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Brzezniak, Zdzislaw orcid.org/0000-0001-8731-6523, Peng, Xuhui and Zhai, Jianliang (2022) Well-posedness and large deviations for 2-D Stochastic Navier-Stokes equations with jumps. Journal of the european mathematical society. pp. 3093-3176. ISSN 1435-9855

https://doi.org/10.4171/JEMS/1214

J. Eur. Math. Soc. 25, 3093–3176 (2023) DOI 10.4171/JEMS/1214

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Well-posedness and large deviations for 2D stochastic Navier–Stokes equations with jumps

Received December 3, 2019

Abstract. The aim of this paper is threefold. Firstly, we prove the existence and uniqueness of a global strong (in both the probabilistic and the PDE senses) H^1_2-valued solution to the 2D stochastic Navier–Stokes equations (SNSEs) driven by a multiplicative Lévy noise under the natural Lipschitz condition on balls and linear growth assumptions on the jump coefficient. Secondly, we prove a Girsanov-type theorem for Poisson random measures and apply this result to a study of the well-posedness of the corresponding stochastic controlled problem for these SNSEs. Thirdly, we apply these results to establish a Freidlin–Wentzell-type large deviation principle for the solutions of these SNSEs by employing the weak convergence method introduced by Budhiraja et al. (2011, 2013).

Keywords. 2D stochastic Navier–Stokes equations, Lévy processes, Girsanov theorem, strong solutions in the probabilistic and PDE senses, Freidlin–Wentzell-type large deviation principle

Contents

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Mathematics Subject Classification (2020): Primary 60H15; Secondary 60F10, 76M35, 76D06

1. Introduction

In this paper, we concentrate on stochastic Navier–Stokes equations (SNSEs), but we believe that our results can be generalized to other types of stochastic partial differential equations (SPDEs). Since the seminal work [4] by Bensoussan and Temam, a great number of papers have been written on SNSEs driven by Gaussian noise. The questions of the existence and uniqueness of solutions to such equations have been investigated in many papers; see, for example, [13, 23, 35, 36, 47, 52]. The Freidlin–Wentzell-type large deviation principle for 2D SNSEs has been proved in [21, 57]. In a recent paper [61], the authors established the moderate deviation principle for these equations. The ergodic properties of invariant measures of the Markov semigroups generated by SNSEs (in the Gaussian case) and related questions have been studied in [8, 34, 39].

However, some real-world models of financial, economic, physical, chemical, and biological phenomena cannot be well described by Gaussian noise. For example, in some circumstances, some large moves and unpredictable events can be captured by jump type noises. In recent years, SPDEs driven by jump type Lévy noise have become extremely popular in modelling these phenomena.

Much effort has been put into understanding various properties of SPDEs driven by general Lévy noise. Compared with the Gaussian case, SPDEs driven by pure jump Lévy processes behave in a drastically different manner. Examples are provided by (i) the Burkholder–Davis–Gundy inequality – see, for instance, [41, 69]; (ii) the Girsanov theorem – see, for example, [44, Theorem III.3.24] for semimartingales, [7, Theorem 3.10.21] for Poisson point processes, [7, Theorem 3.9.19] for finite-dimensional Wiener processes, and [22, Appendix A.1.] for infinite dimensions with respect to a cylindrical Wiener process; (iii) the time regularity of solutions – see, for instance, [43] for OU processes driven by a cylindrical Wiener process, [3] on the uniform convergence of random series in Skorokhod space and representations of càdlàg infinitely divisible processes, and [9, 48, 49, 54, 55] for OU processes driven by cylindrical pure jump processes; (iv) ergodicity – see, for example, [6, 33, 55] for the pure jump case; (v) irreducibility – see, for example, [27, 33, 40, 55, 60] for the pure jump case; and (vi) other long-time properties of the solutions to SPDEs driven by jump processes – see [19, 20].

In general, the methods and techniques available for SPDEs driven by Gaussian noise are not suitable for investigating SPDEs driven by jump type noise, and therefore new and sophisticated tools are needed. We refer to the above-mentioned references and references therein for more details.

As an example let us consider SNSEs. Under the classical Lipschitz condition on balls and linear growth assumptions on the noise coefficients, one can prove the existence and uniqueness of a strong solution in both the probabilistic and PDE senses for 2D SNSEs driven by Gaussian noise; see, for example, [36, 47, 51]. However, in order to prove a similar result for the pure jump case, in the existing literature one is required to introduce additional conditions on the jump coefficient G (see Remarks 3.2 and 4.2). The reason for this difference is that the proof of the existence of solutions relies on the use of the Burkholder–Davis–Gundy inequality for the compensated Poisson random measures for the exponent $p \neq 2$ (see [10, 11, 69]). Similar problems arise in the study of martingale solutions; see, for instance, [30, 35, 53] and many more recent papers.

A natural approach to proving the well-posedness of SPDEs driven by jump type Lévy process is to approximate the Poisson random measure N by a sequence of Poisson random measures N_n with finite intensity measures. Dong and Xie [28] used this method to establish the well-posedness of strong solutions in the probabilistic sense for 2D SNSEs driven by Lévy noise. However, to make this method work, one needs to impose additional assumptions to control the "small jumps" (see Remark 3.2). Another basic idea used to prove the well-posedness of SPDEs is based on introducing an appropriate cut-off and then applying the Banach fixed point theorem for the approximated problems. This method has been exploited in a recent paper [5] to establish the existence of strong solutions in the PDE sense for 2D SNSEs driven by Lévy processes of jump type. However, because they relied on the Burkholder–Davis–Gundy inequality with exponent $p \neq 2$ for the compensated Poisson random measure, in addition to the natural Lipschitz and linear growth assumptions, the authors of [5] had to impose additional and unnatural assumptions on the noise coefficient G (see Remark 4.2).

The first aim of this paper is to remove these unnecessary assumptions imposed in [5] and other papers. For this purpose we employ different ideas and techniques. We use the cut-off approximation method and the Banach fixed point theorem, used recently by the first author and Millet [12] and later in [5] to prove the existence of strong solutions in the PDE sense to a class of stochastic hydrodynamical systems driven by a Lévy process. Earlier, a similar idea had been used by De Bouard and Debussche [25, 26]. However, our auxiliary equations are different.

Using these auxiliary equations, we are able to remove the atypical assumptions described above. Our method strongly depends on the cut-off function θ_m introduced in (3.12). To achieve our goals, it is crucial to establish new a priori estimates. We believe that this method can also be used for other systems driven by Lévy noise to weaken the assumptions and, in particular, eliminate those that are not necessary.

Our second, and in fact, main, aim of this paper is to establish a Freidlin–Wentzell-type large deviation principle (LDP) for strong solutions in the PDE sense (obtained in the first part) of 2D SNSEs driven by Lévy processes of jump type.

The large deviation principle for finite-dimensional stochastic differential equations (SDEs) with a Poisson noise term has been studied by several authors [2, 16, 24]. There is not much study of the LDP for infinite-dimensional SDEs driven by Lévy processes of jump type. The first paper was [56] by Röckner and Zhang, where stochastic evolution equations with additive noise was considered. The case of multiplicative noise was studied in [15, 16, 58, 64]. The study of the LDP for SPDEs with highly nonlinear terms has been carried out as well [29, 62, 63, 66, 68]. Concerning 2D SNSEs, it is important to mention that Xu and Zhang [63] studied the LDP for these equations driven by additive Lévy noise, while the recent papers by the third author and collaborators deal with such SNSEs driven by multiplicative Lévy noise [29, 62, 66]. In all these results, the authors consider strong solutions in the probabilistic sense.

To prove our results, we use the weak convergence approach introduced by Budhiraja et al. [15, 17] for Poisson random measures, which has proved to be very effective in the study of the LDP for finite/infinite-dimensional SDEs driven by Lévy processes [2, 15, 16, 29, 62, 64, 66, 68]. In contrast to the existing results, our main object are strong solutions in the PDE sense, and hence we need to find new a priori estimates to establish the tightness of solutions of the perturbed equations; see Lemmata 5.1, 7.2–7.6. We believe that this is nontrivial.

Finally, let us mention the third aim of our article: the well-posedness of controlled SPDEs (7.1). Such a result is a basic step in applying the weak convergence approach. During our study, it has become apparent that although such a result has been used in the previous literature, e.g., [15, 16, 29, 62, 64, 66], it has never been rigorously formulated or proven. We fill this gap by formulating Lemma 7.1 and providing a rigorous proof. The proof heavily depends on a Girsanov-type theorem for Poisson random measures (Theorem 6.1). Although this is apparently a "standard result" (see for instance [44, Theorem III.3.24] for semimartingales and [7, Theorem 3.10.21] for Poisson point processes), it seems hard to find an accessible reference in the literature which would work under our conditions. Therefore in Section 6 we include a complete proof of the version of the Girsanov-type theorem we need. The Girsanov theorem for the Wiener process states that the shifted and the original Wiener measures are equivalent if and only if the shift function belongs to the corresponding Cameron–Martin space. However, in contrast to the Wiener space case, the Girsanov theorem for Poisson random measures is related to invertible and predictable nonlinear transformations ([44, Theorem III.3.24] and [7, Theorem 3.10.21]). These differences lead to many difficulties in proving the variational representation for Poisson functionals, and therefore in applying the weak convergence method for Poisson random measures and the Freidlin–Wentzell-type LDP for SPDEs driven by Lévy processes of jump type; see, for example, [15–17, 31, 67]. This is also one of the main difficulties this paper had to deal with. Let us mention that another application of the Girsanov theorem is its use, in combination with the Yamada–Watanabe theorem, in proving the well-posedness of SPDEs. For instance, see [37] for the case of SPDEs driven by a Wiener process. However, for applications of the Girsanov theorem in the framework of SPDEs defined in terms of Poisson random measures, the literature contains only few results; see for instance, [40], where however no proofs are provided.

The organization of this paper is as follows. Section 2 is to introduce 2D SNSEs. In Sections 3 and 4, we apply a cut-off and the Banach fixed point theorem to establish the existence and uniqueness of strong (in the probabilistic sense and PDE sense, respectively) solutions for 2D SNSEs with Lévy noise, under the Lipschitz condition on balls and linear growth assumptions. We do this for initial data from the space H (Theorems 3.1 and 3.2) and for initial data from the space V (Theorems 4.1 and 4.2). Section 5 is devoted to the formulation of the LDP (Theorem 5.1). This section also contains the proof of Theorem 5.1 provided some auxiliary results hold true. Moreover, we prove the first one of the auxiliary results, the so called first continuity lemma (Proposition 5.3). The remaining auxiliary results are proven in the following sections. Section 6 contains a formulation and proof of a Girsanov-type theorem for Poisson random measures (Theorem 6.1). The last section 7 is devoted to the proof of the second continuity lemma (Proposition 7.1). The paper also contains two appendices. Appendix A contains necessary definitions related to Poisson random measures. Appendix B is devoted to the last auxiliary result, i.e., Lemma 5.1.

2. The stochastic Navier–Stokes equations (SNSEs)

We assume that D is a bounded open domain in $\mathbb{R}^2$, with smooth boundary ∂D. Let us define the following fundamental function space:

$$\mathrm{V} = \{u \in W_0^{1,2}(D, \mathbb{R}^2) : \operatorname{div} u = 0 \text{ weakly in } D\}, \quad \|u\|_{\mathrm{V}}^2 := \int_D |\nabla u(x)|^2 \, dx.$$

Let H be the closure of V in $\mathbb{L}^2(D) := L^2(D, \mathbb{R}^2)$. The space H is a separable Hilbert space endowed with the norm

$$|u|_{\mathrm{H}}^2 := \int_D |u(x)|^2 \, dx.$$

Let $\Pi : \mathbb{L}^2(D) \to \mathrm{H}$ be the orthogonal projection, which is called the *Leray–Helmholtz projection*. Let us define the Stokes operator A in H by

$$\mathrm{A} f = -\Pi \Delta f, \quad f \in \mathscr{D}(\mathrm{A}), \quad \mathscr{D}(\mathrm{A}) := W^{2,2}(D, \mathbb{R}^2) \cap \mathrm{V}.$$

It is well known (e.g., Cattabriga [18]) that A is positive self-adjoint with compact resolvent. Hence, there is an orthonormal basis $\{e_i : i \in \mathbb{N}\}$ of H (we use $\mathbb{N} = \{1, 2, \dots\}$), consisting of eigenvectors of A, with corresponding eigenvalues $\{\lambda_i : i \in \mathbb{N}\}$, i.e., $\mathrm{A}e_i = \lambda_i e_i$, $i \in \mathbb{N}$, such that $\lambda_i > 0$ for all i and $\lambda_i \nearrow \infty$. In this paper, the space $\mathscr{D}(\mathrm{A})$ is endowed with the norm

$$\|u\|_{\mathscr{D}(\mathrm{A})} := |\mathrm{A}u|_{\mathrm{H}}, \quad u \in \mathscr{D}(\mathrm{A}).$$

It is also well known that

$$\mathrm{V} = \mathscr{D}(\mathrm{A}^{1/2}) \quad \text{and} \quad \|u\|_{\mathrm{V}}^2 = |\mathrm{A}^{1/2} u|_{\mathrm{H}}^2, \quad u \in \mathrm{V}.$$

Let $\mathrm{B} : \mathscr{D}(\mathrm{B}) \to \mathrm{H}$, where $\mathscr{D}(\mathrm{B}) \subset \mathrm{H} \times \mathrm{V}$ is the bilinear operator defined as

$$\mathrm{B}(u, v) = \Pi[(u \cdot \nabla) v].$$

Without danger of ambiguity, by B we also denote the corresponding quadratic function

$$\mathrm{B}(u) := \mathrm{B}(u, u).$$

It is well known [59] that the Navier–Stokes equations can be formulated in the following abstract form:

$$du(t) + \mathrm{A}u(t)\,dt + \mathrm{B}(u(t))\,dt = f(t)\,dt, \quad u(0) = u_0, \tag{2.1}$$

where $u_0 \in \mathrm{H}$ and $f \in L^2_{\mathrm{loc}}([0,\infty), \mathrm{V}')$ denote respectively the initial data and the external force, with V' being the dual space of V.

Considering the Gelfand triple

$$\mathrm{V} \subset \mathrm{H} \cong \mathrm{H}' \subset \mathrm{V}',$$

one can show that there exist unique extensions $\mathscr{A}$ and $\mathscr{B}$ of A and B respectively such that

$$\mathscr{A} : \mathrm{V} \to \mathrm{V}', \quad \mathscr{B} : \mathrm{V} \times \mathrm{V} \to \mathrm{V}'$$

are bounded linear and bilinear maps respectively. In what follows, in agreement with the practice of almost all papers on NSEs, these extensions are denoted by the original symbols A and B.

In the following lemma, we list some useful and well-known equalities and inequalities for the bilinear map B. Some of these are only true because we assume that $D \subset \mathbb{R}^2$. In this list, C denotes a generic constant.

Lemma 2.1. *If $u, v, z \in \mathrm{V}$, then*

$$
\begin{aligned}
&{}_{\mathrm{V}'}\langle \mathrm{B}(u,v), z\rangle_{\mathrm{V}} = -\,{}_{\mathrm{V}'}\langle \mathrm{B}(u,z), v\rangle_{\mathrm{V}}, \quad {}_{\mathrm{V}'}\langle \mathrm{B}(u,v), v\rangle_{\mathrm{V}} = 0,\\
&|{}_{\mathrm{V}'}\langle \mathrm{B}(u,v), z\rangle_{\mathrm{V}}| \le 2\|u\|_{\mathrm{V}}^{1/2}|u|_{\mathrm{H}}^{1/2}\|v\|_{\mathrm{V}}^{1/2}|v|_{\mathrm{H}}^{1/2}\|z\|_{\mathrm{V}},\\
&|{}_{\mathrm{V}'}\langle \mathrm{B}(u) - \mathrm{B}(v), u - v\rangle_{\mathrm{V}}| = |{}_{\mathrm{V}'}\langle \mathrm{B}(u-v), v\rangle_{\mathrm{V}}| \le \tfrac{1}{2}\|u-v\|_{\mathrm{V}}^2 + \|v\|_{L^4(D,\mathbb{R}^2)}^4 |u-v|_{\mathrm{H}}^2,\\
&|\mathrm{B}(u,v)|_{\mathrm{H}}^2 \le C|u|_{\mathrm{H}}\|u\|_{\mathrm{V}}\|v\|_{\mathscr{D}(\mathrm{A})}\|v\|_{\mathrm{V}},\\
&\|v\|_{L^4(D,\mathbb{R}^2)}^4 \le 2|v|_{\mathrm{H}}^2\|v\|_{\mathrm{V}}^2.
\end{aligned}
$$

The last inequality [59] is often called the *Ladyzhenskaya inequality*.

In this paper, we consider SNSEs driven by multiplicative Lévy noise in the following abstract form:

$$
\begin{aligned}
&du(t) + \mathrm{A}u(t)\,dt + \mathrm{B}(u(t))\,dt = f(t)\,dt + \int_{\mathrm{Z}} G(u(t-), z)\,\tilde{\eta}(dz, dt),\\
&u_0 \in \mathrm{H}.
\end{aligned} \tag{2.2}
$$

Here we make the following assumptions.

Assumption 2.1. We assume that Z is a locally compact Polish space, and ν is a σ-finite measure on $(\mathrm{Z}, \mathscr{B}(\mathrm{Z}))$, where $\mathscr{B}(\mathrm{Z})$ denotes the Borel σ-field on Z.

We assume that $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathscr{F}_t\}_{t \ge 0}$ is a filtered probability space satisfying the usual conditions, i.e., the family $\mathbb{F}$ is right continuous, and every set A belonging to the $\mathbb{P}$-completion of the σ-field $\mathscr{F}_\infty$ with $\mathbb{P}(A) = 0$ belongs to every $\mathscr{F}_t$, $t \ge 0$.

We also assume that η is a time-homogeneous Poisson random measure on $[0,\infty)\times \mathrm{Z}$ with intensity measure Leb $\otimes\, \nu$ on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where Leb is the Lebesgue measure on $[0,\infty)$.

We define the compensated Poisson random measure $\tilde{\eta}$ by

$$\tilde{\eta}([0,t]\times O) = \eta([0,t]\times O) - t\nu(O), \quad t\geq 0, \tag{2.3}$$

whenever $O \in \mathscr{B}(\mathrm{Z})$ is such that $\nu(O) < \infty$.

Let us point out that the measure Leb $\otimes\, \nu$ is a σ-finite measure on $([0,\infty)\times \mathrm{Z}, \mathscr{B}([0,\infty))\otimes \mathscr{B}(\mathrm{Z}))$.

In the following, if X is a metric space and $I \subset \mathbb{R}$ is a time interval, we denote by $D(I, X)$ the space of all càdlàg paths from I to X.

3. Solutions to SNSEs with initial data in H

Our treatment of SNSEs (2.2) consists of two steps. In the first, we assume that the coefficient G is globally Lipschitz. In the second, we assume that G is Lipschitz on balls and has linear growth.

Below, we present our standing assumptions on the coefficient G in the first step.

Assumption 3.1. We assume that $G : \mathrm{H}\times \mathrm{Z} \to \mathrm{H}$ is a measurable map such that there exist positive constants C_1 and C_2 such that

(G-H1) (Global Lipschitz)

$$\int_{\mathrm{Z}} |G(v_1,z) - G(v_2,z)|^2_{\mathrm{H}}\, \nu(dz) \leq C_1|v_1 - v_2|^2_{\mathrm{H}}, \quad v_1, v_2 \in \mathrm{H}, \tag{3.1}$$

(G-H2) (Linear growth)

$$\int_{\mathrm{Z}} |G(v,z)|^2_{\mathrm{H}}\, \nu(dz) \leq C_2(1 + |v|^2_{\mathrm{H}}), \quad v \in \mathrm{H}. \tag{3.2}$$

Remark 3.1. We note that the linear growth condition (3.2) follows from the global Lipschitz condition (3.1) and the following one, with $C_2 = 2\max\{C_1, \int_{\mathrm{Z}} |G(0,z)|^2_{\mathrm{H}}\, \nu(dz)\}$:

$$\int_{\mathrm{Z}} |G(0,z)|^2_{\mathrm{H}}\, \nu(dz) < \infty. \tag{3.3}$$

First, we prove the following existence result in the natural setting.

Theorem 3.1. *Assume that Assumption 3.1 holds. Then, for all $u_0 \in \mathrm{H}$ and $f \in L^2_{\mathrm{loc}}([0,\infty), \mathrm{V}')$, there exists a unique $\mathbb{F}$-progressively measurable process u such that*

(1) $u \in D([0,\infty), \mathrm{H}) \cap L^2_{\mathrm{loc}}([0,\infty), \mathrm{V})$, *$\mathbb{P}$-a.s.,*

(2) *the following equality holds, for all* $t \in [0,\infty)$, $\mathbb{P}$*-a.s., in* V'*:*

$$u(t) = u_0 - \int_0^t \mathrm{A}u(s)\,ds - \int_0^t \mathrm{B}(u(s))\,ds + \int_0^t f(s)\,ds + \int_0^t \int_{\mathrm{Z}} G(u(s-),z)\,\tilde{\eta}(dz,ds). \tag{3.4}$$

Moreover, the solution u satisfies the following estimate: for any $T > 0$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |u(t)|^2_{\mathrm{H}}\Big) + \mathbb{E}\Big(\int_0^T \|u(s)\|^2_{\mathrm{V}}\,ds\Big) \le C_T\Big(1 + |u_0|^2_{\mathrm{H}} + \int_0^T \|f(s)\|^2_{\mathrm{V}'}\,ds\Big).$$

Remark 3.2. Assumption 3.1 is a fairly standard assumption when one considers the existence and uniqueness of solutions to SPDEs driven by multiplicative Gaussian noise. However, for the case of Lévy noise, the literature results always require additional assumptions on G besides Assumption 3.1. For example, in [28], the authors assume that there exists a sequence $(\mathrm{Z}_m)_{m\in\mathbb{N}}$ of measurable subsets of Z with $\mathrm{Z}_m \nearrow \mathrm{Z}$ and $\nu(\mathrm{Z}_m) < \infty$ such that, for any $k > 0$,

$$\sup_{|v|_{\mathrm{H}}\le k} \int_{\mathrm{Z}_m^c} |G(v,z)|^2_{\mathrm{H}}\,\nu(dz) \to 0 \quad \text{as } m \to \infty, \tag{3.5}$$

while in [10] and [11], it is assumed that there exusts $K > 0$ such that

$$\int_{\mathrm{Z}} |G(v,z)|^4_{\mathrm{H}}\,\nu(dz) \le K(1 + |v|^4_{\mathrm{H}}), \quad v \in \mathrm{H}. \tag{3.6}$$

Similarly, Motyl [53] assumed that for each $p \in \{1,2,2+\gamma,4,4+2\gamma\}$, where γ is some positive constant, there exists a constant $c_p > 0$ such that

$$\int_{\mathrm{Z}} |G(v,z)|^p_{\mathrm{H}}\,\nu(dz) \le c_p(1 + |v|^p_{\mathrm{H}}), \quad v \in \mathrm{H}.$$

Hence, our Theorem 3.1 improves the existing results in the literature.

In the second step, we relax the global Lipschitz condition in Assumption 3.1 and consider the following assumptions.

Assumption 3.2. We assume that $G : \mathrm{H}\times\mathrm{Z} \to \mathrm{H}$ is a measurable map such that

(G-H1-local) (Lipschitz on balls) For every $\hbar > 0$, there exists a constant $C_\hbar > 0$ such that, for all $v_1, v_2 \in \mathrm{H}$ with $|v_1|_{\mathrm{H}} \vee |v_2|_{\mathrm{H}} \le \hbar$,

$$\int_{\mathrm{Z}} |G(v_1,z) - G(v_2,z)|^2_{\mathrm{H}}\,\nu(dz) \le C_\hbar |v_1 - v_2|^2_{\mathrm{H}}, \tag{3.7}$$

and G satisfies the assumption (G-H2)(Linear growth), i.e., (3.2) holds.

Let us now formulate our main theorem in this relaxed framework.

Theorem 3.2. *Suppose that Assumption 3.2 holds. Then, for every* $u_0 \in \mathrm{H}$ *and* $f \in L^2_{\mathrm{loc}}([0,\infty), \mathrm{V}')$*, there exists a unique* $\mathbb{F}$*-progressively measurable process* u *such that*

(1) $u \in D([0,\infty), \mathrm{H}) \cap L^2_{\mathrm{loc}}([0,\infty), \mathrm{V})$, $\mathbb{P}$*-a.s.,*

(2) *the following equality holds, for all* $t \in [0,\infty)$, $\mathbb{P}$*-a.s., in* V'*:*

$$u(t) = u_0 - \int_0^t \mathrm{A}u(s)\,ds - \int_0^t \mathrm{B}(u(s))\,ds + \int_0^t f(s)\,ds + \int_0^t \int_{\mathrm{Z}} G(u(s-), z)\,\tilde{\eta}(dz, ds).$$

Proof. The proof of Theorem 3.2 is based on the proof of Theorem 3.1 and the standard truncation procedure, and it is essentially the same as [1, proof of Theorem 3.1], keeping in mind that for any $u, v \in \mathrm{V}$, ${}_{\mathrm{V}'}\langle \mathrm{B}(u,v), v\rangle_{\mathrm{V}} = 0$. The proof proceeds as follows.

For any $k \in \mathbb{N}$, we define a map G_k by

$$G_k : \mathrm{H} \times \mathrm{Z} \ni (y,z) \mapsto G\left(\frac{|y|_{\mathrm{H}} \wedge k}{|y|_{\mathrm{H}}} y, z\right) \in \mathrm{H},$$

where we put $\frac{|y|_{\mathrm{H}} \wedge k}{|y|_{\mathrm{H}}} = 1$ when $y = 0$. Since G satisfies Assumption 3.2, we observe that the map G_k satisfies Assumption 3.1. Hence, for every $k > |u_0|_{\mathrm{H}}$, there exists by Theorem 3.1 a unique $\mathbb{F}$-progressively measurable process X^k such that

- $X^k \in D([0,\infty), \mathrm{H}) \cap L^2_{\mathrm{loc}}([0,\infty), \mathrm{V})$, $\mathbb{P}$-a.s.,
- the following equality holds, for all $t \in [0,\infty)$, $\mathbb{P}$-a.s., in V',

$$X^k(t) = u_0 - \int_0^t \mathrm{A}X^k(s)\,ds - \int_0^t \mathrm{B}(X^k(s))\,ds + \int_0^t f(s)\,ds + \int_0^t \int_{\mathrm{Z}} G_k(X^k(s-), z)\,\tilde{\eta}(dz, ds).$$

Define a random time σ_k by

$$\sigma_k := \inf\{t \geq 0 : |X^k(t)|_{\mathrm{H}} > k\}, \tag{3.8}$$

where, for the whole paper, we adopt the convention that $\inf \emptyset = \infty$. By [32, Theorem 2.1.6] it follows that σ_k is a stopping time. It is not difficult to see that σ_k is increasing in k, and

$$X^{k+1}(t) = X^k(t), \quad t \in [0, \sigma_k).$$

This enables us to define a stopping time $\sigma := \lim_{k\to\infty} \sigma_k$ and a process $u = \{u(t),\ t \in [0,\sigma)\}$ as follows:

$$u(t) := X^k(t), \quad t \in [0, \sigma_k).$$

It is easy to see that $u(t)$, $t \in [0,\sigma)$, is a local solution of problem (2.2). To complete the proof, we need only show that $\mathbb{P}(\sigma = \infty) = 1$.

By the Itô formula (see, e.g., [38] and [10]), we have

$$
\begin{aligned}
&|u(t\wedge\sigma_k)|_{\mathrm{H}}^2 + 2\int_0^{t\wedge\sigma_k} \|u(s)\|_{\mathrm{V}}^2\,ds \\
&\quad = |u_0|_{\mathrm{H}}^2 + 2\int_0^{t\wedge\sigma_k} {}_{\mathrm{V}'}\langle f(s), u(s)\rangle_{\mathrm{V}}\,ds + 2\int_0^{t\wedge\sigma_k}\int_{\mathrm{Z}} \langle G(u(s-),z), u(s-)\rangle_{\mathrm{H}}\,\widetilde{\eta}(dz,ds) \\
&\qquad + \int_0^{t\wedge\sigma_k}\int_{\mathrm{Z}} |G(u(s-),z)|_{\mathrm{H}}^2\,\eta(dz,ds), \quad \mathbb{P}\text{-a.s., for } t\geq 0.
\end{aligned}
$$

Noting that the process $\int_0^{t\wedge\sigma_k}\int_{\mathrm{Z}} \langle G(u(s-),z), u(s-)\rangle_{\mathrm{H}}\,\widetilde{\eta}(dz,ds)$, $t\geq 0$, is a martingale, we infer that

$$
\mathbb{E}\int_0^{t\wedge\sigma_k}\int_{\mathrm{Z}} \langle G(u(s-),z), u(s-)\rangle_{\mathrm{H}}\,\widetilde{\eta}(dz,ds) = 0.
$$

Thus, it follows by the linear growth condition (3.2), which is a part of Assumption 3.2, that there exists $C>0$ such that for all k,

$$
\begin{aligned}
&\mathbb{E}|u(t\wedge\sigma_k)|_{\mathrm{H}}^2 + \mathbb{E}\int_0^{t\wedge\sigma_k} \|u(s)\|_{\mathrm{V}}^2\,ds \\
&\qquad \leq |u_0|_{\mathrm{H}}^2 + \int_0^t \|f(s)\|_{\mathrm{V}'}^2\,ds + C\mathbb{E}\int_0^t (1+|u(s\wedge\sigma_k)|_{\mathrm{H}}^2)\,ds, \quad t\geq 0.
\end{aligned}
$$

Therefore, by applying Gronwall's lemma, we deduce that

$$
\mathbb{E}|u(t\wedge\sigma_k)|_{\mathrm{H}}^2 \leq \Big(|u_0|_{\mathrm{H}}^2 + \int_0^t \|f(s)\|_{\mathrm{V}'}^2\,ds + Ct\Big)e^{Ct}, \quad t\geq 0,
$$

which further gives

$$
\mathbb{P}(\sigma_k\leq t) \leq \frac{\mathbb{E}(|u(t\wedge\sigma_k)|_{\mathrm{H}}^2 \mathbb{1}_{\sigma_k\leq t})}{k^2} \leq \frac{(|u_0|_{\mathrm{H}}^2 + \int_0^t \|f(s)\|_{\mathrm{V}'}^2\,ds + Ct)e^{Ct}}{k^2}, \quad t\geq 0.
$$

Letting $k\to\infty$, we obtain

$$
\mathbb{P}(\sigma\leq t) = 0, \quad t\geq 0.
$$

Since $t\geq 0$ is arbitrary, we must have $\mathbb{P}(\sigma=\infty)=1$.

The proof of Theorem 3.2 is complete. ■

To prove Theorem 3.1, we first introduce the following notation (used throughout the paper) and state three preliminary and auxiliary results: Lemmata 3.1 and 3.2, and Corollary 3.1.

The following notation is useful. For $T\geq 0$,

$$
\Upsilon_T(\mathrm{H}) = D([0,T],\mathrm{H}) \cap L^2([0,T],\mathrm{V}). \tag{3.9}
$$

It is standard that the space $\Upsilon_T(\mathrm{H})$ endowed with the norm $\|\cdot\|_{\Upsilon_T(\mathrm{H})}$ defined by

$$
\|y\|_{\Upsilon_T(\mathrm{H})} = \sup_{s\in[0,T]} |y(s)|_{\mathrm{H}} + \Big(\int_0^T \|y(s)\|_{\mathrm{V}}^2\,ds\Big)^{1/2} \tag{3.10}
$$

is a Banach space.

Let $\Lambda_T(\mathrm{H})$ be the space of all H-valued càdlàg $\mathbb{F}$-progressively measurable processes $y : [0,T] \times \Omega \to \mathrm{V}$ such that $\mathbb{P}$-a.s. its trajectories belong to the space $\Upsilon_T(\mathrm{H})$ and

$$\|y\|^2_{\Lambda_T(\mathrm{H})} := \mathbb{E}\biggl(\sup_{s\in[0,T]} |y(s)|^2_{\mathrm{H}} + \int_0^T \|y(s)\|^2_{\mathrm{V}}\, ds\biggr) < \infty. \tag{3.11}$$

For every $m \in \mathbb{N}$, let us fix a function $\theta_m : [0,\infty) \to [0,1]$ satisfying

$$\begin{cases} \theta_m \in C^2[0,\infty), \qquad \sup_{t\in[0,\infty)} |\theta'_m(t)| \le C_1 < \infty, \\ \mathbb{1}_{[0,m]} \le \theta_m \le \mathbb{1}_{[0,m+1]}, \end{cases} \tag{3.12}$$

for some constant $C_1 > 0$ which is independent of m. We also set

$$\phi = \theta_1.$$

Let us also define, for every $\delta > 0$, a function $\phi_\delta : [0,\infty) \to [0,1]$ by

$$\phi_\delta(r) = \phi(\delta r), \quad r \in [0,\infty).$$

It can be easily seen that every function ϕ_δ satisfies the following conditions:

$$\begin{cases} \phi_\delta \in C^2[0,\infty), \qquad \sup_{t\in[0,\infty)} |\phi'_\delta(t)| \le C_1\delta, \\ \mathbb{1}_{[0,1/\delta]} \le \phi_\delta \le \mathbb{1}_{[0,2/\delta]}. \end{cases} \tag{3.13}$$

We are now ready to state the first of the three promised auxiliary results.

Lemma 3.1. *Assume that $T > 0$, $m \in \mathbb{N}$, $M \in \Upsilon_T(\mathrm{H})$, $u_0 \in \mathrm{H}$ and $f \in L^2([0,T],\mathrm{V}')$. Then there exists a function $Y \in C([0,T],\mathrm{H}) \cap L^2([0,T],\mathrm{V})$ satisfying*

$$\begin{aligned} &dY(t) + \mathrm{A}Y(t)\,dt + \theta_m(\|Y + M\|_{\Upsilon_t(\mathrm{H})})\mathrm{B}(Y(t) + M(t))\,dt = f(t)\,dt, \\ &Y(0) = u_0. \end{aligned} \tag{3.14}$$

Proof. The proof is divided into three steps.

Step 1. Let us fix $T > 0$, $m \in \mathbb{N}$, $M \in \Upsilon_T(\mathrm{H})$, $u_0 \in \mathrm{H}$, and $f \in L^2([0,T],\mathrm{V}')$. We will use the Picard iteration method to prove that there exists a number $\delta_0 > 0$ depending only on m, and there exists $X \in C([0,T],\mathrm{H}) \cap L^2([0,T],\mathrm{V})$ which solves the following auxiliary deterministic evolution equation with $\delta = \delta_0$:

$$\begin{aligned} &X'(t) + \mathrm{A}X(t) \\ &\qquad + \theta_m(\|X + M\|_{\Upsilon_t(\mathrm{H})})\phi_\delta(\|X + M\|_{L^2([0,t],\mathrm{V})})\mathrm{B}(X(t) + M(t)) = f(t), \\ &X(0) = u_0. \end{aligned} \tag{3.15}$$

Let us fix $y_0 \in C([0,T],\mathrm{H}) \cap L^2([0,T],\mathrm{V})$ with $y_0(0) = u_0$ (for instance $y_0(t) = e^{-t\mathrm{A}}u_0$, $t \in [0,T]$). Suppose that for $n \in \mathbb{N}$ a function $y_n \in C([0,T],\mathrm{H}) \cap L^2([0,T],\mathrm{V})$

such that $y_n(0) = u_0$ is given. Let us observe that it is not difficult to prove that there exists a unique $y_{n+1} \in C([0,T],\mathrm{H}) \cap L^2([0,T],\mathrm{V})$ solving the linear evolution equation

$$\begin{cases} y'_{n+1}(t) + \mathrm{A}y_{n+1}(t) + \theta_m(\|y_n + M\|_{\Upsilon_t(\mathrm{H})})\phi_\delta(\|y_n + M\|_{L^2([0,t],\mathrm{V})}) \\ \quad \cdot \mathrm{B}(y_n(t) + M(t), y_{n+1}(t) + M(t)) = f(t), \quad t \in [0,T], \\ y_{n+1}(0) = u_0. \end{cases} \tag{3.16}$$

Our aim is to show that $\{y_n : n \in \mathbb{N}\}$ is a Cauchy sequence in $C([0,T],\mathrm{H}) \cap L^2([0,T],\mathrm{V})$.

We now estimate the norm of the difference $y_{n+1} - y_n$ for $n \geq 1$. We cannot do this for $n = 0$.

Given four functions $x_i \in C([0,T],\mathrm{H}) \cap L^2([0,T],\mathrm{V})$, $i = 1,\ldots,4$, we set, for $t \in [0,T]$,

$$\begin{aligned} &\Pi(x_1,x_2,x_3,x_4)(t) \\ &\qquad = \theta_m(\|x_1 + M\|_{\Upsilon_t(\mathrm{H})})\phi_\delta(\|x_2 + M\|_{L^2([0,t],\mathrm{V})})\mathrm{B}(x_3(t) + M(t), x_4(t) + M(t)) \end{aligned}$$

and

$$\Xi(x_1,x_2)(t) = \theta_m(\|x_1 + M\|_{\Upsilon_t(\mathrm{H})})\phi_\delta(\|x_2 + M\|_{L^2([0,t],\mathrm{V})}).$$

By [59, Lemma III.1.2] we have

$$\begin{aligned} &|y_{n+1}(t) - y_n(t)|^2_{\mathrm{H}} + 2\int_0^t \|y_{n+1}(s) - y_n(s)\|^2_{\mathrm{V}}\,ds \\ &= -2\int_0^t {}_{\mathrm{V}'}\langle \Pi(y_n,y_n,y_n,y_{n+1})(s) - \Pi(y_{n-1},y_{n-1},y_{n-1},y_n)(s), y_{n+1}(s) - y_n(s)\rangle_{\mathrm{V}}\,ds \\ &= -2\int_0^t I(s)\,ds, \quad t \in [0,T], \end{aligned} \tag{3.17}$$

where, with the processes I_1 and I_2 defined, for $s \in [0,T]$, by

$$\begin{aligned} I_1(s) &= \Xi(y_n,y_n)(s)\,{}_{\mathrm{V}'}\langle \mathrm{B}(y_n(s) - y_{n-1}(s), y_n(s) + M(s)), y_{n+1}(s) - y_n(s)\rangle_{\mathrm{V}}, \\ I_2(s) &= \big(\Xi(y_n,y_n)(s) - \Xi(y_{n-1},y_{n-1})(s)\big) \\ &\qquad \cdot {}_{\mathrm{V}'}\langle \mathrm{B}(y_{n-1}(s) + M(s), y_n(s) + M(s)), y_{n+1}(s) - y_n(s)\rangle_{\mathrm{V}}, \end{aligned}$$

we have

$$\begin{aligned} I(s) &= {}_{\mathrm{V}'}\langle \Pi(y_n,y_n,y_n,y_{n+1})(s) - \Pi(y_n,y_n,y_n,y_n)(s), y_{n+1}(s) - y_n(s)\rangle_{\mathrm{V}} \\ &\quad + {}_{\mathrm{V}'}\langle \Pi(y_n,y_n,y_n,y_n)(s) - \Pi(y_n,y_n,y_{n-1},y_n)(s), y_{n+1}(s) - y_n(s)\rangle_{\mathrm{V}} \\ &\quad + {}_{\mathrm{V}'}\langle \Pi(y_n,y_n,y_{n-1},y_n)(s) - \Pi(y_{n-1},y_{n-1},y_{n-1},y_n)(s), y_{n+1}(s) - y_n(s)\rangle_{\mathrm{V}} \\ &= 0 + I_1(s) + I_2(s), \quad s \in [0,T]. \end{aligned} \tag{3.18}$$

To estimate $I(s)$, for a fixed $s \in [0,T]$, we will consider three cases, with Case 1 being divided into three subcases. Each case will contain a calculation of a certain “partial” integral $\int_0^t |I(s)|\,ds$.

