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Algorithms and complexity for geodetic sets on partial grids [*]

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Abstract

A set S of vertices of a graph G is a *geodetic set* if every vertex of G lies in a shortest path between some pair of vertices of S. The Minimum Geodetic Set (MGS) problem is to find a geodetic set with minimum cardinality of a given graph. A *grid embedding* of a graph is a set of points in two dimensions with integer coordinates such that each point in the set represents a vertex of the graph and, for each edge, the points corresponding to its endpoints are at Euclidean distance 1. A graph is a *partial grid* if it has a grid embedding. In this paper, we first prove that Minimum Geodetic Set remains NP-hard even for subcubic partial grids of arbitrary girth. This jointly strengthens three existing hardness results: for bipartite graphs (Dourado et al., Discrete. Math, 2010), subcubic graphs (Bueno et al., Inf. Process. Lett., 2018) [4], and planar graphs (Chakraborty et al., CALDAM, 2020).

The *area* of an internal face is the number of integer points lying on the boundary or interior of the face. A graph is a *solid grid* if it has a grid embedding such that all interior faces have area exactly four. To complement the above hardness result, we design a linear-time algorithm for Minimum Geodetic Set on solid grids, improving on a 3-approximation algorithm by Chakraborty et al. (CALDAM, 2020).

Our results hold for Edge Geodetic Set as well. A set S of vertices of a graph G is a *geodetic set* if every edge of G lies in a shortest path between some pair of vertices of S. The Minimum Edge Geodetic Set (MEGS) problem is to find an edge geodetic set with minimum cardinality of a given graph. As corollaries, we obtain that MEGS remains NP-hard on partial grids and is linear-time solvable on solid grids.

Keywords: Geodetic set, partial grids, solid grids, NP-hardness, linear time algorithm.

1 Introduction

A simple undirected graph G has vertex set $V(G)$ and edge set $E(G)$. For two vertices $u, v \in V(G)$, let $I(u, v)$ denote the set of all vertices in G that lie in some shortest path between u and v. For a subset S of vertices of a graph G, let $I(S) = \bigcup_{u,v \in S} I(u, v)$. We say that $T \subseteq V(G)$ is *covered* by S if $T \subseteq I(S)$. A set of vertices S is a *geodetic set* if $V(G)$ is covered by S. The *geodetic number*, denoted as $gn(G)$, is the minimum integer k such that G has a geodetic set of cardinality k. Given a graph G, the Minimum Geodetic Set

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(MGS) problem, introduced in [19], is to compute a geodetic set of G with minimum cardinality. In this paper, we study the computational complexity of Minimum Geodetic Set in subclasses of planar graphs. Minimum Geodetic Set is a natural graph covering problem that falls in the class of problems dealing with the important geometric notion of *convexity*: see [17, 27] for some general discussion of graph convexities. The setting of Minimum Geodetic Set is quite natural, and it can be applied to facility location problems such as the optimal determination of bus routes in a public transport network etc. [7, 33]. See also [16] for further applications. The aim of this paper is to study Minimum Geodetic Set on *partial grids*. A *grid embedding* of a graph is a set of points in two dimensions with integer coordinates such that each point in the set represents a vertex of the graph and, for each edge, the points corresponding to its endpoints are at Euclidean distance 1. A graph is a *partial grid* if it has a grid embedding. The *area* of an internal face is the number of integer points on the boundary or interior of the face. A graph is a *solid grid* if it has a grid embedding such that all interior faces have area exactly four.

The algorithmic complexity of Minimum Geodetic Set has been studied intensively. Minimum Geodetic Set is known to be NP-hard on *chordal* graphs [14], and *(chordal) bipartite* graphs [11–13], *subcubic* graphs [4], *planar* graphs [7], *co-bipartite* graphs [16]. From the perspective of parameterized complexity, Minimum Geodetic Set remains W[1]-hard for the parameters solution size, *feedback vertex set number*, and *pathwidth*, all three parameters combined [22]. From the viewpoint of approximability, Minimum Geodetic Set remains LOG-$\mathcal{APX}$ hard even on subcubic bipartite graphs [10]. In this paper, we jointly strengthen three existing NP-hardness results: for bipartite graphs [11], subcubic graphs [4], and planar graphs [7], by proving the following theorem.

Theorem 1. Minimum Geodetic Set *is NP-hard for subcubic partial grids of girth at least g, for any fixed integer $g \geq 4$.*

Note that partial grids are subclasses of many other important graph classes such as *disk* graphs, *rectangle intersection* graphs, etc. [9, 30]. Hence, our result implies that Minimum Geodetic Set remains NP-hard on the aforementioned graph classes.

On the positive side, polynomial-time algorithms to solve Minimum Geodetic Set are known for *cographs* [11], *split* graphs [11,14] and more generally *well-partitioned chordal* graphs [1], *ptolemaic* graphs [17] and more generally *distance-hereditary* graphs [21], *block-cactus* graphs [16], *outerplanar* graphs [25] and *proper interval* graphs [16]. The problem is fixed-parameter-tractable for parameters *tree-depth*, *modular-width* and *feedback edge set number* [22]. Chakraborty et al. [7] gave a 3-approximation algorithm for Minimum Geodetic Set on solid grids. We improve this as follows.

Theorem 2. *There is a linear-time algorithm for* Minimum Geodetic Set *on solid grids.*

We note that researchers have designed polynomial-time algorithms for various problems on solid grids [18, 23, 31].

Next, we establish that our results also hold for a related problem, Minimum Edge Geodetic Set. A set S of vertices of a graph is an *edge geodetic set* if every edge lies in some shortest path between some vertices in S. Note that an edge geodetic set is also a geodetic set. Given a graph G, the problem Minimum Edge Geodetic Set, introduced independently in [3] and [29], asks to compute an edge geodetic set with minimum cardinality. The computational complexity of Minimum Edge Geodetic Set has been heavily studied [2, 8, 10, 28]. In particular, we have the following results.

Corollary 1. Minimum Edge Geodetic Set *is NP-hard for subcubic partial grids of girth at least g, for any fixed integer $g \geq 4$.*

Corollary 2. *There is a linear-time algorithm for* Minimum Edge Geodetic Set *on solid grids.*

Organisation: In Section 2, we prove Theorem 1 and Corollary 1. In Section 3, we prove Theorem 2 and Corollary 2.

