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Determination of the time-dependent effective ion collision frequency from an integral observation

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Abstract

Identification of materials physical properties is very important because they are in general unknown. Furthermore, their direct experimental measurement could be costly and inaccurate. In such a situation, a cheap and efficient alternative is to mathematically formulate an inverse, but difficult, problem that can be solved, in general, numerically; the challenge being that the problem is, in general, nonlinear and ill-posed. In this paper, the reconstruction of a lower-order unknown time-dependent coefficient in a Cahn-Hilliard-type fourth-order equation from an additional integral observation, which has application to characterising the nonlinear saturation of the collisional trapped-ion mode in a tokamak, is investigated. The local existence and uniqueness of the solution to such inverse problem is established by utilizing the Rothe method. Moreover, the continuous dependence of the unknown coefficient upon the measured data is derived. Next, the Tikhonov regularization method is applied to recover the unknown coefficient from noisy measurements. The stability estimate of the minimizer is derived by investigating an auxiliary linear fourth-order inverse source problem. Henceforth, the variational source condition can be verified. Then, the convergence rate is obtained under such source condition.

Keywords: Inverse problem; Ill-posed problem; Rothe method; Tikhonov regularization method; Cahn-Hilliard equation

1 Introduction

In coefficient identification problems, the unknown material properties appear as coefficients of a partial different equation (PDE), or a system of PDEs, governing the physical phenomenon/scenario under investigation. These coefficients may be constants or functions of time, space or of the main dependent variable(s) and, typically, they represent a conductivity, a capacity/storage, a convection/advection or reaction/absorption property. The governing PDE is usually of second-order involving both space and time variables. Less work has been performed on the identification of coefficients in higher-order PDEs such as those of the fourth-order governing models of elastic beams (Euler-Bernoulli) or plates (Kirchhoff-Love) [3]. More

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complicated fourth-order PDEs such as the the Kuramoto–Sivashinsky (KS) equation, which, in one-dimension written in a form suitable for fluid dynamics applications, reads as

$$u_t + (\sigma(x)u_{xx})_{xx} + D(x)u_{xx} + uu_x = f(x,t), \quad (x,t) \in Q_T := (0,1) \times (0,T), \tag{1.1}$$

where $T > 0$ is a given final time of interest, u represents the dependent variable, f is a source, force or load, σ and D represent the diffusion and anti-diffusion coefficients, respectively, governing the phase of turbulence in reaction-diffusion systems or flame propagation, have also been considered. Equation (1.1) is supplied with an initial condition at $t = 0$ and with physical boundary conditions, e.g., in case of Dirichlet boundary conditions $u(x,t)$ and $u_x(x,t)$ are prescribed at $x \in \{0,1\}$ and for all $t \in (0,T)$. The inverse coefficient problem for this model, consisting in the determination of the unknown coefficients $\sigma(x)$ or $D(x)$ in (1.1), was considered in [4, 12, 27, 11]. In [4], the anti-diffusion coefficient $D(x)$ in (1.1) was retrieved from the measurements of the Neumann partial boundary data $u_{xx}(0,t)$ and $u_{xxx}(0,t)$ and of the data $u(x,T_0)$ at a fixed time $T_0 \in (0,T)$. By applying the Bukhgeım–Klibanov method [5], the Lipschitz stability was established. Such anti-diffusion coefficient was also investigated in [12] but from the internal measurements $u|_{\omega\times(0,T)}$ for $\omega \subset (0,1)$ and $u(x,T_0)$, and the Lipschitz stability was deduced by invoking similar arguments. Recently, the anti-diffusion coefficient $D(x)$ in a linear Kuramoto–Sivashinsky equation, namely, no nonlinear term uu_x in (1.1), was recovered from the final measurement $u(x,T)$ in [11], where such inverse problem was reformulated as a nonlinear regularized optimization, and the local uniqueness and stability of the minimizer were proved under suitable optimality conditions. In addition, the numerical solution of $D(x)$ was obtained utilizing an iterative algorithm together with the finite element method (FEM). The space-dependent diffusion coefficient $\sigma(x)$ in (1.1) was identified in [27] from the same additional observation of [4], and the Lipschitz stability was obtained locally using the Bukhgeım–Klibanov method and Carleman estimates.

Inverse source/load linear problems for recovering the free term f are not discussed herein, but we mention [13, 14] for the Euler-Bernoulli equation and [21, 22] for the more general $2m$-order ($m \in \mathbb{N}^*$) parabolic equation

$$u_t + (-1)^m a(t)\partial_x^{2m} u = f(x,t), \quad (x,t) \in Q_T. \tag{1.2}$$

Taking $m = 2$ in (1.2), the inverse problem concerning the reconstruction of the time-dependent coefficient $a(t)$ in the resulting fourth-order parabolic equation $u_t + a(t)u_{xxxx} = f(x,t)$, governing thermal grooving, was recently investigated in [6]. The local well-posedness of the classical solution to this inverse problem was established by using Fourier analysis and the Banach fixed point theorem, and the numerical reconstruction of $a(t)$ was realised based on a time-discrete method with predictor-corrector scheme and the finite difference method (FDM).

The backward (in time) problem for the more general fourth-order PDEs of Cahn-Hilliard type, namely,

$$u_t + \nabla^4 u + \alpha u + \underline{\beta} \cdot \nabla u + \nabla \cdot (\gamma \nabla u) = f(\underline{x}, t, u, \nabla u, \nabla^2 u), \quad (\underline{x}, t) \in \Omega \times (0,T), \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$, describing the process of isothermal phase separation in a binary alloy of fluids [8] was recently considered in [30]. We finally mention the fourth-order nonlinear PDE

$$u_t + A_1\nabla^2 u + A_2\nabla^4 u + A_3\nabla \cdot (|\nabla u|^2 \nabla u) = f(\underline{x}, t), \quad (\underline{x}, t) \in \Omega \times (0,T), \tag{1.4}$$

where $A_1\nabla^2 u$, $A_2\nabla^4 u$ and $A_3\nabla\cdot(|\nabla u|^2\nabla u)$ represent the diffusion due to evaporation-condensation, capillarity-driven surface diffusion and hopping of atoms, respectively, that was considered in [23] as a model for the epitaxial growth of nanoscale thin films in materials science.

Much closer related to our paper is a particular one-dimensional form ($\Omega = (0, l)$) of the Cahn-Hilliard equation (1.3), see [25],

$$u_t + \sigma u_{xxxx} + u_{xx} + uu_x + qu = f(x,t), \quad (x,t) \in Q_T := (0,l) \times (0,T), \tag{1.5}$$

governing the growth and saturation of the fluctuating potential $u(x,t)$ in plasma confinement of toroidal tokamak devices. In (1.5), the coefficient $\sigma > 0$ is a measure of the relative strength of Landau's damping compared to the electron collisional growth (taken, for simplicity, to be unity), and, compared to equation (6) of [25], we have also applied the scaling $\psi \mapsto u/2$. Also, $q = \nu_- \nu_+ / \omega_0^2$, where ω_0 is the the mode frequency, ν_- is the effective electron collision frequency and ν_+ is the effective ion collision frequency. It is this latter quantity that is assumed to be time-dependent and unknown such that the inverse problem that is investigated in this paper requires finding the coefficient $q(t)$ along with the potential $u(x,t)$ in the equation

$$u_t + u_{xxxx} + u_{xx} + uu_x + q(t)u = f(x,t), \quad (x,t) \in Q_T. \tag{1.6}$$

Associated to this equation, we consider an initial specified condition

$$u(x,0) = g(x), \quad x \in (0,l), \tag{1.7}$$

and, for simplicity, homogeneous Dirichlet boundary conditions

$$u(0,t) = u(l,t) = u_x(0,t) = u_x(l,t) = 0, \quad t \in [0,T]. \tag{1.8}$$

In order to compensate for the missing time-dependent coefficient $q(t)$ we consider the additional time-dependent measurement of the weighted mass/energy of the system given by

$$\int_0^l \rho(x) u(x,t) dx = \phi(t), \quad t \in [0,T], \tag{1.9}$$

where $\rho(x)$ is a given weight function.

First, the inverse problem (1.6)–(1.9) is reformulated as a variational system for the pair $(u(x,t), q(t))$. Then, a discrete form of the variational system is generated by using the backward Euler's method together with a sequence of nonlinear fourth-order elliptic equations. Under certain assumptions on the input data, the unique solvability of these nonlinear elliptic equations are obtained by Schaefer's fixed point theorem, and from the *a priori* estimates of the approximations we obtain that there exists a unique solution (u, q) to (1.6)–(1.9) by Rothe's method [20]. Moreover, the convergence rate estimate of the approximations and the stability of solution are obtained.

Next, the recovery of the unknown time-dependent coefficient $q(t)$ from the noisy integral observation ϕ^ϵ of ϕ is considered by utilizing the Tikhonov regularization method, where here $\epsilon \geq 0$ denotes the noisy level of measured data. The well-posedness of the minimizer to the Tikhonov functional is obtained and we focus on the convergence rate estimates to the minimizer of the Tikhonov functional. Convergence rates to Tikhonov regularization for second-order inverse parabolic problems have been investigated extensively, e.g., [7, 9, 10, 15] and the literature cited therein. The arguments can be extended to our fourth-order PDE problem. The variational source condition (VSC) (e.g. [17, 18, 28]) is established (without needing the Frechet derivative of the forward operator that is required by the classical theory [9]) to obtain the convergence rate estimate using the previously derived stability estimate for the inverse problem (1.6)–(1.9). Consequently, we obtain the convergence rates for the Tikhonov regularization of the inverse problem for a proper choice of the regularization parameter.

The whole paper is novel since the inverse problem concerning the determination of the time-dependent coefficient $q(t)$ in equation (1.6) from the integral measurement (1.9) along with the initial and boundary conditions (1.7) and (1.8) is formulated and analysed for the first time. Previous analyses [16, 31] on the identification of the lower-order time-dependent potential coefficient entering the parabolic second-order linear equation $u_t - u_{xx} + q(t)u = f$ cannot be directly applied to our model not only because the elliptic leading operator is the higher-order bilaplacian but also because the governing equation (1.6) is nonlinear. From the practical point of view, the present formulation would enable the determination of the transient effective ion collision frequency from an integral observation of the fluctuating potential. The well-posedness of such problem is established by using the Rothe technique. The other point to mention is that the VSC is verified based on the stability estimate for the inverse problem derived by considering the solvability of an auxiliary inverse source problem for a linear fourth-order parabolic equation. Then, the convergence rate estimates for the Tikhonov regularization are obtained.

The paper is organized as follows. Section 2 illustrates the well-posedness of the inverse problem (1.6)–(1.9) by the time-discrete scheme and the Rothe method. For noisy measured data in (1.9), the Tikhonov regularization is described in Section 3, and the VSC is verified. From this, convergence rate estimates are obtained. Finally, Section 4 highlights the conclusions of the work.

2 Well-posedness of the inverse problem

2.1 Preliminaries

Let us denote $\mathcal{D} := H^4(0,l) \cap H_0^2(0,l)$. The Wirtinger inequality (which in higher-dimensions is known as the Poincare inequality) will be utilized, i.e.,

$$\|v\| \leq \frac{l}{\pi}\|v'\|, \quad \forall v \in H_0^1(0,l), \tag{2.1}$$

where, throughout the paper, $\|\cdot\|$ denotes the norm of $L^2(0,l)$. By applying (2.1) twice we also obtain that

$$\|\chi''\|^2 - \|\chi'\|^2 \geq \frac{(\pi^2 - l^2)\pi^2}{l^4}\|\chi\|^2, \quad \forall \chi \in H_0^2(0,l), \tag{2.2}$$

$$\|\chi''\|^2 - \|\chi'\|^2 \geq \frac{(\pi^2 - l^2)}{\pi^2}\|\chi''\|^2, \quad \forall \chi \in H_0^2(0,l). \tag{2.3}$$

In the sequel, we shall assume that $0 < l < \pi$ such that the right-hand sides of (2.2) and (2.3) are positive constants (at this stage it is not known whether the analysis can be extended to allow $l \geq \pi$ or to higher dimensions, although [23] may be a good starting point for the analysis). Also, the Schaefer fixed point theorem, stated below, will be utilized in the proof of Lemma 2.1.

Theorem 2.1 (Schaefer's fixed point theorem)**.** *Suppose X is a Banach space and let $\mathbb{A} : X \to X$ be a continuous and compact operator. Assume further that the set $\{u \in X | u = \lambda \mathbb{A} u \text{ for some } \lambda \in [0,1]\}$ is bounded. Then $\mathbb{A}$ has a fixed point.*

By employing the semi-group theory and Banach fixed point theorem, see [27], we have the following well-posedness result of the direct problem (1.6)–(1.8).

Theorem 2.2 ([27])**.** *Suppose that $q \in L^\infty(0,T)$, $f \in L^2(Q_T)$ and $g \in H_0^2(0,l)$. Then the problem (1.6)–(1.8) has a unique solution $u \in C([0,T]; H_0^2(0,1)) \cap L^2(0,T; H^4(0,l)) =: \mathcal{V}$.*

2.2 Variational form

For $\phi \in C^1[0,T]$, multiplying (1.6) by $\rho \in \mathcal{D}$ and integrating with respect to x over $(0,l)$, we obtain

$$\phi'(t) + \langle \rho'''' + \rho'', u(\cdot,t)\rangle + \langle \rho, u(\cdot,t)u_x(\cdot,t)\rangle + q(t)\phi(t) = \langle \rho, f(\cdot,t)\rangle\,, \quad t \in [0,T], \tag{2.4}$$

where $\langle \cdot,\cdot\rangle$ denotes the inner product in $L^2(0,l)$. Supposing further that $\phi(t) \neq 0$ for all $t \in [0,T]$, the unknown quantity $q(t)$ can be expressed as:

$$q(t) = \frac{\langle \rho, f(\cdot,t)\rangle - \phi'(t) - \langle \rho'''' + \rho'', u(\cdot,t)\rangle - \langle \rho, u(\cdot,t)u_x(\cdot,t)\rangle}{\phi(t)}, \quad t \in [0,T]. \tag{2.5}$$

The variational form of (1.6)–(1.8) is given by $u(x,0) = g(x)$ for $x \in (0,l)$, and for any $\chi \in H_0^2(0,l)$,

$$\begin{aligned}\langle u_t(\cdot,t),\chi\rangle + \langle u_{xx}(\cdot,t), \chi'' + \chi\rangle + \langle u(\cdot,t)u_x(\cdot,t),\chi\rangle + q(t)\,\langle u(\cdot,t),\chi\rangle \\ = \langle f(\cdot,t),\chi\rangle\,, \quad t \in [0,T].\end{aligned} \tag{2.6}$$

Hence, the inverse problem (1.6)–(1.9) recasts in reconstructing the pair $(u(x,t), q(t))$ satisfying the system (2.5) and (2.6).

2.3 Time-discretization

The time interval $[0,T]$ is discretized into $K \in \mathbb{N}^*$ equidistant sub-intervals of step $\tau = T/K$ and denote $t_k = k\tau$ for $k = \overline{0,K}$. Applying (2.5) at $t = t_0 = 0$ yields

$$q_0 = \frac{\langle \rho, f_0\rangle - \phi_0' - \langle \rho'''' + \rho'', g\rangle - \langle \rho, gg'\rangle}{\phi_0}, \tag{2.7}$$

where $f_0(x) := f(x,0)$, $\phi_0 := \phi(0)$ and $\phi_0' := \phi'(0)$. Also, connected to (2.5) let us denote $f_k(x) := f(x,t_k)$, $\phi_k := \phi(t_k)$, $\phi_k' := \phi'(t_k)$ and

$$q_k := \frac{\langle \rho, f_k\rangle - \phi_k' - \langle \rho'''' + \rho'', u_{k-1}\rangle - \left\langle \rho, u_{k-1}u_{k-1}'\right\rangle}{\phi_k}, \quad k = \overline{1,K}, \tag{2.8}$$

where $u_0(x) = g(x)$ and, for $k = \overline{1,K}$, $u_k(x)$ is the solution of the following problem:

$$\begin{cases} \delta_t u_k + u_k'''' + u_k'' + u_k u_k' + q_k u_k = f_k, & x \in (0,l), \\ u_k(0) = u_k(l) = u_k'(0) = u_k'(l) = 0, \end{cases} \tag{2.9}$$

where $\delta_t u_k := \frac{u_k - u_{k-1}}{\tau}$. Note that q_k is not $q(t_k)$ but merely an approximation of it.

As in [13], consider the discrete variational problem consisting of finding the pairs $(u_k, q_k) \in H_0^2(0,l) \times \mathbb{R}_+$ for $k = \overline{1,K}$, satisfying (2.8) and the weak form of (2.9) given by

$$\langle \delta_t u_k, \chi\rangle + \langle u_k'', \chi'' + \chi\rangle + \langle u_k u_k', \chi\rangle + q_k\,\langle u_k,\chi\rangle = \langle f_k,\chi\rangle\,, \quad \forall \chi \in H_0^2(0,l). \tag{2.10}$$

In order to obtain the well-posedness of (2.8)–(2.9), we consider the auxiliary nonlinear problem

$$\begin{cases} U'''' + U'' + UU' + QU + \frac{1}{\tau}U = F(x), & x \in (0,l), \\ U(0) = U(l) = U'(0) = U'(l) = 0. \end{cases} \tag{2.11}$$

Lemma 2.1. *Let $F \in L^2(0,l)$, $\frac{(l^2-\pi^2)\pi^2}{l^4} \le Q \in \mathbb{R}$ and $\tau > 0$. Then there exists a solution $U \in H_0^2(0,l)$ to the problem (2.11). Furthermore, a positive constant τ_0 exists such that for any $\tau \in (0,\tau_0]$, the solution is unique.*

Proof. For the following linear problem with $r \in L^2(0,l)$:

$$\begin{cases} w'''' + w'' + \frac{1}{\tau}w + Qw = r(x), \quad x \in (0,l), \\ w(0) = w(l) = w'(0) = w'(l) = 0, \end{cases} \tag{2.12}$$

its variational form is given by:

$$\langle w'', \chi'' \rangle + \langle w'', \chi \rangle + \left(\frac{1}{\tau} + Q\right) \langle w, \chi \rangle = \langle r, \chi \rangle, \quad \forall \chi \in H_0^2(0,l). \tag{2.13}$$

For the bilinear functional $a(w,\chi) := \langle w'', \chi'' \rangle + \langle w'', \chi \rangle + \left(\frac{1}{\tau} + Q\right) \langle w, \chi \rangle$, using (2.3), we can obtain that

$$\begin{aligned} |a(w,\chi)| \le& \|w''\|\|\chi''\| + \|w''\|\|\chi\| + \left(\frac{1}{\tau} + Q\right)\|w\|\|\chi\| \le C(\tau,Q)\|w\|_{H_0^2(0,l)}\|\chi\|_{H_0^2(0,l)}, \\ a(w,w) =& \|w''\|^2 + \langle w'', w \rangle + \left(\frac{1}{\tau} + Q\right)\|w\|^2 = \|w''\|^2 - \|w'\|^2 + \left(\frac{1}{\tau} + Q\right)\|w\|^2 \\ \ge& \frac{(\pi^2 - l^2)}{\pi^2}\|w''\|^2 + \left(\frac{1}{\tau} + Q\right)\|w\|^2 \ge C(\tau,Q)\|w\|^2_{H_0^2(0,l)}, \end{aligned}$$

where, throughout the proof, $C = C(\tau,Q)$ denotes some positive constant that depends only on τ and Q. These inequalities imply that the bilinear functional $a(\cdot,\cdot)$ is continuous and coercive. Furthermore, $\mathcal{F}(\chi) := \langle r, \chi \rangle$ is a linear and bounded functional on $H_0^2(0,l)$ since $|\mathcal{F}(\chi)| \le \|r\|\|\chi\|$. Hence, the existence of a unique solution $w \in H_0^2(0,l)$ to (2.13) is guaranteed due to the Lax-Milgram theorem.