Case 1. Assume that

$$\|y_n + M\|_{L^2([0,s],\mathrm{V})} \le 3/\delta \quad \text{and} \quad \|y_{n-1} + M\|_{L^2([0,s],\mathrm{V})} \le 3/\delta.$$

Subcase 1.1. Assume further that

$$\|y_n + M\|_{\Upsilon_s(\mathrm{H})} \le m + 2 \quad \text{and} \quad \|y_{n-1} + M\|_{\Upsilon_s(\mathrm{H})} > m + 2.$$

The definition of θ_m implies that in this subcase

$$I(s) = {}_{\mathrm{V}'}\langle \Pi(y_n, y_n, y_n, y_{n+1})(s), y_{n+1}(s) - y_n(s)\rangle_{\mathrm{V}}$$

and

$$\begin{aligned}
&\int_0^t |I(s)| \mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})} \vee \|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})} \le 3/\delta\}} \mathbb{1}_{\{\|y_n+M\|_{\Upsilon_s(\mathrm{H})} \le m+2\}} \\
&\qquad\qquad \cdot \mathbb{1}_{\{\|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})} > m+2\}}\, ds \\
&= \int_0^t |I(s)| \mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})} \vee \|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})} \le 3/\delta\}} \mathbb{1}_{\{\|y_n+M\|_{\Upsilon_s(\mathrm{H})} \le m+1\}} \\
&\qquad\qquad \cdot \mathbb{1}_{\{\|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})} > m+2\}}\, ds. \qquad (3.19)
\end{aligned}$$

For any $s \in [0,t]$ such that

$$\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})} \vee \|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})} \le 3/\delta\}} \mathbb{1}_{\{\|y_n+M\|_{\Upsilon_s(\mathrm{H})} \le m+1\}} \cdot \mathbb{1}_{\{\|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})} > m+2\}} = 1,$$

we have

$$\|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{H})} = \|(y_n + M) - (y_{n-1} + M)\|_{\Upsilon_s(\mathrm{H})} \ge 1, \qquad (3.20)$$

and for any $\varepsilon > 0$,

$$\begin{aligned}
|I(s)| &\le \left|{}_{\mathrm{V}'}\langle \mathrm{B}(y_n(s) + M(s), y_{n+1}(s) + M(s)), -y_n(s) - M(s)\rangle_{\mathrm{V}}\right| \\
&= \left|{}_{\mathrm{V}'}\langle \mathrm{B}(y_n(s) + M(s), y_{n+1}(s) - y_n(s)), -y_n(s) - M(s)\rangle_{\mathrm{V}}\right| \\
&\le 2|y_n(s) + M(s)|_{\mathrm{H}}^{1/2} \|y_n(s) + M(s)\|_{\mathrm{V}}^{1/2} |y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^{1/2} \\
&\quad \cdot \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^{1/2} \|y_n(s) + M(s)\|_{\mathrm{V}} \cdot \|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{H})} \\
&\le \frac{3}{2}\varepsilon^{4/3} \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^{2/3} \|y_n(s) + M(s)\|_{\mathrm{V}}^{4/3} \|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{H})}^{4/3} \\
&\quad + \frac{(m+2)^2}{2\varepsilon^4} \|y_n(s) + M(s)\|_{\mathrm{V}}^2 |y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^2 \\
&\le \frac{1}{2}\varepsilon^{4/3} \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^2 + \varepsilon^{4/3} \|y_n(s) + M(s)\|_{\mathrm{V}}^2 \|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{H})}^2 \\
&\quad + \frac{(m+2)^2}{2\varepsilon^4} \|y_n(s) + M(s)\|_{\mathrm{V}}^2 |y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^2. \qquad (3.21)
\end{aligned}$$

In the second "≤" of (3.21), we have used (3.20).

By inequalities (3.19) and (3.21), we get

$$\int_0^t |I(s)|\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}\vee\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}\le 3/\delta\}}\mathbb{1}_{\{\|y_n+M\|_{\Upsilon_s(\mathrm{H})}\le m+2\}} \cdot \mathbb{1}_{\{\|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})}>m+2\}}\,ds$$
$$\le \frac{1}{2}\varepsilon^{4/3}\int_0^t \|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^2\,ds$$
$$+\varepsilon^{4/3}\|y_n-y_{n-1}\|_{\Upsilon_t(\mathrm{H})}^2\int_0^t \|y_n(s)+M(s)\|_{\mathrm{V}}^2\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}\le 3/\delta\}}\,ds$$
$$+\frac{(m+2)^2}{2\varepsilon^4}$$
$$\times\int_0^t \|y_n(s)+M(s)\|_{\mathrm{V}}^2\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}\le 3/\delta\}}(s)\,ds\sup_{s\in[0,t]}|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^2$$
$$\le \frac{1}{2}\varepsilon^{4/3}\int_0^t \|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^2\,ds+\varepsilon^{4/3}\|y_n-y_{n-1}\|_{\Upsilon_t(\mathrm{H})}^2\cdot\frac{9}{\delta^2}$$
$$+\frac{(m+2)^2}{2\varepsilon^4}\cdot\frac{9}{\delta^2}\sup_{s\in[0,t]}|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^2$$
$$\le\left(\frac{1}{2}\varepsilon^{4/3}+\frac{(m+2)^2}{2\varepsilon^4}\cdot\frac{9}{\delta^2}\right)\|y_{n+1}-y_n\|_{\Upsilon_t(\mathrm{H})}^2+\frac{9}{\delta^2}\varepsilon^{4/3}\|y_n-y_{n-1}\|_{\Upsilon_t(\mathrm{H})}^2. \tag{3.22}$$

Subcase 1.2. Assume further that

$$\|y_n+M\|_{\Upsilon_s(\mathrm{H})}>m+2 \quad\text{and}\quad \|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})}\le m+2.$$

Similar to Subcase 1.1, we now have

$$I(s)=-{}_{\mathrm{V}'}\big\langle\Pi(y_{n-1},y_{n-1},y_{n-1},y_n)(s),y_{n+1}(s)-y_n(s)\big\rangle_{\mathrm{V}},$$
$$\int_0^t |I(s)|\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}\vee\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}\le 3/\delta\}}\mathbb{1}_{\{\|y_n+M\|_{\Upsilon_s(\mathrm{H})}>m+2\}} \cdot \mathbb{1}_{\{\|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})}\le m+2\}}\,ds$$
$$=\int_0^t |I(s)|\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}\vee\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}\le 3/\delta\}}\mathbb{1}_{\{\|y_n+M\|_{\Upsilon_s(\mathrm{H})}>m+2\}} \cdot \mathbb{1}_{\{\|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})}\le m+1\}}\,ds, \tag{3.23}$$

and for any $s\in[0,t]$ such that

$$\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}\vee\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}\le 3/\delta\}} \cdot \mathbb{1}_{\{\|y_n+M\|_{\Upsilon_s(\mathrm{H})}>m+2\}}\mathbb{1}_{\{\|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})}\le m+1\}}=1,$$

we have

$$\|y_n-y_{n-1}\|_{\Upsilon_s(\mathrm{H})}=\|(y_n+M)-(y_{n-1}+M)\|_{\Upsilon_s(\mathrm{H})}\ge 1, \tag{3.24}$$

and, for any $\varepsilon>0$,

$$
\begin{aligned}
|I(s)| &\le \big|{}_{\mathrm{V}'}\langle \mathrm{B}(y_{n-1}(s)+M(s), y_n(s)+M(s)), y_{n+1}(s)-y_n(s)\rangle_{\mathrm{V}}\big| \\
&\le 2|y_{n-1}(s)+M(s)|_{\mathrm{H}}^{1/2}\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^{1/2}|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^{1/2} \\
&\quad \cdot \|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^{1/2}\|y_n(s)+M(s)\|_{\mathrm{V}} \\
&\le \frac{1}{2\varepsilon^4}|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^2|y_{n-1}(s)+M(s)|_{\mathrm{H}}^2\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2 \\
&\quad + \frac{3}{2}\varepsilon^{4/3}\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^{2/3}\|y_n(s)+M(s)\|_{\mathrm{V}}^{4/3} \\
&\le \frac{1}{2\varepsilon^4}(m+2)^2|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^2\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2 \\
&\quad + \frac{1}{2}\varepsilon^{4/3}\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^2 + \varepsilon^{4/3}\|y_n(s)+M(s)\|_{\mathrm{V}}^2 \\
&\le \frac{1}{2\varepsilon^4}(m+2)^2|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^2\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2 \\
&\quad + \frac{1}{2}\varepsilon^{4/3}\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^2 + 2\varepsilon^{4/3}\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2 \\
&\quad + 2\varepsilon^{4/3}\|y_n(s)-y_{n-1}(s)\|_{\mathrm{V}}^2 \\
&\le \frac{1}{2\varepsilon^4}(m+2)^2|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^2\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2 + \frac{1}{2}\varepsilon^{4/3}\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^2 \\
&\quad + 2\varepsilon^{4/3}\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2 \cdot \|y_n-y_{n-1}\|_{\Upsilon_s(\mathrm{H})}^2 + 2\varepsilon^{4/3}\|y_n(s)-y_{n-1}(s)\|_{\mathrm{V}}^2.
\end{aligned}
\tag{3.25}
$$

In the last inequality "$\le$" of (3.25), we have used inequality (3.24).

By (3.23) and (3.25), we get, similar to inequality (3.22),

$$
\begin{aligned}
&\int_0^t |I(s)|\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}\vee\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}\le 3/\delta\}} \\
&\qquad\qquad \cdot \mathbb{1}_{\{\|y_n+M\|_{\Upsilon_s(\mathrm{H})}>m+2\}}\mathbb{1}_{\{\|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})}\le m+2\}}\,ds \\
&\le \left(\frac{\varepsilon^{4/3}}{2}+\frac{9(m+2)^2}{2\varepsilon^4\delta^2}\right)\|y_{n+1}-y_n\|_{\Upsilon_t(\mathrm{H})}^2 + 2\varepsilon^{4/3}\left(1+\frac{9}{\delta^2}\right)\|y_n-y_{n-1}\|_{\Upsilon_t(\mathrm{H})}^2.
\end{aligned}
\tag{3.26}
$$

Subcase 1.3. Assume further that $\|y_n+M\|_{\Upsilon_s(\mathrm{H})}\le m+2$ and $\|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})}\le m+2$. Under these assumptions, we infer that for any $\varepsilon>0$,

$$
\begin{aligned}
|I_1(s)| &\le \big|{}_{\mathrm{V}'}\langle \mathrm{B}(y_n(s)-y_{n-1}(s), y_n(s)+M(s)), y_{n+1}(s)-y_n(s)\rangle_{\mathrm{V}}\big| \\
&\le 2\|y_n(s)-y_{n-1}(s)\|_{\mathrm{V}}^{1/2}|y_n(s)-y_{n-1}(s)|_{\mathrm{H}}^{1/2}\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^{1/2} \\
&\quad \cdot |y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^{1/2}\|y_n(s)+M(s)\|_{\mathrm{V}} \\
&\le \varepsilon\|y_n(s)-y_{n-1}(s)\|_{\mathrm{V}}\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}} \\
&\quad + \frac{2}{\varepsilon}|y_n(s)-y_{n-1}(s)|_{\mathrm{H}}|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}\|y_n(s)+M(s)\|_{\mathrm{V}}^2 \\
&\le \varepsilon\|y_n(s)-y_{n-1}(s)\|_{\mathrm{V}}^2 + \varepsilon\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^2 \\
&\quad + \varepsilon\|y_n(s)+M(s)\|_{\mathrm{V}}^2|y_n(s)-y_{n-1}(s)|_{\mathrm{H}}^2 \\
&\quad + \frac{4}{\varepsilon^3}|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^2\|y_n(s)+M(s)\|_{\mathrm{V}}^2.
\end{aligned}
\tag{3.27}
$$

For $I_2(s) = I_{2,1}(s) + I_{2,1}(s)$, where

$$I_{2,1}(s) = \big(\Xi(y_n, y_n)(s) - \Xi(y_{n-1}, y_n)(s)\big) \cdot {}_{\mathrm{V}'}\big\langle \mathrm{B}(y_{n-1}(s) + M(s), y_n(s) + M(s)), y_{n+1}(s) - y_n(s)\big\rangle_{\mathrm{V}}$$

and

$$I_{2,2}(s) = \big(\Xi(y_{n-1}, y_n)(s) - \Xi(y_{n-1}, y_{n-1})(s)\big) \cdot {}_{\mathrm{V}'}\big\langle \mathrm{B}(y_{n-1}(s) + M(s), y_n(s) + M(s)), y_{n+1}(s) - y_n(s)\big\rangle_{\mathrm{V}},$$

we have, for any $\varepsilon > 0$,

$$\begin{aligned}|I_{2,1}(s)| &\le C_1 \|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{H})} |y_{n-1}(s) + M(s)|_{\mathrm{H}}^{1/2} \|y_{n-1}(s) + M(s)\|_{\mathrm{V}}^{1/2} \\ &\quad \cdot |y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^{1/2} \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^{1/2} \|y_n(s) + M(s)\|_{\mathrm{V}} \\ &\le \frac{3}{4}\varepsilon^{4/3} \|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{H})}^{4/3} \|y_n(s) + M(s)\|_{\mathrm{V}}^{4/3} \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^{2/3} \\ &\quad + \frac{C_1^4}{4\varepsilon^4} |y_{n-1}(s) + M(s)|_{\mathrm{H}}^2 \|y_{n-1}(s) + M(s)\|_{\mathrm{V}}^2 |y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^2 \\ &\le \frac{1}{4}\varepsilon^{4/3} [2\|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{H})}^2 \|y_n(s) + M(s)\|_{\mathrm{V}}^2 + \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^2] \\ &\quad + \frac{C_1^4}{4\varepsilon^4} (m+2)^2 \|y_{n-1}(s) + M(s)\|_{\mathrm{V}}^2 |y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^2, \end{aligned} \tag{3.28}$$

and

$$\begin{aligned}|I_{2,2}(s)| &\le C_1 \delta \|y_n - y_{n-1}\|_{L^2([0,s],\mathrm{V})} |y_{n-1}(s) + M(s)|_{\mathrm{H}}^{1/2} \|y_{n-1}(s) + M(s)\|_{\mathrm{V}}^{1/2} \\ &\quad \cdot |y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^{1/2} \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^{1/2} \|y_n(s) + M(s)\|_{\mathrm{V}} \\ &\le \frac{3\delta}{4}\varepsilon^{4/3} \|y_n - y_{n-1}\|_{L^2([0,s],\mathrm{V})}^{4/3} \|y_n(s) + M(s)\|_{\mathrm{V}}^{4/3} \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^{2/3} \\ &\quad + \frac{C_1^4\delta}{4\varepsilon^4} |y_{n-1}(s) + M(s)|_{\mathrm{H}}^2 \|y_{n-1}(s) + M(s)\|_{\mathrm{V}}^2 |y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^2 \\ &\le \frac{1}{2}\delta^{3/2}\varepsilon^{4/3} \|y_n - y_{n-1}\|_{L^2([0,s],\mathrm{V})}^2 \|y_n(s) + M(s)\|_{\mathrm{V}}^2 + \frac{1}{4}\varepsilon^{4/3} \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^2 \\ &\quad + \frac{C_1^4\delta}{4\varepsilon^4} (m+2)^2 \|y_{n-1}(s) + M(s)\|_{\mathrm{V}}^2 |y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^2. \end{aligned} \tag{3.29}$$

Similar to inequality (3.22), by inequalities (3.27)–(3.29), we infer that

$$\begin{aligned}&\int_0^t |I(s)| \mathbb{1}_{\{\|y_n + M\|_{L^2([0,s],\mathrm{V})} \vee \|y_{n-1} + M\|_{L^2([0,s],\mathrm{V})} \le 3/\delta\}} \mathbb{1}_{\{\|y_n + M\|_{\Upsilon_s(\mathrm{H})} \le m+2\}} \\ &\qquad\qquad \cdot \mathbb{1}_{\{\|y_{n-1} + M\|_{\Upsilon_s(\mathrm{H})} \le m+2\}} \, ds \\ &\le \left(\varepsilon + \frac{36}{\varepsilon^3\delta^2} + \frac{1}{2}\varepsilon^{4/3} + \frac{C_1^4}{4\varepsilon^4}(m+2)^2 \frac{9}{\delta^2} + \frac{9}{4}\frac{(m+2)^2}{\varepsilon^4\delta}\right) \|y_{n+1} - y_n\|_{\Upsilon_t(\mathrm{H})}^2 \\ &\quad + \left(\varepsilon + \frac{9\varepsilon}{\delta^2} + \frac{9\varepsilon^{4/3}}{2\delta^2} + \frac{9\varepsilon^{4/3}}{2\delta^{1/2}}\right) \|y_n - y_{n-1}\|_{\Upsilon_t(\mathrm{H})}^2. \end{aligned} \tag{3.30}$$

The proof of Case 1 is complete. ∎

Case 2. Assume that $\|y_n + M\|_{L^2([0,s],\mathrm{V})} \leq 3/\delta$ and $\|y_{n-1} + M\|_{L^2([0,s],\mathrm{V})} > 3/\delta$. In this case, by the definitions of functions θ_m and ϕ_δ, we have

$$I(s) = {}_{\mathrm{V}'}\big\langle \Pi(y_n, y_n, y_n, y_{n+1})(s), y_{n+1}(s) - y_n(s)\big\rangle_{\mathrm{V}},$$

and

$$\begin{aligned}\int_0^t & |I(s)| \mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}\leq 3/\delta\}}(s) \mathbb{1}_{\{\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}>3/\delta\}}(s)\, ds \\ &= \int_0^t |I(s)| \mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}\leq 2/\delta\}}(s) \mathbb{1}_{\{\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}>3/\delta\}}(s) \\ &\qquad \cdot \mathbb{1}_{\{\|y_n+M\|_{\Upsilon_s(\mathrm{H})}\leq m+1\}}\, ds. \end{aligned} \tag{3.31}$$

For any $s \in [0, t]$ such that

$$\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}\leq 2/\delta\}}(s) \mathbb{1}_{\{\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}>3/\delta\}}(s) \mathbb{1}_{\{\|y_n+M\|_{\Upsilon_s(\mathrm{H})}\leq m+1\}} = 1,$$

we have

$$\delta\|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{H})} \geq \delta\|(y_n + M) - (y_{n-1} + M)\|_{L^2([0,s],\mathrm{V})} \geq 1, \tag{3.32}$$

and for any $p \in (0, 1/2)$ and $\varepsilon > 0$,

$$\begin{aligned} |I(s)| &\leq \big| {}_{\mathrm{V}'}\big\langle \mathrm{B}(y_n(s) + M(s), y_{n+1}(s) + M(s)), -y_n(s) - M(s)\big\rangle_{\mathrm{V}}\big| \\ &= \big| {}_{\mathrm{V}'}\big\langle \mathrm{B}(y_n(s) + M(s), y_{n+1}(s) - y_n(s)), y_n(s) + M(s)\big\rangle_{\mathrm{V}}\big| \\ &\leq 2|y_n(s) + M(s)|_{\mathrm{H}}^{1/2}\|y_n(s) + M(s)\|_{\mathrm{V}}^{1/2}|y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^{1/2} \\ &\quad \cdot \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^{1/2}\|y_n(s) + M(s)\|_{\mathrm{V}} \cdot \delta\|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{H})} \\ &\leq \frac{3}{2}\varepsilon^{4/3}\delta^{4(1-p)/3}\|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^{2/3}\|y_n(s) + M(s)\|_{\mathrm{V}}^{4/3}\|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{H})}^{4/3} \\ &\quad + \frac{1}{2\varepsilon^4}\delta^{4p}\|y_n(s) + M(s)\|_{\mathrm{V}}^2|y_n(s) + M(s)|_{\mathrm{H}}^2|y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^2 \\ &\leq \frac{1}{2}\varepsilon^{4/3}\|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^2 + \varepsilon^{4/3}\delta^{2(1-p)}\|y_n(s) + M(s)\|_{\mathrm{V}}^2\|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{H})}^2 \\ &\quad + \frac{1}{2\varepsilon^4}\delta^{4p}(m+1)^2\|y_n(s) + M(s)\|_{\mathrm{V}}^2|y_{n+1}(s) - y_n(s)|_{\mathrm{H}}^2. \end{aligned} \tag{3.33}$$

In the second "≤" of (3.33), we have used (3.32).

Similar to (3.22), by (3.31) and (3.33),

$$\begin{aligned}\int_0^t & |I(s)| \mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}\leq 3/\delta\}}(s) \mathbb{1}_{\{\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}>3/\delta\}}(s)\, ds \\ &\leq \left(\frac{1}{2}\varepsilon^{4/3} + \frac{9}{2\varepsilon^4}(m+1)^2\delta^{4p-2}\right)\|y_{n+1} - y_n\|_{\Upsilon_t(\mathrm{H})}^2 + 9\frac{\varepsilon^{4/3}}{\delta^{2p}}\|y_n - y_{n-1}\|_{\Upsilon_t(\mathrm{H})}^2. \end{aligned} \tag{3.34}$$

The proof of Case 2 is complete. ■

Case 3. Assume that $\|y_n + M\|_{L^2([0,s],\mathrm{V})} > 3/\delta$ and $\|y_{n-1} + M\|_{L^2([0,s],\mathrm{V})} \le 3/\delta$. In this case, similar to Case 2, by the definitions of functions θ_m and ϕ_δ, we have

$$I(s) = -{}_{\mathrm{V}'}\big\langle \Pi(y_{n-1}, y_{n-1}, y_{n-1}, y_n)(s), y_{n+1}(s) - y_n(s)\big\rangle_{\mathrm{V}},$$

and

$$\begin{aligned}\int_0^t |I(s)|\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}>3/\delta\}}(s)\mathbb{1}_{\{\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}\le 3/\delta\}}(s)\,ds \\ = \int_0^t |I(s)|\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}>3/\delta\}}(s)\mathbb{1}_{\{\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}\le 2/\delta\}}(s) \\ \cdot \mathbb{1}_{\{\|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})}\le m+1\}}\,ds. \qquad (3.35)\end{aligned}$$

For any $s \in [0,t]$ such that

$$\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}>3/\delta\}}(s)\mathbb{1}_{\{\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}\le 2/\delta\}}(s)\mathbb{1}_{\{\|y_{n-1}+M\|_{\Upsilon_s(\mathrm{H})}\le m+1\}} = 1,$$

we have

$$\delta^2\|y_n - y_{n-1}\|^2_{\Upsilon_s(\mathrm{H})} \ge \delta^2\|(y_n + M) - (y_{n-1} + M)\|^2_{L^2([0,s],\mathrm{V})} \ge 1 \qquad (3.36)$$

and, for any $p \in (0, 1/2)$ and $\varepsilon > 0$,

$$\begin{aligned}|I(s)| &\le \big|{}_{\mathrm{V}'}\big\langle \mathrm{B}(y_{n-1}(s) + M(s), y_n(s) + M(s)), y_{n+1}(s) - y_n(s)\big\rangle_{\mathrm{V}}\big| \\ &\le 2|y_{n-1}(s)+M(s)|_{\mathrm{H}}^{1/2}\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^{1/2}|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^{1/2} \\ &\quad \cdot\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^{1/2}\|y_n(s)+M(s)\|_{\mathrm{V}} \\ &\le \frac{3}{2}\delta^{-4p/3}\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^{2/3}\|y_n(s)+M(s)\|_{\mathrm{V}}^{4/3} \\ &\quad +\frac{1}{2}\delta^{4p}\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2|y_{n-1}(s)+M(s)|_{\mathrm{H}}^2|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^2 \\ &\le \frac{1}{2}\delta^{4p}(m+1)^2\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^2 \\ &\quad +\frac{1}{2}\varepsilon^3\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^2+\frac{1}{\varepsilon^{3/2}\delta^{2p}}\|y_n(s)+M(s)\|_{\mathrm{V}}^2 \\ &\le \frac{1}{2}\delta^{4p}(m+1)^2\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^2 \\ &\quad +\frac{1}{2}\varepsilon^3\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^2+2\frac{1}{\varepsilon^{3/2}\delta^{2p}}[\|y_n(s)-y_{n-1}(s)\|_{\mathrm{V}}^2+\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2] \\ &\le \frac{1}{2}\delta^{4p}(m+1)^2\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2|y_{n+1}(s)-y_n(s)|_{\mathrm{H}}^2+\frac{1}{2}\varepsilon^3\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}^2 \\ &\quad +2\frac{1}{\varepsilon^{3/2}\delta^{2p}}\|y_n(s)-y_{n-1}(s)\|_{\mathrm{V}}^2+2\frac{1}{\varepsilon^{3/2}\delta^{2p}}\|y_{n-1}(s)+M(s)\|_{\mathrm{V}}^2\cdot\|y_n-y_{n-1}\|^2_{\Upsilon_s(\mathrm{H})}\delta^2. \end{aligned} \qquad (3.37)$$

In the last "≤" in (3.37), we have used (3.36).

Similarly to (3.22), by (3.35) and (3.37),

$$\int_0^t |I(s)|\mathbb{1}_{\{\|y_n+M\|_{L^2([0,s],\mathrm{V})}>3/\delta\}}\mathbb{1}_{\{\|y_{n-1}+M\|_{L^2([0,s],\mathrm{V})}\le 3/\delta\}}\,ds$$
$$\le\left(\frac{1}{2}\varepsilon^3+2(m+1)^2\delta^{4p-2}\right)\|y_{n+1}-y_n\|^2_{\Upsilon_t(\mathrm{H})}+\frac{2}{\varepsilon^{3/2}\delta^{2p}}\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{H})}. \tag{3.38}$$

The proof of Case 3 is complete. ■

Combining (3.22), (3.26), (3.30), (3.34), and (3.38), there exist constants $C>0$ and $l_m>0$, for any $\varepsilon>0$ and $p\in(0,1/2)$, such that

$$\int_0^t |I(s)|\,ds$$
$$\le l_m\left(\varepsilon+\frac{1}{\varepsilon^3\delta^2}+\varepsilon^{4/3}+\frac{1}{\varepsilon^4\delta^2}+\frac{1}{\varepsilon^4\delta}+\varepsilon^3+\delta^{4p-2}+\varepsilon^{-4}\delta^{4p-2}\right)\|y_{n+1}-y_n\|^2_{\Upsilon_t(\mathrm{H})}$$
$$+C\left(\varepsilon+\frac{\varepsilon}{\delta^2}+\frac{\varepsilon^{4/3}}{\delta^2}+\frac{\varepsilon^{4/3}}{\delta^{1/2}}+\varepsilon^{4/3}+\frac{\varepsilon^{4/3}}{\delta^{2p}}+\frac{1}{\varepsilon^{3/2}\delta^{2p}}\right)\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{H})}. \tag{3.39}$$

Choose $p=1/4$. Letting ε small enough first, and then δ large enough, we see that there exist $\varepsilon_0,\delta_0>0$ such that

$$l_m\left(\varepsilon_0+\frac{1}{\varepsilon_0^3\delta_0^2}+\varepsilon_0^{4/3}+\frac{1}{\varepsilon_0^4\delta_0^2}+\frac{1}{\varepsilon_0^4\delta_0}+\varepsilon_0^3+\delta_0^{4p-2}+\frac{\delta_0^{4p-2}}{\varepsilon_0^4}\right)$$
$$+C\left(\varepsilon_0+\frac{\varepsilon_0}{\delta_0^2}+\frac{\varepsilon_0^{4/3}}{\delta_0^2}+\frac{\varepsilon_0^{4/3}}{\delta_0^{1/2}}+\varepsilon_0^{4/3}+\frac{\varepsilon_0^{4/3}}{\delta_0^{2p}}+\frac{1}{\varepsilon_0^{3/2}\delta_0^{2p}}\right)\le\frac{1}{16}. \tag{3.40}$$

Set $\delta=\delta_0$ in (3.15). By (3.17), (3.39), and (3.40), we arrive at

$$\sup_{s\in[0,t]}|y_{n+1}(s)-y_n(s)|^2_{\mathrm{H}}+2\int_0^t\|y_{n+1}(s)-y_n(s)\|^2_{\mathrm{V}}\,ds$$
$$\le\frac{1}{8}[\|y_{n+1}-y_n\|^2_{\Upsilon_t(\mathrm{H})}+\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{H})}]. \tag{3.41}$$

Since

$$\frac{1}{2}\|y_{n+1}-y_n\|^2_{\Upsilon_t(\mathrm{H})}\le\sup_{s\in[0,t]}|y_{n+1}(s)-y_n(s)|^2_{\mathrm{H}}+\int_0^t\|y_{n+1}(s)-y_n(s)\|^2_{\mathrm{V}}\,ds,$$

by (3.41) we infer that

$$\|y_{n+1}-y_n\|^2_{\Upsilon_T(\mathrm{H})}\le\tfrac{1}{3}\|y_n-y_{n-1}\|^2_{\Upsilon_T(\mathrm{H})}, \tag{3.42}$$

which implies that $\{y_n:n\in\mathbb{N}\}$ is a Cauchy sequence in $C([0,T],\mathrm{H})\cap L^2([0,T],\mathrm{V})$; we denote its limit by Y^1. Using classical arguments, it is not difficult to prove that Y^1 is a solution of problem (3.15) with $\delta=\delta_0$. ■

Step 2. Let δ_0 be as in Step 1, and set

$$t_1:=\inf\{t\in[0,T]:\|Y^1+M\|_{L^2([0,t],\mathrm{V})}\ge 1/\delta_0\}.$$

Since by (3.13), $\phi_\delta(r) = 1$ if $r \in [0, 1/\delta]$, it is easy to show that Y^1 is a solution of problem (3.14) on $[0, t_1]$. If $t_1 = T$ then the proof of Lemma 3.1 is finished. Otherwise, let us consider the following deterministic time-inhomogeneous evolution equation:

$$\begin{aligned} X'(t) + \mathrm{A}X(t) &= f(t) - \theta_m(\|X + M\|_{\Upsilon_t(\mathrm{H})}) \\ &\qquad \cdot \phi_{\delta_0}(\|X + M\|_{L^2([t_1,t],\mathrm{V})}) \cdot \mathrm{B}(X(t) + M(t)), \quad t > t_1, \\ X(t) &= Y^1(t), \quad t \in [0, t_1]. \end{aligned} \tag{3.43}$$

Using a similar argument to that in Step 1, we can find a solution Y^2 to problem (3.43). As at the beginning of this step, we set

$$t_2 := \inf\{t \in [t_1, T] : \|Y^2 + M\|_{L^2([t_1,t],\mathrm{V})} \geq 1/\delta_0\},$$

and see that Y^2 is a solution of problem (3.14) on $[0, t_2]$. If $t_2 = T$ then the proof of Lemma 3.1 is finished. Otherwise, by induction, we construct two sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{Y^n\}_{n\in\mathbb{N}}$ satisfying

- $0 < t_1 < t_2 < \cdots,$
- $Y^n \in C([0, T], \mathrm{H}) \cap L^2([0, T], \mathrm{V})$ and $Y^{n+1}(t) = Y^n(t)$ on $t \in [0, t_n]$,
- Y^n is a solution of problem (3.14) for $t \in [0, t_n]$,
- $t_{n+1} := \inf\{t \in [t_n, T] : \|Y^{n+1} + M\|_{L^2([t_n,t],\mathrm{V})} \geq 1/\delta_0\}$.

The proof of Lemma 3.1 is concluded once we prove that $t_n = T$ for some $n \in \mathbb{N}$. This is done in the next step. ■

Step 3. Assume that $X \in C([0, \tau], \mathrm{H}) \cap L^2([0, \tau], \mathrm{V})$, for some $\tau > 0$, is a solution of the deterministic problem (3.14). By the Lions–Magenes lemma ([46]; [59, Lemma III.1.2]), we have this: For every $t \in [0, \tau]$,

$$\begin{aligned} &|X(t)|_{\mathrm{H}}^2 + 2\int_0^t \|X(s)\|_{\mathrm{V}}^2\, ds \\ &\quad = |u_0|_{\mathrm{H}}^2 - 2\int_0^t \theta_m(\|X + M\|_{\Upsilon_s(\mathrm{H})})\, {}_{\mathrm{V}'}\langle \mathrm{B}(X(s) + M(s)), X(s)\rangle_{\mathrm{V}}\, ds \\ &\qquad + 2\int_0^t {}_{\mathrm{V}'}\langle f(s), X(s)\rangle_{\mathrm{V}}\, ds \\ &\quad \leq |u_0|_{\mathrm{H}}^2 + \int_0^t \|X(s)\|_{\mathrm{V}}^2\, ds + 8\int_0^t \theta_m^2(\|X + M\|_{\Upsilon_s(\mathrm{H})})\|\mathrm{B}(X(s) + M(s))\|_{\mathrm{V}'}^2\, ds \\ &\qquad + 8\int_0^t \|f(s)\|_{\mathrm{V}'}^2\, ds \\ &\quad \leq |u_0|_{\mathrm{H}}^2 + \int_0^t \|X(s)\|_{\mathrm{V}}^2\, ds \\ &\qquad + 8\int_0^t \theta_m^2(\|X + M\|_{\Upsilon_s(\mathrm{H})})|X(s) + M(s)|_{\mathrm{H}}^2\|X(s) + M(s)\|_{\mathrm{V}}^2\, ds + 8\int_0^t \|f(s)\|_{\mathrm{V}'}^2\, ds \\ &\quad \leq |u_0|_{\mathrm{H}}^2 + \int_0^t \|X(s)\|_{\mathrm{V}}^2\, ds + 8(m+1)^4 + 8\int_0^\tau \|f(s)\|_{\mathrm{V}'}^2\, ds. \end{aligned}$$

Hence,

$$\sup_{t\in[0,\tau]} |X(t)|_{\mathrm{H}}^2 + \int_0^\tau \|X(s)\|_{\mathrm{V}}^2\,ds \le |u_0|_{\mathrm{H}}^2 + 8(m+1)^4 + 8\int_0^\tau \|f(s)\|_{\mathrm{V}'}^2\,ds.$$

This implies that $t_n = T$ for some $n \in \mathbb{N}$. Then Y^n is the solution sought in Lemma 3.1. ■

The proof of Lemma 3.1 is complete. ■

The following lemma implies that the solution of problem (3.14) is unique (see Corollary 3.1); this lemma will be used later. Recall that the space $\Lambda_T(\mathrm{H})$ (and its norm) was defined around equality (3.11).

Lemma 3.2. *Assume that $m \in \mathbb{N}$. Assume that for all $u_0 \in \mathrm{H}$ and $f \in L^2([0,T],\mathrm{V}')$ and $y \in \Lambda_T(\mathrm{H})$, there exists an element $u = \Phi^y \in \Lambda_T(\mathrm{H})$ satisfying*

$$du(t) + \mathrm{A}u(t)\,dt + \theta_m(\|u\|_{\Upsilon_t(\mathrm{H})})\mathrm{B}(u(t))\,dt = f(t)\,dt + \int_{\mathrm{Z}} G(y(t-),z)\,\tilde{\eta}(dz,dt),$$
$$u(0) = u_0. \tag{3.44}$$

Then there exists a constant $C_m > 0$ such that

$$\|\Phi^{y_1} - \Phi^{y_2}\|_{\Lambda_T(\mathrm{H})}^2 \le C_m T \|y_1 - y_2\|_{\Lambda_T(\mathrm{H})}^2, \quad y_1, y_2 \in \Lambda_T(\mathrm{H}). \tag{3.45}$$

Remark. The above result is not true without the smoothing function θ_m.

Proof of Lemma 3.2. For simplicity, define $u_1 = \Phi^{y_1}$ and $u_2 = \Phi^{y_2}$. Set $U = u_1 - u_2$. By the Itô formula, we have

$$\begin{aligned}
&|U(t)|_{\mathrm{H}}^2 + 2\int_0^t \|U(s)\|_{\mathrm{V}}^2\,ds \\
&\quad = -2\int_0^t {}_{\mathrm{V}'}\big\langle \theta_m(\|u_1\|_{\Upsilon_s(\mathrm{H})})\mathrm{B}(u_1(s)) - \theta_m(\|u_2\|_{\Upsilon_2^H})\mathrm{B}(u_2(s)), U(s)\big\rangle_{\mathrm{V}}\,ds \\
&\qquad + 2\int_0^t \int_{\mathrm{Z}} \big\langle G(y_1(s-),z) - G(y_2(s-),z), U(s-)\big\rangle_{\mathrm{H}}\,\tilde{\eta}(dz,ds) \\
&\qquad + \int_0^t \int_{\mathrm{Z}} |G(y_1(s-),z) - G(y_2(s-),z)|_{\mathrm{H}}^2\,\eta(dz,ds) \\
&\quad =: J_1(t) + J_2(t) + J_3(t), \quad t \in [0,T].
\end{aligned} \tag{3.46}$$

Concerning J_1, we have

$$\begin{aligned}
|J_1(t)| &\le \frac{1}{2}\int_0^t \|U(s)\|_{\mathrm{V}}^2\,ds \\
&\quad + 2\int_0^t \big\|\theta_m(\|u_1\|_{\Upsilon_s(\mathrm{H})})\mathrm{B}(u_1(s)) - \theta_m(\|u_2\|_{\Upsilon_s(\mathrm{H})})\mathrm{B}(u_2(s))\big\|_{\mathrm{V}'}^2\,ds.
\end{aligned} \tag{3.47}$$

Set

$$K(s) := \left\|\theta_m(\|u_1\|_{\Upsilon_s(\mathrm{H})})\mathrm{B}(u_1(s)) - \theta_m(\|u_2\|_{\Upsilon_s(\mathrm{H})})\mathrm{B}(u_2(s))\right\|_{\mathrm{V}'}^2, \quad s \in [0,T].$$

We distinguish four cases to find appropriate bounds for K. By the property of θ_m and the Minkowski inequality, we have the following estimates. Let us fix $s \in [0,T]$.