2 Hardness for partial grids

We now prove Theorem 1. Let $\mathcal{PG}(3, g)$ denote the class of subcubic partial grids of girth at least g. Given a graph G, a subset $S \subseteq V(G)$ is a *vertex cover* of G if every edge in $E(G)$ has at least one end-vertex in S. The problem Minimum Vertex Cover is to compute a vertex cover of an input graph G with minimum cardinality. To prove Theorem 1, we reduce the NP-complete Minimum Vertex Cover on cubic planar graphs [26] to Minimum Geodetic Set on graphs in $\mathcal{PG}(3, g)$.

2.1 Important lemma

We use a result of Valiant [32] which was stated by Clark et al. [9] in the following form.

Theorem 3 ([9,32])**.** *A planar graph G with maximum degree 4 can be embedded in the plane using $O(|V(G)|)$ area in such a way that its vertices are at integer coordinates and its edges are drawn so that they are made up of line segments of the form $x = i$ or $y = j$, for integers i and j.*

In the above theorem, "area" refers to the number of integer points of the grid that are covered by the edges or the internal faces of the embedding of the graph. A graph H is an *equal subdivision* of a graph G if there exists an integer ℓ such that H can be obtained by replacing each edge of G by a path of length ℓ. Below we use Theorem 3 to prove the following.

Lemma 4. *Let G be a planar graph with maximum degree 4. Then there exists a partial grid graph H, which is an equal subdivision of G and contains at most $O(|V(G)|^3)$ vertices.*

Proof. First, we introduce the following definition. Consider a square S whose sides are of length a units for some even integer a. Observe that given any even integer b that satisfies $a \leq b \leq (a+1)^2 - 1$, it is possible to find a rectilinear path P whose length is b units, P lies entirely inside S and the left and right endpoints of P are the left bottom and right bottom corners of S, respectively. We call P a "zigzag expansion of the bottom segment" of S. See Figure 1 for an example.

Now we describe the proof of the lemma. Using Theorem 3, we can get an embedding $\mathcal{R}$ such that its vertices are at integer coordinates and its edges are drawn so that they are made up of line segments of the form $x = i$ or $y = j$. Furthermore, the dimensions of the grid, as well as the lengths of the edges, are $O(|V(G)|)$. So, let $k_1 n$ be the maximum length of an edge in the grid embedding obtained through Theorem 3, where $n = |V(G)|$. (If k_1 is not an integer, consider $k_1 = \lceil k_1 \rceil$.) The minimum length of an edge is at least 1. We scale up the embedding, both, horizontally and vertically by a factor of $8nk_1$. Now the longest edge has a length $\ell = 8k_1^2 n^2$, and the shortest edge has a length of at least $8k_1 n$. Note that due to the scaling factor, all edges have even lengths. Furthermore, due to the scaling, any previously maximal

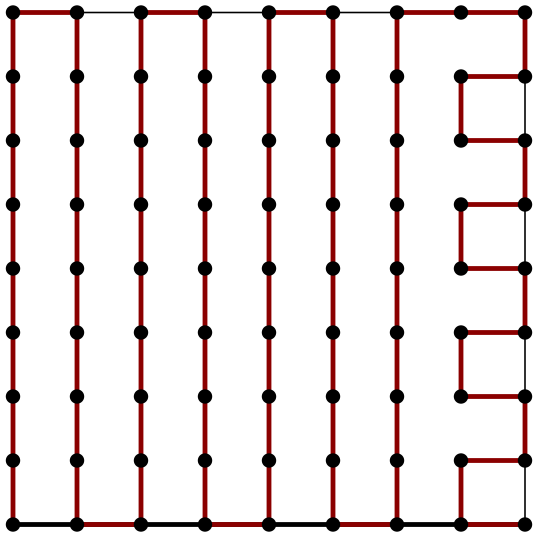

Figure 1: Here $a = 8$. We display a zigzag expansion of the bottom segment of length 80 in color red. We display the lattice points for convenience.

horizontal or vertical unit segment (say s) of an edge has expanded to a length of at least $8k_1n$ and has squares (S_1, S_2) of dimension $8k_1n$ on both sides. Consider the square $S \in \{S_1, S_2\}$ that lies completely above (resp., on the right) of a horizontal (resp., vertical) segment s. (The fact that S lies completely above (resp., on the right) of s can be assumed without loss of generality). We say that S is the *expansion square* of the segment s. Moreover, observe that although a horizontal (resp., vertical) segment s has a unique expansion square, a square S can simultaneously be an expansion square of some horizontal segment s as well as some vertical segment s'.

Now, consider an expansion square S of a horizontal segment s. Inside S, we define two disjoint *zigzag squares* Z_h and Z_v in the following manner. For ease of presentation, let the coordinates of S be as follows: the bottom left point is $(0, 0)$, and the top right point is $(8k_1n, 8k_1n)$. Now, the zigzag square Z_h has the bottom left point at $(4k_1n, 0)$ and top right point at $(8k_1n - 2, 4k_1n - 2)$, and the zigzag square Z_v has the bottom left point at $(0, 4k_1n)$ and top right point at $(4k_1n - 2, 8k_1n - 2)$. See Figure 2 for an illustration. Observe that both squares Z_h and Z_v have a side of length $4k_1n - 2$ each. Therefore, in the zigzag square Z_h (resp., Z_v), we can find a zigzag expansion of the bottom segment of Z_h (resp., left segment of Z_v) of any even length b, where b satisfies $4k_1n - 2 \leq b \leq (4k_1n - 1)^2 - 1 = 16k_1^2n^2 - 8k_1n$ (here, $a = 4k_1n - 2$). Moreover, it is easy to observe that the squares Z_h and Z_v are indeed disjoint.

Since the maximum difference between the lengths of the longest and shortest edges is $8k_1^2n^2 - 8k_1n$, observe that, a zigzag expansion of a horizontal (resp., vertical) segment s in the zigzag square Z_h (resp. Z_v) increases the length of the edge to the maximum length ℓ. Moreover, since each segment uses its unique and disjoint zigzag square, these zigzag paths have no intersection. After equalizing the length of all edges in the embedding $\mathcal{R}$, an equal subdivision H of G can be found by replacing each edge of G by a path of length ℓ. Clearly, H is a partial grid. Since the length of each edge is $O(n^2)$, and there are $O(n)$ edges, the total number of vertices in H is $O(|V(G)|^3)$. □

Next, we have the following lemma.

