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Definable $(\omega, 2)$-theorem for families with VC-codensity less than 2

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Abstract

Let $\mathcal{S}$ be a family of nonempty sets with VC-codensity less than 2. We prove that, if $\mathcal{S}$ has the $(\omega, 2)$-property (for any infinitely many sets in $\mathcal{S}$, at least 2 among them intersect), then $\mathcal{S}$ can be partitioned into finitely many subfamilies, each with the finite intersection property. If $\mathcal{S}$ is definable in some first-order structure, then these subfamilies can be chosen definable too.

This is a strengthening of the case $q = 2$ of the definable (p, q)-conjecture in model theory [Sim15b] and the Alon-Kleitman-Matoušek (p, q)-theorem in combinatorics [Mat04].

1 Introduction

Given a family of sets $\mathcal{S}$, a boolean atom is a maximal nonempty intersection of sets in the closure of $\mathcal{S}$ under complements. The dual shatter function $\pi^*_{\mathcal{S}} : \omega \to \omega$ of $\mathcal{S}$ sends each n to the maximum number of boolean atoms of any subfamily of $\mathcal{S}$ of size n.

For cardinals $p \geq q > 1$, a family of sets $\mathcal{S}$ has the (p, q)-property if it does not contain the empty set and, for any p sets in $\mathcal{S}$, there exists a subfamily among them of size q with nonempty intersection.

Using ideas from Alon and Kleitman [AK92], Matoušek proved the following in [Mat04, Theorem 4].

Theorem A (Alon-Kleitman-Matoušek (p, q)-theorem[1]). *Let $q \geq 2$ be an integer and $\mathcal{S}$ be a family of sets whose dual shatter function satisfies $\pi^*_{\mathcal{S}}(n) \in$*

[1]While classically the Alon-Kleitman-Matoušek (p, q)-theorem is stated for finite $\mathcal{F}$, a straightforward application of first-order logic compactness shows that this is equivalent to the infinite version presented here (see the proof of [Sim15b, Proposition 2.5]).

$o(n^q)$ (that is, $\lim_{n\to\infty} \pi^*_{\mathcal{S}}(n)/n^q = 0$). For any integer $p \geq q$, there exists some $m < \omega$ such that, if $\mathcal{F}$ is a subfamily of $\mathcal{S}$ with the (p, q)-property, then $\mathcal{F}$ can be partitioned into at most m subfamilies, each with the finite intersection property.

For notational conventions and some model theoretic definitions in this paper we refer the reader to Section 2.1 and to [Sim15a].

Chernikov and Simon [CS15] used Theorem A to study NIP theories. In [CS15, Problem 29] they asked whether a definable version of it holds in this setting. This has evolved to be known as the definable (p, q)-conjecture [Sim15b, Conjecture 2.15]. Specifically, the conjecture (which was put forward before the connection with the (p, q)-theorem was established) states that any NIP formula which is non-dividing over a model M belongs to a (finitely) consistent M-definable family. By means of first-order logic compactness, as well as Theorem A, this can be restated as follows.

Conjecture B (Definable (p, q)-conjecture[2])**.** *Let $q \geq 2$ be an integer, M be an L-structure and $\varphi(x, y)$ be an $L(M)$-formula (which we identify with the family of sets $\{\varphi(M, a) : a \in M^{|y|}\}$) with dual shatter function $\pi^*_{\varphi}(n) \in o(n^q)$. If there exists an integer $p \geq q$ such that $\varphi(x, y)$ has the (p, q)-property, then there exists some $m < \omega$ and $L(M)$-formulas $\sigma_1(y), \ldots, \sigma_m(y)$ such that $\cup_i \sigma_i(M) = M^{|y|}$ and, for every $i \leq m$, the family $\{\varphi(x, a) : a \in \sigma_i(M)\}$ is consistent.*

Conjecture B, which can be seen as a definable non-uniform version of Theorem A, is known to hold in certain cases. Simon [Sim14] proved it in dp-minimal theories for formulas $\varphi(x, y)$ with $|x| \leq 2$, and in any theory for formulas that extend to an invariant type of dp-rank 1. In [Sim15b], he proved it in NIP theories of small or medium directionality. Simon and Starchenko [SS14, Theorem 5] proved a stronger version of the conjecture for a class of dp-minimal theories that includes those that are linearly ordered, unpackable VC-minimal, or have definable Skolem functions. Recently, Boxall and Kestner [BK18] proved, using Theorem A and the work on NIP forking of Chernikov and Kaplan [CK12], Conjecture B in distal NIP theories. While this paper was under review, Itay Kaplan [Kap22] presented a proof of a uniform version of Conjecture B for formulas in NIP theories.