(1) Assume that $\|u_1\|_{\Upsilon_s(\mathrm{H})} \vee \|u_2\|_{\Upsilon_s(\mathrm{H})} \le m+1$. In this case, we have

$$\begin{aligned} K(s) &\le C\Big[\|\mathrm{B}(u_1(s)) - \mathrm{B}(u_2(s))\|_{\mathrm{V}'}^2 \\ &\qquad + \left|\theta_m(\|u_1\|_{\Upsilon_s(\mathrm{H})}) - \theta_m(\|u_2\|_{\Upsilon_s(\mathrm{H})})\right|^2 \|\mathrm{B}(u_2(s))\|_{\mathrm{V}'}^2\Big] \\ &\le C|U(s)|_{\mathrm{H}}\|U(s)\|_{\mathrm{V}}\big[|u_1(s)|_{\mathrm{H}}\|u_1(s)\|_{\mathrm{V}} + |u_2(s)|_{\mathrm{H}}\|u_2(s)\|_{\mathrm{V}}\big] \\ &\quad + C|u_2(s)|_{\mathrm{H}}^2\|u_2(s)\|_{\mathrm{V}}^2\|U\|_{\Upsilon_s(\mathrm{H})}^2 \\ &\le \frac{1}{4}\|U(s)\|_{\mathrm{V}}^2 + C\|U\|_{\Upsilon_s(\mathrm{H})}^2\big[|u_1(s)|_{\mathrm{H}}^2\|u_1(s)\|_{\mathrm{V}}^2 + |u_2(s)|_{\mathrm{H}}^2\|u_2(s)\|_{\mathrm{V}}^2\big]. \end{aligned}$$

(2) Assume that $\|u_1\|_{\Upsilon_s(\mathrm{H})} \le m+1$ and $\|u_2\|_{\Upsilon_s(\mathrm{H})} \ge m+1$. In this case,

$$\begin{aligned} K(s) &= \left\|\theta_m(\|u_1\|_{\Upsilon_s(\mathrm{H})})\mathrm{B}(u_1(s))\right\|_{\mathrm{V}'}^2 \\ &= \left|\theta_m(\|u_1\|_{\Upsilon_s(\mathrm{H})}) - \theta_m(\|u_2\|_{\Upsilon_s(\mathrm{H})})\right|^2 \|\mathrm{B}(u_1(s))\|_{\mathrm{V}'}^2 \\ &\le C|u_1(s)|_{\mathrm{H}}^2\|u_1(s)\|_{\mathrm{V}}^2\|U\|_{\Upsilon_s(\mathrm{H})}^2. \end{aligned}$$

(3) Assume that $\|u_1\|_{\Upsilon_s(\mathrm{H})} \ge m+1$ and $\|u_2\|_{\Upsilon_s(\mathrm{H})} \le m+1$. In this case, much as in case (2), we get

$$K(s) \le C|u_2(s)|_{\mathrm{H}}^2\|u_2(s)\|_{\mathrm{V}}^2\|U\|_{\Upsilon_s(\mathrm{H})}^2.$$

(4) Assume that $\|u_1\|_{\Upsilon_s(\mathrm{H})} \wedge \|u_2\|_{\Upsilon_s(\mathrm{H})} \ge m+1$. In this case, we have $K(s) = 0$.

Hence we infer that

$$\begin{aligned} K(s) &\le \frac{1}{4}\|U(s)\|_{\mathrm{V}}^2 + C\|U\|_{\Upsilon_s(\mathrm{H})}^2 \\ &\cdot \big[|u_1(s)|_{\mathrm{H}}^2\|u_1(s)\|_{\mathrm{V}}^2 \cdot \mathbb{1}_{[0,m+1]}(\|u_1\|_{\Upsilon_s(\mathrm{H})}) + |u_2(s)|_{\mathrm{H}}^2\|u_2(s)\|_{\mathrm{V}}^2 \cdot \mathbb{1}_{[0,m+1]}(\|u_2\|_{\Upsilon_s(\mathrm{H})})\big]. \end{aligned} \tag{3.48}$$

Set

$$\Theta(t) := \sup_{s\in[0,t]} |U(s)|_{\mathrm{H}}^2 + \int_0^t \|U(s)\|_{\mathrm{V}}^2\,ds, \qquad t \in [0,T].$$

Substituting (3.48) into (3.47), and then into (3.46), noticing that $\|U\|_{\Upsilon_s(\mathrm{H})}^2 \le 2\Theta(s)$, we have

$$\begin{aligned} \Theta(T) &\le C\int_0^T \Theta(s)\big[|u_1(s)|_{\mathrm{H}}^2\|u_1(s)\|_{\mathrm{V}}^2 \cdot \mathbb{1}_{[0,m+1]}(\|u_1\|_{\Upsilon_s(\mathrm{H})}) \\ &\quad + |u_2(s)|_{\mathrm{H}}^2\|u_2(s)\|_{\mathrm{V}}^2 \cdot \mathbb{1}_{[0,m+1]}(\|u_2\|_{\Upsilon_s(\mathrm{H})})\big]\,ds + \sup_{t\in[0,T]}|J_2(t)| + J_3(T). \end{aligned} \tag{3.49}$$

Gronwall's lemma implies that

$$\Theta(T) \leq \Big(\sup_{t\in[0,T]} |J_2(t)| + J_3(T)\Big) \cdot e^{C\int_0^T [|u_1(s)|_{\mathrm{H}}^2 \|u_1(s)\|_{\mathrm{V}}^2 \cdot \mathbb{1}_{[0,m+1]}(\|u_1\|_{\Upsilon_s(\mathrm{H})}) + |u_2(s)|_{\mathrm{H}}^2 \|u_2(s)\|_{\mathrm{V}}^2 \cdot \mathbb{1}_{[0,m+1]}(\|u_2\|_{\Upsilon_s(\mathrm{H})})]\, ds}$$
$$\leq C_m \Big(\sup_{t\in[0,T]} |J_2(t)| + J_3(T)\Big). \tag{3.50}$$

By the Burkholder–Davis–Gundy inequality (see [45, Theorem 23.12]) and assumption (G-H1) (see (3.1)), we get in a standard way the inequality

$$C_m \mathbb{E}\Big(\sup_{t\in[0,T]} |J_2(t)|\Big) \leq \frac{1}{2}\|U\|_{\Lambda_T(\mathrm{H})}^2 + C_m T \|y_1 - y_2\|_{\Lambda_T(\mathrm{H})}^2. \tag{3.51}$$

Moreover, applying (G-H1) again, we have

$$\mathbb{E}(J_3(T)) \leq CT\|y_1 - y_2\|_{\Lambda_T(\mathrm{H})}^2. \tag{3.52}$$

Summing the inequalities (3.50)–(3.52), we deduce that

$$\|U\|_{\Lambda_T(\mathrm{H})}^2 \leq C_m T \|y_1 - y_2\|_{\Lambda_T(\mathrm{H})}^2.$$

This proves (3.45), and thus the proof of Lemma 3.2 is complete. ■

Corollary 3.1. *Under the assumptions of Lemma 3.1, the solution of problem (3.14) is unique.*

Proof. Suppose that Y_1 and Y_2 are two solutions of problem (3.14). By the Lions–Magenes lemma, we infer that for every $t \geq 0$,

$$|Y_1(t) - Y_2(t)|_{\mathrm{H}}^2 + 2\int_0^t \|Y_1(s) - Y_2(s)\|_{\mathrm{V}}^2\, ds$$
$$= -2\int_0^t {}_{\mathrm{V}'}\big\langle \theta_m(\|Y_1 + M\|_{\Upsilon_s(\mathrm{H})})\mathrm{B}(Y_1(s) + M(s))$$
$$- \theta_m(\|Y_2 + M\|_{\Upsilon_s(\mathrm{H})})\mathrm{B}(Y_2(s) + M(s)), Y_1(s) - Y_2(s)\big\rangle_{\mathrm{V}}\, ds. \tag{3.53}$$

Set $u_1 = Y_1 + M$ and $u_2 = Y_2 + M$. The above equality implies that

$$|u_1(t) - u_2(t)|_{\mathrm{H}}^2 + 2\int_0^t \|u_1(s) - u_2(s)\|_{\mathrm{V}}^2\, ds$$
$$= -2\int_0^t {}_{\mathrm{V}'}\big\langle \theta_m(\|u_1\|_{\Upsilon_s(\mathrm{H})})\mathrm{B}(u_1(s)) - \theta_m(\|u_2\|_{\Upsilon_s(\mathrm{H})})\mathrm{B}(u_2(s)), u_1(s) - u_2(s)\big\rangle_{\mathrm{V}}\, ds. \tag{3.54}$$

We observe that the above equality is a special case of (3.46) with $G \equiv 0$. Therefore, the proof of Lemma 3.2 implies that $u_1 = u_2$. Hence we infer that $Y_1 = Y_2$. ■

Finally, we are ready to finish the proof of the main result in this section. We use the Banach fixed point theorem to prove this result.

Proof of Theorem 3.1. The proof is divided into three steps.

Step 1: Uniqueness. For the uniqueness part of Theorem 3.1, we refer to [11] or [10].

Step 2: Local existence. Consider the auxiliary problem

$$du_n(t)+\mathrm{A}u_n(t)\,dt+\theta_n(\|u_n\|_{\Upsilon_t(\mathrm{H})})\mathrm{B}(u_n(t))\,dt = f(t)\,dt+\int_{\mathrm{Z}} G(u_n(t-),z)\,\widetilde{\eta}(dz,dt),$$
$$u_n(0)=u_0. \tag{3.55}$$

We fix $T>0$. For any $y\in\Lambda_T(\mathrm{H})$, Lemma 3.1 and Corollary 3.1 imply that there exists a unique element $u_n=\Phi^y\in\Lambda_T(\mathrm{H})$ satisfying

$$du_n(t)+\mathrm{A}u_n(t)\,dt+\theta_n(\|u_n\|_{\Upsilon_t(\mathrm{H})})\mathrm{B}(u_n(t))\,dt = f(t)\,dt+\int_{\mathrm{Z}} G(y(t-),z)\,\widetilde{\eta}(dz,dt),$$
$$u_n(0)=u_0. \tag{3.56}$$

Indeed, it is known that there exists a unique $M\in\Lambda_T(\mathrm{H})$ satisfying the equation

$$dM(t)+\mathrm{A}M(t)\,dt=\int_{\mathrm{Z}} G(y(t-),z)\,\widetilde{\eta}(dz,dt),\quad t\geq 0,\quad M(0)=0,$$

and the inequality

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|M(t)|^2_{\mathrm{H}}\Big)+\mathbb{E}\bigg(\int_0^T\|M(t)\|^2_{\mathrm{V}}\,dt\bigg)\leq C_T\Big[\mathbb{E}\Big(\sup_{t\in[0,T]}|y(t)|^2_{\mathrm{H}}\Big)+1\Big].$$

Hence, Lemma 3.1 and Corollary 3.1 imply that for any $\omega\in\Omega$, there exists a unique element $X(\omega)\in C([0,T],\mathrm{H})\cap L^2([0,T],\mathrm{V})$ solving

$$dX(t)+\mathrm{A}X(t)\,dt+\theta_n(\|X+M\|_{\Upsilon_t(\mathrm{H})})\mathrm{B}(X(t)+M(t))\,dt=f(t)\,dt,$$
$$X(0)=u_0.$$

One can show that u is a solution to (3.56) iff $u=X+M$. For uniqueness, we refer to Lemma 3.2. Moreover, Lemma 3.2 implies that there exists a constant $C_n>0$ such that

$$\|\Phi^{y_1}-\Phi^{y_2}\|^2_{\Lambda_T(\mathrm{H})}\leq C_nT\|y_1-y_2\|^2_{\Lambda_T(\mathrm{H})},\quad y_1,y_2\in\Lambda_T(\mathrm{H}). \tag{3.57}$$

Let $T_n=\frac{1}{2C_n}$. In view of (3.57) and by using the Banach fixed point theorem, we infer that there exists a unique element $u_n^1\in\Lambda_{T_n}(\mathrm{H})$ that is a solution of (3.55) for $t\in[0,T_n]$. Repeating the above proof, and observing that T_n does not depend on the initial datum, we can find a unique element $u_n^2:=\{u_n^2(t),\ t\in[0,2T_n]\}\in\Lambda_{2T_n}(\mathrm{H})$ solving the problem

$$du_n(t)+\mathrm{A}u_n(t)\,dt+\theta_n(\|u_n\|_{\Upsilon_t(\mathrm{H})})\mathrm{B}(u_n(t))\,dt$$
$$=f(t)\,dt+\int_{\mathrm{Z}} G(u_n(t-),z)\,\widetilde{\eta}(dz,dt),\quad t\in[T_n,2T_n],$$
$$u_n(t)=u_n^1(t),\quad t\in[0,T_n].$$

It is not difficult to see that u_n^2 is a solution of problem (3.55) on $[0,2T_n]$. By induction, we can construct a unique element $u_n\in\Lambda_T(\mathrm{H})$ which is a solution of problem (3.55) for $t\in[0,T]$, where $T>0$ is arbitrary.

Define a stopping time

$$\tau_n = \inf\{t \geq 0 : \|u_n\|_{\Upsilon_t(\mathrm{H})} > n\}. \tag{3.58}$$

By definition, $\theta_n(\|u_n\|_{\Upsilon_t(\mathrm{H})}) = 1$ for any $t \in [0, \tau_n)$, hence $\{u_n(t), t \in [0, \tau_n)\}$ is a local solution of problem (2.2). Thus, by the uniqueness of solutions to problem (2.2), we infer that

$$u_{n+1}(t) = u_n(t), \quad t \in [0, \tau_n \wedge \tau_{n+1}), \ \mathbb{P}\text{-a.s.}$$

Hence, the sequence $(\tau_n)_{n=1}^{\infty}$ is nondecreasing. We set $\tau_{\max} := \lim_{n\to\infty} \tau_n$, and we observe that $\tau_{\max}$ is also a stopping time.

Now we can define a local solution $\{u(t), t \in [0, \tau_{\max})\}$ of problem (2.2) by

$$u(t) = u_n(t), \quad t \in [0, \tau_n).$$

Using an argument similar to the proof of [5, Theorem 3.5], we can prove that

$$\lim_{t \nearrow \tau_{\max}} \|u\|_{\Upsilon_t(\mathrm{H})} = \infty \quad \text{on } \{\omega \in \Omega : \tau_{\max} < \infty\}, \ \mathbb{P}\text{-a.s.} \tag{3.59}$$

Step 3: Global existence. We will prove that

$$\mathbb{P}(\tau_{\max} = \infty) = 1. \tag{3.60}$$

It is sufficient to prove that for every $T > 0$, $\mathbb{P}(\tau_{\max} \geq T) = 1$. For the rest of this proof we fix $T > 0$.

In this step, we do not use the Lipschitz assumption (3.1) but only the linear growth assumption (3.2).

By the Itô formula, we have

$$\begin{aligned}
&|u(t \wedge \tau_n)|_{\mathrm{H}}^2 + 2\int_0^{t\wedge\tau_n} \|u(s)\|_{\mathrm{V}}^2\, ds \\
&\quad = |u_0|_{\mathrm{H}}^2 + 2\int_0^{t\wedge\tau_n} {}_{\mathrm{V}'}\langle f(s), u(s)\rangle_{\mathrm{V}}\, ds + 2\int_0^{t\wedge\tau_n}\int_{\mathrm{Z}} \langle G(u(s-), z), u(s-)\rangle_{\mathrm{H}}\, \tilde{\eta}(dz, ds) \\
&\quad\quad + \int_0^{t\wedge\tau_n}\int_{\mathrm{Z}} |G(u(s-), z)|_{\mathrm{H}}^2\, \eta(dz, ds), \quad t \in [0, T].
\end{aligned}$$

The Burkholder–Davis–Gundy inequality yields

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,T]} |u(t \wedge \tau_n)|_{\mathrm{H}}^2 + \int_0^{T\wedge\tau_n} \|u(s)\|_{\mathrm{V}}^2\, ds\Big) \\
&\quad \leq |u_0|_{\mathrm{H}}^2 + \int_0^T \|f(s)\|_{\mathrm{V}'}^2\, ds + 2\mathbb{E}\Big(\sup_{t\in[0,T]} \Big|\int_0^{t\wedge\tau_n}\int_{\mathrm{Z}} \langle G(u(s-), z), u(s-)\rangle_{\mathrm{H}}\, \tilde{\eta}(dz, ds)\Big|\Big) \\
&\quad\quad + \mathbb{E}\Big(\int_0^{T\wedge\tau_n}\int_{\mathrm{Z}} |G(u(s-), z)|_{\mathrm{H}}^2\, \eta(dz, ds)\Big) \\
&\quad \leq |u_0|_{\mathrm{H}}^2 + \int_0^T \|f(s)\|_{\mathrm{V}'}^2\, ds + \frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T]} |u(t\wedge\tau_n)|_{\mathrm{H}}^2\Big) \\
&\quad\quad + C\int_0^T \mathbb{E}\Big(\sup_{l\in[0,t]} |u(l\wedge\tau_n)|_{\mathrm{H}}^2\Big)\, dt + CT.
\end{aligned} \tag{3.61}$$

Applying Gronwall's lemma, we infer that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|u(t\wedge\tau_n)|_{\mathrm{H}}^2+\int_0^{T\wedge\tau_n}\|u(s)\|_{\mathrm{V}}^2\,ds\Big)\le C_T\Big(1+|u_0|_{\mathrm{H}}^2+\int_0^T\|f(s)\|_{\mathrm{V}'}^2\,ds\Big).$$

Letting $n\to\infty$, so that $\tau_n\nearrow\tau_{\max}$, we deduce that

$$\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau_{\max})}|u(t)|_{\mathrm{H}}^2+\int_0^{T\wedge\tau_{\max}}\|u(s)\|_{\mathrm{V}}^2\,ds\Big)\le C_T\Big(1+|u_0|_{\mathrm{H}}^2+\int_0^T\|f(s)\|_{\mathrm{V}'}^2\,ds\Big).\tag{3.62}$$

This leads to

$$\sup_{t\in[0,T\wedge\tau_{\max})}|u(t)|_{\mathrm{H}}^2+\int_0^{T\wedge\tau_{\max}}\|u(s)\|_{\mathrm{V}}^2\,ds<\infty,\quad \mathbb{P}\text{-a.s.}$$

The above implies that the function $[0,T\wedge\tau_{\max})\ni t\ni\|u\|_{\Upsilon_t(\mathrm{H})}^2$ is $\mathbb{P}$-a.s. bounded, which in turn in view of (3.59) implies that $\tau_{\max}\ge T$, $\mathbb{P}$-a.s., as required. The proof of Theorem 3.1 is thus complete. ■

4. Solutions to SNSEs with initial data in V

Now we consider SNSEs with more regular data. For this purpose, we formulate the following assumptions.

Assumption 4.1. The function $G:\mathrm{V}\times\mathrm{Z}\to\mathrm{V}$ is measurable such that there exist constants $C_1,C_2>0$ such that

(G-V1) (Lipschitz in V)

$$\int_{\mathrm{Z}}\|G(v_1,z)-G(v_2,z)\|_{\mathrm{V}}^2\,\nu(dz)\le C_1\|v_1-v_2\|_{\mathrm{V}}^2,\quad v_1,v_2\in\mathrm{V},\tag{4.1}$$

(G-V2) (Linear growth in V)

$$\int_{\mathrm{Z}}\|G(v,z)\|_{\mathrm{V}}^2\,\nu(dz)\le C_1(1+\|v\|_{\mathrm{V}}^2),\quad v\in\mathrm{V}.\tag{4.2}$$

(G-VH2) (Linear growth in H)

$$\int_{\mathrm{Z}}|G(v,z)|_{\mathrm{H}}^2\,\nu(dz)\le C_2(1+|v|_{\mathrm{H}}^2),\quad v\in\mathrm{V}.\tag{4.3}$$

In this section, we will prove the following result.

Theorem 4.1. *Assume that a function G satisfies Assumption 4.1. Then for all $u_0\in\mathrm{V}$ and $f\in L^2_{loc}([0,\infty),\mathrm{H})$, there exists a unique $\mathbb{F}$-progressively measurable process u such that*

(1) $u\in D([0,\infty),\mathrm{V})\cap L^2_{loc}([0,\infty),\mathscr{D}(\mathrm{A}))$, $\mathbb{P}$*-a.s.,*

(2) *for all* $t \in [0,\infty)$, $\mathbb{P}$*-a.s. in* H,

$$u(t) = u_0 - \int_0^t \mathrm{A}u(s)\,ds - \int_0^t \mathrm{B}(u(s))\,ds + \int_0^t f(s)\,ds + \int_0^t \int_{\mathrm{Z}} G(u(s-),z)\,\tilde{\eta}(dz,ds). \tag{4.4}$$

Remark 4.1. The Lipschitz property of G with respect to the V-norm does not imply the Lipschitz property of G with respect to the H-norm. Hence, the uniqueness part of Theorem 4.1 is not a consequence of Theorem 3.1, and requires an independent proof.

Remark 4.2. In [5], the authors considered the existence and uniqueness of solutions defined as in Theorem 4.1 for stochastic hydrodynamical systems with Lévy noise, including 2D Navier–Stokes euations. They assumed that the function G is globally Lipschitz in the sense that there exists $K > 0$ such that for $p = 1, 2$,

$$\int_{\mathrm{Z}} \|G(v_1,z) - G(v_2,z)\|_{\mathrm{V}}^{2p}\,\nu(dz) \le K\|v_1 - v_2\|_{\mathrm{V}}^{2p}, \quad v_1, v_2 \in \mathrm{V},$$
$$\int_{\mathrm{Z}} |G(v_1,z) - G(v_2,z)|_{\mathrm{H}}^{2p}\,\nu(dz) \le K|v_1 - v_2|_{\mathrm{H}}^{2p}, \quad v_1, v_2 \in \mathrm{H}.$$

It is easy to see that our assumptions are weaker.

Let us also mention that in the Gaussian case, stochastic Navier–Stokes equations, respectively Euler equations, for initial data in V have been studied in [36, 51], respectively in [14].

Remark 4.3. In Section 3, we proved two existence results. The first one, Theorem 3.1, holds under the global Lipschitz assumptions on the coefficient G. The second one, Theorem 3.2, holds under the assumption that G is Lipschitz on balls in H and has linear growth. The bulk of the proof was devoted to the proof of the former result, as the latter follows from the former by a standard procedure.

In the same vein, in the present section, we first formulate Theorem 4.1 which holds under the assumption that G is globally Lipschitz with respect to V. This result is supplemented by Theorem 4.2 below, in which we assume that G is Lipschitz on balls in V. The latter result can be deduced from the former by a standard truncation procedure.

Assumption 4.2. A map $G : \mathrm{V} \times \mathrm{Z} \to \mathrm{V}$ is measurable and such that

(G-V1-local) (Lipschitz on balls) for every $\hbar > 0$, there exists a constant $C_\hbar > 0$ such that, for all $v_1, v_2 \in \mathrm{V}$ with $\|v_1\|_{\mathrm{V}} \vee \|v_2\|_{\mathrm{V}} \le \hbar$,

$$\int_{\mathrm{Z}} \|G(v_1,z) - G(v_2,z)\|_{\mathrm{V}}^2\,\nu(dz) \le C_\hbar\|v_1 - v_2\|_{\mathrm{V}}^2, \tag{4.5}$$

and the assumptions (G-V2) and (G-VH2) hold.

Theorem 4.2. *Assume that Assumption 4.2 holds. Then for all* $u_0 \in \mathrm{V}$ *and* $f \in L^2_{\mathrm{loc}}([0,\infty),\mathrm{H})$*, there exists a unique* $\mathbb{F}$*-progressively measurable process* u *such that*

(1) $u \in D([0,\infty),\mathrm{V}) \cap L^2_{\mathrm{loc}}([0,\infty),\mathscr{D}(\mathrm{A}))$, $\mathbb{P}$*-a.s.,*

(2) *for all* $t \in [0,\infty)$, $\mathbb{P}$*-a.s., in* H,

$$u(t) = u_0 - \int_0^t \mathrm{A}u(s)\,ds - \int_0^t \mathrm{B}(u(s))\,ds + \int_0^t f(s)\,ds \\ + \int_0^t \int_{\mathrm{Z}} G(u(s-),z)\,\tilde{\eta}(dz,ds). \tag{4.6}$$

Proof. The proof is similar to that for Theorem 3.2.

For any $k \in \mathbb{N}$, we define an auxiliary function G_k by

$$G_k : \mathrm{V} \times \mathrm{Z} \ni (y,z) \mapsto G\Big(\frac{\|y\|_{\mathrm{V}} \wedge k}{\|y\|_{\mathrm{V}}} y, z\Big) \in \mathrm{V},$$

where we set $\frac{\|y\|_{\mathrm{V}}\wedge k}{\|y\|_{\mathrm{V}}} = 1$ when $y = 0$. Since, by our assumptions, G satisfies Assumption 4.2, we can easily show that G_k satisfies Assumption 4.1. From now on we fix $k > \|u_0\|_{\mathrm{V}}$. By Theorem 4.1, there exists a unique $\mathbb{F}$-progressively measurable process X^k such that

- $X^k \in D([0,\infty),\mathrm{V}) \cap L^2_{\mathrm{loc}}([0,\infty),\mathscr{D}(\mathrm{A}))$, $\mathbb{P}$-a.s.,
- for all $t \in [0,\infty)$, $\mathbb{P}$-a.s., in H,

$$X^k(t) = u_0 - \int_0^t \mathrm{A}X^k(s)\,ds - \int_0^t \mathrm{B}(X^k(s))\,ds + \int_0^t f(s)\,ds \\ + \int_0^t \int_{\mathrm{Z}} G_k(X^k(s-),z)\,\tilde{\eta}(dz,ds).$$

Similarly to (3.8) we define a stopping time

$$\sigma_k := \inf\Big\{t \geq 0 : \sup_{s\in[0,t]} \|X^k(s)\|_{\mathrm{V}} > k\Big\}.$$

It is not difficult to see that σ_k is increasing in k, and $X^{k+1}(t) = X^k(t)$, $t \in [0,\sigma_k)$. We also define a stopping time $\sigma := \lim_{k\to\infty} \sigma_k$. The property above enables us to define $u(t)$ for $t \in [0,\sigma)$ as follows:

$$u(t) := X^k(t), \quad t \in [0,\sigma_k).$$

It is easy to see that $u(t)$, $t \in [0,\sigma)$, is a local solution of problem (4.6). To complete the proof, we need only show that $\mathbb{P}(\sigma = \infty) = 1$. For this purpose, we use condition (G-VH2) from Assumption 4.2.

Following the argument we used in the proof of (3.62), we can find $C_T > 0$ such that

$$\mathbb{E}\Big(\sup_{t\in[0,T\wedge\sigma)} |u(t)|^2_{\mathrm{H}}\Big) + \mathbb{E}\Big(\int_0^{T\wedge\sigma} \|u(t)\|^2_{\mathrm{V}}\,dt\Big) \leq C_T. \tag{4.7}$$

Define an additional stopping time $\tilde{\tau}_N$ by

$$\tilde{\tau}_N := \inf\Big\{t \geq 0 : \sup_{s\in[0,t]} |u(s)|^2_{\mathrm{H}} + \int_0^t \|u(s)\|^2_{\mathrm{V}}\,ds \geq N\Big\} \wedge T \wedge \sigma,$$

and set $\tau_{N,k} := \widetilde{\tau}_N \wedge \sigma_k$. By the Itô formula and Lemma 2.1, we have, for $t \geq 0$,

$$
\begin{aligned}
&\|u(t)\|_{\mathrm{V}}^2 + 2\int_0^t \|u(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds \\
&\quad = \|u_0\|_{\mathrm{V}}^2 - 2\int_0^t \langle \mathrm{B}(u(s)), \mathrm{A}u(s)\rangle_{\mathrm{H}}\,ds + 2\int_0^t \langle f(s), \mathrm{A}u(s)\rangle_{\mathrm{H}}\,ds \\
&\qquad + 2\int_0^t\int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(u(s-),z), u(s-)\rangle_{\mathrm{V}}\,\widetilde{\eta}(dz,ds) + \int_0^t\int_{\mathrm{Z}} \|G(u(s-),z)\|_{\mathrm{V}}^2\,\eta(dz,ds) \\
&\quad \leq \|u_0\|_{\mathrm{V}}^2 + \int_0^t \|u(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds + C\int_0^t \|u(s)\|_{\mathrm{V}}^4 |u(s)|_{\mathrm{H}}^2\,ds + 2\int_0^t |f(s)|_{\mathrm{H}}^2\,ds \\
&\qquad + 2\int_0^t\int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(u(s-),z), u(s-)\rangle_{\mathrm{V}}\,\widetilde{\eta}(dz,ds) + \int_0^t\int_{\mathrm{Z}} \|G(u(s-),z)\|_{\mathrm{V}}^2\,\eta(dz,ds).
\end{aligned}
$$

Applying Gronwall's lemma, we infer that

$$
\begin{aligned}
&\|u(t\wedge\tau_{N,k})\|_{\mathrm{V}}^2 + \int_0^{t\wedge\tau_{N,k}} \|u(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds \\
&\leq e^{C\int_0^{t\wedge\tau_{N,k}} |u(s)|_{\mathrm{H}}^2 \|u(s)\|_{\mathrm{V}}^2\,ds} \\
&\cdot \Big(\|u_0\|_{\mathrm{V}}^2 + 2\int_0^T |f(s)|_{\mathrm{H}}^2\,ds + \sup_{s\in[0,T]} \Big| \int_0^{s\wedge\tau_{N,k}} \int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(u(l-),z), u(l-)\rangle_{\mathrm{V}}\,\widetilde{\eta}(dz,dl) \Big| \\
&\qquad\qquad + \int_0^{T\wedge\tau_{N,k}} \int_{\mathrm{Z}} \|G(u(s-),z)\|_{\mathrm{V}}^2\,\eta(dz,ds) \Big) \\
&\leq e^{CN^2} \Big(\|u_0\|_{\mathrm{V}}^2 + 2\int_0^T |f(s)|_{\mathrm{H}}^2\,ds \\
&\qquad + \sup_{s\in[0,T]} \Big| \int_0^{s\wedge\tau_{N,k}} \int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(u(l-),z), u(l-)\rangle_{\mathrm{V}}\,\widetilde{\eta}(dz,dl) \Big| \\
&\qquad + \int_0^{T\wedge\tau_{N,k}} \int_{\mathrm{Z}} \|G(u(s-),z)\|_{\mathrm{V}}^2\,\eta(dz,ds) \Big), \quad t\in[0,T].
\end{aligned} \tag{4.8}
$$

By the Burkholder–Davis–Gundy inequality and the assumption (G-V2), i.e., (4.2), we get

$$
\begin{aligned}
&e^{CN^2}\mathbb{E}\Big(\sup_{s\in[0,T]} \Big| \int_0^{s\wedge\tau_{N,k}} \int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(u(l-),z), u(l-)\rangle_{\mathrm{V}}\,\widetilde{\eta}(dz,dl) \Big| \Big) \\
&\quad \leq Ce^{CN^2}\mathbb{E}\Big(\Big| \int_0^{T\wedge\tau_{N,k}} \int_{\mathrm{Z}} \|G(u(s-),z)\|_{\mathrm{V}}^2 \|u(s-)\|_{\mathrm{V}}^2\,\eta(dz,ds) \Big|^{1/2} \Big) \\
&\quad \leq \frac{1}{2}\mathbb{E}\Big(\sup_{s\in[0,T]} \|u(s\wedge\tau_{N,k})\|_{\mathrm{V}}^2 \Big) + Ce^{CN^2}\mathbb{E}\Big(\int_0^{T\wedge\tau_{N,k}} \int_{\mathrm{Z}} \|G(u(s),z)\|_{\mathrm{V}}^2\,\nu(dz)\,ds \Big) \\
&\quad \leq \frac{1}{2}\mathbb{E}\Big(\sup_{s\in[0,T]} \|u(s\wedge\tau_{N,k})\|_{\mathrm{V}}^2 \Big) + Ce^{CN^2}\int_0^T \mathbb{E}(1 + \|u(s\wedge\tau_{N,k})\|_{\mathrm{V}}^2)\,ds.
\end{aligned} \tag{4.9}
$$

Applying assumption (G-V2) again, we infer that

$$\mathbb{E}\Big(\int_0^{T\wedge\tau_{N,k}}\int_{\mathrm{Z}}\|G(u(s-),z)\|_{\mathrm{V}}^2\,\eta(dz,ds)\Big)$$
$$\le C\int_0^T \mathbb{E}(1+\|u(s\wedge\tau_{N,k})\|_{\mathrm{V}}^2)\,ds. \tag{4.10}$$

Inserting inequalities (4.9) and (4.10) into (4.8), and then using Gronwall's lemma, we infer that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|u(t\wedge\tau_{N,k})\|_{\mathrm{V}}^2\Big)+\mathbb{E}\Big(\int_0^{T\wedge\tau_{N,k}}\|u(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds\Big)$$
$$\le C_{N,T}\Big(\|u_0\|_{\mathrm{V}}^2+\int_0^T|f(s)|_{\mathrm{H}}^2\,ds+1\Big).$$

Letting $k\to\infty$, we get

$$\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tilde{\tau}_N\wedge\sigma)}\|u(t)\|_{\mathrm{V}}^2\Big)+\mathbb{E}\Big(\int_0^{T\wedge\tilde{\tau}_N\wedge\sigma}\|u(t)\|_{\mathscr{D}(\mathrm{A})}^2\,dt\Big)$$
$$\le C_{N,T}\Big(\|u_0\|_{\mathrm{V}}^2+\int_0^T|f(s)|_{\mathrm{H}}^2\,ds+1\Big).$$

This implies that

$$\sup_{t\in[0,T\wedge\tilde{\tau}_N\wedge\sigma)}\|u(t)\|_{\mathrm{V}}^2+\int_0^{T\wedge\tilde{\tau}_N\wedge\sigma}\|u(t)\|_{\mathscr{D}(\mathrm{A})}^2\,dt<\infty,\quad \mathbb{P}\text{-a.s.} \tag{4.11}$$

For a fixed $T>0$, we set

$$\Omega_N:=\{\omega\in\Omega:\tilde{\tau}_N=T\wedge\sigma\}.$$

Then $\Omega_N\subset\Omega_{N+1}$. By (4.7) and (4.11), we deduce that $\lim_{N\to\infty}\mathbb{P}(\Omega_N)=1$, and

$$\sup_{t\in[0,T\wedge\sigma)}\|u(t)\|_{\mathrm{V}}^2+\int_0^{T\wedge\sigma}\|u(t)\|_{\mathscr{D}(\mathrm{A})}^2\,dt<\infty\quad \text{on } \Omega_N,\ \mathbb{P}\text{-a.s.}$$

Hence

$$\sup_{t\in[0,T\wedge\sigma)}\|u(t)\|_{\mathrm{V}}^2+\int_0^{T\wedge\sigma}\|u(t)\|_{\mathscr{D}(\mathrm{A})}^2\,dt<\infty,\quad \mathbb{P}\text{-a.s.},$$

which yields

$$\mathbb{P}(\sigma\ge T)=1,\quad \forall T>0.$$

The proof of Theorem 4.2 is complete. ■

Similar to Section 3, we first introduce symbols which will be used later. Then we state three auxiliary results: Lemmata 4.1 and 4.2, and Corollary 4.1.

In this section, we set, for $T \geq 0$,

$$\Upsilon_T(\mathrm{V}) = D([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A})). \tag{4.12}$$

Note that the definition of the space above differs from (3.9). It is easy to see that the space $\Upsilon_T(\mathrm{V})$ endowed with the norm

$$\|y\|_{\Upsilon_T(\mathrm{V})} = \sup_{s\in[0,T]} \|y(s)\|_{\mathrm{V}} + \left(\int_0^T \|y(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds\right)^{1/2} \tag{4.13}$$

is a Banach space.

Let $\Lambda_T(\mathrm{V})$ be the space of all V-valued càdlàg $\mathbb{F}$-progressively measurable processes y whose a.a. trajectories belong to the space $\Upsilon_T(\mathrm{V})$ and such that

$$\|y\|^2_{\Lambda_T(\mathrm{V})} := \mathbb{E}\left(\sup_{s\in[0,T]} \|y(s)\|^2_{\mathrm{V}} + \int_0^T \|y(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds\right) < \infty. \tag{4.14}$$

We point out that the space $\Lambda_T(\mathrm{H})$ introduced earlier around (3.11) differs from the current space $\Lambda_T(\mathrm{V})$.

Recall that the auxiliary function $\theta_m(\cdot)$ has been introduced in (3.12) (and used, for instance, in Lemma 3.1).

We are now ready to state the first of the three auxiliary results we need to prove Theorem 4.1.

Lemma 4.1. *Assume that $T > 0$ and $m \in \mathbb{N}$. Then for all $u_0 \in \mathrm{V}$, $f \in L^2([0,T],\mathrm{H})$, and $J \in \Upsilon_T(\mathrm{V})$, there exists a function $y \in C([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A}))$ satisfying*

$$\begin{aligned} &y'(t) + \mathrm{A}y(t) + \theta_m(\|y+J\|_{\Upsilon_t(\mathrm{V})})\mathrm{B}(y(t)+J(t)) = f(t), \quad t \in (0,T),\\ &y(0) = u_0. \end{aligned} \tag{4.15}$$

Proof. Fix $T > 0$ and $m \in \mathbb{N}$. Also fix $u_0 \in \mathrm{V}$, $f \in L^2([0,T],\mathrm{H})$ and $J \in \Upsilon_T(\mathrm{V})$. We use the Picard iterative method again to prove this result.

Fix $y_0 \in C([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A}))$ such that $y_0(0) = u_0$. For instance, we can take $y_0(t) = e^{-t\mathrm{A}}u_0$, $t \in [0,T]$.

It is not difficult to prove that, given $y_n \in C([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A}))$, $n \in \mathbb{N}$, there exists a unique $y_{n+1} \in C([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A}))$ satisfying the deterministic initial value problem

$$\begin{aligned} &y'_{n+1}(t) + \mathrm{A}y_{n+1}(t) + \theta_m(\|y_n+J\|_{\Upsilon_t(\mathrm{V})})\mathrm{B}(y_n(t)+J(t), y_{n+1}(t)+J(t)) = f(t),\\ &y_{n+1}(0) = u_0. \end{aligned} \tag{4.16}$$

We will show that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A}))$. We now estimate the norm in $C([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A}))$ of the difference $y_{n+1} - y_n$ for $n \geq 1$. To do so, set, for $x_i \in C([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A}))$, $i = 1,2,3$,

$$\Xi(x_1,x_2,x_3)(s) = \theta_m(\|x_1+J\|_{\Upsilon_s(\mathrm{V})})\mathrm{B}(x_2(s)+J(s), x_3(s)+J(s)), \quad s \in [0,T].$$

By the Lions–Magenes lemma, we have

$$\|y_{n+1}(t) - y_n(t)\|_{\mathrm{V}}^2 + 2\int_0^t \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})}^2 \, ds = -2\int_0^t K(s)\, ds, \qquad (4.17)$$

where

$$K(s) = \big\langle \Xi(y_n, y_n, y_{n+1})(s) - \Xi(y_{n-1}, y_{n-1}, y_n)(s), \mathrm{A}(y_{n+1}(s) - y_n(s))\big\rangle_{\mathrm{H}}$$

with $\langle\cdot,\cdot\rangle_{\mathrm{H}}$ denoting the scalar product in H.