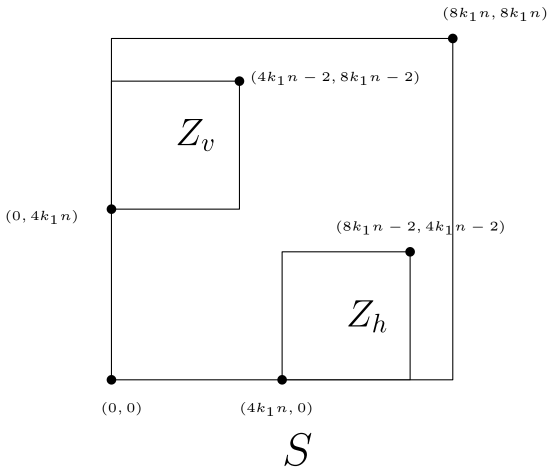

Figure 2: Illustration for zigzag squares.

Lemma 5. *Let G be a graph having an edge geodetic set S. Let H be an equal full subdivision of G. Then S is an edge geodetic set of H.*

Proof. Let uv be an edge of H. Observe that uv must belong to some path that was introduced to replace an edge e of G. Let $u_e, v_e \in S$ be such that e belongs to a shortest path P between u_e and v_e. Let P' be the path obtained by replacing each edge of P with a path having ℓ edges. Observe that P' is a shortest path between u_e and v_e in H. Hence, uv belongs to a shortest path between u_e and v_e in H. Thus, S is an edge geodetic set of H. □

We note that the converse of Lemma 5 may not be true in general. See Figure 3 for an example.

2.2 Overview of the reduction

Given a cubic planar graph G, first, we construct a planar graph $f_1(G)$ having maximum degree at most 6. To construct $f_1(G)$, we essentially replace the vertices of G with a gadget G_v (shown in Figure 4). We show that G has a vertex cover of size k if and only if $f_1(G)$ has an (edge) geodetic set of size $3|V(G)| + k$. Then we construct a partial grid $f_2(G)$ such that the (edge) geodetic numbers of $f_2(G)$ and $f_1(G)$ are equal. To construct $f_2(G)$, first, we apply Lemma 4 on G to get a partial grid H which is an equal subdivision of G. Let us call the vertices of G that are also present in H as the *original vertices.* Then we replace the original vertices of H with the gadget shown in Figure 5, which is motivated by the construction of G_v and exhibits the same properties needed to conclude the result. Finally, if the girth of the resulting graph is less than the value of g in the statement of Theorem 1, then we consider an equal subdivision of $f_2(G)$ that has girth at least g. We show using Lemma 5 and some other observations that the (edge) geodetic set of the constructed graph remains unchanged, and conclude.

2.3 Reduction and proofs for $f_1(G)$: The first step

From a cubic planar graph G with a given planar embedding $\mathcal{R}$, we construct the graph $f_1(G)$ as follows. Each vertex v of G will be replaced by a *vertex gadget* G_v which is shown in Figure 4. The edges outside of

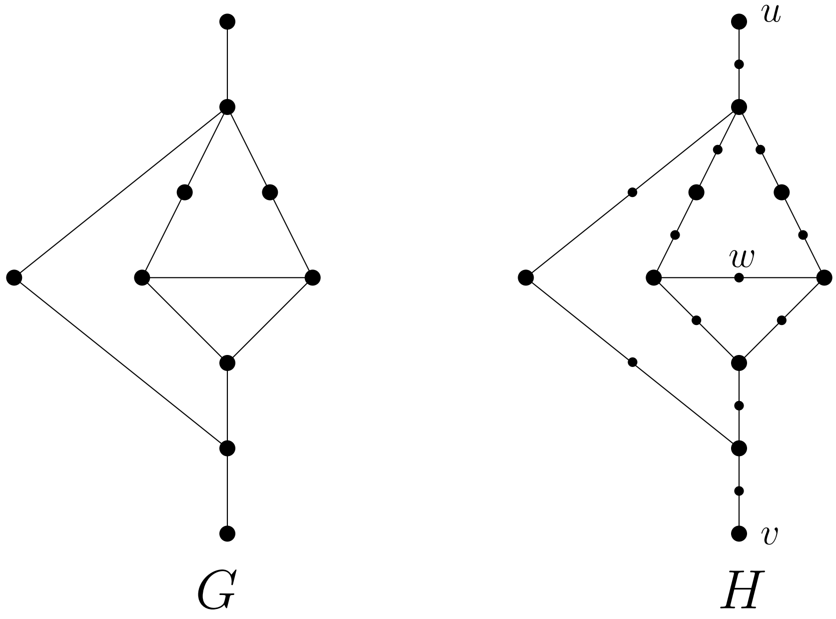

Figure 3: Here, H is an equal subdivision of G. Observe that while the edge geodetic number of G is at least 4, $\{u, v, w\}$ is an edge geodetic set of H of size 3.

the vertex-gadgets will depend on $\mathcal{R}$. We assume that the edges incident with each vertex v of G are labeled e_i^v with $0 \leq i < 3$, in such a way that the numbering increases counterclockwise around v with respect to the embedding (thus the edge vw will have two labels: e_i^v and e_j^w). Consider two vertices v and w that are adjacent in G, and let e_i^v and e_j^w be the two labels of edge vw in G. Add the edges $\{t_i^v, t_j^w\}$, $\{y_{i,i+1}^v, y_{j-1,j}^w\}$ and $\{y_{i-1,i}^v, y_{j,j+1}^w\}$ (See Figure 4). All indices are taken modulo 3. There are no other edges in $f_1(G)$. Observe that $f_1(G)$ has maximum degree at most 6 and girth 4.

Lemma 6. *The graph G has a vertex cover D of size k if and only if $f_1(G)$ has a geodetic set of size $3|V(G)| + k$.*

Proof. We construct a geodetic set S of $f_1(G)$ of size $3|V(G)| + k$ as follows. For each vertex v in G, we add the three vertices $z_{0,1}^v, z_{1,2}^v, z_{2,0}^v$ of G_v to S. If v is in D, then we also add the vertex c^v to S. Let $Q = \{(0,1), (1,2), (2,0)\}$.