In this paper we prove a strengthening of both Conjecture B and (the non-uniform version of) Theorem A in the case where $q = 2$. In particular,

[2] In the literature the conjecture is commonly found with the stronger assumption that the whole structure is NIP [Sim15b, Conjecture 5.1]. Kaplan [Kap22, Corollary 4.9] has recently presented a proof of this version of the conjecture.

we show that Conjecture B holds when $q = 2$, and that we may furthermore weaken the $(p, 2)$-property to the $(\omega, 2)$-property in the statements of Conjecture B and the case $\mathcal{S} = \mathcal{F}$ of Theorem A.

Theorem C (Definable $(\omega, 2)$-theorem)**.** *Let M be an L-structure and $\varphi(x, y)$ be an $L(M)$-formula with dual shatter function $\pi^*_\varphi(n) \in o(n^2)$ (e.g VC-codensity of $\varphi(x, y)$ is less than 2). If $\varphi(x, y)$ has the $(\omega, 2)$-property, then there exist some $m < \omega$ and $L(M)$-formulas $\sigma_1(y), \dots, \sigma_m(y)$ such that $\cup_i \sigma_i(M) = M^{|y|}$ and, for every $i \leq m$, the family $\{\varphi(x, a) : a \in \sigma_i(M)\}$ is consistent.*

Since any family of sets can be witnessed as a definable family in some structure, the following corollary is immediate.

Corollary D ($(\omega, 2)$-theorem)**.** *Let $\mathcal{S}$ be a family of sets with $\pi^*_\mathcal{S}(n) \in o(n^2)$. If $\mathcal{S}$ has the $(\omega, 2)$-property, then it can be partitioned into finitely many subfamilies, each with the finite intersection property.*

Our proof of Theorem C is elementary in that it avoids the use of both the Alon-Kleitman-Matousek (p, q)-theorem (as well as its related fractional Helly theorem) and the work of Shelah, Simon and others on NIP theories.

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2 Preliminaries

2.1 Notation

Throughout we fix two structures $M \preccurlyeq U$ in some language L, where U realizes every type over M. For any $A \subseteq U$, let $L(A)$ denote the expansion of L by formulas with parameters in A.

Given a (partitioned) formula $\varphi(x, y)$, some $b \in U^{|y|}$ and $A \subseteq U^{|x|}$, let $\varphi(A, b) = \{a \in A : U \models \varphi(a, b)\}$. For $A \subseteq U$, we write $\varphi(A, b)$ instead of $\varphi(A^{|x|}, b)$. By "definable set" we mean "definable set in M possibly with parameters", i.e. a set of the form $\varphi(M)$ for some $L(M)$-formula $\varphi(x)$.

We apply notions such as the (p, q)-property and dual shatter function to formulas $\varphi(x, y)$ by adopting the usual convention of identifying them with the family of sets $\{\varphi(M, a) : a \in M^{|y|}\}$. In the context of formulas, we refer to the finite intersection property as being (finitely) consistent, and to being pairwise disjoint as being pairwise inconsistent.

Given a formula $\varphi(x, y)$ and $A \subseteq U^{|y|}$, by a *φ-type over A* we mean a maximal consistent collection $p(x)$ of formulas in $\{\varphi(x, a), \neg\varphi(x, a) : a \in A\}$.

Throughout, n,m, i, j, k and l are positive integers.

2.2 Preliminary results

We present some preliminary lemmas on φ-types for formulas $\varphi(x, y)$ with $\pi^*_\varphi(n) \in o(n^2)$.

