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Inner generalized Weyl algebras and their simplicity criteria

V. V. Bavula

Abstract

The aim of the paper is to introduce a new class of rings – the *inner generalized Weyl algebras* (IGWA) – and to give simplicity criteria for them. For each IGWA A a derivative series of IGWAs, $A \to A' \to A'' \to \cdots \to A^{(\alpha)} \to \cdots$, is attached where α is an arbitrary ordinal. In general, all rings $A^{(\alpha)}$ are distinct. A new construction of rings, the *inner* (σ, τ, a)-*extension* of a ring, is introduced (where σ and τ are endomorphisms of a ring D and $a \in D$).

Key Words: generalized Weyl algebra, inner generalized Weyl algebra, simplicity criterion, normal element, defining relations, grading, endomorphism, automorphism, regular element, centre, fixed ring, ideal, module.

Mathematics subject classification 2020: 16D30, 16P40, 16D25, 16U70, 16W50, 16P50, 16S85.

1 Introduction

Generalized Weyl algebras $D(\sigma, a)$ with central element a.

Definition, [1]–[8]. Let D be a ring, σ be a ring automorphism of D, a be a *central* element of D. The **generalized Weyl algebra** of rank 1 (GWA, for short) $D(\sigma, a) = D[x, y; \sigma, a]$ is a ring generated by the ring D and two elements x and y that are subject to the defining relations:

$$xd = \sigma(d)x \text{ and } yd = \sigma^{-1}(d)y \text{ for all } d \in D, \ \ yx = a \text{ and } xy = \sigma(a). \qquad (1)$$

In 1987, I introduced the generalized Weyl algebras of arbitrary rank when I was an algebra postgradute student at Taras Shevchenko National University of Kyiv, the Department of Algebra and Mathematical Logic, and they were the subject of my PhD thesis "Generalized Weyl algebras and their representations" submitted at the end of 1990 (defended at the beginning of 1991).

Generalized Weyl algebras with two endomorphisms and a left normal element a. In [9], a more general construction of GWAs is introduced.

Definition, [9]. Let D be a ring, σ and τ be ring endomorphisms of D, and $a \in D$. Suppose that

$$\tau\sigma(a) = a, \ \ ad = \tau\sigma(d)a \text{ and } \sigma(a)d = \sigma\tau(d)\sigma(a) \text{ for all } d \in D. \qquad (2)$$

The **generalized Weyl algebra** (GWA) of rank 1, $A = D(\sigma, \tau, a) = D[x, y; \sigma, \tau, a]$, is a ring generated by D, x and y subject to the defining relations:

$$xd = \sigma(d)x \text{ and } yd = \tau(d)y \text{ for all } d \in D, \ \ yx = a \text{ and } xy = \sigma(a). \qquad (3)$$

The ring D is called the *base ring* of the GWA A. The endomorphisms σ, τ and the element a are called the *defining endomorphisms* and the *defining element* of the GWA A, respectively. By (2), the elements a and $\sigma(a)$ are left normal in D. An element d of a ring D is called *left* (resp., *right*) *normal* if $dD \subseteq Dd$ (resp., $Dd \subseteq dD$). An element $d \in D$ is called a *normal* element if $Dd = dD$. To distinguish 'old' GWAs from the 'new' ones the former are called the *classical* GWAs. Every classical GWA is a GWA as the conditions in (2) trivially hold if a is central and $\tau = \sigma^{-1}$.

It is an experimental fact that many popular algebras of small Gelfand-Kirillov dimension are GWAs, [1]–[8]: the first Weyl algebra A_1 and its quantum analogue, the *quantum plane*, the *quantum sphere*, $Usl(2)$, $U_q sl(2)$, the *Heisenberg* algebra, *Witten's* and *Woronowicz's* deformations, Noetherian down-up algebras, etc.

Every GWA $A = \bigoplus_{i\in\mathbb{Z}} Dv_i$ is a free *left* D-module where $v_0 = 1$, $v_i = x^i$ and $v_{-i} = y^i$ for $i \geq 1$, [9]. The *opposite* ring to a GWA is called the *right GWA*, [9]. Since $A^{op} = \bigoplus_{i\in\mathbb{Z}} v_i D^{op}$ is a free *right* D-module the adjective 'right' is added.

Inner generalized Weyl algebras. The aim of the paper is to introduce the inner generalized Weyl algebras.

Definition. Let D be a ring, σ and τ be ring endomorphisms of D, and $a \in D$. Suppose that

$$\sigma(a) = \tau(a) \text{ and } \sigma(a)\tau(d) = \sigma(d)\sigma(a) \text{ for all } d \in D. \tag{4}$$

The **inner generalized Weyl algebra** (IGWA) of rank 1, $A = D(\sigma, \tau, a)_{\rm in} = D[x, y; \sigma, \tau, a]_{\rm in}$, is a ring generated by D, x and y subject to the defining relations:

$$xd = \sigma(d)x \text{ and } dy = y\tau(d) \text{ for all } d \in D, \quad yx = a \text{ and } xy = \sigma(a). \tag{5}$$

The ring D is called the *base ring* of the IGWA A. The endomorphisms σ, τ and the element a are called the *defining endomorphisms* and the *defining element* of the IGWA A, respectively.

The following identities in the ring A explain the origin of the conditions in (4):

$$y\tau(a) = ay = yxy = y\sigma(a) \text{ and } \sigma(d)\sigma(a) = \sigma(d)xy = xdy = xy\tau(d) = \sigma(a)\tau(d).$$

Notice that if $\sigma = \tau$ then the conditions in (4) hold automatically.

For a GWA (resp., a right GWA), all the elements of the base ring D can be moved to the left (resp., right) but for an IGWA this is not the case, in general. There are elements of D that are locked between the variables y and x, like $y^i dx^i$, that cannot be moved neither to the left nor right. This is the reason why the adjective 'inner' is added in the definition of IGWA.

Since we cannot move coefficients (i.e., elements of D) to one side and the conditions on the element a in (4) are less restrictive than in (2) (in (4) there are 2 conditions but in (2) there are 3) properties of IGWAs are less predictable and different in comparison to the ones of GWAs (Theorem 1.1). The proof of existence of IGWAs (Theorem 1.1) is much more complicated and involved than to the proof for GWAs [9, Theorem 2.2]. In Section 4, the IGWAs of rank $n > 1$ are introduced.

Existence of inner generalized Weyl algebras. Theorem 1.1 shows consistency of the conditions in (4) and the defining relations (5). So, for an arbitrary choice of σ, τ and a that satisfy (4) the IGWA $A = D[x, y; \sigma, \tau, a]_{\rm in}$ exists. We denote by $\overline{D}$ the image of the ring D under the homomorphism

$$\nu : D \to A, \quad d \mapsto d. \tag{6}$$

In general, $\ker(\nu) \neq 0$. In Section 2, an ideal D_0 of the ring D is introduced, see (21), such that $D_0 \subseteq \ker(\nu)$, and Proposition 2.2 is an explicit description of the ideal D_0. Theorem 2.11.(1) shows that $\ker(\nu) = D_0$.

The sum

$$D' := \sum_{i\geq 0} y^i \overline{D} x^i \tag{7}$$

is a subring of A such that $\overline{D} \subseteq D'$.

Theorem 1.1 *The IGWA $A = D[x, y; \sigma, \tau, a]_{\rm in}$ is a $\mathbb{Z}$-graded ring $A = \bigoplus_{i\in\mathbb{Z}} A_i$ where $A_0 = D' \neq 0$, $A_i = D'x^i \simeq {}_{D'}D'$ and $A_{-i} = y^i D' \simeq D'_{D'}$ for all $i \geq 1$. Furthermore, $D' \simeq \overline{D'}$, an isomorphism of D-bimodules given by (23), where the D-bimodule $\overline{D'}$ is defined in (22).*

The idea of the proof of Theorem 1.1 is to study, first, properties of the D-bimodule $\overline{D'} := {}'D/{}'\mathcal{T}$ (see Section 2) where

$${}'D \simeq \bigoplus_{i\geq 0} {}^{\tau^i}D^{\sigma^i}$$

is an infinite sum of twisted D-bimodules and $'\mathcal{T}$ is an explicit D-subbimodule of $'D$, see (20). The reason for that is the fact that the D-bimodules D' and $\overline{D'}$ are isomorphic (Theorem 1.1) and properties of the IGWA A are mainly determined by the ring D'. In Section 2, the D-bimodule $\overline{D'}$ is studied in detail. As a result we have plenty of information about the ring D'. An explicit description of elements of the ring D' (and an explicit description of a basis of D' where D is an algebra) is given in Theorem 2.6. The ideal D_0 is σ- and τ-invariant (Corollary 2.9). This fact implies that σ and τ yield ring endomorphisms of the factor ring

$$\overline{D} = D/D_0$$

(see (32)) that can be extended to ring endomorphisms σ and τ of the ring D', respectively (see (33)). *The IGWA $A = D[x, y; \sigma, \tau, a]_{\rm in}$ is canonically isomorphic to the IGWA $\overline{A} = \overline{D}[x, y; \sigma, \tau, \overline{a}]_{\rm in}$ where $\overline{a} = \nu(a)$* (Theorem 2.16). Theorem 2.16 implies that the ring $\overline{D}$ is a *subring* of the ring $\overline{A}$. So, in order to study IGWAs without loss of generality we can assume that the ring D is a *subring* of A.

Theorem 2.11 gives a criterion for the ring homomorphism $\nu : D \to D'$ to be a mono-, epi- or isomorphism. Properties of the rings D and D' are almost unrelated, in general. For example, for a free algebra in infinitely many variables $D = K\langle x_1, x_2, \ldots\rangle$ over a field K we can have $D' = K$ and $A \simeq K[x, x^{-1}]$ is a commutative K-algebra (Corollary 2.20). For each natural number $n \geq 0$, the sum $\sum_{i=0}^{n} y^i \overline{D} x^i$ is a subring of D'. So, in general, the ring D' is not finitely generated over $\overline{D}$ (Theorem 2.22). Lemma 2.18 describes 4 important classes of IGWAs, it is a source of examples of IGWAs. Plenty of examples of IGWAs are given at the end of Section 2 (Corollary 2.20, Lemma 2.21, Theorem 2.22, Theorem 2.23, Lemma 2.24 and Corollary 3.12).

The derivative series of IGWAs $A^{(\alpha)}$ associated with an IGWA A. For each IGWA $A = D[x, y; \sigma, \tau, a]_{\rm in}$ we can attached the IGWA,

$$A' = D'[x_1, y_1; \sigma_1, \tau_1, a]_{\rm in},$$

its *first derivative*, where σ_1 and τ_1 are extensions of σ and τ to D'. Repeating the process repeatedly, we obtain the *derivative series* of IGWAs,

$$A \to A' \to A'' \to \cdots \to A^{(\alpha)} \to \cdots,$$

for all ordinals α. In general, the IGWAs $A^{(\alpha)}$ are distinct (Lemma 2.24). Lemma 2.24 describes the rings $A^{(\alpha)}$ in an explicit way for the IGWA $A = D[x, y; \sigma, \tau, 0]_{\rm in}$ for arbitrary σ and τ.

Connections of IGWAs with GWAs. In general, the classes of IGWAs and GWAs are distinct (Lemma 2.19, Corollary 2.20 and Lemma 2.21) but their intersection is a large class as the following proposition demonstrates.

Proposition 1.2 *Let $A = D[x, y; \sigma, \tau, a]_{\rm in}$ be an IGWA. If the endomorphisms τ (resp., σ) of D is an automorphism then $A = D[x, y; \sigma, \tau^{-1}, a]$ is a GWA (resp., A is a right GWA).*

Simplicity criterion for IGWAs (general case). Let D be a ring and σ be its ring endomorphism. The subring of D, $D^\sigma = \{d \in D \,|\, \sigma(d) = d\}$, is called the *ring of σ-invariants*, and each element of D^σ is called a *σ-invariant*. An element $d \in D$ is called *left* (resp., *right*) *regular* if $d_1 d = 0$ (resp., $dd_1 = 0$), where $d_1 \in D$, implies that $d_1 = 0$. A left and right regular element is called a *regular element*. Every left normal, left regular element d of D yields a ring endomorphism of D:

$$\omega_d : D \to D, \;\; d_1 \mapsto \omega_d(d_1), \;\; \text{where} \;\; dd_1 = \omega_d(d_1)d. \tag{8}$$

Every right normal, right regular element d of D yields a ring endomorphism of D:

$$\omega'_d : D \to D, \;\; d_1 \mapsto \omega'_d(d_1), \;\; \text{where} \;\; d_1 d = d\omega'_d(d_1). \tag{9}$$

Theorem 1.3 gives a simplicity criterion for IGWAs.

Theorem 1.3 *Let $A = D[x,y;\sigma,\tau,a]_{\rm in}$. The following statements are equivalent:*

1. *A is a simple ring.*
2. *(a) The elements a and $\sigma(a)$ are regular in D',*

   (b) For all nonzero ideals I of D', $I' := I + \sum_{i\geq 1}\left(y^iIx^i + D'\sigma^i(I)\sigma^i(a)\cdots\sigma(a)D'\right) = D$.

   (c) None of the ring endomorphisms σ^n $(n \geq 1)$ of D' is equal to the ring endomorphism ω_d (see (8)) where d is a σ-invariant, regular, left normal element of D'.
3. *(a) The elements a and $\tau(a)$ are regular in D',*

   (b) For all nonzero ideals I of D', $I' := I + \sum_{i\geq 1}\left(y^iIx^i + D'\tau(a)\cdots\tau^i(a)\tau^i(I)D'\right) = D$.

   (c) None of the ring endomorphisms τ^n $(n \geq 1)$ of D' is equal to the ring endomorphism ω'_d (see (9)) where d is a τ-invariant, regular, right normal element of D'.

If one of the equivalent conditions holds then $D \subseteq D'$, σ and τ are monomorphisms of D', the elements $\sigma^i(a)$ are left regular, the elements $\tau^i(a)$ are right regular, and the elements a and $\sigma^i(a)\cdots\sigma(a) = \tau(a)\cdots\tau^i(a)$ are regular in the ring D' for all $i \geq 1$.

Remark. In view of (10) and (11), the set I' in statement 2(b) is equal to the set I' in statement 3(b).

In [11], Jespers proved that if a group G is an abelian group and a ring R is G-graded, then the ring R is a simple ring if and only if it is *graded-simple* (i.e., every G-graded ideal of R is R) and the centre of R is a field (see also the papers of Jespers [12], and Nystedt and Oinert [15] for further generalizations).

Simplicity criteria for inner generalized Weyl algebras where either the endomorphism σ or τ is an epimorphism of D. Let D be a ring and σ be its ring endomorphism. An ideal I of D is called *σ-stable* if $\sigma(I) = I$. The ring D is called a *σ-simple* ring iff 0 and D are the only σ-stable ideals of the ring D. An endomorphism σ is *inner* if $\sigma = \omega_u$ for some unit $u \in D$ where $\omega_u(d) = udu^{-1}$ for all $d \in D$. Theorem 1.4 (resp., Theorem 1.5) is a simplicity criterion for an IGWA where σ (resp., τ) is an epimorphism of D.

Theorem 1.4 *Let $A = D[x,y;\sigma,\tau,a]_{\rm in}$. Suppose that σ is an epimorphism of D. Then the following statements are equivalent:*

1. *A is a simple ring.*
2. *(a) The elements a and $\sigma(a)$ are regular in $\overline{D}$,*

   (b) $\overline{D}$ is a σ-simple ring,

   (c) for all $i \geq 1$, σ^i is not an inner automorphism of the ring $\overline{D}$, and

   (d) for all $i \geq 1$, $a\overline{D} + \sigma^i(a)\overline{D} = \overline{D}$.

If one of the equivalent conditions holds then $D' = \overline{D}$, σ is an automorphism and $\tau = \omega'_{\sigma(a)}\sigma$ is a monomorphism of $\overline{D}$, $S_x^{-1}A \simeq \overline{D}[x,x^{-1};\sigma]$ is a skew Laurent polynomial ring ($x^{\pm 1}d = \sigma^{\pm 1}(d)x^{\pm 1}$ for all $d \in \overline{D}$, and $S_x := \{x^i \mid i \geq 0\}$), the elements a and $\sigma(a)$ are right normal in $\overline{D}$.

Theorem 1.5 *Let $A = D[x,y;\sigma,\tau,a]_{\rm in}$. Suppose that τ is an epimorphism of D. Then the following statements are equivalent:*

1. *A is a simple ring.*
2. *(a) The elements a and $\sigma(a)$ are regular in $\overline{D}$,*

   (b) $\overline{D}$ is a τ-simple ring,

   (c) for all $i \geq 1$, τ^i is not an inner automorphism of the ring $\overline{D}$, and

   (d) for all $i \geq 1$, $\overline{D}a + \overline{D}\tau^i(a) = \overline{D}$.

If one of the equivalent conditions holds then $D' = \overline{D}$, τ is an automorphism and $\sigma = \omega_{\sigma(a)}\tau$ is a monomorphism of $\overline{D}$, $AS_y^{-1} \simeq \overline{D}[y, y^{-1}; \tau]_r$ is a right skew Laurent polynomial ring ($dy^{\pm 1} = y^{\pm 1}\tau^{\pm 1}(d)$ for all $d \in \overline{D}$), the elements a and $\tau(a) = \sigma(a)$ are left normal in $\overline{D}$.

Lemma 2.19 shows that in Theorem 1.4 and Theorem 1.5, $\overline{D} \neq D$, in general. A particular case of Theorem 1.4 (resp., Theorem 1.5) where σ (resp., τ) is an automorphism of the ring D is Corollary 3.8 (resp., Corollary 3.9). Corollary 3.10 is the case where both endomorphisms σ and τ of D are automorphisms.

In Section 3, a criterion is given for an IGWA to be a domain (Proposition 3.4). Necessary and sufficient conditions are found for the elements x and y of an IGWA A to be regular elements (Proposition 3.3).

The centre of an IGWA. The next theorem describes the centre of an IGWA.

Theorem 1.6 *Let $A = D[x, y; \sigma, \tau, a]_{\rm in}$. Then the centre of A, $Z(A) = \bigoplus_{i \geq 1} y^i \mathcal{Z}_{-i} \oplus \bigoplus_{i \geq 0} \mathcal{Z}_i x^i$, is a $\mathbb{Z}$-graded subring of A where $\mathcal{Z}_0 = Z(D)^{\sigma,\tau} := \{d \in D' \,|\, \sigma(d) = d, \tau(d) = d\}$ and, for $i \geq 1$, $\mathcal{Z}_i = \{\alpha \in D'^{\sigma} \,|\, y\alpha x = \alpha\sigma^i(a), d\alpha = \alpha\sigma^i(d)$ for all $d \in D'\}$ and $\mathcal{Z}_{-i} = \{\beta \in D'^{\tau} \,|\, y\beta x = \tau^i(a)\beta, \beta d = \tau^i(d)\beta$ for all $d \in D'\}$.*

Involutions on GWAs. An anti-isomorphism $*$ of a ring R ($(ab)^* = b^*a^*$ for all $a, b \in R$) is called an *involution* if $a^{**} = a$ for all elements $a \in R$.

Proposition 1.7 *Let $A = D[x, y; \sigma, \tau, a]_{\rm in}$. Suppose that the ring D is equipped with an involution $*$ such that $\tau = *\sigma*$ and $a^* = a$. Then the involution $*$ can be extended to an involution $*$ of the ring A by the rule $x^* = y$ and $y^* = x$.*

2 Inner generalized Weyl algebras

Throughout the paper $A = D[x, y; \sigma, \tau, a]_{\rm in}$ is an IGWA. The aim of this section is to prove that the construction of inner generalized Weyl algebras is consistent (Theorem 1.1). In the first half of the section, we study the D-bimodule $\overline{D'}$ in great detail. As we mentioned already in the Introduction, the reason for this is the fact that $\overline{D'} \simeq D'$ as D-bimodules. In particular, we show that $D' \neq 0$ (Corollary 2.3.(2)) and this is the main reason for $A \neq 0$.

The elements $(n, -n)$ of D where $n \geq 1$. For each natural number $n \geq 1$, let $(n, -n) := \sigma^n(a) \cdots \sigma(a)$. Then, by (4),

$$(n, -n) = \sigma^n(a)\sigma^{n-1}(a) \cdots \sigma(a) = \tau(a)\tau^2(a) \cdots \tau^n(a). \tag{10}$$

Proof. The case $n = 1$, $\sigma(a) = \tau(a)$, is given, see (4). Now, we obtain the equality by induction on n:

$$\sigma^{n+1}(a) \cdots \sigma(a) = \sigma((n, -n))\sigma(a) \overset{(4)}{=} \sigma(a)\tau((n, -n)) \overset{(4)}{=} \tau(a) \cdots \tau^{n+1}(a). \quad \square$$

Each element $(n, -n)$ can be written in 2^n different ways similar to (10), see (12). To prove this claim we need one more property of the elements $(n, -n)$, see (11). For all elements $d \in D$ and $n \geq 1$,

$$\sigma^n(d)(n, -n) = (n, -n)\tau^n(d). \tag{11}$$

Proof. The case $n = 1$, $\sigma(d)\sigma(a) = \sigma(a)\tau(d)$, is given, see (4). Now, by induction on n we obtain the equality:

$$\begin{aligned} \sigma^n(d)(n, -n) &= \sigma^{n-1}(\sigma(d)\sigma(a))(n-1, -n+1) \overset{(4)}{=} \sigma^{n-1}(\sigma(a)\tau(d))(n-1, -n+1) \\ &= \sigma^n(a)\sigma^{n-1}(\tau(d))(n-1, -n+1) \overset{\rm ind.}{=} \sigma^n(a)(n-1, -n+1)\tau^n(d) = (n, -n)\tau^n(d). \quad \square \end{aligned}$$

Let $\Pi = \{+, -\}$. For each natural number $n \geq 2$ and an element $\varepsilon = (\varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_2) \in \Pi^{n-1}$, consider the following elements of the ring D,

$$a_{n,\varepsilon} := \mu_n^{\varepsilon_n}\mu_{n-1}^{\varepsilon_{n-1}} \cdots \mu_2^{\varepsilon_2}\sigma(a) \;\text{ and }\; a'_{n,\varepsilon} := \mu_n^{\varepsilon_n}\mu_{n-1}^{\varepsilon_{n-1}} \cdots \mu_2^{\varepsilon_2}\tau(a)$$

where the maps $\mu_i^{\varepsilon_i} : D \to D$ are given by the rule:

$$\mu_i^{\varepsilon}(d) = \begin{cases} \sigma^i(a)d & \text{if } \varepsilon = +, \\ d\tau^i(a) & \text{if } \varepsilon = -. \end{cases}$$

For example, $a_{5,(+,-,-,+)} = \sigma^5(a)\sigma^2(a)\sigma(a)\tau^3(a)\tau^4(a)$ and $a'_{5,(+,-,-,+)} = \sigma^5(a)\sigma^2(a)\tau(a)\tau^3(a)\tau^4(a)$; $a_{n,(+,\dots,+)} = \sigma^n(a)\cdots\sigma(a)$ and $a'_{n,(-,\dots,-)} = \tau(a)\cdots\tau^n(a)$.

Then, for all $n \geq 2$ and $\varepsilon \in \Pi^{n-1}$,

$$(n,-n) = a_{n,\varepsilon} = a'_{n,\varepsilon}. \tag{12}$$

Proof. Since $\sigma(a) = \tau(a)$, $a_{n,\varepsilon} = a'_{n,\varepsilon}$ for all $\varepsilon \in \Pi^{n-1}$. To finish the proof it suffices to show that $(n,-n) = a_{n,\varepsilon}$ for all $\varepsilon \in \Pi^{n-1}$. For $n = 2$, the elements $a_{2,+} = \sigma^2(a)\sigma(a)$ and $a_{2,-} = \sigma(a)\tau^2(a) = \tau(a)\tau^2(a)$ are equal, by (10). Suppose that $n > 2$. Then, we obtain the result by induction on n:

$$\begin{aligned} a_{n,\varepsilon} &= \mu_n^{\varepsilon_n}((n-1,-n+1)) = \begin{cases} \sigma^n(a)(n-1,-n+1) & \text{if } \varepsilon_n = +, \\ (n-1,-n+1)\tau^n(a) & \text{if } \varepsilon_n = -, \end{cases} = \begin{cases} \sigma^n(a)\cdots\sigma(a) & \text{if } \varepsilon_n = +, \\ \tau(a)\cdots\tau^n(a) & \text{if } \varepsilon_n = -, \end{cases} \\ &\overset{(10)}{=} (n,-n). \quad \square \end{aligned}$$

IGWA is a $\mathbb{Z}$-graded ring. Let $A = D[x,y;\sigma,\tau,a]_{\rm in}$. The defining relations of the IGWA A show that the algebra A admits a $\mathbb{Z}$-grading where the elements y, $d \in D$, and x have graded degree -1, 0 and 1, respectively:

$$A = \bigoplus_{i \in \mathbb{Z}} A_i \;\; \text{where} \;\; A_i = \begin{cases} D'x^i & \text{if } i \geq 1, \\ D' & \text{if } i = 0, \\ y^{|i|}D' & \text{if } i \leq -1, \end{cases} \tag{13}$$

and $D' = \sum_{i \geq 0} y^i D x^i$ is a subring of A, the zero graded component of A. There is a natural ring homomorphism (see (6)),

$$\nu : D \to D', \;\; d \mapsto d, \tag{14}$$

which is neither a monomorphism nor an epimorphism, in general (Theorem 2.11). Abusing notation, an element $d + \ker(\nu)$ of D', where $d \in D$, is written as d (see (14)). We identify the image of ν,

$$\overline{D} = {\rm im}(\nu),$$

with the factor ring $D/\ker(\nu)$. The ring D' is a D-bimodule. The rings D' and D have very complicated relations (see Theorem 2.11). The multiplication in the ring D' is given by the rule: For all natural numbers $i, j \geq 0$ and elements $d_1, d_2 \in D$,

$$y^i d_1 x^i \cdot y^j d_2 x^j = \begin{cases} y^i d_1 \sigma^{i-j}((j,-j)d_2)x^i & \text{if } i \geq j, \\ y^j \tau^{j-i}(d_1(i,-i))d_2 x^j & \text{if } i < j. \end{cases} \tag{15}$$

In more detail, for $i \geq 1$, $x^i y^i = x^{i-1}\sigma(a)y^{i-1} = \sigma^i(a)x^{i-1}y^{i-1} = \cdots = \sigma^i(a)\cdots\sigma(a) = (i,-i)$, and then

$$\begin{aligned} i \geq j: &\quad y^i d_1 x^i \cdot y^j d_2 x^j = y^i d_1 x^{i-j}(j,-j)d_2 x^j = y^i d_1 \sigma^{i-j}((j,-j)d_2)x^i \\ i < j: &\quad y^i d_1 x^i \cdot y^j d_2 x^j = y^i d_1 (i,-i) y^{j-i} d_2 x^j = y^j \tau^{j-i}(d_1(i,-i))d_2 x^j. \;\; \square \end{aligned}$$

In particular, for all integers $i, j \geq 0$,

$$y^i D x^i \cdot y^j D x^j \subseteq y^k D x^k \;\; \text{where} \;\; k = \max\{i,j\}. \tag{16}$$

So, the ring D' contains the descending chain of *ideals*:

$$D' = D'_{\geq 0} \supseteq D'_{\geq 1} \supseteq \cdots \supseteq D'_{\geq n} \supseteq \cdots \supseteq D'_{\infty} := \bigcap_{n \geq 0} D'_n \;\; \text{where} \;\; D'_{\geq n} := \sum_{i \geq n} y^i D x^i. \tag{17}$$

Proposition 2.12 is a criterion for the ideals $D'_{\geq n}$ to be distinct/equal. Furthermore, the ring D' contains the ascending chain of *subrings*:

$$\overline{D} = D'_{\leq 0} \subseteq D'_{\leq 1} \subseteq \cdots \subseteq D'_{\leq n} \subseteq \cdots \subseteq D' = \bigcup_{n\geq 0} D'_{\leq n} \;\text{ where }\; D'_{\leq n} := \sum_{0\leq i\leq n} y^i D x^i. \tag{18}$$

By (16), each subring $D'_{\leq n}$ is a D-*subbimodule* of D'.