Now fix $s \in [0, T]$. To estimate $K(s)$, we consider three cases. Each case contains a calculation of a certain "partial" integral $\int_0^t |K(s)|\, ds$.

Case 1. Assume that $\|y_n + J\|_{\Upsilon_s(\mathrm{V})} \vee \|y_{n-1} + J\|_{\Upsilon_s(\mathrm{V})} \le m + 2$. Then

$$\begin{aligned}
|K(s)| \le{}& |\theta_m(\|y_n + J\|_{\Upsilon_s(\mathrm{V})}) - \theta_m(\|y_{n-1} + J\|_{\Upsilon_s(\mathrm{V})})| \\
&\cdot \big|\big\langle \mathrm{B}(y_{n-1}(s) + J(s), y_n(s) + J(s)), \mathrm{A}(y_{n+1}(s) - y_n(s))\big\rangle_{\mathrm{H}}\big| \\
&+ \big|\big\langle \mathrm{B}(y_n(s) + J(s), y_{n+1}(s) - y_n(s)), \mathrm{A}(y_{n+1}(s) - y_n(s))\big\rangle_{\mathrm{H}}\big| \\
&+ \big|\big\langle \mathrm{B}(y_n(s) - y_{n-1}(s), y_n(s) + J(s)), \mathrm{A}(y_{n+1}(s) - y_n(s))\big\rangle_{\mathrm{H}}\big| \\
=:{}& I_1(s) + I_2(s) + I_3(s).
\end{aligned}$$

By Lemma 2.1 and the definition of θ_m,

$$\begin{aligned}
I_1(s) &\le C\|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{V})} |y_{n-1}(s) + J(s)|_{\mathrm{H}}^{1/2} \|y_{n-1}(s) + J(s)\|_{\mathrm{V}}^{1/2} \\
&\quad \cdot \|y_n(s) + J(s)\|_{\mathrm{V}}^{1/2} \|y_n(s) + J(s)\|_{\mathscr{D}(\mathrm{A})}^{1/2} \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})} \\
&\le C_m \|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{V})} \|y_n(s) + J(s)\|_{\mathscr{D}(\mathrm{A})}^{1/2} \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})} \\
&\le \varepsilon \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})}^2 + \frac{C_m}{\varepsilon} \|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{V})}^2 \|y_n(s) + J(s)\|_{\mathscr{D}(\mathrm{A})}, \qquad (4.18) \\
I_2(s) &\le C|y_n(s) + J(s)|_{\mathrm{H}}^{1/2} \|y_n(s) + J(s)\|_{\mathrm{V}}^{1/2} \\
&\quad \cdot \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^{1/2} \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})}^{3/2} \\
&\le C_m \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^{1/2} \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})}^{3/2} \\
&\le \varepsilon^{4/3} \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})}^2 + \frac{C_m}{\varepsilon^4} \|y_{n+1}(s) - y_n(s)\|_{\mathrm{V}}^2, \qquad (4.19)
\end{aligned}$$

and

$$\begin{aligned}
I_3(s) &\le C|y_n(s) - y_{n-1}(s)|_{\mathrm{H}}^{1/2} \|y_n(s) - y_{n-1}(s)\|_{\mathrm{V}}^{1/2} \\
&\quad \cdot \|y_n(s) + J(s)\|_{\mathrm{V}}^{1/2} \|y_n(s) + J(s)\|_{\mathscr{D}(\mathrm{A})}^{1/2} \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})} \\
&\le C_m \|y_n(s) - y_{n-1}(s)\|_{\mathrm{V}} \|y_n(s) + J(s)\|_{\mathscr{D}(\mathrm{A})}^{1/2} \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})} \\
&\le \varepsilon \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})}^2 + \frac{C_m}{\varepsilon} \|y_n(s) - y_{n-1}(s)\|_{\mathrm{V}}^2 \|y_n(s) + J(s)\|_{\mathscr{D}(\mathrm{A})}.
\end{aligned} \qquad (4.20)$$

Therefore, since

$$\int_0^t \|y_n(s) + J(s)\|_{\mathscr{D}(\mathrm{A})} \mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+2\}}(s)\, ds \le C_m t^{1/2},$$

by (4.18)–(4.20) we deduce

$$\begin{aligned}
&\int_0^t |K(s)| \mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\vee\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}\le m+2\}}(s)\, ds \\
&\quad \le (2\varepsilon + \varepsilon^{4/3}) \int_0^t \|y_{n+1}(s) - y_n(s)\|^2_{\mathscr{D}(\mathrm{A})}\, ds + \frac{C_m}{\varepsilon^4} t \|y_{n+1} - y_n\|^2_{\Upsilon_t(\mathrm{V})} \\
&\qquad + \frac{C_m}{\varepsilon} \|y_n - y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})} \int_0^t \|y_n(s) + J(s)\|_{\mathscr{D}(\mathrm{A})} \mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+2\}}(s)\, ds \\
&\quad \le \left(2\varepsilon + \varepsilon^{4/3} + \frac{C_m}{\varepsilon^4} t\right) \|y_{n+1} - y_n\|^2_{\Upsilon_t(\mathrm{V})} + \frac{C_m}{\varepsilon} t^{1/2} \|y_n - y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})}, \quad t \in [0,T].
\end{aligned} \tag{4.21}$$

Case 2. Assume that $\|y_n + J\|_{\Upsilon_s(\mathrm{V})} \le m+2$ and $\|y_{n-1} + J\|_{\Upsilon_s(\mathrm{V})} > m+2$. Then the definition of θ_m implies that

$$K(s) = \theta_m(\|y_n + J\|_{\Upsilon_s(\mathrm{V})}) \big\langle \mathrm{B}(y_n(s) + J(s), y_{n+1}(s) + J(s)), \mathrm{A}(y_{n+1}(s) - y_n(s)) \big\rangle_{\mathrm{H}},$$

and

$$\begin{aligned}
&\int_0^t |K(s)| \mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+2\}}(s) \mathbb{1}_{\{\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}> m+2\}}(s)\, ds \\
&\qquad = \int_0^t |K(s)| \mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+1\}}(s) \mathbb{1}_{\{\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}> m+2\}}(s)\, ds.
\end{aligned} \tag{4.22}$$

For any $s \in [0,t]$ such that

$$\mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+1\}}(s) \mathbb{1}_{\{\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}> m+2\}}(s) = 1,$$

we have

$$\|y_n - y_{n-1}\|_{\Upsilon_s(\mathrm{V})} \ge \|y_{n-1} + J\|_{\Upsilon_s(\mathrm{V})} - \|y_n + J\|_{\Upsilon_s(\mathrm{V})} \ge 1, \tag{4.23}$$

and by Lemma 2.1,

$$\begin{aligned}
|K(s)| &\le C |y_n(s) + J(s)|^{1/2}_{\mathrm{H}} \|y_n(s) + J(s)\|^{1/2}_{\mathrm{V}} \\
&\qquad \cdot \|y_{n+1}(s) + J(s)\|^{1/2}_{\mathrm{V}} \|y_{n+1}(s) + J(s)\|^{1/2}_{\mathscr{D}(\mathrm{A})} \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})} \\
&\le C_m \|y_{n+1}(s) + J(s)\|^{1/2}_{\mathrm{V}} \|y_{n+1}(s) + J(s)\|^{1/2}_{\mathscr{D}(\mathrm{A})} \|y_{n+1}(s) - y_n(s)\|_{\mathscr{D}(\mathrm{A})} \\
&\le \varepsilon \|y_{n+1}(s) - y_n(s)\|^2_{\mathscr{D}(\mathrm{A})} + \frac{C_m}{\varepsilon} \|y_{n+1}(s) + J(s)\|_{\mathrm{V}} \|y_{n+1}(s) + J(s)\|_{\mathscr{D}(\mathrm{A})}.
\end{aligned} \tag{4.24}$$

For the second term of the right hand side, we have

$$
\begin{aligned}
&\frac{C_m}{\varepsilon}\|y_{n+1}(s)+J(s)\|_{\mathrm{V}}\|y_{n+1}(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}\\
&\le \frac{C_m}{\varepsilon}(\|y_n(s)+J(s)\|_{\mathrm{V}}+\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}})\\
&\quad\cdot\big(\|y_n(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}+\|y_{n+1}(s)-y_n(s)\|_{\mathscr{D}(\mathrm{A})}\big)\\
&\le \frac{C_m}{\varepsilon}\big(\|y_n(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}\|y_n-y_{n-1}\|^2_{\Upsilon_s(\mathrm{V})}+\|y_{n+1}(s)-y_n(s)\|_{\mathscr{D}(\mathrm{A})}\|y_n-y_{n-1}\|_{\Upsilon_s(\mathrm{V})}\\
&\quad+\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}\|y_n(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}+\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}\|y_{n+1}(s)-y_n(s)\|_{\mathscr{D}(\mathrm{A})}\big)\\
&\le 2\varepsilon\|y_{n+1}(s)-y_n(s)\|^2_{\mathscr{D}(\mathrm{A})}+\frac{C_m}{\varepsilon^3}(\|y_n-y_{n-1}\|^2_{\Upsilon_s(\mathrm{V})}+\|y_{n+1}(s)-y_n(s)\|^2_{\mathrm{V}})\\
&\quad+\frac{C_m}{\varepsilon}(\|y_n(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}\|y_n-y_{n-1}\|^2_{\Upsilon_s(\mathrm{V})}+\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}\|y_n(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}).
\end{aligned}
\tag{4.25}
$$

In the second "≤" in (4.25), we have used (4.23) and $\|y_n(s)+J(s)\|_{\mathrm{V}}\le\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+1$.

Considering (4.22), (4.24), and (4.25) together, we deduce

$$
\begin{aligned}
&\int_0^t |K(s)|\mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+2\}}(s)\mathbb{1}_{\{\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}>m+2\}}(s)\,ds\\
&\le 3\varepsilon\int_0^t \|y_{n+1}(s)-y_n(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds+\frac{C_m}{\varepsilon^3}t\sup_{s\in[0,t]}\|y_{n+1}(s)-y_n(s)\|^2_{\mathrm{V}}\\
&\quad+\frac{C_m}{\varepsilon^3}\sup_{s\in[0,t]}\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})}\,t\\
&\quad+\frac{C_m}{\varepsilon}\sup_{s\in[0,t]}\|y_{n+1}(s)-y_n(s)\|_{\mathrm{V}}\\
&\quad\cdot\int_0^t\|y_n(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}\mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+1\}}(s)\mathbb{1}_{\{\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}>m+2\}}(s)\,ds\\
&\quad+\frac{C_m}{\varepsilon}\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})}\int_0^t\|y_n(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}\mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+1\}}(s)\,ds\\
&\le \|y_{n+1}-y_n\|^2_{\Upsilon_t(\mathrm{V})}\left(3\varepsilon+\frac{C_m}{\varepsilon^3}t+\varepsilon^2\right)+\frac{C_m}{\varepsilon^3}t\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})}\\
&\quad+\frac{C_m}{\varepsilon^4}\bigg(\int_0^t\|y_n(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}\mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+1\}}(s)\mathbb{1}_{\{\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}>m+2\}}(s)\\
&\qquad\qquad\cdot\|y_n-y_{n-1}\|_{\Upsilon_s(\mathrm{V})}\,ds\bigg)^2\\
&\quad+\frac{C_m}{\varepsilon}\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})}\left(\int_0^t\|y_n(s)+J(s)\|^2_{\mathscr{D}(\mathrm{A})}\mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+1\}}(s)\,ds\right)^{1/2}t^{1/2}
\end{aligned}
$$

$$\le \|y_{n+1}-y_n\|^2_{\Upsilon_t(\mathrm{V})}\left(3\varepsilon+\frac{C_m}{\varepsilon^3}t+\varepsilon^2\right)+\frac{C_m}{\varepsilon^3}t\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})}$$
$$+\frac{C_m}{\varepsilon^4}\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})}t\left(\int_0^t \|y_n(s)+J(s)\|^2_{\mathscr{D}(\mathrm{A})}\mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}\le m+1\}}(s)ds\right)$$
$$+\frac{C_m}{\varepsilon}\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})}t^{1/2}$$
$$\le \|y_{n+1}-y_n\|^2_{\Upsilon_t(\mathrm{V})}\left(3\varepsilon+\frac{C_m}{\varepsilon^3}t+\varepsilon^2\right)$$
$$+\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})}\left(\frac{C_m}{\varepsilon^3}t+\frac{C_m}{\varepsilon}t^{1/2}+\frac{C_m}{\varepsilon^4}t\right). \tag{4.26}$$

In the second "$\le$" in (4.26), we have used (4.23).

Case 3. Assume that $\|y_n+J\|_{\Upsilon_s(\mathrm{V})}>m+2$ and $\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}\le m+2$. The definition of θ_m implies that

$$K(s)$$
$$=-\theta_m(\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})})\langle \mathrm{B}(y_{n-1}(s)+J(s),y_n(s)+J(s)),\mathrm{A}(y_{n+1}(s)-y_n(s))\rangle_{\mathrm{H}},$$

and

$$\int_0^t |K(s)|\mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}>m+2\}}(s)\mathbb{1}_{\{\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}\le m+2\}}(s)\,ds$$
$$=\int_0^t |K(s)|\mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}>m+2\}}(s)\mathbb{1}_{\{\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}\le m+1\}}(s)\,ds. \tag{4.27}$$

For any $s\in[0,t]$ such that

$$\mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}>m+2\}}(s)\mathbb{1}_{\{\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}\le m+1\}}(s)=1,$$

we have

$$\|y_n-y_{n-1}\|_{\Upsilon_s(\mathrm{V})}\ge \|y_n+J\|_{\Upsilon_s(\mathrm{V})}-\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}\ge 1, \tag{4.28}$$

and by Lemma 2.1 again,

$$|K(s)|\le C|y_{n-1}(s)+J(s)|^{1/2}_{\mathrm{H}}\|y_{n-1}(s)+J(s)\|^{1/2}_{\mathrm{V}}$$
$$\cdot\|y_n(s)+J(s)\|^{1/2}_{\mathrm{V}}\|y_n(s)+J(s)\|^{1/2}_{\mathscr{D}(\mathrm{A})}\|y_{n+1}(s)-y_n(s)\|_{\mathscr{D}(\mathrm{A})}$$
$$\le C_m\|y_n(s)+J(s)\|^{1/2}_{\mathrm{V}}\|y_n(s)+J(s)\|^{1/2}_{\mathscr{D}(\mathrm{A})}\|y_{n+1}(s)-y_n(s)\|_{\mathscr{D}(\mathrm{A})}$$
$$\le \varepsilon\|y_{n+1}(s)-y_n(s)\|^2_{\mathscr{D}(\mathrm{A})}$$
$$+\frac{C_m}{\varepsilon}\|y_n(s)+J(s)\|_{\mathrm{V}}\|y_n(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}. \tag{4.29}$$

Using similar arguments to (4.25), we have

$$\begin{aligned}
&\frac{C_m}{\varepsilon}\|y_n(s)+J(s)\|_{\mathrm{V}}\|y_n(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}\\
&\quad\le \frac{C_m}{\varepsilon}\big(\|y_{n-1}(s)+J(s)\|_{\mathrm{V}}+\|y_n(s)-y_{n-1}(s)\|_{\mathrm{V}}\big)\\
&\qquad\cdot\big(\|y_{n-1}(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}+\|y_n(s)-y_{n-1}(s)\|_{\mathscr{D}(\mathrm{A})}\big)\\
&\quad\le \frac{C_m}{\varepsilon}\Big(\|y_{n-1}(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}\|y_n-y_{n-1}\|^2_{\Upsilon_s(\mathrm{V})}\\
&\qquad+\|y_n(s)-y_{n-1}(s)\|_{\mathscr{D}(\mathrm{A})}\|y_n-y_{n-1}\|_{\Upsilon_s(\mathrm{V})}\\
&\qquad+\|y_n(s)-y_{n-1}(s)\|_{\mathrm{V}}\|y_{n-1}(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}\\
&\qquad+\|y_n(s)-y_{n-1}(s)\|_{\mathrm{V}}\|y_n(s)-y_{n-1}(s)\|_{\mathscr{D}(\mathrm{A})}\Big)\\
&\quad\le 2\varepsilon\|y_n(s)-y_{n-1}(s)\|^2_{\mathscr{D}(\mathrm{A})}+\frac{C_m}{\varepsilon^3}\big(\|y_n-y_{n-1}\|^2_{\Upsilon_s(\mathrm{V})}+\|y_n(s)-y_{n-1}(s)\|^2_{\mathrm{V}}\big)\\
&\qquad+\frac{C_m}{\varepsilon}\|y_{n-1}(s)+J(s)\|_{\mathscr{D}(\mathrm{A})}\|y_n-y_{n-1}\|^2_{\Upsilon_s(\mathrm{V})}. \qquad (4.30)
\end{aligned}$$

In the second and third "$\le$" of (4.30), we have used (4.28) and

$$\begin{aligned}
\|y_{n-1}(s)+J(s)\|_{\mathrm{V}}&\le\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}\le m+1,\\
\|y_n(s)-y_{n-1}(s)\|_{\mathrm{V}}&\le\|y_n-y_{n-1}\|^2_{\Upsilon_s(\mathrm{V})}.
\end{aligned}$$

Combining (4.27), (4.29), and (4.30), and using the same idea as in (4.26), we deduce

$$\begin{aligned}
&\int_0^t |K(s)|\mathbb{1}_{\{\|y_n+J\|_{\Upsilon_s(\mathrm{V})}>m+2\}}(s)\mathbb{1}_{\{\|y_{n-1}+J\|_{\Upsilon_s(\mathrm{V})}\le m+2\}}(s)\,ds\\
&\qquad\le\varepsilon\|y_{n+1}-y_n\|^2_{\Upsilon_t(\mathrm{V})}+\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})}\left(3\varepsilon+\frac{C_m}{\varepsilon^3}t+\frac{C_m}{\varepsilon}t^{1/2}\right). \qquad (4.31)
\end{aligned}$$

We have now finished the estimates for $K(s)$ in the three cases.

The statements made in (4.21), (4.26), and (4.31), combined with equality (4.17), allow us to state that for all $\varepsilon>0$ and $t\in[0,T]$,

$$\begin{aligned}
&\|y_{n+1}(t)-y_n(t)\|^2_{\mathrm{V}}+2\int_0^t\|y_{n+1}(s)-y_n(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds\\
&\qquad\le C_m\|y_{n+1}-y_n\|^2_{\Upsilon_t(\mathrm{V})}\left(\varepsilon+\varepsilon^{4/3}+\frac{t}{\varepsilon^4}+\varepsilon^2+\frac{t}{\varepsilon^3}\right)\\
&\qquad\quad+C_m\|y_n-y_{n-1}\|^2_{\Upsilon_t(\mathrm{V})}\left(\varepsilon+\frac{t}{\varepsilon^4}+\frac{t^{1/2}}{\varepsilon}+\frac{t}{\varepsilon^3}\right). \qquad (4.32)
\end{aligned}$$

Since, by the definition of $\Upsilon_t(\mathrm{V})$,

$$\frac{1}{2}\|y_{n+1}-y_n\|^2_{\Upsilon_t(\mathrm{V})}\le\sup_{s\in[0,t]}\|y_{n+1}(s)-y_n(s)\|^2_{\mathrm{V}}+\int_0^t\|y_{n+1}(s)-y_n(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds,$$

we can choose ε and $t_0>0$ small enough such that for all $n\ge1$,

$$\|y_{n+1}-y_n\|^2_{\Upsilon_{t_0}(\mathrm{V})}\le\tfrac{1}{3}\|y_n-y_{n-1}\|^2_{\Upsilon_{t_0}(\mathrm{V})}. \qquad (4.33)$$

This implies that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,t_0],\mathrm{V})\cap L^2([0,t_0],\mathscr{D}(\mathrm{A}))$. Therefore, it has a unique limit in that space, which we denote by y^1. It is rather standard (if not obvious) that y^1 is a solution of (4.15) on $[0,t_0]$. Observe that the constant t_0 does not depend on the initial data.

Next, we consider

$$\begin{aligned} &y'(t)+\mathrm{A}y(t)+\theta_m(\|y+J\|_{\Upsilon_t(\mathrm{V})})\mathrm{B}(y(t)+J(t))=f(t), \quad t>t_0,\\ &y(t)=y^1(t), \quad t\in[0,t_0]. \end{aligned} \tag{4.34}$$

Repeating the above arguments, we can solve (4.34) on $[0,2t_0]$, and denote its solution by $y^2:=\{y^2(t),\ t\in[0,2t_0]\}$. It is not difficult to prove that y^2 is a solution of (4.15) on $[0,2t_0]$. Then, by induction, we can solve (4.15) on $[0,3t_0]$, $[0,4t_0]$, and so on. We finally obtain a solution $y\in C([0,T],\mathrm{V})\cap L^2([0,T],\mathscr{D}(\mathrm{A}))$ of (4.15) for any fixed $T>0$. The proof of Lemma 4.1 is complete. ■

Although the uniqueness of solutions to equation (4.15) follows from the existence proof, for completeness sake, we give an independent proof of this property (see also Corollary 4.1). The following lemma is a preliminary step in this direction. It will also be used later.

Let us recall that the space $\Lambda_T(\mathrm{V})$ (and its norm) was defined around equality (4.14).

Lemma 4.2. *Let $n\in\mathbb{N}$ and $T>0$. Assume that for all $u_0\in\mathrm{V}$, $f\in L^2([0,T],\mathrm{H})$ and $y\in\Lambda_T(\mathrm{V})$, there exists an element $u=\Phi^y\in\Lambda_T(\mathrm{V})$ satisfying*

$$\begin{aligned} &du(t)+\mathrm{A}u(t)\,dt+\theta_n(\|u\|_{\Upsilon_t(\mathrm{V})})\mathrm{B}(u(t))\,dt\\ &\quad=f(t)\,dt+\int_{\mathrm{Z}}G(y(t-),z)\,\tilde{\eta}(dz,dt),\\ &u(0)=u_0. \end{aligned} \tag{4.35}$$

Then there exist a positive constant C and a function $L_n:(0,\infty)\to(0,\infty)$ such that $\lim_{T\to0}L_n(T)=1$ and

$$\|\Phi^{y_1}-\Phi^{y_2}\|^2_{\Lambda_T(\mathrm{V})}\le CL_n^2(T)T\|y_1-y_2\|^2_{\Lambda_T(\mathrm{V})}, \quad y_1,y_2\in\Lambda_T(\mathrm{V}). \tag{4.36}$$

Proof. Fix $n\in\mathbb{N}$ and $T>0$. Assume that $u_0\in\mathrm{V}$, $f\in L^2([0,T],\mathrm{H})$ and $y_1,y_2\in\Lambda_T(\mathrm{V})$.

For simplicity, let us set $u_1=\Phi^{y_1}$, $u_2=\Phi^{y_2}$, and $u=u_1-u_2$. By the Itô formula, we have

$$\begin{aligned} &\|u(t)\|^2_{\mathrm{V}}+2\int_0^t\|u(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds\\ &=-2\int_0^t\langle\theta_n(\|u_1\|_{\Upsilon_s(\mathrm{V})})\mathrm{B}(u_1(s))-\theta_n(\|u_2\|_{\Upsilon_s(\mathrm{V})})\mathrm{B}(u_2(s)),\mathrm{A}u(s)\rangle_{\mathrm{H}}\,ds\\ &\quad+2\int_0^t\int_{\mathrm{Z}}{}_{\mathrm{V}'}\langle G(y_1(s-),z)-G(y_2(s-),z),u(s-)\rangle_{\mathrm{V}}\,\tilde{\eta}(dz,ds)\\ &\quad+\int_0^t\int_{\mathrm{Z}}\|G(y_1(s-),z)-G(y_2(s-),z)\|^2_{\mathrm{V}}\,\eta(dz,ds)\\ &=:J_1(t)+J_2(t)+J_3(t). \end{aligned} \tag{4.37}$$

For the first term, J_1, we have

$$|J_1(t)| \leq \frac{1}{2}\int_0^t \|u(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds + 2\int_0^t K(s)\,ds \tag{4.38}$$

where

$$K(s) := \left|\theta_n(\|u_1\|_{\Upsilon_s(\mathrm{V})})\mathrm{B}(u_1(s)) - \theta_n(\|u_2\|_{\Upsilon_s(\mathrm{V})})\mathrm{B}(u_2(s))\right|^2_{\mathrm{H}}, \qquad s \in [0, T].$$

To find suitable bounds on K, we consider four cases. By the property of θ_n (see (3.12)) and the Minkowski inequality, we have the following estimates.

(1) For $\|u_1\|_{\Upsilon_s(\mathrm{V})} \vee \|u_2\|_{\Upsilon_s(\mathrm{V})} \leq n+1$, we have

$$\begin{aligned}
K(s) &\leq C\big[|\mathrm{B}(u_1(s)) - \mathrm{B}(u_2(s))|^2_{\mathrm{H}} + \left|\theta_n(\|u_1\|_{\Upsilon_s(\mathrm{V})}) - \theta_n(\|u_2\|_{\Upsilon_s(\mathrm{V})})\right|^2 |\mathrm{B}(u_2(s))|^2_{\mathrm{H}}\big]\\
&\leq C\big[|u_1(s)|_{\mathrm{H}}\|u_1(s)\|_{\mathrm{V}}\|u(s)\|_{\mathrm{V}}\|u(s)\|_{\mathscr{D}(\mathrm{A})} + \|u_2(s)\|_{\mathrm{V}}\|u_2(s)\|_{\mathscr{D}(\mathrm{A})}|u(s)|_{\mathrm{H}}\|u(s)\|_{\mathrm{V}}\big]\\
&\quad + C|u_2(s)|_{\mathrm{H}}\|u_2(s)\|^2_{\mathrm{V}}\|u_2(s)\|_{\mathscr{D}(\mathrm{A})}\|u\|^2_{\Upsilon_s(\mathrm{V})}\\
&\leq \tfrac{1}{4}\|u(s)\|^2_{\mathscr{D}(\mathrm{A})}\\
&\quad + C\|u\|^2_{\Upsilon_s(\mathrm{V})}\big[|u_1(s)|^2_{\mathrm{H}}\|u_1(s)\|^2_{\mathrm{V}} + \|u_2(s)\|_{\mathrm{V}}\|u_2(s)\|_{\mathscr{D}(\mathrm{A})} + \|u_2(s)\|^3_{\mathrm{V}}\|u_2(s)\|_{\mathscr{D}(\mathrm{A})}\big].
\end{aligned}$$

(2) For $\|u_1\|_{\Upsilon_s(\mathrm{V})} \leq n+1$ and $\|u_2\|_{\Upsilon_s(\mathrm{V})} \geq n+1$, we have

$$\begin{aligned}
K(s) &= \left|\theta_n(\|u_1\|_{\Upsilon_s(\mathrm{V})})\mathrm{B}(u_1(s))\right|^2_{\mathrm{H}}\\
&= \left|\theta_n(\|u_1\|_{\Upsilon_s(\mathrm{V})}) - \theta_n(\|u_2\|_{\Upsilon_s(\mathrm{V})})\right|^2 |\mathrm{B}(u_1(s))|^2_{\mathrm{H}}\\
&\leq C|u_1(s)|_{\mathrm{H}}\|u_1(s)\|^2_{\mathrm{V}}\|u_1(s)\|_{\mathscr{D}(\mathrm{A})}\|u\|^2_{\Upsilon_s(\mathrm{V})}.
\end{aligned}$$

(3) For $\|u_1\|_{\Upsilon_s(\mathrm{V})} \geq n+1$ and $\|u_2\|_{\Upsilon_s(\mathrm{V})} \leq n+1$, similar to case (2), we get

$$K(s) \leq C|u_2(s)|_{\mathrm{H}}\|u_2(s)\|^2_{\mathrm{V}}\|u_2(s)\|_{\mathscr{D}(\mathrm{A})}\|u\|^2_{\Upsilon_s(\mathrm{V})}.$$

(4) For $\|u_1\|_{\Upsilon_s(\mathrm{V})} \wedge \|u_2\|_{\Upsilon_s(\mathrm{V})} \geq n+1$, we have

$$K(s) = 0.$$

Hence,

$$K(s) \leq \tfrac{1}{4}\|u(s)\|^2_{\mathscr{D}(\mathrm{A})} + C\|u\|^2_{\Upsilon_s(\mathrm{V})}\,\Xi(s), \qquad s \in [0, T], \tag{4.39}$$

where

$$\begin{aligned}
\Xi(s) := {}& \big(|u_1(s)|^2_{\mathrm{H}}\|u_1(s)\|^2_{\mathrm{V}} + \|u_1(s)\|^3_{\mathrm{V}}\|u_1(s)\|_{\mathscr{D}(\mathrm{A})}\big)\mathbb{1}_{[0,n+1]}(\|u_1\|_{\Upsilon_s(\mathrm{V})})\\
&+ \big(\|u_2(s)\|_{\mathrm{V}}\|u_2(s)\|_{\mathscr{D}(\mathrm{A})} + \|u_2(s)\|^3_{\mathrm{V}}\|u_2(s)\|_{\mathscr{D}(\mathrm{A})}\big)\mathbb{1}_{[0,n+1]}(\|u_2\|_{\Upsilon_s(\mathrm{V})}),\\
&\qquad s \in [0, T].
\end{aligned}$$

Set

$$\Theta(t) := \sup_{s\in[0,t]} \|u(t)\|^2_{\mathrm{V}} + \int_0^t \|u(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds.$$

Substituting (4.39) into (4.38), and then into (4.37), and noting that $\|u\|^2_{\Upsilon_s(\mathrm{V})} \le 2\Theta(s)$, $s \in [0, T]$, we deduce that

$$\Theta(t) \le C \int_0^t \Theta(s)\Xi(s)\,ds + \sup_{s\in[0,t]} |J_2(s)| + J_3(t), \qquad t \in [0, T]. \tag{4.40}$$

Then Gronwall's lemma implies that

$$\begin{aligned} \Theta(T) &\le \Big(\sup_{t\in[0,T]} |J_2(t)| + J_3(T)\Big) \cdot e^{C\int_0^T \Xi(s)\,ds} \\ &\le \Big(\sup_{t\in[0,T]} |J_2(t)| + J_3(T)\Big) \cdot e^{C(n^4T+n^4T^{1/2}+n^2T^{1/2})}. \end{aligned} \tag{4.41}$$

Set

$$L_n(T) = e^{C(n^4T+n^4T^{1/2}+n^2T^{1/2})}. \tag{4.42}$$

By the Burkholder–Davis–Gundy inequality and Assumption 4.1, we have

$$L_n(T)\mathbb{E}\Big(\sup_{t\in[0,T]} |J_2(t)|\Big) \le \tfrac{1}{2}\|u\|^2_{\Lambda_T(\mathrm{V})} + CL_n(T)^2T\|y_1 - y_2\|^2_{\Lambda_T(\mathrm{V})}, \tag{4.43}$$

and

$$\mathbb{E}(J_3(T)) \le CT\|y_1 - y_2\|^2_{\Lambda_T(\mathrm{V})}. \tag{4.44}$$

Summing up (4.41), (4.43), and (4.44) we have

$$\|u\|^2_{\Lambda_T(\mathrm{V})} \le CL_n^2(T)T\|y_1 - y_2\|^2_{\Lambda_T(\mathrm{V})}.$$

Notice that in view of the definition (4.42), $L_n(T) \to 1$ as $T \to 0$. The proof of Lemma 4.2 is thus complete. ∎

Using arguments similar to those for Corollary 3.1, by Lemma 4.2 we have

Corollary 4.1. *Under the same assumptions as in Lemma 4.1, the solution of problem (4.15) is unique.*

Now we are in a position to prove Theorem 4.1. But before doing so, for the benefit of the reader, we make the following remark. Conditions (2.15) and (2.16) in [5], the auxiliary problems (3.4)[1] and (3.13) and their proofs of the existence and uniqueness of solutions correspond to properties (3.12), the auxiliary problems (4.46) and (4.45) and the proofs of the existence and uniqueness of solutions in the present paper.

Proof of Theorem 4.1. We also use the Banach fixed point theorem to give this proof in three steps.

[1]Note that (3.4) in [5] should read as follows: $du_n(t) + \mathrm{A}u_n(t)dt = -(\mathrm{B}_n^T v)(t)dt + \int_{\mathrm{Z}} G(z, v(t-))\,\tilde{\eta}(dz, dt)$.

Step 1: Local existence. Consider the auxiliary problem

$$\begin{aligned} &du_n(t) + \mathrm{A}u_n(t)\,dt + \theta_n(\|u_n\|_{\Upsilon_t(\mathrm{V})})\mathrm{B}(u_n(t))\,dt \\ &\quad = f(t)\,dt + \int_{\mathrm{Z}} G(u_n(t-),z)\,\widetilde{\eta}(dz,dt), \\ &u_n(0) = u_0. \end{aligned} \tag{4.45}$$

Fix $T > 0$. For any $y \in \Lambda_T(\mathrm{V})$, there exists a unique element $u = \Phi^y$ such that $u \in D([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A})) = \Upsilon_T(\mathrm{V})$, $\mathbb{P}$-a.s., and

$$\begin{aligned} &du(t) + \mathrm{A}u(t)\,dt + \theta_n(\|u\|_{\Upsilon_t(\mathrm{V})})\mathrm{B}(u(t))\,dt \\ &\quad = f(t)\,dt + \int_{\mathrm{Z}} G(y(t-),z)\,\widetilde{\eta}(dz,dt), \\ &u(0) = u_0. \end{aligned} \tag{4.46}$$

This can be seen as follows. It is known that there exists a unique $\mathbb{F}$-progressively measurable process $\mathbf{J} \in \Upsilon_T(\mathrm{V})$ satisfying the stochastic Langevin equation

$$\begin{aligned} &d\mathbf{J}(t) + \mathrm{A}\mathbf{J}(t)\,dt = \int_{\mathrm{Z}} G(y(t-),z)\,\widetilde{\eta}(dz,dt), \\ &\mathbf{J}(0) = 0. \end{aligned} \tag{4.47}$$

Moreover, this process, called an Ornstein–Uhlenbeck process, satisfies

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|\mathbf{J}(t)\|^2_{\mathrm{V}}\Big) + \mathbb{E}\bigg(\int_0^T \|\mathbf{J}(t)\|^2_{\mathscr{D}(\mathrm{A})}\,dt\bigg) \le CT\Big(\mathbb{E}\Big(\sup_{t\in[0,T]}\|y(t)\|^2_{\mathrm{V}}\Big) + 1\Big).$$

For any $\omega \in \Omega$, consider the following deterministic PDE:

$$\begin{aligned} &dx(t) + \mathrm{A}x(t) + \theta_n(\|x+\mathbf{J}\|_{\Upsilon_t(\mathrm{V})})\mathrm{B}(x(t)+\mathbf{J}(t))\,dt = f(t)\,dt, \\ &x(0) = u_0. \end{aligned}$$

By Lemma 4.1 and Corollary 4.1, this PDE has a unique solution $x \in \Upsilon_T(\mathrm{V})$. One can show that the process u defined by $u = x + \mathbf{J}$ is a solution to (4.46). For the uniqueness, we refer to Lemma 4.2.

Now we prove that $u \in \Lambda_T(\mathrm{V})$. Applying the Itô formula, we get

$$\begin{aligned} &\|u(t)\|^2_{\mathrm{V}} + 2\int_0^t \|u(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds \\ &\quad = \|u_0\|^2_{\mathrm{V}} - 2\int_0^t \big\langle \theta_n(\|u\|_{\Upsilon_s(\mathrm{V})})\mathrm{B}(u(s)), \mathrm{A}u(s)\big\rangle_{\mathrm{H}}\,ds \\ &\qquad + 2\int_0^t \langle f(s), \mathrm{A}u(s)\rangle_{\mathrm{H}}\,ds + 2\int_0^t\int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(y(s-),z), u(s)\rangle_{\mathrm{V}}\,\widetilde{\eta}(dz,ds) \\ &\qquad + \int_0^t\int_{\mathrm{Z}} \|G(y(s-),z)\|^2_{\mathrm{V}}\,\eta(dz,ds) \\ &\quad = \sum_{i=1}^5 J_i(t). \end{aligned} \tag{4.48}$$

By Lemma 2.1, we have

$$\begin{aligned}|J_2(t)| &\leq \frac{1}{2}\int_0^t \|u(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds + 2\int_0^t |\theta_n(\|u\|_{\Upsilon_s(\mathrm{V})})\mathrm{B}(u(s))|^2_{\mathrm{H}}\,ds\\ &\leq \frac{1}{2}\int_0^t \|u(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds + C\int_0^t \theta_n^2(\|u\|_{\Upsilon_s(\mathrm{V})})|u(s)|_{\mathrm{H}}\|u(s)\|^2_{\mathrm{V}}\|u(s)\|_{\mathscr{D}(\mathrm{A})}\,ds\\ &\leq \frac{1}{2}\int_0^t \|u(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds + Cn^4t^{1/2}. \end{aligned}\tag{4.49}$$

It is easy to see that

$$J_3(t) \leq \frac{1}{2}\int_0^t \|u(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds + 2\int_0^t |f(s)|^2_{\mathrm{H}}\,ds, \quad t\in[0,T], \tag{4.50}$$

and

$$\mathbb{E}(J_5(T)) \leq CT\Big(1+\mathbb{E}\Big(\sup_{s\in[0,T]}\|y(s)\|^2_{\mathrm{V}}\Big)\Big). \tag{4.51}$$

By the Burkholder–Davis–Gundy inequality, we get

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|J_4(t)|\Big) \leq \frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T]}\|u(t)\|^2_{\mathrm{V}}\Big) + CT\Big(1+\mathbb{E}\Big(\sup_{s\in[0,T]}\|y(s)\|^2_{\mathrm{V}}\Big)\Big). \tag{4.52}$$

Combining (4.48)–(4.52), we have

$$\begin{aligned}&\mathbb{E}\Big(\sup_{t\in[0,T]}\|u(t)\|^2_{\mathrm{V}}\Big) + \mathbb{E}\Big(\int_0^T \|u(s)\|^2_{\mathscr{D}(\mathrm{A})}\,ds\Big)\\ &\qquad\leq C_{n,T}\Big(\|u_0\|^2_{\mathrm{V}} + 1 + \mathbb{E}\Big(\sup_{s\in[0,T]}\|y(s)\|^2_{\mathrm{V}}\Big)\Big).\end{aligned}$$

We have shown that $u\in\Lambda_T(\mathrm{V})$, and this implies that $\Phi^{\cdot}:\Lambda_T(\mathrm{V})\to\Lambda_T(\mathrm{V})$ is well-defined.

By Lemma 4.2, there exist a positive constant C and $L_n(T)\to 1$ as $T\to 0$ such that

$$\|\Phi^{y_1}-\Phi^{y_2}\|^2_{\Lambda_T(\mathrm{V})} \leq CL_n^2(T)T\|y_1-y_2\|^2_{\Lambda_T(\mathrm{V})}, \quad y_1,y_2\in\Lambda_T(\mathrm{V}). \tag{4.53}$$

Using arguments similar to the proof of Theorem 3.1, we can construct a unique $u_n\in\Lambda_T(\mathrm{V})$ for any $T>0$ that is a solution of (4.45). However, we do not know whether the solution is unique.