Lemma 2.1. *Let $\varphi(x, y)$ be an $L(M)$-formula such that $\pi^*_\varphi(n) \in o(n^2)$. Suppose that there exists some $b \in U^{|y|}$ such that $\varphi(M, b) = \emptyset$. Then there exists $\theta(y) \in \mathrm{tp}(b/M)$ such that the elements of $\varphi(U, b)$ realize only finitely many φ-types over $\theta(M)$.*

Proof. Let $\varphi(x, y)$ and $b \in U^{|y|}$ be as in the lemma. We assume that, for any $\theta(y) \in \mathrm{tp}(b/M)$, the elements of $\varphi(U, b)$ realize infinitely many φ-types over $\theta(M)$. We prove the lemma by showing that, for every n,

$$\pi^*_\varphi(n) \geq \sum_{i=1}^{n} i = \frac{n^2 + n}{2}. \tag{1}$$

In particular, it follows that $\pi^*_\varphi(n) \notin o(n^2)$.

We construct a sequence $(a_n : 1 \leq n < \omega)$ in $M^{|y|}$ and a set $\{c_{i,j} : 1 \leq i < \omega,\ 1 \leq j \leq i\}$ in $M^{|x|}$ with the following property. For every n and distinct pairs (i, j), (i', j'), with $i, i' \leq n$, $j \leq i$ and $j' \leq i'$, it holds that

$$\varphi(c_{i,j}, \{a_1, \ldots, a_n\}) \neq \varphi(c_{i',j'}, \{a_1, \ldots, a_n\}). \tag{2}$$

That is, for every n, the set $\{c_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq i\}$ witnesses that

$$|\{\varphi(c, \{a_1, \ldots, a_n\}) : c \in M^{|x|}\}| \geq \sum_{i=1}^{n} i,$$

which in turn shows that the elements $\{a_1, \ldots, a_n\}$ witness Equation (1). Specifically, the set $\{c_{i,j} : 1 \leq i < \omega, 1 \leq j \leq i\}$ will have the following two properties:

(i) $\neg\varphi(c_{i',j'}, a_i)$ and $\varphi(c_{i,j}, a_i)$ holds for all $i' < i$, $j' \leq i'$, $j \leq i$,

(ii) $\varphi(c_{i,j}, \{a_1, \dots, a_{i-1}\}) \neq \varphi(c_{i,j'}, \{a_1, \dots, a_{i-1}\})$ for all $i \geq 2$, $j < j' \leq i$.

It is easy to see that condition (2) follows from (i) and (ii).

For every n and $a_1, \dots, a_n$ in $M^{|y|}$, let $s(a_1, \dots, a_n)$ denote the number of boolean atoms C of $\{\varphi(U, a_1), \dots, \varphi(U, a_n)\}$ satisfying that $\varphi(C, b) \neq \emptyset$. We construct our sequence in such a way that $s(a_1, \dots, a_n) \geq n + 1$ for every n.

We proceed to build sets $\{a_i : 1 \leq i \leq n\}$ and $\{c_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq i\}$ by induction on n.

Case $n = 1$.

Since, by assumption, the elements of $\varphi(U, b)$ realize infinitely many φ-types over M, there must be some $a \in M^{|y|}$ such that

$$\varphi(U, b) \cap \varphi(U, a) \neq \emptyset \text{ and } \varphi(U, b) \setminus \varphi(U, a) \neq \emptyset.$$

Let a_1 be any such a. Let $c_{1,1}$ be any element in $\varphi(M, a_1)$. Observe that $s(a_1) = 2$.

Induction $n > 1$.

Suppose we have a sequence $(a_1, \dots, a_{n-1})$ in $M^{|y|}$ as desired. Since $s(a_1, \dots, a_{n-1}) \geq n$, there are n distinct boolean atoms $C_1, \dots, C_n$ of the family $\{\varphi(U, a_1), \dots, \varphi(U, a_{n-1})\}$ containing each elements from $\varphi(U, b)$. Let

$$\theta(M) = \{a \in M^{|y|} : \neg\varphi(c_{i,j}, a),\ \varphi(C_k, a) \neq \emptyset \text{ for } j \leq i < n,\ k \leq n\}.$$

Since $\varphi(M, b) = \emptyset$, note that $b \in \theta(U)$. Consequently, by assumption, the elements of $\varphi(U, b)$ realize infinitely many φ-types over $\theta(M)$. In particular, there must exist some boolean atom C of $\{\varphi(U, a_1), \dots, \varphi(U, a_{n-1})\}$ satisfying that the elements of $\varphi(C, b)$ realize more than one φ-type over $\theta(M)$. Let $a_n \in \theta(M)$ witness this, i.e. $\varphi(C, b) \cap \varphi(U, a_n) \neq \emptyset$ and $\varphi(C, b) \setminus \varphi(U, a_n) \neq \emptyset$. It then follows that $s(a_1, \dots, a_n) \geq n + 1$.