Let R be a ring, $\mathrm{Aut}(R)$ be the group of ring automorphisms of R, $\mathrm{End}(R)$ be the semigroup of ring endomorphisms of R, $\alpha \in \mathrm{End}(R)$ and M be an R-module. The $\mathbb{Z}$-module M has another structure of R-module given by the rule:

$$r \cdot m = \alpha(r) m \;\text{ for all }\; r \in R \;\text{ and }\; m \in M.$$

The new module ${}^{\alpha}M$ is called *the R-module M twisted by the endomorphism α*. If N be an R-bimodule and $\alpha, \beta \in \mathrm{End}(R)$ then in a similar way the twisted R-bimodule ${}^{\alpha}N^{\beta}$ is defined:

$$r_1 \cdot n \cdot d_2 = \alpha(d_1) n \beta(d_2) \;\text{ for all elements }\; n \in N \;\text{ and }\; d_1, d_2 \in R.$$

The D-bimodules $'D$ and $'\mathcal{T}$. The D-bimodules $'D$ and $'\mathcal{T}$ that we are going to introduce are instrumental in the proof of Theorem 1.1. Let

$$'D := \bigoplus_{i\geq 0} y^i D x^i \tag{19}$$

be a direct sum of D-bimodules where $y^0 D x^0 := D$ and for $i \geq 1$ the D-bimodule $y^i D x^i$ is, by definition, the twisted D-bimodule ${}^{\tau^i}D^{\sigma^i}$, i.e., the map

$${}^{\tau^i}D^{\sigma^i} \to y^i D x^i, \;\; d \mapsto y^i d x^i$$

is an isomorphism of $\mathbb{Z}$-modules and, for all elements $d, d_1, d_2 \in D$,

$$d_1 \cdot y^i d x^i \cdot d_2 = y^i \tau^i(d_1) d \sigma^i(d_2) x^i.$$

Let $\mathcal{T}$ be the $\mathbb{Z}$-submodule of D generated by the set $\tau(D)\sigma(D)$. Each element of $\mathcal{T}$ is a finite sum of the type $\sum_{i=1}^n \tau(d_i)\sigma(d'_i)$ for some elements $d_i, d'_i \in D$. Abusing the notation we write $\mathcal{T} = \tau(D)\sigma(D)$. Let

$$\widetilde{D} := D/\mathcal{T}.$$

If we treat the $\mathbb{Z}$-module D as the twisted D-bimodule ${}^{\tau}D^{\sigma}$ then $\mathcal{T}$ is a D-subbimodule of ${}^{\tau}D^{\sigma}$ (since $\tau(D)\mathcal{T}\sigma(D) \subseteq \mathcal{T}$), and then $\widetilde{D}$ is the D-bimodule ${}^{\tau}D^{\sigma}/\mathcal{T}$.

For each natural number i, there is a $\mathbb{Z}$-homomorphism

$$s_i : {}'D \to {}'D, \;\; d' \mapsto y^i d' x^i \;\; (\text{i.e., } y^j d x^j \mapsto y^{i+j} d x^{i+j} \;\text{ for all } j \geq 0 \text{ and } d \in D).$$

For all $i \geq 1$, $s_i = s_1^i$. The map s_i is an injection and $\mathrm{im}(s_i) = \bigoplus_{j\geq i} y^j D x^j$ is a D-subbimodule of $'D$ which is isomorphic to ${}^{\tau^i}({}'D)^{\sigma^i}$, that is $s_i : {}^{\tau^i}{}'D^{\sigma^i} \to \mathrm{im}(s_i)$ is a D-bimodule isomorphism. Clearly,

$$'D = \mathrm{im}(s_0) \supseteq \mathrm{im}(s_1) \supset \cdots \supset \mathrm{im}(s_n) \supset \cdots$$

is a strictly descending chain of D-subbimodules of $'D$ such that $\bigcap_{n\geq 0} \mathrm{im}(s_n) = 0$. The D-bimodule $'D$ contains the ascending chain of D-subbimodules

$$D = {}'D_{\leq 0} \subset {}'D_{\leq 1} \subset \cdots \subset {}'D_{\leq n} \subset \cdots \subset {}'D = \bigcup_{n\geq 0} {}'D_{\leq n} \;\text{ where }\; {}'D_{\leq n} := \bigoplus_{0\leq i\leq n} y^i D x^i.$$

Definition. Let $'\mathcal{T}$ be a $\mathbb{Z}$-submodule of $'D$ generated by the set of elements

$$\{ y^i \tau(d_1)\sigma(d_2) x^i - y^{i-1} d_1 a d_2 x^{i-1} \mid d_1, d_2 \in D, i \geq 1 \}. \tag{20}$$

The idea of introducing $'\mathcal{T}$ is inspired by the following relations in the IGWA $A = D[x, y; \sigma, \tau, a]_{\rm in}$:

$$y^i\tau(d_1)\sigma(d_2)x^i = y^{i-1}\cdot y\tau(d_1)\sigma(d_2)x\cdot x^{i-1} \stackrel{(5)}{=} y^{i-1}\cdot d_1yxd_2\cdot x^{i-1} \stackrel{(5)}{=} y^{i-1}d_1ad_2x^{i-1}.$$

In fact, *$'\mathcal{T}$ is a D-subbimodule of $'D$*: For all elements $d_1', d_2' \in D$,

$$d_1'(y^i\tau(d_1)\sigma(d_2)x^i - y^{i-1}d_1ad_2x^{i-1})d_2' = y^i\tau(\tau^{i-1}(d_1')d_1)\sigma(d_2\sigma^{i-1}(d_2'))x^i - y^{i-1}\tau^{i-1}(d_1')d_1ad_2\sigma^{i-1}(d_2')x^{i-1}.$$

For all $i \geq 0$, $s_i('\mathcal{T}) \subseteq {}'\mathcal{T}$. The D-bimodule $'\mathcal{T}$ admits the induced ascending filtration $\{'\mathcal{T}_{\leq n}\}_{n\geq 0}$ where

$$'\mathcal{T}_{\leq n} := {}'D_{\leq n} \cap {}'\mathcal{T}.$$

Clearly, $'\mathcal{T} = \bigcup_{n\geq 0} {}'\mathcal{T}_{\leq n}$ and each $'\mathcal{T}_{\leq n}$ is a D-subbimodule of $'\mathcal{T}$. The zero component of this filtration,

$$D_0 := {}'\mathcal{T}_{\leq 0} = D \cap {}'\mathcal{T}, \tag{21}$$

which is an ideal of D, is a key to study the structure of the ring D' (the zero component of the IGWA A) and to prove that $A \neq 0$ (Theorem 1.1).

The D-bimodule $'\mathcal{T}$ has another ascending filtration $\{'\mathcal{T}_m\}_{m\geq 1}$ where $'\mathcal{T}_m$ is a $\mathbb{Z}$-submodule of $'\mathcal{T}$ generated by the elements in (20) where $1 \leq i \leq m$. Clearly, $'\mathcal{T}_m$ is a D-subbimodule of $'\mathcal{T}$ such that $'\mathcal{T}_m \subseteq {}'\mathcal{T}_{\leq m}$ and $'\mathcal{T} = \bigcup_{m\geq 1} {}'\mathcal{T}_m$.

Definition. Let us consider the factor D-bimodule

$$\overline{D'} := {}'D/{}'\mathcal{T}. \tag{22}$$

Clearly, the $\mathbb{Z}$-homomorphism

$$\iota : \overline{D'} \to D', \;\; y^idx^i + {}'\mathcal{T} \mapsto y^idx^i \;\; (d \in D, \; i \geq 0) \tag{23}$$

is a D-bimodule *epimorphism*. We will see that ι is an isomorphism of D-bimodules, $\overline{D'} \simeq D'$ (Theorem 1.1). *This isomorphism is used in the proofs of almost all statements about the ring D'.* So, first we study the D-bimodule $\overline{D'}$ in great detail (Lemma 2.1 – Theorem 2.8). These results are used in the proof of Theorem 1.1 which also shows that the map ι is an isomorphism. As soon as this fact is proven then Lemma 2.1 – Theorem 2.8 become statements about the ring D' (and its ideals).

Since $s_i('\mathcal{T}) \subseteq {}'\mathcal{T}$ for all $i \geq 1$, we have the induced maps

$$\overline{s}_i : \overline{D'} \to \overline{D'}, \;\; y^jdx^j + {}'\mathcal{T} \mapsto y^{i+j}dx^{i+j} + {}'\mathcal{T}$$

where $d \in D$ and $j \geq 0$. Notice that $\overline{s}_i = \overline{s}_1^i$.

The maps θ and ϕ. The $\mathbb{Z}$-homomorphism, where $\otimes := \otimes_{\mathbb{Z}}$,

$$\theta : D \otimes D \stackrel{\tau\otimes\sigma}{\longrightarrow} D \otimes D \stackrel{\cdot}{\longrightarrow} {}^{\tau}D^{\sigma}, \;\; d \otimes e \mapsto \tau(d)\sigma(e) \tag{24}$$

is a D-bimodule homomorphism since for all elements $d_1, d_2 \in D$ and $\delta \in D \otimes D$,

$$\theta(d_1\delta d_2) = \tau(d_1)\theta(\delta)\sigma(d_2) = d_1 \cdot \theta(\delta) \cdot d_2.$$

Clearly, $\mathrm{im}(\theta) = \mathcal{T}$ is a D-subbimodule of ${}^{\tau}D^{\sigma}$. Let $\mathcal{K} = \ker(\theta)$. Then $\ker(\tau)\otimes D + D\otimes \ker(\sigma) \subseteq \mathcal{K}$ and we have the short exact sequence of D-bimodules

$$0 \to \mathcal{K} \to D \otimes D \stackrel{\theta}{\to} \mathcal{T} \to 0. \tag{25}$$

Let us consider the D-bimodule homomorphism

$$\phi := \cdot a \cdot : D \otimes D \to D, \;\; d \otimes e \mapsto dae. \tag{26}$$

Its image $\mathrm{im}(\phi) = DaD = (a)$ is an ideal of the ring D. Let $\mathcal{K}_a := \ker(\phi)$. Then $\mathrm{l.ann}_D(a) \otimes D + D \otimes \mathrm{r.ann}_D(a) \subseteq \mathcal{K}_a$ and we have the s.e.s. of D-bimodules

$$0 \to \mathcal{K}_a \to D \otimes D \stackrel{\phi}{\to} (a) \to 0. \tag{27}$$

The following lemma reveals connections between the endomorphisms σ and τ, the ring D and the maps ϕ and θ. The equalities of the lemma are used in many proofs of the paper.

Lemma 2.1 *Let $A = D[x,y;\sigma,\tau,a]_{\rm in}$ and $\mathbb{K} := \ker(\sigma)\cap\ker(\tau)$ which is an ideal of the ring D. Then*

1. *For all elements $\delta\in D\otimes D$, $\sigma\phi(\delta) = \sigma(a)\theta(\delta)$ and $\tau\phi(\delta) = \theta(\delta)\sigma(a) = \theta(\delta)\tau(a)$.*
2. *$\phi(\mathcal{K})\subseteq {\rm im}(\phi)\cap\mathbb{K} = (a)\cap\mathbb{K}$.*

Proof. 1. Let $\delta = \sum_i \alpha_i\otimes\beta_i\in D\otimes D$ where $\alpha_i,\beta_i\in D$. Then

$$\sigma\phi(\delta) = \sum_i \sigma(\alpha_i)\sigma(a)\sigma(\beta_i) \overset{(4)}{=} \sigma(a)\sum_i \tau(\alpha_i)\sigma(\beta_i) = \sigma(a)\theta(\delta),$$

$$\tau\phi(\delta) = \sum_i \tau(\alpha_i)\tau(a)\tau(\beta_i) \overset{(4)}{=} \sum_i \tau(\alpha_i)\sigma(a)\tau(\beta_i) \overset{(4)}{=} \left(\sum_i \tau(\alpha_i)\sigma(\beta_i)\right)\cdot\sigma(a) = \theta(\delta)\sigma(a).$$

2. By statement 1, $\sigma\phi(\mathcal{K}) = \sigma(a)\theta(\mathcal{K}) = 0$ and $\tau\phi(\mathcal{K}) = \theta(\mathcal{K})\sigma(a) = 0$ since $\mathcal{K} = \ker(\theta)$. Hence, $\phi(\mathcal{K})\subseteq {\rm im}(\phi)\cap\mathbb{K} = (a)\cap\mathbb{K}$ (since ${\rm im}(\phi) = (a)$). $\square$

For each natural number $i\geq 1$, the sum

$$T_i = \sum_{\delta\in D\otimes D} (y^i\theta(\delta)x^i - y^{i-1}\phi(\delta)x^{i-1})$$

is a D-subbimodule of $'\mathcal{T}$. For all $i,j\geq 0$, $s_i(T_j) = T_{i+j}$. In particular, $T_{i+1} = s_1(T_i)$. By the definition of the sets $'\mathcal{T}$ and $'\mathcal{T}_m$ ($m\geq 1$): $'\mathcal{T} = \sum_{i\geq 1} T_i$ and $'\mathcal{T}_m = \sum_{i=1}^m T_i$. So, each element of $'\mathcal{T}_m$ is a sum

$$\sum_{i=1}^m \left(y^i\theta(\delta_i)x^i - y^{i-1}\phi(\delta_i)x^{i-1}\right) \text{ for some elements } \delta_1,\ldots,\delta_m\in D\otimes D.$$

The D-bimodule $'\mathcal{T}$ contains the descending chain of D-subbimodules $\{T_{\geq m}\}_{m\geq 0}$ where $T_{\geq m} := \sum_{i\geq m} T_i$. For all $m\geq 1$, $'\mathcal{T} = {}'\mathcal{T}_m + T_{\geq m+1}$.

Let R be a ring and $\sigma : R\to R$ be a ring endomorphism. Then

$$\ker(\sigma)\subseteq\ker(\sigma^2)\subseteq\cdots\subseteq\ker(\sigma^i)\subseteq\cdots$$

is an ascending chain of ideals of R, and their union

$$\mathcal{K}(\sigma) := \mathcal{K}_R(\sigma) := \bigcup_{i\geq 1}\ker(\sigma^i) \tag{28}$$

is an ideal of R such that $\sigma(\mathcal{K}(\sigma))\subseteq\mathcal{K}(\sigma)$ (since $\sigma(\ker(\sigma)) = 0$ and $\sigma(\ker(\sigma^i))\subseteq\ker(\sigma^{i-1})$ for all $i\geq 1$). Let $R(\sigma) := R/\mathcal{K}(\sigma)$. Then the map

$$\overline{\sigma} : R(\sigma)\to R(\sigma),\ \ r+\mathcal{K}(\sigma)\mapsto\sigma(r)+\mathcal{K}(\sigma) \tag{29}$$

is a ring *monomorphism*.

Description of the ideals $D_{0,m}$ of the ring D where $m\geq 1$. The ideal $D_0 = D\cap{}'\mathcal{T}$ of the ring D admits the induced ascending filtration $\{D_{0,m}\}_{m\geq 1}$ where

$$D_{0,m} := D_0\cap{}'\mathcal{T}_m = D\cap{}'\mathcal{T}_m$$

is an ideal of the ring D (since $D_0 = D\cap{}'\mathcal{T}\subseteq{}'\mathcal{T} = \bigcup_{m\geq 1}{}'\mathcal{T}_m$). So,

$$D_{0,1}\subseteq D_{0,2}\subseteq\cdots\subseteq D_{0,m}\subseteq\cdots\subseteq D_0 = \bigcup_{m\geq 1} D_{0,m}$$

is an ascending chain of ideals of the ring D.

The next proposition presents an explicit description of the ideals D_0 and $D_{0,m}$, where $m\geq 1$, of the ring D.

Proposition 2.2 *1.* $D_{0,1}=\phi(\mathcal{K})\subseteq(a)\cap\mathbb{K}$.

2. *For all* $m\geq 2$, $D_{0,m}=\{\phi(\delta_1)\,|$ *there exists an element* $(\delta_1,\dots,\delta_m)\in(D\otimes D)^m$ *such that* $\delta_m\in\mathcal{K}$ *and* $\theta(\delta_i)=\phi(\delta_{i+1})$ *for* $i=1,\dots,m-1\}$. *So,* $D_0=\{\phi(\delta_1)\,|$ *there exists an element* $(\delta_1,\dots,\delta_m)\in(D\otimes D)^m$ *for some* $m\geq 1$ *such that* $\delta_m\in\mathcal{K}$ *and* $\theta(\delta_i)=\phi(\delta_{i+1})$ *for* $i=1,\dots,m-1\}$.

3. *Let an element* $(\delta_1,\dots,\delta_m)\in(D\otimes D)^m$ *be as in statement 2. Then for all natural numbers* i,j *such that* $1\leq i,j\leq m$ *and* $i+j\leq m$,

   (a) $\sigma^j\phi(\delta_i)=\sigma^j(a)\cdots\sigma(a)\phi(\delta_{i+j})$ *and* $\tau^j\phi(\delta_i)=\phi(\delta_{i+j})\tau(a)\cdots\tau^j(a)$.

   (b) $\sigma^{m-1}\phi(\delta_1)=\sigma^{m-1}(a)\cdots\sigma(a)\phi(\delta_m)$ *and* $\tau^{m-1}\phi(\delta_1)=\phi(\delta_m)\tau(a)\cdots\tau^{m-1}(a)$.

   (c) $\sigma^{j+1}\phi(\delta_i)=\sigma^{j+1}(a)\cdots\sigma(a)\theta(\delta_{i+j})$ *and* $\tau^{j+1}\phi(\delta_i)=\theta(\delta_{i+j})\tau(a)\cdots\tau^{j+1}(a)$.

   (d) *If, in addition,* $i+j=m$ *then* $\sigma^{j+1}\phi(\delta_i)=0$ *and* $\tau^{j+1}\phi(\delta_i)=0$.

4. *For all* $m\geq 2$,

   (a) $\sigma^{m-1}(D_{0,m})\subseteq\sigma^{m-1}(a)\cdots\sigma(a)\phi(\mathcal{K})=\sigma^{m-1}(a)\cdots\sigma(a)D_{0,1}\subseteq\sigma^{m-1}(a)\cdots\sigma(a)\cdot\big((a)\cap\mathbb{K}\big)$. *In particular,*

$$D_{0,m}\subseteq(\sigma^{m-1})^{-1}\left(\sigma^{m-1}(a)\cdots\sigma(a)\phi(\mathcal{K})\right).$$

   (b) $\tau^{m-1}(D_{0,m})\subseteq\phi(\mathcal{K})\tau(a)\cdots\tau^{m-1}(a)=D_{0,1}\tau(a)\cdots\tau^{m-1}(a)\subseteq\big((a)\cap\mathbb{K}\big)\tau(a)\cdots\tau^{m-1}(a)$. *In particular,*

$$D_{0,m}\subseteq(\tau^{m-1})^{-1}\left(\phi(\mathcal{K})\tau(a)\cdots\tau^{m-1}(a)\right).$$

   (c) *Furthermore,* $\sigma^m(D_{0,m})=0$ *and* $\tau^m(D_{0,m})=0$, *i.e.,* $D_{0,m}\subseteq\ker(\sigma^m)\cap\ker(\tau^m)$.

5. $D_0\subseteq\mathcal{K}(\sigma)\cap\mathcal{K}(\tau)$ *where* $\mathcal{K}(\sigma):=\bigcup_{i\geq 1}\ker(\sigma^i)$ *and* $\mathcal{K}(\tau):=\bigcup_{i\geq 1}\ker(\tau^i)$. *In particular, if either* σ *or* τ *is a monomorphism then* $D_0=0$.

Proof. 1. An element $d=y\theta(\delta_1)x-\phi(\delta_1)$ of $'\mathcal{T}_1$ (where $\delta_1\in D\otimes D$) belongs to the set $D_{0,1}=D\cap{}'\mathcal{T}_1$ iff $\theta(\delta_1)=0$ iff $\delta_1\in\mathcal{K}$ iff $d\in\phi(\mathcal{K})$. So, $D_{0,1}=\phi(\mathcal{K})$. By Lemma 2.1.(2), $\phi(\mathcal{K})\subseteq(a)\cap\mathbb{K}$.

2. Similarly, by (19), an element $d=\sum_{i=1}^m(y^i\theta(\delta_i)x^i-y^{i-1}\phi(\delta_i)x^{i-1})$ of $'\mathcal{T}_m$ (where $\delta_i\in D\otimes D$) belongs to the set $D_{0,m}$ iff

$$y^i\theta(\delta_i)x^i=y^i\phi(\delta_{i+1})x^i\ \text{ for }\ i=1,\dots,m-1\ \text{ and }\ y^m\theta(\delta_m)x^m=0$$

iff the conditions in statement 2 hold, and in this case $d=-\phi(\delta_1)$ ($-D_{0,m}=D_{0,m}$ since the set $D_{0,m}$ is an ideal of the ring D).

3(a). The statement (a) follows from Lemma 2.1.(1) by induction on j: For $j=1$, $\sigma\phi(\delta_i)=\sigma(a)\theta(\delta_i)=\sigma(a)\phi(\delta_{i+1})$ and $\tau\phi(\delta_i)=\theta(\delta_i)\tau(a)$. For $j>1$, using induction we finish the proof,

$$\begin{aligned}\sigma^j\phi(\delta_i)&=\sigma\left(\sigma^{j-1}(a)\cdots\sigma(a)\phi(\delta_{i+j-1})\right)=\sigma^j(a)\cdots\sigma^2(a)\sigma\phi(\delta_{i+j-1})=\sigma^j(a)\cdots\sigma(a)\theta(\delta_{i+j-1})\\&=\sigma^j(a)\cdots\sigma(a)\phi(\delta_{i+j}),\\ \tau^j\phi(\delta_i)&=\tau\left(\phi(\delta_{i+j-1})\tau(a)\cdots\tau^{j-1}(a)\right)=\tau\phi(\delta_{i+j-1})\tau^2(a)\cdots\tau^j(a)=\theta(\delta_{i+j})\tau(a)\cdots\tau^j(a).\end{aligned}$$

3(b). The statement (b) is a particular case of the statement (a).

3(c). Apply the endomorphism σ (resp., τ) to the first (resp., second) equality in the statement (a) and then use Lemma 2.1.(1) to obtain the result.

3(d). The statement (d) follows from the statement (c) since $\delta_m\in\ker(\theta)$.

4. By Lemma 2.1.(2) and the statement 3(b), for all $m\geq 2$,

$$\begin{aligned}\sigma^{m-1}(D_{0,m})&\subseteq\sigma^{m-1}(a)\cdots\sigma(a)\phi(\mathcal{K})=\sigma^{m-1}(a)\cdots\sigma(a)D_{0,1}\subseteq\sigma^{m-1}(a)\cdots\sigma(a)\cdot\big((a)\cap\mathbb{K}\big),\\ \tau^{m-1}(D_{0,m})&\subseteq\phi(\mathcal{K})\tau(a)\cdots\tau^{m-1}(a)=D_{0,1}\tau(a)\cdots\tau^{m-1}(a)\subseteq\big((a)\cap\mathbb{K}\big)\tau(a)\cdots\tau^{m-1}(a).\end{aligned}$$

Now, by applying the endomorphisms σ and τ to the first and the second inclusion of the statement 4, respectively, we obtain that $\sigma^m(D_{0,m}) = 0$ and $\tau^m(D_{0,m}) = 0$ (since $\mathbb{K} = \ker(\sigma) \cap \ker(\tau)$).

5. Since $D_0 = \bigcup_{m\geq 1} D_{0,m}$, statement 5 follows from statement 4(c). $\square$

Proposition 2.2.(4) is a very effective tool in finding the ideals D_0 and $D_{0,m}$ ($m \geq 1$) of the ring D. The next corollary is the reason why the IGWA $A \neq \{0\}$ (Theorem 1.1).

Corollary 2.3 *1. $D_0 \neq D$.*

2. $\overline{D'} \neq 0$.

Proof. 1. Recall that $D = \bigcup_{m\geq 1} D_{0,m}$. Suppose that $D_0 = D$. Then $1 \in D_{0,m}$ for some $m \geq 1$ and then, by Proposition 2.2.(4),

$$1 = \sigma^{m+1}(1) \in \sigma^{m+1}(D_{0,m}) = 0,$$

a contradiction. Therefore, $D_0 \neq D$.