Define a stopping time τ_n by

$$\tau_n = \inf\{t\geq 0 : \|u_n\|_{\Upsilon_t(\mathrm{V})} > n\}. \tag{4.54}$$

Then $\theta_n(\|u_n\|_{\Upsilon_t(\mathrm{V})})=1$ for any $t\in[0,\tau_n)$, hence $\{u_n(t),\ t\in[0,\tau_n)\}$ is a local solution of problem (4.4).

Step 2: Local uniqueness. We need a proof of uniqueness not relying on the uniqueness from Theorem 3.1; see Remark 4.1.

Assume that $\{U_1(t),\ t \in [0,\sigma_1)\}$ and $\{U_2(t),\ t \in [0,\sigma_2)\}$ are two local solutions of (4.4). Fix $R > 0$. Define

$$\sigma_R^i = \inf\{t > 0 : \|U_i\|_{\Upsilon_t(\mathrm{V})} > R\} \wedge \sigma_i, \quad i = 1,2.$$
$$\sigma = \sigma_1 \wedge \sigma_2, \quad \sigma_R = \sigma_R^1 \wedge \sigma_R^2.$$

It is known that $\sigma_i, \sigma_R^i, i = 1,2, \sigma$, and σ_R are stopping times.

Now we prove that

$$U_1 = U_2 \quad \text{on } [0,\sigma). \tag{4.55}$$

Let $M(t) = U_1(t) - U_2(t)$. By the Itô formula,

$$\begin{aligned}
&\|M(t)\|_{\mathrm{V}}^2 + 2\int_0^t \|M(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds \\
&\quad = -2\int_0^t \langle \mathrm{B}(U_1(s)) - \mathrm{B}(U_2(s)), \mathrm{A}M(s)\rangle_{\mathrm{H}}\,ds \\
&\qquad + 2\int_0^t\int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(U_1(s-),z) - G(U_2(s-),z), M(s-)\rangle_{\mathrm{V}}\,\tilde{\eta}(dz,ds) \\
&\qquad + \int_0^t\int_{\mathrm{Z}} \|G(U_1(s-),z) - G(U_2(s-),z)\|_{\mathrm{V}}^2\,\eta(dz,ds) \\
&\quad = I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{4.56}$$

By Lemma 2.1,

$$\begin{aligned}
|I_1(t)| &\le \frac{1}{2}\int_0^t \|M(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds + 2\int_0^t |\mathrm{B}(U_1(s)) - \mathrm{B}(U_2(s))|_{\mathrm{H}}^2\,ds \\
&\le \frac{1}{2}\int_0^t \|M(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds + C\int_0^t |U_1(s)|_{\mathrm{H}}\|U_1(s)\|_{\mathrm{V}}\|M(s)\|_{\mathrm{V}}\|M(s)\|_{\mathscr{D}(\mathrm{A})}\,ds \\
&\quad + C\int_0^t \|U_2(s)\|_{\mathrm{V}}\|U_2(s)\|_{\mathscr{D}(\mathrm{A})}\|M(s)\|_{\mathrm{V}}|M(s)|_{\mathrm{H}}\,ds \\
&\le \int_0^t \|M(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds \\
&\quad + C\int_0^t \|M(s)\|_{\mathrm{V}}^2\big[|U_1(s)|_{\mathrm{H}}^2\|U_1(s)\|_{\mathrm{V}}^2 + \|U_2(s)\|_{\mathscr{D}(\mathrm{A})}\|U_2(s)\|_{\mathrm{V}}\big]\,ds.
\end{aligned} \tag{4.57}$$

In view of inequality (4.57), by Gronwall's lemma applied to (4.56), we infer that for all $t \in [0,T]$,

$$\begin{aligned}
&\|M(t\wedge\sigma_R)\|_{\mathrm{V}}^2 + \int_0^{t\wedge\sigma_R} \|M(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds \\
&\le e^{C\int_0^{t\wedge\sigma_R} |U_1(s)|_{\mathrm{H}}^2\|U_1(s)\|_{\mathrm{V}}^2 + \|U_2(s)\|_{\mathscr{D}(\mathrm{A})}\|U_2(s)\|_{\mathrm{V}}\,ds}\Big[\sup_{s\in[0,T]} |I_2(s\wedge\sigma_R)| + I_3(T\wedge\sigma_R)\Big] \\
&\le e^{CTR^4 + CR^2T^{1/2}}\Big[\sup_{s\in[0,T]} |I_2(s\wedge\sigma_R)| + I_3(T\wedge\sigma_R)\Big].
\end{aligned} \tag{4.58}$$

Next, by the Burkholder–Davis–Gundy inequality and Assumption 4.1 we infer that for any $\delta > 0$,

$$\mathbb{E}\Big(\sup_{s\in[0,T]} |I_2(s\wedge\sigma_R)|\Big) \le \delta\mathbb{E}\Big(\sup_{s\in[0,T]} \|M(s\wedge\sigma_R)\|_{\mathrm{V}}^2\Big) + C_\delta\mathbb{E}\Big(\int_0^{T\wedge\sigma_R} \|M(s)\|_{\mathrm{V}}^2\,ds\Big) \tag{4.59}$$

and

$$\mathbb{E}(I_3(T\wedge\sigma_R)) \le C\mathbb{E}\Big(\int_0^{T\wedge\sigma_R} \|M(s)\|_{\mathrm{V}}^2\,ds\Big). \tag{4.60}$$

Combining inequalities (4.58)–(4.60), and applying Gronwall's lemma, we deduce that

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|M(t\wedge\sigma_R)\|_{\mathrm{V}}^2\Big) + \mathbb{E}\Big(\int_0^{T\wedge\sigma_R} \|M(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds\Big) = 0.$$

Since $\lim_{R\nearrow\infty}\sigma_R = \sigma$, by taking first the limit as $R\nearrow\infty$ and then the limit as $T\nearrow\infty$, we infer that

$$\mathbb{E}\Big(\sup_{t\in[0,\sigma)} \|M(t)\|_{\mathrm{V}}^2\Big) + \mathbb{E}\Big(\int_0^{\sigma} \|M(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds\Big) = 0,$$

which implies the uniqueness of the local solution.

Step 3: Global existence. Let us recall that τ_n has been defined in (4.54). By Step 2 the sequence $\{\tau_n\}_{n=1}^{\infty}$ is nondecreasing and

$$u_{n+1}(t) = u_n(t), \quad t\in[0,\tau_n), \quad \mathbb{P}\text{-a.s.}$$

Put $\tau_{\max} := \lim_{n\to\infty}\tau_n$. By [32, Proposition 2.1.2], the random time $\tau_{\max}$ is a stopping time. As in the proof of Theorem 3.1, we can define a process $\{u(t),\, t\in[0,\tau_{\max})\}$ by

$$u(t) = u_n(t), \quad t\in[0,\tau_n),$$

This process is a local solution of (4.4), and it satisfies (see (3.59))

$$\lim_{t\nearrow\tau_{\max}} \|u\|_{\Upsilon_t(\mathrm{V})} = \infty \quad \text{on } \{\omega : \tau_{\max} < \infty\},\ \mathbb{P}\text{-a.s.}$$

Using an argument similar to the one used in the proof of Theorem 4.2, we can prove that

$$\mathbb{P}(\tau_{\max} = \infty) = 1.$$

This concludes the proof of the global existence, and hence Theorem 4.1 is established.

■

5. Large deviation principle (LDP)

Fix $T > 0$. In this section, we establish a Freidlin–Wentzell LDP for problem (2.2) on $\Upsilon_T(\mathrm{V})$ defined in (4.12), i.e.,

$$\Upsilon_T(\mathrm{V}) := D([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A})).$$

In the following, the space $D([0,T],\mathrm{V})$ is equipped with the Skorokhod topology.

5.1. Description of the problem and the statement of the main result

We first introduce the problem and then state the precise assumptions on the coefficients, followed by the main result.

Let us recall that Z is a locally compact Polish space. We set

$$\mathrm{Z}_T = [0,T]\times \mathrm{Z}, \quad \mathrm{Y} = \mathrm{Z}\times[0,\infty), \quad \mathrm{Y}_T = [0,T]\times\mathrm{Z}\times[0,\infty).$$

We let $M_T = \mathscr{M}(\mathrm{Z}_T)$ be the space of all nonnegative measures ϑ on $(\mathrm{Z}_T, \mathscr{B}(\mathrm{Z}_T))$ such that $\vartheta(K) < \infty$ for every compact subset K of Z_T.

We endow the set M_T with the weakest topology, denoted by $\mathscr{T}(M_T)$, such that for every $g \in C_c(\mathrm{Z}_T)$ (where $C_c(\mathrm{Z}_T)$ is the space of real continuous functions on Z_T with compact support), the map

$$M_T \ni \vartheta \mapsto \int_{\mathrm{Z}_T} g(z,s)\,\vartheta(dz,ds) \in \mathbb{R}$$

is continuous. Analogously, we define $\mathbb{M}_T = \mathscr{M}(\mathrm{Y}_T)$ and $\mathscr{T}(\mathbb{M}_T)$. It is known (see [17, Section 1]) that both $(M_T, \mathscr{T}(M_T))$ and $(\mathbb{M}_T, \mathscr{T}(\mathbb{M}_T))$ are Polish spaces.

In the present paper, we denote

$$\bar{\Omega} = \mathbb{M}_T, \quad \mathscr{G} := \mathscr{T}(\mathbb{M}_T).$$

Fix a σ-finite measure ν on $(\mathrm{Z}, \mathscr{B}(\mathrm{Z}))$ such that $\nu(K) < \infty$ for every compact subset K of Z. By [42, Section I.8], there exists a unique probability measure $\mathbb{Q}$ on $(\bar{\Omega}, \mathscr{G})$ on which the canonical/identity map

$$\mathrm{N} : \bar{\Omega} \ni \mathrm{m} \mapsto \mathrm{m} \in \mathbb{M}_T$$

is a Poisson random measure (PRM) on Y_T with intensity measure $\mathrm{Leb}(dt)\otimes\nu(dz)\otimes \mathrm{Leb}(dr)$, over the probability space $(\bar{\Omega}, \mathscr{G}, \mathbb{Q})$.

We also introduce the following notation:

$$\begin{aligned}
\mathscr{G}_t &= \text{ the } \mathbb{Q}\text{-completion of } \sigma\{\mathrm{N}((0,s]\times O) : s\in[0,t],\ O\in\mathscr{B}(\mathrm{Y})\}, \quad t\in[0,T],\\
\mathbb{G} &= (\mathscr{G}_t)_{t\in[0,T]},\\
\mathscr{P} &= \text{ the } \mathbb{G}\text{-predictable } \sigma\text{-field on } [0,T]\times\bar{\Omega},\\
\bar{\mathbb{A}} &= \text{the class of all } (\mathscr{P}\otimes\mathscr{B}(\mathrm{Z}))\text{-measurable functions } \varphi : \mathrm{Z}_T\times\bar{\Omega}\to[0,\infty).
\end{aligned}$$

It can be shown that N is a time-homogeneous PRM on Y, with intensity measure $\mathrm{Leb}(dt)\otimes\ \nu(dz)\otimes\mathrm{Leb}(dr)$, over the (filtered) probability space $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{Q})$; see Appendix A. The corresponding compensated PRM is denoted by $\widetilde{\mathrm{N}}$.

For every function $\varphi \in \bar{\mathbb{A}}$, let us define a counting process N^φ on Z by

$$\begin{aligned}
\mathrm{N}^\varphi((0,t]\times O) &:= \int_{(0,t]\times O\times(0,\infty)} \mathbb{1}_{[0,\varphi(s,z)]}(r)\,\mathrm{N}(ds,dz,dr),\\
&= \int_{(0,t]\times O\times(0,\infty)} \mathbb{1}_{\{(s,z,r):r\le\varphi(s,z)\}}(s,z,r)\,\mathrm{N}(ds,dz,dr)\\
&= \int_{(0,t]\times O\times(0,\infty)} \mathbb{1}_{[r,\infty)}(\varphi(s,z))\,\mathrm{N}(ds,dz,dr), \quad t\in[0,T],\ O\in\mathscr{B}(\mathrm{Z}). \qquad (5.1)
\end{aligned}$$

We observe that $\mathrm{N}^\varphi : \bar{\Omega}\to\mathscr{M}(\mathrm{Z}_T) = M_T$.

Analogously, we define a process $\tilde{\mathrm{N}}^{\varphi}$:

$$\begin{aligned}\tilde{\mathrm{N}}^{\varphi}((0,t]\times O) &:= \int_{(0,t]\times O\times(0,\infty)} \mathbb{1}_{[0,\varphi(s,z)]}(r)\,\tilde{\mathrm{N}}(ds,dz,dr),\\ &= \int_{(0,t]\times O\times(0,\infty)} \mathbb{1}_{\{(s,z,r):\, r\le\varphi(s,z)\}}(s,z,r)\,\tilde{\mathrm{N}}(ds,dz,dr)\\ &= \int_{(0,t]\times O\times(0,\infty)} \mathbb{1}_{[r,\infty)}(\varphi(s,z))\,\tilde{\mathrm{N}}(ds,dz,dr), \quad t\in[0,T],\ O\in\mathscr{B}(\mathrm{Z}). \end{aligned}\tag{5.2}$$

For any Borel function $g:\mathrm{Z}_T\to[0,\infty)$,

$$\int_{(0,t]\times\mathrm{Z}} g(s,z)\,\tilde{\mathrm{N}}^{\varphi}(ds,dz) = \int_{(0,t]\times\mathrm{Z}\times(0,\infty)} \mathbb{1}_{[0,\varphi(s,z)]}(r)g(s,z)\,\tilde{\mathrm{N}}(ds,dz,dr). \tag{5.3}$$

Note that if φ is a constant function a with value $a\in[0,\infty)$, then

$$\begin{aligned}\mathrm{N}^{a}((0,t]\times O) &= \mathrm{N}\big((0,t]\times O\times(0,a]\big), \quad t\in[0,T],\ O\in\mathscr{B}(\mathrm{Z}),\\ \tilde{\mathrm{N}}^{a}((0,t]\times O) &= \tilde{\mathrm{N}}\big((0,t]\times O\times(0,a]\big), \quad t\in[0,T],\ O\in\mathscr{B}(\mathrm{Z}).\end{aligned}$$

We finish this introduction with the following two simple observations.

Proposition 5.1. *In the above framework, for every $a>0$, the map*

$$\mathrm{N}^{a}:\bar{\Omega}\to\mathscr{M}(\mathrm{Z}_T)=M_T \tag{5.4}$$

is a Poisson random measure on Z_T with intensity measure $\mathrm{Leb}(dt)\otimes a\,\nu(dz)$ *and $\tilde{\mathrm{N}}^{a}$ is equal to the corresponding compensated Poisson random measure.*

Proposition 5.2. *In the above framework, suppose that $\varphi,\psi\in\bar{\mathbb{A}}$, $t_0\in[0,T]$, and a Borel set $O\subset\mathrm{Z}$ are such that*

$$\varphi(s,z,\omega)=\psi(s,z,\omega) \quad \text{for } (s,z,\omega)\in[0,t_0]\times O\times\bar{\Omega}.$$

Then

$$\mathrm{N}^{\varphi}((0,t]\times C)=\mathrm{N}^{\psi}((0,t]\times C) \quad \text{for } t\in[0,t_0],\ C\in O\cap\mathscr{B}(\mathrm{Z}). \tag{5.5}$$

Let us fix $\varepsilon>0$, $u_0\in\mathrm{V}$ and $f\in L^2([0,T],\mathrm{H})$. Consider the following SPDE on the given probability space $(\bar{\Omega},\mathscr{G},\mathbb{G},\mathbb{Q})$:

$$\begin{aligned} &du^{\varepsilon}(t)+\mathrm{A}u^{\varepsilon}(t)\,dt+\mathrm{B}(u^{\varepsilon}(t))\,dt\\ &\quad = f(t)\,dt+\varepsilon\int_{\mathrm{Z}} G(u^{\varepsilon}(t-),z)\,\tilde{\mathrm{N}}^{1/\varepsilon}(dz,dt), \quad t\in[0,T],\\ &u^{\varepsilon}(0)=u_0. \end{aligned}\tag{5.6}$$

By Theorem 4.2 there exists a unique solution u^{ε} to problem (5.6) whose trajectories a.s. belong to the space $\Upsilon_T(\mathrm{V})$ (see (4.12)). In particular, u^{ε} induces an $\Upsilon_T(\mathrm{V})$-valued random variable. In this section, we aim to establish an LDP for the laws of the family $\{u^{\varepsilon}\}_{\varepsilon>0}$ on $\Upsilon_T(\mathrm{V})$.

For our main result, we use the following notation. Denote, for $N > 0$,

$$S^N = \{g : \mathrm{Z}_T \to [0,\infty) : g \text{ is Borel measurable and } L_T(g) \le N\}, \quad (5.7)$$
$$\mathbb{S} = \bigcup_{N\ge 1} S^N,$$

where for a Borel measurable function $g : \mathrm{Z}_T \to [0,\infty)$, we set

$$L_T(g) := \int_0^T \int_{\mathrm{Z}} \big(g(t,z)\log g(t,z) - g(t,z) + 1\big)\,\nu(dz)\,dt. \quad (5.8)$$

A function $g \in S^N$ can be identified with a measure $\nu^g \in M_T$, defined by

$$\nu^g(O) = \int_O g(t,z)\,\nu(dz)\,dt, \quad O \in \mathscr{B}(\mathrm{Z}_T).$$

This identification induces a topology on S^N, under which S^N is a compact space (see [15, Appendix]). Throughout this section, we use this topology on S^N.

Let us finally define

$$\mathscr{H} := \Big\{h : \mathrm{Z} \to \mathbb{R} : h \text{ is Borel measurable and there exists } \delta > 0 \text{ such that}$$
$$\int_\Gamma e^{\delta h^2(z)}\,\nu(dz) < \infty \text{ for all } \Gamma \in \mathscr{B}(\mathrm{Z}) \text{ with } \nu(\Gamma) < \infty\Big\}, \quad (5.9)$$

and

$$L^2(\nu) := \Big\{h : \mathrm{Z} \to [0,\infty) : h \text{ is Borel measurable and } \int_{\mathrm{Z}} h^2(z)\,\nu(dz) < \infty\Big\}.$$

In many parts of this section, we use the following assumption.

Assumption 5.1. There exist functions $L_\hbar, L_i \in \mathscr{H} \cap L^2(\nu)$, for $\hbar > 0$, $i = 2,3$, such that

(LDP-01) (Lipschitz on balls) for every $\hbar > 0$ and $v_1, v_2 \in \mathrm{V}$ with $\|v_1\|_{\mathrm{V}} \vee \|v_2\|_{\mathrm{V}} \le \hbar$,

$$\|G(v_1,z) - G(v_2,z)\|_{\mathrm{V}} \le L_\hbar(z)\|v_1 - v_2\|_{\mathrm{V}}, \quad z \in \mathrm{Z},$$

(LDP-02) (Linear growth in V)

$$\|G(v,z)\|_{\mathrm{V}} \le L_2(z)(1 + \|v\|_{\mathrm{V}}), \quad v \in \mathrm{V},\ z \in \mathrm{Z},$$

(LDP-03) (Linear growth in H)

$$|G(v,z)|_{\mathrm{H}} \le L_3(z)(1 + |v|_{\mathrm{H}}), \quad v \in \mathrm{V},\ z \in \mathrm{Z}.$$

Remark 5.1. A word of warning is due here. Quite often, the Lipschitz property is formulated differently. See, for instance, inequality (3.1) in Assumption 3.1.

Remark 5.2. Because the functions $L_\hbar$, L_2, and L_3 belong to $L^2(\nu)$, Assumption 5.1 implies Assumption 4.2.

We now state the main result of this section. We use the convention that $\inf(\emptyset) = \infty$.

Theorem 5.1. *Assume that Assumption 5.1 holds, $f \in L^2([0,T],\mathrm{H})$, and $u_0 \in \mathrm{V}$. Then the family $\{u^\varepsilon\}_{\varepsilon>0}$ satisfies an LDP on $\Upsilon_T(\mathrm{V})$ with the good rate function I defined by*

$$I(k) := \inf\{L_T(g) : g \in \mathbb{S},\, u^g = k\}, \quad k \in \Upsilon_T(\mathrm{V}), \tag{5.10}$$

where for $g \in \mathbb{S}$, u^g is the unique solution of the deterministic PDE

$$\begin{aligned} &\frac{du^g(t)}{dt} + \mathrm{A}u^g(t) + \mathrm{B}(u^g(t)) = f(t) + \int_{\mathrm{Z}} G(u^g(t),z)(g(t,z)-1)\,\nu(dz), \\ &u^g(0) = u_0. \end{aligned} \tag{5.11}$$

Remark 5.3. By Theorem 3.2, and using arguments similar to the proof of Theorem 5.1, it is not difficult to improve the results on Freidlin–Wentzell-type LDP for strong solutions in the probabilistic sense of 2D SNSEs driven by Lévy processes of jump type. We only state the result here, and omit the proof.

Assumption 5.2. There exist functions $\Upsilon_\hbar \in \mathscr{H} \cap L^2(\nu)$ for $\hbar > 0$ and $\Upsilon \in \mathscr{H} \cap L^2(\nu)$ such that

(LDP-01-P) (Lipschitz on balls in H) for all $\hbar > 0$ and $v_1, v_2 \in \mathrm{H}$ with $\|v_1\|_{\mathrm{H}} \vee \|v_2\|_{\mathrm{H}} \le \hbar$,

$$\|G(v_1,z) - G(v_2,z)\|_{\mathrm{H}} \le \Upsilon_\hbar(z)\|v_1 - v_2\|_{\mathrm{H}}, \quad z \in \mathrm{Z},$$

(LDP-02-P) (Linear growth in H)

$$\|G(v,z)\|_{\mathrm{H}} \le \Upsilon(z)(1 + \|v\|_{\mathrm{H}}), \quad v \in \mathrm{H},\ z \in \mathrm{Z}.$$

Theorem 5.2. *Assume that Assumption 5.2 holds, $f \in L^2([0,T],\mathrm{V}')$, and $u_0 \in \mathrm{H}$. Then the family $\{u^\varepsilon\}_{\varepsilon>0}$ satisfies an LDP on $D([0,T],\mathrm{H}) \cap L^2([0,T],\mathrm{V})$ with the good rate function J defined by*

$$J(k) := \inf\{L_T(g) : g \in \mathbb{S},\, u^g = k\}, \quad k \in D([0,T],\mathrm{H}) \cap L^2([0,T],\mathrm{V}),$$

where for $g \in \mathbb{S}$, u^g is the unique solution of the deterministic PDE

$$\begin{aligned} &\frac{du^g(t)}{dt} + \mathrm{A}u^g(t) + \mathrm{B}(u^g(t)) = f(t) + \int_{\mathrm{Z}} G(u^g(t),z)(g(t,z)-1)\,\nu(dz), \\ &u^g(0) = u_0. \end{aligned}$$

Let us point out that the results (see [29, 62, 66]) on this topic assume that the global Lipschitz condition in H with condition (LDP-02-P) holds, i.e.,

- (Global Lipschitz in H) There exist functions $\widetilde{\Upsilon} \in \mathscr{H} \cap L^2(\nu)$ such that, for all $v_1, v_2 \in \mathrm{H}$,

$$\|G(v_1,z) - G(v_2,z)\|_{\mathrm{H}} \le \widetilde{\Upsilon}(z)\|v_1 - v_2\|_{\mathrm{H}}, \quad z \in \mathrm{Z}.$$

Before we can embark on the proof of Theorem 5.1, we need to establish the well-posedness of equation (5.11). It is a consequence of the following result, whose proof is postponed to Appendix B.

Lemma 5.1. *Assume that* $N \in \mathbb{N}$. *Then, for all* $u_0 \in \mathrm{V}$, $f \in L^2([0,T],\mathrm{H})$, *and* $g \in S^N$, *there exists a unique solution* $u^g \in C([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A}))$ *of problem* (5.11). *Moreover, for any* $\rho, R > 0$, *there exists a positive constant* $C_N = C_{N,\rho,R}$ *such that for every* $g \in S^N$ *and all* $u_0 \in \mathrm{V}$ *and* $f \in L^2([0,T],\mathrm{H})$ *such that* $\|u_0\|_{\mathrm{V}} \le \rho$ *and* $\|f\|_{L^2([0,T],\mathrm{H})} \le R$, *the following estimate is satisfied:*

$$\sup_{t\in[0,T]} \|u^g(t)\|_{\mathrm{V}}^2 + \int_0^T \|u^g(t)\|_{\mathscr{D}(\mathrm{A})}^2 \, dt \le C_N. \tag{5.12}$$

Proof of Theorem 5.1. By applying Theorem 4.2 to problem (5.6), in view of [65, Theorem 8], we infer that there exists a family $\{\mathscr{G}^\varepsilon\}_{\varepsilon>0}$, where $\mathscr{G}^\varepsilon : M_T \to \Upsilon_T(\mathrm{V})$ is a measurable map such that for every $\varepsilon > 0$, the following condition holds:

- If η is a time-homogeneous Poisson random measure on Z with intensity $\varepsilon^{-1}\,\nu(dz)$, i.e., a Poisson random measure on Z_T with intensity $\mathrm{Leb}(dt) \otimes \varepsilon^{-1}\,\nu(dz)$, on a stochastic basis $(\Omega^1, \mathscr{F}^1, \mathbb{P}^1, \mathbb{F}^1)$ with $\mathbb{F}^1 = \{\mathscr{F}^1_t, t \in [0,T]\}$ satisfying the usual conditions, then the process Y^ε defined by $Y^\varepsilon := \mathscr{G}^\varepsilon(\varepsilon\eta)$ is the unique solution of

$$\begin{aligned} &dY^\varepsilon(t) + \mathrm{A}Y^\varepsilon(t)\,dt + \mathrm{B}(Y^\varepsilon(t))\,dt \\ &\qquad = f(t)\,dt + \varepsilon \int_{\mathrm{Z}} G(Y^\varepsilon(t-),z)\big(\eta(dz,dt) - \varepsilon^{-1}\,\nu(dz)\,dt\big), \\ &Y^\varepsilon(0) = u_0. \end{aligned} \tag{5.13}$$

The statements in the condition mean that Y^ε induces (in a natural way) an $\mathbb{F}^1$-progressively measurable process (for which we do not introduce a separate notation) which satisfies

(a1) the trajectories of Y^ε belong to $\Upsilon_T(\mathrm{V})$, $\mathbb{P}^1$-a.s.,

(a2) the following equality holds in H: for all $t \in [0,T]$, $\mathbb{P}^1$-a.s.,

$$\begin{aligned} Y^\varepsilon(t) &= u_0 - \int_0^t \mathrm{A}Y^\varepsilon(s)\,ds - \int_0^t \mathrm{B}(Y^\varepsilon(s))\,ds + \int_0^t f(s)\,ds \\ &\quad + \varepsilon \int_0^t \int_{\mathrm{Z}} G(Y^\varepsilon(s-),z)\big(\eta(dz,ds) - \varepsilon^{-1}\nu(dz)\,ds\big). \end{aligned} \tag{5.14}$$

Therefore, since by Proposition 5.1, $N^{\varepsilon^{-1}}$ is a Poisson random measure on Z_T with intensity measure $\mathrm{Leb}(dt) \otimes \varepsilon^{-1}\,\nu(dz)$, we deduce the following result which will be used later on.

Corollary 5.1. *In the above framework, the unique solution of problem* (5.6) *on the probability space* $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{Q})$ *is given by*

$$u^\varepsilon := \mathscr{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}}). \tag{5.15}$$

Moreover, Lemma 5.1 implies that, for every $g \in \mathbb{S}$, there is a unique solution $u^g \in \Upsilon_T(\mathrm{V})$ of equation (5.11). This allows us to define a map

$$\mathscr{G}^0 : \mathbb{S} \ni g \mapsto u^g \in \Upsilon_T(\mathrm{V}). \tag{5.16}$$

We apply [15, Theorem 2.4] to finish the proof of Theorem 5.1. According to [15], it is sufficient to verify two claims. The first one is the following.

Claim LDP-1. *For all $N \in \mathbb{N}$, if $g_n, g \in S^N$ are such that $g_n \to g$ as $n \to \infty$, then*

$$\mathscr{G}^0(g_n) \to \mathscr{G}^0(g), \quad i.e., \quad u^{g_n} \to u^g \quad in\ \Upsilon_T(\mathrm{V}).$$

To state the second claim, we introduce additional notations.

Let us fix an increasing sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets of Z such that

$$\bigcup_{n=1}^{\infty} K_n = \mathrm{Z}. \tag{5.17}$$

Define

$$\bar{\mathbb{A}}_b = \bigcup_{n=1}^{\infty} \{\varphi \in \bar{\mathbb{A}} : \varphi(t,z,\omega) \in [1/n, n] \text{ if } (t,z,\omega) \in [0,T] \times K_n \times \bar{\Omega} \\ \text{and } \varphi(t,z,\omega) = 1 \text{ if } (t,z,\omega) \in [0,T] \times K_n^c \times \bar{\Omega}\}, \tag{5.18}$$

where the class $\bar{\mathbb{A}}$ was introduced at the beginning of this subsection. We also set

$$\mathscr{U}^N := \{\varphi \in \bar{\mathbb{A}}_b : \varphi(\cdot,\cdot,\omega) \in S^N \text{ for } \mathbb{Q}\text{-a.a. } \omega \in \bar{\Omega}\}, \quad \mathscr{U} := \bigcup_{N=1}^{\infty} \mathscr{U}^N. \tag{5.19}$$

Claim LDP-2. *For all $N \in \mathbb{N}$, if $\varepsilon_n \to 0$ and $\varphi_{\varepsilon_n}, \varphi \in \mathscr{U}^N$ are such that φ_{ε_n} converges in law to φ, then*

$$\mathscr{G}^{\varepsilon_n}(\varepsilon_n N^{\varepsilon_n^{-1}\varphi_{\varepsilon_n}}) \text{ converges in law to } \mathscr{G}^0(\varphi) \text{ in } \Upsilon_T(\mathrm{V}).$$

The verification of Claim LDP-1 will be given in Proposition 5.3 in the following subsection. Claim LDP-2 will be established in Proposition 7.1. Assuming these claims have been proven, the proof of Theorem 5.1 is complete. ∎

5.2. *The first continuity lemma*

To verify Claim LDP-1, it is sufficient to prove the following result.

Proposition 5.3 (The first continuity lemma). *For all $N \in \mathbb{N}$, let $g_n, g \in S^N$ be such that $g_n \to g$ in S^N as $n \to \infty$. Then*

$$\mathscr{G}^0(g_n) \to \mathscr{G}^0(g) \quad in\ \Upsilon_T(\mathrm{V}).$$

Proof. Recall the definition of $\mathscr{G}^0$ in (5.16). Let u^{g_n} be the solution of (5.11) with g replaced by g_n. For simplicity, set $u_n = u^{g_n} = \mathscr{G}^0(g_n)$ and $u = u^g = \mathscr{G}^0(g)$. To prove our result, we will show that

$$u_n \to u \quad \text{in } \Upsilon_T(\mathrm{V}).$$

Fix $\alpha \in (0, 1/2)$. Let $W^{\alpha,2}([0,T], \mathrm{V}')$ be the Sobolev space consisting of all $h \in L^2([0,T], \mathrm{V}')$ satisfying

$$\int_0^T \int_0^T \frac{\|h(t) - h(s)\|^2_{\mathrm{V}'}}{|t-s|^{1+2\alpha}}\, dt\, ds < \infty,$$

endowed with the norm

$$\|h\|^2_{W^{\alpha,2}([0,T],\mathrm{V}')} = \int_0^T \|h(t)\|^2_{\mathrm{V}'}\, dt + \int_0^T \int_0^T \frac{\|h(t) - h(s)\|^2_{\mathrm{V}'}}{|t-s|^{1+2\alpha}}\, dt\, ds. \tag{5.20}$$

By Lemma 5.1 and using arguments similar to [66, proof of (4.8)], we can deduce that

$$\sup_{n \geq 1} \|u_n\|^2_{W^{\alpha,2}([0,T],\mathrm{V}')} \leq \widetilde{C}_N < \infty. \tag{5.21}$$

Moreover, since by [35, Theorem 2.1] (see also [59]), the embedding

$$L^2([0,T], \mathscr{D}(\mathrm{A})) \cap W^{\alpha,2}([0,T], \mathrm{V}') \hookrightarrow L^2([0,T], \mathrm{V}) \tag{5.22}$$

is compact, by Lemma 5.1 and (5.21) we infer that there exists $\tilde{u} \in L^2([0,T], \mathscr{D}(\mathrm{A})) \cap L^\infty([0,T], \mathrm{V})$ and a subsequence (for simplicity, also denoted by u_n) such that

(P1) $u_n \to \tilde{u}$ weakly in $L^2([0,T], \mathscr{D}(\mathrm{A}))$,

(P2) $u_n \to \tilde{u}$ in the weak-* topology of $L^\infty([0,T], \mathrm{V})$, and

$$\sup_{n \geq 1} \sup_{t \in [0,T]} \|u_n(t)\|_{\mathrm{V}} + \sup_{t \in [0,T]} \|\tilde{u}(t)\|_{\mathrm{V}} =: \hbar_0 < \infty,$$

(P3) $u_n \to \tilde{u}$ strongly in $L^2([0,T], \mathrm{V})$.

Now we prove that the limit function $\tilde{u}$ is a solution of equation (5.11). By the uniqueness of this solution, we infer $\tilde{u} = u = u^g$. The proof seems to be classical, but it is not, because of the nonstandard terms.

Let ψ be a continuously differentiable V-valued function on $[0,T]$ with $\psi(T) = 0$. We multiply $u_n(t)$ scalarly in H by $\psi(t)$, and then integrate by parts. This leads to the following equation:

$$\begin{aligned} -\int_0^T \langle u_n(t), \psi'(t)\rangle_{\mathrm{H}}\, dt &+ \int_0^T {}_{\mathrm{V}'}\langle u_n(t), \psi(t)\rangle_{\mathrm{V}}\, dt \\ &= \langle u_0, \psi(0)\rangle_{\mathrm{H}} - \int_0^T {}_{\mathrm{V}'}\langle \mathrm{B}(u_n(t)), \psi(t)\rangle_{\mathrm{V}}\, dt + \int_0^T \langle f(t), \psi(t)\rangle_{\mathrm{H}}\, dt \\ &\quad + \int_0^T \int_{\mathrm{Z}} \langle G(u_n(t), z), \psi(t)\rangle_{\mathrm{H}} (g_n(t,z) - 1)\, \nu(dz)\, dt. \end{aligned} \tag{5.23}$$

Keeping in mind properties (P1)–(P3) above and arguing similarly to [59, proof of Theorem III.3.1], we see that

$$\lim_{n\to\infty}\Big[-\int_0^T\langle u_n(t),\psi'(t)\rangle_{\mathrm{H}}\,dt+\int_0^T {}_{\mathrm{V}'}\langle u_n(t),\psi(t)\rangle_{\mathrm{V}}\,dt$$
$$-\langle u_0,\psi(0)\rangle_{\mathrm{H}}+\int_0^T {}_{\mathrm{V}'}\langle \mathrm{B}(u_n(t)),\psi(t)\rangle_{\mathrm{V}}\,dt-\int_0^T\langle f(t),\psi(t)\rangle_{\mathrm{H}}\,dt\Big]$$
$$=-\int_0^T\langle \tilde u(t),\psi'(t)\rangle_{\mathrm{H}}\,dt+\int_0^T {}_{\mathrm{V}'}\langle \tilde u(t),\psi(t)\rangle_{\mathrm{V}}\,dt$$
$$-\langle u_0,\psi(0)\rangle_{\mathrm{H}}+\int_0^T {}_{\mathrm{V}'}\langle \mathrm{B}(\tilde u(t)),\psi(t)\rangle_{\mathrm{V}}\,dt-\int_0^T\langle f(t),\psi(t)\rangle_{\mathrm{H}}\,dt. \tag{5.24}$$

What concerns us is the last term in (5.23). Since $g_n\to g$ in S^N, by [15, Lemma 3.11] we infer that

$$\lim_{n\to\infty}\int_0^T\int_{\mathrm{Z}}\langle G(\tilde u(t),z),\psi(t)\rangle_{\mathrm{H}}(g_n(t,z)-1)\,\nu(dz)\,dt$$
$$=\int_0^T\int_{\mathrm{Z}}\langle G(\tilde u(t),z),\psi(t)\rangle_{\mathrm{H}}(g(t,z)-1)\,\nu(dz)\,dt. \tag{5.25}$$

Next, for $\delta>0$, we set

$$A_{n,\delta}:=\{t\in[0,T]:\|u_n(t)-\tilde u(t)\|_{\mathrm{V}}\ge\delta\}.$$

Since $u_n\to\tilde u$ strongly in $L^2([0,T],\mathrm{V})$, by applying the Chebyshev inequality we infer that

$$\lim_{n\to\infty}\mathrm{Leb}_{[0,T]}(A_{n,\delta})\le\lim_{n\to\infty}\frac{1}{\delta^2}\int_0^T\|u_n(t)-\tilde u(t)\|_{\mathrm{V}}^2\,dt=0, \tag{5.26}$$

where $\mathrm{Leb}_{[0,T]}$ is the Lebesgue measure on $[0,T]$.

Fix $\delta>0$. Then, by Assumption 5.1 and assertion (P2), we infer that

$$\Big|\int_0^T\int_{\mathrm{Z}}\langle G(u_n(t),z)-G(\tilde u(t),z),\psi(t)\rangle_{\mathrm{H}}(g_n(t,z)-1)\,\nu(dz)\,dt\Big|$$
$$\le|\psi|_{L^\infty([0,T],\mathrm{V})}\int_0^T\|u_n(t)-\tilde u(t)\|_{\mathrm{V}}\int_{\mathrm{Z}}L_{\hbar_0}(z)|g_n(t,z)-1|\,\nu(dz)\,dt$$
$$\le 2\hbar_0|\psi|_{L^\infty([0,T],\mathrm{V})}\int_{A_{n,\delta}}\int_{\mathrm{Z}}L_{\hbar_0}(z)|g_n(t,z)-1|\,\nu(dz)\,dt$$
$$+\delta|\psi|_{L^\infty([0,T],\mathrm{V})}\int_0^T\int_{\mathrm{Z}}L_{\hbar_0}(z)|g_n(t,z)-1|\,\nu(dz)\,dt, \tag{5.27}$$

where $\hbar_0$ is the positive constant appearing in (P2).

In what follows, we use the following result; see [66, (3.3) of Lemma 3.1], [64, Remark 2], or [15, (3.5) of Lemma 3.4].

