^*. Let H be the hole given by u_1-Q-u_2-P-u_1. Then, v has two non-adjacent neighbors in H. Because G is clean and theta-free, it follows that (H, v) is not a proper wheel or a short pyramid. Therefore, (H, v) is a twin wheel, and since by definition of canonical star separation v has no neighbor in B, $Q = u_1$-a-u_2 for some vertex $a \in A^*$, and a is the clone of v in H. Since every path from a to B intersects $N[v]$, it follows that (H, v) is a-poor, so (H, v, a) is a terminal twin wheel in G, a contradiction. ■

6. Graphs with no star cutset

In this section, we show that if G is a C_4-free odd-signable graph with bounded degree and no star cutset, then G has bounded treewidth. A partition (X_1, X_2) of the vertex set of a graph G is a *2-join* if for $i = 1, 2$ there exist disjoint nonempty $A_i, B_i \subseteq X_i$ satisfying the following:

- A_1 is complete to A_2, B_1 is complete to B_2, and there are no other edges between X_1 and X_2;
- for $i = 1, 2$, $G[X_i]$ contains a path with one end in A_i, one end in B_i and interior in $X_i \setminus (A_i \cup B_i)$ and $G[X_i]$ is not a path.

We say that $(X_1, X_2, A_1, B_1, A_2, B_2)$ is a *split* of the 2-join (X_1, X_2). A *long pyramid* is a pyramid all of whose three paths are of length at least 2. An *extended nontrivial basic* graph R is defined as follows:

- $V(R) = V(L) \cup \{x, y\}$.
- L is the line graph of a tree T.
- x and y are adjacent, and $\{x, y\} \cap V(L) = \emptyset$.
- L contains at least two maximal cliques of size at least 3.
- The vertices of L corresponding to the edges incident with vertices of degree 1 in T are called *leaf vertices*. Each leaf vertex of L is adjacent to exactly one of $\{x, y\}$ and no other vertex of L is adjacent to a vertex of $\{x, y\}$.
- These are the only edges in R.

We observe that in order to prove the decomposition theorem for C_4-free odd-signable graphs, extended nontrivial basic graphs are defined in a more complicated way in [12], but for what we want to prove here the above definition suffices. Let $\mathcal{B}^*$ be the class of graphs that consists of cliques, holes, long pyramids and extended nontrivial basic graphs.

Theorem 6.1. ([12]) *A C_4-free odd-signable graph either belongs to $\mathcal{B}^*$ or it has a star cutset or a 2-join.*

Let G be a graph and $(X_1, X_2, A_1, B_1, A_2, B_2)$ a split of a 2-join of G. The *blocks of decomposition* of G with respect to (X_1, X_2) are graphs G_1 and G_2 defined as follows. Block G_1 is obtained from $G[X_1]$ by adding a *marker path* $P_2 = a_2$-…-b_2 of length 3 such that a_2 is complete to A_1, b_2 is complete to B_1, and these are the only edges between P_2 and X_1. Block G_2 is obtained analogously from $G[X_2]$ by adding a marker path $P_1 = a_1$-…-b_1.

The following lemma follows from the proofs of Lemmas 3.5 and 3.7 in [22].

Lemma 6.2. ([22]) *Let G be a C_4-free graph with no star cutset, let (X_1, X_2) be a 2-join of G, and G_1 and G_2 the corresponding blocks of decomposition. Then G_1 and G_2 do not have star cutsets.*

Below, we prove that if G is a C_4-free odd-signable graph and $(X_1, X_2, A_1, B_1, A_2, B_2)$ is a split of a 2-join of G, then the blocks of decomposition of G with respect to (X_1, X_2) are also C_4-free odd-signable.