2. Since, by statement 1, $0 \neq D/D_0 = D/(D \cap {}'\mathcal{T}) \subseteq \overline{D'}$, statement 2 follows. $\square$

Lemma 2.4 *For all $m \geq 1$, ${}'\mathcal{T}_{\leq m} = {}'\mathcal{T}_m + y^m D_0 x^m$.*

Proof. Notice that ${}'\mathcal{T}_m \subseteq {}'\mathcal{T}_{\leq m}$ and ${}'\mathcal{T} = {}'\mathcal{T}_m + T_{\geq m+1}$, and so

$${}'\mathcal{T}_{\leq m} = {}'\mathcal{T} \cap {}'D_{\leq m} = ({}'\mathcal{T}_m + T_{\geq m+1}) \cap {}'D_{\leq m} = {}'\mathcal{T}_m + T_{\geq m+1} \cap {}'D_{\leq m}.$$

We repeat the argument of the proof of Proposition 2.2.(2) to show that

$$\begin{aligned} T_{\geq m+1} \cap {}'D_{\leq m} \;\; = \;\; & \{\phi(\delta_{m+1}) \,|\, \text{there is an element } (\delta_{m+1}, \ldots, \delta_{m+n}) \in (D \otimes D)^n \\ & \text{for some } n \geq 1 \text{ such that } \delta_{m+n} \in \mathcal{K} \text{ and } \theta(\delta_{m+i}) = \phi(\delta_{m+i+1}) \\ & \text{for } i = 1, \ldots, n-1\} = y^m D_0 x^m, \end{aligned}$$

by Proposition 2.2.(2). $\square$

For each natural number $m \geq 0$, the intersection of two D-bimodules ${}'\mathcal{T}_{[m]} := {}'\mathcal{T} \cap y^m D x^m$ is also a D-bimodule. The direct sum

$${}'\mathcal{T}_{[\infty]} := \bigoplus_{m\geq 1} {}'\mathcal{T}_{[m]}$$

is the largest 'homogeneous' D-subbimodule of ${}'\mathcal{T}$ for the direct sum decomposition ${}'D = \bigoplus_{m\geq 0} y^m D x^m$. For each $m \geq 1$,

$${}'\mathcal{T}_{[m]} \supseteq {}'\mathcal{T}_{[m]} \cap {}'\mathcal{T}_m = y^m D x^m \cap {}'\mathcal{T}_m = y^m L_m x^m$$

for some $\mathbb{Z}$-submodule L_m of D such that $\tau^m(D) L_m \sigma^m(D) \subseteq L_m$. The set L_m is described in Proposition 2.5.(2).

Let R be a ring. For an element $r \in R$, the sets $\text{l.ann}_R(r) = \{s \in R \,|\, sr = 0\}$ and $\text{r.ann}_R(r) = \{s \in R \,|\, rs = 0\}$ are called the *left* and *right annihilator* of the element r in R, respectively. An element $r \in R$ is called a *left* (resp., *right*) *regular element* if $\text{l.ann}_R(r) = 0$ (resp., $\text{r.ann}_R(r) = 0$). The sets of left and right regular elements of the ring R are denoted by ${}'\mathcal{C}_R$ and $\mathcal{C}'_R$, respectively. Their intersection $\mathcal{C}_R = {}'\mathcal{C}_R \cap \mathcal{C}'_R$ is the set of *regular elements* of R, the set of non-zero-divisors.

Proposition 2.5.(1,2) is an explicit description of the sets ${}'\mathcal{T}_{[m]}$ and L_m.

Proposition 2.5 *1. ${}'\mathcal{T}_{[0]} = D_0$ and ${}'\mathcal{T}_{[1]} = y(D_0 + L_1)x$ where $L_1 = \theta(\mathcal{K}_a)$.*

2. For all $m \geq 2$, ${}'\mathcal{T}_{[m]} = y^m(D_0 + L_m)x^m$ and $L_m = \{\theta(\delta_m) \,|\, \text{there is an element } (\delta_1, \ldots, \delta_m) \in (D \otimes D)^m$ such that $\delta_1 \in \mathcal{K}_a$ and $\theta(\delta_i) = \phi(\delta_{i+1})$ for $i = 1, \ldots, m-1\}$.

3. Let an element $(\delta_1, \ldots, \delta_m) \in (D \otimes D)^m$ be as in statement 2. Then for all natural numbers i, j such that $1 \leq i, j \leq m$ and $i + j \leq m$, *(a)* $\sigma^j\phi(\delta_i)=\sigma^j(a)\cdots\sigma(a)\phi(\delta_{i+j})$ *and* $\tau^j\phi(\delta_i)=\phi(\delta_{i+j})\tau(a)\cdots\tau^j(a)$.

(b) $\sigma^{j+1}\phi(\delta_i)=\sigma^{j+1}(a)\cdots\sigma(a)\theta(\delta_{i+j})$ *and* $\tau^{j+1}\phi(\delta_i)=\theta(\delta_{i+j})\tau(a)\cdots\tau^{j+1}(a)$.

(c) In particular, for $i=1$ *and* $j=1,\dots,m-1$,

$$\begin{aligned} 0 &= \sigma^j\phi(\delta_1)=\sigma^j(a)\cdots\sigma(a)\phi(\delta_{j+1}) \text{ and } 0=\tau^j\phi(\delta_1)=\phi(\delta_{j+1})\tau(a)\cdots\tau^j(a),\\ 0 &= \sigma^{j+1}\phi(\delta_1)=\sigma^{j+1}(a)\cdots\sigma(a)\theta(\delta_{j+1}) \text{ and } 0=\tau^{j+1}\phi(\delta_1)=\theta(\delta_{j+1})\tau(a)\cdots\tau^{j+1}(a).\end{aligned}$$

4. *For all* $m\geq 1$, $\sigma^m(a)\cdots\sigma(a)L_m=0$ *and* $L_m\tau(a)\cdots\tau^m(a)=0$, *i.e.,*

$$L_m\subseteq \mathrm{l.ann}_D(m,-m)\cap \mathrm{r.ann}_D(m,-m).$$

5. *If one of the conditions (a)–(c) below holds then* $L_m=0$ *for* $m\geq 1$:

(a) $\mathrm{l.ann}_D(m,-m)\cap \mathrm{r.ann}_D(m,-m)=0$ *for* $m\geq 1$,

(b) all elements $(m,-m)$, *where* $m\geq 1$, *are either left regular or right regular in* D,

(c) all elements $\sigma^m(a)$, *where* $m\geq 1$, *or all elements* $\tau^m(a)$, *where* $m\geq 1$, *are either left regular or right regular in* D.

Proof. 1–2. For $m=0$, $'\mathcal{T}_{[0]}={}'\mathcal{T}\cap D=D_0$.

For $m\geq 1$, $'\mathcal{T}_{[m]}\cap{}'\mathcal{T}_m=y^mL_mx^m$, and using Lemma 2.4, we see that

$$'\mathcal{T}_{[m]}={}'\mathcal{T}_{[m]}\cap{}'\mathcal{T}_{\leq m}={}'\mathcal{T}_{[m]}\cap({}'\mathcal{T}_m+y^mD_0x^m)=y^mD_0x^m+{}'\mathcal{T}_{[m]}\cap{}'\mathcal{T}_m=y^m(D_0+L_m)x^m.$$

Notice that $'\mathcal{T}_{[1]}\cap{}'\mathcal{T}_1=y\theta(\mathcal{K}_a)x$, and statements 1 and 2 follow.

3(a). Repeat the proof of Proposition 2.2.(3a).

3(b). Apply the endomorphism σ (resp., τ) to the first (resp., second) equality in the statement (a) and then use Lemma 2.1.(1) to obtain the result.

3(c). The statement (c), as a particular case of the statement (b), is obvious (since $\delta_1\in\ker(\phi)$).

4. For $m=1$, $L_1=\theta(\mathcal{K}_a)$, by statement 1. Now using Lemma 2.1.(1), we see that

$$\sigma(a)\theta(\mathcal{K}_a)=\sigma\phi(\mathcal{K}_a)=0 \text{ and } \theta(\mathcal{K}_a)\tau(a)=\tau\phi(\mathcal{K}_a)=0.$$

For $m\geq 2$, the result follows from statement 2 and the statement 3(c) when $j=m-1$. By (10), $(m,-m)=\sigma^m(a)\cdots\sigma(a)=\tau(a)\cdots\tau^m(a)$. Hence,

$$L_m\subseteq \mathrm{l.ann}_D(m,-m)\cap \mathrm{r.ann}_D(m,-m).$$

5. Statement 5 follows from statement 4. $\square$

The associated graded D-bimodule $\mathrm{gr}(\overline{D'})$**.** The D-bimodule $\overline{D'}={}'D/{}'\mathcal{T}$ admits the induced filtration $\{\overline{D'}_{\leq m}:=({}'D_{\leq m}+{}'\mathcal{T})/{}'\mathcal{T}\}_{m\geq 0}$ from the D-bimodule $'D$. Theorem 2.6 is an explicit description of the associated graded D-bimodule

$$\mathrm{gr}(\overline{D'}):=\bigoplus_{m\geq 0}\overline{D'}_{\leq m}/\overline{D'}_{\leq m-1}$$

where $\overline{D'}_{\leq -1}:=0$. By the very definition, all the sets $\overline{D'}_{\leq m}$ are D-bimodules, hence so are $\overline{D'}_{\leq m}/\overline{D'}_{\leq m-1}$.

Theorem 2.6 $\mathrm{gr}(\overline{D'})=D/D_0\oplus\bigoplus_{m\geq 1}y^m(D/(D_0+\mathcal{T}))x^m$, *i.e.,* $\overline{D'}_{\leq 0}\simeq D/D_0$ *and for* $m\geq 1$, $\overline{D'}_{\leq m}/\overline{D'}_{\leq m-1}=y^mDx^m/y^m(D_0+\mathcal{T})x^m\simeq y^m(D/(D_0+\mathcal{T}))x^m$.

Proof. For $m=0$, $\overline{D'}_{\leq 0}/\overline{D'}_{\leq -1}=\overline{D'}_{\leq 0}=({}'D_{\leq 0}+{}'\mathcal{T})/{}'\mathcal{T}=(D+{}'\mathcal{T})/{}'\mathcal{T}\simeq D/D\cap{}'\mathcal{T}\simeq D/D_0$. For $m\geq 1$,

$$\begin{aligned}\overline{D'}_{\leq m}/\overline{D'}_{\leq m-1} &\simeq ({}'D_{\leq m}+{}'\mathcal{T})/({}'D_{\leq m-1}+{}'\mathcal{T})\simeq {}'D_{\leq m}/({}'D_{\leq m-1}+{}'D_{\leq m}\cap{}'\mathcal{T})\\ &= {}'D_{\leq m}/({}'D_{\leq m-1}+{}'\mathcal{T}_{\leq m})\overset{L.2.4}{=}{}'D_{\leq m}/({}'D_{\leq m-1}+{}'\mathcal{T}_m+y^mD_0x^m)\\ &= {}'D_{\leq m}/({}'D_{\leq m-1}+y^m\theta(D)x^m+y^mD_0x^m)\simeq y^mDx^m/y^m(D_0+\mathcal{T})x^m\\ &\simeq y^m(D/(D_0+\mathcal{T}))x^m.\ \square\end{aligned}$$

The associated graded D-bimodule $\mathrm{gr}_\geq(\overline{D'})$. The D-bimodule $'D$ admits a descending D-bimodule filtration $\{'D_{\geq m}\}_{m\geq 0}$ where $'D_{\geq m}=\bigoplus_{i\geq m} y^iDx^i$, $'D_{\geq 0}={}'D$ and $\bigcap_{m\geq 0} D_{\geq m}=0$. Then the D-subbimodule $'\mathcal{T}$ of $'D$ inherits the induced descending D-bimodule filtration $\{'\mathcal{T}_{\geq m}:={}'\mathcal{T}\cap D_{\geq m}\}_{m\geq 0}$, $'\mathcal{T}_{\geq 0}={}'\mathcal{T}$ and $\bigcap_{m\geq 0}{}'\mathcal{T}_{\geq m}=0$. Similarly, the factor D-bimodule $\overline{D'}={}'D/{}'\mathcal{T}$ inherits the induced descending D-bimodule filtration

$$\{\overline{D'}_{\geq m}:=({}'D_{\geq m}+{}'\mathcal{T})/{}'\mathcal{T}\}_{m\geq 0},$$

where $\overline{D'}_{\geq 0}=\overline{D'}$. The next Lemma gives an explicit description of the filtration $\{'\mathcal{T}_{\geq m}\}$.

Lemma 2.7 *$'\mathcal{T}_{\geq 0}={}'\mathcal{T}$ and $'\mathcal{T}_{\geq m}=y^mL_mx^m+T_{\geq m+1}$ for all $m\geq 1$ (the sets L_m are defined in Proposition 2.5.(1,2)). In particular, $'\mathcal{T}_{\geq 1}=y\theta(\mathcal{K}_a)x^m+T_{\geq 2}$.*

Proof. For $m\geq 1$,

$$\begin{aligned}{}'\mathcal{T}_{\geq m} &= {}'\mathcal{T}\cap{}'D_{\geq m}=({}'\mathcal{T}_m+T_{\geq m+1})\cap{}'D_{\geq m}={}'\mathcal{T}_m\cap{}'D_{\geq m}+T_{\geq m+1}\\ &= {}'\mathcal{T}_m\cap y^mDx^m+T_{\geq m+1}={}'\mathcal{T}_m\cap{}'\mathcal{T}_{[m]}+T_{\geq m+1}=y^mL_mx^m+T_{\geq m+1}\end{aligned}$$

since $'\mathcal{T}_m\cap y^mDx^m={}'\mathcal{T}_m\cap{}'\mathcal{T}\cap y^mDx^m={}'\mathcal{T}_m\cap{}'\mathcal{T}_{[m]}=y^mL_mx^m$. $\square$

Theorem 2.8 is an explicit description of the associated graded D-bimodule associated with the descending D-bimodule filtration $\{\overline{D'}_{\geq m}\}$.

Theorem 2.8 *$\mathrm{gr}_\geq(\overline{D'})\simeq D/(a)\oplus\bigoplus_{m\geq 1}y^m(D/((a)+L_m)x^m$, i.e., $\overline{D'}_{\geq 0}/\overline{D'}_{\geq 1}\simeq D/(a)$ and $\overline{D'}_{\geq m}/\overline{D'}_{\geq m+1}\simeq y^mDx^m/y^m((a)+L_m)x^m\simeq y^m(D/((a)+L_m)x^m$.*

Proof. For $m=0$,

$$\begin{aligned}\overline{D'}_{\geq 0}/\overline{D'}_{\geq 1} &\simeq ({}'D+{}'\mathcal{T})/({}'D_{\geq 1}+{}'\mathcal{T})\simeq{}'D/({}'D_{\geq 1}+{}'\mathcal{T})\simeq({}'D/{}'D_{\geq 1})/(({}'D_{\geq 1}+{}'\mathcal{T})/{}'D_{\geq 1})\\ &\simeq D/(({}'D_{\geq 1}\oplus\mathrm{im}(\varphi))/{}'D_{\geq 1})\simeq D/\mathrm{im}(\varphi)\simeq D/(a)\end{aligned}$$

since $\mathrm{im}(\varphi)=(a)$. For $m\geq 1$,

$$\begin{aligned}\overline{D'}_{\geq m}/\overline{D'}_{\geq m+1} &\simeq ({}'D_{\geq m}+{}'\mathcal{T})/({}'D_{\geq m+1}+{}'\mathcal{T})\simeq{}'D_{\geq m}/({}'D_{\geq m+1}+{}'D_{\geq m}\cap{}'\mathcal{T})\\ &\simeq {}'D_{\geq m}/({}'D_{\geq m+1}+{}'\mathcal{T}_{\geq m})\overset{L.2.7}{=}{}'D_{\geq m}/({}'D_{\geq m+1}+y^mL_mx^m+T_{\geq m+1})\\ &= {}'D_{\geq m}/({}'D_{\geq m+1}+y^mL_mx^m+y^m(a)x^m)\simeq y^mDx^m/y^m((a)+L_m)x^m\\ &\simeq y^m(D/((a)+L_m)x^m.\ \square\end{aligned}$$

Proof of Theorem 1.1. Recall that $\overline{D'}\neq 0$ (Corollary 2.3.(2)). Let

$$A':=\bigoplus_{i\in\mathbb{Z}}A'_i$$

be a direct sum of left D-modules where $A'_0=\overline{D'}$, $A'_i=\overline{D'}x^i\simeq{}_D\overline{D'}$ and $A'_{-i}=y^i\overline{D'}\simeq{}^{\tau^i}\overline{D'}$ for $i\geq 1$. Recall that $\overline{D'}={}'D/{}'\mathcal{T}$. The idea of the proof of the theorem is to define the structure of left A-module on A'. That is to define the action of the elements x and y on A' and to show that the defining relations (5) of the IGWA A holds. This would prove existence of the GWA A (since $A\subseteq\mathrm{End}_{\mathbb{Z}}(A')$) and give

$${}_AA\simeq{}_AA'.$$

The action of the elements x and y on A' is given below but first we prove properties (i)–(vi), see below. Let us extend the ring endomorphism σ of D to a $\mathbb{Z}$-module endomorphism σ of $'D$ by the rule: For all elements $d\in D$ and $j\geq 1$,

$$\sigma(y^jdx^j)=y^{j-1}\tau^{j-1}\sigma(a)dx^{j-1}=y^{j-1}\tau^j(a)dx^{j-1}. \tag{30}$$

The origin of this extension is the following identities in the algebra A:

$$x\cdot y^j dx^j = \sigma(a)y^{j-1}dx^j = y^{j-1}\tau^{j-1}\sigma(a)dx^{j-1}\cdot x = y^{j-1}\tau^j(a)dx^{j-1}\cdot x$$

since $\sigma(a)=\tau(a)$.

(i) $\sigma('\mathcal{T})\subseteq{}'\mathcal{T}$: For all elements $d_1, d_2\in D$ and $j\geq 2$,

$$\sigma(y\tau(d_1)\sigma(d_2)x - d_1ad_2) \overset{(30)}{=} \sigma(a)\tau(d_1)\sigma(d_2) - \sigma(d_1)\sigma(a)\sigma(d_2) \overset{(4)}{=} \sigma(d_1)\sigma(a)\sigma(d_1) - \sigma(d_1)\sigma(a)\sigma(d_2) = 0,$$

$$\sigma(y^j\tau(d_1)\sigma(d_2)x^j - y^{j-1}d_1ad_2x^{j-1}) \overset{(30)}{=} y^{j-1}\tau(\tau^{j-1}(a)d_1)\sigma(d_2)x^{j-1} - y^{j-2}\tau^{j-1}(a)d_1ad_2x^{j-2}\in{}'\mathcal{T}.$$

By the statement (i), the $\mathbb{Z}$-module endomorphism σ of $'D$ yields the $\mathbb{Z}$-module endomorphism σ of the factor module $\overline{D'} = {}'D/{}'\mathcal{T}$ by the rule

$$\sigma('d+{}'\mathcal{T}) = \sigma('d)+{}'\mathcal{T}\ \text{ for all }\ 'd\in{}'D.$$

(ii) *For all elements $d\in D$ and $'d\in{}'D$, $\sigma(d\cdot{}'d)=\sigma(d)\sigma('d)$*: This is obvious for elements $'d\in D$ since σ is a ring homomorphism of D. For all elements $d_1\in D$ and $j\geq 1$,

$$\begin{aligned}\sigma(d\cdot y^jd_1x^j) &= \sigma(y^j\tau^j(d)d_1x^j) = y^{j-1}\tau^{j-1}\sigma(a)\tau^j(d)d_1x^{j-1} = y^{j-1}\tau^{j-1}(\sigma(a)\tau(d))d_1x^{j-1}\\ &\overset{(5)}{=} y^{j-1}\tau^{j-1}(\sigma(d)\sigma(a))d_1x^{j-1} = y^{j-1}\tau^{j-1}(\sigma(d))\cdot\tau^{j-1}\sigma(a)d_1x^{j-1}\\ &= \sigma(d)\cdot y^{j-1}\tau^{j-1}\sigma(a)d_1x^{j-1} = \sigma(d)\cdot\sigma(y^jd_1x^j),\end{aligned}$$

and the statement (ii) follows.

(iii) *For all elements $d\in D$ and $d'\in\overline{D'}$, $\sigma(dd')=\sigma(d)\sigma(d')$*: The statement (iii) follows from the statement (ii).

Let us extend the ring homomorphism τ of the ring D to a $\mathbb{Z}$-module endomorphism τ of $'D$ by the rule: For all elements $d\in D$ and $j\geq 1$,

$$\tau(y^jdx^j) = y^{j+1}dx^{j+1}. \tag{31}$$

(iv) $\tau('\mathcal{T})\subseteq{}'\mathcal{T}$: Trivial.

By the statement (iv), the $\mathbb{Z}$-module endomorphism τ of $'D$ yields the $\mathbb{Z}$-module endomorphism of the factor module $\overline{D'} = {}'D/{}'\mathcal{T}$ by the rule

$$\tau('d+{}'\mathcal{T}) = \tau('d)+{}'\mathcal{T}\ \text{ for all }\ 'd\in{}'D.$$

(v) *For all elements $d\in D$ and $'d\in{}'D$, $\tau(\tau(d)\cdot{}'d)=d\tau('d)$*: This is obvious for $'d\in D$ since τ is a ring homomorphism of D,

$$\tau(\tau(d)\cdot{}'d) = y\tau(d)\cdot{}'dx = d\cdot y\,'dx = d\tau('d).$$

For all $d_1\in D$ and $j\geq 1$,

$$\tau(\tau(d)\cdot y^jd_1x^j)) = \tau(y^j\tau^{j+1}(d)d_1x^j) = y^{j+1}\tau^{j+1}(d)d_1x^{j+1} = d\cdot y^{j+1}d_1x^{j+1} = d\cdot\tau(y^jd_1x^j).$$

(vi) *For all elements $d\in D$ and $d'\in\overline{D'}$, $\tau(\tau(d)\cdot d')=d\tau(d')$*: The statement (vi) follows from the statements (iv) and (v).

Let us define the multiplication of the elements x and y on A' by the rule: For all elements $d'\in\overline{D'}$, $i\geq 0$ and $k\geq 1$,

$$x\cdot\begin{cases}d'x^i\\ y^kd'\end{cases} = \begin{cases}\sigma(d')x^{i+1}\\ y^{k-1}\tau^{k-1}\sigma(a)d'\end{cases}\quad\text{and}\quad y\cdot\begin{cases}d'x^k\\ y^id'\end{cases} = \begin{cases}\tau(d')x^{k-1}\\ y^{i+1}d'.\end{cases}$$

Let us verify that all the relations in (5) hold (we keep the notation as above):

(vii) $xd = \sigma(d)x$:

$$xd \cdot \begin{cases} d'x^i \\ y^k d' \end{cases} = \begin{cases} \sigma(dd')x^{i+1} \\ y^{k-1}\tau^{k-1}\sigma(a)\tau^k(d)d' \end{cases} \overset{(iii)}{=} \begin{cases} \sigma(d)\sigma(d')x^{i+1} \\ y^{k-1}\tau^{k-1}(\sigma(a)\tau(d))d' \end{cases} \overset{(5)}{=} \begin{cases} \sigma(d)\sigma(d')x^{i+1} \\ y^{k-1}\tau^{k-1}(\sigma(d)\sigma(a))d' \end{cases}$$
$$= \sigma(d) \cdot \begin{cases} \sigma(d')x^{i+1} \\ y^{k-1}\tau^{k-1}\sigma(a)d' \end{cases} = \sigma(d)x \cdot \begin{cases} d'x^i \\ y^k d'. \end{cases}$$

(viii) $dy = y\tau(d)$:

$$dy \cdot \begin{cases} d'x^k \\ y^i d' \end{cases} = d \cdot \begin{cases} \tau(d')x^{k-1} \\ y^{i+1}d' \end{cases} = \begin{cases} d\tau(d')x^{k-1} \\ y^{i+1}\tau^{i+1}(d)d' \end{cases} \overset{(vi)}{=} \begin{cases} \tau(\tau(d)d')x^{k-1} \\ y^{i+1}\tau^{i+1}(d)d' \end{cases}$$
$$= y \cdot \begin{cases} \tau(d)d'x^k \\ y^i\tau^{i+1}(d)d' \end{cases} = y\tau(d) \cdot \begin{cases} d'x^k \\ y^i d'. \end{cases}$$

(ix) $yx = a$: Notice that $y\sigma(d')x = ad'$ in $\overline{D'}$. Now,

$$yx \cdot \begin{cases} d'x^i \\ y^k d' \end{cases} = y \cdot \begin{cases} \sigma(d')x^{i+1} \\ y^{k-1}\tau^{k-1}\sigma(a)d' \end{cases} \overset{(5)}{=} \begin{cases} y\sigma(d')x \cdot x^i \\ y^k\tau^k(a)d' \end{cases} = \begin{cases} ad'x^i \\ ay^k d' \end{cases} = a \cdot \begin{cases} d'x^i \\ y^k d'. \end{cases}$$

(x) $xy = \sigma(a)$:

$$xy \cdot \begin{cases} d'x^k \\ y^i d' \end{cases} = x \cdot \begin{cases} yd'x \cdot x^{k-1} \\ y^{i+1}d' \end{cases} = \begin{cases} \sigma(a)d'x^k \\ y^i\tau^i\sigma(a)d' \end{cases} = \sigma(a) \cdot \begin{cases} d'x^k \\ y^i d'. \end{cases}$$

This finishes the proof of consistency of (4) and (5). The left A-module $A' = A1$ is a $\mathbb{Z}$-graded A-module, by the very definition. By (23),

$$\overline{D'} \simeq D'.$$

Then, by (13), ${}_AA' \simeq {}_AA$, and the theorem follows. $\square$

Till the end of the paper, *we identify the D-bimodules D' and $\overline{D'}$ via (23), i.e.,*

$$D' = \overline{D'} \text{ (Theorem 1.1).}$$

As a result, the filtrations $\{D'_{\leq m}\}$ and $\{D'_{\geq m}\}$ on D' coincides with the filtrations $\{\overline{D'}_{\leq m}\}$ and $\{\overline{D'}_{\geq m}\}$ on $\overline{D'}$, respectively. So, Theorem 2.6 and Theorem 2.8 are results about the filtrations $\{D'_{\leq m}\}$ and $\{D'_{\geq m}\}$ of D' (since $D_0 = 0$ in A).

Corollary 2.9 *$\sigma(D_0) \subseteq D_0$ and $\tau(D_0) \subseteq D_0$.*

Proof. In the ring A, we have the equalities, $\sigma(D_0)x = xD_0 = 0$ and $y\tau(D_0) = D_0y = 0$. By Theorem 1.1, $\sigma(D_0) \subseteq D \cap {}'\mathcal{T} = D_0$. $\square$

By Corollary 2.9, the ring endomorphisms σ, τ of D yield the following ring endomorphisms of $\overline{D} = D/D_0$ denoted by the same symbols:

$$\sigma, \tau : \overline{D} \to \overline{D}, \;\; \sigma(d + D_0) = \sigma(d) + D_0 \;\text{ and }\; \tau(d + D_0) = \tau(d) + D_0. \tag{32}$$

Remark. In general, the endomorphisms σ and τ of the ring $\overline{D}$ are not monomorphisms: Suppose that $\ker(\sigma) = 0$ and $\ker(\tau) \neq 0$. Then $D_0 = 0$, by Proposition 2.2.(5), i.e., $\overline{D} = D$ and $\ker_{\overline{D}}(\tau) \neq 0$.