Finally, by definition of $\theta(M)$, we have that $\varphi(C_j, a_n) \neq \emptyset$ for every $j \leq n$. For any $j \leq n$, let $c_{n,j}$ be an element in $\varphi(C_j, a_n) \cap M^{|x|}$. Then clearly $\{c_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq i\}$ satisfies condition (ii). By definition of $\theta(M)$, note that it also satisfies condition (i). □

Lemma 2.2. *Let $\varphi(x, y)$ be an $L(M)$-formula such that $\pi^*_\varphi(n) \in o(n^2)$. Suppose that there exists some $b \in U^{|y|}$ such that, for any $\sigma(y) \in \mathrm{tp}(b/M)$, the family $\{\varphi(x, a) : a \in \sigma(M)\}$ fails to be consistent. Then there exists $\theta(y) \in \mathrm{tp}(b/M)$ such that the elements of $\varphi(U, b)$ realize only finitely many φ-types over $\theta(M)$ and, moreover, for any such type $p(x)$ exactly one of the following two conditions holds.*

(a) $\{a \in \theta(M) : \varphi(x,a) \in p(x)\} = \emptyset$.

(b) *For every* $\theta'(y) \in \mathrm{tp}(b/M)$*, the set* $\{a \in \theta'(M) : \varphi(x,a) \in p(x)\}$ *is not definable (in* M*).*

Proof. Note that, by definition of b, for any $c \in M^{|x|}$ we have $\varphi(c,y) \notin \mathrm{tp}(b/M)$. So $\varphi(M,b) = \emptyset$. We apply Lemma 2.1. Hence let $\theta_0(y) \in \mathrm{tp}(b/M)$ be such that the elements of $\varphi(U,b)$ realize only finitely many φ-types over $\theta_0(M)$. Since otherwise the lemma is trivial we may assume that $\varphi(U,b) \neq \emptyset$. We denote these types by $p_1(x), \dots, p_m(x)$.

Let $F \subseteq \{1, \dots, m\}$ be the set of i satisfying that there exists a formula $\theta_i(y) \in \mathrm{tp}(b/M)$ such that the set $\sigma_i(M) = \{a \in \theta_i(M) : \varphi(x,a) \in p_i(x)\}$ is definable. Observe that, for any $i \in F$, since $\{\varphi(x,a) : a \in \sigma_i(M)\}$ is consistent, by definition of b it holds that $b \notin \sigma_i(M)$. Finally let $\theta(y)$ be given by

$$\theta_0(y) \wedge \bigwedge_{i \in F} (\theta_i(y) \wedge \neg\sigma_i(y)).$$

Since $\theta(M) \subseteq \theta_0(M)$, the φ-types over $\theta(M)$ realized in $\varphi(U,b)$ are exactly the restrictions $p_i(x)|_{\theta(M)}$ of the types $p_i(x)$ to $\theta(M)$, for $i \leq m$. We have ensured that, for any $i \in F$, the type $p_i(x)|_{\theta(M)}$ is the (necessarily unique) type described by condition (a). On the other hand, by definition of F, for any $j \in \{1, \dots, m\} \setminus F$ the type $p_j(x)|_{\theta(M)}$ satisfies condition (b). □

Lemma 2.3. *Let* $\varphi(x,y)$*,* $b \in U^{|y|}$*,* $\theta(y) \in \mathrm{tp}(b/M)$ *and* $p(x)$ *be such that they satisfy condition* (b) *in Lemma 2.2. Then, for any* $L(M)$*-formula* $\lambda(x)$ *satisfying that* $\varphi(U,b) \subseteq \lambda(U)$*, there exists some* $a \in \theta(M)$ *such that*

$$\varphi(U,a) \subseteq \lambda(U) \text{ and } \varphi(x,a) \in p(x).$$

Proof. Let $\theta'(M)$ be the set of $a \in \theta(M)$ with $\varphi(U,a) \subseteq \lambda(U)$. Observe that $\theta'(y) \in \mathrm{tp}(b/M)$. Then, by condition (b) in Lemma 2.2, the set $\{a \in \theta'(M) : \varphi(x,a) \in p(x)\}$ is nonempty. Let a be any element in the set. □

3 Proof of the main result

We prove Theorem C through the next proposition.