Taking $\chi = w$ in the variational form (2.13), employing (2.3) and that $\langle r, w \rangle \le \|r\|\|w\| \le \frac{1}{2\tau}\|w\|^2 + \frac{\tau}{2}\|r\|^2$, we have

$$\frac{(\pi^2 - l^2)}{\pi^2}\|w''\|^2 + \left(\frac{1}{2\tau} + Q\right)\|w\|^2 \le \frac{\tau}{2}\|r\|^2, \tag{2.14}$$

which means that the following estimate holds:

$$\|w\|_{H_0^2(0,l)} \le C(\tau,Q)\|r\|. \tag{2.15}$$

The first equation in (2.12) can be written as $w'''' = r - w'' - \left(Q + \frac{1}{\tau}\right)w$, and therefore

$$\|w''''\|^2 \le C(\tau,Q)(\|r\|^2 + \|w''\|^2 + \|w\|^2). \tag{2.16}$$

Then, (2.14)–(2.16) imply that $w \in \mathcal{D}$ and

$$\|w\|_{\mathcal{D}} \le C(\tau,Q)\|r\|. \tag{2.17}$$

Given a function $U \in H_0^2(0,l)$, setting $r = F - UU' \in L^2(0,l)$, then (2.12) defines an operator $\mathbb{B}U = w$ mapping from $H_0^2(0,l)$ to $\mathcal{D} \subset H_0^2(0,l)$. Meanwhile, using that $\|U\|_{L^\infty(0,l)} \le \sqrt{l}\|U'\|$, from (2.15) we have

$$\begin{aligned} \|\mathbb{B}U\|_{D(\mathbb{A})} \le C(\tau,Q)(\|UU'\| + \|F\|) \le& C(\tau,Q)(\sqrt{l}\|U'\|^2 + \|F\|) \\ \le& C(\tau,Q,l)(\|U\|^2_{H_0^2(0,l)} + \|F\|). \end{aligned}$$

Suppose that the sequence $\{U_n\}_{n=0}^{\infty} \subset H_0^2(0,l)$ converges to $U \in H_0^2(0,l)$. Then the above estimate yields that the sequence $\{w_n := \mathbb{B}U_n\}_{n=0}^{\infty} \subset \mathcal{D}$ is bounded in $\mathcal{D}$. Hence, there exists a subsequence, still denoted by $\{w_n\}_{n=0}^{\infty}$, that converges to a function $w \in H_0^2(0,l)$ strongly in $H_0^2(0,l)$, which shows that $\mathbb{B} : H_0^2(0,l) \mapsto H_0^2(0,l)$ is compact. For any $\chi \in H_0^2(0,l)$, we have

$$\langle w_n'', \chi'' \rangle + \langle w_n'', \chi \rangle + \left(\frac{1}{\tau} + Q\right) \langle w_n, \chi \rangle = \langle F, \chi \rangle - \langle U_n U_n', \chi \rangle .$$

For the last term in the right-hand side of the above identity, the Sobolev embedding $H_0^2(0,l) \hookrightarrow L^{\infty}(0,l)$ and the convergence of $\{U_n\}_{n=0}^{\infty}$ imply that

$$\begin{aligned}\langle U_n U_n', \chi \rangle =& \langle (U_n - U) U_n', \chi \rangle + \langle U U_n', \chi \rangle \\ \leq& \|U_n - U\| \|U_n'\| \|\chi\|_{L^{\infty}(0,1)} + \langle U U_n', \chi \rangle \to \langle U U', \chi \rangle , \quad \text{as } n \to \infty.\end{aligned}$$

Consequently, we obtain that

$$\langle w'', \chi'' \rangle + \langle w'', \chi \rangle + \left(\frac{1}{\tau} + Q\right) \langle w, \chi \rangle = \langle F, \chi \rangle - \langle U U', \chi \rangle ,$$

i.e., $w = \mathbb{B}U$, which indicates that $\mathbb{B}U_n \to \mathbb{B}U$ strongly in $H_0^2(0,l)$, namely $\mathbb{B}$ is continuous.

Let $U \in H_0^2(0,l)$ and $\lambda \in [0,1]$, and consider the problem $U = \lambda \mathbb{B} U$, i.e.,

$$U'''' + U'' + \left(\frac{1}{\tau} + Q\right) u = \lambda F - \lambda U U'.$$

Multiplying the above identity by U and integrating over $(0,l)$, using Young's inequality, (2.3) and $\int_0^l U^2 U' dx = 0$, as in deriving (2.14), we have

$$\frac{(\pi^2 - l^2)}{\pi^2} \|U''\|^2 + \left(\frac{1}{2\tau} + Q\right) \|U\|^2 \leq \frac{\tau}{2} \|F\|^2,$$

that is, $\|U\|_{H_0^2(0,1)} \leq C(\tau, Q, l) \|F\|$ for any $\lambda \in [0,1]$. Therefore, Schaefer's fixed point Theorem 2.1 yields that the nonlinear problem (2.11) has a solution $U \in H_0^2(0,l)$.

For uniqueness, suppose that there exists two solutions U_1 and U_2 of (2.11). Then $U := U_1 - U_2$ solves the problem given by

$$\begin{cases} U'''' + U'' + U_2 U' + \left(\frac{1}{\tau} + Q + U_1'\right) U = 0, \quad x \in (0,l), \\ \qquad U(0) = U(l) = U'(0) = U'(l) = 0. \end{cases} \tag{2.18}$$

By an analogous method to the above, and utilizing $\|U_i'\|_{L^{\infty}(0,l)} \leq \sqrt{l} \|U_i''\| \leq \pi \sqrt{\frac{\tau l}{2(\pi^2 - l^2)}} \|F\|$ for $i = 1, 2$, $\int_0^l U_2 U U' dx = -\frac{1}{2} \int_0^l U_2' U^2 dx$ and the inequality (2.2), we then have

$$\begin{aligned}\left(\frac{(\pi^2 - l^2)\pi^2}{l^4} + \frac{1}{\tau} + Q\right) \|U\|^2 \leq& \left(\|U_1'\|_{L^{\infty}(0,l)} + \frac{1}{2} \|U_2'\|_{L^{\infty}(0,l)}\right) \|U\|^2 \\ \leq& \pi \sqrt{\frac{9\tau l}{8(\pi^2 - l^2)}} \|F\| \|U\|^2.\end{aligned} \tag{2.19}$$

Thus, for every $\tau \leq \tau_0 := 2 \sqrt[3]{\frac{(\pi^2 - l^2)}{9\pi^2 l \|F\|^2}} > 0$, this leads to $\frac{(\pi^2 - l^2)\pi^2}{l^4} + \frac{1}{\tau} + Q - 3\pi \sqrt{\frac{\tau l}{8(\pi^2 - l^2)}} \|F\| > 0$ and $\|U\| = 0$. Hence, the solution $U \in H_0^2(0,l)$ to (2.11) is unique. $\square$

Lemma 2.2. *Assume that $\phi \in C^1[0,T]$ and $\phi^+ \geq \phi(t) \geq \phi_- > 0$ for $t \in [0,T]$, with the positive constants ϕ^+ and ϕ_-, $\rho \in \mathcal{D}$, $f \in L^\infty(0,T;L^2(0,l))$ and $g \in H_0^2(0,l)$ such that*

$$\langle \rho, f(\cdot,t)\rangle - \phi'(t) \geq (\|\rho''\| + \|\rho''''\|)\left(\|g\| + \frac{\sqrt{t}}{2}\|f\|_{L^\infty(0,t;L^2(0,l))}\right)$$
$$+\frac{1}{2}\|\rho'\|_{L^\infty(0,l)}\left(\|g\|^2 + \frac{t}{4}\|f\|^2_{L^\infty(0,t;L^2(0,l))}\right), \quad \forall t \in [0,T]. \tag{2.20}$$

Then there exists $(u_k, q_k) \in H_0^2(0,l) \times [0,\infty)$ for $k = \overline{0,K}$ which solves the time-discretization system (2.8)–(2.9)*. Furthermore, there exists a positive constant τ_0 such that for any $\tau \in (0,\tau_0]$, the solution of* (2.8)–(2.9) *is unique.*

Proof. Clearly, $u_0 = g$ and q_0 given by (2.7) are well- and uniquelly-defined. Also, applying (2.20) at $t = 0$ along with (2.7) results in $q_0 \geq 0$. Also, q_1 is non-negative by (2.8) and (2.20). Taking $\chi = u_k$ in (2.10), and supposing $q_k \geq 0$ for $k \geq 2$, applying Young's inequality, inequality (2.2) and $\int_0^l u_k^2 u_k' dx = 0$, we have

$$\frac{1}{\tau}\|u_k\|^2 + \left(\frac{\pi^2(\pi^2 - l^2)}{l^4} + q_k\right)\|u_k\|^2 \leq \frac{1}{2\tau}\|u_{k-1}\|^2 + \frac{1}{2\tau}\|u_k\|^2 + \frac{1}{8}\|f_k\|^2 + 2\|u_k\|^2,$$

which implies that $\|u_k\|^2 \leq \|u_{k-1}\|^2 + \frac{\tau}{4}\|f_k\|^2$ for $k \geq 1$. We thus obtain that

$$\|u_k\|^2 \leq \|u_0\|^2 + \frac{\tau}{4}\sum_{i=1}^{k}\|f_k\|^2 \leq \|g\|^2 + \frac{t_k}{4}\|f\|^2_{L^\infty(0,t_k;L^2(0,l))}. \tag{2.21}$$

Then by the discrete form (2.8) and the condition (2.20), we have that

$$q_{k+1} \geq \frac{\int_0^1 \rho f_{k+1}dx - \phi'_{k+1} - (\|\rho''\| + \|\rho''''\|)\|u_k\| - \frac{1}{2}\|\rho'\|_{L^\infty(0,l)}\|u_k\|^2}{\phi^+}$$
$$\geq \frac{\int_0^1 \rho f_{k+1}dx - \phi'_{k+1} - (\|\rho''\| + \|\rho''''\|)\left(\|g\| + \frac{\sqrt{t_k}}{2}\|f\|_{L^\infty(0,t_k;L^2(0,l))}\right)}{\phi^+}$$
$$-\frac{\frac{1}{2}\|\rho'\|_{L^\infty(0,l)}\left(\|g\|^2 + \frac{t_k}{4}\|f\|^2_{L^\infty(0,t_k;L^2(0,l))}\right)}{\phi^+} \geq 0,$$

which implies that $q_k \geq 0$ for all $k = \overline{1,K}$, by mathematical induction. Meanwhile, an upper bound of q_k for $k = \overline{1,K}$ is

$$|q_k| \leq \frac{\|\phi\|_{C^1[0,T]} + (\|\rho''\| + \|\rho''''\|)\left(\|g\| + \frac{\sqrt{T}}{2}\|f\|_{L^\infty(0,T;L^2(0,l))}\right)}{\phi_-}$$
$$+\frac{\|\rho'\|_{L^\infty(0,l)}\left(\|g\|^2 + \frac{T}{4}\|f\|^2_{L^\infty(0,T;L^2(0,l))}\right)}{2\phi_-}. \tag{2.22}$$

By employing the results in Lemma 2.1 for $k = \overline{1,K}$, the nonlinear problem (2.9) has a solution $u_k \in H_0^2(0,1)$ since $q_k \geq 0$ and $\frac{1}{\tau}u_{k-1} + f_k \in L^2(0,l)$. According to the arguments of Lemma 2.1, the uniqueness of $u_k \in H_0^2(0,l)$ for $k = \overline{1,K}$ is guaranteed when the following inequality holds:

$$\frac{(\pi^2 - l^2)\pi^2}{l^4} + \frac{1}{\tau} + q_k - 3\pi\sqrt{\frac{\tau l}{8(\pi^2 - l^2)}}\left(\frac{1}{\tau}\|u_{k-1}\| + \|f_k\|\right) > 0.$$

On using (2.21) this inequality is guaranteed if

$$\alpha_1\sqrt{\tau} + \alpha_2\tau\sqrt{\tau} - 1 \le 0,$$

where $\alpha_1 := 3\pi\sqrt{\frac{l}{8(\pi^2-l^2)}}\left(\|g\| + \frac{\sqrt{T}}{2}\|f\|_{L^\infty(0.T;L^2(0,l))}\right)$ and $\alpha_2 := 3\pi\sqrt{\frac{l}{8(\pi^2-l^2)}}\|f\|_{L^\infty(0,T;L^2(0,l))}$. Choosing $\tau_0 := \min\{1, (\alpha_1+\alpha_2)^{-2}\}$, the above inequality holds for any $\tau \in (0, \tau_0]$, and consequently, the solution u_k for $k = \overline{1,K}$ is unique. $\square$

Remark 2.1. *Although condition* (2.20) *may look too restrictive it was needed in the proof of Lemma 2.2 to ensure that the discrete values* $(q_k)_{k=\overline{0,K}}$ *of the coefficient* q *are non-negative, according to physical reality. Moreover, it can be checked using, for example, the symbolic computational package Maple, that the following example satisfies the assumptions of Theorem 2.2:*

$$l = 1, \quad T = 10^{-6}, \quad q(t) = 500, \quad g(x) = x^2(1-x)^2, \quad \rho(x) = x^2(1-x)^2,$$
$$f(x,t) = 2x^3(2x-1)(x-1)^3e^{-2t} + 499e^{-t}\left(x^4 - 2x^3 + x^2 + \frac{24}{499}\right),$$
$$u(x,t) = e^{-t}x^2(1-x)^2, \quad \phi(t) = e^{-t}/630.$$

The requirement for the sufficient smallness of the final time T is commonly encountered in inverse coeficient problems for parabolic equations whose unique solvability can, in general, be established only in a small neighbourhood of the initial time $t = 0$, see, e.g., [19]. Nevertheless it would be interesting to investigate in the future the possibility of replacing the condition (2.20) by a more natural one, using a different method which ensures that the inverse problem (1.6)–(1.9) is still uniquely solvable.

In the next lemma some *a priori* estimates are proved. These will be used to prove the existence of a solution to the problem (2.5)-(2.6) (which by other means will also be shown to be unique). The results of the lemma below will also be used to prove the convergence of the approximate solution towards the unique solution of the problem (2.5)-(2.6) and to derive error estimates.

Lemma 2.3. *Under the hypotheses of Lemma 2.2, there exist two positive constants* $C = C(T, g, f, \rho, \phi)$ *and* τ_0 *such that for any* $\tau \in (0, \tau_0]$ *and* $j = \overline{1,K}$,

$$\max_{i=\overline{1,K}} |q_i|^2 \le C, \tag{2.23}$$

$$\|u_j\|^2 + \sum_{k=1}^{j}\|u_k - u_{k-1}\|^2 + \tau\sum_{k=1}^{j}\|u_k''\|^2 \le C, \tag{2.24}$$

$$\tau\sum_{k=1}^{j}\|\delta_t u_k\|^2 + \|u_j''\|^2 + \sum_{k=1}^{j}\|u_k'' - u_{k-1}''\|^2 \le C. \tag{2.25}$$

Proof. It is obvious that the estimate (2.23) holds by the upper bound (2.22) of q_k. For $k = \overline{1,j}$ and $j = \overline{1,K}$, setting $\chi = u_k\tau$ in (2.10) and using that $\int_0^l u_k^2 u_k' dx = 0$, we have

$$\langle u_k - u_{k-1}, u_k\rangle + \tau\|u_k''\|^2 - \tau\|u_k'\|^2 + q_k\tau\|u_k\|^2 = \langle f_k, u_k\rangle\,\tau.$$

The summation of such identity for $k = \overline{1,j}$ yields that

$$\sum_{k=1}^{j}\langle u_k - u_{k-1}, u_k\rangle + \tau\sum_{k=1}^{j}(\|u_k''\|^2 - \|u_k'\|^2) + \tau\sum_{k=1}^{j} q_k\|u_k\|^2 = \sum_{k=1}^{j}\langle f_k, u_k\rangle\,\tau. \tag{2.26}$$

Abel's lemma gives that

$$\sum_{k=1}^{j} \langle u_k - u_{k-1}, u_k \rangle = \frac{1}{2}\left(\|u_j\|^2 - \|g\|^2 + \sum_{k=1}^{j} \|u_k - u_{k-1}\|^2 \right), \tag{2.27}$$

and Young's inequality leads to

$$\tau \sum_{k=1}^{j} \langle f_k, u_k \rangle \le \frac{1}{2}\|f\|^2_{L^\infty(0,T;L^2(0,l))} + \frac{\tau}{2}\sum_{k=1}^{j} \|u_k\|^2. \tag{2.28}$$

The non-negativity of q_k, the inequality (2.3) and (2.26)-(2.28) imply that

$$\begin{aligned} \|u_j\|^2 + \sum_{k=1}^{j} \|u_k - u_{k-1}\|^2 + \frac{2\tau(\pi^2 - l^2)}{\pi^2} \sum_{k=1}^{j} \|u_k''\|^2 \\ \le \|f\|^2_{L^\infty(0,T;L^2(0,l))} + \|g\|^2 + \tau \sum_{k=1}^{j} \|u_k\|^2. \end{aligned}$$

We then obtain

$$\|u_j\|^2 + \sum_{k=1}^{j} \|u_k - u_{k-1}\|^2 + \tau \sum_{k=1}^{j} \|u_k''\|^2 \le C_0 + C_0 \tau \overline{C} \sum_{k=1}^{j} \|u_k\|^2, \tag{2.29}$$

where $C_0 := \max\{\|f\|^2_{L^\infty(0,T;L^2(0,l))} + \|g\|^2, 1\} > 0$. Then, taking $\tau_0 := \frac{1}{1+C_0}$, we have $1 - \tau C_0 \ge 1 - C_0\tau_0 > 0$ for $\tau \in (0, \tau_0]$, and

$$\begin{aligned} \|u_j\|^2 + \sum_{k=1}^{j} \|u_k - u_{k-1}\|^2 + \tau \sum_{k=1}^{j} \|u_k''\|^2 \\ \le \frac{C_0}{1 - C_0\tau_0} + \frac{C_0\tau\overline{C}}{1 - C_0\tau_0} \sum_{k=1}^{j-1} \left(\|u_k\|^2 + \sum_{i=1}^{k} \|u_i - u_{i-1}\|^2 + \tau \sum_{i=1}^{k} \|u_i''\|^2 \right). \end{aligned} \tag{2.30}$$

Hence the estimate (2.24) can be derived by the discrete Gronwall inequality with

$$C := \frac{C_0}{1 - C_0\tau_0} + \frac{C_0^2 \overline{C} T}{(1 - C_0\tau_0)^2} e^{\frac{C_0}{1 - C_0\tau_0}}.$$