In view of (32), Theorem 2.6 can be written as follows.

Corollary 2.10 $\mathrm{gr}(D') = \overline{D} \oplus \bigoplus_{m\geq 1} y^m\big(\overline{D}/(\tau(\overline{D})\sigma(\overline{D}))\big)x^m$, *i.e.,* $D'_{\leq 0} = \overline{D}$ *and for* $m \geq 1$, $D'_{\leq m}/D'_{\leq m-1} = y^m\overline{D}x^m/y^m(\tau(\overline{D})\sigma(\overline{D}))x^m \simeq y^m\big(\overline{D}/(\tau(\overline{D})\sigma(\overline{D}))\big)x^m$.

Connection between rings D and D'. Recall that we have the ring homomorphism $\nu : D \to D'$, see (14). Theorem 2.11 below describes the kernel and the image of ν, it also gives a criterion for the homomorphism ν to be an isomorphism.

Theorem 2.11 *1. $\ker(\nu) = D_0$ and $\mathrm{im}(\nu) \simeq D/D_0$.*

2. The homomorphism ν is a monomorphism (i.e., $D \subseteq D'$) iff $D_0 = 0$.

3. The homomorphism ν is an epimorphism iff $D = D_0 + \mathcal{T}$. If the homomorphism ν is an epimorphism then $D' \simeq D/D_0$.

4. The homomorphism ν is a isomorphism (i.e., $D = D'$) iff $D_0 = 0$ and $D = \mathcal{T}$.

Proof. 1. Recall that we identified the D-bimodules D' and $\overline{D'}$ (Theorem 1.1). Then $\nu : D \to D' = {}'D/{}'\mathcal{T}$, and so $\ker(\nu) = D \cap {}'\mathcal{T} = D_0$. Hence, $\mathrm{im}(\nu) \simeq D/D_0$.

2. Statement 2 follows from statement 1.

3. Statement 3 follows from Theorem 2.6 and Theorem 1.1.

4. Statement 4 follows from statements 2 and 3. $\Box$

Proposition 2.12 *1. $D' = D'_{\geq 1}$ iff $D = (a)$ iff $D' = D'_{\geq m}$ for all $m \geq 1$.*

2. Given $m \geq 1$. Then $D'_{\geq m} = D'_{\geq m+1}$ iff $D = (a) + L_m$.

Proof. Statements 1 and 2 follow from Theorem 2.8 bearing in mind that $D' = \overline{D'}$ as D-bimodules. $\Box$

The subrings A_+ and A_- of A. By Theorem 1.1, the IGWA $A = D[x, y; \sigma, \tau, a]_{\rm in}$ contains the skew polynomial ring $A_+ = D'[x; \sigma]$ (where $xd = \sigma(d)x$ for $d \in D'$) and the right the skew polynomial ring $A_- = D'[y; \tau]_r$ (where $dy = y\tau(d)$ for $d \in D'$) where the ring endomorphisms σ and τ of the ring D' are extensions of the ring endomorphisms σ and τ of the ring $\overline{D}$ (see (32)) and are given by the rule: For all $j \geq 1$ and $d \in \overline{D}$,

$$\sigma(y^j dx^j) = y^{j-1}\tau^j(a)dx^{j-1} = \sigma(a)y^{j-1}dx^{j-1} \;\text{ and }\; \tau(y^j dx^j) = y^{j-1}d\sigma^j(a)x^{j-1} = y^{j-1}dx^{j-1}\tau(a). \tag{33}$$

In more detail, $x \cdot y^j dx^j = y^{j-1}\tau^j(a)dx^{j-1} \cdot x$ and $y^j dx^j \cdot y = y \cdot y^{j-1}d\sigma^j(a)x^{j-1}$, and (33) follows from Theorem 1.1. For all numbers $1 \leq i \leq j$,

$$\begin{aligned} \sigma^i(y^j dx^j) &= y^{j-i}\tau^{j-i}((i,-i))dx^{j-i} = (i,-i)y^{j-i}dx^{j-i}, \\ \tau^i(y^j dx^j) &= y^{j-i}d\sigma^{j-i}((i,-i))x^{j-i} = y^{j-i}dx^{j-i}(i,-i). \end{aligned}$$

Since $D' = {}'D/{}'\mathcal{T}$, for all numbers $1 \leq i \leq j$ and all elements $d' \in D'$,

$$\sigma^i(y^j d' x^j) = (i,-i)y^{j-i}d'x^{j-i} \;\text{ and }\; \tau^i(y^j d' x^j) = y^{j-i}d'x^{j-i}(i,-i). \tag{34}$$

The subrings A_+ and A_- are homogeneous subrings of A such that

$$A = A_+ + A_- \;\text{ and }\; A_+ \cap A_- = D'.$$

The multiplication in the IGWA A is given by the rule: For all elements $d_1', d_2' \in D'$, $i, j \geq 1$ and $s, t \geq 0$:

$$d_1'x^i \cdot y^j d_2' = \begin{cases} d_1'\sigma^{i-j}((j,-j)d_2')x^{i-j} & \text{if } i \geq j, \\ y^{j-i}\tau^{j-i}(d_1'(i,-i))d_2' & \text{if } i \leq j, \end{cases} \;\text{ and }\; y^j d_2' \cdot d_1' x^i = \begin{cases} y^j d_2' d_1' x^j \cdot x^{i-j} & \text{if } i \geq j, \\ y^{j-i} \cdot y^i d_2' d_1' x^i & \text{if } i \leq j, \end{cases}$$

$$d_1'x^s \cdot d_2'x^t = d_1'\sigma^s(d_2')x^{s+t} \;\text{ and }\; y^s d_1' \cdot y^t d_2' = y^{s+t}\tau^t(d_1')d_2'.$$

Corollary 2.13 *1. If $\sigma(a) \in {}'\mathcal{C}_D$ then $\ker(\sigma) \supseteq \ker(\tau)$.*

2. If $\sigma(a) \in \mathcal{C}_D'$ then $\ker(\sigma) \subseteq \ker(\tau)$.

3. *If* $\sigma(a) \in \mathcal{C}_D$ *then* $\ker(\sigma) = \ker(\tau)$.

Proof. By (4), $\sigma(d)\sigma(a) = \sigma(a)\tau(d)$ for all elements $d \in D$. Now, statements 1–3 follow. $\Box$
Abusing the notation, we can write

$$A = D[x, y; \sigma, \tau, a]_{\rm in} = \sum_{i,j\geq 0} y^i D x^j \ \text{ and } \ D' = \sum_{i\geq 0} y^i D x^i. \tag{35}$$

IGWAs and (σ, τ)-skew polynomial rings. Let us introduce a new class of rings.

Definition. Let R be a ring, $\sigma, \tau \in \mathrm{End}(R)$ and t be a variable. The ring $D = R[t; \sigma, \tau]$ which is (freely) generated by R and t subject to the defining relations:

$$\sigma(r)t = t\tau(r) \ \text{ for all elements } \ r \in R,$$

is called a (σ, τ)-*skew polynomial ring.*

If τ (resp., σ) is an automorphism of R then $D \simeq R[t; \sigma\tau^{-1}]$ is a skew polynomial ring (resp., $D \simeq R[t; \tau\sigma^{-1}]_r$ is a right skew polynomial ring). In general, a (σ, τ)-skew polynomial ring is far from being a (right) skew polynomial ring: Let $R = K[s]$ be a polynomial algebra over a ring K, $\sigma = \mathrm{id}$ be the identity automorphism of R, and $\tau \in \mathrm{End}_K(R)$ be given by the rule $\tau(s) = 0$. Then $K[s][t; \mathrm{id}, \tau] \simeq K\langle s, t\rangle/(st)$. Theorem 2.14 is a source of many non-trivial examples of IGWAs.

Theorem 2.14 *Let* $D = R[t; \sigma, \tau]$ *be a* (σ, τ)*-skew polynomial ring such that* $(\sigma\tau(r) - \tau\sigma(r))t = 0$ *and* $t(\sigma\tau(r) - \tau\sigma(r)) = 0$ *for all elements* $r \in R$ *(eg,* $\sigma\tau = \tau\sigma$*). Then*

1. *The ring endomorphisms* σ *and* τ *of the ring* R *can be extended to ring endomorphisms of the ring* D *by the rule:* $\sigma(t) = t$ *and* $\tau(t) = t$. *For all* $n \geq 1$ *and* $r \in R$, $t^n\tau^n(r) = \sigma^n(r)t^n$.

2. *Let* $u \in R^{\sigma,\tau} \cap Z(R)$ *be a unit and* $n \geq 1$ *be a natural number where* $R^{\sigma,\tau} := \{r \in R \,|\, \sigma(r) = r,\ \tau(r) = r\}$. *Then* $A = D[x, y; \sigma^n, \tau^n, ut^n]_{\rm in}$ *is an IGWA.*

Proof. 1. Since for all $r \in R$,

$$\begin{aligned}\sigma(\sigma(d)t) &= \sigma\sigma(d)t = t\tau\sigma(d) = t\sigma\tau(d) = \sigma(t\tau(d)),\\ \tau(\sigma(d)t) &= \tau\sigma(d)t = \sigma\tau(d)t = t\tau\tau(d) = \tau(t\tau(d)),\end{aligned}$$

the first part of statement 1 follows.

We prove the equality $t^n\tau^n(r) = \sigma^n(r)t^n$ by induction on n. The initial case when $n = 1$ is obvious (since the ring R is a (σ, τ)-ring). Suppose that $n \geq 2$ and the equality holds for all $n' < n$. Now,

$$\begin{aligned}t^n\tau^n(r) &= t \cdot t^{n-1}\tau^{n-1}\tau(r) = t \cdot \sigma^{n-1}\tau(r) \cdot t^{n-1} = t \cdot \sigma^{n-2}\big(\sigma\tau(r)t\big) \cdot t^{n-2}\\ &= t \cdot \sigma^{n-2}\big(\tau\sigma(r)t\big) \cdot t^{n-2} = t \cdot \sigma^{n-3}\big(\sigma\tau\sigma(r)t\big) \cdot t^{n-2} = t \cdot \sigma^{n-3}\big(\tau\sigma^2(r)t\big) \cdot t^{n-2}\\ &= \cdots = t \cdot \tau\sigma^{n-1}(r)t \cdot t^{n-2} = \sigma^n(r)t^n.\end{aligned}$$

2. We have to verify that for the element $a = ut^n$ the two conditions in (4) hold. Clearly, $\sigma^n(a) = ut^n = \tau^n(a)$. For all elements $r \in R$,

$$\begin{aligned}\sigma^n(a)\tau^n(r) &= ut^n\tau^n(r) = u\sigma^n(r)t^n = \sigma^n(r) \cdot ut^n = \sigma^n(r)a = \sigma^n(r)\sigma^n(a),\\ \sigma^n(a)t &= ut^nt = \sigma(u)t \cdot t^n = t\tau(u)t^n = t \cdot ut^n = ta = t\sigma^n(a),\end{aligned}$$

as required. $\Box$

Example. Let K be a ring. We have seen above that the ring $D = K\langle s, t\rangle/(st) \simeq K[s][t; \mathrm{id}, \tau]$ is a (σ, τ)-skew polynomial ring where $\tau(s) = 0$. Let $u \in Z(K)$ be a unit. Since the endomorphisms id and τ commute, by Theorem 2.14,

$$A = D[x, y; \mathrm{id}, \tau^n, ut^n]_{\rm in}$$

is an IGWA.

The ideals $\overline{D}_{\geq m}$ $(m\geq 0)$ of the ring $\overline{D}$. The descending chain of ideals $\{D'_{\geq m}\}_{m\in\mathbb{N}}$ of the ring D' induces the descending chain of ideals of the subring $\overline{D}$ of D', $\{\overline{D}_{\geq m}\}_{m\in\mathbb{N}}$, where

$$\overline{D}_{\geq m}:=\overline{D}\cap D'_{\geq m}.$$

In particular, $\overline{D}_{\geq 0}=\overline{D}\cap D'=\overline{D}$. The next proposition presents an explicit description of the ideals $\overline{D}_{\geq m}$.

Proposition 2.15 *1. $\overline{D}_{\geq 1}=(a)=\overline{D}a\overline{D}$.*

2. For all $m\geq 2$, $\overline{D}_{\geq m}=(\Delta_{m-1}+D_0)/D_0$ where $\Delta_{m-1}:=D\cap({}'\mathcal{T}_{m-1}+y^{m-1}(a)x^{m-1})$, $\Delta_1=\{\phi(\delta_1)\,|\,\delta_1\in D\otimes D$ such that $\theta(\delta_1)\in(a)\}$ and for $m\geq 3$, $\Delta_{m-1}=\{\phi(\delta_1)\,|$ there exists an element $(\delta_1,\dots,\delta_{m-1})\in(D\otimes D)^{m-1}$ such that $\theta(\delta_{m-1})\in(a)$ and $\theta(\delta_i)=\phi(\delta_{i+1})$ for $i=1,\dots,m-2\}$.

3. Let an element $(\delta_1,\dots,\delta_m)\in(D\otimes D)^m$ $(m\geq 2)$ be as in statement 2 ($\theta(\delta_m)\in(a)$ and $\theta(\delta_i)=\phi(\delta_{i+1})$ for $i=1,\dots,m-1$). Then for all natural numbers i,j such that $1\leq i,j\leq m$ and $i+j\leq m$,

(a) $\sigma^j\phi(\delta_i)=\sigma^j(a)\cdots\sigma(a)\phi(\delta_{i+j})$ and $\tau^j\phi(\delta_i)=\phi(\delta_{i+j})\tau(a)\cdots\tau^j(a)$.

(b) $\sigma^{j+1}\phi(\delta_i)=\sigma^{j+1}(a)\cdots\sigma(a)\theta(\delta_{i+j})$ and $\tau^{j+1}\phi(\delta_i)=\theta(\delta_{i+j})\tau(a)\cdots\tau^{j+1}(a)$.

4. For all $m\geq 2$, $\sigma^m(\overline{D}_{\geq m})\subseteq\sigma^m(a)\cdots\sigma(a)\cdot(a)$ and $\phi^m(\overline{D}_{\geq m})=(a)\cdot\tau(a)\cdots\tau^m(a)$.

Proof. 1. Recall that $\mathrm{im}(\phi)=DaD$. Now,

$$\begin{aligned}\overline{D}_{\geq 1}&\simeq(D+{}'\mathcal{T})\cap({}'D_{\geq 1}+{}'\mathcal{T})/{}'\mathcal{T}=(D\cap({}'D_{\geq 1}+{}'\mathcal{T})+{}'\mathcal{T})/{}'\mathcal{T}\\&=(D\cap(\mathrm{im}(\phi)\oplus{}'D_{\geq 1})+{}'\mathcal{T})/{}'\mathcal{T}\simeq(\mathrm{im}(\phi)+{}'\mathcal{T})/{}'\mathcal{T}=(DaD+{}'\mathcal{T})/{}'\mathcal{T}=\overline{D}a\overline{D}.\end{aligned}$$

2. For $m\geq 2$,

$$\begin{aligned}\overline{D}_{\geq m}&\simeq(D+{}'\mathcal{T})\cap({}'D_{\geq m}+{}'\mathcal{T})/{}'\mathcal{T}=(D\cap({}'D_{\geq m}+{}'\mathcal{T})+{}'\mathcal{T})/{}'\mathcal{T}=(D\cap({}'D_{\geq m}+{}'\mathcal{T}_m)+{}'\mathcal{T})/{}'\mathcal{T}\\&=(D\cap({}'D_{\geq m}\oplus({}'\mathcal{T}_{m-1}+y^{m-1}(a)x^{m-1}))+{}'\mathcal{T})/{}'\mathcal{T}\ \ (\text{since }\ \mathrm{im}(\phi)=(a))\\&=(D\cap({}'\mathcal{T}_{m-1}+y^{m-1}(a)x^{m-1}))+{}'\mathcal{T})/{}'\mathcal{T}\qquad(\text{since }\ D'_{\leq m-1}\cap D'_{\geq m}=0)\\&\simeq(D\cap({}'\mathcal{T}_{m-1}+y^{m-1}(a)x^{m-1}))/(D_0\cap({}'\mathcal{T}_{m-1}+y^{m-1}(a)x^{m-1}))\ \ (\text{since }\ D\cap{}'\mathcal{T}=D_0)\\&\simeq(D\cap({}'\mathcal{T}_{m-1}+y^{m-1}(a)x^{m-1}))+D_0)/D_0=(\Delta_{m-1}+D_0)/D_0.\end{aligned}$$

3. Repeat the proof of Proposition 2.2.(3).

4. Statement 4 follows from statement 3(b) for $j=m-1$, $i=1$ and the explicit description of the sets Δ_{m-1} given in statement 2. $\square$

By (33), $\sigma(D'_{\geq m})=\sigma(a)D'_{\geq m-1}\subseteq D'_{\geq m-1}$ and $\tau(D'_{\geq m})=D'_{\geq m-1}\tau(a)\subseteq D'_{\geq m-1}$ for all $m\geq 1$. Hence, for all numbers $m\geq i\geq 1$,

$$\sigma^i(D'_{\geq m})=\sigma^i(a)\cdots\sigma(a)D'_{\geq m-i}\ \text{ and }\ \tau^i(D'_{\geq m})=D'_{\geq m-i}\tau(a)\cdots\tau^i(a).\qquad(36)$$

In particular, for all $i=m\geq 1$,

$$\sigma^i(D'_{\geq m})=\sigma^i(a)\cdots\sigma(a)D'\ \text{ and }\ \tau^i(D'_{\geq m})=D'\tau(a)\cdots\tau^i(a).$$

It follows that $\sigma^m(D'_{\geq m})$ (resp., $\tau^m(D'_{\geq m})$), where $m\geq 1$, is an ideal of the ring D' iff the element $(m,-m)$ is right (resp., left) normal in the ring D'.

The ideal of $\overline{D}$, $\overline{D}_{\geq\infty}:=\bigcap_{m\geq 0}\overline{D}_{\geq m}$ is a (σ,τ)-invariant ideal, i.e.,

$$\sigma(\overline{D}_{\geq\infty})\subseteq\overline{D}_{\geq\infty}\ \text{ and }\ \tau(\overline{D}_{\geq\infty})\subseteq\overline{D}_{\geq\infty}\qquad(37)$$

since $\sigma(\overline{D}_{\geq\infty}) \subseteq \sigma(\overline{D}\cap D'_{\geq m}) \subseteq \sigma(\overline{D})\cap\sigma(D'_{\geq m}) \subseteq \overline{D}\cap D'_{m-1}$ and $\tau(\overline{D}_{\geq\infty}) \subseteq \tau(\overline{D})\cap\tau(D'_{\geq m}) \subseteq \overline{D}\cap D'_{m-1}$ for all $m\geq 1$.

The IGWA $\overline{A}=\overline{D}[x,y;\sigma,\tau,\overline{a}]$. Recall that $\overline{D}=\mathrm{im}(\nu)\simeq D/D_0$ is a subring of D' (Theorem 2.11.(1)). By Corollary 2.9, we have the IGWA $\overline{A}:=\overline{D}[x,y;\sigma,\tau,\overline{a}]$ where the endomorphisms σ and τ of D are defined in (32) and $\overline{a}=a+D_0$. In fact, the ring $\overline{A}$ is canonically isomorphic to A.

Theorem 2.16 *The map $\overline{A}\rightarrow A$, $x\mapsto x$, $y\mapsto y$, $\overline{d}\mapsto\overline{d}$, where $\overline{d}\in\overline{D}$, is a ring isomorphism. In particular, $\overline{D}$ is a subring of $\overline{A}$ and $(\overline{D})_0=0$.*

Proof. The map $A\rightarrow\overline{A}$, $x\mapsto x$, $y\mapsto y$, $d\mapsto d+D_0$, where $d\in D$, is a ring homomorphism which, by the very definition, is the inverse of the ring homomorphism $\overline{A}\rightarrow A$. Hence, $\overline{A}\simeq A$, $\overline{D}$ is a subring of $\overline{A}$ and $(\overline{D})_0=0$. $\Box$

So, in order to study IGWAs without loss of generality we can assume that the ring D is a *subring* of D'.

The opposite ring of an IGWA. Let R be a ring. The *opposite ring R^{op}* of R is a ring that is equal to R as an abelian group but the multiplication in R^{op} is given by the rule $r\cdot s=sr$. The defining relations of an IGWA are *left-right symmetric* in the sense that the opposite of an IGWA is again an IGWA: Let $A=D[x,y;\sigma,\tau,a]_{\rm in}$. Then

$$A^{op}=D^{op}[y,x;\tau,\sigma,a]_{\rm in}. \tag{38}$$

So, as a class, IGWAs are left-right symmetric. It suffices to study, say, only left properties of them. Then the right ones are obtained automatically by using (38), and vice versa. The opposite of a GWA is the, so-called, right GWA, see [9].

Proof of Proposition 1.2. In view of (38), it suffices to consider the case when τ is an automorphism. Let $A'=D[x,y;\sigma,\tau^{-1},a]$. Since the rings A and A' are generated by D, x and y, it suffices to show that they have the same defining ideal of relations and the same conditions on the element a:

$$\begin{aligned}
\sigma(a) &= \tau(a) \quad\Leftrightarrow\quad \tau^{-1}\sigma(a)=a,\\
\sigma(a)\tau(d) &= \sigma(d)\sigma(a) \Leftrightarrow \sigma(a)d=\sigma\tau^{-1}(d)\sigma(a)\Leftrightarrow ad=\tau^{-1}\sigma(d)a,\\
dy &= y\tau(d) \quad\Leftrightarrow\quad yd=\tau^{-1}(d)y.
\end{aligned}$$

Therefore, $A=A'$. $\Box$

If the defining endomorphisms of a GWA are automorphisms then the GWA is a right GWA where the defining endomorphisms are automorphisms, and vice versa, see [9] for details. So,

- *when the defining endomorphisms are automorphisms we have*

$$\text{IGWA} = \text{GWA} = \text{right GWA}.$$

Left and right normal elements, [9]. Let D be a ring. An element $a\in D$ is called a *left* (resp., *right*) *normal* element of D if $aD\subseteq Da$ (resp., $Da\subseteq aD$). If a is a left normal element in D then the left ideal Da is an ideal of D. Similarly, if a is a right normal element in D then the right ideal aD is an ideal of D. An element $a\in D$ is *normal* if $aD=Da$, i.e., a is left and right normal. Let $\cdot a:=\cdot a_D:D\rightarrow D$, $d\mapsto da$, and $\mathfrak{a}:=\ker(\cdot a)$. In particular, $\mathfrak{a}a=0$. Similarly, let $a\cdot:=a_D\cdot:D\rightarrow D$, $d\mapsto ad$, and $\mathfrak{b}:=\ker(a\cdot)$. In particular, $a\mathfrak{b}=0$. If the element a is left normal then $\mathfrak{b}$ is an ideal of the ring D: $a\cdot D\mathfrak{b}D\subseteq Da\mathfrak{b}D=0$. If the element a is right normal then $\mathfrak{a}$ is an ideal of the ring D: $D\mathfrak{a}D\cdot a\subseteq D\mathfrak{a}aD=0$. The sets $\mathbb{L}_a:=\{d\in D\,|\,da=ad'$ for some $d'\in D\}$ and $\mathbb{R}_a:=\{d\in D\,|\,ad=d'a$ for some $d'\in D\}$ are subrings of D such that $\mathfrak{a}\subseteq\mathbb{L}_a$ and $\mathfrak{b}\subseteq\mathbb{R}_a$. Furthermore, $\mathfrak{a}$ is an ideal of $\mathbb{L}_a$ ($\mathbb{L}_a\mathfrak{a}\mathbb{L}_a\cdot a\subseteq\mathbb{L}_a\mathfrak{a}\,aD=0$, and so $\mathbb{L}_a\mathfrak{a}\mathbb{L}_a\subseteq\mathfrak{a}$) and $\mathfrak{b}$ is an ideal of $\mathbb{R}_a$ ($a\cdot\mathbb{R}_a\mathfrak{b}\mathbb{R}_a\subseteq Da\mathfrak{b}\mathbb{R}_a=0$). If a is a left (resp., right) normal element of D then $\mathbb{L}_a a=aD$ (resp., $Da=a\mathbb{R}_a$).

Suppose that $a\in D$ is a left normal element. Then, for each element $d\in D$, $ad=d_la$ for some element $d_l\in\mathbb{L}_a$ which is unique up to adding $\mathfrak{a}$ ($d_la=(d_l+\mathfrak{a})a$). Hence, the map

$$\omega_a:D/\mathfrak{b}\rightarrow\mathbb{L}_a/\mathfrak{a},\ \ d+\mathfrak{b}\mapsto d_l+\mathfrak{a}, \tag{39}$$

is a ring *isomorphism* and we can write $ad = \omega_a(d)a$ for all $d \in D$. The left normal element a is a normal element iff $\mathbb{L}_a = D$. If the element a is a normal element then $\mathbb{L}_a = D$ and the map $\omega_a : D/\mathfrak{b} \to D/\mathfrak{a}$, $d + \mathfrak{b} \mapsto d_l + \mathfrak{a}$, is a ring isomorphism.

Similarly, suppose that the element $a \in D$ is a right normal element. Then, for each element $d \in D$, $da = ad_r$ for some element $d_r \in \mathbb{R}_a$ which is unique up to adding $\mathfrak{b}$ ($ad_r = a(d_r + \mathfrak{b})$). Hence, the map

$$\omega'_a : D/\mathfrak{a} \to \mathbb{R}_a/\mathfrak{b}, \quad d + \mathfrak{a} \mapsto d_r + \mathfrak{b}, \tag{40}$$

is a ring *isomorphism* and we can write $da = a\omega'_a(d)$ for all $d \in D$.

A right normal element a is a normal element iff $\mathbb{R}_a = D$. If the element a is a normal element then $\mathbb{R}_a = D$ and the map $\omega'_a : R/\mathfrak{a} \to R/\mathfrak{b}$, $d + \mathfrak{a} \mapsto d_r + \mathfrak{b}$, is a ring isomorphism. So, for a normal element a of D,

$$ad = \omega_a(d)a \ \text{ and } \ da = a\omega'_a(d) \ \text{ for all } \ d \in D. \tag{41}$$

Lemma 2.17 *([9, Lemma 2.1].) If the element $a \in D$ is a normal element then the maps $\omega_a : D/\mathfrak{b} \to D/\mathfrak{a}$ and $\omega'_a : D/\mathfrak{a} \to D/\mathfrak{b}$ are ring isomorphisms such that $\omega'_a = \omega_a^{-1}$. If, in addition, a is a regular element then the maps $\omega_a, \omega'_a : D \to D$ are ring isomorphisms such that $\omega'_a = \omega_a^{-1}$.*

Proof of Theorem 1.6. The ring A is a $\mathbb{Z}$-graded ring, hence so is its centre and we have the sum as in the theorem for some subsets $\mathcal{Z}_i$ of D' for $i \in \mathbb{Z}$.