Lemma 6.3. *Let G be a C_4-free odd-signable graph with no star cutset, let (X_1, X_2) be a 2-join of G, and G_1 and G_2 the corresponding blocks of decomposition. Then G_1 and G_2 are C_4-free odd-signable.*

Proof. By constructions of the blocks, clearly G_1 and G_2 are C_4-free. So by Theorem 1.7 it suffices to show that if G_1 contains an even wheel, theta or a prism Σ, then G contains an even wheel, theta or a prism. Let $(X_1, X_2, A_1, B_1, A_2, B_2)$ be the split of (X_1, X_2), and let $P_2 = a_2\text{-}\dots\text{-}b_2$ be the marker path of G_1. We may assume that $\Sigma \cap P_2 \neq \emptyset$, since otherwise we are done. Suppose that A_2 is complete to B_2. By definition of 2-join, either $X_2 \setminus (A_2 \cup B_2) \neq \emptyset$, or, without loss of generality, $|B_2| \geq 2$. So for $u \in B_2$, $S = A_2 \cup B_1 \cup \{u\}$ is a star cutset in G separating $X_1 \setminus B_1$ from $X_2 \setminus (A_2 \cup \{u\})$. Therefore, A_2 is not complete to B_2, so let $a \in A_2$ and $b \in B_2$ be such that ab is not an edge. By definition of 2-join, there exists a path Q_2 in $G[X_2]$ whose one end is in A_2, the other in B_2 and whose interior is in $X_2 \setminus (A_2 \cup B_2)$.

First suppose that $\Sigma = (H, x)$ is an even wheel. If $H \subseteq X_1$ then without loss of generality $x = a_2$, and hence (H, a) is an even wheel in G. So we may assume that $H \cap P_2 \neq \emptyset$. It follows that without loss of generality, $H \cap P_2 \in \{\{a_2\}, \{a_2, b_2\}, P_2\}$. It follows that $x \in X_1$. If $H \cap P_2 = \{a_2\}$ then let $H' = (H \setminus \{a_2\}) \cup \{a\}$; if $H \cap P_2 = \{a_2, b_2\}$ then let $H' = (H \setminus \{a_2, b_2\}) \cup \{a, b\}$; and if $H \cap P_2 = P_2$ then let $H' = (H \setminus P_2) \cup Q_2$. Then clearly (H', x) is an even wheel in G.

Now assume that Σ is a theta or a prism. Let R_1, R_2, R_3 be the three paths of Σ. Note that any two of the paths induce a hole, and assume up to symmetry that out of the three holes so created, the hole $H = R_1 \cup R_2$ has the largest intersection with P_2. Then without loss of generality $H \cap P_2 = \{a_2\}, \{a_2, b_2\}$ or P_2. If $H \cap P_2 = \{a_2\}$ then let $H' = (H \setminus \{a_2\}) \cup \{a\}$; if $H \cap P_2 = \{a_2, b_2\}$ then let $H' = (H \setminus \{a_2, b_2\}) \cup \{a, b\}$; and if $H \cap P_2 = P_2$ then let $H' = (H \setminus P_2) \cup Q_2$. Then clearly H' is a hole in G. By the choice of H it follows that $|R_3 \cap P_2| \leq 1$ and hence either $R_3 \subseteq X_1$, or $H \cap P_2 = \{a_2\}$ and $R_3 \cap P_2 = \{b_2\}$. In the first case clearly $H' \cup R_3$ is a theta or a prism, so assume that $H \cap P_2 = \{a_2\}$ and $R_3 \cap P_2 = \{b_2\}$. Then, up to symmetry, $a_2 \in R_2$. But then it follows that the hole $R_2 \cup R_3$ has a larger intersection with P_2 than H, a contradiction. ■

Let G be a graph. A *flat path* in G is a path of G of length at least 2 whose interior vertices all have degree 2 in G and whose ends do not have a common neighbor outside this path. A *leaf* in a graph is a vertex of degree at most 1. Let $\mathcal{D}$ be a class of graphs and $\mathcal{B} \subseteq \mathcal{D}$. Given a graph $G \in \mathcal{D}$, a rooted tree T_G is a *2-join decomposition tree for G with respect to $\mathcal{B}$* if the following hold:

- Each vertex of T_G is a pair $(H, \mathcal{M})$ where H is a graph in $\mathcal{D}$ and $\mathcal{M}$ is a set of vertex-disjoint flat paths of H.
- The root of T_G is $(G, \emptyset)$.
- Each non-leaf vertex of T_G is $(G', \mathcal{M}')$ where G' has a 2-join (X_1, X_2) such that the edges between X_1 and X_2 do not belong to any flat path in $\mathcal{M}'$. Let $\mathcal{M}_1$ (respectively $\mathcal{M}_2$) be the set of all flat paths of $\mathcal{M}'$ that belong to $G[X_1]$ (respectively $G[X_2]$). Let G_1 and G_2 be the blocks of decomposition of G' with respect to 2-join (X_1, X_2) with marker paths P_2 and P_1 respectively. The vertex $(G', \mathcal{M}')$ has two children, which are $(G_1, \mathcal{M}_1 \cup \{P_2\})$ and $(G_2, \mathcal{M}_2 \cup \{P_1\})$.
- Each leaf vertex of T_G is $(G', \mathcal{M}')$ where $G' \in \mathcal{B}$.

The following theorem follows from Lemma 4.6 in [22].

Theorem 6.4. ([22]) *Let G be a graph and let $\mathcal{M}$ be a set of vertex-disjoint flat paths of G. Then one of the following holds:*

(i) G has no 2-join.
(ii) There exists a 2-join (X_1, X_2) of G such that for every path $P \in \mathcal{M}$, $P \subseteq X_1$ or $P \subseteq X_2$.
(iii) G or a block of decomposition with respect to some 2-join of G has a star cutset.

The following lemma shows that C_4-free odd-signable graphs with no star cutset have 2-join decomposition trees with respect to $\mathcal{B}^*$.

Lemma 6.5. *If G is a C_4-free odd-signable graph with no star cutset then G has a 2-join decomposition tree with respect to $\mathcal{B}^*$.*

Proof. If G is a C_4-free odd-signable graph that has no star cutset then, by Lemmas 6.2 and 6.3, blocks of decomposition of G with respect to every 2-join are C_4-free odd-signable and have no star cutset. So by repeated application of Theorem 6.4 there is a 2-join decomposition tree for G in which the leaves correspond to C_4-free odd-signable graphs that have no star cutset and no 2-join, and hence by Theorem 6.1 are graphs from $\mathcal{B}^*$, i.e. the result holds. ■

The *rankwidth* of a graph G, denoted by $\text{rw}(G)$, is a property of G similar to treewidth. The definition of rankwidth can be found in [18] (where it was first defined). The following theorem bounds the rankwidth of graphs that have a 2-join decomposition tree with respect to $\mathcal{B}^*$.

Theorem 6.6. ([16, 17]) *If $\mathcal{D}$ is a class of graphs such that every $G \in \mathcal{D}$ has a 2-join decomposition tree with respect to $\mathcal{B}^*$, then $rw(G) \leq 3$.*

Corollary 6.7. *If G is a C_4-free odd-signable graph with no star cutset then $rw(G) \leq 3$.*

Proof. Follows from Theorem 6.6 and Lemma 6.5. ■

The following theorem bounds the treewidth of G by a function of the rankwidth of G for graphs G with no subgraph isomorphic to $K_{r,r}$, where $K_{r,r}$ is a complete bipartite graph with r vertices in both sides of the bipartition.

Theorem 6.8. ([13]) *If G is a graph that has no subgraph isomorphic to $K_{r,r}$, then $tw(G)+1 \leq 3(r-1)(2^{rw(G)+1}-1)$.*

Finally, we show that the treewidth of G is bounded by a function of δ.

Corollary 6.9. *If G is a C_4-free odd-signable graph with maximum degree δ and no star cutset then $tw(G) \leq 45\delta - 1$.*

Proof. Follows from Corollary 6.7 and Theorem 6.8. ■

7. Balanced separators in C_4-free odd-signable graphs

Let δ be a positive integer and let G be a C_4-free odd-signable graph with maximum degree δ. In this section, we prove Theorem 1.5, showing that G has a balanced separator. We begin by stating a helpful lemma showing that if G has bounded treewidth, then G has a balanced separator.