Analogously, taking $\chi = \tau\delta_t u_k$ in (2.10) and summing it over $k = \overline{1, j}$, we obtain

$$\begin{aligned} \tau \sum_{k=1}^{j} \|\delta_t u_k\|^2 + \tau \sum_{k=1}^{j} \langle u_k'', \delta_t u_k'' \rangle \\ = -\tau \sum_{k=1}^{j} \langle u_k'', \delta_t u_k \rangle - \tau \sum_{k=1}^{j} \langle u_k u_k', \delta_t u_k \rangle - \tau \sum_{k=1}^{j} q_k \langle u_k, \delta_t u_k \rangle + \tau \sum_{k=1}^{j} \langle f_k, \delta_t u_k \rangle. \end{aligned} \tag{2.31}$$

Using Abel's lemma again, the second term of the left-hand side in (2.31) becomes

$$\tau \sum_{k=1}^{j} \langle u_k'', \delta_t u_k'' \rangle = \sum_{k=1}^{j} \langle u_k'', u_k'' - u_{k-1}'' \rangle = \frac{1}{2}\left(\|u_j''\|^2 - \|g''\|^2 + \sum_{k=1}^{j} \|u_k'' - u_{k-1}''\|^2 \right), \tag{2.32}$$

and applying Young's inequality and the estimates (2.23) and (2.24) to the first, second and fourth terms of the right-hand side of (2.31), we have

$$\left|\tau\sum_{k=1}^{j}\langle u_k'', \delta_t u_k\rangle\right| \le \frac{\tau}{8}\sum_{k=1}^{j}\|\delta_t u_k\|^2 + 2\tau\sum_{k=1}^{j}\|u_k''\|^2 \le \frac{\tau}{8}\sum_{k=1}^{j}\|\delta_t u_k\|^2 + C, \quad (2.33)$$

$$\left|\tau\sum_{k=1}^{j}q_k\langle u_k, \delta_t u_k\rangle\right| \le \frac{\tau}{8}\sum_{k=1}^{j}\|\delta_t u_k\|^2 + 2\tau\max_{k=\overline{1,j}}|q_k|^2\sum_{k=1}^{j}\|u_k\|^2 \le \frac{\tau}{8}\sum_{k=1}^{j}\|\delta_t u_k\|^2 + C, \quad (2.34)$$

$$\left|\tau\sum_{k=1}^{j}\langle f_k, \delta_t u_k\rangle\right| \le \frac{\tau}{8}\sum_{k=1}^{j}\|\delta_t u_k\|^2 + 2\tau\sum_{k=1}^{j}\|f_k\|^2 \le \frac{\tau}{8}\sum_{k=1}^{j}\|\delta_t u_k\|^2 + 2\|f\|^2_{L^2(Q_T)}. \quad (2.35)$$

Finally, using $\|u_k'\|_{L^\infty(0,l)} \le \|u_k''\|$ and (2.24), the second term in the right-hand side of (2.31) can be estimated as follows:

$$\begin{aligned}\left|\tau\sum_{k=1}^{j}\langle u_k u_k', \delta_t u_k\rangle\right| &\le \frac{\tau}{8}\sum_{k=1}^{j}\|\delta_t u_k\|^2 + 2\tau\sum_{k=1}^{j}\|u_k u_k'\|^2 \le \frac{\tau}{8}\sum_{k=1}^{j}\|\delta_t u_k\|^2 + 2\tau\sum_{k=1}^{j}\|u_k\|^2\|u_k''\|^2\\ &\le \frac{\tau}{8}\sum_{k=1}^{j}\|\delta_t u_k\|^2 + C\tau\sum_{k=1}^{j}\|u_k''\|^2 \le \frac{\tau}{8}\sum_{k=1}^{j}\|\delta_t u_k\|^2 + C.\end{aligned}$$

Combining the above inequalities, we obtain that

$$\tau\sum_{k=1}^{j}\|\delta_t u_k\|^2 + \|u_j''\|^2 + \sum_{k=1}^{j}\|u_k'' - u_{k-1}''\|^2 \le \|g''\|^2 + 4\|f\|^2_{L^2(Q_T)} + C,$$

which concludes that the estimate (2.25) holds. □

2.4 Existence and uniqueness

In this section, the existence and uniqueness of solution to the problem (2.5) and (2.6) shall be investigated by invoking the Rothe's method [20]. Let us introduce the piecewise constant in time functions

$$\bar{u}_K : [0,l]\times(-\tau,T] \to \mathbb{R} \text{ defined by } \bar{u}_K(x,t) = \begin{cases} u_0(x), & t \in (-\tau, 0],\\ u_k(x), & t\in(t_{k-1},t_k],\ k=\overline{1,K}\end{cases} \quad (2.36)$$

and

$$\bar{q}_K : [0,T] \to \mathbb{R}_+ \text{ defined by } \bar{q}_K(t) = \begin{cases} q_0 & \text{if } t=0,\\ q_k & \text{if } t\in(t_{k-1},t_k],\ k=\overline{1,K},\end{cases} \quad (2.37)$$

where q_0 and q_k for $k=\overline{1,K}$ are given by (2.7) and (2.8), respectively. We also define

$$\bar{f}_K : [0,l]\times[0,T] \to \mathbb{R} \text{ defined by } \bar{f}_K(x,t) = \begin{cases} f_0(x) & \text{if } t=0,\\ f_k(x) & \text{if } t\in(t_{k-1},t_k],\ k=\overline{1,K},\end{cases} \quad (2.38)$$

$$\bar{\phi}_K : [0,T] \to \mathbb{R} \text{ defined by } \bar{\phi}_K(t) = \begin{cases} \phi_0 & \text{if } t=0,\\ \phi_k & \text{if } t\in(t_{k-1},t_k],\ k=\overline{1,K},\end{cases} \quad (2.39)$$

and

$$\bar{\phi}'_K : [0,T] \to \mathbb{R} \text{ defined by } \bar{\phi}'_K(t) = \begin{cases} \phi'_0 & \text{if } t = 0, \\ \phi'_k & \text{if } t \in (t_{k-1}, t_k],\ k = \overline{1, K}, \end{cases} \tag{2.40}$$

Let us also introduce the piecewise linear in time functions

$$U_K : [0,l] \times [0,T] \to \mathbb{R} \text{ defined by}$$
$$U_K(x,t) = \begin{cases} u_0(x), & t = 0, \\ u_{k-1}(x) + (t - t_{k-1})\delta_t u_k(x), & t \in (t_{k-1}, t_k],\ k = \overline{1, K} \end{cases} \tag{2.41}$$

and

$$Q_K : [0,T] \to \mathbb{R}_+ \text{ defined by}$$
$$Q_K(t) = \begin{cases} q_0 & \text{if } t = 0, \\ q_{k-1} + (t - t_{k-1})\delta_t q_k & \text{if } t \in (t_{k-1}, t_k],\ k = \overline{1, K}, \end{cases} \tag{2.42}$$

where $\delta_t q_k := \frac{q_k - q_{k-1}}{\tau}$. From (2.8) and (2.37) we can remark that

$$\bar{q}_K(t) = \frac{\langle \rho, \bar{f}_K(\cdot,t) \rangle - \bar{\phi}'_K(t) - \langle \rho'''' + \rho'', \bar{u}_K(\cdot, t-\tau) \rangle - \langle \rho, \bar{u}_K(\cdot, t-\tau)\partial_x \bar{u}_K(\cdot, t-\tau) \rangle}{\bar{\phi}_K(t)}, \tag{2.43}$$

Also, the variational form (2.6) can be approximated by

$$\begin{aligned} \langle \partial_t U_K(\cdot,t), \chi \rangle + \langle \partial_{xx} \bar{u}_K(\cdot,t), \chi'' + \chi \rangle + \langle \bar{u}_K(\cdot,t)\partial_x \bar{u}_K(\cdot,t), \chi \rangle \\ + \bar{q}_K(t) \langle \bar{u}_K(\cdot,t), \chi \rangle = \langle \bar{f}_K(\cdot,t), \chi \rangle, \end{aligned} \tag{2.44}$$

for all $t \in (0,T]$ and $\chi \in H_0^2(0,l)$, and $U_K(x,0) = u_0(x) = g(x)$.

Remark 2.2. *Suppose $\phi \in H^2(0,T)$ and $f \in H^1(0,T; L^2(0,l))$. Then, for $t \in (0,T]$,*

$$\begin{aligned} \int_0^t \|\bar{f}_K(\cdot,s) - f(\cdot,s)\|^2 ds &\le \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \|f_k(\cdot) - f(\cdot,s)\|^2 ds \\ &= \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \left\| \int_s^{t_k} f_t(\cdot,\varsigma) d\varsigma \right\|^2 ds \le T\tau \|f_t\|^2_{L^2(Q_T)}, \end{aligned} \tag{2.45}$$

$$\begin{aligned} \int_0^t |\bar{\phi}_K(s) - \phi(s)|^2 ds &\le \sum_{k=1}^K \int_{t_{k-1}}^{t_k} |\phi_k - \phi(s)|^2 ds \\ &= \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \left| \int_s^{t_k} \phi'(\varsigma) d\varsigma \right|^2 ds \le T\tau^2 \|\phi'\|^2_{L^\infty(0,T)}, \end{aligned} \tag{2.46}$$

$$\begin{aligned} \int_0^t |\bar{\phi}'_K(s) - \phi'(s)|^2 ds &\le \sum_{k=1}^K \int_{t_{k-1}}^{t_k} |\phi'_k - \phi'(s)|^2 ds \\ &= \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \left| \int_s^{t_k} \phi''(\varsigma) d\varsigma \right|^2 ds \le T\tau \|\phi''\|^2_{L^2(0,T)}, \end{aligned} \tag{2.47}$$

where we have used that $K\tau = T$. The estimates (2.24), (2.25) *and the definition* (2.36) *imply that*

$$\int_0^t \|\bar{u}_K(\cdot, s-\tau) - \bar{u}_K(\cdot, s)\|^2 ds \le \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \|u_k - u_{k-1}\|^2 ds = \tau^3 \sum_{k=1}^K \|\delta_t u_k\|^2 \le C\tau^2, \quad (2.48)$$

$$\begin{aligned}\int_0^t \|\partial_x \bar{u}_K(\cdot, s-\tau) - \partial_x \bar{u}_K(\cdot, s)\|^2 ds &\le \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \|u_k' - u_{k-1}'\|^2 ds \\ &\le C\tau \sum_{k=1}^K \|u_k'' - u_{k-1}''\|^2 \le C\tau, \quad (2.49)\end{aligned}$$

$$\int_0^t \|\bar{\partial}_{xx} \bar{u}_K(\cdot, s-\tau) - \partial_{xx} \bar{u}_K(\cdot, s)\|^2 ds \le \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \|u_k'' - u_{k-1}''\|^2 ds \le C\tau. \quad (2.50)$$

By (2.36) *and* (2.41) *it is easy to see that*

$$\|\bar{u}_K(\cdot, t) - U_K(\cdot, t)\| = \|u_k - u_{k-1} - (t - t_{k-1})\delta_t u_k\| \le \|u_k - u_{k-1}\|, \quad \forall t \in (t_{k-1}, t_k] \quad (2.51)$$

and similarly

$$\|\partial_x \bar{u}_K(\cdot, t) - \partial_x U_K(\cdot, t)\| \le \|u_k' - u_{k-1}'\|, \quad \|\partial_{xx} \bar{u}_K(\cdot, t) - \partial_{xx} U_K(\cdot, t)\| \le \|u_k'' - u_{k-1}''\|. \quad (2.52)$$

These three inequalities and the analogous method used to prove (2.48)–(2.50) *yield that*

$$\int_0^t \|\bar{u}_K(\cdot, s) - U_K(\cdot, s)\|^2 ds \le C\tau^2, \quad (2.53)$$

$$\int_0^t \|\partial_x \bar{u}_K(\cdot, s) - \partial_x U_K(\cdot, s)\|^2 ds \le C\tau, \quad (2.54)$$

$$\int_0^t \|\partial_{xx} \bar{u}_K(\cdot, s) - \partial_{xx} U_K(\cdot, s)\|^2 ds \le C\tau. \quad (2.55)$$

Theorem 2.3. *Assume that $\phi \in H^2(0,T)$ and $\phi^+ \ge \phi(t) \ge \phi_- > 0$ for $t \in [0,T]$, with positive constants ϕ^+ and ϕ_-, $\rho \in \mathcal{D}$, $f \in H^1(0,T; L^2(0,l))$ and $g \in H_0^2(0,l)$ such that the inequality* (2.20) *holds. Then there exists a solution (u,q) to* (1.7), (2.5) *and* (2.6) *with $q \in \mathcal{Q} := \{q \in L^\infty(0,T); q(t) \ge 0,\ t \in [0,T]\}$, $u \in C([0,T]; L^2(0,l)) \cap L^\infty(0,T; H_0^2(0,l)) =: \mathcal{U}$ and $u_t \in L^2(Q_T)$.*

Proof. From the definitions (2.36) and (2.41), and the estimates (2.24) and (2.25), we have

$$\max_{t\in[0,T]} \|\bar{u}_K(\cdot, t)\|_{H_0^2(0,l)}^2 + \|\partial_t U_K\|_{L^2(Q_T)}^2 \le C. \quad (2.56)$$

Also, from (2.51), (2.52) and (2.56) we can deduce that

$$\max_{t\in[0,T]} \|U_K(\cdot, t)\|_{H_0^2(0,l)}^2 \le C. \quad (2.57)$$

Employing the arguments in [13, 20], and using the continuous embedding of $H_0^2(0,l) \hookrightarrow L^2(0,l)$, there exist a function $u(x,t) \in C([0,T]; L^2(0,l)) \cap L^\infty(0,T; H_0^2(0,l))$ with $u_t \in L^2(Q_T)$, and the subsequences $\{U_K(x,t)\}_{K\in\mathbb{N}^*}$ and $\{\bar{u}_K(x,t)\}_{K\in\mathbb{N}^*}$, still denoted by the same symbols, such that

$$\begin{cases} U_K \to u & \text{in} \quad C([0,T]; L^2(0,l)), \\ \partial_t U_K \rightharpoonup u_t & \text{in} \quad L^2(Q_T), \\ U_K(\cdot,t) \rightharpoonup u(\cdot,t) & \text{in} \quad H_0^2(0,l),\ \forall t \in [0,T], \\ \bar{u}_K(\cdot,t) \rightharpoonup u(\cdot,t) & \text{in} \quad H_0^2(0,l),\ \forall t \in [0,T], \end{cases} \quad (2.58)$$

as $K \to \infty$. The inequality (2.53) means that $\{U_K(x,t)\}_{K\in\mathbb{N}^*}$ and $\{\bar{u}_K(x,t)\}_{K\in\mathbb{N}^*}$ share the same limit in $L^2(Q_T)$, and $\bar{u}_K$ converges to u strongly in $L^2(Q_T)$, and the strong convergence of $\partial_x\bar{u}_K$ to u_x in $L^2(Q_T)$ can be obtained by (2.54). In addition, the boundedness of $\bar{q}_K(t)$ in $L^2(0,T)$ can be ensured by the estimate (2.23). Thus there exist a subsequence, still denoted by $\{\bar{q}_K(t)\}_{K\in\mathbb{N}^*}$, and a function $q(t) \in L^2(0,T)$, such that $\bar{q}_K \rightharpoonup q$ in $L^2(0,T)$.

For any $t \in [0,T]$, applying (2.58), we obtain that $\bar{u}_K(\cdot,t)$ converges to $u(\cdot,t)$ strongly in $H_0^1(0,1)$, and

$$
\begin{aligned}
|\langle(\bar{u}_K(\cdot,t)-u(\cdot,t))\partial_x\bar{u}_K(\cdot,t),\chi\rangle| \le \|\bar{u}_K(\cdot,t)-u(\cdot,t)\|\|\partial_x\bar{u}_K(\cdot,t)\|\|\chi\|_{L^\infty(0,1)} \to 0,\\
\langle\bar{u}_K(\cdot,t)\partial_x\bar{u}_K(\cdot,t),\chi\rangle = \langle(\bar{u}_K(\cdot,t)-u(\cdot,t))\partial_x\bar{u}_K(\cdot,t),\chi\rangle\\
+\langle u(\cdot,t)\partial_x\bar{u}_K(\cdot,t),\chi\rangle \to \langle u(\cdot,t)\partial_x u(\cdot,t),\chi\rangle,
\end{aligned}
$$

as $K \to \infty$. Analogously, the convergences result of $\{\bar{q}_K(t)\}_{K\in\mathbb{N}^*}$ in $L^2(0,T)$ and $\{\bar{u}_K(x,t)\}_{K\in\mathbb{N}^*}$ in $L^2(Q_T)$ imply that

$$
\begin{aligned}
\int_0^t \bar{q}_K(s)\langle\bar{u}_K(\cdot,s)-u(\cdot,s),\chi\rangle\,ds \le \|\bar{q}_K\|_{L^2(0,T)}\|\bar{u}_K-u\|_{L^2(Q_T)}\|\chi\| \to 0,\\
\int_0^t \bar{q}_K(s)\langle\bar{u}_K(\cdot,s),\chi\rangle\,ds = \int_0^t \bar{q}_K(s)\langle\bar{u}_K(\cdot,s)-u(\cdot,s),\chi\rangle\,ds\\
+\int_0^t \bar{q}_K(s)\langle u(\cdot,s),\chi\rangle\,ds \to \int_0^t q(s)\langle u(\cdot,s),\chi\rangle\,ds,
\end{aligned}
$$

as $K \to \infty$. Integrating (2.44) over $(0,t)$ with $t \in (0,T]$, we have

$$
\begin{aligned}
&\int_0^t \langle\partial_s U_K(\cdot,s),\chi\rangle\,ds + \int_0^t \langle\partial_{xx}\bar{u}_K(\cdot,s),\chi''+\chi\rangle\,ds + \int_0^t \langle\bar{u}_K(\cdot,s)\partial_x\bar{u}_K(\cdot,s),\chi\rangle\,ds\\
&+\int_0^t \bar{q}_K(s)\langle\bar{u}_K(\cdot,s),\chi\rangle\,ds = \int_0^t \langle\bar{f}_K(\cdot,s),\chi\rangle\,ds,
\end{aligned}
$$

Passing K to ∞, and using (2.58) and the above convergence results, the above identity yields

$$
\begin{aligned}
&\int_0^t \langle u_s(\cdot,s),\chi\rangle\,ds + \int_0^t \langle\partial_{xx}u(\cdot,s),\chi''+\chi\rangle\,ds + \int_0^t \langle u(\cdot,s)\partial_x u(\cdot,s),\chi\rangle\,ds\\
&+\int_0^t q(s)\langle u(\cdot,s),\chi\rangle\,ds = \int_0^t \langle f(\cdot,s),\chi\rangle\,ds.
\end{aligned}
$$

Then the pair (u,q) solves (2.6) by differentiating the above identity with respect to t.