Let us show that S is indeed a geodetic set. First, we observe that in any vertex gadget G_v that is part of $f_1(G)$, the unique shortest path between two distinct vertices $z_{i,j}^v$, $z_{i',j'}^v$ has length 4 and goes through vertices $y_{i,j}^v$, t_k^v and $y_{i',j'}^v$ (where $\{k\} = \{i,j\} \cap \{i',j'\}$). Thus, it only remains to show that the vertices $\{c^v, x_{i,j}^v\}$ ($(i,j) \in Q$) belong to some shortest path of vertices of S. Assume that v is a vertex of G in D. The shortest paths between c^v and $z_{i,j}^v$ have length 3 and one of them goes through vertex $x_{i,j}^v$. Thus, all vertices of G_v belong to some shortest path between vertices of S. Now, consider a vertex $w \notin D$ of G. Since G is a cubic planar graph, all three neighbours of w, say, w_1, w_2, w_3 must lie in D. Let $C = \{c^{w_1}, c^{w_2}, c^{w_3}\}$ and $Z = \{z_{0,1}^w, z_{1,2}^w, z_{2,0}^w\}$. Observe that all vertices of G_w lie in the set $I(C \cup Z)$. Therefore, S is a geodetic set.

For the converse, assume we have a geodetic set S' of $f_1(G)$ of size $3|V(G)| + k$. We will show that G has a vertex cover of size k. First of all, observe that all the $3|V(G)|$ vertices of type $z_{i,j}^v$ are necessarily in S',

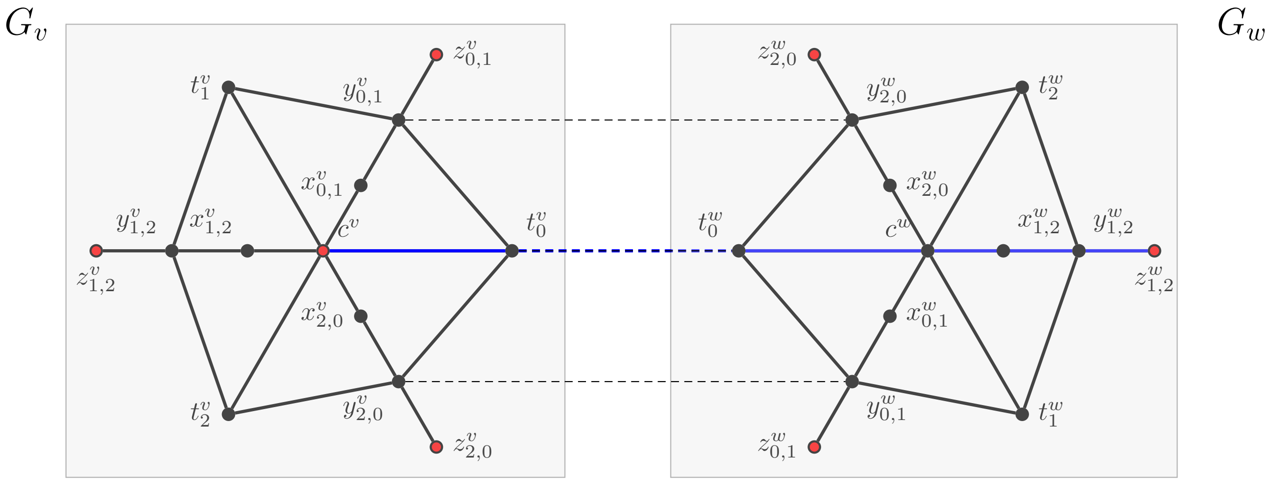

Figure 4: Illustration of vertex gadgets in the reduction in the proof of Theorem 1. The dashed lines indicate edges between two vertex gadgets. The red vertices illustrate the vertices of a solution. The blue path illustrates an example of a shortest path between c^v and $z_{1,2}^w$.

since they have degree 1. As observed earlier, the shortest paths between those vertices already go through all vertices of type t_i^v and $y_{i,j}^v$. However, no other vertex lies on a shortest path between two such vertices: these shortest paths always go through the boundary 6-cycle of the vertex-gadgets. Let S_0' be the set of the remaining k vertices of S'. These vertices are there to cover the vertices of types c^v and $x_{i,j}^v$. We construct a subset D' of $V(G)$ as follows: D' contains those vertices v of G whose vertex-gadget G_v contains a vertex of S_0'. We claim that D' is a vertex cover of G. Suppose by contradiction that there is an edge vw of G such that neither G_v nor G_w contains any vertex of S_0'. Without loss of generality assume that e_0^v and e_0^w are the two labels of the edge vw. We prove the following claim:

Claim 7. *Let P be the vertices of degree one of $f_1(G)$, $A_1 = \{z_{1,2}^v, z_{2,0}^v, z_{0,1}^v\}$ and $A_2 = \{x_{1,2}^v, x_{0,1}^v, x_{2,0}^v\}$. Then $I(P) \cap A_2 = \emptyset$. Moreover, for any vertex u' of G that is not a neighbour of v in G, and any vertex $a' \in G_{u'}$, $I(a' \cup A_1) \cap A_2 = \emptyset$.*

Proof. Clearly no vertex of A_2 lies in any shortest path between vertices of A_1. Now let w' be a neighbour of v in G and consider the set $A_3 = \{z_{1,2}^{w'}, z_{2,0}^{w'}, z_{0,1}^{w'}\}$. For any vertex $b \in A_1$ and a vertex $b' \in A_3$, observe that all shortest paths between b and b' always pass through the vertices of the boundary 6-cycle of G_v. Therefore such shortest paths will not contain any vertex of A_2 and hence $I(A_1 \cup A_3) \cap A_2 = \emptyset$. Now consider u' which is not a neighbour of v in G, and any vertex $a' \in G_{u'}$. Then, any shortest path between a' and a vertex $a'' \in A_1$, will contain a vertex of the form $y_{i,j}^v$ and therefore, will not contain any vertex of A_2. This concludes the proof. □

Let a and b be the neighbours of v different from w in G. Then, $I(P \cup V(G_a) \cup V(G_b) \cup V(G_c) \cup V(G_d))$ where c and d are neighbours of w in G does not contain $x_{1,2}^v$. Due to the above claim, for any vertex u' that is not a neighbour of v in G, and any vertex $a' \in G_{u'}$, the shortest path between a' and any vertex of $\{z_{1,2}^v, z_{2,0}^v, z_{0,1}^v\}$ does not contain any vertex of the form $x_{i,j}^v$. Hence, $x_{1,2}^v$ is not covered by any pair of vertices in S', a contradiction. Therefore, D' is a vertex cover of G. □