Proposition 3.1. *Let* $\varphi(x,y)$ *be an* $L(M)$*-formula with* $\pi^*_\varphi(n) \in o(n^2)$ *and suppose that there exists* $b \in U^{|y|}$ *such that, for any* $\sigma(y) \in \mathrm{tp}(b/M)$*, the family* $\{\varphi(x,a) : a \in \sigma(M)\}$ *fails to be consistent. Let* $\chi(x)$ *be an* $L(M)$*-formula such that* $\varphi(U,b) \subseteq \chi(U)$*. Then there exists some* $a \in M^{|y|}$ *such that*

$$\varphi(U,a) \subseteq \chi(U)$$

and moreover

$$\varphi(U,a)\cap\varphi(U,b)=\emptyset.$$

Proof. By Lemma 2.2, there exists some $\theta(y)\in \mathrm{tp}(b/M)$ such that the elements of $\varphi(U,b)$ realize only finitely many φ-types over $\theta(M)$, and furthermore for any such type condition (a) or condition (b) in the lemma holds. By passing from $\theta(M)$ to $\theta(M)\cap\{a\in M^{|y|}:\varphi(U,a)\subseteq\chi(U)\}$ if necessary, we may also assume that every $a\in\theta(M)$ satisfies that $\varphi(U,a)\subseteq\chi(U)$. In particular, to prove Proposition 3.1 it suffices to find some $a\in\theta(M)$ such that $\varphi(U,a)\cap\varphi(U,b)=\emptyset$. Since otherwise the result is trivial we may assume that $\varphi(U,b)\neq\emptyset$.

Let $p_1(x),\dots,p_l(x)$ denote the distinct φ-types over $\theta(M)$ realized by elements of $\varphi(U,b)$. We prove Proposition 3.1 by finding some $a\in\theta(M)$ such that $\varphi(x,a)\notin p_i(x)$ for every $i\leq l$. If $l=1$ and $p_1(x)$ is the (unique) type described by condition (a) in Lemma 2.2, then clearly it suffices to take any $a\in\theta(M)$ and we are done. We assume this is not the case.

Let the numbering of the types $p_i(x)$ be such that, for some fixed $k\in\{l-1,l\}$, the types $p_i(x)$ for $1\leq i\leq k$ satisfy condition (b) and the possibly remaining type $p_i(x)$ for $k<i\leq l$ satisfies condition (a) in Lemma 2.2. Hence, either $k=l$ or otherwise $1\leq k=l-1$ and the type $p_l(x)$ satisfies that $\varphi(x,a)\notin p_l(x)$ for every $a\in\theta(M)$. In either case it suffices to find some $a\in\theta(M)$ with $\varphi(x,a)\notin p_i(x)$ for every $1\leq i\leq k$.

Now let us fix, for every $1\leq i\leq k$, an $L(M)$-formula $\chi_i(x)$ satisfying the following conditions:

- $p_i(x)\models\chi_i(x)$ for every $i<k$,
- $p_j(x)\models\chi_k(x)$ for all $k\leq j\leq l$,
- $\chi_i(U)\cap\chi_j(U)=\emptyset$ for every $i<j\leq k$.

We define, for any $1\leq m\leq k$ and elements $a_1,\dots,a_{m-1}\in M^{|y|}$, a set $\psi_m(M,a_1,\dots,a_{m-1})\subseteq\theta(M)$ as follows.