For any $t \in (0,T]$, utilizing (2.45) and (2.46), we get

$$
\begin{aligned}
&\left|\int_0^t \left(\frac{\langle\rho,\bar{f}_K(\cdot,s)\rangle}{\bar{\phi}_K(s)} - \frac{\langle\rho,f(\cdot,s)\rangle}{\phi(s)}\right)ds\right|\\
\le&\int_0^t \left|\frac{\langle\rho,\bar{f}_K(\cdot,s)\rangle}{\bar{\phi}_K(s)} - \frac{\langle\rho,f(\cdot,s)\rangle}{\phi(s)}\right|ds \le \frac{1}{\phi_-^2}\int_0^t |\phi(s)\langle\rho,\bar{f}_K(\cdot,s)\rangle - \bar{\phi}_K(s)\langle\rho,f(\cdot,s)\rangle|ds\\
\le&\frac{1}{\phi_-^2}\int_0^t |\phi(s)\langle\rho,\bar{f}_K(\cdot,s)-f(\cdot,s)\rangle + (\phi(s)-\bar{\phi}_K(s))\langle\rho,f(\cdot,s)\rangle|ds\\
\le&\frac{\|\rho\|}{\phi_-^2}\left(\|\phi\|_{L^2(0,t)}\int_0^t \|\bar{f}_K(\cdot,s)-f(\cdot,s)\|ds + \|\bar{\phi}_K-\phi\|\int_0^t \|f(\cdot,s)\|ds\right) \le C\sqrt{\tau}.
\end{aligned}
\tag{2.59}
$$

Similarly, by (2.46) and (2.47), the following result can be derived:

$$\left|\int_0^t \left(\frac{\bar{\phi}'_K}{\bar{\phi}_K} - \frac{\phi'}{\phi}\right) ds\right| \le \int_0^t \left|\frac{\bar{\phi}'_K}{\bar{\phi}_K} - \frac{\phi'}{\phi}\right| ds \le C\sqrt{\tau}. \tag{2.60}$$

Furthermore, (2.46) and (2.48) lead to

$$\begin{aligned}
&\left|\int_0^t \left(\frac{\langle \rho'''' + \rho'', \bar{u}_K(\cdot, s-\tau)\rangle}{\bar{\phi}_K(s)} - \frac{\langle \rho'''' + \rho'', u(\cdot, s)\rangle}{\phi(s)}\right) ds\right| \\
&\le \int_0^t \left|\frac{\langle \rho'''' + \rho'', \bar{u}_K(\cdot, s-\tau)\rangle}{\bar{\phi}_K(s)} - \frac{\langle \rho'''' + \rho'', u(\cdot, s)\rangle}{\phi(s)}\right| ds \\
&\le \frac{1}{\phi_-^2} \int_0^t \Big|\phi \left\langle \rho'''' + \rho'', \bar{u}_K(\cdot, s-\tau) - \bar{u}_K(\cdot, s)\right\rangle + \phi \left\langle \rho'''' + \rho'', \bar{u}_K(\cdot, s) - u(\cdot, s)\right\rangle \\
&\qquad + (\phi - \bar{\phi}_K) \left\langle \rho'''' + \rho'', u(\cdot, s)\right\rangle \Big| ds \\
&\le \frac{\|\rho\|_{\mathcal{D}}}{\phi_-^2} \Big[\|\phi\|_{L^2(0,t)} \left(\int_0^t \|\bar{u}_K(\cdot, s-\tau) - \bar{u}_K(\cdot, s)\| ds + \int_0^t \|\bar{u}_K(\cdot, s) - u(\cdot, s)\| ds\right) \\
&\qquad + \|\phi - \bar{\phi}_K\|_{L^2(0,t)} \int_0^t \|u(\cdot, s)\| ds\Big] \le C\tau + C\int_0^t \|\bar{u}_K(\cdot, s) - u(\cdot, s)\| ds.
\end{aligned} \tag{2.61}$$

Finally, from (2.46), (2.48), (2.49) and (2.56) we arrive at

$$\begin{aligned}
&\left|\int_0^t \left(\frac{\langle \rho, \bar{u}_K(\cdot, s-\tau)\partial_x \bar{u}_K(\cdot, s-\tau)\rangle}{\bar{\phi}_K(s)} - \frac{\langle \rho, u(\cdot, s)\partial_x u(\cdot, s)\rangle}{\phi(s)}\right) ds\right| \\
&\le \frac{1}{\phi_-^2} \Big\{\int_0^t \phi \Big| \left\langle \rho, (\bar{u}_K(\cdot, s-\tau) - \bar{u}_K(\cdot, s))\partial_x \bar{u}_K(\cdot, s-\tau)\right\rangle \\
&\qquad + \left\langle \rho, (\bar{u}_K(\cdot, s) - u(\cdot, s))\partial_x \bar{u}_K(\cdot, s-\tau)\right\rangle \Big| ds \\
&+ \int_0^t \Big|\phi \left\langle \rho, u(\cdot, s)(\partial_x \bar{u}_K(\cdot, s-\tau) - \partial_x \bar{u}_K(\cdot, s))\right\rangle + \phi \left\langle \rho, u(\cdot, s)(\partial_x \bar{u}_K(\cdot, s) - \partial_x u(\cdot, s))\right\rangle \\
&\qquad + (\phi - \bar{\phi}_K) \left\langle \rho, u(\cdot, s)\partial_x u(\cdot, s)\right\rangle \Big| ds\Big\} \\
&\le C\int_0^t \|\bar{u}_K(\cdot, s) - u(\cdot, s)\| ds + C\int_0^t \phi \Big| \left\langle \rho' u(\cdot, s) + \rho \partial_x u(\cdot, s), \bar{u}_K(\cdot, s) - u(\cdot, s)\right\rangle \Big| ds \\
&\qquad + C\sqrt{\tau} \le C\left(\sqrt{\tau} + \int_0^t \|\bar{u}_K(\cdot, s) - u(\cdot, s)\| ds\right).
\end{aligned} \tag{2.62}$$

Integrating (2.43) over $(0, t)$ for any $t \in (0, T]$, passing K to ∞, and using the estimates (2.59)–(2.62), the strong convergence of $\bar{u}_K$ to u in $L^2(0, T; H_0^1(0, l))$ yields

$$\int_0^t q(s) ds = \int_0^t \frac{\langle \rho, f(\cdot, s)\rangle - \phi'(s) - \langle \rho'''' + \rho'', u(\cdot, s)\rangle - \langle \rho, u(\cdot, s)u'(\cdot, s)\rangle}{\phi(s)} ds.$$

Differentiating this with respect to t, the formulation (2.5) is achieved. Meanwhile, the function q belonging to $\mathcal{Q}$ can be ensured by the regularity of the functions u, ϕ and f, together with the condition (2.20). □

Theorem 2.4. *Suppose that the assumptions in Theorem 2.3 are fulfilled. Then the solution (u, q) to (1.7), (2.5) and (2.6) obtained in Theorem 2.3 is unique.*

Proof. Let $(u^{(1)}, q^{(1)})$ and $(u^{(2)}, q^{(2)})$ be two solutions to the problem (2.5) and (2.6). Then, by Theorem 2.3, we have

$$0 \le q^{(i)}(t) \le C, \quad t \in [0,T], \tag{2.63}$$

$$\max_{t\in[0,T]} \|u^{(i)}(\cdot,t)\|_{H_0^2(0,l)} + \|u_t^{(i)}\|_{L^2(Q_T)} \le C, \quad i = 1,2, \tag{2.64}$$

and the functions $\mathrm{u} := u^{(1)} - u^{(2)}$ and $\mathrm{q} = q^{(1)} - q^{(2)}$ solve the problem given by:

$$\begin{aligned} \mathrm{q}(t) &= \frac{-\langle \rho'''' + \rho'', \mathrm{u}(\cdot,t)\rangle - \left\langle \rho, \mathrm{u}(\cdot,t)\partial_x u^{(1)}(\cdot,t) + u^{(2)}(\cdot,t)\partial_x \mathrm{u}(\cdot,t)\right\rangle}{\phi(t)} \\ &= \frac{-\langle \rho'''' + \rho'', \mathrm{u}(\cdot,t)\rangle - \left\langle \rho\partial_x u^{(1)}(\cdot,t), \mathrm{u}(\cdot,t)\right\rangle + \left\langle \rho' u^{(2)}(\cdot,t) + \rho\partial_x u^{(2)}(\cdot,t), \mathrm{u}(\cdot,t)\right\rangle}{\phi(t)}, \end{aligned} \tag{2.65}$$

$$\begin{aligned} &\langle \mathrm{u}_t(\cdot,t), \chi\rangle + \langle \partial_{xx}\mathrm{u}(\cdot,t), \chi + \chi''\rangle + \left\langle \mathrm{u}(\cdot,t)\partial_x u^{(1)}(\cdot,t) + u^{(2)}(\cdot,t)\partial_x \mathrm{u}(\cdot,t), \chi\right\rangle \\ &\qquad + \mathrm{q}(t)\left\langle u^{(1)}(\cdot,t), \chi\right\rangle + q^{(2)}(t)\langle \mathrm{u}(\cdot,t), \chi\rangle = 0, \quad \forall \chi \in H_0^2(0,l) \end{aligned} \tag{2.66}$$

and $\mathrm{u}(x,0) = 0$ for all $x \in (0,l)$. Integrating (2.65) and using integration by parts, inequality (2.64) and that $\rho \in \mathcal{D}$, yield

$$\int_0^t |\mathrm{q}(s)|^2 ds \le C \int_0^t \|\mathrm{u}(\cdot,s)\|^2 ds, \quad t \in [0,T]. \tag{2.67}$$

For fixed $t \in [0,T]$, taking $\chi = \mathrm{u}(\cdot,t)$ in (2.66) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\mathrm{u}(\cdot,t)\|^2 + \langle \partial_{xx}\mathrm{u}(\cdot,t), \partial_{xx}\mathrm{u}(\cdot,t) + \mathrm{u}(\cdot,s)\rangle \\ &\quad + \left\langle \mathrm{u}(\cdot,t)\partial_x u^{(1)}(\cdot,t) + u^{(2)}(\cdot,t)\partial_x \mathrm{u}(\cdot,t), \mathrm{u}(\cdot,t)\right\rangle \\ &+ \mathrm{q}(t)\left\langle u^{(1)}(\cdot,t), \mathrm{u}(\cdot,t)\right\rangle + \|\mathrm{q}^{(2)}(t)\mathrm{u}(\cdot,t)\|^2 = 0, \quad t \in [0,T]. \end{aligned} \tag{2.68}$$

Using integration by parts and (1.8), we obtain that the third term in (2.68) can be rewritten as

$$\begin{aligned} &\left\langle \mathrm{u}(\cdot,t)\partial_x u^{(1)}(\cdot,t) + u^{(2)}(\cdot,t)\partial_x \mathrm{u}(\cdot,t), \mathrm{u}(\cdot,t)\right\rangle = \left\langle \mathrm{u}(\cdot,t)\partial_x u^{(1)}(\cdot,t), \mathrm{u}(\cdot,t)\right\rangle \\ &+ \left\langle u^{(2)}(\cdot,t)\partial_x \mathrm{u}(\cdot,t), \mathrm{u}(\cdot,t)\right\rangle = \left\langle \partial_x u^{(1)}(\cdot,t), \mathrm{u}^2(\cdot,t)\right\rangle + \left\langle u^{(2)}(\cdot,t), \frac{1}{2}\partial_x(\mathrm{u}^2(\cdot,t))\right\rangle \\ &= \left\langle \partial_x u^{(1)}(\cdot,t) - \frac{1}{2}\partial_x u^{(2)}(\cdot,t), \mathrm{u}^2(\cdot,t)\right\rangle. \end{aligned}$$

Also, using (2.1) for the second term in (2.68), we have

$$\langle \partial_{xx}\mathrm{u}(\cdot,t), \partial_{xx}\mathrm{u}(\cdot,t) + \mathrm{u}(\cdot,t)\rangle = \|\partial_{xx}\mathrm{u}(\cdot,t)\|^2 - \|\partial_x \mathrm{u}(\cdot,t)\|^2 \ge \frac{(\pi^2 - l^2)}{l^2}\|\partial_x \mathrm{u}(\cdot,t)\|^2 \ge 0.$$

Integrating (2.68), and using (2.63) and (2.64), and the Cauchy-Schwarz inequality we obtain

$$\|\mathrm{u}(\cdot,t)\|^2 \le C\int_0^t \|\mathrm{u}(\cdot,s)\|^2 ds, \quad t \in [0,T],$$

for some positive constant C. Finally, from Gronwall's inequality it follows that $\mathrm{u} \equiv 0$ in Q_T and (2.67) implies that $\mathrm{q} \equiv 0$ in $[0,T]$. This concludes the uniqueness proof. $\square$

2.5 Convergence

The convergence of the approximations obtained in (2.8)–(2.9) to the exact solution q in (2.5), together with a convergence order, are presented in the following theorem.

Theorem 2.5. *Suppose that the assumptions in Theorem 2.3 are satisfied. Then there exists a constant $\tau_0 > 0$ such that for any $\tau \in (0, \tau_0]$,*

$$\|\bar{q}_K - q\|^2_{L^2(0,T)} \le C\tau. \tag{2.69}$$

Proof. Applying similar approaches to those used in (2.59)–(2.62), from (2.5) and (2.43) it follows that

$$\begin{aligned} &\int_0^t |\bar{q}_K(s) - q(s)|^2 ds \\ = &\int_0^t \left| \frac{\langle \rho, \bar{f}_K(\cdot, s)\rangle - \bar{\phi}'_K - \langle \rho'''' + \rho'', \bar{u}_K(\cdot, s-\tau)\rangle - \langle \rho, \bar{u}_K(\cdot, s-\tau)\partial_x \bar{u}_K(\cdot, s-\tau)\rangle}{\bar{\phi}_K(s)} \right. \\ &\left. - \frac{\langle \rho, f(\cdot, s)\rangle - \phi' - \langle \rho'''' + \rho'', u(\cdot, s)\rangle - \langle \rho, u(\cdot, s)\partial_x u(\cdot, s)\rangle}{\phi(s)} \right|^2 ds \\ &\le C\left(\int_0^t \|\bar{u}_K(\cdot, s) - u(\cdot, s)\|^2 ds + \tau \right). \end{aligned} \tag{2.70}$$

Subtracting the variational formulation (2.6) from the discrete form (2.44), letting, for fixed $t \in [0, T]$, $\chi(\cdot) = U_K(\cdot, t) - u(\cdot, t)$, and integrating the result over $[0, t]$, we obtain

$$\begin{aligned} &\frac{1}{2}\|U_K - u\|^2 + \int_0^t \|\partial_{xx} U_K - \partial_{xx} u\|^2 ds - \int_0^t \|\partial_x U_K - \partial_x u\|^2 ds \\ &+ \int_0^t q\|U_K - u\|^2 ds = \int_0^t (q - \bar{q}_K)\langle \bar{u}_K, U_K - u\rangle\, ds - \int_0^t q\langle \bar{u}_K - U_K, U_K - u\rangle\, ds \\ &+ \int_0^t \langle \bar{f}_K - f, U_K - u\rangle\, ds - \int_0^t \langle \partial_{xx}\bar{u}_K - \partial_{xx} U_K, (\partial_{xx} U_K - \partial_{xx} u) + (U_K - u)\rangle\, ds \\ &- \int_0^t \langle \bar{u}_K \partial_x \bar{u}_K - u\partial_x u, U_K - u\rangle\, ds =: \sum_{j=1}^5 I_j. \end{aligned} \tag{2.71}$$

For the integrals I_1 to I_5, using Young's inequality and (2.45), (2.53)–(2.56), we have

$$|I_1| \leq \frac{1}{2}\int_0^t |q-\bar{q}_K|^2\|\bar{u}_K\|^2 ds + \frac{1}{2}\int_0^t \|U_K-u\|^2 ds \leq C\int_0^t |q-\bar{q}_K|^2 ds + \frac{1}{2}\int_0^t \|U_K-u\|^2 ds,$$

$$|I_2| \leq \frac{1}{2}\int_0^t |q|^2\|\bar{u}_K-U_K\|^2 ds + \frac{1}{2}\int_0^t \|U_K-u\|^2 ds \leq C\left(\tau^2+\int_0^t \|U_K-u\|^2 ds\right),$$

$$|I_3| \leq \frac{1}{2}\int_0^t \|\bar{f}_K-f\|^2 ds + \frac{1}{2}\int_0^t \|U_K-u\|^2 ds \leq C\left(\tau+\int_0^t \|U_K-u\|^2 ds\right),$$

$$\begin{aligned}|I_4| &\leq \frac{3}{2}\int_0^1 \|\partial_{xx}\bar{u}_K-\partial_{xx}U_K\|^2 ds + \frac{1}{4}\int_0^t \|\partial_{xx}U_K-\partial_{xx}u\|^2 ds + \frac{1}{2}\int_0^t \|U_K-u\|^2 ds \\ &\leq C\left(\tau+\int_0^t \|U_K-u\|^2 ds\right) + \frac{1}{4}\int_0^t \|\partial_{xx}U_K-\partial_{xx}u\|^2 ds,\end{aligned}$$

$$\begin{aligned}|I_5| &\leq \left|\int_0^t \langle(\bar{u}_K-U_K)\partial_x\bar{u}_K, U_K-u\rangle\, ds\right| + \left|\int_0^t \langle U_K(\partial_x\bar{u}_K-\partial_xU_K), U_K-u\rangle\, ds\right| \\ &\quad + \left|\int_0^t \langle(U_K-u)\partial_xU_K, U_K-u\rangle\, ds\right| + \left|\int_0^t \langle u(\partial_xU_K-\partial_xu), U_K-u\rangle\, ds\right| \\ &\leq C\left(\tau+\int_0^t \|U_K-u\|^2 ds\right) + \frac{1}{4}\int_0^t \|\partial_xU_K-\partial_xu\|^2 ds.\end{aligned}$$

Consequently, the estimates of the integrals I_1 to I_5, (2.1) and (2.70) lead to

$$\|U_K(\cdot,t)-u(\cdot,t)\|^2 + \int_0^t \|\partial_xU_K-\partial_xu\|^2 ds \leq C\left(\tau+\int_0^t \|U_K-u\|^2 ds\right).$$

Finally, Gronwall's inequality yields that

$$\max_{t\in[0,T]} \|U_K(\cdot,t)-u(\cdot,t)\|^2 \leq C\tau. \tag{2.72}$$

Therefore, the convergence rate estimate (2.69) can be obtained by combining (2.53) and (2.72) with (2.70), and the proof is complete. □

2.6 Continuous dependence upon the data

Let $\{u^{(i)}(x,t), q^{(i)}(t)\}$ for $i=1,2$, be the solutions of the inverse problem (1.6)–(1.9) equipped with the same initial data $g(x)$ and source term $f(x,t)$, but with different integral-type measured data $\phi^{(1)}$ and $\phi^{(2)}$. We can then write the time-discrete system (2.7)–(2.9) as follows:

$$q_0^{(i)} = \frac{\langle\rho, f_0\rangle - (\phi_0^{(i)})' - \langle\rho''''+\rho'', g\rangle - \langle\rho, gg'\rangle}{\phi_0^{(i)}}, \tag{2.73}$$

$$q_k^{(i)} = \frac{\langle\rho, f_k\rangle - (\phi_k^{(i)})' - \left\langle\rho''''+\rho'', u_{k-1}^{(i)}\right\rangle - \left\langle\rho, u_{k-1}^{(i)}(u_{k-1}^{(i)})'\right\rangle}{\phi_k^{(i)}}, \tag{2.74}$$

$$\begin{cases}\delta_t u_k^{(i)} + (u_k^{(i)})'''' + (u_k^{(i)})'' + u_k^{(i)}(u_k^{(i)})' + q_k^{(i)}u_k^{(i)} = f_k, & x\in(0,l), \\ u_k^{(i)}|_{x=0,l} = (u_k^{(i)})'|_{x=0,l} = 0,\end{cases} \tag{2.75}$$

where for $i=1,2$, $u_0^{(i)}(x)=g(x)$, $\phi_k^{(i)} := \phi^{(i)}(t_k)$ and $(\phi_k^{(i)})' := (\phi^{(i)})'(t_k)$ for $k=\overline{0,K}$, and $\delta_t u_k^{(i)} = \frac{u_k^{(i)}-u_{k-1}^{(i)}}{\tau}$ for $k=\overline{1,K}$.