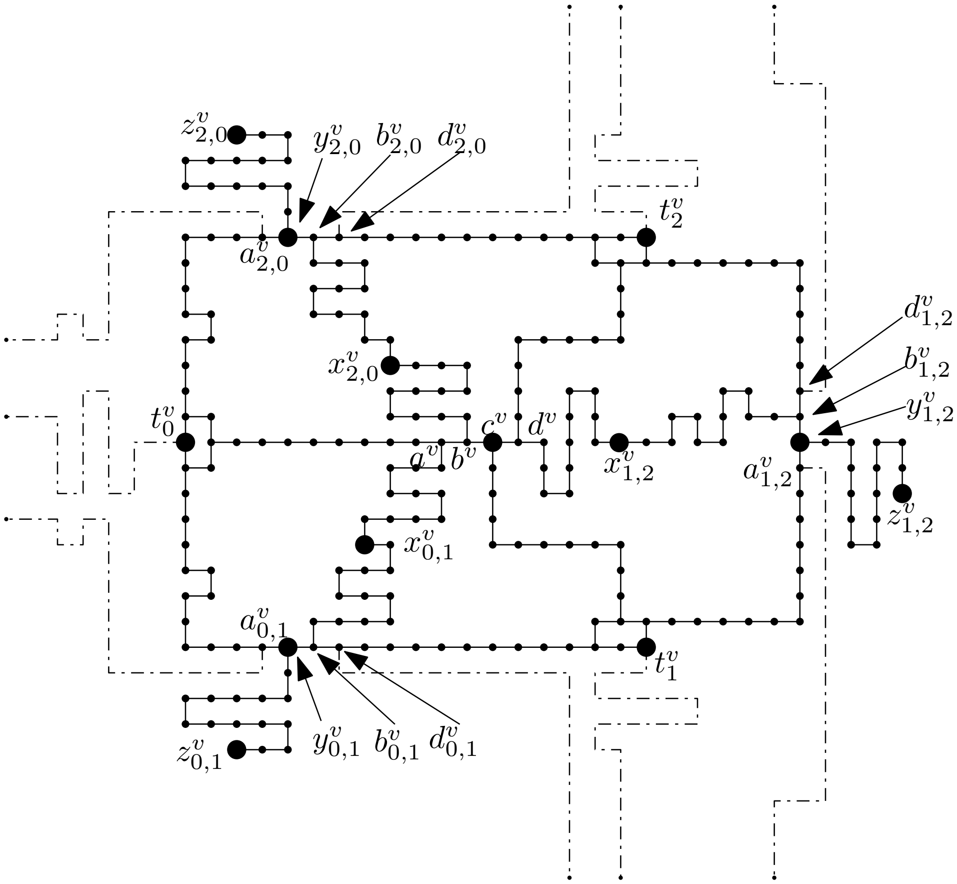

Figure 5: Construction of $f_2(G)$.

2.4 Reduction and proof for $f_2(G)$: The final step

In this section, we shall construct a partial grid $f_2(G)$ from a cubic planar graph and prove Theorem 1. We shall use the gadget shown in Figure 5. As before, for each vertex v of G we shall consider a new copy of the graph shown in Figure 5 and call it G'_v. Roughly speaking, G'_v is designed such that it is a subcubic partial grid graph, and exhibits properties similar to that of G_v. First, we consider the following observations concerning our gadget G'_v that will be useful for us.

Observation 8. *Consider the graph concerning the gadget G'_v and consider two vertices p and q. Then all of the following hold (and are evident from the construction):*

1. *Let $p = c^v$. Then, if $q \in \{y^v_{0,1}, y^v_{2,0}, y^v_{1,2}\}$, then $d(p, q) = 26$. And, if $q \in \{x^v_{0,1}, x^v_{2,0}, x^v_{1,2}\}$, then $d(p, q) = 13$. And, if $q \in \{t^v_0, t^v_1, t^v_2\}$, then $d(p, q) = 14$. Furthermore, each $x^v_{i,j} \in I(p, y^v_{i,j})$.*
2. *If $p, q \in \{y^v_{0,1}, y^v_{2,0}, y^v_{1,2}\}$, then $d(p, q) = 28$. Similarly, if $p, q \in \{t^v_0, t^v_1, t^v_2\}$, then $d(p, q) = 28$.*
3. *Let $p = t^v_0$. Then, if $q \in \{y^v_{0,1}, y^v_{2,0}\}$, then $d(p, q) = 14$. If $q = y^v_{1,2}$, then $d(p, q) = 40$.*

Next, we have the following observation that is crucial for our proof.

Observation 9. *Consider the graph G'_v. Then, all the following hold:*

1. *Let $S_1 = \{z^v_{0,1}, z^v_{2,0}, z^v_{1,2}, c^v\}$. Then, S_1 covers every edge of G'_v. Moreover, $S_1 \setminus \{c^v\}$ covers every vertex on the boundary face of G'_v.*

2. *Let $S_2 = \{z^v_{0,1}, z^v_{2,0}, z^v_{1,2}, t^v_0, t^v_1, t^v_2\}$. Then S_2 covers every edge of G'_v.*

3. *Consider the set $S_3 = \{y^v_{0,1}, y^v_{2,0}, y^v_{1,2}, t^v_0, t^v_1, t^v_2, a^v_{0,1}, d^v_{0,1}, a^v_{2,0}, d^v_{2,0}, a^v_{1,2}, d^v_{1,2}\}$. Then, $x^v_{0,1} \notin I(S_3 \setminus \{t^v_2\})$. Similarly, $x^v_{2,0} \notin I(S_3 \setminus \{t^v_1\})$ and $x^v_{1,2} \notin I(S_3 \setminus \{t^v_0\})$.*

Proof. The proofs of items (1) and (2) follow directly from the construction and can be verified easily using Observation 8. To prove (3), we show that only that $x^v_{0,1} \notin I(S_3 \setminus \{t^v_2\})$ since the proof of $x^v_{2,0} \notin I(S_3 \setminus \{t^v_1\})$ and $x^v_{1,2} \notin I(S_3 \setminus \{t^v_0\})$ uses identical arguments. Targeting a contradiction, assume that there are two vertices $p, q \in S_3 \setminus \{t^v_2\}$ such that there is a shortest path between p and q, say P, containing the vertex $x^v_{0,1}$. Since all vertices in $S_3 \setminus \{t^v_2\}$ are on the boundary of G'_v, note that to cover the vertex $x^v_{0,1}$, P must contain the $a^v, b^v_{0,1}$-path containing $x^v_{0,1}$ as a subpath, which has length 23.