- For every function $\Im\in\mathscr{H}\cap L^2(\nu)$ and every $\varepsilon>0$ there exists $\beta>0$ such that for every $O\in\mathscr{B}([0,T])$ with $\mathrm{Leb}_{[0,T]}(O)\le\beta$,

$$\sup_{h\in S^N}\int_O\int_{\mathrm{Z}}\Im(z)|h(s,z)-1|\,\nu(dz)\,ds\le\varepsilon. \tag{5.28}$$

Hence, by (5.26)–(5.28) and (B.3) in Appendix B, we have

$$\limsup_{n\to\infty}\left|\int_0^T\int_Z \langle G(u_n(t),z)-G(\tilde u(t),z),\psi(t)\rangle_H (g_n(t,z)-1)\,\nu(dz)\,dt\right| \le \delta C_{\hbar_0,N}|\psi|_{L^\infty([0,T],V)}. \tag{5.29}$$

Since $\delta>0$ can be chosen arbitrarily small, this implies

$$\lim_{n\to\infty}\left|\int_0^T\int_Z \langle G(u_n(t),z)-G(\tilde u(t),z),\psi(t)\rangle_H (g_n(t,z)-1)\,\nu(dz)\,dt\right| = 0. \tag{5.30}$$

We observe that the above proof of (5.29) yields the following stronger result, which we use later on: for every $\Im\in\mathscr{H}\cap L^2(\nu)$,

$$\lim_{n\to\infty}\sup_{k\in S^N}\int_0^T \|u_n(s)-\tilde u(s)\|_V\int_Z \Im(z)|k(s,z)-1|\,\nu(dz)\,ds=0. \tag{5.31}$$

Combining (5.24), (5.25), and (5.30), we arrive at

$$\begin{aligned}
-\int_0^T \langle\tilde u(t),\psi'(t)\rangle_H\,dt + \int_0^T {}_{V'}\langle\tilde u(t),\psi(t)\rangle_V\,dt
&= \langle u_0,\psi(0)\rangle_H - \int_0^T {}_{V'}\langle B(\tilde u(t)),\psi(t)\rangle_V\,dt + \int_0^T\langle f(t),\psi(t)\rangle_H\,dt\\
&\quad + \int_0^T\int_Z \langle G(\tilde u(t),z),\psi(t)\rangle_H (g(t,z)-1)\,\nu(dz)\,dt.
\end{aligned} \tag{5.32}$$

From this, following [59, Sect. 3, Chapter III, proof of Theorems 3.1 and 3.2], we can conclude that $\tilde u$ is a solution of (5.11) as claimed, and then by uniqueness, $\tilde u=u=u^g$.

At the final stage of our proof of Proposition 5.3, we prove that

$$u_n\to u \quad \text{in } C([0,T],V)\cap L^2([0,T],\mathscr{D}(A)).$$

For this purpose, let $v_n=u_n-u$. Then by [59, Lemma III.1.2] (in V) we get, for all $t\in[0,T]$,

$$\begin{aligned}
&\|v_n(t)\|_V^2 + 2\int_0^t \|v_n(s)\|_{\mathscr{D}(A)}^2\,ds\\
&= -2\int_0^t \langle B(u_n(s))-B(u(s)),Av_n(s)\rangle_H\,ds\\
&\quad + 2\int_0^t\int_Z {}_{V'}\big\langle G(u_n(s),z)(g_n(s,z)-1)-G(u(s),z)(g(s,z)-1),v_n(s)\big\rangle_V\,\nu(dz)\,ds\\
&\le \frac12\int_0^t \|v_n(s)\|_{\mathscr{D}(A)}^2\,ds + 2\int_0^t |B(u_n(s)-B(u(s))|_H^2\,ds\\
&\quad + 2\int_0^t \|v_n(s)\|_V^2\int_Z L_{\hbar_0}(z)|g_n(s,z)-1|\,\nu(dz)\,ds\\
&\quad + 2\int_0^t \|v_n(s)\|_V(1+\|u(s)\|_V)\int_Z L_2(z)(|g_n(s,z)-1|+|g(s,z)-1|)\,\nu(dz)\,ds.
\end{aligned} \tag{5.33}$$

By Lemma 2.1, for all $s \in [0, T]$,

$$\begin{aligned}
&|\mathrm{B}(u_n(s)) - \mathrm{B}(u(s))|_{\mathrm{H}}^2 \\
&\quad \le 2\big(|\mathrm{B}(u_n(s), v_n(s))|_{\mathrm{H}}^2 + |\mathrm{B}(v_n(s), u(s)|_{\mathrm{H}}^2\big) \\
&\quad \le C\big(|u_n(s)|_{\mathrm{H}}\|u_n(s)\|_{\mathrm{V}}\|v_n(s)\|_{\mathrm{V}}\|v_n(s)\|_{\mathscr{D}(\mathrm{A})} + |v_n(s)|_{\mathrm{H}}\|v_n(s)\|_{\mathrm{V}}\|u(s)\|_{\mathrm{V}}\|u(s)\|_{\mathscr{D}(\mathrm{A})}\big) \\
&\quad \le \tfrac{1}{4}\|v_n(s)\|_{\mathscr{D}(\mathrm{A})}^2 + C\|v_n(s)\|_{\mathrm{V}}^2\big(\|u_n(s)\|_{\mathrm{V}}^4 + \|u(s)\|_{\mathrm{V}}\|u(s)\|_{\mathscr{D}(\mathrm{A})}\big).
\end{aligned} \tag{5.34}$$

Substituting (5.34) into (5.33), and by Lemma 5.1, since $u \in L^\infty([0, T], \mathrm{V})$, we obtain

$$\begin{aligned}
&\|v_n(t)\|_{\mathrm{V}}^2 + \int_0^t \|v_n(s)\|_{\mathscr{D}(\mathrm{A})}^2\, ds \\
&\quad \le \int_0^t \|v_n(s)\|_{\mathrm{V}}^2 \\
&\qquad \cdot \left(C\|u_n(s)\|_{\mathrm{V}}^4 + C\|u(s)\|_{\mathrm{V}}\|u(s)\|_{\mathscr{D}(\mathrm{A})} + 2\int_{\mathrm{Z}} L_{\hbar_0}(z)|g_n(s,z) - 1|\,\nu(dz)\right) ds \\
&\qquad + C \sup_{h\in S^N} \int_0^T \|v_n(s)\|_{\mathrm{V}} \int_{\mathrm{Z}} L_2(z)|h(s,z) - 1|\,\nu(dz)\,dt, \quad t \in [0, T].
\end{aligned} \tag{5.35}$$

By Gronwall's lemma, Lemma 5.1 and (B.2) imply that

$$\begin{aligned}
\sup_{t\in[0,T]} \|v_n(t)\|_{\mathrm{V}}^2 + \int_0^T \|v_n(t)\|_{\mathscr{D}(\mathrm{A})}^2\, dt
&\le e^{C\int_0^T (\|u_n(s)\|_{\mathrm{V}}^4 + \|u(s)\|_{\mathrm{V}}\|u(s)\|_{\mathscr{D}(\mathrm{A})} + \int_{\mathrm{Z}} L_{\hbar_0}(z)|g_n(s,z)-1|\,\nu(dz))\,ds} \\
&\quad \cdot \sup_{h\in S^N} \int_0^T \|v_n(s)\|_{\mathrm{V}} \int_{\mathrm{Z}} L_2(z)|h(s,z) - 1|\,\nu(dz)\,dt \\
&\le C_{N,T} \sup_{h\in S^N} \int_0^T \|v_n(s)\|_{\mathrm{V}} \int_{\mathrm{Z}} L_2(z)|h(s,z) - 1|\,\nu(dz)\,dt, \quad t \in [0, T].
\end{aligned}$$

Note that the integral $\int_0^T (\|u_n(s)\|_{\mathrm{V}}^4 + \|u(s)\|_{\mathrm{V}}\|u(s)\|_{\mathscr{D}(\mathrm{A})})\, ds$ is finite in view of Lemma 5.1.

Therefore, by (5.31),

$$\lim_{n\to\infty} \left(\sup_{t\in[0,T]} \|v_n(t)\|_{\mathrm{V}}^2 + \int_0^T \|v_n(t)\|_{\mathscr{D}(\mathrm{A})}^2\, dt\right) = 0. \tag{5.36}$$

The proof of Proposition 5.3 is thus complete. ■

6. A generalization of the Girsanov theorem

The aim of this section is to establish a certain generalization of the Girsanov theorem. This result will then be used in Section 7 to verify Claim LDP-2. First, we state and prove Lemma 6.1. Then we prove Theorem 6.1, which is the main result of this section.

In order to formulate Lemma 6.1, let us recall the sets K_n that were introduced around (5.17), and let us introduce, for $n \in \mathbb{N}$, the following set:

$$\bar{\mathbb{A}}_{b,n} = \{\varphi \in \bar{\mathbb{A}} : \varphi(t,z,\omega) \in [1/n, n] \text{ if } (t,z,\omega) \in [0,T] \times K_n \times \bar{\Omega} \\ \text{and } \varphi(t,z,\omega) = 1 \text{ if } (t,z,\omega) \in [0,T] \times K_n^c \times \bar{\Omega}\}. \tag{6.1}$$

Note that with the notation of (5.18), we have

$$\bar{\mathbb{A}}_b = \bigcup_{n=1}^{\infty} \bar{\mathbb{A}}_{b,n}.$$

The proof of [17, Lemma 2.4] implies the following result; however, since [17] gives no details, we present a detailed proof. This result is important in proving Theorem 6.1.

Lemma 6.1. *Let $n \in \mathbb{N}$ and $\varphi \in \bar{\mathbb{A}}_{b,n}$. Then there exists an $\bar{\mathbb{A}}_{b,n}$-valued sequence $\{\psi_m\}_{m\in\mathbb{N}}$ such that the following properties are satisfied.*

(R1) *For every m, there exist $l \in \mathbb{N}$ and $n_1, \dots, n_l \in \mathbb{N}$, a partition $0 = t_0 < t_1 < \cdots < t_l = T$ and families*

$$\xi_{ij}, \quad i = 1,\dots,l,\ j = 1,\dots,n_i,$$
$$E_{ij}, \quad i = 1,\dots,l,\ j = 1,\dots,n_i,$$

such that the ξ_{ij} are $[1/n, n]$-valued, $\mathscr{G}_{t_{i-1}}$-measurable random variables, and, for each $i = 1,\dots,l$, $\{E_{ij}\}_{j=1}^{n_i}$ is a measurable partition of the set K_n such that

$$\psi_m(t,z,\omega) = \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{l}\sum_{j=1}^{n_i} \mathbb{1}_{(t_{i-1},t_i]}(t)\xi_{ij}(\omega)\mathbb{1}_{E_{ij}}(z) + \mathbb{1}_{K_n^c}(z)\mathbb{1}_{(0,T]}(t) \tag{6.2}$$

for all $(t,z,\omega) \in [0,T] \times \mathrm{Z} \times \bar{\Omega}$.

(R2) $\lim_{m\to\infty} \int_0^T |\psi_m(t,z,\omega) - \varphi(t,z,\omega)|\, dt = 0$ *for $\nu \otimes \mathbb{Q}$-a.a. $(z,\omega) \in \mathrm{Z} \times \bar{\Omega}$.*

Proof. Fix $n \in \mathbb{N}$ and $\varphi \in \bar{\mathbb{A}}_{b,n}$. First, let us remark that "φ_k" in [17, p. 729 line −6] should read

$$\varphi_k(t,z,\omega) = k\left(\frac{1}{k} - t\right)^+ + k\int_{(t-1/k)^+}^{t} \varphi(s,z,\omega)\, ds. \tag{6.3}$$

One can check that

$$\varphi_k(t,z,\omega) = 1 \quad \text{on } (t,z,\omega) \in [0,T] \times K_n^c \times \bar{\Omega}. \tag{6.4}$$

In [17, proof of Lemma 2.4], the following three assertions were proved.

(L1) The process φ_k defined in (6.3) has the following three properties:

(L1.1) $\lim_{k\to\infty} \int_0^T |\varphi_k(t,z,\omega) - \varphi(t,z,\omega)|\, dt = 0$ $\nu \otimes \mathbb{Q}$-a.s. $(z,\omega) \in \mathrm{Z} \times \bar{\Omega}$,

(L1.2) $\varphi_k \in \bar{\mathbb{A}}_{b,n}$,

(L1.3) $[0,\infty) \ni t \mapsto \varphi_k(t,z,\omega)$ is continuous for $\nu \otimes \mathbb{Q}$-a.s. $(z,\omega) \in \mathrm{Z} \times \bar{\Omega}$.

(L2) If, for $k, q \in \mathbb{N}$, we set

$$\varphi_k^q(t,z,\omega) = \mathbb{1}_{\{0\}}(t) + \sum_{m=0}^{\lfloor qT \rfloor} \varphi_k\left(\frac{m}{q}, z, \omega\right) \mathbb{1}_{(m/q,(m+1)/q]}(t), \quad (t,z,\omega) \in [0,T] \times \mathrm{Z} \times \bar{\Omega}.$$

then

(L2.1) $\varphi_k^q \in \bar{\mathbb{A}}_{b,n}$,

(L2.2)

$$\lim_{q\to\infty} \int_0^T |\varphi_k^q(t,z,\omega) - \varphi_k(t,z,\omega)|\, dt = 0,\ \nu \otimes \mathbb{Q}\text{-a.s. } (z,\omega) \in \mathrm{Z} \times \bar{\Omega}.$$

(L3) For all $k, q \in \mathbb{N}$, there exists an $\bar{\mathbb{A}}_{b,n}$-valued sequence $\{\varphi_k^{q,r}\}_{k=1}^\infty$ of processes such that

(L3.1) for every k, $\varphi_k^{q,r}$ satisfies condition **(R1)**,

(L3.2) $\lim_{r\to\infty} \int_0^T |\varphi_k^{q,r}(t,z,\omega) - \varphi_k^q(t,z,\omega)|\, dt = 0$, $\nu \otimes \mathbb{Q}$-a.s. $(z,\omega) \in \mathrm{Z} \times \bar{\Omega}$.

Note that (L2.2) follows easily from (L1.3).

To prove (L3), we repeat the argument from [17, proof of Lemma 2.4].

Note that for fixed q and m, $g(z,\omega) = \varphi_k(m/q, z, \omega)$ is a $\mathscr{B}(\mathrm{Z}) \otimes \mathscr{G}_{m/q}$-measurable map with values in $[1/n, n]$ and $g(z,\omega) = 1$ for $z \in K_n^c$. By a standard approximation procedure, one can find $\mathscr{B}(\mathrm{Z}) \otimes \mathscr{G}_{m/q}$-measurable maps $g_r,\ r \in \mathbb{N}$, with the following properties: $g_r(z,\omega) = \sum_{j=1}^{a(r)} c_j^r(\omega) \mathbb{1}_{E_j^r}(z)$ for $z \in K_n$, where for each r, $\{E_j^r\}_{j=1}^{a(r)}$ is some measurable partition of K_n and for all j, r, $c_j^r(\omega) \in [1/n, n]$ a.s.; $g_r(z,\omega) = 1$ for $z \in K_n^c$; and $g_r \to g$ as $r \to \infty$, $\nu \otimes \mathbb{Q}$-a.s.

Having established the above, it is easy to see that assertion (L3.2) holds.

Moreover, these three assertions imply our result in Lemma 6.1. This can be seen as follows. First of all, it is easy to see that, for any $k, q, r \in \mathbb{N}$,

$$\begin{aligned} \varphi(t,z,\omega) = \varphi_k(t,z,\omega) = \varphi_k^q(t,z,\omega) = \varphi_k^{q,r}(t,z,\omega) \\ = 1 \quad \text{for } (t,z,\omega) \in [0,T] \times K_n^c \times \bar{\Omega}. \end{aligned} \tag{6.5}$$

Hence, we only need to consider the case of $z \in K_n$.

Set

$$\Omega_1 = \left\{(z,\omega) \in K_n \times \bar{\Omega} : \lim_{k\to\infty} \int_0^T |\varphi_k(t,z,\omega) - \varphi(t,z,\omega)|\, dt = 0\right\}.$$

Then assertion (L1.1) implies

$$(\nu \otimes \mathbb{Q})(K_n \times \bar{\Omega} \setminus \Omega_1) = 0. \tag{6.6}$$

For simplicity, we set $O^c = K_n \times \bar{\Omega} \setminus O$ for any $O \subset K_n \times \bar{\Omega}$, and keep in mind that $\nu(K_n) < \infty$.

Let

$$I_m^i = \bigcap_{k\ge i}\Big\{(z,\omega)\in K_n\times\bar\Omega : \int_0^T |\varphi_k(t,z,\omega)-\varphi(t,z,\omega)|\,dt \le 1/m\Big\}.$$

It is easy to see that

$$I_m^i \subset I_m^{i+1}, \quad \bigcup_{i=1}^\infty I_m^i =: \Omega_{1,m}, \quad \Omega_1 \subset \Omega_{1,m}.$$

In view of (6.6), there exists i_m such that

$$(\nu\otimes\mathbb{Q})((I_m^{i_m})^c) \le 1/m^2. \tag{6.7}$$

Set

$$\Omega_{2,k} = \Big\{(z,\omega)\in K_n\times\bar\Omega : \lim_{q\to\infty}\int_0^T |\varphi_k^q(t,z,\omega)-\varphi_k(t,z,\omega)|\,dt = 0\Big\},$$
$$\Omega_{3,k,q} = \Big\{(z,\omega)\in K_n\times\bar\Omega : \lim_{r\to\infty}\int_0^T |\varphi_k^{q,r}(t,z,\omega)-\varphi_k^q(t,z,\omega)|\,dt = 0\Big\}.$$

Let

$$II_m^{k,i} = \bigcap_{q\ge i}\Big\{(z,\omega)\in K_n\times\bar\Omega : \int_0^T |\varphi_k^q(t,z,\omega)-\varphi_k(t,z,\omega)|\,dt \le 1/m\Big\},$$
$$III_m^{k,q,i} = \bigcap_{r\ge i}\Big\{(z,\omega)\in K_n\times\bar\Omega : \int_0^T |\varphi_k^{q,r}(t,z,\omega)-\varphi_k^q(t,z,\omega)|\,dt \le 1/m\Big\}.$$

Similarly to (6.7), there exist $j_{m,k}$ and $l_{m,k,q}$ such that

$$(\nu\otimes\mathbb{Q})((II_m^{k,j_{m,k}})^c) \le 1/m^2, \tag{6.8}$$
$$(\nu\otimes\mathbb{Q})((III_m^{k,q,l_{m,k,q}})^c) \le 1/m^2. \tag{6.9}$$

By the definition of I_m^k, $II_m^{k,q}$, $III_m^{k,q,r}$,

$$\begin{aligned} I_m^k \cap II_m^{k,q} \cap III_m^{k,q,r} &\subset \Big\{(z,\omega)\in K_n\times\bar\Omega : \int_0^T |\varphi_k(t,z,\omega)-\varphi(t,z,\omega)|\,dt \le 1/m\Big\} \\ &\cap \Big\{(z,\omega)\in K_n\times\bar\Omega : \int_0^T |\varphi_k^q(t,z,\omega)-\varphi_k(t,z,\omega)|\,dt \le 1/m\Big\} \\ &\cap \Big\{(z,\omega)\in K_n\times\bar\Omega : \int_0^T |\varphi_k^{q,r}(t,z,\omega)-\varphi_k^q(t,z,\omega)|\,dt \le 1/m\Big\}, \end{aligned}$$

so, for any $(z,\omega)\in I_m^k \cap II_m^{k,q} \cap III_m^{k,q,r}$,

$$\int_0^T |\varphi_k^{q,r}(t,z,\omega)-\varphi(t,z,\omega)|\,dt \le 3/m. \tag{6.10}$$

For

$$k = i_m, \quad q = j_{m,k} = j_{m,i_m}, \quad r = l_{m,k,q} = l_{m,i_m,j_{m,i_m}}, \tag{6.11}$$

set

$$\psi_m = \varphi_k^{q,r}$$

and define

$$\Omega_0 = \bigcup_{i=1} \bigcap_{m \geq i} \{I_m^k \cap II_m^{k,q} \cap III_m^{k,q,r}\}.$$

In the following, we prove that ψ_m satisfies **(R1)** and **(R2)**, while keeping in mind (6.5). The claim for **(R1)** is obvious. For **(R2)**, it is easy to see $\Omega_0 \subset K_n \times \bar{\Omega}$, so we only need to prove that

$$(\nu \otimes \mathbb{Q})(\Omega_0^c) = 0, \tag{6.12}$$

$$\lim_{m\to\infty} \int_0^T |\psi_m(t,z,\omega) - \varphi(t,z,\omega)|\, dt \quad \text{on } (z,\omega) \in \Omega_0. \tag{6.13}$$

We use convention (6.11) to get

$$\bigcap_{m=i}^{\infty} \{I_m^k \cap II_m^{k,q} \cap III_m^{k,q,r}\} \nearrow \Omega_0 \quad \text{as } k \to \infty,$$

and so, by (6.7)–(6.9), for every $i \in \mathbb{N}$,

$$\begin{aligned}(\nu \otimes \mathbb{Q})(\Omega_0^c) &\leq (\nu \otimes \mathbb{Q})\Big(\Big(\bigcap_{m\geq i} \{I_m^k \cap II_m^{k,q} \cap III_m^{k,q,r}\}\Big)^c\Big) \\ &\leq \sum_{m=i}^{\infty} \big((\nu \otimes \mathbb{Q})((I_m^k)^c) + (\nu \otimes \mathbb{Q})((II_m^{k,q})^c) + (\nu \otimes \mathbb{Q})((III_m^{k,q,r})^c)\big) \\ &\leq \sum_{m=i}^{\infty} \frac{3}{m^2}.\end{aligned} \tag{6.14}$$

Since $\sum_{m=i}^{\infty} 3/m^2 \to 0$ as $i \to \infty$, (6.12) is proved.

Next, we prove (6.13). Fix $(z,\omega) \in \Omega_0$. Then, using (6.11) again, there exists $i_0 \in \mathbb{N}$ such that

$$(z,\omega) \in \bigcap_{m=i_0}^{\infty} \{I_m^k \cap II_m^{k,q} \cap III_m^{k,q,r}\}.$$

Thus by (6.10),

$$\int_0^T |\psi_m(t,z,\omega) - \varphi(t,z,\omega)|\, dt \leq 3/m \quad \text{for all } m \geq i_0. \tag{6.15}$$

Hence (6.13) follows, and thus **(R2)** holds. Consequently, the proof of Lemma 6.1 is complete. ∎

Fix $\varepsilon > 0$ and $\varphi_\varepsilon \in \bar{\mathbb{A}}_b$, and set $\psi_\varepsilon = 1/\varphi_\varepsilon$. In view of the definition (5.18) of $\bar{\mathbb{A}}_b$, it is easy to see that $\psi_\varepsilon \in \bar{\mathbb{A}}_b$. In particular, there exists $n \in \mathbb{N}$ such that

$$\psi_\varepsilon(t,z,\omega) \in [1/n, n] \quad \text{if } (t,z,\omega) \in [0,T] \times K_n \times \bar{\Omega},$$

where K_n is a compact subset of Z from (5.17) and

$$\psi_\varepsilon(t,z,\omega) = 1 \quad \text{if } (t,z,\omega) \in [0,T] \times K_n^c \times \bar{\Omega}.$$

Combining this with the fact that $\nu(K_n) < \infty$, we infer the following four assertions, grouped for convenience in Theorem 6.1.

Theorem 6.1. *In the framework introduced above, the following hold:*

(S1) *The process* $\mathscr{M}_t^\varepsilon(\psi_\varepsilon)$*,* $t \in [0, T]$*, defined by*

$$\begin{aligned}\mathscr{M}_t^\varepsilon(\psi_\varepsilon) &= \exp\biggl(\int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]} \log(\psi_\varepsilon(s,z))\,\mathrm{N}(ds,dz,dr) \\ &\qquad + \int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]} (-\psi_\varepsilon(s,z)+1)\,\nu(dz)\,ds\,dr\biggr) \\ &= \exp\biggl(\int_{(0,t]\times K_n\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]} \log(\psi_\varepsilon(s,z))\,\mathrm{N}(ds,dz,dr) \\ &\quad + \int_{(0,t]\times K_n\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]} (-\psi_\varepsilon(s,z)+1)\,\nu(dz)\,ds\,dr\biggr), \quad t\in[0,T], \end{aligned} \tag{6.16}$$

is a $\mathbb{G}$*-martingale on* $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{Q})$.

(S2) *The formula*

$$\mathbb{P}_T^\varepsilon(O) = \int_O \mathscr{M}_T^\varepsilon(\psi_\varepsilon)\,d\mathbb{Q}, \quad O \in \mathscr{G},$$

defines a probability measure on $(\bar{\Omega}, \mathscr{G})$.

(S3) *The measures* $\mathbb{Q}$ *and* $\mathbb{P}_T^\varepsilon$ *are equivalent.*

(S4) *The laws on* M_T *of the following two random variables are equal:* (i) $\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}$ *defined on* $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{P}_T^\varepsilon)$ *and* (ii) $\varepsilon N^{\varepsilon^{-1}}$ *defined on* $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{Q})$.

Recall that the two processes appearing in (S4) were introduced in (5.1).

Although Theorem 6.1 is a "standard result" (see for instance [44, Theorem III.3.24] for semimartingales and [7, Theorem 3.10.21] for Poisson point processes), it seems hard to find an accessible reference which would work under our conditions. Therefore, we give a detailed proof.

Proof of Theorem 6.1. Since assertion (S2) is implied by (S1), we only prove (S1), (S3), and (S4). We divide the proof into three steps.

Step 1. Assume that φ_ε is a step process (see Lemma 6.1), i.e., there exist $l, n_1, \dots, n_l \in \mathbb{N}$, a partition

$$0 = t_0 < t_1 < \cdots < t_l = T,$$

$[1/n, n]$-valued random variables

$$\xi_{ij}, \quad i = 1, \dots, l,\ j = 1, \dots, n_i,$$

such that ξ_{ij} is $\mathscr{G}_{t_{i-1}}$-measurable, and, for each $i = 1, \dots, l$, a disjoint measurable partition $\{E_{ij}\}_{j=1}^{n_i}$ of K_n, such that for all $(t, z, \omega) \in [0, T] \times \mathrm{Z} \times \bar{\Omega}$,

$$\varphi_\varepsilon(t,z,\omega) = \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{l}\sum_{j=1}^{n_i} \mathbb{1}_{(t_{i-1},t_i]}(t)\xi_{ij}(\omega)\mathbb{1}_{E_{ij}}(z) + \mathbb{1}_{K_n^c}(z)\mathbb{1}_{(0,T]}(t).$$

Then, for any $O \in \mathscr{B}(\mathrm{Z})$, we have

$$
\begin{aligned}
N^{\varepsilon^{-1}\varphi_\varepsilon}(T,O) &= N^{\varepsilon^{-1}\varphi_\varepsilon}(t_l,O) \\
&= \int_0^{t_l}\int_O\int_0^\infty 1\!\!1_{(0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)\,\mathrm{N}(ds,dz,dr) \\
&= N^{\varepsilon^{-1}\varphi_\varepsilon}(t_{l-1},O) \\
&\quad + \int_{t_{l-1}}^{t_l}\int_{O\cap K_n^c}\int_0^\infty 1\!\!1_{(0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)\,\mathrm{N}(ds,dz,dr) \\
&\quad + \sum_{j=1}^{n_l}\int_{t_{l-1}}^{t_l}\int_{O\cap E_{lj}}\int_0^\infty 1\!\!1_{(0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)\,\mathrm{N}(ds,dz,dr) \\
&= N^{\varepsilon^{-1}\varphi_\varepsilon}(t_{l-1},O) \\
&\quad + \int_{t_{l-1}}^{t_l}\int_{O\cap K_n^c}\int_0^\infty 1\!\!1_{(0,\varepsilon^{-1}]}(r)\,\mathrm{N}(ds,dz,dr) \\
&\quad + \sum_{j=1}^{n_l}\int_{t_{l-1}}^{t_l}\int_{O\cap E_{lj}}\int_0^\infty 1\!\!1_{(0,\varepsilon^{-1}\xi_{lj}]}(r)\,\mathrm{N}(ds,dz,dr).
\end{aligned}
$$

Moreover, for the process $\mathscr{M}_t^\varepsilon(\psi_\varepsilon)$ defined in (6.16), we have

$$
\begin{aligned}
\mathscr{M}_T^\varepsilon(\psi_\varepsilon) &= \exp\biggl(\int_{(0,T]\times\mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}\log(\psi_\varepsilon(s,z))\,\mathrm{N}(ds,dz,dr) \\
&\qquad + \int_{(0,T]\times\mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(-\psi_\varepsilon(s,z)+1)\,\nu(dz)\,ds\,dr\biggr) \\
&= \exp\biggl(\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}\log(\psi_\varepsilon(s,z))\,\mathrm{N}(ds,dz,dr) \\
&\qquad + \int_{(0,T]\times K_n\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(-\psi_\varepsilon(s,z)+1)\,\nu(dz)\,ds\,dr\biggr) \\
&= \exp\biggl(\int_{(0,t_{l-1}]\times K_n\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}\log(\psi_\varepsilon(s,z))\,\mathrm{N}(ds,dz,dr) \\
&\qquad + \int_{(0,t_{l-1}]\times K_n\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(-\psi_\varepsilon(s,z)+1)\,\nu(dz)\,ds\,dr\biggr) \\
&\quad \cdot\exp\biggl(\sum_{j=1}^{n_l}\biggl[\int_{(t_{l-1},t_l]}\int_{E_{lj}}\int_{(0,\varepsilon^{-1}\xi_{lj}]}\log\Bigl(\frac{1}{\xi_{lj}}\Bigr)\,\mathrm{N}(ds,dz,dr) \\
&\qquad + \int_{(t_{l-1},t_l]}\int_{E_{lj}}\int_{(0,\varepsilon^{-1}\xi_{lj}]}\Bigl(-\frac{1}{\xi_{lj}}+1\Bigr)\,\nu(dz)\,ds\,dr\biggr]\biggr) \\
&= \mathscr{M}_{t_{l-1}}^\varepsilon(\psi_\varepsilon)\cdot\exp\biggl(\sum_{j=1}^{n_l}\biggl[\int_{(t_{l-1},t_l]}\int_{E_{lj}}\int_{(0,\varepsilon^{-1}\xi_{lj}]}\log\Bigl(\frac{1}{\xi_{lj}}\Bigr)\,\mathrm{N}(ds,dz,dr) \\
&\qquad + \int_{(t_{l-1},t_l]}\int_{E_{lj}}\int_{(0,\varepsilon^{-1}\xi_{lj}]}\Bigl(-\frac{1}{\xi_{lj}}+1\Bigr)\,\nu(dz)\,ds\,dr\biggr]\biggr). \qquad (6.17)
\end{aligned}
$$

Hence, for any $\xi \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\big(e^{\sqrt{-1}\,\xi N^{\varepsilon^{-1}\varphi_\varepsilon}(T,O)}\cdot \mathscr{M}_T^\varepsilon(\psi_\varepsilon)\big)\\
&\qquad = \mathbb{E}^{\mathbb{Q}}\big[\mathbb{E}^{\mathbb{Q}}\big(e^{\sqrt{-1}\,\xi N^{\frac{\varphi_\varepsilon}{\varepsilon}}(T,O)}\cdot \mathscr{M}_T^\varepsilon(\psi_\varepsilon)\;\big|\;\mathscr{G}_{t_{l-1}}\big)\big]\\
&\qquad = \mathbb{E}^{\mathbb{Q}}\big[e^{\sqrt{-1}\,\xi N^{\varepsilon^{-1}\varphi_\varepsilon}(t_{l-1},O)}\cdot \mathscr{M}_{t_{l-1}}^\varepsilon(\psi_\varepsilon)\cdot Y(\cdot;\xi,t_{l-1},t_l)\big], \qquad (6.18)
\end{aligned}$$

where $Y = Y(\cdot;\xi,t_{l-1},t_l)$ is defined by

$$\begin{aligned}
&Y(\cdot;\xi,t_l,t_{l-1})\\
&:= \mathbb{E}^{\mathbb{Q}}\Bigg(\exp\Bigg(\sqrt{-1}\,\xi\Bigg(\int_{t_{l-1}}^{t_l}\int_{O\cap K_n^c}\int_0^\infty \mathbb{1}_{(0,\varepsilon^{-1}]}(r)\,\mathrm{N}(ds,dz,dr)\\
&\qquad + \sum_{j=1}^{n_l}\int_{(t_{l-1},t_l]}\int_{O\cap E_{lj}}\int_0^\infty \mathbb{1}_{(0,\varepsilon^{-1}\xi_{lj}]}(r)\,\mathrm{N}(ds,dz,dr)\Bigg)\Bigg)\\
&\qquad \cdot\exp\Bigg(\sum_{j=1}^{n_l}\Bigg[\int_{(t_{l-1},t_l]}\int_{E_{lj}}\int_{(0,\varepsilon^{-1}\xi_{lj}]}\log\Big(\frac{1}{\xi_{lj}}\Big)\,\mathrm{N}(ds,dz,dr)\\
&\qquad + \int_{(t_{l-1},t_l]}\int_{E_{lj}}\int_{(0,\varepsilon^{-1}\xi_{lj}]}\Big(-\frac{1}{\xi_{lj}}+1\Big)\,\nu(dz)\,ds\,dr\Bigg]\Bigg)\;\Bigg|\;\mathscr{G}_{t_{l-1}}\Bigg).
\end{aligned}$$

By assumptions, each ξ_{lj}, $j = 1,\dots,n_l$, is $\mathscr{G}_{t_{l-1}}$-measurable, so by the properties of the conditional expectation, we infer that, $\mathbb{Q}$-a.s.,

$$Y(\omega,\xi,t_l,t_{l-1}) = K(\omega,\xi,\xi_{l1}(\omega),\xi_{l2}(\omega),\dots,\xi_{ln_l}(\omega),t_l,t_{l-1})$$

where a random variable $K(\omega,\xi,a_1,a_2,\dots,a_{n_l},t_l,t_{l-1})$ is defined by

$$\begin{aligned}
&K(\cdot,\xi,a_1,a_2,\dots,a_{n_l},t_l,t_{l-1})\\
&:= \mathbb{E}^{\mathbb{Q}}\Bigg(\exp\Bigg(\sqrt{-1}\,\xi\Bigg(\int_{t_{l-1}}^{t_l}\int_{O\cap K_n^c}\int_0^\infty \mathbb{1}_{(0,\varepsilon^{-1}]}(r)\,\mathrm{N}(ds,dz,dr)\\
&\qquad + \sum_{j=1}^{n_l}\int_{(t_{l-1},t_l]}\int_{O\cap E_{lj}}\int_0^\infty \mathbb{1}_{(0,\varepsilon^{-1}a_j]}(r)\,\mathrm{N}(ds,dz,dr)\Bigg)\Bigg)\\
&\qquad \cdot\exp\Bigg(\sum_{j=1}^{n_l}\Bigg[\int_{(t_{l-1},t_l]}\int_{E_{lj}}\int_{(0,\varepsilon^{-1}a_j]}\log\Big(\frac{1}{a_j}\Big)\,\mathrm{N}(ds,dz,dr)\\
&\qquad + \int_{(t_{l-1},t_l]}\int_{E_{lj}}\int_{(0,\varepsilon^{-1}a_j]}\Big(-\frac{1}{a_j}+1\Big)\,\nu(dz)\,ds\,dr\Bigg]\Bigg)\;\Bigg|\;\mathscr{G}_{t_{l-1}}\Bigg).
\end{aligned}$$

Note that for any positive constants $a_1, a_2, \dots, a_{n_l}$, we have the identity

$$
\begin{aligned}
&K(\cdot,\xi,a_1,a_2,\dots,a_{n_l},t_l,t_{l-1})\\
&= \exp\Big(\sum_{j=1}^{n_l}(t_l-t_{l-1})\nu(E_{lj})\varepsilon^{-1}a_j\Big(-\frac{1}{a_j}+1\Big)\Big)\\
&\quad\cdot \mathbb{E}^{\mathbb{Q}}\Big(\exp\Big(\sqrt{-1}\,\xi\Big(\int_{t_{l-1}}^{t_l}\int_{O\cap K_n^c}\int_0^\infty \mathbb{1}_{(0,\varepsilon^{-1}]}(r)\,\mathrm{N}(ds,dz,dr)\Big)\Big)\Big)\\
&\quad\cdot \mathbb{E}^{\mathbb{Q}}\Big(\exp\Big(\sum_{j=1}^{n_l}\Big(\sqrt{-1}\,\xi+\log\Big(\frac{1}{a_j}\Big)\Big)\Big)\\
&\qquad\qquad\cdot\int_{(t_{l-1},t_l]}\int_{O\cap E_{lj}}\int_0^\infty \mathbb{1}_{(0,\varepsilon^{-1}a_j]}(r)\,\mathrm{N}(ds,dz,dr)\Big)\\
&\quad\cdot \mathbb{E}^{\mathbb{Q}}\Big(\exp\Big(\sum_{j=1}^{n_l}\log\Big(\frac{1}{a_j}\Big)\int_{(t_{l-1},t_l]}\int_{O^c\cap E_{lj}}\int_0^\infty \mathbb{1}_{(0,\varepsilon^{-1}a_j]}(r)\,\mathrm{N}(ds,dz,dr)\Big)\\
&= \exp\big((t_l-t_{l-1})\nu(O)\varepsilon^{-1}[e^{\sqrt{-1}\,\xi}-1]\big).
\end{aligned}
$$

Summing up, we infer that for $\omega \in \Omega$, $\mathbb{Q}$-a.s., we have

$$
Y(\cdot,\xi,t_l,t_{l-1}) = \exp\big((t_l-t_{l-1})\nu(O)\varepsilon^{-1}[e^{\sqrt{-1}\,\xi}-1]\big). \tag{6.19}
$$

In particular, $Y(\omega,0,t_l,t_{l-1}) = 1$, and hence by (6.18),

$$
\mathbb{E}^{\mathbb{Q}}(\mathscr{M}^\varepsilon_{t_l}(\psi_\varepsilon)\mid \mathscr{G}_{t_{l-1}}) = \mathscr{M}^\varepsilon_{t_{l-1}}(\psi_\varepsilon)Y(\omega,0,t_l,t_{l-1}) = \mathscr{M}^\varepsilon_{t_{l-1}}(\psi_\varepsilon).
$$

Furthermore, by employing the above argument, we can easily verify

$$
\mathbb{E}^{\mathbb{Q}}(\mathscr{M}^\varepsilon_{t_l}(\psi_\varepsilon)\mid \mathscr{G}_t) = \mathscr{M}^\varepsilon_t(\psi_\varepsilon), \quad t\in[0,T].
$$

This implies that the process $\{\mathscr{M}^\varepsilon_t(\psi_\varepsilon),\ t\ge 0\}$ is a $\mathbb{G}$-martingale on $(\bar\Omega,\mathscr{G},\mathbb{G},\mathbb{Q})$. Hence $\mathbb{P}^\varepsilon_T$ is a well-defined probability measure.