Remark 2.3. *Under the hypotheses of Lemma 2.2, then similar methods applied in Lemmas 2.2 and 2.3 yield that, for $i = 1, 2$, there exist two positive constants $C = C(T, g, f, \rho, \phi_-, \phi^+)$ and τ_0 such that for any $\tau \in (0, \tau_0]$, the pair $(u_k^{(i)}, q_k^{(i)}) \in H_0^2(0, l) \times \mathbb{R}_+$, for $k = \overline{1, K}$, solves the system (2.74)–(2.75) uniquely, and the following estimates hold for $j = \overline{1, K}$:*

$$\max_{j=\overline{1,K}} |q_j^{(i)}|^2 \leq C, \tag{2.76}$$

$$\|u_j^{(i)}\|^2 + \sum_{k=1}^{j} \|u_k^{(i)} - u_{k-1}^{(i)}\|^2 + \tau \sum_{k=1}^{j} \|(u_k^{(i)})''\|^2 \leq C, \tag{2.77}$$

$$\tau \sum_{k=1}^{j} \|\delta_t u_k^{(i)}\|^2 + \|(u_j^{(i)})''\|^2 + \sum_{k=1}^{j} \|(u_k^{(i)})'' - (u_{k-1}^{(i)})''\|^2 \leq C. \tag{2.78}$$

Lemma 2.4. *Under the assumptions of Theorem 2.3 there exists a constant $\tau_0 > 0$ such that for any $\tau \in (0, \tau_0]$,*

$$\|\bar{q}_K^{(1)} - \bar{q}_K^{(2)}\|_{L^2(0,T)}^2 \leq C(\|\phi^{(1)} - \phi^{(2)}\|_{H^1(0,T)}^2 + \tau), \tag{2.79}$$

where $\bar{q}_K^{(i)}$ is defined according to (2.43) *(via* (2.36)*).*

Proof. By (2.74), using the estimates (2.24), (2.25) and (2.77), for $t \in [0, T]$, as in the proof of Theorem 2.3, we have

$$\begin{aligned}
&|\bar{q}_K^{(1)}(t) - \bar{q}_K^{(2)}(t)| \\
&\leq C\Big(\left|\bar{\phi}_K^{(2)} - \bar{\phi}_K^{(1)}\right| + \left|\bar{\phi}'^{(1)}_K - \bar{\phi}'^{(2)}_K\right| + \left|(\bar{\phi}_K^{(1)} - \bar{\phi}_K^{(2)})\left\langle \rho'''' + \rho'', \bar{u}_K^{(2)}(\cdot, t - \tau)\right\rangle\right| \\
&\quad + \left|\bar{\phi}_K^{(2)}\langle \rho'''' + \rho'', \bar{w}_K(\cdot, t - \tau)\rangle\right| + \left|(\bar{\phi}_K^{(1)} - \bar{\phi}_K^{(2)})\left\langle \rho, \bar{u}_K^{(2)}(\cdot, t - \tau)\partial_x \bar{u}_K^{(2)}(\cdot, t - \tau)\right\rangle\right| \\
&\quad + \left|\bar{\phi}_K^{(2)}\left\langle \rho, \bar{w}_K(\cdot, t - \tau)\partial_x \bar{u}_K^{(2)}(\cdot, t - \tau)\right\rangle\right| + \left|\bar{\phi}_K^{(2)}\left\langle \rho, \bar{u}_K^{(1)}(\cdot, t - \tau)\partial_x \bar{w}_K(\cdot, t - \tau)\right\rangle\right|\Big) \\
&\leq C(|\bar{\phi}_K^{(1)}(t) - \bar{\phi}_K^{(2)}(t)| + |\bar{\phi}'^{(1)}_K(t) - \bar{\phi}'^{(2)}_K(t)| + \|\bar{w}_K(\cdot, t - \tau)\|),
\end{aligned} \tag{2.80}$$

where $C = C(T, g, f, \rho, \phi_-, \phi^+) > 0$, $w_k := u_k^{(2)} - u_k^{(1)}$, and the functions $\bar{\phi}_K^{(i)}$, $\bar{\phi}'^{(i)}_K$, $\bar{w}_K$ and $\bar{u}_K^{(i)}$ are defined as in Section 2.4. Moreover, $W_K(x, t)$ can also be defined as in (2.41), which together with $\bar{w}_K(x, t)$ they satisfy the problem given by (see (2.44)),

$$\begin{aligned}
&\langle \partial_t W_K, \chi\rangle + \langle \partial_{xx} \bar{w}_K, \chi'' + \chi\rangle + \left\langle \bar{w}_K \partial_x \bar{u}_K^{(2)} + \bar{u}_K^{(1)} \partial_x \bar{w}_K, \chi\right\rangle \\
&\quad + \bar{q}_K^{(1)}\langle \bar{w}_K, \chi\rangle = (\bar{q}_K^{(2)} - \bar{q}_K^{(1)})\left\langle \bar{u}_K^{(2)}, \chi\right\rangle,
\end{aligned} \tag{2.81}$$

for all $t \in (0, T]$ and $\chi \in H_0^2(0, 1)$, and $W_K(x, 0) = 0$. By the estimates (2.23), (2.24) and (2.76), and approaches used in Lemma 2.3, there exists τ_0 such that for any $t \in (0, \tau_0]$ and $j = \overline{1, K}$,

$$\|w_j\|^2 + \sum_{k=1}^{j} \|w_k - w_{k-1}\|^2 + \tau \sum_{k=1}^{j} \|w_k''\|^2 \leq C, \tag{2.82}$$

$$\tau \sum_{k=1}^{j} \|\delta_t w_k\|^2 + \|w_j''\|^2 + \sum_{k=1}^{j} \|w_k'' - w_{k-1}''\|^2 \leq C. \tag{2.83}$$

Consequently, (2.82) implies that

$$\int_0^t \|\bar{w}_K(\cdot, s-\tau)\|^2 ds \le C\left(\int_0^t \|W_K(\cdot, s)\|^2 ds + \int_0^t \|\bar{w}_K(\cdot, s) - W_K(\cdot, s)\|^2 ds \right.$$
$$\left. + \int_0^t \|\bar{w}_K(\cdot, s-\tau) - \bar{w}_K(\cdot, s)\|^2 ds\right)$$
$$\le C\left(\int_0^t \|W_K(\cdot, s)\|^2 ds + 2\tau \sum_{k=1}^K \|w_k - w_{k-1}\|^2\right) \le C\left(\int_0^t \|W_K(\cdot, s)\|^2 ds + \tau\right).$$

Now, using (2.46),

$$\int_0^t |\bar{\phi}_K^{(1)}(s) - \bar{\phi}_K^{(2)}(s)|^2 ds \le C\left(\int_0^t |\phi^{(1)} - \phi^{(2)}|^2 ds + \int_0^t |\bar{\phi}_K^{(1)} - \phi^{(1)} - \bar{\phi}_K^{(2)} + \phi^{(2)}|^2 ds\right)$$
$$\le C\left(\|\phi^{(1)} - \phi^{(2)}\|^2_{L^2(0,T)} + \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \left|\int_s^{t_k} (\phi^{(1)})'(\varsigma) d\varsigma\right|^2 ds + \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \left|\int_s^{t_k} (\phi^{(2)})'(\varsigma) d\varsigma\right|^2 ds\right)$$
$$\le C(\|\phi^{(1)} - \phi^{(2)}\|^2_{L^2(0,T)} + \tau^2),$$

and similarly, using (2.50),

$$\int_0^t |\bar{\phi}'^{(1)}_K - \bar{\phi}'^{(2)}_K|^2 ds \le C(\|(\phi^{(1)})' - (\phi^{(2)})'\|^2_{L^2(0,T)} + \tau).$$

Using the above inequalities, (2.80) becomes

$$\int_0^t |\bar{q}_K^{(1)}(s) - \bar{q}_K^{(2)}(s)|^2 ds \le C\left(\|\phi^{(1)} - \phi^{(2)}\|^2_{H^1(0,T)} + \tau + \int_0^t \|W_K(\cdot, s)\|^2 ds\right). \tag{2.84}$$

Taking $\chi = W_K(\cdot, t)$ for $t \in (0, T]$ in (2.81), and integrating over $(0, t)$, we have

$$\frac{1}{2}\|W_K(\cdot, t)\|^2 + \int_0^t (\|\partial_{xx} W_K\|^2 - \|\partial_x W_K\|^2) ds + \int_0^t \bar{q}_K^{(2)} \|W_K\|^2 ds$$
$$= -\int_0^t \langle \partial_{xx}\bar{w}_K - \partial_{xx} W_K, \partial_{xx} W_K + W_K\rangle ds - \int_0^t (\bar{q}_K^{(1)} - \bar{q}_K^{(2)}) \langle \bar{w}_K, W_K\rangle ds$$
$$- \int_0^t \bar{q}_K^{(2)} \langle \bar{w}_K - W_K, W_K\rangle ds - \int_0^t (\bar{q}_K^{(2)} - \bar{q}_K^{(1)}) \left\langle \bar{u}_K^{(2)}, W_K\right\rangle ds$$
$$- \int_0^t \left\langle \bar{w}_K \partial_x \bar{u}_K^{(2)} + \bar{u}_K^{(1)} \partial_x \bar{w}_K, W_K\right\rangle ds = \sum_{j=1}^5 I_j. \tag{2.85}$$

Now for the integrals I_1 to I_5, using Young's inequality and (2.82), (2.83) and (2.84), we obtain

$$
\begin{aligned}
|I_1| \leq \frac{3}{2}\int_0^t \|\partial_{xx}\bar{w}_K - \partial_{xx}W_K\|^2 ds + \frac{1}{4}\int_0^t \|\partial_{xx}W_K\|^2 ds + \frac{1}{2}\int_0^t \|W_K\|^2 ds \\
\leq C\left(\tau + \int_0^t \|W_K\|^2 ds\right) + \frac{1}{4}\int_0^t \|\partial_{xx}W_K\|^2 ds, \\
|I_2| \leq \frac{1}{2}\int_0^t |\bar{q}_K^{(1)} - \bar{q}_K^{(2)}|^2 \|\bar{w}_K\|^2 ds + \frac{1}{2}\int_0^t \|W_K\|^2 ds \\
\leq C\left(\|\phi^{(1)} - \phi^{(2)}\|^2_{H^1(0,T)} + \tau + \int_0^t \|W_K\|^2 ds\right), \\
|I_3| \leq \frac{1}{2}\int_0^t |\bar{q}_K^{(2)}(s)| \left(\|\bar{w}_K - W_K\|^2 + \|W_K\|^2\right) ds \leq C\left(\tau + \int_0^t \|W_K\|^2\right) ds, \\
|I_4| \leq \frac{1}{2}\int_0^t |\bar{q}_K^{(1)} - \bar{q}_K^{(2)}|^2 \left(\|\bar{u}_K^{(2)}\|^2 + \|W_K\|^2\right) ds \\
\leq C\left(\|\phi^{(1)} - \phi^{(2)}\|^2_{H^1(0,T)} + \tau + \int_0^t \|W_K\|^2 ds\right), \\
|I_5| \leq C\left(\tau + \int_0^t \|W_K\|^2\right) ds,
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
-I_5 = \left\langle \partial_x \bar{u}_K^{(2)} W_K, W_K \right\rangle + \left\langle \partial_x \bar{u}_K^{(2)} (\bar{w}_K - W_K), W_K \right\rangle \\
+ \left\langle \bar{u}_K^{(1)} \partial_x(\bar{w}_K - W_K), W_K \right\rangle + \left\langle \bar{u}_K^{(1)} \partial_x W_K, W_K \right\rangle,
\end{aligned}
$$

$$
\left\langle \bar{u}_K^{(1)} \partial_x W_K, W_K \right\rangle = -\frac{1}{2}\left\langle \partial_x \bar{u}_K^{(1)}, W_K^2 \right\rangle, \quad \|\partial_x(\bar{w}_K - W_K)\|^2 = -\left\langle \bar{w}_K - W_K, \partial_{xx}(\bar{w}_K - W_K).\right\rangle
$$

Finally, employing (2.1), we get

$$
\|W_K(\cdot,t)\|^2 + \int_0^t \|\partial_x W_K(\cdot,s)\|^2 ds \leq C\left(\|\phi^{(1)} - \phi^{(2)}\|^2_{H^1(0,T)} + \tau + \int_0^t \|W_K(\cdot,s)\|^2 ds\right).
$$

Then, $\|W_K(\cdot,t)\|^2 \leq C(\|\phi^{(1)} - \phi^{(2)}\|^2_{H^1(0,T)} + \tau)$ due to Gronwall's inequality, which implies that the estimate (2.79) holds by applying (2.84) again. $\square$

Applying the convergence rate estimate (2.69) of Theorem 2.5, it is easy to see that

$$
\|q^{(i)} - \bar{q}_K^{(i)}\|^2_{L^2(0,T)} \leq C\tau, \quad i = 1, 2.
$$

Consequently, this estimate along with (2.79) and the triangle inequality yield that

$$
\begin{aligned}
\|q^{(1)} - q^{(2)}\|^2_{L^2(0,T)} \leq & \|q^{(1)} - \bar{q}_K^{(1)}\|^2_{L^2(0,T)} + \|q^{(2)} - \bar{q}_K^{(2)}\|^2_{L^2(0,T)} + \|\bar{q}_K^{(1)} - \bar{q}_K^{(2)}\|^2_{L^2(0,T)} \\
\leq & C(\|\phi^{(1)} - \phi^{(2)}\|^2_{H^1(0,T)} + \tau).
\end{aligned}
$$

Since such inequality holds for any $\tau \in (0, \tau_0]$, we therefore have

$$
\|q^{(1)} - q^{(2)}\|_{L^2(0,T)} \leq C\|\phi^{(1)} - \phi^{(2)}\|_{H^1(0,T)}, \tag{2.86}
$$

which implies the following stability result.

Theorem 2.6. *Let the assumptions of Theorem 2.3 be fulfilled. Then the solution q of the inverse problem* (1.6)–(1.9) *depends continuously upon the measured data ϕ in the norms given by expression* (2.86).

3 Tikhonov regularization method

Throughout this section, we assume that $g \in H_0^2(0,l)$, $f \in L^2(Q_T)$, $\rho \in L^2(0,l)$ and $\phi \in H^1(0,T)$. Now, for $q \in L^\infty(0,T)$, from Theorem 2.2 we have that the direct problem (1.6)–(1.8) has a unique weak solution $u \in \mathcal{V}$. Since equation (1.6) can be written as $u_t = f - u_{xxxx} - u_{xx} - uu_x - qu$, this yields that

$$\|u_t\|_{L^2(Q_T)} \leq C \left(\|u\|_{L^2(0,T;H^4(0,l))} + \|f\|_{L^2(Q_T)} \right). \tag{3.1}$$

Thus, we obtain that u also belongs to $H^1(0,T;L^2(0,l))$.

In practical applications, the non-local observation ϕ in (1.9) usually contains noise, i.e., ϕ^ϵ, which is assumed to be in $L^2(0,T)$ and satisfies

$$\|\phi - \phi^\epsilon\|_{L^2(0,T)} \leq \epsilon, \tag{3.2}$$

where $\epsilon \geq 0$ is the amount of noise. We thus reformulate the original inverse problem (1.6)–(1.9) to determine $q(t)$ as follows: find $q \in \mathcal{A}$ such that

$$\mathbb{F}[q](t) := \langle \rho, u(\cdot, t; q) \rangle = \phi^\epsilon(t), \quad \forall t \in (0,T), \tag{3.3}$$

where $u(x,t;q)$ (or $u(q)$) indicates the weak solution of (1.6)–(1.8) which belongs to $\mathcal{V} \cap H^1(0,T;L^2(0,l))$, and $\mathcal{A} := \{q \in L^2(0,T); 0 < q_- \leq q(t) \leq q^+ < \infty \text{ a.e. } (0,T)\}$ denotes the admissible set for the unkown $q(t)$ with two given positive constants q_- and q^+.

We remark that the stability estimate (2.86) expresses the continuous dependence of the solution $q \in L^2(0,T)$ upon the data $\phi \in H^1(0,T)$. However, in real world, ϕ is contaminated with random noise and measured as the non-smooth perturbation $\phi^\epsilon \in L^2(0,T)$ satisfying (3.2). So, in general, $\phi^\epsilon \notin H^1(0,T)$, hence one cannot invoke the stability estimate (2.86). In fact, the following example shows that the inverse problem is ill-posed if the data (1.9) is not in $H^1(0,T)$.

Example of instability. Consider the following example that satisfies the inverse problem (1.6)-(1.9):

$$l = 1, \quad q_n(t) = n, \quad g(x) = x^2(1-x)^2, \quad \rho(x) = x^2(1-x)^2,$$
$$f_n(x,t) = 2x^3(2x-1)(x-1)^3 e^{-2nt} + 24e^{-nt}, \quad u_n(x,t) = e^{-nt}x^2(1-x)^2, \quad \phi_n(t) = e^{-nt}/630,$$

for $n \in \mathbb{N}^*$. Then, it can be seen that f_n, u_n and ϕ_n tend to zero in L^2, as $n \to \infty$, but the solution $q_n(t)$ tends to ∞, as $n \to \infty$. The reason for why the $H^1(0,T)$-stability estimate (2.86) cannot be applied is that, $\|\phi_n'\|^2_{L^2(0,T)} = n(1-e^{-2nT})/793800 \to \infty$, as $n \to \infty$.

The above example shows that the inverse problem (1.6)-(1.9) is ill-posed if the input measured data $\phi^\epsilon \in L^2(0,T) \backslash H^1(0,T)$ and therefore regularization needs to be employed in order to obtain a stable solution. In this section, we develop the Tikhonov regularization method which consists of solving the nonlinear operator equation (3.3) by minimizing

$$\mathcal{J}_{\beta(\epsilon)}(q) := \frac{1}{2}\|\mathbb{F}[q] - \phi^\epsilon\|^2_{L^2(0,T)} + \frac{\beta}{2}\|q - q^*\|^2_{L^2(0,T)}, \tag{3.4}$$

where $\beta = \beta(\epsilon) > 0$ is the regularization parameter, which depends on the amount of noise ϵ present in (3.2), and $q^* \in \mathcal{A}$ is an a priori estimate of the unknown q, which plays the role of a good initial guess to the minimizer of (3.4). The notation $q_{\beta(\epsilon)}$ shall be applied to denote the minimizer of (3.4), which approximates the solution of the inverse problem (1.6)–(1.9). In addition, we utilize the notation

$$q^\dagger := \operatorname*{arg\,min}_{q \in \mathcal{A}; \mathbb{F}[q] = \phi} \|q - q^*\|_{L^2(0,T)} \tag{3.5}$$

to denote the q^*-minimizing solution of (3.3) (with $\epsilon = 0$).