Thus, the length of $P \geq \min\{d(p, a^v) + d(q, b^v_{0,1}), d(q, a^v) + d(p, b^v_{0,1})\} + 23$. We will verify that this contradicts that P is an isometric path for every choice of $p, q \in S_3 \setminus \{t^v_2\}$. First, it can be easily verified that $d(p, q) \leq 42$. Now, consider the following choices of p, q:

1. One of p, q (without loss of generality, say p) is from $\{a^v_{2,0}, y^v_{2,0}, d^v_{2,0}, a^v_{1,2}, y^v_{1,2}, d^v_{1,2}\}$: In this case, note that $\min\{d(p, a^v),\ d(p, b^v_{0,1})\} \geq 26$. Therefore, $|P| \geq 26 + 23 = 49$, which contradicts that P is an isometric path (since $d(p, q) \leq 42$).

2. One of p, q is t^v_0: Without loss of generality, let us assume that $p = t^v_0$ and $q \in \{a^v_{0,1}, y^v_{0,1}, d^v_{0,1}, t^v_1\}$. Note that $d(p, q) \leq 28$. Moreover, $\min\{d(p, a^v), d(p, b^v_{0,1})\} \geq 12$, and hence $|P| \geq 35$. This contradicts that P is an isometric path (since $d(p, q) \leq 28$).

3. $p, q \in \{a^v_{0,1}, y^v_{0,1}, d^v_{0,1}, t^v_1\}$: Note that in this case, $d(p, q) \leq 15$, which contradicts that P is an isometric path (since $|P| \geq 23$).

This completes our proof. ☐

Finally, to complete the reduction of $f_2(G)$, first, we apply Lemma 4 on G to get a partial grid H which is an equal subdivision of G. Let us call the vertices of G in H as the *original vertices*. Note that if $uw \in E(G)$, then there exists exactly one path in H with endpoints u and w that does not contain any other original vertices. We call these paths as *external paths*. Moreover, note that each external path has the same length; let that length be ℓ. Then, we replace each original vertex v of H with G'_v to get $f_2(G)$, and for an edge uw of G, we connect G'_u and G'_w with three paths of equal length ℓ in $f_2(G)$, as we did in the case of $f_1(G)$ (where we added three edges). Note that, at this step, a further equal subdivision of H (by a constant) may be necessary to "accommodate" G'_v. Furthermore, note that, due to scaling, for any $uw \in E(G)$, the three "external paths" between the gadgets G'_u and G'_w can be navigated following the "path pattern" of the external path between u and w in H (since any two external paths are "sufficiently" far apart from each other in H other than at the endpoints). Now, for a gadget G_v in $f_1(G)$, if some external edges were having $y^v_{2,0}, t^v_0, y^v_{0,1}$ as endpoints, then the corresponding external paths will have $a^v_{2,0}, t^v_0, a^v_{0,1}$, respectively as endpoints in G'_v in $f_2(G)$ (see Figure 5). Similarly, if the endpoints were $y^v_{2,0}, t^v_2, y^v_{1,2}$ in $f_1(G)$, then they will be $d^v_{2,0}, t^v_2, d^v_{1,2}$, respectively in $f_2(G)$; and if the endpoints were $y^v_{1,2}, t^v_1, y^v_{0,1}$ in $f_1(G)$, then they will be $a^v_{1,2}, t^v_1, d^v_{0,1}$, respectively in $f_2(G)$. This completes our construction.

Completion of the proof of Theorem 1. Clearly, G'_v is a partial grid for each $v \in G$ (Figure 5). Moreover, it is not difficult to verify that $f_2(G)$ is a partial grid that has maximum degree 3. Let D be a vertex cover of G with cardinality k. We construct a set S of cardinality $3|V(G)| + k$ as follows. For each vertex v in G, we add the three vertices with labels $z^v_{i,j}$ $(0 \leq i < j \leq 2)$ to S. If v is in D, we also add vertex c^v to S. From the construction of $f_2(G)$, Observation 9, and using arguments similar to the ones used in the proof of the Lemma 6, G has a vertex cover of size k if and only if $f_2(G)$ has an (edge) geodetic set of size $3|V(G)| + k$.

Now, if the value of g in the statement of Theorem 1 is less than the girth of $f_2(G)$, then from the previous discussions, we have that Minimum Geodetic Set is NP-hard for graphs in $\mathcal{PG}(3, g)$. Otherwise, we replace every edge of $f_2(G)$ with a path of length g. Call this modified graph $f_3(G)$, and observe that $f_3(G) \in \mathcal{PG}(3, g)$. Due to Lemma 5, S is a geodetic set of $f_3(G)$ of cardinality $3|V(G)| + k$.

Now we shall argue the converse. First we have the following observation whose proof would follow a similar line of argumentation to that of Observation 9.

Observation 10. *Consider an equal subdivision of the graph* G'_v. *Consider the set* $W = \{y^v_{0,1}, y^v_{2,0}, y^v_{1,2}, t^v_0, t^v_1, t^v_2, a^v_{0,1}, d^v_{0,1}, a^v_{2,0}, d^v_{2,0}, a^v_{1,2}, d^v_{1,2}\}$. *Then,* $x^v_{0,1} \notin I(W \setminus \{t^v_2\})$. *Similarly,* $x^v_{2,0} \notin I(W \setminus \{t^v_1\})$ *and* $x^v_{1,2} \notin I(W \setminus \{t^v_0\})$.

Let S' be a geodetic set of size $3|V(G)| + k$ of $f_3(G)$. Let $S'_0 \subset S'$ be the set of vertices whose degree is greater than one in $f_3(G)$. Now using the above observation and arguments similar to that of Lemma 6 we can show that G has a vertex cover of size k. This completes the proof. Our proof also implies Corollary 1.