Inserting identity (6.19) into (6.18), we arrive at

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\varepsilon_T}\big(e^{\sqrt{-1}\,\xi N^{\varepsilon^{-1}\varphi_\varepsilon}(T,O)}\big)\\
&\qquad = \mathbb{E}^{\mathbb{Q}}\big[e^{\sqrt{-1}\,\xi N^{\varepsilon^{-1}\varphi_\varepsilon}(t_{l-1},O)}\cdot \mathscr{M}^\varepsilon_{t_{l-1}}(\psi_\varepsilon)\big]e^{(t_l-t_{l-1})\nu(O)\varepsilon^{-1}[e^{\sqrt{-1}\,\xi}-1]}.
\end{aligned}
$$

By induction, we get

$$
\mathbb{E}^{\mathbb{P}^\varepsilon_T}\big(e^{\sqrt{-1}\,\xi N^{\varepsilon^{-1}\varphi_\varepsilon}(T,O)}\big) = \exp\big(T\nu(O)\varepsilon^{-1}[e^{\sqrt{-1}\,\xi}-1]\big).
$$

We have proved that if φ_ε is a step process, then the law of $\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}$ on $(\bar\Omega,\mathscr{G},\mathbb{G},\mathbb{P}^\varepsilon_T)$ is equal to the law of $\varepsilon N^{\varepsilon^{-1}}$ on $(\bar\Omega,\mathscr{G},\mathbb{G},\mathbb{Q})$. The proof of Step 1 is now complete. ■

Step 2. The general case. Let us assume that $\varphi_\varepsilon \in \bar{\mathbb{A}}_b$. Then by (5.18) there exists $n\in\mathbb{N}$ such that $\varphi_\varepsilon \in \bar{\mathbb{A}}_{b,n}$. Hence, by Lemma 6.1, there exists a sequence $\psi_m \in \bar{\mathbb{A}}_{b,n}$, $m = 1,2,\dots$, satisfying conditions **(R1)** and **(R2)** with φ replaced by φ_ε.

Applying Step 1 to ψ_m, we get

- for any $O \in \mathscr{B}(\mathrm{Z})$ with $\nu(O) < \infty$,

$$\mathbb{E}^{\mathbb{Q}}\Big(\exp\big(\sqrt{-1}\,\xi N^{\varepsilon^{-1}\psi_m}(T,O)\big)\mathscr{M}_T^{\varepsilon}\Big(\frac{1}{\psi_m}\Big)\Big) = \exp\big(T\nu(O)\varepsilon^{-1}(e^{\sqrt{-1}\,\xi}-1)\big), \tag{6.20}$$

- for any $0 \le t_1 < t_2 \le T$,

$$\mathbb{E}^{\mathbb{Q}}\Big(\mathscr{M}_{t_2}^{\varepsilon}\Big(\frac{1}{\psi_m}\Big)\,\Big|\,\mathscr{G}_{t_1}\Big) = \mathscr{M}_{t_1}^{\varepsilon}\Big(\frac{1}{\psi_m}\Big), \quad \mathbb{Q}\text{-a.s.} \tag{6.21}$$

In order to prove our results, we first prove that there exists a subsequence, for simplicity still denoted by m, such that

$$\lim_{m\to\infty}\mathbb{E}^{\mathbb{Q}}\Big(\Big|\mathscr{M}_t^{\varepsilon}\Big(\frac{1}{\psi_m}\Big) - \mathscr{M}_t^{\varepsilon}(\varphi_\varepsilon)\Big|\Big) = 0, \tag{6.22}$$

and, for any $O \in \mathscr{B}(\mathrm{Z})$ satisfying $\nu(O) < \infty$,

$$\lim_{m\to\infty}\mathbb{E}^{\mathbb{Q}}\Big(\sup_{t\in[0,T]}|N^{\varepsilon^{-1}\psi_m}(t,O) - N^{\varepsilon^{-1}\varphi_\varepsilon}(t,O)|\Big) = 0. \tag{6.23}$$

We have divided the argument into four parts. Recall that $\bar{\mathbb{A}}_{b,n}$ was defined in (6.1).

Part 1. For any $\psi \in \bar{\mathbb{A}}_{b,n}$, we have

$$\int_0^T |\psi(s,z)|\,ds \le nT \quad \text{for } (z,\omega) \in \mathrm{Z}\times\bar{\Omega}, \tag{6.24}$$

and

$$\begin{aligned}
&\Big|\int_{(0,T]\times\mathrm{Z}\times[0,\varepsilon^{-1}\psi(s,z)]}\log\Big(\frac{1}{\psi(s,z)}\Big)\mathrm{N}(ds,dz,dr) \\
&\qquad + \int_{(0,T]\times\mathrm{Z}\times[0,\varepsilon^{-1}\psi(s,z)]}\Big(-\frac{1}{\psi(s,z)}+1\Big)\nu(dz)\,ds\,dr\Big| \\
&= \Big|\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}\psi(s,z)]}\log\Big(\frac{1}{\psi(s,z)}\Big)\mathrm{N}(ds,dz,dr) \\
&\qquad + \int_{(0,T]\times K_n\times[0,\varepsilon^{-1}\psi(s,z)]}\Big(-\frac{1}{\psi(s,z)}+1\Big)\nu(dz)\,ds\,dr\Big| \\
&\le \int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]}\log n\,\mathrm{N}(ds,dz,dr) + \int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]}(n+1)\,\nu(dz)\,ds\,dr.
\end{aligned} \tag{6.25}$$

It is easy to see that, since $\nu(K_n) < \infty$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big(\exp\Big(\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]}\log n\,\mathrm{N}(ds,dz,dr) \\
&\qquad + \int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]}(n+1)\,\nu(dz)\,ds\,dr\Big)\Big) < \infty.
\end{aligned} \tag{6.26}$$

By inequality (6.24), we have

$$\int_0^T |\psi_m(s,z)-\varphi_\varepsilon(s,z)|\,ds \le 2nT, \quad (z,\omega)\in \mathrm{Z}\times\bar{\Omega}. \tag{6.27}$$

Part 2. The following inequality holds:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big(\sup_{t\in[0,T]}|N^{\varepsilon^{-1}\psi_m}(t,O)-N^{\varepsilon^{-1}\varphi_\varepsilon}(t,O)|\Big)\\
&= \mathbb{E}^{\mathbb{Q}}\Big(\sup_{t\in[0,T]}\Big|\int_0^t\int_O\int_0^{\varepsilon^{-1}n}\big(\mathbb{1}_{[0,\varepsilon^{-1}\psi_m(s,z)]}(r)-\mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)\big)\,\mathrm{N}(ds,dz,dr)\Big|\Big)\\
&\le \mathbb{E}^{\mathbb{Q}}\Big(\int_0^T\int_O\int_0^{\varepsilon^{-1}n}|\mathbb{1}_{[0,\varepsilon^{-1}\psi_m(s,z)]}(r)-\mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)|\,\mathrm{N}(ds,dz,dr)\Big)\\
&= \mathbb{E}^{\mathbb{Q}}\Big(\int_0^T\int_{O\cap K_n}\int_0^{\varepsilon^{-1}n}|\mathbb{1}_{[0,\varepsilon^{-1}\psi_m(s,z)]}(r)-\mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)|\,dr\,\nu(dz)\,ds\Big)\\
&\le \mathbb{E}^{\mathbb{Q}}\Big(\int_0^T\int_{O\cap K_n}\varepsilon^{-1}|\psi_m(s,z)-\varphi_\varepsilon(s,z)|\,\nu(dz)\,ds\Big).
\end{aligned} \tag{6.28}$$

Part 3. The following inequality holds:

$$\begin{aligned}
&\Big|\int_{(0,T]\times\mathrm{Z}\times[0,\varepsilon^{-1}\psi_m(s,z)]}\Big(-\frac{1}{\psi_m(s,z)}+1\Big)\nu(dz)\,ds\,dr\\
&\qquad\qquad -\int_{(0,T]\times\mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}\Big(-\frac{1}{\varphi_\varepsilon(s,z)}+1\Big)\nu(dz)\,ds\,dr\Big|\\
&=\Big|\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}\psi_m(s,z)]}\Big(-\frac{1}{\psi_m(s,z)}+1\Big)\nu(dz)\,ds\,dr\\
&\qquad\qquad -\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}\Big(-\frac{1}{\varphi_\varepsilon(s,z)}+1\Big)\nu(dz)\,ds\,dr\Big|\\
&=\Big|\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]}\mathbb{1}_{[0,\varepsilon^{-1}\psi_m(s,z)]}(r)\Big(-\frac{1}{\psi_m(s,z)}+1\Big)\\
&\qquad\qquad -\mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)\Big(-\frac{1}{\varphi_\varepsilon(s,z)}+1\Big)\nu(dz)\,ds\,dr\Big|\\
&\le \int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]}|\mathbb{1}_{[0,\varepsilon^{-1}\psi_m(s,z)]}(r)-\mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)|\,\nu(dz)\,ds\,dr\\
&\quad+\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]}|\mathbb{1}_{[0,\varepsilon^{-1}\psi_m(s,z)]}(r)-\mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)|\frac{1}{\psi_m(s,z)}\,\nu(dz)\,ds\,dr\\
&\quad+\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]}\mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)\Big|\frac{1}{\psi_m(s,z)}-\frac{1}{\varphi_\varepsilon(s,z)}\Big|\,\nu(dz)\,ds\,dr\\
&\le \varepsilon^{-1}(1+n+n^3)\int_{(0,T]\times K_n}|\psi_m(s,z)-\varphi_\varepsilon(s,z)|\,\nu(dz)\,ds.
\end{aligned} \tag{6.29}$$

Part 4. Using arguments similar to those used in Parts 2 and 3, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\bigg(\sup_{t\in[0,T]}\bigg|\int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\psi_m(s,z)]}\log\Big(\frac{1}{\psi_m(s,z)}\Big)\,\mathrm{N}(ds,dz,dr)\\
&\qquad\qquad -\int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}\log\Big(\frac{1}{\varphi_\varepsilon(s,z)}\Big)\,\mathrm{N}(ds,dz,dr)\bigg|\bigg)\\
&=\mathbb{E}^{\mathbb{Q}}\bigg(\sup_{t\in[0,T]}\bigg|\int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}n]}\mathbb{1}_{[0,\varepsilon^{-1}\psi_m(s,z)]}(r)\log\Big(\frac{1}{\psi_m(s,z)}\Big)\\
&\qquad\qquad -\mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)\log\Big(\frac{1}{\varphi_\varepsilon(s,z)}\Big)\,\mathrm{N}(ds,dz,dr)\bigg|\bigg)\\
&\le\mathbb{E}^{\mathbb{Q}}\bigg(\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]}\bigg|\mathbb{1}_{[0,\varepsilon^{-1}\psi_m(s,z)]}(r)\log\Big(\frac{1}{\psi_m(s,z)}\Big)\\
&\qquad\qquad -\mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)\log\Big(\frac{1}{\varphi_\varepsilon(s,z)}\Big)\bigg|\,\mathrm{N}(ds,dz,dr)\bigg)\\
&=\mathbb{E}^{\mathbb{Q}}\bigg(\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]}\bigg|\mathbb{1}_{[0,\varepsilon^{-1}\psi_m(s,z)]}(r)\log\Big(\frac{1}{\psi_m(s,z)}\Big)\\
&\qquad\qquad -\mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}(r)\log\Big(\frac{1}{\varphi_\varepsilon(s,z)}\Big)\bigg|\,\nu(dz)\,ds\,dr\bigg)\\
&\le\varepsilon^{-1}C_n\mathbb{E}^{\mathbb{Q}}\bigg(\int_{(0,T]\times K_n}|\psi_m(s,z)-\varphi_\varepsilon(s,z)|\,\nu(dz)\,ds\bigg).
\end{aligned}\tag{6.30}$$

Keeping in mind $\nu(K_n)<\infty$, we combine assertion **(R2)** from Lemma 6.1, the estimates in (6.27), and Parts 2, 3 and 4. Doing so, and applying the Lebesgue dominated convergence theorem (DCT), we get equality (6.23), as well as

$$\begin{aligned}
&\lim_{m\to\infty}\mathbb{E}^{\mathbb{Q}}\bigg(\sup_{t\in[0,T]}\bigg|\int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\psi_m(s,z)]}\log\Big(\frac{1}{\psi_m(s,z)}\Big)\,\mathrm{N}(ds,dz,dr)\\
&\qquad -\int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}\log\Big(\frac{1}{\varphi_\varepsilon(s,z)}\Big)\,\mathrm{N}(ds,dz,dr)\bigg|\bigg)=0,
\end{aligned}\tag{6.31}$$

and, $\mathbb{Q}$-a.s.,

$$\begin{aligned}
&\lim_{m\to\infty}\bigg|\int_{(0,T]\times \mathrm{Z}\times[0,\varepsilon^{-1}\psi_m(s,z)]}\Big(-\frac{1}{\psi_m(s,z)}+1\Big)\,\nu(dz)\,ds\,dr\\
&\qquad -\int_{(0,T]\times \mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}\Big(-\frac{1}{\varphi_\varepsilon(s,z)}+1\Big)\,\nu(dz)\,ds\,dr\bigg|=0.
\end{aligned}\tag{6.32}$$

Let us observe that in view of (6.31), there exists a subsequence, for simplicity still denoted by m, such that $\mathbb{Q}$-a.s.,

$$\begin{aligned}
&\lim_{m\to\infty}\bigg(\sup_{t\in[0,T]}\bigg|\int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\psi_m(s,z)]}\log\Big(\frac{1}{\psi_m(s,z)}\Big)\,\mathrm{N}(ds,dz,dr)\\
&\qquad -\int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}\log\Big(\frac{1}{\varphi_\varepsilon(s,z)}\Big)\,\mathrm{N}(ds,dz,dr)\bigg|\bigg)=0.
\end{aligned}\tag{6.33}$$

Combining (6.32), (6.33), (6.25), (6.26), (6.16), and the definition of $\mathscr{M}_T^\varepsilon(\cdot)$, and again employing the Lebesgue DCT, we infer that (6.22) holds.

Having proved (6.22) and (6.23), we are now in a position to prove (S1) and (S4).

Since (S1) can be immediately obtained from (6.22) and (6.21), we now prove (S4).

By (6.23), there exists a subsequence, for simplicity still denoted by m, such that

$$\lim_{m\to\infty} N^{\varepsilon^{-1}\psi_m}(t,O) = N^{\varepsilon^{-1}\varphi_\varepsilon}(t,O), \quad \mathbb{Q}\text{-a.s.}$$

Combining this result with (6.22) and Part 1, using the Lebesgue DCT again, we have

$$\begin{aligned}
&\Big|\mathbb{E}^{\mathbb{Q}}\Big(\exp\big(\sqrt{-1}\,\xi N^{\varepsilon^{-1}\psi_m}(T,O)\big)\mathscr{M}_T^\varepsilon\Big(\frac{1}{\psi_m}\Big)\Big) \\
&\qquad\qquad\qquad - \mathbb{E}^{\mathbb{Q}}\big(\exp\big(\sqrt{-1}\,\xi N^{\varepsilon^{-1}\varphi_\varepsilon}(T,O)\big)\mathscr{M}_T^\varepsilon(\varphi_\varepsilon)\big)\Big| \\
&\le \mathbb{E}^{\mathbb{Q}}\Big(\Big|\mathscr{M}_T^\varepsilon\Big(\frac{1}{\psi_m}\Big) - \mathscr{M}_T^\varepsilon(\varphi_\varepsilon)\Big|\Big) \\
&\quad + \mathbb{E}^{\mathbb{Q}}\big(\big|\exp\big(\sqrt{-1}\,\xi N^{\varepsilon^{-1}\varphi_\varepsilon}(T,O)\big) - \exp\big(\sqrt{-1}\,\xi N^{\varepsilon^{-1}\psi_m}(T,O)\big)\big|\mathscr{M}_T^\varepsilon(\varphi_\varepsilon)\big) \\
&\le \mathbb{E}^{\mathbb{Q}}\Big(\Big|\mathscr{M}_T^\varepsilon\Big(\frac{1}{\psi_m}\Big) - \mathscr{M}_T^\varepsilon(\varphi_\varepsilon)\Big|\Big) \\
&\quad + \mathbb{E}^{\mathbb{Q}}\Big(\big|\exp\big(\sqrt{-1}\,\xi N^{\varepsilon^{-1}\varphi_\varepsilon}(T,O)\big) - \exp\big(\sqrt{-1}\,\xi N^{\varepsilon^{-1}\psi_m}(T,O)\big)\big| \\
&\cdot \exp\Big(\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]} \log n\, \mathrm{N}(ds,dz,dr) + \int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]} (n+1)\,\nu(dz)\,ds\,dr\Big)\Big).
\end{aligned}$$

Since RHS $\to 0$ as $m\to\infty$, we infer, by recalling (6.20), that

$$\mathbb{E}^{\mathbb{Q}}\big(\exp\big(\sqrt{-1}\,\xi N^{\varepsilon^{-1}\varphi_\varepsilon}(T,O)\big)\mathscr{M}_T^\varepsilon(\varphi_\varepsilon)\big) = \exp\big(T\nu(O)\varepsilon^{-1}(e^{\sqrt{-1}\,\xi}-1)\big)$$

for any $O\in\mathscr{B}(\mathrm{Z})$ such that $\nu(O)<\infty$, which implies assertion (S4). The proof of Step 2 is now complete. ∎

Step 3. Proof of (S3). We observe that, by (6.16) and arguments similar to those for (6.25), $\mathbb{Q}$-a.s.,

$$\begin{aligned}
\exp\Big(&\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]} -\log n\, \mathrm{N}(ds,dz,dr) \\
&\qquad + \int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]} [-(n+1)]\,\nu(dz)\,ds\,dr\Big) \le \mathscr{M}_T^\varepsilon(\psi_\varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
\exp\Big(&\int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]} \log n\, \mathrm{N}(ds,dz,dr) \\
&\qquad + \int_{(0,T]\times K_n\times[0,\varepsilon^{-1}n]} (n+1)\,\nu(dz)\,ds\,dr\Big) \ge \mathscr{M}_T^\varepsilon(\psi_\varepsilon).
\end{aligned}$$

Using the facts that $\nu(K_n)<\infty$ and that $\int_{(0,t]\times K_n\times[0,\varepsilon^{-1}n]} \mathrm{N}(ds,dz,dr)$ only has finite jumps on $[0,T]$, $\mathbb{Q}$-a.s., we conclude that the probability measures $\mathbb{Q}$ and $\mathbb{P}_T^\varepsilon$ are equivalent, which is (S3). This completes the proof of Step 3, and the whole proof of Theorem 6.1. ∎

7. Verification of Claim LDP-2

The main result of this section is Proposition 7.1, in which we prove Claim LDP-2. To this end, we first prove that the process $X^\varepsilon := \mathscr{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ is the unique solution of the controlled SPDE (7.1), given in Lemma 7.1. Then to prove Claim LDP-2, we only need to prove some a priori estimates and establish the tightness of the laws of the processes X^ε, $\varepsilon > 0$, which we do in Lemmata 7.2–7.6. The key to proving Lemma 7.1 is a Girsanov-type theorem for Poisson random measures, which is formulated within Theorem 6.1.

This section is divided into two subsections. In the second one, we formulate and prove Proposition 7.1 from which Claim LDP-2 follows. In the first subsection, we prove Lemma 7.1 and find necessary estimates.

7.1. A representation result and a priori estimates

Fix $u_0 \in \mathrm{V}$ and $f \in L^2([0,T],\mathrm{H})$. Assume that the control φ_ε belongs to the set $\mathscr{U}$ (see (5.19)). Let us consider the following controlled SPDE:

$$\begin{aligned} &dX^\varepsilon(t) + \mathrm{A}X^\varepsilon(t)dt + \mathrm{B}(X^\varepsilon(t))\,dt \\ &\quad = f(t)\,dt + \varepsilon\int_{\mathrm{Z}} G(X^\varepsilon(t-),z)\,\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,dt) - \varepsilon^{-1}\,\nu(dz)\,dt\big), \qquad (7.1) \\ &\quad = f(t)\,dt + \int_{\mathrm{Z}} G(X^\varepsilon(t),z)(\varphi_\varepsilon(t,z)-1)\,\nu(dz)\,dt \\ &\qquad + \varepsilon\int_{\mathrm{Z}} G(X^\varepsilon(t-),z)\,\tilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz,dt), \qquad (7.2) \\ &X^\varepsilon(0) = u_0. \end{aligned}$$

Note that below, in (7.9) for example, we use the second version of the above equation, i.e., (7.2). We also observe that it is easy to see that the integral $\varepsilon\int_0^t\int_{\mathrm{Z}} G(X^\varepsilon(s-),z)\,(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds) - \varepsilon^{-1}\,\nu(dz)\,ds)$ exists.

Recall the definition of $\mathscr{G}^\varepsilon$ in the proof of Theorem 5.1 (around (5.15)). By Corollary 5.1 the process

$$u^\varepsilon = \mathscr{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}}) \qquad (7.3)$$

is the unique solution of problem (5.6) on the probability space $(\bar\Omega,\mathscr{G},\mathbb{G},\mathbb{Q})$.

We prove the following fundamental result.

Lemma 7.1. *Assuming $\varepsilon > 0$, for every process $\varphi_\varepsilon \in \bar{\mathbb{A}}_b$ defined on $(\bar\Omega,\mathscr{G},\mathbb{G},\mathbb{Q})$, the process X^ε defined by*

$$X^\varepsilon = \mathscr{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}) \qquad (7.4)$$

is the unique solution of (7.1).

Proof. Fix $\varepsilon > 0$ and a process $\varphi_\varepsilon \in \bar{\mathbb{A}}_b$ defined on $(\bar\Omega,\mathscr{G},\mathbb{G},\mathbb{Q})$. Define X^ε by (7.4). Then by assertion (S4) in Theorem 6.1 and the definition of $\mathscr{G}^\varepsilon$, we infer that the process X^ε is the unique solution of (7.1) on $(\bar\Omega,\mathscr{G},\mathbb{G},\mathbb{P}^\varepsilon_T)$, that is, (C1) X^ε is $\mathbb{G}$-progressively measurable,

(C2) trajectories of X^ε belong to $\Upsilon_T(\mathrm{V})$, $\mathbb{P}^\varepsilon_T$-a.s.,

(C3) in H, for all $t\in[0,T]$, $\mathbb{P}^\varepsilon_T$-a.s.,

$$X^\varepsilon(t)=u_0-\int_0^t \mathrm{A}X^\varepsilon(s)\,ds-\int_0^t \mathrm{B}(X^\varepsilon(s))\,ds+\int_0^t f(s)\,ds \\ +\varepsilon\int_0^t\int_{\mathrm{Z}} G(X^\varepsilon(s-),z)\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds)-\varepsilon^{-1}\,\nu(dz)\,ds\big). \tag{7.5}$$

Next, we prove that X^ε is the unique solution of (7.1) on $(\bar{\Omega},\mathscr{G},\mathbb{G},\mathbb{Q})$; that is,

(C1-0) X^ε is $\mathbb{G}$-progressively measurable,

(C2-0) trajectories of X^ε belong to $\Upsilon_T(\mathrm{V})$, $\mathbb{Q}$-a.s.,

(C3-0) in H, for all $t\in[0,T]$, $\mathbb{Q}$-a.s.,

$$X^\varepsilon(t)=u_0-\int_0^t \mathrm{A}X^\varepsilon(s)\,ds-\int_0^t \mathrm{B}(X^\varepsilon(s))\,ds+\int_0^t f(s)\,ds \\ +\varepsilon\int_0^t\int_{\mathrm{Z}} G(X^\varepsilon(s-),z)\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds)-\varepsilon^{-1}\,\nu(dz)\,ds\big). \tag{7.6}$$

Note that despite the two measures $\mathbb{Q}$ and $\mathbb{P}^\varepsilon_T$ being equivalent, equality (7.6) does not follow from (7.5) without additional justification. We provide this justification below. The proof is divided into two steps.

Step 1. We prove that the process X^ε satisfies (C1-0)–(C3-0). Obviously, condition (C1) coincides with (C1-0). In view of (S3), the measures $\mathbb{Q}$ and $\mathbb{P}^\varepsilon_T$ are equivalent, so condition (C2) implies (C2-0).

We are now in a position to prove that condition (C3-0) holds as well. Fix n (in view of the definition (5.18) of $\bar{\mathbb{A}}_b$). Observe that equality (7.1) can be rewritten as

$$\begin{aligned} &dX^\varepsilon(t)+\mathrm{A}X^\varepsilon(t)\,dt+\mathrm{B}(X^\varepsilon(t))\,dt\\ &\qquad=f(t)\,dt+\varepsilon\int_{K_n} G(X^\varepsilon(t-),z)\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,dt)-\varepsilon^{-1}\,\nu(dz)\,dt\big)\\ &\qquad\quad+\varepsilon\int_{K_n^c} G(X^\varepsilon(t-),z)\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,dt)-\varepsilon^{-1}\,\nu(dz)\,dt\big)\\ &\qquad=f(t)\,dt+\varepsilon\int_{K_n} G(X^\varepsilon(t-),z)\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,dt)-\varepsilon^{-1}\,\nu(dz)\,dt\big)\\ &\qquad\quad+\varepsilon\int_{K_n^c} G(X^\varepsilon(t-),z)\big(N^{\varepsilon^{-1}}(dz,dt)-\varepsilon^{-1}\,\nu(dz)\,dt\big),\\ &X^\varepsilon_0=u_0. \end{aligned}$$

The second equality follows from Proposition 5.2 because $\varphi_\varepsilon(s,z,\omega)=1$ if $(s,z,\omega)\in[0,T]\times K_n^c\times\bar{\Omega}$.

Thus, we have the following:

(V1) Since $\nu(K_n) < \infty$, by (S4) there exists $\Omega_1 \subset \bar{\Omega}$ with $\mathbb{P}^\varepsilon_T(\Omega_1) = 1$ such that for any $\omega \in \Omega_1$, the process $\{N^{\varepsilon^{-1}\varphi_\varepsilon}((0,t] \times K_n),\ t \in [0,T]\}$ has only finite jumps. Hence for any $\omega \in \Omega_1$, the integrals

$$\int_0^t \int_{K_n} G(X^\varepsilon(s-), z)\, N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) \quad \text{and} \quad \int_0^t \int_{K_n} G(X^\varepsilon(s-), z)\, \nu(dz)\, ds$$

are well-defined as Lebesgue–Stieltjes integrals.

(V2) Similarly, for any $m > n$, the integrals

$$\int_0^t \int_{K_n^c \cap K_m} G(X^\varepsilon(s-), z)\, N^{\varepsilon^{-1}}(dz, ds) \text{ and } \int_0^t \int_{K_n^c \cap K_m} G(X^\varepsilon(s-), z)\, \nu(dz)\, ds$$

are well-defined as Lebesgue–Stieltjes integrals, $\mathbb{P}^\varepsilon_T$-a.s.

(V3) By [42, Chapter II, Section 3, pp. 59–63] and by the definition of the integral

$$\int_0^t \int_{K_n^c} G(X^\varepsilon(s-), z) \left(N^{\varepsilon^{-1}}(dz, ds) - \varepsilon^{-1}\, \nu(dz)\, ds\right)$$

on the probability space $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{P}^\varepsilon_T)$, there exist $\Omega_2 \subset \bar{\Omega}$ with $\mathbb{P}^\varepsilon_T(\Omega_2) = 1$ and a subsequence $\{m_k\}$ such that for any $\omega \in \Omega_2$,

$$\lim_{m_k \to \infty} \int_0^t \int_{K_n^c \cap K_{m_k}} G(X^\varepsilon(s-), z) \left(N^{\varepsilon^{-1}}(dz, ds) - \varepsilon^{-1}\, \nu(dz)\, ds\right)$$
$$= \int_0^t \int_{K_n^c} G(X^\varepsilon(s-), z) \left(N^{\varepsilon^{-1}}(dz, ds) - \varepsilon^{-1}\, \nu(dz)\, ds\right).$$

Arguing as in the proof of assertions (V1)–(V3), and using the equality

$$\begin{aligned} N^{\varepsilon^{-1}\varphi_\varepsilon}((0,t] \times K_n) &= \int_0^t \int_{K_n} \int_0^\infty \mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(z,s)]}(r)\, \mathrm{N}(dz, ds, dr) \\ &= \int_0^t \int_{K_n} \int_0^{\varepsilon^{-1} n} \mathbb{1}_{[0,\varepsilon^{-1}\varphi_\varepsilon(z,s)]}(r)\, \mathrm{N}(dz, ds, dr), \end{aligned}$$

we infer three facts:

(V1-0) There exists $\Omega_3 \subset \bar{\Omega}$ with $\mathbb{Q}(\Omega_3) = 1$ such that $\{N^{\varepsilon^{-1}\varphi_\varepsilon}((0,t] \times K_n),\ t \in [0,T]\}$ has only finite jumps for any $\omega \in \Omega_3$. Hence for any $\omega \in \Omega_3$, $\int_0^t \int_{K_n} G(X^\varepsilon(s-), z)\, N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds)$ and $\int_0^t \int_{K_n} G(X^\varepsilon(s-), z)\, \nu(dz)\, ds$ are well-defined as Lebesgue–Stieltjes integrals.

(V2-0) For any $m > n$, the integrals $\int_0^t \int_{K_n^c \cap K_m} G(X^\varepsilon(s-), z)\, N^{\varepsilon^{-1}}(dz, ds)$ and $\int_0^t \int_{K_n^c \cap K_m} G(X^\varepsilon(s-), z)\, \nu(dz)\, ds$ are well-defined $\mathbb{Q}$-a.s. as Lebesgue–Stieltjes integrals.

(V3-0) By the definition of

$$\int_0^t \int_{K_n^c} G(X^\varepsilon(s-), z) \left(N^{\varepsilon^{-1}}(dz, ds) - \varepsilon^{-1}\, \nu(dz)\, ds\right)$$

on $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{Q})$, there exist $\Omega_4 \subset \bar{\Omega}$ with $\mathbb{Q}(\Omega_4) = 1$ and a subsequence of $\{m_k\}$ from (V3) (for simplicity still denoted by m_k) such that for any $\omega \in \Omega_4$,

$$\lim_{m_k \to \infty} \int_0^t \int_{K_n^c \cap K_{m_k}} G(X^\varepsilon(s-), z) \big(N^{\varepsilon^{-1}}(dz, ds) - \varepsilon^{-1} \nu(dz)\, ds\big) \\ = \int_0^t \int_{K_n^c} G(X^\varepsilon(s-), z) \big(N^{\varepsilon^{-1}}(dz, ds) - \varepsilon^{-1} \nu(dz)\, ds\big).$$

By (C3) and (V1)–(V3), there exists $\Omega_5 \subset \bar{\Omega}$ such that $\mathbb{P}_T^\varepsilon(\Omega_5) = 1$ and, for any $\omega \in \Omega_5$,

$$-\lim_{m_k \to \infty} \int_0^t \int_{K_n^c \cap K_{m_k}} G(X^\varepsilon(s-), z) \big(N^{\varepsilon^{-1}}(dz, ds) - \varepsilon^{-1} \nu(dz)\, ds\big) \\ = X^\varepsilon(t) - u_0 + \int_0^t \mathrm{A} X^\varepsilon(s)\, ds + \int_0^t \mathrm{B}(X^\varepsilon(s))\, ds - \int_0^t f(s)\, ds \\ - \varepsilon \int_0^t \int_{K_n} G(X^\varepsilon(s-), z) \big(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) - \varepsilon^{-1} \nu(dz)\, ds\big) \quad \text{in H}.$$

Since, by assertion (S3) in Theorem 6.1, the measures $\mathbb{Q}$ and $\mathbb{P}_T^\varepsilon$ are equivalent, we deduce that $\mathbb{Q}(\bigcap_{i=1}^5 \Omega_i) = 1$. Moreover, since by (V1) and (V1-0) the right side of the above equality is pathwise well-defined for $\omega \in \Omega_1 \cap \Omega_3$, we infer that for any $\omega \in \bigcap_{i=1}^5 \Omega_i$,

$$-\lim_{m_k \to \infty} \int_0^t \int_{K_n^c \cap K_{m_k}} G(X^\varepsilon(s-), z) \big(N^{\varepsilon^{-1}}(dz, ds) - \varepsilon^{-1} \nu(dz)\, ds\big) \\ = X^\varepsilon(t) - u_0 + \int_0^t \mathrm{A} X^\varepsilon(s)\, ds + \int_0^t \mathrm{B}(X^\varepsilon(s))\, ds - \int_0^t f(s)\, ds \\ - \varepsilon \int_0^t \int_{K_n} G(X^\varepsilon(s-), z) \big(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) - \varepsilon^{-1} \nu(dz)\, ds\big) \quad \text{in H}.$$

Combining this last equality with (V3-0) completes the proof of claim (C3-0) and of Step 1.

Step 2. We prove that the solution of (7.1) on $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{Q})$ is unique. Assume that Y^ε is another solution of (7.1) on $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{Q})$, that is, (C1-0)–(C3-0) are satisfied with X^ε replaced by Y^ε. By arguments similar to those for Step 1, Y^ε is a solution of (7.1) on $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{P}_T^\varepsilon)$, and the uniqueness of solution to (7.1) on $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{P}_T^\varepsilon)$ implies that $Y^\varepsilon = X^\varepsilon$ $\mathbb{P}_T^\varepsilon$-a.s. Since the measures $\mathbb{Q}$ and $\mathbb{P}_T^\varepsilon$ are equivalent,

$$Y^\varepsilon = X^\varepsilon, \quad \mathbb{Q}\text{-a.s.}$$

Thus the proof of Lemma 7.1 is complete. ■

Now we give some a priori estimates to be used later. For simplicity, $\mathbb{E}^{\mathbb{Q}}$ will be denoted by $\mathbb{E}$. Recall that the norm $\|\cdot\|^2_{W^{\alpha,2}([0,T],\mathrm{V}')}$ was introduced in equality (5.20).

Lemma 7.2. *For every $N \in \mathbb{N}$, there exist constants $C_N > 0$ and $\varepsilon_N \in (0, 1]$, and for every $\alpha \in (0, 1/2)$ there exists a constant $C_{\alpha,N} > 0$ such that for every process $\varphi_\varepsilon \in \mathscr{U}^N$ and every $\varepsilon \in (0, \varepsilon_N]$, the process X^ε defined by (7.4) satisfies*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|X^\varepsilon(t)\|_{\mathrm{H}}^2 + \int_0^T \|X^\varepsilon(t)\|_{\mathrm{V}}^2\, dt\Big) \le C_N, \tag{7.7}$$

$$\mathbb{E}(\|X^\varepsilon\|^2_{W^{\alpha,2}([0,T],\mathrm{V}')}) \le C_{\alpha,N}. \tag{7.8}$$

Proof. Fix $\varepsilon > 0$, $N \in \mathbb{N}$, and $\varphi_\varepsilon \in \mathscr{U}^N$. Let X^ε be defined by (7.4). By Lemma 7.1, this process is the unique solution of (7.1) on $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{Q})$.

Therefore, we can apply the Itô formula to deduce that

$$\begin{aligned}
&|X^\varepsilon(t)|_{\mathrm{H}}^2 + 2\int_0^t \|X^\varepsilon(s)\|_{\mathrm{V}}^2\, ds \\
&\qquad = |u_0|_{\mathrm{H}}^2 + 2\int_0^t {}_{\mathrm{V}'}\langle f(s), X^\varepsilon(s)\rangle_{\mathrm{V}}\, ds \\
&\qquad\quad + 2\int_0^t \int_{\mathrm{Z}} \langle G(X^\varepsilon(s), z), X^\varepsilon(s)\rangle_{\mathrm{H}} (\varphi_\varepsilon(s,z) - 1)\, \nu(dz)\, ds \\
&\qquad\quad + 2\varepsilon \int_0^t \int_{\mathrm{Z}} \langle G(X^\varepsilon(s-), z), X^\varepsilon(s-)\rangle_{\mathrm{H}}\, \widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) \\
&\qquad\quad + \varepsilon^2 \int_0^t \int_{\mathrm{Z}} |G(X^\varepsilon(s-), z)|_{\mathrm{H}}^2\, N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) \\
&\qquad \le |u_0|_{\mathrm{H}}^2 + \int_0^t \|X^\varepsilon(s)\|_{\mathrm{V}}^2\, ds + \int_0^t \|f(s)\|_{\mathrm{V}'}^2\, ds \\
&\qquad\quad + 2\int_0^t (1 + 2|X^\varepsilon(s)|_{\mathrm{H}}^2) \int_{\mathrm{Z}} L_3(z)|\varphi_\varepsilon(s,z) - 1|\, \nu(dz)\, ds \\
&\qquad\quad + 2\varepsilon \int_0^t \int_{\mathrm{Z}} \langle G(X^\varepsilon(s-), z), X^\varepsilon(s-)\rangle_{\mathrm{H}}\, \widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) \\
&\qquad\quad + \varepsilon^2 \int_0^t \int_{\mathrm{Z}} |G(X^\varepsilon(s-), z)|_{\mathrm{H}}^2\, N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds).
\end{aligned} \tag{7.9}$$

Set

$$J_1(t) := 2\varepsilon \int_0^t \int_{\mathrm{Z}} \langle G(X^\varepsilon(s-), z), X^\varepsilon(s-)\rangle_{\mathrm{H}}\, \widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds)$$

$$J_2(t) := \varepsilon^2 \int_0^t \int_{\mathrm{Z}} |G(X^\varepsilon(s-), z)|_{\mathrm{H}}^2\, N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds).$$

Applying Gronwall's lemma and (B.2), we get

$$\begin{aligned}
&\sup_{t\in[0,T]} |X^\varepsilon(t)|_{\mathrm{H}}^2 + \int_0^T \|X^\varepsilon(t)\|_{\mathrm{V}}^2\, dt \\
&\qquad \le C_N\Big(|u_0|_{\mathrm{H}}^2 + \int_0^T \|f(s)\|_{\mathrm{V}'}^2\, ds + 1 + \sup_{t\in[0,T]} |J_1(t)| + J_2(T)\Big).
\end{aligned} \tag{7.10}$$

The Burkholder–Davis–Gundy inequality implies that

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{t\in[0,T]}|J_1(t)|\Big) &\le C\varepsilon\mathbb{E}\Big(\int_0^T\int_Z |X^\varepsilon(s-)|_H^2|G(X^\varepsilon(s-),z)|_H^2\, N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds)\Big)^{1/2}\\
&\le \varepsilon^{1/2}\mathbb{E}\Big(\sup_{t\in[0,T]}|X^\varepsilon(t)|_H^2\Big)\\
&\quad + C\varepsilon^{1/2}\mathbb{E}\Big(\int_0^T (1+|X^\varepsilon(s)|_H^2)\int_Z L_3^2(z)\varphi_\varepsilon(s,z)\,\nu(dz)\,ds\Big)\\
&\le C_N\varepsilon^{1/2}\mathbb{E}\Big(\sup_{t\in[0,T]}|X^\varepsilon(t)|_H^2\Big) + C_N\varepsilon^{1/2}.
\end{aligned}\tag{7.11}
$$

To deduce the last inequality above, we use the fact (see [15, Lemma 3.4, (3.3)]) that for any fixed $\Im\in\mathscr{H}\cap L^2(\nu)$ (for the definition of $\mathscr{H}$, see (5.9)),

$$
C_{\Im,N} := \sup_{k\in S^N}\int_0^T\int_Z \Im^2(z)(k(s,z)+1)\,\nu(dz)\,ds<\infty. \tag{7.12}
$$

As in (7.11), we get

$$
\mathbb{E}(|J_2(T)|)\le C_N\varepsilon + C_N\varepsilon\mathbb{E}\Big(\sup_{t\in[0,T]}|X^\varepsilon(t)|_H^2\Big). \tag{7.13}
$$

Substituting (7.11) and (7.13) into (7.10), and then choosing $\varepsilon_N>0$ small enough, we get (7.7). Using the arguments that prove [66, (4.67)], we infer (7.8). This completes the proof of Lemma 7.2. ∎

Let us define a stopping time $\tau_{\varepsilon,M}$ by[2]

$$
\tau_{\varepsilon,M} := \inf\Big\{t\ge 0: \sup_{s\in[0,t]}|X^\varepsilon(s)|_H^2 + \int_0^t \|X^\varepsilon(s)\|_V^2\,ds > M\Big\},\quad M>0. \tag{7.14}
$$

Before we continue with our estimates, let us state the following simple but useful corollary from the previous result and the Chebyshev inequality.