3.1 Well-posedness

In this section, the existence, stability and convergence, namely the well-posedness of the minimizer to the Tikhonov functional (3.4) are addressed.

Theorem 3.1. *There exists at least a minimizer $q_{\beta(\epsilon)} \in \mathcal{A}$ to the Tikhonov functional* (3.4).

Proof. Since $\inf_{q\in\mathcal{A}} \mathcal{J}_{\beta(\epsilon)}(q)$ is finite, there exists a minimizing sequence $\{q_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\lim_{n\to\infty} \mathcal{J}_{\beta(\epsilon)}(q_n) = \inf_{q\in\mathcal{A}} \mathcal{J}_{\beta(\epsilon)}(q)$, which implies that the sequence $\{q_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T)$. Hence, a subsequence (denoted by $\{q_n\}_{n\in\mathbb{N}}$ again) of $\{q_n\}_{n\in\mathbb{N}}$ and $\bar{q} \in L^2(0,T)$ exist such that $q_n \rightharpoonup \bar{q}$ in $L^2(0,T)$, as $n \to \infty$. Meanwhile, $\bar{q} \in \mathcal{A}$ due to $\{q_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and the convexity and closedness of $\mathcal{A}$. Then, since the sequence $\{u_n := u(x,t;q_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{V} \cap H^1(0,T;L^2(0,l))$, there exist a subsequence still denoted by $\{u_n\}_{n\in\mathbb{N}}$ and an element $\bar{u} \in \mathcal{V} \cap H^1(0,T;L^2(0,1))$ such that $u_n \rightharpoonup \bar{u}$ in $L^2(0,T;H^4(0,l))$, $u_n \to \bar{u}$ in $L^2(Q_T)$ and $(u_n)_t \rightharpoonup \bar{u}_t$ in $L^2(Q_T)$. The weak solution $u_n = u(q_n)$ of (1.6)–(1.8) with $q = q_n \in \mathcal{A}$ satisfies the identity:

$$\int_{Q_T} [u_t(q_n)\eta + u_{xx}(q_n)\eta_{xx} + u_{xx}(q_n)\eta + u(q_n)u_x(q_n)\eta + q_n u(q_n)\eta]\, dxdt = \int_{Q_T} f\eta dxdt,$$
$$\forall \eta \in L^2(0,T;H_0^2(0,l)).$$

For the last two terms in the left-hand side of the above identity, we have

$$\begin{aligned}
&\left|\int_{Q_T} (u(q_n)-\bar{u})u_x(q_n)\eta dxdt\right| \le \|u(q_n)-\bar{u}\|_{L^2(Q_T)}\|u_x(q_n)\|_{L^\infty(Q_T)}\|\eta\|_{L^2(Q_T)} \to 0,\\
&\left|\int_{Q_T} q_n(u(q_n)-\bar{u})\eta dxdt\right| \le q^+\|u(q_n)-\bar{u}\|_{L^2(Q_T)}\|\eta\|_{L^2(Q_T)} \to 0,\\
&\int_{Q_T} u(q_n)u_x(q_n)\eta dxdt = \int_{Q_T} \bar{u}u_x(q_n)\eta dxdt + \int_{Q_T} (u(q_n)-\bar{u})u_x(q_n)\eta dxdt \to \int_{Q_T} \bar{u}\bar{u}_x\eta dxdt,\\
&\int_{Q_T} q_n u(q_n)\eta dxdt = \int_{Q_T} q_n\bar{u}\eta dxdt + \int_{Q_T} q_n(u(q_n)-\bar{u})\eta dxdt \to \int_{Q_T} \bar{q}\bar{u}\eta dxdt,
\end{aligned}$$

as $n \to \infty$. Consequently, the weak convergence results of $\{q_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ lead to

$$\int_{Q_T} (\bar{u}_t\eta + \bar{u}_{xx}\eta_{xx} + \bar{u}\eta + \bar{u}\bar{u}_x\eta + \bar{q}\bar{u}\eta)dxdt = \int_{Q_T} f\eta dxdt,$$

which implies that $\bar{u} = u(x,t;\bar{q})$. Then applying the weak lower semi-continuity of L^2-norm, we have

$$\begin{aligned}
\mathcal{J}_{\beta(\epsilon)}(\bar{q}) =& \frac{1}{2}\|\mathbb{F}[\bar{q}] - \phi^\epsilon\|^2_{L^2(0,T)} + \frac{\beta}{2}\|\bar{q} - q^*\|^2_{L^2(0,T)}\\
\le& \liminf_{n\to\infty}\left\{\frac{1}{2}\|\mathbb{F}[q_n] - \phi^\delta\|^2_{L^2(0,T)} + \frac{\beta}{2}\|q_n - q^*\|^2_{L^2(0,T)}\right\} \le \mathcal{J}_{\beta(\epsilon)}(q),
\end{aligned}$$

hence $\bar{q} \in \mathcal{A}$ is a minimizer of (3.4). □

The continuous dependence (stability) of the Tikhonov minimizer q^ϵ_β on the measurement ϕ^ϵ, and the convergence of minimizers to a q^*-minimizer solution of (3.3), as ϵ tends to zero, are given in the following two theorems. Both theorems can be proved by employing the arguments in [17, 18] for instance, and the proof procedure is omitted.

Theorem 3.2. *Suppose that $\{\phi_n\}$ is a sequence converging to ϕ^ϵ in $L^2(0,T)$, and $\{q_n\}$ is a sequence of minimizers to the problem:*

$$\min_{q\in\mathcal{A}}\left\{\frac{1}{2}\|\mathbb{F}[q]-\phi_n\|^2_{L^2(0,T)}+\frac{\beta}{2}\|q-q^*\|^2_{L^2(0,T)}\right\}.$$

Then, $\{q_n\}$ has a subsequence which converges to a minimizer of $\mathcal{J}_{\beta(\epsilon)}$ in $L^2(0,T)$.

Theorem 3.3. *Assume that the sequence $\{\epsilon_n\}$ converges to zero and that $\phi^{\epsilon_n}\in L^2(0,T)$ fulfills $\|\phi-\phi^{\epsilon_n}\|_{L^2(0,T)}\leq\epsilon_n$. In addition, suppose that the regularization parameter $\beta_n:=\beta(\epsilon_n)$ is chosen so as to satisfy*

$$\beta_n\to 0,\quad \frac{\epsilon_n^2}{\beta_n}\to 0,\quad \text{as } n\to\infty.$$

Let $\{q_{\beta_n}\}$ be a sequence of minimizers to the problems:

$$\min_{q\in\mathcal{A}}\left\{\frac{1}{2}\|\mathbb{F}[q]-\phi^{\epsilon_n}\|^2_{L2(0,T)}+\frac{\beta_n}{2}\|q-q^*\|^2_{L^2(0,T)}\right\}.$$

Then the sequence $\{q_{\beta_n}\}$ has a subsequence converging in $L^2(0,T)$ to a q^*-minimizer $q^\dagger$ defined in equation (3.5). *Moreover, if the q^*-minimizer $q^\dagger$ is unique, then $q_{\beta_n}\to q^\dagger$ in $L^2(0,T)$, as $n\to\infty$.*

3.2 Convergence rates for Tikhonov regularization

In this section, we establish convergence rate estimates for the minimizer $q_{\beta(\epsilon)}\in\mathcal{A}$ of the Tikhonov functional (3.4). According to the general convergence theory and convergence rate estimates to nonlinear ill-posed problems presented in [9], the minimizer $q_{\beta(\epsilon)}\in\mathcal{A}$ converges to $q^\dagger$, as $\epsilon\to 0$, together with the following convergence rate estimate:

$$\|q_{\beta(\epsilon)}-q^\dagger\|_{L^2(0,T)}=\mathcal{O}(\sqrt{\epsilon}),\tag{3.6}$$

under *a priori* choice of the regularization parameter $\beta\sim\epsilon$, when the following hypotheses on the nonlinear operator equation (3.3) are fulfilled: (a) $\mathbb{F}:\mathcal{A}\to H^1(0,T)$ is weakly (sequentially) closed, i.e., for any sequence $\{q_n\}_{n\in\mathbb{N}}\subset\mathcal{A}$, $\{\mathbb{F}[q_n]\}_{n\in\mathbb{N}}$ converges weakly to $\mathbb{F}[q]$ if $\{q_n\}_{n\in\mathbb{N}}$ converges to $q\in\mathcal{A}$ weakly, and such property of $\mathbb{F}$ can be obtained by applying the arguments in Theorem 3.1; (b) $\mathbb{F}$ is Fréchet differentiable, see Lemma 3.3, and its Fréchet derivative $\mathbb{F}'[q]$ is Lipschitz continuous with a Lipschitz constant $\mathrm{L}>0$; (c) there exists a function $\omega\in L^2(0,T)$ such that the source condition:

$$q^\dagger-q^*=\mathbb{F}'[q^\dagger]^*\omega\tag{3.7}$$

holds, where ω is small enough, in the following sense:

$$\mathrm{L}\|\omega\|_{L^2(0,T)}<1.\tag{3.8}$$

Note that such small enough condition in (c) appears to be extremely restrictive and difficult to verify. In order to avoid the smallness condition, the nonlinear ill-posed problem (3.3) has to be investigated in Banach space settings under certain assumptions, see e.g., [17, 18, 28]. One technique is the so-called *variational source condition* (VSC), and for the Tikhonov functional (3.4) considered in this paper, such inequality takes the following form:

$$\left\langle q^\dagger-q^*,q^\dagger-q\right\rangle_{L^2(0,T)}\leq\gamma_1\|q-q^\dagger\|^2_{L^2(0,T)}+\gamma_2\Psi\left(\|\mathbb{F}[q]-\mathbb{F}[q^\dagger]\|_{H^1(0,T)}\right),\quad\forall q\in\mathcal{A},\tag{3.9}$$

where $\gamma_1 \in [0, 1)$ and $\gamma_2 \geq 0$ are two constants, and $\Psi : (0, \infty) \to (0, \infty)$ is a continuous and strictly increasing function, which satisfies $\lim_{t\to 0} \Psi(t) = 0$. Compared to the general convergence theory in [9], neither the Fréchet differentiability of the operator $\mathbb{F}[q]$ nor the smallness condition (3.16) is required, when the VSC (3.9) is satisfied. For instance, in the inverse radiativity problems for elliptic and parabolic equations investigated in [7], some new VSCs were verified rigorously, from which convergence rates of the Tikhonov regularized solutions were deduced. Moreover, a new convergence theory with a simpler and weaker source condition without requiring the Fréchet differentiability of $\mathbb{F}$, the Lipschitz continuity of $\mathbb{F}'$ or the smallness requirement (3.16) was developed in [10, 15].

In the present work, the VSC (3.9) for the Tikhonov regularization of the inverse problem (1.6)–(1.9) shall be verified, and henceforth the convergence rate estimate of $q_{\beta(\epsilon)}$ to $q^\dagger$ will be obtained under a suitable choice of the regularization parameter $\beta = \beta(\epsilon)$. We first present a stability estimate in Theorem 3.4.

Lemma 3.1. *Assume $f \in L^2(Q_T)$, $g \in H_0^2(0, l)$ and $q \in \mathcal{A}$. Then the solution $u \in \mathcal{V} \cap H^1(0, T; L^2(0, l))$ of the direct problem* (1.6)–(1.8) *satisfies the following estimate:*

$$\|u\|_{H^1(0,T;L^2(0,l))} + \|u\|_{\mathcal{V}} \leq C, \tag{3.10}$$

where C is a positive constant depending on T, q^+, f and g.

Proof. Multiplying (1.6) by u, integrating with respect to x over $(0, 1)$ and using that $\int_0^l u^2 u_x dx = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot, t)\|^2 + \|u_{xx}(\cdot, t)\|^2 - \|u_x(\cdot, t)\|^2 \leq \int_0^l f(x, t)u(x, t)dx, \quad \forall t \in [0, T]. \tag{3.11}$$

Then, using equation (2.1) it yields that

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot, t)\|^2 + \frac{\pi^2 - l^2}{\pi^2}\|u_{xx}(\cdot, t)\|^2 \leq \|f(\cdot, t)\|\|u(\cdot, t)\|, \quad \forall t \in [0, T]. \tag{3.12}$$

Denoting $Q_t := (0, l) \times (0, t)$, from (3.12) it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u(\cdot, t)\|^2 \leq \|f(\cdot, t)\|\|u(\cdot, t)\| \implies \frac{d}{dt}\|u(\cdot, t)\| \leq \|f(\cdot, t)\| \\ \implies &\|u(\cdot, t)\| - \|g\| \leq \int_0^t \|f(\cdot, \tau)\|d\tau \leq \sqrt{t}\|f\|_{L^2(Q_t)} \leq \sqrt{t}\|f\|_{L^2(Q_T)} \\ &\implies \|u(\cdot, t)\| \leq \sqrt{t}\|f\|_{L^2(Q_T)} + \|g\|, \quad \forall t \in [0, T], \end{aligned} \tag{3.13}$$

Squaring (3.13) and integrating with respect to t we obtain

$$\|u\|^2_{L^2(Q_t)} \leq \int_0^t \left(2\tau\|f\|^2_{L^2(Q_T)} + 2\|g\|^2\right) d\tau = t^2\|f\|^2_{L^2(Q_T)} + 2t\|g\|^2. \tag{3.14}$$

Also, integrating (3.12) with respect to t we obtain

$$\begin{aligned} \frac{1}{2}\|u(\cdot, t)\|^2 - \frac{1}{2}\|g\|^2 + \frac{\pi^2 - l^2}{\pi^2}\|u_{xx}(\cdot, t)\|^2_{L^2(Q_t)} &\leq \int_0^t \|f(\cdot, \tau)\|\|u(\cdot, \tau)\|d\tau \\ &\leq \frac{1}{2}\|f\|^2_{L^2(Q_t)} + \frac{1}{2}\|u\|^2_{L^2(Q_t)}. \end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15) yields

$$\begin{aligned}\|u_{xx}\|^2_{L^2(Q_t)} &\le \frac{\pi^2(t^2+1)}{2(\pi^2-l^2)}\|f\|^2_{L^2(Q_T)} + \frac{\pi^2(2t+1)}{2(\pi^2-l^2)}\|g\|^2 \\ &= A\left[(t^2+1)\|f\|^2_{L^2(Q_T)} + (2t+1)\|g\|^2\right], \quad \forall t \in [0,T], \end{aligned} \tag{3.16}$$

where $A := \frac{\pi^2}{2(\pi^2-l^2)}$. In addition, using the inequality

$$u_x^2(x,t) \le l\|u_{xx}(\cdot,t)\|^2, \tag{3.17}$$

from (3.16), we also have that

$$\|u_x\|^2_{L^2(0,t;L^\infty(0,l))} \le Al\left[(t^2+1)\|f\|^2_{L^2(Q_T)} + (2t+1)\|g\|^2\right], \quad \forall t \in [0,T]. \tag{3.18}$$

Multiplying (1.6) by u_{xxxx} and integrating the result with respect to x, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_{xx}(\cdot,t)\|^2 + \|u_{xxxx}(\cdot,t)\|^2 = \int_0^l f u_{xxxx}dx - \int_0^l u_{xx}u_{xxxx}dx \\ - \int_0^l u u_x u_{xxxx} - q(t)\int_0^l u u_{xxxx}dx,\end{aligned}$$

and applying Young's inequality,

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}(\cdot,t)\|^2 \le \|f(\cdot,t)\|^2 + \|u_{xx}(\cdot,t)\|^2 + \int_0^l u^2u_x^2dx + (q^+)^2\|u(\cdot,t)\|^2. \tag{3.19}$$

Note that

$$\int_0^t\int_0^l u^2u_x^2dxds \le \|u_x\|^2_{L^2(0,t;L^\infty(0,l))}\|u\|^2_{L^\infty(0,t;L^2(0,l))}, \quad \forall t \in [0,T].$$

Then integrating (3.19) with respect to t and using (3.13) and (3.18) imply

$$\begin{aligned}\|u_{xx}(\cdot,t)\|^2 \le \|g''\|^2 + 2\Big(\|f\|^2_{L^2(Q_T)} + \|u_{xx}\|^2_{L^2(Q_t)} \\ +\|u\|^2_{L^\infty(0,t;L^2(0,1))}\|u_x\|^2_{L^2(0,t;L^\infty(0,1))} + (q^+)^2\|u\|^2_{L^2(Q_t)}\Big) \le 2\|f\|^2_{L^2(Q_T)} + \|g''\|^2 \\ +2A\left[1+2l\left(t\|f\|^2_{L^2(Q_T)} + \|g\|^2\right)\right]\left((t^2+1)\|f\|^2_{L^2(Q_T)} + (2t+1)\|g\|^2\right) \\ +2(q^+)^2\left(t^2\|f\|^2_{L^2(Q_T)} + 2t\|g\|^2\right) \\ \le C\left((\|f\|^2_{L^2(Q_T)} + \|g\|^2)^2 + \|f\|^2_{L^2(Q_T)} + \|g\|^2_{H_0^2(0,1)}\right),\end{aligned} \tag{3.20}$$

where $C > 0$ is a constant depending on q^+, l and T. A similar method yields that

$$\|u_{xxxx}\|^2_{L^2(Q_T)} \le C\left((\|f\|^2_{L^2(Q_T)} + \|g\|^2)^2 + \|f\|^2_{L^2(Q_T)} + \|g\|^2_{H_0^2(0,l)}\right). \tag{3.21}$$

Hence, the estimate (3.10) can be derived from (3.18), the above inequalities and (3.1). $\square$

Lemma 3.2. *Suppose that $f \in L^2(Q_T)$, $g \in H_0^2(0,l)$, $a \in L^\infty(0,T;W^{1,\infty}(0,l))$ and $b \in L^\infty(Q_T)$. Then the problem:*

$$\begin{cases} u_t + u_{xxxx} + u_{xx} + a(x,t)u_x + b(x,t)u = f(x,t), & (x,t) \in Q_T, \\ u(0,t) = u(l,t) = u_x(0,t) = u_x(l,t) = 0, & t \in [0,T], \\ u(x,0) = g(x), & x \in (0,l) \end{cases} \tag{3.22}$$

has a unique weak solution $u \in \mathcal{V} \cap H^1(0,T;L^2(0,l))$, and the following estimate holds:

$$\|u\|_{H^1(0,T;L^2(0,l))} + \|u\|_{\mathcal{V}} \le C\left(\|f\|_{L^2(Q_T)} + \|g\|_{H_0^2(0,l)}\right), \tag{3.23}$$

where C is a positive constant depending on l, T, a and b.