For $m=k$, let $\psi_k(M,a_1,\dots,a_{k-1})$ denote the set of all $a\in\theta(M)$ such that

$$\varphi(U,a)\subseteq\bigcup_{i=1}^{k-1}(\varphi(U,a_i)\cap\chi_i(U))\cup\chi_k(U).$$

For $m<k$, let $\psi_m(M,a_1,\dots,a_{m-1})$ denote the set of all $a\in\theta(M)$ such that

$$\varphi(U,a)\subseteq\bigcup_{i=1}^{m-1}(\varphi(U,a_i)\cap\chi_i(U))\cup\bigcup_{i=m}^{k}\chi_i(U)$$

and moreover there exists two elements $a', a'' \in \psi_{m+1}(M, a_1, \ldots, a_{m-1}, a)$, with

$$\varphi(U, a') \cap \varphi(U, a'') \cap \chi_{m+1}(U) = \emptyset.$$

Claim 3.2. *For any $m \leq k$, the sets $\psi_m(M, a_1, \ldots, a_{m-1})$ are definable uniformly (in M) over the parameters $a_i \in M^{|y|}$, $i < m$.*

Proof. For any given $m \leq k$, let (A_m) be the statement that the sets $\psi_m(M, a_1, \ldots, a_{m-1})$ are definable uniformly over the parameters $a_i \in M^{|y|}$, $i < m$. Statement (A_k) clearly holds by definition. Then, for any $m < k$, (A_m) follows easily from (A_{m+1}) and the definition of sets $\psi_m(M, a_1, \ldots, a_{m-1})$. $\square_{\text{Claim}}$

We now prove two claims regarding the set $\psi_1(M)$ that will yield Proposition 3.1, by showing the existence of some $a \in \theta(M)$ with $\varphi(x, a) \notin p_i(x)$ for every $i \leq k$.

Claim 3.3. *There exist $a, a' \in \psi_1(M)$ such that*

$$\varphi(U, a) \cap \varphi(U, a') \cap \chi_1(U) = \emptyset.$$

Proof. For any $m \leq k$ consider the following two statements (I_m) and (II_m):

$(\mathbf{I}_m)$ Let $a_i \in M^{|y|}$ be such that $\varphi(x, a_i) \in p_i(x)$, for $i < m$, and let $a \in \theta(M)$. Suppose that

$$\varphi(U, a) \subseteq \bigcup_{i=1}^{m-1} (\varphi(U, a_i) \cap \chi_i(U)) \cup \bigcup_{i=m}^{k} \chi_i(U)$$

and

$$\varphi(x, a) \in p_m(x).$$

Then

$$a \in \psi_m(M, a_1, \ldots, a_{m-1}).$$

$(\mathbf{II}_m)$ Let $a_i \in M^{|y|}$ be such that $\varphi(x, a_i) \in p_i(x)$, for $i < m$. Then there exist

$$a, a' \in \psi_m(M, a_1, \ldots, a_{m-1})$$

such that

$$\varphi(U, a) \cap \varphi(U, a') \cap \chi_m(U) = \emptyset.$$

We prove (I_m) and (II_m) for every $m \leq k$ using a reverse induction on m. Claim 3.3 is then given by (II_1).

Trivially (I_k) holds by definition of $\psi_k(M, a_1, \ldots, a_{k-1})$, even without the condition $\varphi(x, a) \in p_k(x)$. We prove the remaining statements as follows.

For $m \leq k$, we derive (II_m) from (I_m) using Claim 3.2. For $m < k$, we derive (I_m) from (II_{m+1}).

Proof of ($\mathbf{I}_m$) $\Rightarrow$ ($\mathbf{II}_m$) for $m \leq k$.

Let $\varphi(x, a_i) \in p_i(x)$ for $i < m$. Let $\theta'(M)$ be the set of all $a \in \theta(M)$ such that

$$\varphi(U, a) \subseteq \bigcup_{i=1}^{m-1} (\varphi(U, a_i) \cap \chi_i(U)) \cup \bigcup_{i=m}^{k} \chi_i(U).$$