Corollary 7.1. *In the framework above, we have*

$$
\begin{aligned}
\mathbb{Q}(\tau_{\varepsilon,M}<T)&\le C_N/M, && M>0,\\
\mathbb{Q}(\|X^\varepsilon\|^2_{W^{\alpha,2}([0,T],V')}\ge R^2)&\le C_{\alpha,N}/R^2, && R,M>0.
\end{aligned}\tag{7.15}
$$

We have the following estimate.

Lemma 7.3. *For all $N\in\mathbb{N}$ and $M>0$, there exist constants $C_{N,M}>0$ and $\varepsilon_{N,M}\in(0,1]$ such that for every process $\varphi_\varepsilon\in\mathscr{U}^N$ and every $\varepsilon\in(0,\varepsilon_{N,M}]$, the process X^ε defined by (7.4) satisfies*

$$
\sup_{\varepsilon\in(0,\varepsilon_{N,M})}\mathbb{E}\Big(\sup_{t\in[0,T]}\|X^\varepsilon(t\wedge\tau_{\varepsilon,M})\|_V^2 + \int_0^{T\wedge\tau_{\varepsilon,M}}\|X^\varepsilon(s)\|^2_{\mathscr{D}(A)}\,ds\Big)\le C_{N,M}. \tag{7.16}
$$

[2]In fact, this stopping time depends on X^ε so it depends on both ε and φ_ε. Hence, it should be denoted $\tau_{X^\varepsilon,M}$ or $\tau_{\varphi_\varepsilon,\varepsilon,M}$. Since these two are cumbersome, we decided not to use them. In the same vein, X^ε should be denoted by $X^{\varepsilon,\varphi_\varepsilon}$, but we use the simpler notation.

Before we give the proof of Lemma 7.3, we record the following result which is immediate from Lemma 7.3 and the Chebyshev inequality.

Corollary 7.2. *In the framework above, for all $M, R > 0$,*

$$\mathbb{Q}\left(\tau_{\varepsilon,M} \geq T, \sup_{t\in[0,T]} \|X^\varepsilon(t)\|_{\mathrm{V}}^2 + \int_0^T \|X^\varepsilon(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds \geq R^2\right) \leq \frac{C_{N,M}}{R^2}. \tag{7.17}$$

Proof. Notice that $\tau_{\varepsilon,M} \geq T$ iff $T \wedge \tau_{\varepsilon,M} = T$. Thus, if $\tau_{\varepsilon,M} \geq T$ then $t \wedge \tau_{\varepsilon,M} = t$ for all $t \in [0,T]$. ■

Proof of Lemma 7.3. We argue as in [13, proof of Lemma 7.3]. By the Itô formula and Lemma 2.1, we have, for all $t \in [0,T]$,

$$\begin{aligned}
&\|X^\varepsilon(t\wedge\tau_{\varepsilon,M})\|_{\mathrm{V}}^2 + 2\int_0^{t\wedge\tau_{\varepsilon,M}} \|X^\varepsilon(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds \\
&\quad= \|u_0\|_{\mathrm{V}}^2 - 2\int_0^{t\wedge\tau_{\varepsilon,M}} \langle \mathrm{B}(X^\varepsilon(s)), \mathrm{A}X^\varepsilon(s)\rangle_{\mathrm{H}}\,ds + 2\int_0^{t\wedge\tau_{\varepsilon,M}} \langle f(s), \mathrm{A}X^\varepsilon(s)\rangle_{\mathrm{H}}\,ds \\
&\qquad+ 2\int_0^{t\wedge\tau_{\varepsilon,M}}\int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(X^\varepsilon(s),z), X^\varepsilon(s)\rangle_{\mathrm{V}}(\varphi_\varepsilon(s,z)-1)\,\nu(dz)\,ds \\
&\qquad+ 2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(X^\varepsilon(s-),z), X^\varepsilon(s-)\rangle_{\mathrm{V}}\,\widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds) \\
&\qquad+ \varepsilon^2\int_0^{t\wedge\tau_{\varepsilon,M}}\int_{\mathrm{Z}} \|G(X^\varepsilon(s-),z)\|_{\mathrm{V}}^2\,N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds) \\
&\quad\leq \|u_0\|_{\mathrm{V}}^2 + \int_0^{t\wedge\tau_{\varepsilon,M}} \|X^\varepsilon(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds + C\int_0^{t\wedge\tau_{\varepsilon,M}} \|X^\varepsilon(s)\|_{\mathrm{V}}^4|X^\varepsilon(s)|_{\mathrm{H}}^2\,ds \\
&\qquad+ 2\int_0^{t\wedge\tau_{\varepsilon,M}} |f(s)|_{\mathrm{H}}^2\,ds \\
&\qquad+ 2\int_0^{t\wedge\tau_{\varepsilon,M}} (1+2\|X^\varepsilon(s)\|_{\mathrm{V}}^2)\int_{\mathrm{Z}} L_2(z)|\varphi_\varepsilon(s,z)-1|\,\nu(dz)\,ds \\
&\qquad+ 2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(X^\varepsilon(s-),z), X^\varepsilon(s-)\rangle_{\mathrm{V}}\,\widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds) \\
&\qquad+ \varepsilon^2\int_0^{t\wedge\tau_{\varepsilon,M}}\int_{\mathrm{Z}} \|G(X^\varepsilon(s-),z)\|_{\mathrm{V}}^2\,N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds).
\end{aligned} \tag{7.18}$$

We define

$$\begin{aligned}
J_1(t) &:= 2\varepsilon\int_0^t\int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(X^\varepsilon(s-),z), X^\varepsilon(s-)\rangle_{\mathrm{V}}\,\widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds), \\
J_2(t) &:= \varepsilon^2\int_0^t\int_{\mathrm{Z}} \|G(X^\varepsilon(s-),z)\|_{\mathrm{V}}^2\,N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds),
\end{aligned}$$

for $t \in [0,T]$.

By (B.2),

$$\begin{aligned}
&\|X^\varepsilon(t\wedge\tau_{\varepsilon,M})\|_{\mathrm{V}}^2+\int_0^{t\wedge\tau_{\varepsilon,M}}\|X^\varepsilon(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds\\
&\le\|u_0\|_{\mathrm{V}}^2+2\int_0^{t\wedge\tau_{\varepsilon,M}}|f(s)|_{\mathrm{H}}^2\,ds+\sup_{t\in[0,T]}|J_1(t\wedge\tau_{\varepsilon,M})|+J_2(T\wedge\tau_{\varepsilon,M})+C_N\\
&\quad+\int_0^{t\wedge\tau_{\varepsilon,M}}\|X^\varepsilon(s)\|_{\mathrm{V}}^2\Big(C\|X^\varepsilon(s)\|_{\mathrm{V}}^2|X^\varepsilon(s)|_{\mathrm{H}}^2+4\int_{\mathrm{Z}}L_2(z)|\varphi_\varepsilon(s,z)-1|\,\nu(dz)\Big)\,ds
\end{aligned}$$

for $t\in[0,T]$. By (B.2) again and the definition of $\tau_{\varepsilon,M}$, Gronwall's lemma implies that

$$\begin{aligned}
&\sup_{t\in[0,T]}\|X^\varepsilon(t\wedge\tau_{\varepsilon,M})\|_{\mathrm{V}}^2+\int_0^{t\wedge\tau_{\varepsilon,M}}\|X^\varepsilon(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds\\
&\le e^{M^2+C_N}\Big(\|u_0\|_{\mathrm{V}}^2+2\int_0^T|f(s)|_{\mathrm{H}}^2\,ds+\sup_{t\in[0,T]}|J_1(t\wedge\tau_{\varepsilon,M})|+J_2(T\wedge\tau_{\varepsilon,M})+C_N\Big)
\end{aligned}\tag{7.19}$$

for $t\in[0,T]$. Similar to (7.11) and (7.13), we get

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|J_1(t\wedge\tau_{\varepsilon,M})|\Big)\le C_N\varepsilon^{1/2}\mathbb{E}\Big(\sup_{t\in[0,T]}\|X^\varepsilon(t\wedge\tau_{\varepsilon,M})\|_{\mathrm{V}}^2\Big)+C_N\varepsilon^{1/2},\tag{7.20}$$

and

$$\mathbb{E}(|J_2(T\wedge\tau_{\varepsilon,M})|)\le C_N\varepsilon+C_N\varepsilon\mathbb{E}\Big(\sup_{t\in[0,T]}\|X^\varepsilon(t\wedge\tau_{\varepsilon,M})\|_{\mathrm{V}}^2\Big).\tag{7.21}$$

Substituting (7.20) and (7.21) into (7.19), and then choosing $\varepsilon_{N,M}>0$ small enough, we get (7.16). ■

Our next result is a tightness result.

Lemma 7.4. *For every $N\in\mathbb{N}$, for any fixed subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\varepsilon_k\to0$, and for every $\mathscr{U}^N$-valued sequence φ_{ε_k}, the laws of $\{X^{\varepsilon_k}\}_{k\in\mathbb{N}}$ are tight on the Hilbert space $L^2([0,T],\mathrm{V})$.*

Proof. Assume that $N\in\mathbb{N}$. Fix $\eta>0$, and choose $M>0$ such that

$$C_N/M<\eta/2,$$

where C_N is the constant of Lemma 7.2. Let ε_N and $\varepsilon_{N,M}$ be as in Lemmata 7.2 and 7.3.

Without loss of generality, we can assume that $\varepsilon_k\in(0,\varepsilon_N\wedge\varepsilon_{N,M})$ for all $k\in\mathbb{N}$.

Fix an auxiliary number $\alpha\in(0,1/2)$. Since the embedding $\mathscr{D}(\mathrm{A})\subset\mathrm{V}$ is compact, by [35, Theorem 2.1] the embedding

$$\Lambda=L^2([0,T],\mathscr{D}(\mathrm{A}))\cap W^{\alpha,2}([0,T],\mathrm{V}')\hookrightarrow L^2([0,T],\mathrm{V})$$

is also compact. Define

$$\|g\|_{\Lambda}^2 = \int_0^T \|g(t)\|_{\mathscr{D}(\mathrm{A})}^2 \, dt + \|g\|_{W^{\alpha,2}([0,T],\mathrm{V}')}^2, \quad g \in \Lambda.$$

Choose $R > 0$ such that

$$\frac{2C_{N,M} + 2C_{\alpha,N}}{R^2} < \frac{\eta}{2},$$

where the constants $C_{\alpha,N}$ and $C_{N,M}$ are those that appear in Lemmata 7.2 and 7.3.

Since the set

$$K_R = \{g \in \Lambda : \|g\|_{\Lambda} \le R\}$$

is relatively compact in $L^2([0,T],\mathrm{V})$, it is sufficient to show that

$$\mathbb{Q}(X^{\varepsilon_k} \notin K_R) < \eta \quad \text{for all } k.$$

Indeed, by Lemma 7.2 and Corollaries 7.1 and 7.2 , we infer that

$$\begin{aligned}\mathbb{Q}(X^{\varepsilon_k} \notin K_R) &\le \mathbb{Q}(\tau_{\varepsilon_k,M} < T) + \mathbb{Q}((\tau_{\varepsilon_k,M} \ge T) \cap (X^{\varepsilon_k} \notin K_R)) \\ &\le \frac{C_N}{M} + \frac{2C_{N,M} + 2C_{\alpha,N}}{R^2} < \eta.\end{aligned}$$

■

Using arguments similar to those proving [66, Lemma 4.5], we get

Lemma 7.5. *There exists $\varrho > 1$ such that for every $N \in \mathbb{N}$, for any fixed subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\varepsilon_k \to 0$, and for every $\mathscr{U}^N$-valued sequence φ_{ε_k}, the laws of $\{X^{\varepsilon_k}\}_{k\in\mathbb{N}}$ are tight on the Skorokhod space $D([0,T],\mathscr{D}(\mathrm{A}^{-\varrho}))$.*

Next, consider a family φ_ε, $\varepsilon \in (0,1]$, of $\mathscr{U}^N$-valued processes, for some fixed $N \in \mathbb{N}$. For each ε, let Y^ε be the unique solution of the (auxiliary) stochastic Langevin equation

$$Y^\varepsilon(t) = \int_0^t \mathrm{A}Y^\varepsilon(s)\, ds + \varepsilon \int_0^t \int_{\mathrm{Z}} G(X^\varepsilon(s-), z)\, \widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds). \tag{7.22}$$

In this situation, we have the following.

Lemma 7.6. *In the above framework, if $\eta > 0$, then*

$$\lim_{\varepsilon\to 0} \mathbb{Q}\left(\sup_{t\in[0,T]} \|Y^\varepsilon(t)\|_{\mathrm{V}}^2 + \int_0^T \|Y^\varepsilon(s)\|_{\mathscr{D}(\mathrm{A})}^2\, ds \ge \eta \right) = 0. \tag{7.23}$$

Proof. Fix $\eta > 0$. Suppose we have proved that for every $M > 0$,

$$\mathbb{E}\left(\sup_{t\in[0,T\wedge\tau_{\varepsilon,M}]} \|Y^\varepsilon(t)\|_{\mathrm{V}}^2 + \int_0^{T\wedge\tau_{\varepsilon,M}} \|Y^\varepsilon(s)\|_{\mathscr{D}(\mathrm{A})}^2\, ds \right) \le \varepsilon C_N C_{N,M}, \quad \varepsilon \in (0, \varepsilon_{N,M}), \tag{7.24}$$

where the stopping time $\tau_{\varepsilon,M}$ is defined in (7.14) and $C_{N,M}$ and $\varepsilon_{N,M}$ are as in Lemma 7.3. Then we can conclude the proof as follows.

First, we set

$$\Lambda_\eta := \Big\{\omega \in \Omega : \sup_{t\in[0,T]} \|Y^\varepsilon(t)\|_{\mathrm{V}}^2 + \int_0^T \|Y^\varepsilon(s)\|_{\mathscr{D}(\mathrm{A})}^2\, ds \ge \eta\Big\}.$$

Hence, by Lemma 7.2 and (7.24), for all $M > 0$ and $\varepsilon \in (0, \varepsilon_{N,M})$,

$$\begin{aligned}\mathbb{Q}(\Lambda_\eta) &\le \mathbb{Q}(\tau_{\varepsilon,M} < T) + \mathbb{Q}((\tau_{\varepsilon,M} \ge T)\cap \Lambda_\eta)\\ &\le \frac{C_N}{M} + \frac{\varepsilon C_N C_{N,M}}{\eta},\end{aligned}$$

which, by a standard argument, implies (7.23).

Thus, we only have to show inequality (7.24). Let us fix $M > 0$. By the Itô formula,

$$\begin{aligned}\|Y^\varepsilon(t)\|_{\mathrm{V}}^2 + 2\int_0^t \|Y^\varepsilon(s)\|_{\mathscr{D}(\mathrm{A})}^2\, ds\\ &= 2\varepsilon \int_0^t\int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(X^\varepsilon(s-), z), Y^\varepsilon(s-)\rangle_{\mathrm{V}}\, \widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds)\\ &\quad + \varepsilon^2 \int_0^t\int_{\mathrm{Z}} \|G(X^\varepsilon(s-), z)\|_{\mathrm{V}}^2\, N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds).\end{aligned}$$

By Assumption 5.1 and (7.12),

$$\begin{aligned}&\varepsilon^2\mathbb{E}\Big(\int_0^{T\wedge\tau_{\varepsilon,M}}\int_{\mathrm{Z}} \|G(X^\varepsilon(s-), z)\|_{\mathrm{V}}^2\, N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds)\Big)\\ &\quad\le \varepsilon\mathbb{E}\Big(\int_0^{T\wedge\tau_{\varepsilon,M}}\int_{\mathrm{Z}} \|G(X^\varepsilon(s), z)\|_{\mathrm{V}}^2\varphi_\varepsilon(s,z)\,\nu(dz)\,ds\Big)\\ &\quad\le 2\varepsilon\mathbb{E}\Big(\int_0^{T\wedge\tau_{\varepsilon,M}}\int_{\mathrm{Z}} (1 + \|X^\varepsilon(s)\|_{\mathrm{V}}^2)L_2^2(z)\varphi_\varepsilon(s,z)\,\nu(dz)\,ds\Big)\\ &\quad\le \varepsilon C_N\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau_{\varepsilon,M}]} \|X^\varepsilon(t)\|_{\mathrm{V}}^2\Big) + \varepsilon C_N.\end{aligned}$$

Applying the Burkholder–Davis–Gundy inequality and (7.12) again, we get

$$\begin{aligned}&2\varepsilon\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau_{\varepsilon,M}]}\Big|\int_0^t\int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(X^\varepsilon(s-), z), Y^\varepsilon(s-)\rangle_{\mathrm{V}}\, \widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds)\Big|\Big)\\ &\le \varepsilon C\mathbb{E}\Big(\Big|\int_0^{T\wedge\tau_{\varepsilon,M}}\int_{\mathrm{Z}} \|G(X^\varepsilon(s-), z)\|_{\mathrm{V}}^2\|Y^\varepsilon(s-)\|_{\mathrm{V}}^2\, N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds)\Big|^{1/2}\Big)\\ &\le \frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau_{\varepsilon,M}]} \|Y^\varepsilon(t)\|_{\mathrm{V}}^2\Big) + \varepsilon C\mathbb{E}\Big(\int_0^{T\wedge\tau_{\varepsilon,M}}\int_{\mathrm{Z}} \|G(X^\varepsilon(s), z)\|_{\mathrm{V}}^2\varphi_\varepsilon(s,z)\,\nu(dz)\,ds\Big)\\ &\le \frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau_{\varepsilon,M}]} \|Y^\varepsilon(t)\|_{\mathrm{V}}^2\Big) + \varepsilon C_N\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau_{\varepsilon,M}]} \|X^\varepsilon(t)\|_{\mathrm{V}}^2\Big) + \varepsilon C_N.\end{aligned}$$

Combining the above three estimates and Lemma 7.3, we deduce (7.24). ∎

7.2. *The second continuity lemma*

This subsection is devoted to proving Claim LDP-2 formulated in the proof of Theorem 5.1. We need the following result.

Proposition 7.1 (The second continuity lemma). *Let $\varphi_{\varepsilon_n}, \varphi \in \mathscr{U}^N$ be such that φ_{ε_n} converges in law to φ as $\varepsilon_n \to 0$. Then the processes $\mathscr{G}^{\varepsilon_n}(\varepsilon_n N^{\varepsilon_n^{-1}\varphi_{\varepsilon_n}})$ converge in law on $\Upsilon_T(\mathrm{V})$ to the process $\mathscr{G}^0(\varphi)$.*

Proof. Fix a natural number N, a sequence ε_n such that $\varepsilon_n \to 0$, and a $\mathscr{U}^N$-valued sequence $\{\varphi_{\varepsilon_n}\}_{n\in\mathbb{N}}$, such that φ_{ε_n} converges in law to φ for some $\varphi \in \mathscr{U}^N$.

By Lemma 7.1, the process

$$X^{\varepsilon_n} := \mathscr{G}^{\varepsilon_n}(\varepsilon_n N^{\varepsilon_n^{-1}\varphi_{\varepsilon_n}})$$

is the unique solution of problem (7.1) with ε and φ replaced by ε_n and φ_{ε_n} respectively. Recall that the process Y^ε, for $\varepsilon > 0$, was defined in (7.22).

By Lemmata 7.4–7.6,

(1) the laws of the processes $\{X^{\varepsilon_n}\}_{n\in\mathbb{N}}$ are tight on $L^2([0,T],\mathrm{V}) \cap D([0,T],\mathscr{D}(\mathrm{A}^{-\varrho}))$;

(2) the sequence $\{Y^{\varepsilon_n}\}_{n\in\mathbb{N}}$ converges in probability to 0 in $\Upsilon_T(\mathrm{V})$.

Set

$$\Gamma_T = \left[L^2([0,T],\mathrm{V}) \cap D([0,T],\mathscr{D}(\mathrm{A}^{-\varrho}))\right] \otimes \Upsilon_T(\mathrm{V}) \otimes S^N.$$

Let $(X, 0, \varphi)$ be any limit point of the tight family $\{(X^{\varepsilon_n}, Y^{\varepsilon_n}, \varphi_{\varepsilon_n})\}_{n\in\mathbb{N}}$. By the Skorokhod representation theorem, there exists a stochastic basis $(\Omega^1, \mathbb{F}^1, \mathbb{P}^1)$ and, on this basis, Γ_T-valued random variables $(X_1, 0, \varphi_1)$, $(X_1^n, Y_1^n, \varphi_1^n)$, $n \in \mathbb{N}$, such that

(a) $(X_1, 0, \varphi_1)$ has the same law as $(X, 0, \varphi)$;

(b) for any $n \in \mathbb{N}$, $(X_1^n, Y_1^n, \varphi_1^n)$ has the same law as $(X^{\varepsilon_n}, Y^{\varepsilon_n}, \varphi_{\varepsilon_n})$;

(c) $\lim_{n\to\infty}(X_1^n, Y_1^n, \varphi_1^n) = (X_1, 0, \varphi_1)$ in Γ_T, $\mathbb{P}^1$-a.s.

Because equations (7.5) and (7.22) are satisfied by the processes $(X^{\varepsilon_n}, Y^{\varepsilon_n}, \varphi_{\varepsilon_n})$, we infer that $(X_1^n, Y_1^n, \varphi_1^n)$ satisfies

$$\begin{aligned} X_1^n(t) - Y_1^n(t) = u_0 - \int_0^t \mathrm{A}(X_1^n(s) - Y_1^n(s))\, ds \int_0^t \mathrm{B}(X_1^n(s))\, ds + \int_0^t f(s)\, ds \\ + \int_0^t \int_{\mathrm{Z}} G(X_1^n(s), z)(\varphi_1^n(s,z) - 1)\, \nu(dz)\, ds, \quad t \in [0,T]. \end{aligned} \tag{7.25}$$

Hence, by deterministic results and (b), we conclude that

$$\begin{aligned} &\mathbb{P}^1\big(X_1^n - Y_1^n \in C([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A}))\big) \\ &\qquad = \mathbb{Q}\big(X^{\varepsilon_n} - Y^{\varepsilon_n} \in C([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A}))\big) = 1. \end{aligned}$$

Since

$$\lim_{n\to 0}\left(\sup_{t\in[0,T]} \|Y_1^n(t)\|_{\mathrm{V}}^2 + \int_0^T \|Y_1^n(s)\|_{\mathscr{D}(\mathrm{A})}^2\, ds\right) = 0, \quad \mathbb{P}^1\text{-a.s.}, \tag{7.26}$$

by applying arguments similar to the proof of Proposition 5.3, we can show that (X_1, φ_1) satisfies

$$X_1(t) = u_0 - \int_0^t \mathrm{A} X_1(s)\, ds - \int_0^t \mathrm{B}(X_1(s))\, ds + \int_0^t f(s)\, ds + \int_0^t \int_{\mathrm{Z}} G(X_1(s), z)(\varphi_1(s,z) - 1)\, \nu(dz)\, ds.$$

The maximal regularity property of the solutions to the deterministic 2D Navier–Stokes equations, taking into account Lemma B.1, imply that, $\mathbb{P}^1$-a.s.,

$$X_1 \in C([0,T], \mathrm{V}) \cap L^2([0,T], \mathscr{D}(\mathrm{A})).$$

By (7.26) and (c), using arguments as in the proof of (5.36), we get

$$\lim_{n\to 0}\Big(\sup_{t\in[0,T]} \|X_1^n(t) - X_1(t)\|_{\mathrm{V}}^2 + \int_0^T \|X_1^n(s) - X_1(s)\|_{\mathscr{D}(\mathrm{A})}^2\, ds \Big) = 0, \quad \mathbb{P}^1\text{-a.s.}$$

Hence, by (5.16), which is the definition of $\mathscr{G}^0$, X^{ε_n} converges in law to $\mathscr{G}^0(\varphi)$, which implies the desired result. ∎

Appendix A. Poisson random measures

Recall the following definition, which is taken from [42, Definition I.8.1]; see also [10].

Definition A.1. A time-homogeneous Poisson random measure on $\mathrm{Y} = \mathrm{Z} \times [0,\infty)$ (i.e., a Poisson random measure on $\mathrm{Y}_T = [0,T] \times \mathrm{Z} \times [0,\infty)$) over the probability space $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{Q})$ *with intensity measure* $\mathrm{Leb}_{[0,T]} \otimes \nu \otimes \mathrm{Leb}_{[0,\infty)}$ is a measurable function

$$\eta : (\bar{\Omega}, \mathscr{G}) \to \mathscr{M}(\mathrm{Y}_T) = \mathbb{M}_T$$

satisfying the following conditions:

(1) for each $U \in \mathscr{B}([0,T]) \otimes \mathscr{B}(\mathrm{Y})$, $\eta(U) := i_U \circ \eta : \bar{\Omega} \to \overline{\mathbb{N}}$ [3] is a Poisson random variable with parameter[4] $\mathbb{E}\eta(U)$;

(2) η is *independently scattered*, i.e., if the sets $U_j \in \mathscr{B}([0,T]) \otimes \mathscr{B}(\mathrm{Y})$, $j = 1, \dots, n$, are disjoint, then the random variables $\eta(U_j)$, $j = 1, \dots, n$, are mutually independent;

(3) for all $U \in \mathscr{B}(\mathrm{Y})$ and $I \in \mathscr{B}([0,T])$,

$$\mathbb{E}[\eta(I \times U)] = (\mathrm{Leb}_{[0,T]} \otimes \nu \otimes \mathrm{Leb}_{[0,\infty)})(I \times U) = \mathrm{Leb}_{[0,T]}(I)(\nu \otimes \mathrm{Leb}_{[0,\infty)})(U);$$

(4) for each $U \in \mathscr{B}(\mathrm{Y})$, the $\overline{\mathbb{N}}$-valued process

$$(0,\infty) \times \Omega \ni (t,\omega) \mapsto \eta(\omega)(U \times (0,t])$$

is $\mathbb{G}$-adapted, and its increments are independent of the past, i.e., the increment between times t and s with $t > s > 0$ is independent of the σ-field $\mathscr{G}_s$.

[3] $\overline{\mathbb{N}} := \mathbb{N} \cup \{0\} \cup \{\infty\}$.

[4] If $\mathbb{E}\eta(U) = \infty$, then obviously $\eta(U) = \infty$ a.s.

Similarly, we have

Definition A.2. A time-homogeneous Poisson random measure on Z (or Poisson random measure on $Z_T = [0,T] \times Z$) over the probability space $(\bar{\Omega}, \mathscr{G}, \mathbb{G}, \mathbb{Q})$ *with intensity measure* $\mathrm{Leb}_{[0,T]} \otimes \nu$ is a measurable function

$$\eta : (\bar{\Omega}, \mathscr{G}) \to \mathscr{M}(Z_T) = M_T$$

satisfying the following conditions:

(1) for each $U \in \mathscr{B}([0,T]) \otimes \mathscr{B}(Z)$, $\eta(U) := i_U \circ \eta : \bar{\Omega} \to \overline{\mathbb{N}}$ is a Poisson random variable with parameter $\mathbb{E}\eta(U)$;

(2) η is independently scattered, i.e., if the sets $U_j \in \mathscr{B}([0,T]) \otimes \mathscr{B}(Z)$, $j = 1, \ldots, n$, are disjoint, then the random variables $\eta(U_j)$, $j = 1, \ldots, n$, are mutually independent;

(3) for all $U \in \mathscr{B}(Z)$ and $I \in \mathscr{B}([0,T])$,

$$\mathbb{E}[\eta(I \times U)] = (\mathrm{Leb}_{[0,T]} \otimes \nu)(I \times U) = \mathrm{Leb}_{[0,T]}(I)\nu(U);$$

(4) for each $U \in \mathscr{B}(Z)$, the $\overline{\mathbb{N}}$-valued process

$$(0,\infty) \times \Omega \ni (t,\omega) \mapsto \eta(\omega)(U \times (0,t])$$

is $\mathbb{G}$-adapted, and its increments are independent of the past, i.e., the increment between times t and s with $t > s > 0$ is independent of the σ-field $\mathscr{G}_s$.

Appendix B. Proof of Lemma 5.1

This section is devoted to the proof of Lemma 5.1, which, for the convenience of the reader, we state again.

Lemma B.1. *Assume that $N \in \mathbb{N}$. Then, for all $u_0 \in V$, $f \in L^2([0,T],H)$, and $g \in S^N$, there exists a unique solution $u^g \in C([0,T],V) \cap L^2([0,T],\mathscr{D}(A))$ of problem (5.11). Moreover, for any ρ, $R > 0$, there exists a positive constant $C_N = C_{N,\rho,R}$ such that for every $g \in S^N$ and all $u_0 \in V$ and $f \in L^2([0,T],H)$ such that $\|u_0\|_V \le \rho$ and $\|f\|_{L^2([0,T],H)} \le R$,*

$$\sup_{t\in[0,T]} \|u^g(t)\|_V^2 + \int_0^T \|u^g(t)\|_{\mathscr{D}(A)}^2 \, dt \le C_N. \tag{B.1}$$

Proof. Fix $N \in \mathbb{N}$, $u_0 \in V$, $f \in L^2([0,T],H)$, and $g \in S^N$. Define an auxiliary function

$$F(t,y) := \int_Z G(y,z)(g(t,z) - 1)\,\nu(dz), \quad t \in [0,T],\ y \in V.$$

By Assumption 5.1, for all $t \in [0,T]$, $\hbar > 0$, and $y, y_1, y_2 \in \mathrm{V}$ with $\|y_1\|_{\mathrm{V}} \vee \|y_2\|_{\mathrm{V}} \le \hbar$,

$$\|F(t,y_1) - F(t,y_2)\|_{\mathrm{V}} \le \int_{\mathrm{Z}} L_\hbar(z)|g(t,z)-1|\,\nu(dz)\,\|y_1 - y_2\|_{\mathrm{V}},$$
$$\|F(t,y)\|_{\mathrm{V}} \le \int_{\mathrm{Z}} L_2(z)|g(t,z)-1|\,\nu(dz)\,(1+\|y\|_{\mathrm{V}}),$$
$$\|F(t,y)\|_{\mathrm{H}} \le \int_{\mathrm{Z}} L_3(z)|g(t,z)-1|\,\nu(dz)\,(1+\|y\|_{\mathrm{H}}).$$

By [15, Lemma 3.4],

$$C_{1,N} := \max_{i=2,3} \sup_{g\in S^N} \int_0^T \int_{\mathrm{Z}} L_i(z)|g(t,z)-1|\,\nu(dz)\,dt < \infty, \tag{B.2}$$

and, for every $\hbar > 0$,

$$C_{\hbar,N} := \sup_{g\in S^N} \int_0^T \int_{\mathrm{Z}} L_\hbar(z)|g(t,z)-1|\,\nu(dz)\,dt < \infty. \tag{B.3}$$

Combining the above five inequalities and using an argument similar to those for Theorems 4.1 and 4.2, we can deduce that there exists a unique solution $u^g \in C([0,T],\mathrm{V}) \cap L^2([0,T],\mathscr{D}(\mathrm{A}))$ of equation (5.11).

Now we are ready to prove (B.1). We begin with a priori estimates in the space H. Note that we only use the assumption "Linear growth in V" and "Linear growth in H" in Assumption 5.1 to get (B.1). By Assumption 5.1 and the Lions–Magenes lemma, we have

$$\begin{aligned}
&|u^g(t)|_{\mathrm{H}}^2 + 2\int_0^t \|u^g(s)\|_{\mathrm{V}}^2\,ds \\
&= |u_0|_{\mathrm{H}}^2 + 2\int_0^t \langle f(s), u^g(s)\rangle_{\mathrm{H}}\,ds + 2\int_0^t \int_{\mathrm{Z}} \langle G(u^g(s),z), u^g(s)\rangle_{\mathrm{H}} (g(s,z)-1)\,\nu(dz)\,ds \\
&\le |u_0|_{\mathrm{H}}^2 + \int_0^t \|u^g(s)\|_{\mathrm{V}}^2\,ds + \int_0^t \|f(s)\|_{\mathrm{V}'}^2\,ds \\
&\quad + 2\int_0^t (1+2|u^g(s)|_{\mathrm{H}}^2) \int_{\mathrm{Z}} L_3(z)|g(s,z)-1|\,\nu(dz)\,ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|u^g(t)|_{\mathrm{H}}^2 + \int_0^t \|u^g(s)\|_{\mathrm{V}}^2\,ds \\
&\quad \le |u_0|_{\mathrm{H}}^2 + \int_0^T \|f(s)\|_{\mathrm{V}'}^2\,ds + 2\int_0^T \int_{\mathrm{Z}} L_3(z)|g(s,z)-1|\,\nu(dz)\,ds \\
&\qquad + 4\int_0^t |u^g(s)|_{\mathrm{H}}^2 \int_{\mathrm{Z}} L_3(z)|g(s,z)-1|\,\nu(dz)\,ds.
\end{aligned} \tag{B.4}$$

By applying Gronwall's lemma, we get

$$\sup_{t\in[0,T]} |u^g(t)|_{\mathrm{H}}^2 + \int_0^T \|u^g(t)\|_{\mathrm{V}}^2\,dt \le \left(|u_0|_{\mathrm{H}}^2 + \int_0^T \|f(s)\|_{\mathrm{V}'}^2\,ds + 2\int_0^T\int_{\mathrm{Z}} L_3(z)|g(s,z)-1|\,\nu(dz)\,ds\right) \cdot e^{4\int_0^T\int_{\mathrm{Z}} L_3(z)|g(s,z)-1|\,\nu(dz)\,ds}.$$

Applying (B.2), we get

$$\sup_{g\in S^N}\left(\sup_{t\in[0,T]} |u^g(t)|_{\mathrm{H}}^2 + \int_0^T \|u^g(t)\|_{\mathrm{V}}^2\,dt\right) \le K_{N,H} < \infty. \tag{B.5}$$

Now, by Assumption 5.1 and [59, Lemma III.1.2], in the space V we have

$$\begin{aligned}
&\|u^g(t)\|_{\mathrm{V}}^2 + 2\int_0^t \|u^g(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds \\
&= \|u_0\|_{\mathrm{V}}^2 + 2\int_0^t \langle \mathrm{B}(u^g(s)), \mathrm{A}u^g(s)\rangle_{\mathrm{H}}\,ds + 2\int_0^t \langle f(s), \mathrm{A}u^g(s)\rangle_{\mathrm{H}}\,ds \\
&\quad + 2\int_0^t\int_{\mathrm{Z}} {}_{\mathrm{V}'}\langle G(u^g(s),z), u^g(s)\rangle_{\mathrm{V}}(g(s,z)-1)\,\nu(dz)\,ds \\
&\le \|u_0\|_{\mathrm{V}}^2 + \int_0^t \|u^g(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds + C\int_0^t \|u^g(s)\|_{\mathrm{V}}^4 |u^g(s)|_{\mathrm{H}}^2\,ds + 2\int_0^t |f(s)|_{\mathrm{H}}^2\,ds \\
&\quad + 2\int_0^t (1+2\|u^g(s)\|_{\mathrm{V}}^2)\int_{\mathrm{Z}} L_2(z)|g(s,z)-1|\,\nu(dz)\,ds.
\end{aligned} \tag{B.6}$$

Hence, we find that

$$\begin{aligned}
&\|u^g(t)\|_{\mathrm{V}}^2 + \int_0^t \|u^g(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds \\
&\le \|u_0\|_{\mathrm{V}}^2 + 2\int_0^T |f(s)|_{\mathrm{H}}^2\,ds + 2\int_0^T\int_{\mathrm{Z}} L_2(z)|g(s,z)-1|\,\nu(dz)\,ds \\
&\quad + \int_0^t \|u^g(s)\|_{\mathrm{V}}^2\left(C\|u^g(s)\|_{\mathrm{V}}^2|u^g(s)|_{\mathrm{H}}^2 + 4\int_{\mathrm{Z}} L_2(z)|g(s,z)-1|\,\nu(dz)\right)ds.
\end{aligned}$$

Therefore, by applying Gronwall's lemma, we deduce that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|u^g(t)\|_{\mathrm{V}}^2 + \int_0^T \|u^g(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds \\
&\le \left(\|u_0\|_{\mathrm{V}}^2 + 2\int_0^T |f(s)|_{\mathrm{H}}^2\,ds + 2\int_0^T\int_{\mathrm{Z}} L_2(z)|g(s,z)-1|\,\nu(dz)\,ds\right) \\
&\quad \cdot e^{C\int_0^T \|u^g(s)\|_{\mathrm{V}}^2|u^g(s)|_{\mathrm{H}}^2\,ds + 4\int_0^T\int_{\mathrm{Z}} L_2(z)|g(s,z)-1|\,\nu(dz)\,ds}.
\end{aligned}$$

Thus, in view of (B.2) and (B.5), we know that

$$\sup_{g\in S^N}\left(\sup_{t\in[0,T]} \|u^g(t)\|_{\mathrm{V}}^2 + \int_0^T \|u^g(s)\|_{\mathscr{D}(\mathrm{A})}^2\,ds\right) \le K_{N,\mathrm{V}} < \infty.$$

This completes the proof of Lemma 5.1. ■

Acknowledgments. The first named author would like to thank Utpal Manna for discussions. The third named author would like to thank Markus Riedle for his kind help in London. All authors would like to thank Alexei Daletski, Carl Chalk, John Herman and Tomasz Zastawniak for careful reading of the manuscript and many useful comments.

Funding. Zdzisław Brzeźniak was partially supported by the National Science Foundation of China (No. 11501509) and by a Royal Society grant "Stochastic Landau–Lifshitz–Gilbert equation with Lévy noise and ferromagnetism". Xuhui Peng is supported by National Natural Science Foundation of China (No. 12071123) and the Scientific Research Fund of Hunan Provincial Education Department (No. 20A329). Jianliang Zhai is supported by the National Natural Science Foundation of China (Nos. 12131019, 11971456, 11721101).

The third named author acknowledges funding by K.C. Wong Foundation for a year-long fellowship at King's College London, during which this work was started.

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