Proof. The unique solvability of the linear problem (3.22) can be obtained by using the arguments in [27]. Multiplying by u and u_{xxxx} the first equation in (3.22), respectively, and integrating with respect to x, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 + \|u_{xx}(\cdot,t)\|^2 - \|u_x(\cdot,t)\|^2 = \int_0^l fudx - \int_0^l au_xudx - \int_0^l bu^2dx\\
=&\int_0^l fudx + \int_0^l \left(\frac{1}{2}a_x - b\right)u^2dx,\\
&\frac{1}{2}\frac{d}{dt}\|u_{xx}(\cdot,t)\|^2 + \|u_{xxxx}(\cdot,t)\|^2\\
=&\int_0^l fu_{xxxx}dx - \int_0^l u_{xx}u_{xxxx}dx - \int_0^l au_xu_{xxxx}dx - \int_0^l buu_{xxxx}dx.
\end{aligned}$$

Then, by Young's inequality (with weights 2 and 1/8) and (2.3), we have

$$\begin{aligned}
&\frac{d}{dt}\|u(\cdot,t)\|^2 \le \|f(\cdot,t)\|^2 + (1+K_1+2K_2)\|u(\cdot,t)\|^2,\\
&\frac{2(\pi^2-l^2)}{\pi^2}\|u_{xx}\|^2_{L^2(Q_T)} \le \|f\|^2_{L^2(Q_T)} + (1+K_1+2K_2)\|u\|^2_{L^2(Q_T)} + \|g\|^2,\\
&\left\{\|u_{xx}(\cdot,t)\|^2, \|u_{xxxx}\|^2_{L^2(Q_T)}\right\}\\
\le& 4\left(\|f\|^2_{L^2(Q_T)} + \|u_{xx}\|^2_{L^2(Q_T)} + K_1^2\|u_x\|^2_{L^2(Q_T)} + K_2^2\|u\|^2_{L^2(Q_T)}\right) + \|g''\|^2,
\end{aligned}$$

where $K_1 := \|a_x\|_{L^\infty(Q_T)}$ and $K_2 := \|b\|_{L^\infty(Q_T)}$. Then, Gronwall's inequality implies that

$$\begin{aligned}
&\|u(\cdot,t)\|^2 \le e^{T(1+K_1+2K_2)}\left(\|f\|^2_{L^2(Q_T)} + \|g\|^2\right),\\
&\|u_{xx}\|^2_{L^2(Q_T)} \le A(Te^{T(1+K_1+2K_2)}+1)\left(\|f\|^2_{L^2(Q_T)} + \|g\|^2\right),\\
&\left\{\|u_{xx}(\cdot,t)\|^2, \|u_{xxxx}\|^2_{L^2(Q_T)}\right\} \le C\left(\|f\|^2_{L^2(Q_T)} + \|g\|^2\right) + \|g''\|^2, \quad \forall t \in [0,T],
\end{aligned}$$

where $C > 0$ depends on l, T, K_1 and K_2. Consequently, the estimate (3.23) can be deduced by the above three inequalities and (3.1). $\square$

Lemma 3.3. *The nonlinear mapping $\mathbb{F} : \mathcal{A} \to H^1(0,T)$, defined by (3.3), is Fréchet differentiable.*

Proof. Taking any $\delta q \in L^2(0,T)$ such that $q + \delta q \in \mathcal{A}$, then the function $\delta u := u(q+\delta q) - u(q)$ solves the problem given by:

$$\begin{cases}
(\delta u)_t + (\delta u)_{xxxx} + (\delta u)_{xx} + u(q)(\delta u)_x + u_x(q+\delta q)\delta u & \\
+(q+\delta q)\delta u = -\delta q u(q), & (x,t) \in Q_T,\\
\delta u(0,t) = \delta u(l,t) = (\delta u)_x(0,t) = (\delta u)_x(l,t) = 0, & t \in [0,T],\\
\delta u(x,0) = 0, & x \in (0,l).
\end{cases} \tag{3.24}$$

Since q, $q+\delta q \in \mathcal{A}$, then $u(q)$, $u(q+\delta q) \in L^\infty(0,T;W^{1,\infty}(0,l))$ by the Sobolev embedding $H_0^2(0,l) \hookrightarrow L^\infty(0,l)$ and Lemma 2.2. Hence, the problem (3.24) has a unique solution $\delta u \in \mathcal{V} \cap H^1(0,T;L^2(0,l))$. By Lemma 2.2, we can deduce that $\|u(q)\|_{\mathcal{V}} \le C$ and $\|u(q+\delta q)\|_{\mathcal{V}} \le C$, which yields that $\|u_x(q)\|_{L^\infty(Q_T)} \le C$ and $\|u_x(q+\delta q)\|_{L^\infty(Q_T)} \le C$, with a positive constant C depending on l, T, q^+, f and g. Hence, by the estimate (3.23) in Lemma 3.2, the solution δu of problem (3.24) satisfies the estimate:

$$\begin{aligned}&\|\delta u\|_{H^1(0,T;L^2(0,l))} + \|\delta u\|_{\mathcal{V}} \le C\|\delta q u(q)\|_{L^2(Q_T)}\\ &\le C\|\delta q\|_{L^2(0,T)}\|u(q)\|_{L^\infty(0,T;L^2(0,l))} \le C\|\delta q\|_{L^2(0,T)},\end{aligned} \tag{3.25}$$

and C is a positive constant independent of δq. Consider now the problem:

$$\begin{cases} v_t + v_{xxxx} + v_{xx} + u(q)v_x + u_x(q)v + qv = -u(q)\delta q, & (x,t) \in Q_T,\\ v(0,t) = v(l,t) = v_x(0,t) = v_x(l,t) = 0, & t \in [0,T],\\ v(x,0) = 0, & x \in (0,l). \end{cases} \tag{3.26}$$

Analogous arguments presented above for the problem (3.24) can be employed for the problem (3.26). Then such problem has a unique solution $v \in \mathcal{V} \cap H^1(0,T;L^2(0,l))$, and

$$\|v\|_{H^1(0,T;L^2(0,l))} + \|v\|_{\mathcal{V}} \le C\|\delta q\|_{L^2(0,T)}. \tag{3.27}$$

Thus, $\langle \rho, v(\delta q)\rangle$ defines a linear bounded operator mapping from $L^2(0,T)$ to $H^1(0,T)$, since

$$\|\langle \rho, v(\delta q)\rangle\|_{H^1(0,T)} \le \int_0^l |\rho(x)|\|v(x,\cdot)\|_{H^1(0,T)}dx \le \|\rho\|\|v\|_{H^1(0,T;L^2(0,l))} \le C\|\delta q\|_{L^2(0,T)}.$$

Now the function $w =: \delta u - v$ satisfies the problem:

$$\begin{cases} w_t + w_{xxxx} + w_{xx} + u(q)w_x + u_x(q+\delta q)w + qw = -(\delta u)_x v - \delta q\delta u, & (x,t) \in Q_T,\\ w(0,t) = w(l,t) = w_x(0,t) = w_x(l,t) = 0, & t \in [0,T],\\ w(x,0) = 0, & x \in (0,l). \end{cases}$$

Clearly, $-(\delta u)_x v - \delta q \delta u \in L^2(Q_T)$ since $\delta q \in L^2(0,T)$, $\delta u \in \mathcal{V}$ and $v \in \mathcal{V}$. Hence, the above problem has a unique solution $w \in \mathcal{V} \cap H^1(0,T;L^2(0,l))$, and using (3.25) and (3.27),

$$\|w\|_{H^1(0,T;L^2(0,l))} + \|w\|_{\mathcal{V}} \le C\|(\delta u)_x v + \delta q\delta u\|_{L^2(Q_T)} \le C\|\delta q\|^2_{L^2(0,T)}.$$

We thus obtain that

$$\begin{aligned}&\|\mathbb{F}[q+\delta q] - \mathbb{F}[q] - \langle \rho, v(\delta q)\rangle\|_{H^1(0,T)} = \left\|\int_0^l \rho(x)w(x,\cdot)dx\right\|_{H^1(0,T)}\\ &\le \int_0^1 |\rho(x)|\|w(x,\cdot)\|_{H^1(0,T)}dx \le \|\rho\|\|w\|_{H^1(0,T;L^2(0,l))} \le C\|\delta q\|^2_{L^2(0,T)},\end{aligned} \tag{3.28}$$

where C is a positive constant depending on T, q^+, f, g and ρ. Consequently,

$$\lim_{\|\delta q\|_{L^2(0,T)} \to 0} \frac{\|\mathbb{F}[q+\delta q] - \mathbb{F}[q] - \langle \rho, v(\cdot,t;\delta q)\rangle\|_{H^1(0,T)}}{\|\delta q\|_{L^2(0,T)}} = 0.$$

Therefore, the nonlinear operator $\mathbb{F}$ is Fréchet differentiable, and $\mathbb{F}'[q](\delta q) := \langle \rho, v(\delta q)\rangle$ is its Fréchet differential, where $v(\delta q)$ solves the problem (3.26). $\square$

In order to establish the stability estimate for the inverse problem (1.6)–(1.9), we consider the problem (3.24) with $\delta q = q^\dagger - q$ for $q \in \mathcal{A}$, namely,

$$\begin{cases} (\delta u)_t + (\delta u)_{xxxx} + (\delta u)_{xx} + u(q)(\delta u)_x + u_x(q^\dagger)\delta u + q\delta u = -u(q^\dagger)\delta q, & (x,t) \in Q_T, \\ \delta u(0,t) = \delta u(l,t) = (\delta u)_x(0,t) = (\delta u)_x(l,t) = 0, & t \in [0,T], \\ \delta u(x,0) = 0, & x \in (0,l), \end{cases} \tag{3.29}$$

to determine the function pair $(\delta u, \delta q)$ from the additional data:

$$\langle \rho, \delta u(\cdot,t)\rangle = \mathbb{F}[q^\dagger](t) - \mathbb{F}[q](t) \tag{3.30}$$

The problem (3.29) and (3.30) to find the unknown pair $(\delta u, \delta q)$ can be regraded as an inverse source problem for a linear fourth-order parabolic equation.

Theorem 3.4. *Suppose that $f \in L^2(Q_T)$, $g \in H_0^2(0,l)$, $\rho \in L^2(0,l)$ and $\phi \in H^1(0,T)$. Assume also that $|\phi(t)| \geq \phi_0 > 0$ for all $t \in [0,T]$, with a positive constant ϕ_0, and f, g satisfy*

$$\frac{3M}{2} \leq \frac{\pi^2(\pi^2 - l^2)}{l^4}, \tag{3.31}$$

where M is defined in (3.34) *below. This inequality is satisfied for the example given in Remark 2.1. Then, the solution $(\delta u, \delta q) \in \mathcal{V} \cap H^1(0,T;L^2(0,l)) \times L^2(0,T)$ to the inverse source problem* (3.29) *and* (3.30) *exists uniquely. Moreover, the following stability estimate holds:*

$$\|\overline{q} - q\|_{L^2(0,T)} \leq C\|\mathbb{F}[q^\dagger] - \mathbb{F}[q]\|_{H^1(0,T)}, \tag{3.32}$$

where $C = C(l,T,q^+,\phi_0,\rho,f,g)$ is a positive constant.

Proof. For any $t \in (0,T]$, the inequalities (3.17) and (3.20) yield that

$$\{|u_x(q)|, |u_x(q^\dagger)|\} \leq M, \tag{3.33}$$

with

$$\begin{aligned} M =& \sqrt{2(q^+)^2T^2 + 2T^2 + 4}\|f\|_{L^2(Q_T)} + \sqrt{4(q^+)^2T + 4T + 2}\|g\| + \|g''\| \\ &+ \sqrt{4T^3 + 6T^2 + 6T + 2}\|f\|^2_{L^2(Q_T)} + \sqrt{6T^2 + 10T + 6}\|g\|^2. \end{aligned} \tag{3.34}$$

Hence, the condition (3.31) implies that

$$\left|\frac{1}{2}u_x(q) - u_x(q^\dagger)\right| \leq \frac{3}{2}M \leq \frac{\pi^2(\pi^2 - l^2)}{l^4}. \tag{3.35}$$

Multiplying (3.29) by ρ, integrating with respect to x and using that $\mathbb{F}[q^\dagger] = \phi$, we have

$$\delta q = -\frac{(\delta\mathbb{F}[q])' + \big\langle \rho, (\delta u)_{xxxx} + (\delta u)_{xx} + u(q)(\delta u)_x + u_x(q^\dagger)\delta u + q\delta u\big\rangle}{\phi}, \tag{3.36}$$

where $\delta u \in \mathcal{V} \cap H^1(0,T;L^2(0,l))$ is the solution of (3.29), by utilizing Lemma 3.2 and q, $q^\dagger \in \mathcal{A}$, $u(q)$, $u(q^\dagger) \in \mathcal{V}$. Such identity yields the operator equation:

$$\delta q = \mathbb{C}[\delta q], \tag{3.37}$$

where the operator $\mathbb{C} : L^2(0,T) \to L^2(0,T)$ is defined by

$$\mathbb{C}[\delta q] := -\frac{(\delta\mathbb{F}[q])' + \big\langle \rho, (\delta u)_{xxxx} + (\delta u)_{xx} + u(q)(\delta u)_x + u_x(q^\dagger)\delta u + q\delta u\big\rangle}{\phi}. \tag{3.38}$$

For any $\delta q^{(1)}, \delta q^{(2)} \in L^2(0,T)$, let $\delta u^{(1)}$ and $\delta u^{(2)}$ denote the corresponding solutions to the problem (3.29), respectively. Then, the function $w := \delta u^{(1)} - \delta u^{(2)}$ solves the problem given by

$$\begin{cases} w_t + w_{xxxx} + w_{xx} + u(q)w_x + u_x(q^\dagger)w + qw = -F(t)u(q^\dagger), & (x,t) \in Q_T, \\ w(0,t) = w(l,t) = w_x(0,t) = w_x(l,t) = 0, & t \in [0,T], \\ w(x,0) = 0, & x \in (0,l), \end{cases} \tag{3.39}$$

where $F(t) := \delta q^{(1)}(t) - \delta q^{(2)}(t)$. Multiplying the first equation in (3.39) by u and integrating with respect to x, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w(\cdot,t)\|^2 + \|w_{xx}(\cdot,t)\|^2 - \|w_x(\cdot,t)\|^2 \\ =& -\int_0^l F(t)u(q^\dagger)w dx + \int_0^l \left(\frac{1}{2}u_x(q) - u_x(q^\dagger) - q(t)\right) w^2 dx. \end{aligned} \tag{3.40}$$

Then, using (2.1), (2.2) and (3.35) in (3.40) yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w(\cdot,t)\|^2 + \left(\frac{\pi^2(\pi^2 - l^2)}{l^4} - \frac{3}{2}M + q_-\right)\|w(\cdot,t)\|^2 \\ &\le -\int_0^l F(t)u(q^\dagger)w dx \le |F(t)|\|u(q^\dagger)(\cdot,t)\|\|w(\cdot,t)\|. \end{aligned} \tag{3.41}$$

For $t \in (0,t^*]$ with $t^* \in (0,1)$, using (3.13) and (3.41) we then obtain that

$$\begin{aligned} &\|w(\cdot,t)\| \le \sqrt{t}\|F\|_{L^2(0,t^*)}\|u(q^\dagger)\|_{L^\infty(0,t^*;L^2(0,l))} \\ &\le \sqrt{t}\|F\|_{L^2(0,t^*)}(\sqrt{t^*}\|f\|_{L^2(Q_T)} + \|g\|) \le C\sqrt{t}\|F\|_{L^2(0,t^*)}, \end{aligned} \tag{3.42}$$

with $C = C(f,g) > 0$. In addition, using (2.3), (3.33), (3.40) and (3.41), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w(\cdot,t)\|^2 + \frac{1}{2}\|w_{xx}(\cdot,t)\|^2 \le& -F(t)\int_0^l u(q^\dagger)w dx + \int_0^l \left(\frac{1}{2}u_x(q) - u_x(q^\dagger)\right) w^2 dx \\ \le& |F(t)|\|u(q^\dagger)(\cdot,t)\|\|w(\cdot,t)\| + \frac{3}{2}M\|w(\cdot,t)\|^2, \end{aligned}$$

which by integration and using (3.13) and (3.42) yield

$$\|w_{xx}\|^2_{L^2(Q_t)} \le Ct\|F\|^2_{L^2(0,t^*)} + \frac{3}{2}Mt^2\|F\|^2_{L^2(0,t^*)} \le Ct\|F\|^2_{L^2(0,t^*)}, \quad t \in (0,t^*]. \tag{3.43}$$

Multiplying the first equation in (3.39) by u_{xxxx}, integrating with respect to x and using Young's inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w_{xx}(\cdot,t)\|^2 + \|w_{xxxx}(\cdot,t)\|^2 \\ =& -F(t)\int_0^l u(q^\dagger)w_{xxxx}dx - \int_0^l w_{xx}w_{xxxx}dx - \int_0^l u(q)w_x w_{xxxx}dx \\ &- \int_0^l (u_x(q^\dagger) + q)ww_{xxxx}dx \le -F(t)\int_0^l u_{xx}(q^\dagger)w_{xx}dx + \|w_{xx}(\cdot,t)\|^2 \\ &+\|u(q)(\cdot,t)\|_{L^\infty(0,l)}\|w_x(\cdot,t)\|^2 + (q^+ + \|u_x(q^\dagger)\|_{L^\infty(0,l)})\|w(\cdot,t)\|^2 + \frac{3}{4}\|w_{xxxx}(\cdot,t)\|^2. \end{aligned}$$

Integrating with respect to t, after some computations/estimations, we get

$$\|w_{xxxx}\|_{L^2(Q_t)}^2 \le Ct\|F\|_{L^2(0,t^*)}^2, \quad \forall t \in (0, t^*], \tag{3.44}$$

where $C = C(q^+, f, g) > 0$. Consequently, there exists a positive constant $c = c(q^+, \phi_0, \rho, f, g)$ such that

$$\begin{aligned}
&\|\mathbb{C}[\delta q^{(1)}] - \mathbb{C}[\delta q^{(2)}]\|_{L^2(0,t)} \\
=&\left(\int_0^t \left|\frac{\langle \rho, w_{xxxx} + w_{xx} + u(q)w_x + u_x(q^\dagger)w + qw\rangle}{\phi}\right|^2 dt\right)^{1/2} \\
\le&\frac{2\|\rho\|}{\phi_0}\left(\|w_{xxxx}\|_{L^2(Q_t)} + \|w_{xx}\|_{L^2(Q_t)} + \|u(q)\|_{L^\infty(Q_t)}\|w_x\|_{L^2(Q_t)} + \|u_x(q^\dagger) + q\|_{L^\infty(Q_t)}\|w\|_{L^2(Q_t)}\right) \\
\le&c\sqrt{t}\|\delta q^{(1)} - \delta q^{(2)}\|_{L^2(0,t^*)}.
\end{aligned}$$

Hence, taking $t^* = \min\left\{1, \frac{1}{4c^2}\right\} > 0$, we obtain that

$$\|\mathbb{C}[\delta q^{(1)}] - \mathbb{C}[\delta q^{(2)}]\|_{L^2(0,t^*)} \le \frac{1}{2}\|\delta q^{(1)} - \delta q^{(2)}\|_{L^2(0,t^*)}, \tag{3.45}$$

which means that $\mathbb{C}$ is a contraction operator on $L^2(0,t^*)$. Therefore, the operator equation (3.37) has a unique fixed point $\delta q \in L^2(0,t^*)$, i.e., the inverse problem (3.29) and (3.30) has a unique solution $\delta q \in L^2(0,t^*)$, and $\delta u \in C(0,t^*; H_0^2(0,l)) \cap L^2(0,t^*; H^4(0,l)) \cap H^1(0,t^*; L^2(0,l))$ due to Lemma 3.2.