Note that $\theta'(y) \in \mathrm{tp}(b/M)$. By definition of $p_m(x)$ (see condition (b) in Lemma 2.2), the set A of all $a \in \theta'(M)$ with $\varphi(x, a) \in p_m(x)$ is not definable (in M). By (I_m) note that

$$A \subseteq \psi_m(M, a_1, \ldots, a_{m-1}).$$

By Claim 3.2, the set $\psi_m(M, a_1, \ldots, a_{m-1})$ is definable. Since the subset A is not definable, there must exist some $a \in \psi_m(M, a_1, \ldots, a_{m-1})$ that is not in A, in particular

$$\varphi(x, a) \notin p_m(x).$$

Now, by Lemma 2.3, there exists some $a' \in \theta(M)$ with

$$\varphi(U, a') \subseteq \bigcup_{i=1}^{m-1} (\varphi(U, a_i) \cap \chi_i(U)) \cup (\chi_m(U) \setminus \varphi(U, a)) \cup \bigcup_{i=m+1}^{k} \chi_i(U)$$

such that

$$\varphi(x, a') \in p_m(x).$$

(In the case $m = k = l - 1$ Lemma 2.3 can still be applied because $\varphi(x, a) \notin p_l(x)$ by definition of the type $p_l(x)$.) Once again by (I_m) it follows that

$$a' \in \psi_m(M, a_1, \ldots, a_{m-1}).$$

Finally, by construction note that

$$\varphi(U, a) \cap \varphi(U, a') \cap \chi_m(U) = \emptyset.$$

Proof of ($\mathbf{II}_{m+1}$) $\Rightarrow$ ($\mathbf{I}_m$) for $m < k$.

Let $\varphi(x, a_i) \in p_i(x)$ for $i < m$, and $a \in \theta(M)$ be as described in (I_m). In particular we have that $\varphi(x, a) \in p_m(x)$.

By (II_{m+1}), there exist $a', a'' \in \psi_{m+1}(M, a_1, \ldots, a_{m-1}, a)$ such that

$$\varphi(U, a') \cap \varphi(U, a'') \cap \chi_{m+1}(U) = \emptyset.$$

But then by definition this means that $a \in \psi_m(M, a_1, \ldots, a_{m-1})$. $\square_{\text{Claim}}$

Claim 3.4. *Suppose that there exists some* $a' \in \psi_1(M)$ *with*

$$\varphi(x, a') \notin p_1(x).$$

Then there exists some $a \in \theta(M)$ *satisfying that*

$$\varphi(x, a) \notin p_i(x) \text{ for every } 1 \leq i \leq k.$$

Proof. For any $m \leq k$ consider the following statement (B_m):

($\mathbf{B}_m$) Let $a_i \in M^{|y|}$, $i < m$, be such that there exist $a' \in \psi_m(M, a_1, \ldots, a_{m-1})$, with

$$\varphi(x, a') \notin p_m(x).$$

Then there exists some $a \in \theta(M)$ with

$$\varphi(U, a) \subseteq \bigcup_{i=1}^{m-1} (\varphi(U, a_i) \cap \chi_i(U)) \cup \bigcup_{i=m}^{k} \chi_i(U)$$

satisfying that

$$\varphi(x, a) \notin p_j(x) \text{ for every } m \leq j \leq k.$$

We prove (B_m) for every $m \leq k$ by reverse induction on m. Claim 3.4 then immediately follows from (B_1). Let a_i, for $i < m$, and a' be as in (B_m).

For the base case $m = k$, it clearly suffices to take $a = a'$. We assume that $m < k$ and show that (B_{m+1}) $\Rightarrow$ (B_m).

By definition of $\psi_m(M, a_1, \ldots, a_{m-1})$, there exist $a'', a''' \in \psi_{m+1}(M, a_1, \ldots, a_{m-1}, a')$ with

$$\varphi(U, a'') \cap \varphi(U, a''') \cap \chi_{m+1}(U) = \emptyset.$$

Without loss of generality we may assume that $\varphi(x, a'') \notin p_{m+1}(x)$. By ($\mathrm{B}_{m+1}$), we derive that there exists some $a \in \theta(M)$ such that

$$\varphi(U, a) \subseteq \bigcup_{i=1}^{m-1} (\varphi(U, a_i) \cap \chi_i(U)) \cup (\varphi(U, a') \cap \chi_m(U)) \cup \bigcup_{i=m+1}^{k} \chi_i(U) \quad (3)$$

and

$$\varphi(x, a) \notin p_j(x) \text{ for every } m < j \leq k.$$

However, since $\varphi(x, a') \notin p_m(x)$, then by (3) it must also be that $\varphi(x, a) \notin p_m(x)$. $\square_{\text{Claim}}$