(i) $\mathcal{Z}_0 = Z(D)^{\sigma,\tau}$: Notice that $\mathcal{Z}_0 \subseteq Z(D')$. Let $d \in Z(D')$. Then $d \in \mathcal{Z}_0$ iff $0 = [x, d] = (\sigma(d) - d)x$ and $0 = [d, y] = y(\tau(d) - d)$ iff $d \in Z(D)^{\sigma,\tau}$, by Theorem 1.1.

(ii) $Z(A) \cap A_i = \mathcal{Z}_i x^i$ for $i \geq 1$: Let $\alpha \in D'$ and $d \in D'$. Then $0 = [d, \alpha x^i] = (d\alpha - \alpha\sigma^i(d))x^i \Leftrightarrow d\alpha = \alpha\sigma^i(d)$; $0 = [x, \alpha x^i] = (\sigma(\alpha) - \alpha)x^{i+1} \Leftrightarrow \alpha \in D'^{\sigma}$; and $0 = [y, \alpha x^i] = (y\alpha x - \alpha\sigma^i(a))x^{i-1} \Leftrightarrow y\alpha x = \alpha\sigma^i(a)$, and the statement (ii) follows.

(iii) $Z(A) \cap A_{-i} = y^i \mathcal{Z}_{-i}$: Use similar arguments as in the proof of the statement (ii). □

Examples of IGWAs. Using Lemma 2.18 we can construct plenty of examples of IGWAs.

Lemma 2.18 *1. If $\sigma = \tau$ then Eq. (4) holds.*

2. If the element $\sigma(a)$ is central and regular in D then Eq. (4) is equivalent to $\sigma = \tau$.

3. If the element $\sigma(a)$ is left normal and regular in D then Eq. (4) is equivalent to $\sigma = \omega_{\sigma(a)}\tau$.

4. If the element $\sigma(a)$ is right normal and regular in D then Eq. (4) is equivalent to $\tau = \omega'_{\sigma(a)}\sigma$.

Proof. 1–2. Statements 1 and 2 are obvious.

3. If the element $\sigma(a)$ is a left normal and left regular element then the equality $\sigma(d)\sigma(a) = \sigma(a)\tau(d)$ for $d \in D$ can be written as $(\sigma(d) - \omega_{\sigma(a)}\tau(d))\sigma(a) = 0$, or equivalently, as $\sigma = \omega_{\sigma(a)}\tau$. Taking $d = a$, we have the equality $\sigma(a)\sigma(a) = \sigma(a)\tau(a)$, and so $\sigma(a) = \tau(a)$ since the element $\sigma(a)$ is right regular in D.

4. Use arguments similar to the ones in the proof of statement 3. □

Examples. 1. Let R be a ring and $D = R[t; \gamma]$ be a skew polynomial ring ($tr = \gamma(r)t$ for all $r \in R$) such that γ is a ring monomorphism of R. Then the element t is a left normal, regular element in D. Hence, so is the element $a = ut^n$ where u is a unit of R and $n \geq 1$. Let $\tau \in \mathrm{Aut}(D)$ and $\sigma = \omega_{\sigma(a)}\tau$. Then $\tau(a)$ is a left normal, regular element in D. The equality $\sigma = \omega_{\sigma(a)}\tau$ implies the equalities $\sigma(a)\tau(a) = \omega_{\sigma(a)}\tau(a)\sigma(a) = \sigma(a)\sigma(a)$. Hence, $\tau(a) = \sigma(a)$ since the element $\sigma(a)$ is regular. By Lemma 2.18.(3),

$$A = D[x, y; \omega_{\sigma(a)}\tau, \tau, a = ut^n]_{\mathrm{in}}$$

is an IGWA.

2. Let R be a ring and $D = R[t; \gamma]_r$ be a skew polynomial ring ($rt = t\gamma(r)$ for all $r \in R$) such that γ is a ring monomorphism of R. Then the element t is a right normal, regular element in D. Hence, so is the element $a = ut^n$ where u is a unit of R and $n \geq 1$. Let $\sigma \in \mathrm{Aut}(D)$ and $\tau = \omega'_{\sigma(a)}\sigma$. By Lemma 2.18.(4),

$$A = D[x, y; \sigma, \omega_{\sigma(a)}\sigma, a = ut^n]_{\mathrm{in}}$$

is an IGWA.

Lemma 2.19 *Let $A = D[x, y; \sigma, \sigma, a]_{\rm in}$, see Lemma 2.18.(1). Suppose that a is a central unit of D and σ is an epimorphism of D. Then $D_{0,m} = \ker(\sigma^m)$ for all $m \geq 1$, $D_0 = \mathcal{K}(\sigma)$, $D' \simeq D/D_0 = D(\sigma)$ and $A \simeq D/D_0[x, x^{-1}; \overline{\sigma}]$ where $\mathcal{K}(\sigma)$ and $\overline{\sigma}$ are defined in (29).*

Proof. Let $y' = a^{-1}y$. Then $A = D[x, y'; \sigma, \tau, 1]$ where $\sigma = \tau$ since the element a is a central unit of D. So, we may assume that $a = 1$. Since $\sigma = \tau$ is an epimorphism, $\mathcal{T} = D$, and so

$$D' \simeq D/D_0,$$

by Theorem 2.6. Since $D = \bigcup_{m\geq 1} D_{0,m}$, in order to finish the proof of the lemma it suffices to show that $D_{0,m} = \ker(\sigma^m)$ for all $m \geq 1$. We split the proof into several steps.

(i) $\ker(\theta) = C + \ker(\sigma) \otimes 1$ *where C is a left D-submodule of $D \otimes D$ generated by the elements $d \otimes 1 - 1 \otimes d$ where $d \in D$*: Clearly, $C \subseteq \ker(\theta)$. Using the equality $d_1 \otimes d_2 = d_1(1 \otimes d_2 - d_2 \otimes 1) + d_1d_2 \otimes 1$ in $D \otimes D$ where $d_1, d_2 \in D$, we see that

$$D \otimes D = C + D \otimes 1. \tag{42}$$

Since $C \subseteq \ker(\theta)$ (as $\sigma = \tau$), we have the result.

(ii) $C \subseteq \ker(\phi)$: Trivial (since $a = 1$).

(iii) $\phi(\ker(\theta)) = \ker(\sigma)$: By the statements (i) and (ii), $\phi(\ker(\theta)) = \ker(\sigma) \cdot a = \ker(\sigma)$ since $a = 1$.

(iv) $D_{0,m} = \ker(\sigma^m)$ *for all $m \geq 1$*: Suppose that $m = 1$. Then, by Proposition 2.2.(1) and the statement (iii),

$$D_{0,1} = \phi(\ker(\theta)) = \ker(\sigma).$$

Suppose that $m \geq 2$. By Proposition 2.2.(4c), $D_{0,m} \subseteq \ker(\sigma^m)$. To show that the reverse inclusion holds for each element $d \in \ker(\sigma^m)$ we have to find an element $\delta = (\delta_1, \dots, \delta_m) \in (D \otimes D)^m$ that satisfies the conditions in Proposition 2.2.(2). Let $\delta_1 = d_1 \otimes 1$ where $d_1 = d \in \ker(\sigma^m)$; then $\phi(\delta_1) = d$. Suppose that we have found elements $\delta_i = d_i \otimes 1$ for some elements $d_i \in \ker(\sigma^{m+1-i})$ for $i = 2, \dots, n < m$. Let $\delta_{n+1} = d_{n+1} \otimes 1$ where $d_{n+1} = \sigma(d_n) \in \ker(\sigma^{m-n})$. Then $\theta(d_n) = \sigma(d_n) = \phi(\delta_{n+1})$. By induction on n, we can find the element δ (since $\delta_m = d_m \otimes 1$ where $d_m \in \ker(\sigma)$, $\theta(\delta_m) = 0$). $\square$

Corollary 2.20 *Let D be either a polynomial ring $K[x_1, \dots, x_n, \dots]$ or a free ring $K\langle x_1, \dots, x_n, \dots\rangle$ over a ring K, and $\mathfrak{m} = (x_1, x_2, \dots)$ be the ideal of D generated by the variables x_i. Let σ be a ring endomorphism of D such that $\sigma(K) = K$, $\sigma|_K \in {\rm Aut}(K)$, $\sigma(x_1) = 0$ and $\sigma(x_i) = x_{i-1}$ for all $i \geq 2$. By Lemma 2.18.(1), $A = D[x, y; \sigma, \sigma, a]_{\rm in}$ is an IGWA for an arbitrary choice of a. If a is a central unit of K then $D_0 = \mathfrak{m}$, $D' \simeq D/D_0 \simeq K$ and $A = K[x, x^{-1}; \sigma|_K]$ is a skew Laurent polynomial ring over K.*

Proof. The corollary is a particular case of Lemma 2.19 where $\mathcal{K}(\sigma) = \mathfrak{m}$. $\square$

Lemma 2.21 *Let $D = K[x_1, x_2, \dots]$ be a polynomial ring over a ring K and $A = D[x, y; \sigma, \sigma, a]_{\rm in}$ be an IGWA(Lemma 2.18.(1)).*

1. *Let σ be a K-endomorphism of D given by the rule $\sigma(x_i) = x_{2i}$ for all $i \geq 1$. Then $D' = D \oplus \bigoplus_{i\geq 1} y^i D^+_{odd} x^i$ where $D^+_{odd} = (x_1, x_3, \dots)$ is an ideal of D generated by the variables $x_1, x_3, \dots$.*

2. *Let σ be a K-endomorphism of D given by the rule $\sigma(x_i) = x_{i+n}$ for all $i \geq 1$ when $n \geq 1$ is a natural number. Then $D' = D \oplus \bigoplus_{i\geq 1} y^i P^+_n x^i$ where $P^+_n = (x_1, \dots, x_n)$ is an ideal of D generated by the variables $x_1, \dots, x_n$.*

Proof. In both statements the endomorphism σ is a monomorphism of D. Then, by Proposition 2.2.(5), $D_0 = 0$, and so $D \subseteq D'$. Clearly,

$$\mathcal{T} = \mathrm{im}(\sigma) = \begin{cases} K[x_2, x_4, \ldots] & \text{in statement 1,} \\ K[x_{n+1}, x_{n+2}, \ldots] & \text{in statement 2.} \end{cases}$$

Since

$$D = \begin{cases} D^+_{odd} \oplus \mathcal{T} & \text{in statement 1,} \\ P^+_n \oplus \mathcal{T} & \text{in statement 2,} \end{cases}$$

the lemma follows from Theorem 2.6. In more detail,

$$D/(D_0 + \mathcal{T}) = D/\mathcal{T} = \begin{cases} (D^+_{odd} \oplus \mathcal{T})/\mathcal{T} \simeq D^+_{odd} & \text{in statement 1,} \\ (P^+_n \oplus \mathcal{T})/\mathcal{T} \simeq P^+_n & \text{in statement 2.} \end{cases} \quad \square$$

The elements $(i, -i)$ of D' where $i \in \mathbb{Z}$. For each $i \geq 0$, the elements $(i, -i) = x^i y^i$ and $(-i, i) := y^i x^i$ belong to the ring D' where $(0, 0) := 1$.

- *For all $i \geq -1$, $(i, -i) \in \overline{D}$ (since $(-1, 1) = a$). In general, none of the elements $(-i, i)$, where $i \geq 2$, belongs to the ring $\overline{D}$.*

Example. Take $a = x_1$ in Lemma 2.21.(1). Then for $i \geq 2$, $y^i x^i = y^{i-1} x_1 x^{i-1} \notin \overline{D} = D$. $\square$

For all $0 \leq i \leq n$,

$$\sigma^i(y^n x^n) = (i, -i) y^{n-i} x^{n-i} \quad \text{and} \quad \tau^i(y^n x^n) = y^{n-i} x^{n-i} (i, -i).$$

In particular, for all $n \geq 1$,

$$\sigma^n(y^n x^n) = (n, -n) = \tau^n(y^n x^n).$$

The IGWAs $A = D[x, y; \sigma, \tau, 0]_{\mathrm{in}}$. For $a = 0$, the conditions in (4) hold, and so for an arbitrary choice of σ and τ we have the IGWA $A = D[x, y; \sigma, \tau, 0]_{\mathrm{in}}$. Then $\phi = 0$ and $'\mathcal{T} = \bigoplus_{i \geq 1} y^i \mathcal{T} x^i$ where $\mathcal{T} = \tau(D)\sigma(D)$. Now,

$$D' = {}'D/{}'\mathcal{T} \simeq D \oplus \bigoplus_{i \geq 1} y^i D x^i / y^i \mathcal{T} x^i \simeq D \oplus \bigoplus_{i \geq 1} y^i \widetilde{D} x^i$$

is a direct sum of D-bimodules where $\widetilde{D} = D/\mathcal{T}$, and statement 1 of Theorem 2.22 follows.

Theorem 2.22 *Let $A = D[x, y; \sigma, \tau, 0]_{\mathrm{in}}$. Then*

1. *$D_0 = 0$, $D \subseteq D'$, $D' = D \oplus D'_{\geq 1}$ and $D'_{\geq 1} = \bigoplus_{i \geq 1} y^i \widetilde{D} x^i$, $\widetilde{D} = D/\mathcal{T}$ and $\mathcal{T} = \tau(D)\sigma(D)$; $y^i \widetilde{D} x^i \simeq {}^{\tau^i}(D/\mathcal{T})^{\sigma^i}$ as D-bimodules.*
2. *(a) $\ker_{D'}(\sigma^i) = \ker_D(\sigma^i) \oplus D'_{\geq 1}$ for all $i \geq 1$ and $\mathcal{K}_{D'}(\sigma) = \mathcal{K}_D(\sigma) \oplus D'_{\geq 1}$.*
   (b) $\ker_{D'}(\tau^i) = \ker_D(\tau^i) \oplus D'_{\geq 1}$ for all $i \geq 1$ and $\mathcal{K}_{D'}(\tau) = \mathcal{K}_D(\tau) \oplus D'_{\geq 1}$.
3. *$\tau(D')\sigma(D') = \mathcal{T}$ and $\widetilde{D'} = \widetilde{D} \oplus \bigoplus_{i \geq 1} y^i \widetilde{D} x^i$ where $\widetilde{D'} := D'/\tau(D')\sigma(D')$.*

Proof. Statements 2 and 3 follow from statement 1. $\square$

Remark. Theorem 2.22 shows that if $\ker_D(\sigma) = 0$ (resp., $\ker_D(\tau) = 0$) then $\ker_{D'}(\sigma) \neq 0$ (resp., $\ker_{D'}(\tau) \neq 0$).

The derivative series of IGWAs $A^{(\alpha)}$ associated with an IGWA A. Let $A = D[x, y; \sigma, \tau, a]_{\mathrm{in}}$ be an IGWA. Recall that the ring endomorphisms σ and τ of D are extended respectively to ring endomorphisms σ_1 and τ_1 of D' ('extended' means that for all elements $d \in D$, $\nu\sigma(d) = \sigma_1\nu(d)$ and $\nu\tau(d) = \tau_1\nu(d)$, see (14)).

Definition. The IGWA $A' = D'[x_1, y_1; \sigma_1, \tau_1, a]_{\rm in}$ is called the *first derivative* of the IGWA A (the conditions in (4) hold automatically).

Furthermore, we have the ring homomorphism

$$A \stackrel{f_0}{\to} A', \quad x \mapsto x_1, \quad y \mapsto y_1, \quad d \mapsto \nu(d) \;\; (d \in D). \tag{43}$$

The homomorphism f_0 is a $\mathbb{Z}$-graded homomorphism. Repeating the same construction inductively we obtain the *derivative series* of IGWAs associated with A:

$$A \stackrel{f_0}{\to} A' \stackrel{f_1}{\to} A'' \stackrel{f_2}{\to} \cdots \stackrel{f_{n-1}}{\to} A^{(n)} \stackrel{f_n}{\to} \cdots A^{(\mathbb{N})} \stackrel{f_{\mathbb{N}}}{\to} A^{(\mathbb{N}+1)} \stackrel{f_{\mathbb{N}+1}}{\to} \cdots A^{(\alpha)} \stackrel{f_\alpha}{\to} \cdots \tag{44}$$

where α is an arbitrary ordinal number where

$$A'' = (A')', \ldots, A^{(n)} = (A^{(n-1)})', \ldots, A^{(\mathbb{N})} := \lim_{n} A^{(n)}$$

is the direct limit of rings. Let Γ be the set of all ordinals. If $\alpha \in \Gamma$ is not a limit ordinal, i.e., $\alpha = \beta + 1$, then

$$A^{(\alpha)} := (A^{(\beta)})'.$$

If $\alpha \in \Gamma$ is a limit ordinal then $A^{(\alpha)} := \lim_{\gamma < \alpha} A^{(\gamma)}$, the direct limit of rings. Since the homomorphisms f_α are $\mathbb{Z}$-graded we have the derivative series of rings associated with D:

$$D \stackrel{\nu_0}{\to} D' \stackrel{\nu_1}{\to} D'' \stackrel{\nu_2}{\to} \cdots \stackrel{\nu_{n-1}}{\to} D^{(n)} \stackrel{\nu_n}{\to} \cdots D^{(\mathbb{N})} \stackrel{\nu_{\mathbb{N}}}{\to} D^{(\mathbb{N}+1)} \stackrel{\nu_{\mathbb{N}+1}}{\to} \cdots D^{(\alpha)} \stackrel{\nu_\alpha}{\to} \cdots \tag{45}$$

where $D'' = (D')', \ldots, D^{(n)} = (D^{(n-1)})', \ldots, D^{(\mathbb{N})} := \lim_n D^{(n)}$ is the direct limit of rings. If $\alpha \in \Gamma$ is not a limit ordinal, i.e., $\alpha = \beta + 1$, then

$$D^{(\alpha)} := (D^{(\beta)})'.$$

If $\alpha \in \Gamma$ is a limit ordinal then $D^{(\alpha)} := \lim_{\gamma < \alpha} D^{(\gamma)}$, the direct limit of rings. For all $\alpha \in \Gamma$,

$$A^{(\alpha)} = D^{(\alpha)}[x_\alpha, y_\alpha; \sigma_\alpha, \tau_\alpha, a] \tag{46}$$

and ν_α is the restriction of f_α to $D^{(\alpha)}$. For all ordinals, α and β, $(A^{(\alpha)})^{(\beta)} = A^{(\alpha+\beta)}$ and $(D^{(\alpha)})^{(\beta)} = D^{(\alpha+\beta)}$. Given two ordinals $\alpha, \beta \in \Gamma$ such that $\alpha < \beta$. We have maps

$$D^{(\alpha)} \to D^{(\beta)} \;\text{ and }\; A^{(\alpha)} \to A^{(\beta)}.$$

When for an element $d \in D^{(\alpha)}$ (resp., $d \in A^{(\alpha)}$), we write $d \in D^{(\beta)}$ (resp., $d \in A^{(\beta)}$) we mean its image under the above maps. This notation simplifies many formulae.

For each ordinal $\alpha \geq 2$,

$$D^{(\alpha)} = D + \sum_{\lambda} \sum_{n(\lambda)} y_\lambda^{n(\lambda)} D x_\lambda^{n(\lambda)} \tag{47}$$

where the sums are taken over all l-tuples $\lambda = (\lambda_1, \ldots, \lambda_l) \in \Gamma^l$ such that $\lambda_1 < \cdots < \lambda_l$ where $l \geq 1$ and $n(\lambda) = (n(\lambda_1), \ldots, n(\lambda_l)) \in \mathbb{N}^l \backslash \{(0, \ldots, 0)\}$,

$$x_\lambda^{n(\lambda)} := x_{\lambda_1}^{n(\lambda_1)} \cdots x_{\lambda_l}^{n(\lambda_l)} \;\text{ and }\; y_\lambda^{n(\lambda)} := y_{\lambda_l}^{n(\lambda_l)} \cdots y_{\lambda_1}^{n(\lambda_1)}$$

(in the reverse order), $x_0 := x$ and $y_0 := y$. The 'D' in the sum above means the image of the ring D under the ring homomorphism $D \to D^{(\alpha)}$.

Example. For $\alpha = 2$, $D^{(\alpha)} = D + \sum_{i \geq 1} y^i D x^i + \sum_{j \geq 1} \sum_{i \geq 0} y_1^j y^i D x^i x_1^j$.

The next theorem shows that for all ordinals $\alpha \geq 2$ the sets $\mathcal{K}_{D^{(\alpha)}}(\sigma)$ and $\mathcal{K}_{D^{(\alpha)}}(\tau)$ are very large, in general.

Theorem 2.23 *Let α be an ordinal. Then*

1. *For all $\alpha \geq 2$, $D^{(\alpha)} = D' + \mathcal{K}_{D^{(\alpha)}}(\sigma_\alpha) \cap \mathcal{K}_{D^{(\alpha)}}(\tau_\alpha) = D' + \mathcal{K}_{D^{(\alpha)}}(\sigma_\alpha) = D' + \mathcal{K}_{D^{(\alpha)}}(\tau_\alpha)$.*

2. *For all $i \geq 1$, $\ker_{D^{(\alpha)}}(\sigma_\alpha^i) = \lim_{\beta<\alpha} \ker_{D^{(\beta)}}(\sigma_\beta^i)$ and $\ker_{D^{(\alpha)}}(\tau_\alpha^i) = \lim_{\beta<\alpha} \ker_{D^{(\beta)}}(\tau_\beta^i)$; $\mathcal{K}_{D^{(\alpha)}}(\sigma_\alpha) = \lim_{\beta<\alpha} \mathcal{K}_{D^{(\beta)}}(\sigma_\beta)$ and $\mathcal{K}_{D^{(\alpha)}}(\tau_\alpha) = \lim_{\beta<\alpha} \mathcal{K}_{D^{(\beta)}}(\tau_\beta)$.*

Proof. 2. The first equality follows from the fact that for all ordinals $\gamma \leq \beta$, σ_β is an 'extension' of σ_γ (i.e., sigmas and nus commute in (45)). By a similar reason, the second equality holds.

1. We use induction on α. Let $\alpha = 2$. We have to show that

$$D'' = D' + \mathcal{K}_{D''}(\sigma) \cap \mathcal{K}_{D''}(\tau).$$

Recall that $A = D[x, y; \sigma, \tau, a]_{\rm in} \supseteq D'$ and $A' = D'[x_1, y_1; \sigma_1, \tau_1, a]_{\rm in} \supseteq D''$. For all elements $d' \in D'$ and $i \geq 1$,

$$y_1^i d' x_1^i - y^i d' x^i \in \ker_{D''}(\sigma_1^i) \cap \ker_{D''}(\tau_1^i)$$

since $\sigma_1^i(y_1^i d' x_1^i) = (i, -i)d' = \sigma^i(y^i d' x^i) = \sigma_1^i(y^i d' x^i)$ and $\tau_1^i(y_1^i d' x_1^i) = d'(i, -i) = \tau^i(y^i d' x^i) = \tau_1^i(y^i d' x^i)$, and the equality follows.

Suppose that $\alpha > 2$ and the equalities in statement 1 hold for all ordinals β such that $\beta < \alpha$. If $\alpha = \beta + 1$ for some ordinals β then the result follows from the case $\alpha = 2$ since $D^{(\alpha)} = (D^{(\beta)})'$ (or repeat the arguments above using (47)). If α is the limit ordinals then the result follows from the equalities for $\beta < \alpha$ and statement 2. $\square$

Lemma 2.24 shows that, in general, the procedure of creating the rings $A^{(\alpha)}$ from A never stops.

Lemma 2.24 *Let $A = D[x, y; \sigma, \tau, 0]_{\rm in}$. Then for all ordinals $\alpha, \beta \in \Gamma$ such that $\alpha < \beta$ the maps $D^{(\alpha)} \to D^{(\beta)}$ and $A^{(\alpha)} \to A^{(\beta)}$ are strict inclusions. In more detail,*

1. *For every ordinal $\alpha \geq 2$, $A^{(\alpha)} = D^{(\alpha)}[x_\alpha, y_\alpha; \sigma_\alpha, \tau_\alpha, a]_{\rm in}$ where $D^{(\alpha)} = D \oplus \mathcal{D}^{(\alpha)}$ is a direct sum of D-bimodules, $\mathcal{D}^{(\alpha)} = \bigoplus_\lambda \bigoplus_{n(\lambda)} y_\lambda^{n(\lambda)} \widetilde{D} x_\lambda^{n(\lambda)}$ where the sums are taken over all l-tuples $\lambda = (\lambda_1, \ldots, \lambda_l) \in \Gamma^l$ ($l \geq 1$) such that $\lambda_1 < \cdots < \lambda_l \leq \alpha$ and $n(\lambda) = (n(\lambda_1), \ldots, n(\lambda_l)) \in \mathbb{N}^l \backslash \{(0, \ldots, 0)\}$ ($\widetilde{D} = D/\mathcal{T}$).*

2. *(a) $\ker_{D^{(\alpha)}}(\sigma_\alpha^i) = \ker_D(\sigma^i) \oplus \mathcal{D}^{(\alpha)}$ for all $i \geq 1$ and $\mathcal{K}_{D^{(\alpha)}}(\sigma) = \mathcal{K}_D(\sigma) \oplus \mathcal{D}^{(\alpha)}$.*
   (b) $\ker_{D^{(\alpha)}}(\tau_\alpha^i) = \ker_D(\tau^i) \oplus \mathcal{D}^{(\alpha)}$ for all $i \geq 1$ and $\mathcal{K}_{D^{(\alpha)}}(\tau) = \mathcal{K}_D(\tau) \oplus \mathcal{D}^{(\alpha)}$.