Now, for any $\delta q_0 \in L^2(0,t^*)$, the solution $\delta q \in L^2(0,t^*)$ can be approximated successively by $\delta q_{n+1} = \mathbb{C}[\delta q_n]$ for $n \in \mathbb{N}$, and then

$$\begin{aligned}
\|\delta q - \delta q_n\|_{L^2(0,t^*)} = \|\mathbb{C}[\delta q] - \mathbb{C}[\delta q_{n-1}]\|_{L^2(0,t^*)} &\le \frac{1}{2}\|\delta q - \delta q_{n-1}\|_{L^2(0,t^*)} \\
&\le \cdots \le \frac{1}{2^n}\|\delta q - \delta q_0\|_{L^2(0,t^*)},
\end{aligned}$$

which illustrates that $\delta q_n \to \delta q$ in $L^2(0,t^*)$ as $n \to \infty$. Taking $\delta q = 0$ in the problem (3.29), we have $\delta u = 0$ by Lemma 3.2, and

$$\|\mathbb{C}[0]\|_{L^2(0,t^*)} = \left\|\frac{(\delta\mathbb{F}[q])'}{\phi}\right\|_{L^2(0,t^*)} \le \frac{1}{\phi_0}\|\delta\mathbb{F}[q]\|_{H^1(0,t^*)}.$$

Consequently, setting $\delta q_0 = 0$, we obtain that

$$\begin{aligned}
\|\delta q\|_{L^2(0,t^*)} &= \lim_{n\to\infty}\|\delta q_{n+1}\|_{L^2(0,t^*)} = \lim_{n\to\infty}\|\mathbb{C}[\delta q_n]\|_{L^2(0,t^*)} \\
&= \lim_{n\to\infty}\|\mathbb{C}[\delta q_n] - \mathbb{C}[\delta q_{n-1}] + \mathbb{C}[\delta q_{n-1}] - \mathbb{C}[\delta q_{n-2}] + \cdots + \mathbb{C}[\delta q_0]\|_{L^2(0,t^*)} \\
&\le \sum_{n=1}^{\infty}\|\mathbb{C}[\delta q_n] - \mathbb{C}[\delta q_{n-1}]\|_{L^2(0,t^*)} + \|\mathbb{C}[\delta q_0]\|_{L^2(0,t^*)} \\
&\le \sum_{n=0}^{\infty}\frac{1}{2^n}\|\mathbb{C}[0]\|_{L^2(0,t^*)} = 2\|\mathbb{C}[0]\|_{L^2(0,t^*)} \le \frac{2}{\phi_0}\|\delta\mathbb{F}[q]\|_{H^1(0,t^*)},
\end{aligned} \tag{3.46}$$

where in the last identity we have used mathematical induction to derive that

$$\|\mathbb{C}[\delta q_n] - \mathbb{C}[\delta q_{n-1}]\|_{L^2(0,t^*)} \le \frac{1}{2^n}\|\mathbb{C}[0]\|_{L^2(0,t^*)}, \quad \forall n \in \mathbb{N}^*.$$

Finally, for $t \geq t^*$, the inverse problem (3.29) and (3.30) shall be investigated in the interval $(t^*, 2t^*)$ with the initial data $\delta u(x, t^*) \in H_0^2(0, l)$. Thus, the employment of analogous arguments illustrates that the problem (3.29) and (3.30) has a unique solution in $(t^*, 2t^*)$. Then the solution $(\delta u, \delta q) \in \mathcal{V} \cap H^1(0, T; L^2(0, l)) \times L^2(0, T)$ to the inverse source problem (3.29) and (3.30) exists uniquely by repeating the approach a finite number of times $N^* := [T/t^*] + 1$. Meanwhile, the estimate (3.32) can be obtained by summing up N^* times the inequality (3.46). $\square$

Remark 3.1. *From* (3.28), (3.32) *and the Fréchet differentiability of the operator* $\mathbb{F}$, *the so-called* η*-condition (e.g., [28]) of* $\mathbb{F}$ *can also be derived. This states that for any* $q \in \mathcal{A}$ *there exists a positive constant* $\eta = \eta(T, q^+, \phi_0, \rho, f, g)$ *such that*

$$\begin{aligned} \|\mathbb{F}[q] - \mathbb{F}[q^\dagger] - \mathbb{F}'[q^\dagger](q - q^\dagger)\|_{H^1(0,T)} &\leq C\|q - q^\dagger\|_{L^2(0,T)}^2 \\ &\leq 2q^+\sqrt{T}C\|q - q^\dagger\|_{L^2(0,T)} \leq \eta\|\mathbb{F}[q] - \mathbb{F}[q^\dagger]\|_{H^1(0,T)}. \end{aligned} \tag{3.47}$$

Moreover, for any $q \in \mathcal{A}$, *from* (3.9) *and* (3.32) *we have*

$$\begin{aligned} \left\langle q^\dagger - q^*, q^\dagger - q\right\rangle_{L^2(0,T)} &\leq \|q^* - q^\dagger\|_{L^2(0,T)}\|q^\dagger - q\|_{L^2(0,T)} \\ \leq C\|q^* - q^\dagger\|_{L^2(0,T)}\|\mathbb{F}[q] - \mathbb{F}[q^\dagger]\|_{H^1(0,T)} &= \gamma_2\|\mathbb{F}[q] - \mathbb{F}[q^\dagger]\|_{H^1(0,T)}, \end{aligned} \tag{3.48}$$

which implies that the VSC (3.9) *holds by setting* $\gamma_1 = 0$, $\gamma_2 = C\|q^\dagger - q^*\|_{L^2(0,T)} > 0$ *and* $\Psi(t) = t$, *provided the assumptions of Theorem 3.4 are fulfilled. The convergence rate is presented in the following theorem.*

Theorem 3.5. *Under the assumptions of Theorem 3.4, the following convergence rate estimates hold:*

$$\|\mathbb{F}[q_\beta(\epsilon)] - \phi^\epsilon\|_{L^2(0,T)} = \mathcal{O}(\epsilon), \tag{3.49}$$

$$\|q_\beta(\epsilon) - q^\dagger\|_{L^2(0,T)} = \mathcal{O}(\sqrt{\epsilon}), \tag{3.50}$$

for the choice of regularization parameter $\beta \sim \epsilon$.

Proof. Since $q_{\beta(\epsilon)} \in \mathcal{A}$ is the minimizer of $\mathcal{J}_{\beta(\epsilon)}(q)$, we have that $\mathcal{J}_{\beta(\epsilon)}(q_{\beta(\epsilon)}) \leq \mathcal{J}_{\beta(\epsilon)}(q^\dagger)$, which implies that

$$\begin{aligned} &\frac{1}{2}\|\mathbb{F}[q_{\beta(\epsilon)}] - \phi^\epsilon\|_{L^2(0,T)}^2 \\ +\frac{\beta}{2}\|q_{\beta(\epsilon)} - q^*\|_{L^2(0,T)}^2 \leq\; &\frac{1}{2}\|\mathbb{F}[q^\dagger] - \phi^\epsilon\|_{L^2(0,T)}^2 + \frac{\beta}{2}\|q^\dagger - q^*\|_{L^2(0,T)}^2 \\ &\leq \frac{1}{2}\epsilon^2 + \frac{\beta}{2}\|q^\dagger - q^*\|_{L^2(0,T)}^2. \end{aligned} \tag{3.51}$$

This together with (3.5), (3.48) and the triangle inequality imply that

$$\begin{aligned} \frac{1}{2}\|q_{\beta(\epsilon)} - q^\dagger\|_{L^2(0,T)}^2 &= \frac{1}{2}\|q_{\beta(\epsilon)} - q^*\|_{L^2(0,T)}^2 - \frac{1}{2}\|q^\dagger - q^*\|_{L^2(0,T)}^2 \\ &\quad + \left\langle q^\dagger - q^*, q^\dagger - q_{\beta(\epsilon)}\right\rangle_{L^2(0,T)} \\ \leq \frac{1}{2}\|q_{\beta(\epsilon)} &- q^*\|_{L^2(0,T)}^2 - \frac{1}{2}\|q^\dagger - q^*\|_{L^2(0,T)}^2 + \gamma_2\|\mathbb{F}[q_{\beta(\epsilon)}] - \mathbb{F}[q^\dagger]\|_{H^1(0,T)} \\ &\leq \frac{1}{2\beta}\left(\epsilon^2 - \|\mathbb{F}[q_{\beta(\epsilon)}] - \phi^\epsilon\|_{L^2(0,T)}^2\right) + \gamma_2\|\mathbb{F}[q_{\beta(\epsilon)}] - \mathbb{F}[q^\dagger]\|_{H^1(0,T)} \\ &\leq \frac{1}{2\beta}\left(2\epsilon^2 - \|\mathbb{F}[q_{\beta(\epsilon)}] - \mathbb{F}[q^\dagger]\|_{H^1(0,T)}^2\right) + \gamma_2\|\mathbb{F}[q_{\beta(\epsilon)}] - \mathbb{F}[q^\dagger]\|_{H^1(0,T)}. \end{aligned} \tag{3.52}$$

Taking $\beta = \mathcal{O}(\epsilon)$, the above inequalities yield

$$\|\mathbb{F}[q_{\beta(\epsilon)}] - \mathbb{F}[q^\dagger]\|^2_{H^1(0,T)} \le 2\epsilon^2 + C\epsilon\|\mathbb{F}[q_{\beta(\epsilon)}] - \mathbb{F}[q^\dagger]\|_{H^1(0,T)},$$

from which

$$\|\mathbb{F}[q_{\beta(\epsilon)}] - \mathbb{F}[q^\dagger]\|^2_{H^1(0,T)} \le C\epsilon^2.$$

Hence the convergence rate estimate (3.49) can be derived from

$$\|\mathbb{F}[q_{\beta(\epsilon)}] - \phi^\epsilon\|^2_{L^2(0,T)} \le \|\mathbb{F}[q_{\beta(\epsilon)}] - \mathbb{F}[q^\dagger]\|^2_{L^2(0,T)} + \epsilon^2 \le C\epsilon^2.$$

Meanwhile, from (3.51) and the second row in equation (3.52) we have

$$\begin{aligned}\|q_{\beta(\epsilon)} - q^\dagger\|^2_{L^2(0,T)} \le& \|q_{\beta(\epsilon)} - q^*\|^2_{L^2(0,T)} - \|q^\dagger - q^*\|^2_{L^2(0,T)} + 2\gamma_2\|\mathbb{F}[q_{\beta(\epsilon)}] - \mathbb{F}[q^\dagger]\|_{H^1(0,T)}\\ \le& \frac{\epsilon^2}{\beta} + 2\gamma_2\|\mathbb{F}[q_{\beta(\epsilon)}] - \mathbb{F}[q^\dagger]\|_{H^1(0,T)} \le C\epsilon,\end{aligned}$$

which concludes the estimate (3.50). □

Remark 3.2. *The regularization parameter $\beta = \beta(\epsilon)$ chosen above is an a priori choice, which is independent of the additional observation ϕ^ϵ. The so-called Morozov's discrepancy principle (e.g. [1, 2]), which is an a posteriori parameter choice of $\beta = \beta(\epsilon, \phi^\epsilon)$ depending both on the noise level ϵ and the measured data ϕ^ϵ, is defined as follows: for $1 \le \tau_1 \le \tau_2$, choose $\beta = \beta(\epsilon, \phi^\epsilon)$ for some $q_{\beta(\epsilon)} \in \mathcal{A}$, such that $\tau_1\epsilon \le \|\mathbb{F}[q_{\beta(\epsilon)}] - \phi^\epsilon\|_{L^2(0,T)} \le \tau_2\epsilon$. Then, the convergence estimates* (3.49) *and* (3.50) *can be obtained by using the arguments in [2] together with the η-condition* (3.47) *and the VSC* (3.48)*.*

4 Conclusions

The determination of the unknown time-dependent coefficient $q(t)$, which physically represents the effective ion collision efficiency, in the nonlinear equation (1.6) subjected to the initial and boundary conditions (1.7) and (1.8), respectively, from the integral-type measurement (1.9) has been accomplished. Based on a time-discrete scheme, the original inverse problem has been written in a discrete form. Then, the existence and uniqueness of the solution to the inverse problem has been proved rigorously by employing the Rothe's method and the Schaefer's fixed point theorem. The convergence rate of the approximations generated by the time-discrezation and a stability result has also been derived. For measured noisy data, the Tikhonov regularization method has been applied, and the well-posedness of the minimizer have been illustrated. Then, by investigating the solvability of an auxiliary inverse source problem for a fourth-order parabolic problem, a stability estimate for the unknown coefficient has been deduced. We henceforth have obtained the convergence rates of the Tikhonov minimizer by using the VSC generated by the stability estimate. Future work will be concerned with the computational implementation for obtaining a stable numerical solution to the investigated nonlinear inverse coefficient problem.

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References

[1] S. W. Anzengruber and R. Ramlau. Morozov's discrepancy principle and Tikhonov-type functionals with nonlinear operators. *Inverse Problems*, 26:025001, 2010.

[2] S. W. Anzengruber and R. Ramlau. Convergence rates for Morozov's discrepancy principle using variational inequalities. *Inverse Problems*, 27:105007, 2011.

[3] I. T. Ardekani. *Bayesian Damage Identification from Elastostatic Data*. PhD Thesis, Mathematics, University of Auckland, New Zealand, 2020.

[4] L. Baudouin, E. Cerpa, E. Crépeau, and A. Mercado. Lipschitz stability in an inverse problem for the Kuramoto–Sivashinsky equation. *Applicable Analysis*, 92(10):2084–2102, 2013.

[5] A. L. Bukhgeım and M. V. Klibanov. Uniqueness in the large of a class of multidimensional inverse problems. *Dokl. Akad. Nauk SSSR*, 260(2):269–272, 1981.

[6] K. Cao, D. Lesnic, and M. I. Ismailov. Determination of the time-dependent thermal grooving coefficient. *Journal of Applied Mathematics and Computing*, 65:199–221, 2021.

[7] D. H. Chen, D. Jiang, and J. Zou. Convergence rates of Tikhonov regularizations for elliptic and parabolic inverse radiativity problems. *Inverse Problems*, 36:075001, 2020.

[8] C. M. Elliott. The Cahn-Hilliard model for the kinetics of phase separation. In: *Mathematical Models for Phase Cange Problems*, International Series in Numerical Mathematics 88. Birkhauser, Basel, pp.35-73, 1989.

[9] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems*, volume 375. Springer Science & Business Media, 1996.

[10] H. W. Engl and J. Zou. A new approach to convergence rate analysis of Tikhonov regularization for parameter identification in heat conduction. *Inverse Problems*, 16(6):1907–1923, 2000.

[11] D. Gajardo, A. Mercado, and J. C. Munoz. Identification of the anti-diffusion coefficient for the linear Kuramoto-Sivashinsky equation. *Journal of Mathematical Analysis and Applications*, 495:124747, 2021.

[12] P. Gao. A new global Carleman estimate for the one-dimensional Kuramoto–Sivashinsky equation and applications to exact controllability to the trajectories and an inverse problem. *Nonlinear Analysis: Theory, Methods and Applications*, 117:133–147, 2015.

[13] M. Grimmonprez, L. Marin, and K. V. Bockstal. The reconstruction of a solely time-dependent load in a simply supported non-homogeneous Euler-Bernoulli beam. *Applied Mathematical Modelling*, 79:914–933, 2020.

[14] A. Hasanov and O. Baysal. Identification of a temporal load in a cantilever beam from measured boundary bending moment. *Inverse Problems*, 35:105005, 2019.

[15] D. N. Hào and T. N. T. Quyen. Convergence rates for Tikhonov regularization of coefficient identification problems in Laplace-type equations. *Inverse Problems*, 26(12), Article 125014, 2010.

[16] A. Hazanee, D. Lesnic, M. I. Ismailov and N. B. Kerimov. Inverse time-dependent source problems for the heat equation with nonlocal boundary conditions. *Applied Mathematics and Computation*, 346:800-815, 2019.

[17] B. Hofmann, B. Kaltenbacher, C. Poschl, and O. Scherzer. A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators. *Inverse Problems*, 23(3):987–1010, 2007.

[18] K. Ito and B. Jin. *Inverse Problems: Tikhonov Theory and Algorithms.* World Scientific Publishing Co., New Jersey, 2015.

[19] M. Ivanchov *Inverse Problems for Equations of Parabolic Type.* Mathematical Studies, Monograph Series, Vol.10, VNTL Publishers, Lviv, 2003.

[20] J. Kacur. *Method of Rothe in Evolution Equations.* volume 80, Teubner Texte zur Mathematik, Leipzig, 1985.

[21] V. L. Kamynin and T. I. Bukharova. Inverse problems of determination of the right-hand side term in the degenerate higher-order parabolic equation on a plane. In: *International Conference on Numerical Analysis and Its Applications*, pages 391–397, 2016.

[22] V. L. Kamynin and T. I. Bukharova. On inverse problem of determination the right-hand side term in higher order degenerate parabolic equation with integral observation in time. *IOP Conference series: Journal of Physics: Conference Series*, 937:012018, 2017.

[23] B. B. King, O. Stein and M. Winkler. A fourth-order parabolic equation modeling epitaxial thin film growth. *Journal of Mathematical Analysis and Applications*, 286:459–490, 2003.

[24] Y. Kuramoto and T. Tsuzuki. On the formation of dissipative structures in reaction–diffusion systems. *Progress of Theoretical Physics*, 54(3):687–699, 1975.

[25] R. E. Laquey, S. M. Mahajan, P. H. Rutherford and W. M. Tang. Nonlinear saturation of the trapped-ion mode. *Physical Review Letters*, 34(7):391–394, 1975.

[26] W. J. Liu and M. Krstic. Stability enhancement by boundary control in the Kuramoto–Sivashinsky equation. *Nonlinear Analysis: Theory, Methods and Applications*, 43(4):485–507, 2001.

[27] P. G. Meléndez. Lipschitz stability in an inverse problem for the main coefficient of a Kuramoto–Sivashinsky type equation. *Journal of Mathematical Analysis and Applications*, 408(1):275–290, 2013.

[28] T. Schuster, B. Kaltenbacher, B. Hofmann and K. S. Kazimierski. *Regularization Methods in Banach Spaces.* Boston: De Gruyter, 2012.

[29] G. I. Sivashinsky. Nonlinear analysis of hydrodynamic instability in laminar flames—I. derivation of basic equations. *Acta Astronautica*, 4:1177–1206, 1977.

[30] Y. Shang and S. Li. Conditional stability in a backward Cahn-Hilliard equation via a Carleman estimate. *Journal of Inverse and Ill-Posed Problems*, 29(2):159–171, 2021.

[31] D. Trucu, D. B. Ingham and D. Lesnic. Inverse time-dependent perfusion coefficient identification. *Journal of Physics: Conference Series: 4th AIP International Conference and the 1st Congress of the IPIA*, 124, Article 012050, 2008.