We now complete the proof of the proposition. By Claim 3.3, let $a', a'' \in \psi_1(M)$ be two elements such that $\varphi(U, a') \cap \varphi(U, a'') \cap \chi_1(U) = \emptyset$. Without loss of generality we may assume that a' is such that $\varphi(x, a') \notin p_1(x)$. By Claim 3.4 we conclude that there exists some $a \in \theta(M)$ satisfying that $\varphi(x, a) \notin p_i$ for every $i \leq k$, as desired. $\square$

Proof of Theorem C. Let $\varphi(x, y)$ be an $L(M)$-formula with $\pi^*_\varphi(n) \in o(n^2)$. We assume that $\varphi(x, y)$ does not partition into finitely many consistent families and derive that it does not have the $(\omega, 2)$-property, i.e. we build a sequence $(a_n : 1 \leq n < \omega)$ in $M^{|y|}$ such that the family $\{\varphi(x, a_n) : 1 \leq n < \omega\}$ is pairwise inconsistent.

Hence we assume that $\varphi(x, y)$ satisfies that, for any finite collection of $L(M)$-formulas $\{\sigma_i(y) : 1 \leq i \leq m\}$, if the family $\{\varphi(x, a) : a \in \sigma_i(M)\}$ is consistent for every $i \leq m$, then there exists some $a \in M^{|y|}$ such that $a \notin \cup_i \sigma_i(M)$. By model theoretic compactness we may fix some $b \in U^{|y|}$ satisfying that, for any formula $\sigma(y) \in \text{tp}(b/M)$, the family $\{\varphi(x, a) : a \in \sigma(M)\}$ fails to be consistent. We build our sequence $(a_n : 1 \leq n < \omega)$ using Proposition 3.1. In particular it will satisfy that, for every $i < \omega$, it holds that

$$\varphi(U, a_i) \cap \varphi(U, b) = \emptyset \tag{4}$$

We proceed inductively on n.

By Proposition 3.1 (with $\chi(x) :=$ "$x = x$"), let $a_1 \in M^{|y|}$ be any element satisfying (4). Then, for the inductive step, let $(a_1, \ldots, a_{n-1})$ be elements each satisfying (4) and such that the formulas $\varphi(x, a_i)$, for $i < n$, are pairwise inconsistent. Let $\chi(x)$ denote the formula

$$\bigwedge_{i=1}^{n-1} \neg\varphi(x, a_i).$$

Note that $\varphi(U, b) \subseteq \chi(U)$. Now, applying Proposition 3.1, let $a_n \in M^{|y|}$ be an element satisfying (4) and $\varphi(U, a_n) \subseteq \chi(U)$. The family $\{\varphi(x, a_i) : 1 \leq i \leq n\}$ is pairwise inconsistent as desired. $\square$

We end the paper with some questions. We note that, while this paper was under review, Kaplan [Kap22] presented a positive answer to Question (2) for formulas in NIP theories.

Questions 3.5.

(1) Definable (ω, q)-conjecture: *Let $\varphi(x, y)$ be a formula and $q \geq 2$ an integer such that $\pi^*_\varphi(n) \in o(n^q)$. If $\varphi(x, y)$ has the (ω, q)-property, does it partition into finitely many consistent definable subfamilies?*

(2) Uniform definable $(p, 2)$-conjecture 1: *Let $\varphi(x, y)$ and $\psi(y, z)$ be formulas where $\pi^*_\varphi(n) \in o(n^2)$. Given any integer $p \geq 2$, is there an m such that any family of the form $\{\varphi(x, a) : M \models \psi(a, b)\}$, for $b \in M^{|z|}$, with the $(p, 2)$-property partitions into at most m consistent definable subfamilies?*

(3) Uniform definable $(p, 2)$-conjecture 2: *Let $\varphi(x, y)$ be a formula with $\pi^*_\varphi(n) \in o(n^2)$. Given any integer $p \geq 2$, is there an m such that any definable subfamily of $\varphi(x, y)$ with the $(p, 2)$-property partitions into at most m consistent definable subfamilies?*

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