Lemma 7.1 ([10], Lemma 7.19)**.** *Let G be a graph with treewidth at most k and let $w : V(G) \to [0,1]$ be a weight function of G with $w(G) = 1$. Then, G has a $(w, \frac{1}{2}, k+1)$-balanced separator.*

Now, we prove that if G is a clean C_4-free odd-signable graph with maximum degree δ, then G has a balanced separator.

Theorem 7.2. *Let δ, d be positive integers, let $c \in [\frac{1}{2}, 1)$, let $m \in [0,1]$, and let $f(2,\delta) = 2(\delta+1)^2+1$, with $d \geq 47\delta + 2f(2,\delta)\delta - 2$, and $(1-c) + [m + 2f(2,\delta)\delta 2^{\delta}(1-c) + (\delta-1)2^{\delta}(1-c)](\delta+\delta^2) < \frac{1}{2}$. Let G be a connected clean C_4-free odd-signable graph with maximum degree δ and let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$ and $w^{\max} \leq m$. Then, G has a (w, c, d)-balanced separator.*

Proof. Suppose that G does not have a (w, c, d)-balanced separator. Let $\mathcal{T}$ be the set of all twin wheel forcers in G and let β_{2k+1} be the terminal bag of a $\mathcal{T}$-decomposition of G, with $k \leq f(2,\delta)$. It follows from Theorem 4.10 that β_{2k+1} does not have a clique cutset or a $(w', c, d-2k\delta-2(\delta-1))$-balanced separator for some weight function w' with $w'(\beta_{2k+1}) = 1$ and $w'^{\max} \leq w^{\max} + f(2,\delta)\delta 2^{\delta}(1-c) + (\delta-1)2^{\delta}(1-c)$. By Theorem 5.3, β_{2k+1} does not contain a terminal twin wheel.

By Lemma 5.4, β_{2k+1} has no star cutset. Since β_{2k+1} has no star cutset, it follows from Corollary 6.9 that $\text{tw}(\beta_{2k+1}) \leq 45\delta - 1$. By Lemma 7.1, β_{2k+1} has a $(w', \frac{1}{2}, 45\delta)$-balanced separator. Since $d - 2k\delta - 2(\delta - 1) \geq d - 2f(2,\delta)\delta - 2(\delta - 1) \geq 45\delta$ and $c \geq \frac{1}{2}$, it follows that β_{2k+1} has a $(w', c, d - 2k\delta - 2(\delta - 1))$-balanced separator, a contradiction. ■

Finally, we prove Theorem 1.5.

Theorem 1.5. *Let δ, d be positive integers. Let G be a connected C_4-free odd-signable graph with maximum degree δ and let $w : V(G) \to [0,1]$ be a weight function such that $w(G) = 1$. Let $f(2,\delta) = 2(\delta+1)^2 + 1$, and let $c \in [\frac{1}{2}, 1)$. Assume that $d \geq 49\delta + 4f(2,\delta)\delta - 4$ and $(1-c) + [w^{\max} + 3f(2,\delta)\delta 2^{\delta}(1-c) + 2(\delta-1)2^{\delta}(1-c)](\delta + \delta^2) < \frac{1}{2}$. Then, G has a (w,c,d)-balanced separator.*

Proof. Suppose that G does not have a (w,c,d)-balanced separator. Let $\mathcal{F}$ be the set of all strong forcers of G and let β_{2k+1} be the terminal bag for an $\mathcal{F}$-decomposition of G, with $k \leq f(2,\delta)$. By Theorem 4.10, β_{2k+1} is connected and does not have a $(w', c, d - 2k\delta - 2(\delta-1))$-balanced separator for some weight function w' with $w'(\beta_{2k+1}) = 1$ and ${w'}^{\max} \leq w^{\max} + f(2,\delta)\delta 2^{\delta}(1-c) + (\delta-1)2^{\delta}(1-c)$, and by Theorem 4.11, β_{2k+1} is connected and clean. Since β_{2k+1} is clean, it follows from Theorem 7.2 that β_{2k+1} has a $(w', c, d - 2k\delta - 2(\delta-1))$-balanced separator, a contradiction. ■

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