3 A linear-time algorithm for solid grids

We here give a linear-time algorithm for Minimum Geodetic Set on solid grids and prove Theorem 2. In Section 3.1, we define some preliminary notations and state some observations. In Section 3.2, we state Algorithm 1 to solve Minimum Geodetic Set on solid grids. Then, in Section 3.3, we prove that the set returned by our algorithm is indeed a geodetic set (Lemma 23). Afterwards, in Section 3.4, we prove a lower bound on the geodetic number of a solid grid (Lemma 29). Finally, we prove Theorem 2 in Section 3.5. The reader may take the help of Figure 6 to navigate through the proof of Theorem 2.

3.1 Preliminaries

Let G be a solid grid and $\mathcal{R}$ its grid embedding. In this section, whenever we refer to a grid embedding of a solid grid, we refer to a grid embedding whose each interior face has area exactly four. Each vertex of G corresponds to a grid point in $\mathcal{R}$, and each edge of G corresponds to an orthogonal unit segment. For an edge e, $\mathbf{e}$ shall denote the corresponding orthogonal unit segment in $\mathcal{R}$. For a subgraph H, let $\mathcal{R}_H = \bigcup\limits_{e \in E(H)} \mathbf{e}$. Observe that for a cycle C of G, $\mathcal{R}_C$ is a simple, closed, rectilinear curve that divides the plane into two disjoint regions. Let $I(\mathcal{R}_C)$ denote the union of $\mathcal{R}_C$ and "the interior region bounded by $\mathcal{R}_C$". A square whose each side is an orthogonal segment containing exactly two grid points is referred to as a *unit square*. Note that the length of a each side of a unit square is one. For a unit square U whose all four corner points

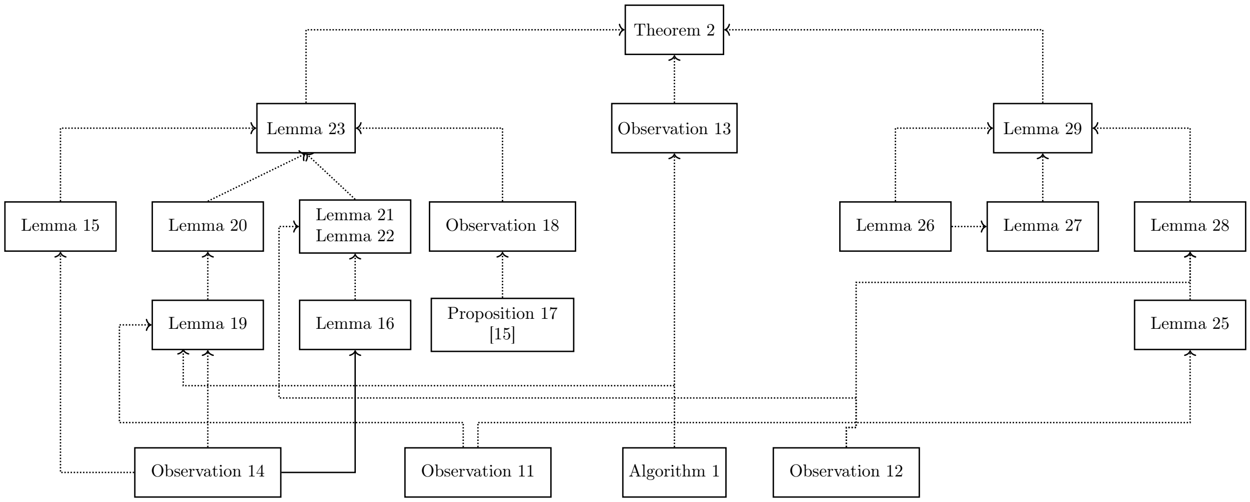

Figure 6: Roadmap of proof of Theorem 2.

are grid points, let $tl(U)$, $tr(U)$, $bl(U)$, $br(U)$ denote the top-left, top-right, bottom-left, and bottom-right corner point of U, respectively. We shall use the following observation which essentially says that there is no "hole" inside the grid embedding of a cycle of a solid grid graph.

Observation 11. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let C be a cycle of G and $\mathcal{R}_C$ be the sub-embedding of C in $\mathcal{R}$. Let U be a unit square whose all four corner points are grid points and $U \subseteq I(\mathcal{R}_C)$. Then, the four corner points $tl(U)$, $tr(U)$, $bl(U)$, $br(U)$ induce a cycle of order four in G.*

Proof. If the four corner points $tl(U)$, $tr(U)$, $bl(U)$, $br(U)$ do not induce a cycle of order four in G, then there would be an interior face in $\mathcal{R}$ whose area would be more than four. This would contradict that $\mathcal{R}$ is a grid embedding of G. ☐

Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. A path P in G is a *vertical path* (resp. *horizontal path*) if x-coordinates (resp. y-coordinates) of all vertices of P are the same. An *isometric path* is a shortest path between its end vertices. We shall also use the following observation.

Observation 12. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let P_1 and P_2 be a vertical path and a horizontal path, respectively. Any path P_3 with $V(P_3) \subseteq V(P_1) \cup V(P_2)$ is an isometric path.*

In the next section, we state our algorithm to compute the minimum geodetic set of a solid grid.

3.2 Algorithm

Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. A vertical (resp. horizontal) path P in G with at least two vertices is a *vertical corner path* (resp. *horizontal corner path*) if (i) no vertex of P is a *cut-vertex*[1], (ii) both end-vertices of P have degree 2 in G, (iii) all other vertices of P have degree 3 in G. A path of G is a *corner path* if it is either a vertical corner path or a horizontal corner path. See Figure 7 for an example. A vertex $v \in V(G)$ is a *corner vertex* if v is an end-vertex of some corner path.

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[1] A vertex of a graph is a *cut vertex* if its removal disconnects the graph.