3. *$\tau(D_\alpha^{(\alpha)})\sigma_\alpha(D^{(\alpha)}) = \mathcal{T}$ and $\widetilde{D^{(\alpha)}} = \widetilde{D} \oplus \mathcal{D}^{(\alpha)}$ (where $\widetilde{D^{(\alpha)}} := D^{(\alpha)}/\tau_\alpha(D^{(\alpha)})\sigma_\alpha(D^{(\alpha)})$).*

Proof. By Theorem 2.22.(1), $D \subset D'$ and $A \subset A' = D'[x_1, y_1; \sigma_1, \tau_1, 0]_{\rm in}$ are strict inclusions and the lemma follows. $\square$

The inner (σ, τ, a)-extension of a ring. *Definition.* Let $A = D[x, y; \sigma, \tau, a]_{\rm in}$. The subring D' of A is called the *inner (σ, τ, a)-extension* of the ring D. By (33), the ring D' admits the endomorphisms σ and τ. We have seen above that repeating the process iteratively we obtain its derivatives, see (45).

Proof of Proposition 1.7. We have to show that $*$ respects (4) and (5). Notice that $\sigma(a)^* = \tau(a^*) = \tau(a) = \sigma(a)$.

(i) $\sigma(a) = \tau(a)$: $\sigma(a)^* = \tau(a)^*$.

(ii) $\sigma(d)\sigma(a) = \sigma(a)\tau(d)$ for all $d \in D$: $(\sigma(d)\sigma(a))^* = \sigma(a)\tau(d^*) = \sigma(d^*)\sigma(a) = \tau(d)^*\sigma(a) = (\sigma(a)\tau(d))^*$.

(iii) $xd = \sigma(d)x$: $(xd)^* = d^*y = y\tau(d^*) = y\sigma(d)^* = (\sigma(d)x)^*$.

(iv) $dy = y\tau(d)$: $(dy)^* = xd^* = \sigma(d^*)x = \tau(d)^*x = (y\tau(d))^*$.

(v) $yx = a$: $(yx)^* = yx = a = a^*$.

(vi) $xy = \sigma(a)$: $(xy)^* = xy = \sigma(a) = \sigma(a)^*$. $\square$

For a *commutative* ring D, the identity map of D is an involution on D which is called the *trivial involution* on D.

Corollary 2.25 *Let $A = D[x, y; \sigma, \sigma, a]_{\rm in}$ and D be a commutative ring. Then the trivial involution on D can be extended to an involution $*$ of D by the rule $x^* = y$ and $y^* = x$.*

Proof. The result follows from Proposition 1.7. $\Box$

3 Simplicity criteria for inner generalized Weyl algebras

In this section, proofs of simplicity criteria for IGWAs are given (Theorem 1.3, Theorem 1.4 and Theorem 1.5). Different approaches are used in the proofs. A criterion is given for an IGWA to be a domain (Proposition 3.4). Necessary and sufficient conditions are found for the elements x and y of an IGWA A to be regular elements (Proposition 3.3).

Ore sets and denominator sets. Let S be a nonempty subset of a ring R. Let ${\rm ass}_l(S) := \{r \in R \,|\, sr = 0 \text{ for some } s = s(r) \in S\}$ and ${\rm ass}_r(S) := \{r \in R \,|\, rs = 0 \text{ for some } s = s(r) \in S\}$. A nonempty subset S of $R \backslash \{0\}$ is called a *multiplicative set* if $SS \subseteq S$ and $1 \in S$. A multiplicative set S is called a *left* (resp., *right*) *Ore set* if the *left* (resp., *right*) *Ore condition* holds: For all elements $s \in S$ and $r \in R$, $Sr \cap Rs \neq \emptyset$ (resp., $rS \cap sR \neq \emptyset$).

It follows from the Ore condition that if S is a left (resp., right) Ore set of R then ${\rm ass}_l(S)$ (resp., ${\rm ass}_r(S)$) is an ideal of the ring R. The sets of left and right Ore sets of R are denoted by ${\rm Ore}_l(R)$ and ${\rm Ore}_r(R)$, respectively. Their intersection ${\rm Ore}(R) = {\rm Ore}_l(R) \cap {\rm Ore}_r(R)$ is the *set of Ore sets* of R.

A left Ore set S of R is called a *left denominator set* of R if ${\rm ass}_l(S) \supseteq {\rm ass}_r(S)$. A right Ore set S of R is called a *right denominator set* of R if ${\rm ass}_l(S) \subseteq {\rm ass}_r(S)$. The sets of left and right denominator sets are denoted ${\rm Den}_l(R)$ and ${\rm Den}_r(R)$, respectively. Their intersection ${\rm Den}(R) = {\rm Den}_l(R) \cap {\rm Den}_r(R)$ is the *set of denominator sets.* For an ideal $\mathfrak{a}$ of R, ${\rm Den}_l(R, \mathfrak{a}) := \{S \in {\rm Den}_l(R) \,|\, {\rm ass}_l(R) = \mathfrak{a}\}$ and ${\rm Den}_r(R, \mathfrak{a}) := \{S \in {\rm Den}_r(R) \,|\, {\rm ass}_r(R) = \mathfrak{a}\}$. For each $S \in {\rm Den}_l(R)$, the ring $S^{-1}R = \{s^{-1}r \,|\, s \in S, r \in R\}$ is called the *left quotient ring* of R at S or the *left localization* of R at S. For each $S \in {\rm Den}_r(R)$, the ring $RS^{-1} = \{rs^{-1} \,|\, s \in S, r \in R\}$ is called the *right quotient ring* of R at S or the *right localization* of R at S. If $S \in {\rm Den}(R)$ then ${\rm ass}_l(S) = {\rm ass}_r(S)$ and $S^{-1}R \simeq RS^{-1}$.

For each natural number $i \geq 1$, let us consider the maps $l_i, r_i : D' \to D'$ given by the rule

$$l_i(d') = (i, -i)d' \;\text{ and }\; r_i(d') = d'(i, -i) \;\text{ for }\; d' \in D'.$$

For a subset S of D', let $l_i^{-1}(S) := \{d' \in D' \,|\, l_i(d') \in S\}$ and $r_i^{-1}(S) := \{d' \in D' \,|\, r_i(d') \in S\}$.

Proposition 3.1 *Let $A = D[x, y; \sigma, \tau, a]_{\rm in}$ be an IGWA, $\mathcal{K}(\sigma) := \mathcal{K}_{D'}(\sigma)$, $\mathcal{K}(\tau) := \mathcal{K}_{D'}(\tau)$ (see (28)) and $S_x = \{x^i \,|\, i \geq 0\}$. Then*

1. *$S_x \in {\rm Den}_l(A, \mathfrak{a})$ where $\mathfrak{a} := \bigoplus_{i\geq 1} y^i l_i^{-1}(\mathcal{K}(\sigma)) \oplus \bigoplus_{i \geq 0} \mathcal{K}(\sigma)x^i$.*

2. *$A/\mathfrak{a} \simeq \bigoplus_{i \geq 1} y^i D'/l_i^{-1}(\mathcal{K}(\sigma)) \oplus D'(\sigma)[x; \overline{\sigma}]$ where $D'(\sigma) := D'/\mathcal{K}(\sigma)$, the endomorphism $\overline{\sigma}$ of $D'(\sigma)$ is a monomorphism, and $\overline{\sigma}(d' + \mathcal{K}(\sigma)) := \sigma(d') + \mathcal{K}(\sigma)$ for all $d' \in D'$, see (29).*

3. *${\rm ass}_r(S_x) = \bigoplus_{i \geq 1} y^i {\rm ker}_{D'}(\overline{s}_i)$ where $\overline{s}_i : D' \to D'$, $d' \mapsto y^i d' x^i$. Furthermore, ${\rm ker}_{D'}(\overline{s}_i) \subseteq {\rm ker}_{D'}(l_i) \subseteq l_i^{-1}(\mathcal{K}(\sigma))$ for all $i \geq 1$.*

4. *The ring $A_x := S_x^{-1}A \simeq S_x^{-1}A_+$ is the skew Laurent polynomial ring $A_{x,0}[x^{\pm 1}; \sigma]$ where $A_{x,0} := \bigcup_{i \geq 0} x^{-i} D'(\sigma) x^i$ and $\sigma(x^{-i}(d' + \mathcal{K}(\sigma))x^i) := x^{-i}(\sigma(d') + \mathcal{K}(\sigma))x^i$ for all $d' \in D'$. The addition and multiplication in the ring $A_{x,0}$ are given in (49). Elements $x^{-i}(d_1' + \mathcal{K}(\sigma))x^i, x^{-j}(d_2' + \mathcal{K}(\sigma))x^j \in D'(\sigma)$, where $i \leq j$, are equal iff $d_2' + \mathcal{K}(\sigma) = \sigma^{j-i}(d_1') + \mathcal{K}(\sigma)$.*

5. *(a) ${\rm ker}_A(x\cdot) = \bigoplus_{i \geq 1} y^i {\rm ker}_{D'}(\tau^i(a)\cdot) \oplus \bigoplus_{i \geq 0} {\rm ker}_{D'}(\sigma)x^i$; $x \in \mathcal{C}_A'$ iff ${\rm ker}_{D'}(\sigma) = 0$ and $\{\tau^i(a) \,|\, i \geq 1\} \subseteq \mathcal{C}_{D'}'$.*

   (b) ${\rm ker}_A(\cdot x) = \bigoplus_{i \geq 1} y^i {\rm ker}_{D'}(\overline{s}_1)$ where $\overline{s}_1 : D' \to D'$, $d' \mapsto yd'x$; $x \in {}'\mathcal{C}_A$ iff ${\rm ker}_{D'}(\overline{s}_1) = 0$.

   (c) $x \in \mathcal{C}_A$ iff ${\rm ker}_{D'}(\sigma) = 0$ and $\{\tau^i(a) \,|\, i \geq 1\} \subseteq \mathcal{C}_{D'}'$.

6. $\mathrm{ass}_l(S_x) = \mathrm{ass}_r(S_x)$ *iff* $\ker_{D'}(\sigma) = 0$. *If* $\mathrm{ass}_l(S_x) = \mathrm{ass}_r(S_x)$ *then their common value is equal to* $\bigoplus_{i\geq 1} y^i \ker_{D'}((i,-i)\cdot)$ *and* $l_i^{-1}(\mathcal{K}(\sigma)) = \ker_{D'}(\overline{s}_i) = \ker_{D'}((i,-i)\cdot)$ *for all* $i \geq 1$.

7. $S_x \in \mathrm{Den}(A)$ *iff* σ *is an automorphism of the ring* D'. *If* $S_x \in \mathrm{Den}(A)$ *then* $\mathrm{ass}(S_x) = \bigoplus_{i\geq 1} y^i \ker_{D'}((i,-i)\cdot)$.

Proof. 3. Let $\mathfrak{b} := \mathrm{ass}_r(S_x)$. Since the ring $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is a $\mathbb{Z}$-graded ring and the element x is a homogeneous element,

$$\mathfrak{b} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{b}_i \ \text{ where } \ \mathfrak{b}_i := \mathfrak{b} \cap A_i.$$

Since the element $x \in A_+ = D'[x;\sigma]$ is a right regular in A_+, we have that $\mathfrak{b}_i = 0$ for all $i \geq 0$. For all $j \geq i \geq 1$ and $d' \in D'$, $y^i d' \cdot x^j = y^i d' x^i \cdot x^{j-i}$. Hence,

$$\mathfrak{b}_{-i} = y^i \ker_{D'}(\overline{s}_i).$$

Suppose that $y^i d' \in \mathfrak{b}_{-i}$. That is $y^i d' x^i = 0$. Then $0 = x^i \cdot y^i d' x^i = (i,-i) d' x^i$, and so $(i,-i)d' = 0$. Therefore, $d' \in \ker_{D'}(l_i) = l_i^{-1}(0) \subseteq l_i^{-1}(\mathcal{K}(\sigma))$, and so $\ker_{D'}(\overline{s}_i) \subseteq l_i^{-1}(\mathcal{K}(\sigma))$.

1(i) $S_x \in \mathrm{Ore}_l(A)$: This fact follows from the inclusions $x^j A_i \subseteq D' x^{i+j}$ where $j \geq 1$ such that $i + j \geq 0$.

(ii) $\mathrm{ass}_l(S_x) = \mathfrak{a}$: By the statement (i), $\mathfrak{a}' := \mathrm{ass}_l(S_x)$ is a homogeneous ideal of the ring A,

$$\mathfrak{a}' = \bigoplus_{i\in\mathbb{Z}} \mathfrak{a}'_i \ \text{ where } \ \mathfrak{a}'_i := \mathfrak{a}' \cap A_i.$$

For all $i \geq 0$ $\mathfrak{a}'_i = \mathcal{K}(\sigma) x^i$ since $x^j \cdot : D' x^i \to D' x^{i+j}$, $d' x^i \mapsto \sigma^j(d') x^{i+j}$. For all $i \geq 1$,

$$\mathfrak{a}'_{-i} = y^i l_i^{-1}(\mathcal{K}(\sigma))$$

since for all $j \geq 0$, $x^{i+j} \cdot : y^i D' \to D' x^j$, $\ y^i d' \mapsto \sigma^j((i,-i)d') x^j$. Therefore, $\mathfrak{a}' = \mathfrak{a}$.

(iii) $S_x \in \mathrm{Den}_l(R, \mathfrak{a})$: By statement 3, $\mathrm{ass}_r(S_x) \subseteq \mathfrak{a}$, and so the statement (iii) follows from the statements (i) and (ii).

2. Statement 2 follows from statement 1.

4. $S_x^{-1} A \simeq S_x^{-1} A_+$: By statement 2, the left $\mathbb{Z}$-graded A_+-module A/A_+ is S_x-torsion, and so $S_x^{-1} A \simeq S_x^{-1} A_+$. Now,

$$S_x^{-1} A_+ = S_x^{-1}(A_+ / A_+ \cap \mathfrak{a}) \simeq S_x^{-1} D(\sigma)'[x;\overline{\sigma}] \simeq A_{x,0}[x^{\pm 1};\sigma]$$

is a skew Laurent polynomial ring where $A_{x,0}$ and σ are as in statement 4 since

$$x \cdot x^{-i}(d' + \mathcal{K}(\sigma)) x^i = x^{-i}(\sigma(d') + \mathcal{K}(\sigma)) x^i \cdot x = \sigma(x^{-i}(d' + \mathcal{K}(\sigma)) x^i) \cdot x.$$

Notice that by (29) the homomorphism $\overline{\sigma} : D'(\sigma) \to D'(\sigma)$, $\overline{d'} = d' + \mathcal{K}(\sigma) \mapsto \sigma(d') + \mathcal{K}(\sigma)$ is a monomorphism. Every element $x^{-i}\overline{d'} x^i$ of the ring $A_{x,0}$, where $i \geq 0$ and $\overline{d'} \in D'(\sigma)$, can be written also as follows

$$x^{-i}\overline{d'} x^i = x^{-i-j} x^j \overline{d'} x^i = x^{-i-j} \overline{\sigma^j(d')} x^{i+j} \ \text{ for } \ j \geq 0. \tag{48}$$

So, the addition and multiplication in the ring $A_{x,0}$ are given by the rule:

$$x^{-i}\overline{d'} x^i + x^{-j}\overline{e'} x^j = x^{-i-j}\left(\overline{\sigma^j(d') + \sigma^i(e')}\right) x^{i+j} \ \text{ and } \ x^{-i}\overline{d'} x^i \cdot x^{-j}\overline{e'} x^j = x^{-i-j}\overline{\sigma^j(d') \cdot \sigma^i(e')} x^{i+j}. \tag{49}$$

Since $x \cdot x^{-i}\overline{d'} x^i = x^{-i}\overline{\sigma(d')} x^i \cdot x = \sigma(x^{-i}\overline{d'} x^i) x$, the ring A_x is the skew polynomial ring $A_{x,0}[x^{\pm 1};\sigma]$.

Clearly, the elements $x^{-i}(d'_1 + \mathcal{K}(\sigma)) x^i, x^{-j}(d'_2 + \mathcal{K}(\sigma)) x^j \in D'(\sigma)$, where $i \leq j$, are equal iff

$$d'_2 + \mathcal{K}(\sigma) = x^{j-i}\big(\sigma^{j-i}(d'_1) + \mathcal{K}(\sigma)\big) x^{-(j-i)} = \sigma^{j-i}(d'_1) + \mathcal{K}(\sigma).$$

5(a) Notice that $\ker_A(x\cdot)$ is a homogeneous right ideal of A. Then the equality in the statement (a) follows from the following equalities: For all $d'\in D'$,

$$x\cdot y^i d' = y^{i-1}\tau^i(a)d' \ \ (i\geq 1) \ \text{ and } \ x\cdot d'x^i = \sigma(d')x^{i+1} \ \ (i\geq 0).$$

Then the 'iff' statement in (a) follows from the equality for $\ker_A(x\cdot)$.

5(b) Notice that $\ker_A(\cdot x)$ is a homogeneous right ideal of A. Then the equality in the statement (b) follows from the following equalities:

$$y^i d'x = y^i\cdot yd'x = y^i\overline{s}_1(d') \ \text{ for all } \ i\geq 1 \ \text{ and } \ d'\in D'.$$

Then, $x\in {}'\mathcal{C}_A$ iff $\ker_{D'}(\overline{s}_1)=0$.

5(c) The statements (c) follows from the statements (a) and (b) and the fact that $\ker_{D'}(\overline{s}_1)=0$ provided that $\tau(a)\in\mathcal{C}'_{D'}$:

$$yd'x=0\Rightarrow 0=x\cdot yd'x=\tau(a)d'x\Rightarrow \tau(a)d'=0\Rightarrow d'=0.$$

6. By statements 1 and 3, $\mathrm{ass}_l(S_x)=\mathrm{ass}_r(S_x)$ iff $\ker_{D'}(\sigma)=0$ and $l_i^{-1}(\mathcal{K}(\sigma))\subseteq\ker_{D'}(\overline{s}_i)$ iff $\ker_{D'}(\sigma)=0$ and for each $i\geq 1$, $(i,-i)d'=0$ for some $d'\in D'$ implies $y^id'x^i=0$ since

$$l_i^{-1}(\mathcal{K}(\sigma))=\{d'\in D'\,|\,\sigma^j((i,-i)d')=0 \ \text{ for some } \ j\}$$

iff $\ker_{D'}(\sigma)=0$ and since the second condition is redundant ($(i,-i)d'=0\Rightarrow 0=x^iy^id'x^i=\sigma^i(y^id'x^i)x^i=0\Rightarrow\sigma^i(y^id'x^i)=0\Rightarrow y^id'x^i=0$ since $\ker_{D'}(\sigma)=0$). In particular,

$$l_i^{-1}(\mathcal{K}(\sigma))=\ker_{D'}(\overline{s}_i)=\ker_{D'}((i,-i)\cdot) \ \text{ for all } \ i\geq 1.$$

Now, if $\mathrm{ass}_l(S_x)=\mathrm{ass}_r(S_x)$ then their common value is $\bigoplus_{i\geq 1}y^i\ker_{D'}((i,-i)\cdot)$.

7. By statement 1, $S_x\in\mathrm{Den}(A)$ iff $S_x\in\mathrm{Den}_r(A)$ (by statement 3) iff S_x is a right Ore set of A and $\mathrm{ass}_l(S_x)=\mathrm{ass}_r(S_x)$ iff S_x is a right Ore set iff σ is monomorphism of the ring D' (by statement 6) iff σ is an epimorphism and σ is a monomorphism of D' iff σ is an isomorphism of D'. If $S_x\in\mathrm{Den}(A)$ then, by statement 6, $\mathrm{ass}(S_x)=\bigoplus_{i\geq 1}y^i\ker_{D'}((i,-i)\cdot)$. $\square$

Proposition 3.2 follows from Proposition 3.1 and (38).

Proposition 3.2 *Let $A=D[x,y;\sigma,\tau,a]_{\rm in}$ be an IGWA, $\mathcal{K}(\tau):=\mathcal{K}_{D'}(\tau)$, $\mathcal{K}(\tau):=\mathcal{K}_{D'}(\tau)$ and $S_y=\{y^i\,|\,i\geq 0\}$. Then*

1. *$S_y\in\mathrm{Den}_r(A,\mathfrak{a}^o)$ where $\mathfrak{a}^o:=\bigoplus_{i\geq 0}y^i\mathcal{K}(\tau)\oplus\bigoplus_{i\geq 1}r_i^{-1}(\mathcal{K}(\tau))x^i$.*

2. *$A/\mathfrak{a}^o\simeq D'(\tau)[x;\overline{\tau}]_r\oplus\bigoplus_{i\geq 1}D'/r_i^{-1}(\mathcal{K}(\tau))\cdot x^i$ where $D'(\tau):=D'/\mathcal{K}(\tau)$, the endomorphism $\overline{\tau}$ of $D'(\tau)$ is a monomorphism, and $\overline{\tau}(d+\mathcal{K}(\tau)):=\tau(d)+\mathcal{K}(\tau)$.*

3. *$\mathrm{ass}_l(S_y)=\bigoplus_{i\geq 1}\ker_{D'}(\overline{s}_i)x^i$ where $\overline{s}_i:D'\to D'$, $d'\mapsto y^id'x^i$. Furthermore, $\ker_{D'}(\overline{s}_i)\subseteq\ker_{D'}(r_i)\subseteq r_i^{-1}(\mathcal{K}(\tau))$ for all $i\geq 1$.*

4. *The ring $A_y:=AS_y^{-1}\simeq A_-S_y^{-1}$ is the right skew Laurent polynomial ring $A_{y,0}[y^{\pm 1};\tau]_r=\bigoplus_{i\in\mathbb{Z}}A_{y,0}y^i$ where $A_{y,0}:=\bigcup_{i\geq 0}y^iD'(\tau)y^{-i}$, $\tau(y^i(d'+\mathcal{K}(\tau))y^{-i}):=y^i(\tau(d')+\mathcal{K}(\tau))y^{-i}$ for all $d'\in D'$, and $\alpha y=y\tau(\alpha)$ for all $\alpha\in A_{y,0}$. The elements $y^i(d_1'+\mathcal{K}(\tau))y^{-i},y^j(d_2'+\mathcal{K}(\tau))y^{-j}\in D'(\tau)$, where $i\leq j$, are equal iff $d_2'+\mathcal{K}(\tau)=\tau^{j-i}(d_1')+\mathcal{K}(\tau)$.*

5. *(a) $\ker_A(\cdot y)=\bigoplus_{i\geq 0}y^i\ker_{D'}(\tau))\oplus\bigoplus_{i\geq 1}\ker_{D'}(\cdot\sigma^i(a))x^i$; $y\in {}'\mathcal{C}_A$ iff $\ker_{D'}(\tau)=0$ and $\{\sigma^i(a)\,|\,i\geq 1\}\subseteq {}'\mathcal{C}_{D'}$.*
   (b) $\ker_A(y\cdot)=\bigoplus_{i\geq 1}\ker_{D'}(\overline{s}_1)x^i$ where $\overline{s}_1:D'\to D'$, $d'\mapsto yd'x$; $y\in\mathcal{C}'_A$ iff $\ker_{D'}(\overline{s}_1)=0$.
   (c) $y\in\mathcal{C}_A$ iff $\ker_{D'}(\tau)=0$ and $\{\sigma^i(a)\,|\,i\geq 1\}\subseteq {}'\mathcal{C}_{D'}$.

6. *$\mathrm{ass}_l(S_x)=\mathrm{ass}_r(S_x)$ iff $\ker_{D'}(\tau)=0$. If $\mathrm{ass}_l(S_x)=\mathrm{ass}_r(S_x)$ then their common value is equal to $\bigoplus_{i\geq 1}\ker_{D'}(\cdot(i,-i))x^i$ and $r_i^{-1}(\mathcal{K}(\tau))=\ker_{D'}(\overline{s}_i)=\ker_{D'}(\cdot(i,-i))$ for all $i\geq 1$.*

7. $S_y \in \mathrm{Den}(A)$ *iff* τ *is an automorphism of the ring* D'. *If* $S_y \in \mathrm{Den}(A)$ *then* $\mathrm{ass}(S_y) = \bigoplus_{i\geq 1} \ker_{D'}(\cdot(i,-i))x^i$.

Regularity criterion for the elements x and y of an IGWA.

Proposition 3.3 *The following statements are equivalent:*

1. $x, y \in \mathcal{C}_A$ *(where* $\mathcal{C}_A$ *is the set of regular elements of* A*).*
2. σ *and* τ *are monomorphisms of the ring* D', $\sigma^i(a) \in {}'\mathcal{C}_{D'}$ *and* $\tau^i(a) \in \mathcal{C}'_{D'}$ *for all* $i \geq 1$.
3. $a, \sigma(a) \in \mathcal{C}_{D'}$.

Proof. ($1 \Leftrightarrow 2$) By Proposition 3.1.(5c) and Proposition 3.2.(5c), $x, y \in \mathcal{C}_A$ iff σ and τ are monomorphisms of the ring D', $\sigma^i(a) \in {}'\mathcal{C}_{D'}$ and $\tau^i(a) \in \mathcal{C}'_{D'}$ for all $i \geq 1$.

($1 \Leftrightarrow 3$) The equivalence follows from the equalities $yx = a$ and $xy = \sigma(a)$. $\square$

A criterion for an IGWA to be a domain.

Proposition 3.4 *An IGWA* $A = D[x, y; \sigma, \tau, a]_{\mathrm{in}}$ *is a domain iff* D' *is a domain and* $a, \sigma(a) \in \overline{D}\backslash\{0\}$.

Proof. ($\Rightarrow$) If A is a domain then so is the ring D' and $a, \sigma(a) \in \overline{D}\backslash\{0\}$ (since $x \neq 0$ and $y \neq 0$ in A, $yx = a \neq 0$ and $\sigma(a) = xy \neq 0$).

($\Leftarrow$) By the assumption $a, \sigma(a) \in \overline{D}\backslash\{0\}$. Hence, $a, \sigma(a) \in \mathcal{C}_{D'}$, and so $x, y \in \mathcal{C}_A$ and σ and τ are monomorphisms of D', by Proposition 3.3. Now, the ring A is a domain (see the expression for multiplication of homogeneous elements in the ring A). $\square$

The next corollary is used in the proof of a simplicity criterion for GWAs (Theorem 1.4 and Theorem 1.5).

Lemma 3.5 *Let* $A = D[x, y; \sigma, \tau, a]_{\mathrm{in}}$, $\mathfrak{a}_i := yD'x + \sigma^i(a)\sigma^{i-1}(D')$ *and* $\mathfrak{b}_i := yD'x + \tau^{i-1}(D')\tau^i(a)$ *for* $i \geq 1$, *and* $n \geq 1$ *be a natural number. Then*

1. $(x^i) = A$ *for* $i = 1, \ldots, n$ *iff* $D' = \mathfrak{a}_i$ *for* $i = 1, \ldots, n$. *In particular,* $(x^i) = A$ *for all* $i \geq 1$ *iff* $D' = \mathfrak{a}_i$ *for all* $i \geq 1$.
2. $(y^i) = A$ *for* $i = 1, \ldots, n$ *iff* $D' = \mathfrak{b}_i$ *for* $i = 1, \ldots, n$. *In particular,* $(y^i) = A$ *for all* $i \geq 1$ *iff* $D' = \mathfrak{b}_i$ *for all* $i \geq 1$.

Proof. 1. ($\Rightarrow$) The ideal (x^i) is a homogeneous ideal of A. Hence,

$$(x^i)_{i-1} := (x^i) \cap D'x^{i-1} = yD' \cdot x^i + x^i \cdot yD' = \big(yD'x + \sigma^i(a)\sigma^{i-1}(D')\big)x^{i-1} = \mathfrak{a}_i x^{i-1}.$$

If $(x^i) = A$ then $D' = \mathfrak{a}_i$.

($\Leftarrow$) For $n = 1$, $(x)_0 = \mathfrak{a}_1 = D'$, and so $(x) = A$. We use induction on n to prove the implication. So, let $n > 1$ and we assume that $D' = \mathfrak{a}_i$ for $i = 1, \ldots, n$. By induction on n, $(x) = \cdots = (x^{n-1}) = A$. Since

$$(x^n)_{n-1} = \mathfrak{a}_n x^{n-1} \ni x^{n-1} \ \ (\text{as } \ \mathfrak{a}_n = D'),$$

we must have $A = (x^{n-1}) \subseteq (x^n) \subseteq (x^{n-1}) = A$, i.e., $(x^n) = A$, as required.

2. By (38), statement 2 follows from statement 1. $\square$

Recall that for each $i \geq 1$, we have the map $\overline{s}_i : D' \to D'$, $d' \mapsto y^i d' x^i$. Since $\overline{s}_i = (\overline{s}_1)^i$,

$$\ker_{D'}(\overline{s}_1) \subseteq \ker_{D'}(\overline{s}_2) \subseteq \cdots \subseteq \ker_{D'}(\overline{s}_i) \subseteq \cdots.$$

Proposition 3.6 describes the kernels $\ker_{D'}(\overline{s}_i)$ for $i \geq 1$.

Proposition 3.6 *For all* $m \geq 1$, $\ker_{D'}(\overline{s}_m) = (L_m + D_0)/D_0 \subseteq \overline{D}$ *where the sets* L_m *are defined in Proposition 2.5.(1). In particular,* $\ker_{D'}(\overline{s}_1) = (\theta(\mathcal{K}_a) + D_0)/D_0$.

Proof. By Theorem 2.6,

$$\ker_{D'}(\overline{s}_m) \subseteq \overline{D} = D/D_0.$$

Therefore,

$$\ker_{D'}(\overline{s}_m) = \{\overline{d} \in \overline{D} \,|\, y^m dx^m \in \mathcal{T}\} = \{d + D_0 \in \overline{D} \,|\, y^m dx^m \in {}'\mathcal{T}_{[m]}\} = (L_m + D_0)/D_0,$$

by Proposition 2.5.(1,2). In particular, $\ker_{D'}(\overline{s}_1) = (\theta(\mathcal{K}_a) + D_0)/D_0$, by Proposition 2.5.(1). □

Lemma 3.7 *1. For all $i \geq 1$, $\ker_{D'}(\overline{s}_i) \subseteq \ker_{D'}((i,-i)\cdot) \cap \ker_{D'}(\cdot(i,-i))$.*

2. $\tau(\ker_{D'}(\cdot a)) + \sigma(\ker_{D'}(a\cdot)) \subseteq \ker_{D'}(\overline{s}_1) \subseteq \ker_{D'}(\sigma(a)\cdot) \cap \ker_{D'}(\cdot\sigma(a))$.

3. If either $(i,-i) \in \mathcal{C}'_{D'}$ for all $i \geq 1$ or $(i,-i) \in {}'\mathcal{C}_{D'}$ for all $i \geq 1$ then $\ker_{D'}(\overline{s}_i) = 0$ for all $i \geq 1$.

Proof. 1. Suppose that $d' \in \ker_{D'}(\overline{s}_i)$, that is $y^i d' x^i = 0$. Multiplying this equality on the left by x^i and on the right by y^i we have the equalities $(i,-i)d'x^i = 0$ and $y^i d'(i,-i) = 0$, and statement 1 follows.

2. The first inclusion follows from the equalities $y\tau(d')x = d'a$ and $y\sigma(d')x = ad'$. The second inclusion is a particular case of statement 1 for $i = 1$.

3. Statement 3 follows from statement 1. □

Let $R = D[x;\sigma]$ be a skew polynomial ring. Let $u = d_n x^n + d_{n-1}x^{n-1} + \cdots + d_m x^m$ be a nonzero element of R where $d_i \in D$, $d_n \neq 0$ and $d_m \neq 0$. The natural number $l(u) = n - m$ is called the (graded) *length* of u.

Proof of Theorem 1.3. $(1 \Rightarrow 2)$ Suppose that the ring A is simple. Recall that $S_x \in \mathrm{Den}_l(A)$ (Proposition 3.1) and $S_y \in \mathrm{Den}_r(A)$ (Proposition 3.2). Then $\mathrm{ass}_l(S_x) = 0$ and $\mathrm{ass}_l(S_y) = 0$, i.e., the elements x and y are regular in A. By Proposition 3.3, the statement (a) holds.

Let I be a nonzero ideal of the ring D'. The zero component $(I)_0$ of the homogeneous ideal $(I) = AIA$ of the ring A is equal to

$$\sum_{i\in\mathbb{Z}} A_i I A_{-i} = I'.$$

Since the ring A is simple and the ideal (I) is a nonzero homogeneous ideal of A, we have $(I)_0 = D$, and so the statement (b) holds.

Finally, suppose that the statement (c) is false, i.e., $\sigma^n = \omega_d$ for some σ-invariant, regular, left normal element d of D', we seek a contradiction. Then

$$dd' = \sigma^n(d')d \;\text{ for all elements }\; d' \in D'.$$

Claim. *The ideal of A generated by the element $u = x^n + d$ is not equal to A.*

Notice that $ux = xu$ (since $\sigma(d) = d$) and $ud' = \sigma^n(d')u$ for all $d' \in D'$ (since $\sigma^n = \omega_d$). This means that the element $u \in A_+ = \bigoplus_{i\geq 0} D'x^i$ is a left normal element of the ring A_+. In particular, $A_+uA_+ = A_+u$ is an ideal of A_+. Suppose that

$$AuA = A.$$

Then $1 \in \sum_{-l\leq i,j\leq l} A_i u A_j$ for some $l \in \mathbb{N}$. Then $x^{2l} = x^l 1 x^l \in A_+uA_+ = A_+u$, and so $x^{2l} = vu$ for some element $v \in A_+$. Then

$$v = x^{2l-n} + \cdots + d_m x^m$$

where $d_m \in D\backslash\{0\}$ and $d_m x^m$ is the least term of v (w.r.t. the $\mathbb{Z}$-grading of A). Then

$$0 = d_m x^m d = d_m \sigma^m(d) x^m = d_m d x^m$$

(since $\sigma(d) = d$), i.e., $d_m d = 0$. The element d is regular and $d_m \neq 0$, hence $d_m d \neq 0$, a contradiction. This means that $0 \neq (u) \neq A$, as claimed. Therefore, the statement (c) holds.

(2 ⇒ 1) Suppose that the conditions (a)–(c) hold. By the statement (a), the elements a and $\sigma(a)$ are regular in D'. By Proposition 3.3, $x, y \in \mathcal{C}_A$ and σ, τ are monomorphisms. Then $x^i, y^i \in \mathcal{C}_A$ for all $i \geq 1$, $D \subseteq D'$ (Proposition 2.2.(5)), $\sigma^i(a) \in {}'\mathcal{C}_{D'}$ (Proposition 3.2.(5c)) and $\tau^i(a) \in \mathcal{C}'_{D'}$ for all $i \geq 1$ (Proposition 3.1.(5c)). For all $i \geq 0$, $(i, -i) = x^i y^i$ and $(-i, i) := y^i x^i$ are regular elements of D'.

Let J be a nonzero ideal of A. We have to show that $J = A$. The ideal J contains a nonzero element, say u, of least possible length, say l.

If $l = 0$ then J contains a nonzero element, say d, of D' (since x^i and y^i are regular element of A for all $i \geq 1$). Let $I = D'dD'$. Then $J \supseteq I' = D$, by the statement (b), and so $J = A$, as required.

Suppose that $l > 1$. Replacing the element u by $x^s u$ or $y^s u$ for some $s \geq 0$, we may assume that $u = u_0 + u_1 y + \cdots + u_l y^l$ for some elements $u_i \in D$ such that $u_0 \neq 0$ and $u_l \neq 0$ (since x^i and y^i are regular elements in A). Let $I = D'u_0D'$, a nonzero ideal of D'. Then

$$J \supseteq \sum_{i \in \mathbb{Z}} A_i u A_{-i} = \sum_{i \in \mathbb{Z}} A_i u_0 A_{-i} + \cdots = I' + \cdots$$

where the three dots means smaller terms, i.e., elements of the set $\bigoplus_{i \geq 1} y^i D'$. By the statement (b), the ideal J contains an element of the form $v = 1 + \cdots$. Then

$$0 \neq w := vx^l = d_0 + d_1 x + \cdots + d_{l-1} x^{l-1} + x^l \in J$$

where $d_i \in D'$ and $d_0 \neq 0$, by the minimality of l. By the minimality of l, the element

$$[x, w] = \sum_{i=0}^{l-1} (\sigma(d_i) - d_i) x^{i+1} \in J$$

must be zero, i.e., $xw = wx$ and $\sigma(d_i) = d_i$ for all $i = 0, 1, \ldots, l-1$. In particular the element d_0 is *σ-invariant*. Similarly,

$$\sigma^l(d)w - wd = \sum_{i=0}^{l-1} (\sigma^l(d) d_i - d_i \sigma^i(d)) x^i \in J \text{ for all } d \in D'.$$

Therefore, $\sigma^l(d)w = wd$ and $\sigma^l(d)d_i = d_i \sigma^i(d)$ for all $i = 0, 1, \ldots, l-1$. In particular,

$$\sigma^l(d) d_0 = d_0 d \text{ for all } d \in D',$$

i.e., the element $d_0 \in D'$ is a *left normal* element of D'. The element d_0 is a *regular* element of D'. Since otherwise we would have either $d'd_0 = 0$ or $d_0 d' = 0$ for some $d' \in D'$. Then either $0 \neq d'w = \sum_{i=1}^{l-1} d' d_i x^i + d' x^l \in J$ or $0 \neq wd' = \sum_{i=1}^{l-1} d_i \sigma^i(d') x^i + \sigma^l(d') x^l \in J$ (since σ is a monomorphism). In both cases, this would contradict the minimality of l. Clearly, $\sigma^l(d_0) = d_0$ (since the element d_0 is a regular element of D' and $d_0 d_0 = \sigma^l(d_0) d_0$, and so $\sigma^l(d_0) = d_0$). Therefore, $\sigma^l = \omega_{d_0}$ where the element $d_0 \in D'$ is a σ-invariant, regular, left normal element of D'. This fact contradicts to the property (c). Hence, $J = A$, as required.

(1 ⇔ 3) This equivalence follows from the equivalence (1 ⇔ 2) by the left-right symmetry of IGWAs, see (38). □

Proof of Theorem 1.4. Since σ is an epimorphism of the ring D, the homomorphism $\nu : D \to D'$ is an epimorphism, by Theorem 2.6, and so $D' \simeq \overline{D}$ and $A \simeq \overline{D}[x, y; \sigma, \tau, a]_{\rm in}$.

(1 ⇔ 2) In view of Theorem 1.3, without loss of generality we can assume that the condition (a) holds for the ring A and we have to show that the ring A is simple iff the conditions (b)–(d) hold. By Proposition 3.3, $x, y \in \mathcal{C}_A$ and σ, τ are monomorphisms of the ring $\overline{D}$. So, σ is an automorphism of the ring $\overline{D}$. Then the elements $\sigma^i(a)$ ($i \geq 1$) are regular in $\overline{D}$. By Proposition 3.1.(7), $S_x \in {\rm Den}(A)$ with ${\rm ass}(S_x) = 0$. By Proposition 3.1.(4),

$$S_x^{-1} A \simeq S_x^{-1} A_+ = S_x^{-1} \overline{D}[x\ \sigma] \simeq \overline{D}[x, x^{-1}; \sigma]$$

since σ is an automorphism of the ring $\overline{D}$. Recall that ([9, Proposition 3.1.(3)]): *If S is a left and right denominator set of a ring R then R is a simple ring iff $S^{-1}R$ is a simple ring,* $\mathrm{ass}(S)=0$ *and $RsR=R$ for all elements $s\in S$.* Therefore, the ring A is simple iff $S_x^{-1}A$ is a simple ring and $(x^i)=A$ for all $i\geq 1$. It is a classical result that the skew polynomial ring $S_x^{-1}A=\overline{D}[x,x^{-1};\sigma]$ is simple iff the conditions (b) and (c) hold, see [13] or [14, Theorem 1.8.5]. By Lemma 3.5.(1), $(x^i)=A$ for all $i\geq 1$ iff for all $i\geq 1$,

$$\overline{D}=y\overline{D}x+\sigma^i(a)\sigma^{i-1}(\overline{D})=yx\sigma^{-1}(\overline{D})+\sigma^i(a)\overline{D}=a\overline{D}+\sigma^i(a)\overline{D}$$

iff the condition (d) holds. The proof of ($1\Leftrightarrow 2$) is complete.

Suppose that the ring A is simple. Then $\sigma\in\mathrm{Aut}(\overline{D})$. Hence, the regular elements a and $\sigma(a)$ are right normal in $\overline{D}$ since

$$\sigma(a)\tau(d)=\sigma(d)\sigma(a)\ \text{ and }\ a\sigma^{-1}\tau(d)=da\ \text{ for all }\ d\in\overline{D}.$$

It follows that $\tau=\omega'_{\sigma(a)}\sigma$ is a monomorphism of $\overline{D}$. $\Box$

Proof of Theorem 1.5. Theorem 1.5 follows from Theorem 1.4 by the left-right symmetry for IGWAs, see (38). $\Box$

Corollary 3.8 is a particular case of Theorem 1.4 where $D_0=0$ since $\sigma\in\mathrm{Aut}(D)$ (Proposition 2.2.(5)).

Corollary 3.8 *Let $A=D[x,y;\sigma,\tau,a]_{\mathrm{in}}$. Suppose that σ is an automorphism of D. Then the following statements are equivalent:*

1. *A is a simple ring.*
2. *(a) The elements a and $\sigma(a)$ are regular in D,*
   (b) D is a σ-simple ring,
   (c) for all $i\geq 1$, σ^i is not an inner automorphism of the ring D, and
   (d) for all $i\geq 1$, $aD+\sigma^i(a)D=D$.

If one of the equivalent conditions holds then $D'=D$, $\tau=\omega'_{\sigma(a)}\sigma$ is a monomorphism of D, $S_x^{-1}A\simeq D[x,x^{-1};\sigma]$ is a skew Laurent polynomial ring ($x^{\pm1}d=\sigma^{\pm1}(d)x^{\pm1}$ for all $d\in D$), and the elements a and $\sigma(a)$ are right normal in D.

Corollary 3.9 is a particular case of Theorem 1.5 where $D_0=0$ since $\tau\in\mathrm{Aut}(D)$ (Proposition 2.2.(5)).

Corollary 3.9 *Let $A=D[x,y;\sigma,\tau,a]_{\mathrm{in}}$. Suppose that τ is an automorphism of D. Then the following statement are equivalent:*

1. *A is a simple ring.*
2. *(a) The elements a and $\sigma(a)$ are regular in D,*
   (b) D is a τ-simple ring,
   (c) for all $i\geq 1$, τ^i is not an inner automorphism of the ring D, and
   (d) for all $i\geq 1$, $Da+D\tau^i(a)=D$.

If one of the equivalent conditions holds then $D'=D$, $\sigma=\omega_{\sigma(a)}\tau$ is a monomorphism of D, $AS_y^{-1}\simeq D[y,y^{-1};\tau]_r$ is a right skew Laurent polynomial ring ($dy^{\pm1}=y^{\pm1}\tau^{\pm1}(d)$ for all $d\in D$), and the elements a and $\tau(a)=\sigma(a)$ are left normal in D.

Corollary 3.10 *Let $A=D[x,y;\sigma,\tau,a]_{\mathrm{in}}$. Suppose that σ and τ are automorphisms of D. Then the following statements are equivalent:*

1. *A is a simple ring.*

2. (a) *The elements a and $\sigma(a)$ are regular in D,*

   (b) *D is a σ-simple ring,*

   (c) *for all $i \geq 1$, σ^i is not an inner automorphism of the ring D, and*

   (d) *for all $i \geq 1$, $aD + \sigma^i(a)D = D$.*

3. (a) *The elements a and $\sigma(a)$ are regular in D,*

   (b) *D is a τ-simple ring,*

   (c) *for all $i \geq 1$, τ^i is not an inner automorphism of the ring D, and*

   (d) *for all $i \geq 1$, $Da + D\tau^i(a) = D$.*

If one of the equivalent conditions holds then $D' = D$, the elements a and $\sigma(a) = \tau(a)$ are normal in D, $\tau = \omega'_{\sigma(a)}\sigma$ and $\sigma = \omega_{\sigma(a)}\tau$, $S_x^{-1}A \simeq D[x, x^{-1}; \sigma]$ is a skew Laurent polynomial ring ($x^{\pm 1}d = \sigma^{\pm 1}(d)x^{\pm 1}$ for all $d \in D$), and $AS_y^{-1} \simeq D[y, y^{-1}; \tau]_r$ is a right skew Laurent polynomial ring ($dy^{\pm 1} = y^{\pm 1}\tau^{\pm 1}(d)$ for all $d \in D$).

Proof. The result follows from Corollary 3.8 and Corollary 3.9. $\square$

Left normal elements in $D[x;\sigma]$ and IGWAs. Let $R = D[x;\sigma]$ be a skew polynomial ring. Let S be a subset of R such that $S \neq \{0\}$. Then $l(S) := \min\{l(u) \mid u \in S\backslash\{0\}\}$ is called the (graded) *length* of S. Clearly, $l(Sx^i) = l(S)$ for all $i \geq 0$. Let

$$\begin{aligned} \mathcal{M}(S) &:= \{u \in S \mid l(u) = l(S)\}, \\ \mathcal{M}_1(S) &:= \{u \in S \mid l(u) = l(S) \text{ and } u = x^n + \cdots\}. \end{aligned} \tag{50}$$

The natural numbers

$$\begin{aligned} \deg_x(\mathcal{M}(S)) &:= \min\{n \in \mathbb{N} \mid \deg_x(u), u \in \mathcal{M}(S)\}, \\ \deg_x(\mathcal{M}_1(S)) &:= \min\{n \in \mathbb{N} \mid \deg_x(u), u \in \mathcal{M}_1(S)\}, \end{aligned} \tag{51}$$

are called the *degrees* of the sets $\mathcal{M}(S)$ and $\mathcal{M}_1(S)$, respectively. Notice that $\mathcal{M}_1(S) \subseteq \mathcal{M}(S)$, and so $\deg_x(\mathcal{M}_1(S)) \geq \deg_x(\mathcal{M}(S))$ provided $\mathcal{M}_1(S) \neq \emptyset$. Clearly, $l(R) = 0$, the set $\mathcal{M}(R)$ contains precisely all nonzero homogeneous elements of R, $\mathcal{M}_1(R) = \{x^i \mid i \geq 0\}$ and $\deg_x(\mathcal{M}_1(R)) = 1$. For each nonzero ideal J of R, Theorem 3.11 describes the set $\mathcal{M}_1(J)$ under a mild condition on the ring R. The sets $\mathcal{M}_1(J)$ are used to construct IGWAs.

Theorem 3.11 *Let $R = D[x;\sigma]$ be a skew polynomial ring such that for each nonzero left ideal I of D such that $I\sigma^i(D) \subseteq I$ for some $i \geq 0$, $D = I'$ where $I' := \sum_{j\geq 0} D\sigma^j(I)$ (e.g., D is a σ-simple ring and σ is an automorphism). Then*

1. *The endomorphism σ of the ring D is a monomorphism.*

2. *Let J be a nonzero ideal of R and $n = \deg_x(\mathcal{M}_1(J))$. Then $\mathcal{M}_1(J) = \{ux^i \mid i \geq 0\}$ for a unique nonzero element $u = x^n + d_{n-1}x^{n-1} + \cdots + d_m x_m$ such that $d_i \in D^\sigma \cap {}'\mathcal{C}_D$ provided $d_i \neq 0$, and $\sigma^n(d)d_i = d_i\sigma^i(d)$ for all $i = m, \dots, n-1$ and $d \in D$. The element u is regular, left normal in R, $ux = xu$, $ud = \sigma^n(d)u$ for all $d \in D$, and $\sigma^n = \omega_{d_i}\sigma^i$ for all nonzero elements d_i (e.g., d_m). The subring of R generated by D and ux^j is isomorphic to the skew polynomial ring $D[ux^j; \sigma^{n+j}]$ where $j \in \mathbb{N}$.*

3. *The ring R is a prime ring.*

Proof. 1. Suppose that σ is not a monomorphism. Then the kernel of σ, $I = \ker(\sigma)$, is a *proper* ideal of D such that $D = I' = DI = I \neq D$, a contradiction. Therefore, σ is a monomorphism.

2. Let $l = l(J)$.

(i) $\mathcal{M}_1(J) \neq \emptyset$: Fix an element $u = d_nx^n + d_{n-1}x^{n-1} + \cdots + d_mx^m$ in $\mathcal{M}(J)$ where $d_i \in D$, $d_n \neq 0$, $d_m \neq 0$ and $l = n - m$. Let $I_s = Dd_s\sigma^s(D)$ (the $(D, \sigma^s(D))$-subbimodule of ${}_DD_{\sigma^s(D)}$

generated by the element d_s). By the assumption, $D = \sum_{j=0}^{k} D\sigma^j(I_n)$ for some $k \geq 0$ (since $1 \in D$). Then the set (written symbolically as)

$$\sum_{j=0}^{k} Dx^j u D x^{k-j} = \left\{ \left(\sum_{j=0}^{k} D\sigma^j(I_n) \right) x^n + \sum_{s=m}^{n-1} \left(\sum_{j=0}^{k} D\sigma^j(I_s) \right) x^{\sigma} \right\} x^k \tag{52}$$

contains an element of length l with highest coefficient 1, as required.

(ii) $\mathcal{M}_1(J) = \{ux^i \,|\, i \geq 0\}$ *where* u *is the element of* $\mathcal{M}_1(J)$ *of least degree in* x: Clearly, $\mathcal{M}_1(J) \supseteq \{ux^i \,|\, i \geq 0\}$. The reverse inclusion follows from the fact that for each $i \geq n$, there is a unique element of degree i that belongs to the set $\mathcal{M}_1(J)$ (since all the elements of the set $\mathcal{M}_1(J)$ are monic and of minimal length).

(iii) $ux = xu$: The statement follows from the inequality $l(ux - xu) < l$ and the minimality of l.

(iv) $ud = \sigma^n(d)u$ *for all* $d \in D$: The statement follows from the inequality $l(ud - \sigma^n(d)u) < l$ and the minimality of l.

(v) $\sigma^n(d)d_i = d_i\sigma^i(d)$ *for all* $d \in D$ *and* $i = m, \dots, n-1$: The statement (v) follows from the statement (iv) be equating the coefficients of x^i in the equality $ud = \sigma^n(d)u$.

(vi) $d_i \in D^{\sigma} \cap {}'\mathcal{C}_D$ *provided* $d_i \neq 0$: By the statement (iii), all $d_i \in D^{\sigma}$. Suppose that $d_i \neq 0$ and $dd_i = 0$ for some $d \neq 0$ of D, we seek a contradiction. Then the element $u' = du$ has length l but the number, say $c(u')$, of nonzero coefficients of u' is strictly smaller than the number, say $c(u)$, of nonzero coefficients of u. Replacing the element u by u' and using the argument in (52) we then obtain an element in $\mathcal{M}_1(J)$, say u'', with $c(u'') \leq c(u') < c(u)$. This fact contradicts to the statement (ii) (since $c(u) = c(ux^i)$ for all $i \geq 0$). Therefore, $d_i \in {}'\mathcal{C}_D$.

(vii) $\sigma^n = \omega_{d_i}\sigma^i$ *for all nonzero elements* d_i (by the statements (v) and (vi)).

(viii) *The element* u *is a regular element of* R: The statement (viii) follows from the fact that σ is a monomorphism and the leading term of the element u is 1.

3. By statement 2, every nonzero ideal of R contains a regular element, and statement 3 follows. Hence, the product of two nonzero ideals of R is a nonzero ideal, and statement 3 follows. $\square$

Definition. The unique element u in Theorem 3.11.(2) is called the *core* of the nonzero ideal J and is denoted by $\mathfrak{c}(J)$.

Using Theorem 3.11, we can construct many examples of IGWAs as the next corollary shows.

Corollary 3.12 *We keep the notation as in Theorem 3.11. In particular,* $u = x^n + \sum_{i=m}^{n-1} d_i x^i$. *Then we have IGWAs* $D[x_1, y_1; \sigma^{n+j}, \sigma^{i+j}, d_i]_{\rm in}$ *and* $R[x_1, y_1; \sigma^{n+k+j}, \sigma^j, ux^k]_{\rm in}$, *where* $\sigma(x) = x$, *for all* $j, k \geq 0$ *and* $d_i \neq 0$.

For a ring R, we denote by $\mathcal{N}_l(R)$ and $\mathcal{N}(R)$ the sets of all left normal and normal elements, respectively. The sets $\mathcal{N}_l(R)$ and $\mathcal{N}(R)$ are multiplicative monoids that contain the group $R^{\times}$ of units of R, and $\mathcal{N}(R) \subseteq \mathcal{N}_l(R)$. A nonzero element of $\mathcal{N}_l(R)$ is called a *left normal irreducible* element of R if it is not a product of two left normal elements and each of them is not a unit. Let $\mathcal{N}_{l,irr}(R)$ be the set of all left normal irreducible elements of R.

Corollary 3.13 *We keep the notation as in Theorem 3.11. If the ideal* J *is a nonzero prime ideal of the ring* R *then its core* $\mathfrak{c}(J)$ *is a left normal irreducible element of* R.

Proof. Suppose that $u = \mathfrak{c}(J)$ is not a left normal irreducible element of R. Then $u = ab$ for some elements $\mathcal{N}_l(R) \backslash R^{\times}$. Then $l(a) < l(u)$ and $l(b) < l(u)$.

It follows from $(a)(b) = RaRbR \subseteq Rab = Ru \subseteq J$ that either $(a) \subseteq J$ or $(b) \subseteq J$ (the ideal J is prime), and so either $a \in J$ or $b \in J$. This fact contradicts to the choice of the element u. $\square$

4 Inner generalized Weyl algebras of rank n

In this section the class of inner generalized Weyl algebras of rank n is introduced.

Inner generalized Weyl algebras of rank n. Let A be a ring and σ be an endomorphism of A. A subring B of A is called *σ-invariant* if $\sigma(B) \subseteq B$. The ring

$$A = D[x_1, y_1; \sigma_1, \tau_1, a_1]_{\rm in} \cdots [x_n, y_n; \sigma_n, \tau_n, a_n]_{\rm in}$$

is called an *iterated IGWA* of rank n. Let $A_i := D[x_1, y_1; \sigma_1, \tau_1, a_1]_{\rm in} \cdots [x_i, y_i; \sigma_i, \tau_i, a_i]_{\rm in}$ for $i = 1, \ldots, n$. Then there is a chain of ring homomorphisms

$$D \to A_1 \to A_2 \to \cdots \to A_{n-1} \to A_n = A. \tag{53}$$

Let $D_i := {\rm im}(D \to A_i)$. Then there is a chain of ring homomorphisms

$$D = D_0 \to D_1 \to D_2 \to \cdots \to D_{n-1} \to D_n. \tag{54}$$

To simplify the notation the image of an element $d \in D_i$ in D_j, where $i \leq j$, is denoted also by d.

Definition. An iterated IGWA $A = D[x_1, y_1; \sigma_1, \tau_1, a_1]_{\rm in} \ldots [x_n, y_n; \sigma_n, \tau_n, a_n]_{\rm in}$ is called the **inner generalized Weyl algebra** of rank n if $a_i \in D_i$, $\sigma_i(D_i) \subseteq D_i$ and $\tau_i(D_i) \subseteq D_i$ for all $i = 1, \ldots, n$; and for all integers $i, j = 1, \ldots, n$ such that $i > j$:

$$\sigma_i(x_j) = \lambda_{ij} x_j, \;\; \sigma_i(y_j) = y_j \lambda'_{ji}, \;\; \tau_i(x_j) = \mu_{ij} x_j, \;\; \tau_i(y_j) = y_j \mu'_{ji},$$

for some elements $\lambda_{ij}, \lambda'_{ij}, \mu_{ij}$ and μ'_{ij} of the ring D_{i-1}. The elements $\Lambda = (\lambda_{ij})$, $\Lambda' = (\lambda'_{ji})$, $M = (\mu_{ij})$ and $M' = (\mu'_{ji})$ are called the *defining coefficients* of A. The n-tuples of endomorphisms $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\tau = (\tau_1, \ldots, \tau_n)$ are called the *defining endomorphisms* of A, and the n-tuple of elements $a = (a_1, \ldots, a_n)$ is called the *defining elements* of A. The IGWA A of rank n is denoted by $A = D[x, y; \sigma, \tau, a, \Lambda, \Lambda', M, M']_{\rm in}$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

The restrictions $\sigma_i|_{D_{i-1}}$ and $\tau_i|_{D_{i-1}}$ are also denoted by σ_i and τ_i, respectively.

An element $\Lambda = (\lambda_{ij})$ (where $1 \leq j < i \leq n$) (resp., $\Lambda' = (\lambda_{ji})$) is called a *lower* (resp., *upper*) *triangular half-matrix* with coefficients in D. The set of all such elements is denoted by $L_n(D)$ (resp., $U_n(D)$).

The next lemma describes IGWAs of rank n via generators and defining relations.

Lemma 4.1 *Let $A = D[x, y; \sigma, \tau, a, \Lambda, \Lambda', M, M']_{\rm in}$ be an IGWA of rank n and $\mathfrak{a}(A) := \ker(D \to A)$. Then*

1. *The ring A is generated by the ring D and the elements $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ subject to the defining relations:*
$$\mathfrak{a}(A) = 0; \tag{55}$$
for each $i = 1, \ldots, n$ and $d \in D_i$,
$$x_i d = \sigma_i(d) x_i, \;\; d y_i = y_i \tau_i(d), \;\; y_i x_i = a_i \;\text{ and }\; x_i y_i = \sigma_i(a_i); \tag{56}$$
for all $i > j$,
$$x_i x_j = \lambda_{ij} x_j x_i, \;\; x_i y_j = y_j \lambda'_{ji} x_i, \;\; x_j y_i = y_i \mu_{ij} x_j \;\text{ and }\; y_j y_i = y_i y_j \mu'_{ji}. \tag{57}$$

2. *The ring $A = \bigoplus_{\gamma \in \mathbb{Z}^n} A_\gamma$ is a $\mathbb{Z}^n$-graded ring where $A_\gamma := \bigoplus_{\{\alpha, \beta \in \mathbb{N}^n \mid -\alpha + \beta = \gamma\}} y^\alpha D x^\beta$, $y^\alpha := y_n^{\alpha_n} \cdots y_1^{\alpha_1}$ (reverse order) and $x^\beta := x_1^{\beta_1} \cdots x_n^{\beta_n}$. In particular, for each $i = 1, \ldots, n$, the ring $A_i = D[x_1, y_1; \sigma_1, \tau_1, a_1]_{\rm in} \cdots [x_i, y_i; \sigma_i, \tau_i, a_i]_{\rm in}$ is a $\mathbb{Z}^i$-graded ring.*

3. *For each $i = 2, \ldots, n$, the endomorphisms σ_i and τ_i of the IGWA A_{i-1} respect the $\mathbb{Z}^{i-1}$-grading of A_{i-1}.*

Remark. Since $\ker(D \to D_s) \subseteq \ker(D \to D_n) = \mathfrak{a}(A)$, the equalities in (56) and (57) make sense due to (55).

Proof. 1. Statement 1 follows from the definition of the IGWA $A = D[x, y; \sigma, \tau, a, \Lambda, \Lambda', M, M']_{\rm in}$.

2. Statement 2 follows from statement 1.

3. Statement 3 follows from statements 1 and 2. $\square$

For the IGWA $A = D[x, y; \sigma, \tau, a, \Lambda, \Lambda', M, M']_{\rm in}$, the next proposition describes conditions that its defining data must satisfy for the ring A to be an iterated IGWA.

Proposition 4.2 *Let $A = D[x, y; \sigma, \tau, a, \Lambda, \Lambda', M, M']_{\rm in}$ be an IGWA of rank n. Then the defining data of the IGWA A define the iterated IGWA $A = D[x_1, y_1; \sigma_1, \tau_1, a_1]_{\rm in} \cdots [x_n, y_n; \sigma_n, \tau_n, a_n]_{\rm in}$ iff the following conditions hold: For all $i = 1, \ldots, n$ and $d \in D$,*

$$\sigma_i(a_i) = \tau_i(a_i), \ \text{ and } \ \sigma_i(d)\sigma_i(a_i) = \sigma_i(a_i)\tau_i(d); \tag{58}$$

for all $i > j$,

$$\sigma_i(a_i)\mu_{ij} = \lambda_{ij}\sigma_j\sigma_i(a_i) \ \text{ and } \ \lambda'_{ji}\sigma_i(a_i) = \tau_j\sigma_i(a_i)\mu'_{ji}; \tag{59}$$

for all $i > j$ and $d \in D$,

$$\lambda_{ij}\sigma_j\sigma_i(d) = \sigma_i\sigma_j(d)\lambda_{ij} \ \text{ and } \ \tau_j\sigma_i(d)\lambda'_{ji} = \lambda'_{ji}\sigma_i\tau_j(d), \tag{60}$$

$$\mu_{ij}\sigma_j\tau_i(d) = \tau_i\sigma_j(d)\mu_{ij} \ \text{ and } \ \tau_j\tau_i(d)\mu'_{ji} = \mu'_{ji}\tau_i\tau_j(d); \tag{61}$$

for all $i > j > k$,

$$\lambda_{ij}\sigma_j(\lambda_{ik})\lambda_{jk} = \sigma_i(\lambda_{jk})\lambda_{ik}\sigma_k(\lambda_{ij}) \ \text{ and } \ \tau_j(\lambda_{ik})\mu_{jk}\sigma_k(\lambda'_{ji}) = \lambda'_{ji}\sigma_i(\mu_{jk})\lambda_{ik}, \tag{62}$$

$$\mu_{ij}\sigma_j(\mu_{ik})\lambda_{jk} = \tau_i(\lambda_{jk})\mu_{ik}\sigma_k(\mu_{ij}) \ \text{ and } \ \tau_j(\mu_{ik})\mu_{jk}\sigma_k(\mu'_{ji}) = \mu'_{ji}\tau_i(\mu_{jk})\mu_{ik}, \tag{63}$$

$$\tau_k(\lambda_{ij})\lambda'_{kj}\sigma_j(\lambda'_{ki}) = \lambda'_{ki}\sigma_i(\lambda'_{kj})\lambda_{ij} \ \text{ and } \ \mu'_{kj}\tau_j(\lambda'_{ki})\lambda'_{ji} = \tau_k(\lambda'_{ji})\lambda'_{ki}\sigma_i(\mu'_{kj}), \tag{64}$$

$$\tau_k(\mu_{ij})\lambda'_{kj}\sigma_j(\mu'_{ki}) = \mu'_{ki}\tau_i(\lambda'_{kj})\mu_{ij} \ \text{ and } \ \mu'_{kj}\tau_j(\mu'_{ki})\mu'_{ji} = \tau_k(\mu'_{ji})\mu'_{ki}\tau_i(\mu'_{kj}); \tag{65}$$

for all $i > j$,

$$\sigma_i(a_j) = y_j\lambda'_{ji}\lambda_{ij}x_j \ \text{ and } \ \tau_i(a_j) = y_j\mu'_{ji}\mu_{ij}x_j, \tag{66}$$

$$\sigma_i\sigma_j(a_j) = \lambda_{ij}\sigma_j(a_j)\lambda'_{ji} \ \text{ and } \ \tau_i\sigma_j(a_j) = \mu_{ji}\sigma_j(a_j)\mu'_{ji}. \tag{67}$$

Remark. The equalities above hold in the corresponding rings: equalities (58)–(65) and (67) hold in the ring D_{i-1} and equalities (66) hold in the ring A_{i-1}.

Proof. The proof is based on the fact that the IGWA A of rank n is a special type of the iterated GWA of rank n and Theorem 1.1. By Lemma 4.1.(1), the ring A is generated by the ring D and the elements $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ subject to the defining relations in (56) and (57) (by the definition of A). The remaining equations in (58)–(67) follow from (4) and (5) bearing in mind the iterated nature of the IGWA A and the definition of the endomorphisms $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n$. Equalities (58) and (59) are the conditions (4) written for all IGWAs A_i, see (53). In more detail, the equalities in (59) are the equalities $\sigma_i(d_{i-1})\sigma_i(a_i) = \sigma_i(a_i)\tau_i(d_{i-1})$ in the IGWA $A_i = A_{i-1}[x_i, y_i; \sigma_i, \tau_i, a_i]_{\rm in}$ for $d_{i-1} = x_j$ and $d_{i-1} = y_j$ $(i > j)$, respectively:

$$d_{i-1} = x_j : \quad \sigma_i(a_i)\mu_{ij}x_j = \sigma_i(a_i)\tau_i(x_j) \stackrel{(4)}{=} \sigma_i(x_j)\sigma_i(a_i) = \lambda_{ij}x_j\sigma_i(a_i) = \lambda_{ij}\sigma_j\sigma_i(a_i)x_j,$$
$$d_{i-1} = y_j : \quad y_j\lambda'_{ji}\sigma_i(a_i) = \sigma_i(y_j)\sigma_i(a_i) \stackrel{(4)}{=} \sigma_i(a_i)\tau_i(y_j) = \sigma_i(a_i)y_j\mu'_{ji} = y_j\tau_j\sigma_i(a_i)\mu'_{ji}.$$

Equalities (60), ... , (67) guarantee that the four defining relations of the IGWA A_j are respected by all the automorphisms σ_i and τ_i for $i > j$. In particular, equalities (60), ... , (65) guarantee that for all $i > j$ the automorphisms σ_i and τ_i respect the equalities $x_jd_{j-1} = \sigma_j(d_{j-1})x_j$ and $d_{j-1}y_j = y_j\tau_j(y_{j-1})$ in the ring A_i where $d_{j-1} = d$ $(d \in D)$, x_k, y_k (where $i > j > k$): For all $i > j$ and $d \in D$,

$$\sigma_i, \ d_{i-1} = d : \quad \sigma_i(x_jd) = \lambda_{ij}x_j\sigma_i(d) = \lambda_{ij}\sigma_j\sigma_i(d)x_j, \ \ \sigma_i(\sigma_j(d)x_j) = \sigma_i\sigma_j(d)\lambda_{ij}x_j,$$
$$\sigma_i(dy_j) = \sigma_i(d)y_j\lambda'_{ji} = y_j\tau_j\sigma_i(d)\lambda'_{ji}, \ \ \sigma_i(y_j\tau_j(d)) = y_j\lambda'_{ji}\sigma_i\tau_j(d).$$
$$\tau_i, \ d_{i-1} = d : \quad \tau_i(x_jd) = \mu_{ij}x_j\tau_i(d) = \mu_{ij}\sigma_j\tau_i(d)x_j, \ \ \tau_i(\sigma_j(d)x_j) = \tau_i\sigma_j(d)\mu_{ij}x_j,$$
$$\tau_i(dy_j) = \tau_i(d)y_j\mu'_{ji} = y_j\tau_j\tau_i(d)\mu'_{ji}, \ \ \tau_i(y_j\tau_j(d)) = y_j\mu'_{ji}\tau_i\tau_j(d).$$

For all $i > j > k$,

$$
\begin{array}{ll}
\sigma_i,\ d_{i-1}=x_k: & \sigma_i(x_jx_k)=\lambda_{ij}x_j\lambda_{ik}x_k=\lambda_{ij}\sigma_j(\lambda_{ik})\lambda_{jk}x_kx_j,\ \ \sigma_i(\lambda_{jk}x_kx_j)=\sigma_i(\lambda_{jk})\lambda_{ik}x_k\lambda_{ij}x_j\\
& =\sigma_i(\lambda_{jk})\lambda_{ik}\sigma_k(\lambda_{ij})x_kx_j,\\
& \sigma_i(x_ky_j)=\lambda_{ik}x_ky_j\lambda'_{ji}=\lambda_{ik}y_j\mu_{jk}x_k\lambda'_{ji}=y_j\tau_j(\lambda_{ik})\mu_{jk}\sigma_k(\lambda'_{ji})x_k,\\
& \sigma_i(y_j\tau_j(x_k))=\sigma_i(y_j\mu_{jk}x_k)=y_j\lambda'_{ji}\sigma_i(\mu_{jk})\lambda_{ik}x_k;\\
\tau_i,\ d_{i-1}=x_k: & \tau_i(x_jx_k)=\mu_{ij}x_j\mu_{ik}x_k=\mu_{ij}\sigma_j(\mu_{ik})\lambda_{jk}x_kx_j,\ \ \tau_i(\sigma_j(x_k)x_j)=\tau_i(\lambda_{jk}x_kx_j)\\
& =\tau_i(\lambda_{jk})\mu_{ik}x_k\mu_{ij}x_j=\tau_i(\lambda_{jk})\mu_{ik}\sigma_k(\mu_{ij})x_kx_j,\\
& \tau_i(x_ky_j)=\mu_{ik}x_ky_j\mu'_{ji}=y_j\tau_j(\mu_{ik})\mu_{jk}x_k\mu'_{ji}=y_j\tau_j(\mu_{ik})\mu_{jk}\sigma_k(\mu'_{ji})x_k,\\
& \tau_i(y_j\tau_j(x_k))=\tau_i(y_j\mu_{jk}x_k)=y_j\mu'_{ji}\tau_i(\mu_{jk})\mu_{ik}x_k;\\
\sigma_i,\ d_{i-1}=y_k: & \sigma_i(x_jy_k)=\lambda_{ij}x_jy_k\lambda'_{ki}=\lambda_{ij}y_k\lambda'_{kj}x_j\lambda'_{ki}=y_k\tau_k(\lambda_{ij})\lambda'_{kj}\sigma_j(\lambda'_{ki})x_j,\\
& \sigma_i(\sigma_j(y_k)x_j)=\sigma_i(y_k\lambda'_{kj}x_j)=y_k\lambda'_{ki}\sigma_i(\lambda'_{kj})\lambda_{ij}x_j;\\
& \sigma_i(y_ky_j)=y_k\lambda'_{ki}y_j\lambda'_{ji}=y_ky_j\tau_j(\lambda'_{ki})\lambda'_{ji}=y_jy_k\mu'_{kj}\tau_j(\lambda'_{ki})\lambda'_{ji},\\
& \sigma_i(y_j\tau_j(y_k))=\sigma_i(y_jy_k\mu'_{kj})=y_j\lambda'_{ji}y_k\lambda'_{ki}\sigma_i(\mu'_{kj})=y_jy_k\tau_k(\lambda'_{ji})\lambda'_{ki}\sigma_i(\mu'_{kj});\\
\tau_i,\ d_{j-1}=y_k: & \tau_i(x_jy_k)=\mu_{ij}x_jy_k\mu'_{ki}=\mu_{ij}y_k\lambda'_{kj}\sigma_j(\mu'_{ki})x_j=y_k\tau_k(\mu_{ij})\lambda'_{kj}\sigma_j(\mu'_{ki})x_j,\\
& \tau_i(\sigma_j(y_k)x_j)=\tau_i(y_k\lambda'_{kj}x_j)=y_k\mu'_{ki}\tau_i(\lambda'_{kj})\mu_{ij}x_j,\\
& \tau_i(y_ky_j)=y_k\mu'_{ki}y_j\mu'_{ji}=y_jy_k\mu'_{kj}\tau_j(\mu'_{ki})\mu'_{ji},\\
& \tau_i(y_j\tau_j(y_k))=\tau_i(y_jy_k\mu'_{kj})=y_j\mu'_{ji}y_k\mu'_{ki}\tau_i(\mu_{kj})=y_jy_k\tau_k(\mu'_{ji})\mu'_{ki}\tau_i(\mu_{kj}).
\end{array}
$$

The equalities (66) and (67) guarantee that, for all $i > j$, the endomorphisms σ_i and τ_i respect the equalities $\sigma_j(a_j) = x_jy_j$ and $a_j = y_jx_j$:

$$
\begin{array}{ll}
\sigma_i: & \sigma_i\sigma_j(a_j)=\lambda_{ij}x_jy_j\lambda'_{ji}=\lambda_{ij}\sigma_j(a_j)\lambda'_{ji},\\
\tau_i: & \tau_i\sigma_j(a_j)=\mu_{ij}x_jy_j\mu'_{ji}=\mu_{ij}\sigma_j(a_j)\mu'_{ji},\\
\sigma_i: & \sigma_i(a_j)=y_j\lambda'_{ji}\lambda_{ij}x_j,\\
\tau_i: & \tau_i(a_j)=y_j\mu'_{ji}\mu_{ij}x_j.\quad\Box
\end{array}
$$

Corollary 4.3 provides an important class if IGWAs of rank n.

Corollary 4.3 *Let D be a ring, $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\tau = (\tau_1, \dots, \tau_n)$ be n-tuples of ring endomorphisms of D, $a = (a_1, \dots, a_n) \in D^n$, $\Lambda = (\lambda_{ij}), M = (\mu_{ij}) \in L_n(D)$ and $\Lambda' = (\lambda'_{ji}), M' = (\mu'_{ji}) \in U_n(D)$ be such that the following equalities hold in the ring D: For all $i = 1, \dots, n$ and $d \in D$,*

$$\sigma_i(a_i) = \tau_i(a_i), \ \text{ and } \ \sigma_i(d)\sigma_i(a_i) = \sigma_i(a_i)\tau_i(d); \tag{68}$$

for all $i > j$,

$$\sigma_i(a_i)\mu_{ij} = \lambda_{ij}\sigma_j\sigma_i(a_i) \ \text{ and } \ \lambda'_{ji}\sigma_i(a_i) = \tau_j\sigma_i(a_i)\mu'_{ji}; \tag{69}$$

for all $i > j$ and $d \in D$,

$$\lambda_{ij}\sigma_j\sigma_i(d) = \sigma_i\sigma_j(d)\lambda_{ij} \ \text{ and } \ \tau_j\sigma_i(d)\lambda'_{ji} = \lambda'_{ji}\sigma_i\tau_j(d), \tag{70}$$

$$\mu_{ij}\sigma_j\tau_i(d) = \tau_i\sigma_j(d)\mu_{ij} \ \text{ and } \ \tau_j\tau_i(d)\mu'_{ji} = \mu'_{ji}\tau_i\tau_j(d); \tag{71}$$

for all $i > j > k$,

$$\lambda_{ij}\sigma_j(\lambda_{ik})\lambda_{jk} = \sigma_i(\lambda_{jk})\lambda_{ik}\sigma_k(\lambda_{ij}) \ \text{ and } \ \tau_j(\lambda_{ik})\mu_{jk}\sigma_k(\lambda'_{ji}) = \lambda'_{ji}\sigma_i(\mu_{jk})\lambda_{ik}, \tag{72}$$

$$\mu_{ij}\sigma_j(\mu_{ik})\lambda_{jk} = \tau_i(\lambda_{jk})\mu_{ik}\sigma_k(\mu_{ij}) \ \text{ and } \ \tau_j(\mu_{ik})\mu_{jk}\sigma_k(\mu'_{ji}) = \mu'_{ji}\tau_i(\mu_{jk})\mu_{ik}, \tag{73}$$

$$\tau_k(\lambda_{ij})\lambda'_{kj}\sigma_j(\lambda'_{ki}) = \lambda'_{ki}\sigma_i(\lambda'_{kj})\lambda_{ij} \ \text{ and } \ \mu'_{kj}\tau_j(\lambda'_{ki})\lambda'_{ji} = \tau_k(\lambda'_{ji})\lambda'_{ki}\sigma_i(\mu'_{kj}), \tag{74}$$

$$\tau_k(\mu_{ij})\lambda'_{kj}\sigma_j(\mu'_{ki}) = \mu'_{ki}\tau_i(\lambda'_{kj})\mu_{ij} \ \text{ and } \ \mu'_{kj}\tau_j(\mu'_{ki})\mu'_{ji} = \tau_k(\mu'_{ji})\mu'_{ki}\tau_i(\mu'_{kj}); \tag{75}$$

for all $i > j$, $\lambda'_{ji}\lambda_{ij} = \sum_s \tau_j(p'_{ji,s})\sigma_j(p_{ij,s})$ and $\mu'_{ji}\mu_{ij} = \sum_t \tau_j(q'_{ji,t})\sigma_j(q_{ij,t})$ for some elements $p'_{ji,s}, \dots, q_{ij,t} \in D$,

$$\sigma_i(a_j) = \sum_s p'_{ji,s} a_j p_{ij,s} \text{ and } \tau_i(a_j) = \sum_t q'_{ji,t} a_j q_{ij,t}, \tag{76}$$

$$\sigma_i\sigma_j(a_j) = \lambda_{ij}\sigma_j(a_j)\lambda'_{ji} \text{ and } \tau_i\sigma_j(a_j) = \mu_{ji}\sigma_j(a_j)\mu'_{ji}. \tag{77}$$

Then

1. *The conditions of Proposition 1.2 hold and we denote by $A = D[x,y;\sigma,\tau,a,\Lambda,\Lambda',M,M']_{\rm in}$ the corresponding IGWA of rank n.*
2. *The IGWA of rank n, $A = D[x,y;\sigma,\tau,a,\Lambda,\Lambda',M,M']_{\rm in}$, is a ring generated by D, $x_1,\dots,x_n$ and $y_1,\dots,y_n$ subject to the defining relations: For all $i = 1,\dots,n$ and $d \in D$,*

$$x_i d = \sigma_i(d)x_i, \;\; dy_i = y_i\tau_i(d), \;\; y_ix_i = a_i \text{ and } x_iy_i = \sigma_i(a_i); \tag{78}$$

for all $i > j$,

$$x_ix_j = \lambda_{ij}x_jx_i, \;\; x_iy_j = y_j\lambda'_{ji}x_i, \;\; x_jy_i = y_i\mu_{ij}x_j \text{ and } y_jy_i = y_iy_j\mu'_{ji}. \tag{79}$$

Proof. 1. The equalities of the corollary imply the conditions of Proposition 1.2. In more detail, equalities (68)–(75) and (77) imply equalities (58)–(65) and (67). The two equalities in (76) imply the two equalities in (66):

$$\sigma_i(a_j) = y_j\lambda'_{ji}\lambda_{ij}x_j = y_j\Big(\sum_s \tau_j(p'_{ji,s})\sigma_j(p_{ij,s})\Big)x_j = \sum_s p'_{ji,s}a_jp_{ij,s},$$
$$\tau_i(a_j) = y_j\mu'_{ji}\mu_{ij}x_j = y_j\Big(\sum_t \tau_j(q'_{ji,t})\sigma_j(q_{ij,t})\Big)x_j = \sum_t q'_{ji,t}a_jq_{ij,t}.$$

2. By statement 1, the defining relations given in statement 2 imply the defining relations in Lemma 4.1.(1) (use the fact that A is an iterative IGWA of rank n and Theorem 1.1 on each step of iteration). $\square$

Example. If $\sigma = (\sigma_1,\dots,\sigma_n)$ is an n-tuple of *commuting* endomorphisms of the ring D, $\tau := \sigma = (\sigma_1,\dots,\sigma_n)$, $a = (a_1,\dots,a_n) \in Z(D)$ and $\sigma_i(a_j) = a_j$ for all $i \neq j$; and $\lambda_{ij} = \lambda'_{ij} = \mu_{ij} = \mu'_{ij} = 1$ for all $i > j$, then we have the IGWA $A = D[x,y;\sigma,\sigma,a]_{\rm in}$ of rank n. The ring A is generated by D, $x_1,\dots,x_n$ and $y_1,\dots,y_n$ subject to the defining relations: For all $i = 1,\dots,n$ and $d \in D$,

$$x_i d = \sigma_i(d)x_i, \;\; dy_i = y_i\tau_i(d), \;\; y_ix_i = a_i \text{ and } x_iy_i = \sigma_i(a_i); \tag{80}$$

for all $i \neq j$,

$$x_ix_j = x_jx_i, \;\; x_iy_j = y_jx_i, \text{ and } y_jy_i = y_iy_j. \tag{81}$$

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School of Mathematics and Statistics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UK
email: v.bavula@sheffield.ac.uk