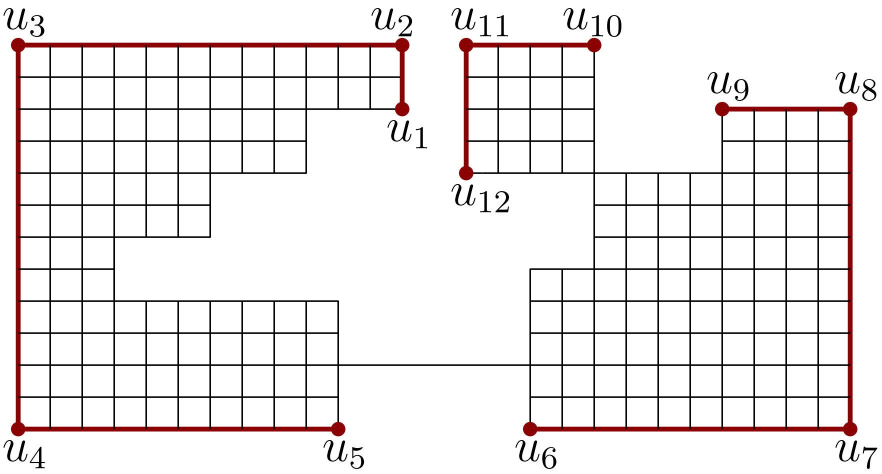

Figure 7: Illustration for corner paths. Here, the horizontal and vertical paths marked in bold red are corner paths. Moreover, $(u_1, u_2, u_3, u_4, u_5)$, (u_6, u_7, u_8, u_9), and (u_{10}, u_{11}, u_{12}) are maximal corner sequences in this solid grid.

Definition 1. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. We say that vertices $u_1, u_2, \ldots, u_k$ form a* corner sequence *if for each $1 \le i \le k-1$, there is a corner path with u_i and u_{i+1} as end-vertices. A corner sequence is* maximal *if it is not a subsequence of any other corner sequence. Two corner sequences are* distinct *from each other if neither is a permutation of the other. For a corner sequence S, let $|S|$ denote the length of S.*

```
Algorithm 1: Pseudocode to find a minimum geodetic set of a solid grid graph
Inputs : A solid grid graph $G$ and its embedding $\mathcal{R}$
Output: A minimum geodetic set of $G$
Let $V_1$ denote the set of all vertices of degree 1;
Find all corner paths and corner vertices.
Let $\mathcal{S}$ be the set of all distinct maximal corner sequences.
Let $D = V_1$.
for each distinct maximal sequence $S \in \mathcal{S}$ do
    Let $S = (u_1, u_2, \ldots, u_k)$.
    Let $f(S) = \{u_2, u_4 \ldots, u_{k-k'}\}$ where $k' = 0$ if $k$ is even and $k' = 1$, otherwise.
    $D = D \cup f(S)$.
return $D$.
```

Figure 8 show an example of a solid grid G and the set of vertices returned by Algorithm 1. Since the embedding is given as part of the input, the running time is $O(n)$, where n is part of the input. The next observation follows immediately from the above algorithm.

Observation 13. *Let D be the set returned by Algorithm 1. Let t be the number of vertices of G with degree one and $\mathcal{S}$ set of all distinct maximal corner sequences. Then, $|D| \le t + \sum_{S \in \mathcal{S}} \lfloor |S|/2 \rfloor$.*

In the next section, we show that the set returned by Algorithm 1 is indeed a geodetic set of the input solid grid.

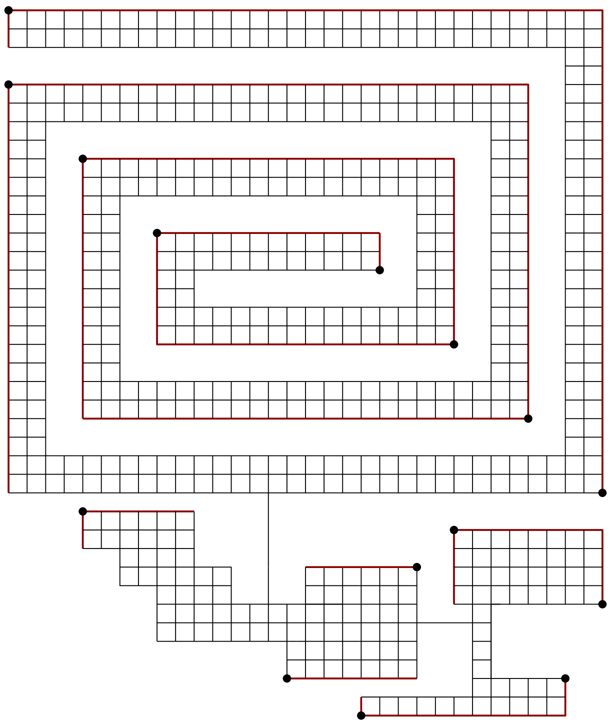

Figure 8: Illustration for execution of Algorithm 1 on a solid grid. The marked vertices are selected by the algorithm. Moreover, the corner paths are represented by bold red paths.

3.3 Feasibility

In this section, we shall show that Algorithm 1 returns a geodetic set. In Section 3.3.1, we shall prove some properties of the boundary of a biconnected component of a solid grid. Finally, in Section 3.3.2, we shall use the developed machinery to establish the feasibility of our algorithm.

3.3.1 Properties of the boundary of a biconnected component of a solid grid

Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let H be a biconnected component of G. For a vertex v of H, let X_v denote the maximal horizontal path of H containing v and Y_v be the maximal vertical path of H containing v. Let $north\,(v)$ and $south\,(v)$ denote the end-vertices of Y_v that have the highest and lowest y-coordinates, respectively. Similarly, let $east\,(v)$ and $west\,(v)$ denote the end-vertices of X_v that have the highest and lowest x-coordinates, respectively. Observe that the vertices $north\,(v)\,, west\,(v)\,, south\,(v)\,, east\,(v)$ lie on the boundary of $\mathcal{R}_H$. Moreover, if v does not lie in the boundary of $\mathcal{R}_H$, then $v, north\,(v)\,, west\,(v)\,, south\,(v)\,, east\,(v)$ are all distinct vertices.

Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let H be a biconnected component of G. For a vertex v of H, let $Tchain\,(v)$ denote the subgraph induced by the set of vertices encountered in a counter-clockwise traversal of the boundary of $\mathcal{R}_H$ from $east\,(v)$ to $west\,(v)$. Let $Bchain\,(v)$ denote the subgraph induced by the set of vertices encountered in a counter-clockwise traversal of the boundary of $\mathcal{R}_H$ from $west\,(v)$ to $east\,(v)$. Let $Lchain\,(v)$ denote the set of vertices encountered in a counter-clockwise traversal of the boundary of $\mathcal{R}_H$ from $north\,(v)$ to $south\,(v)$. Let $Rchain\,(v)$ denote the set of vertices encountered

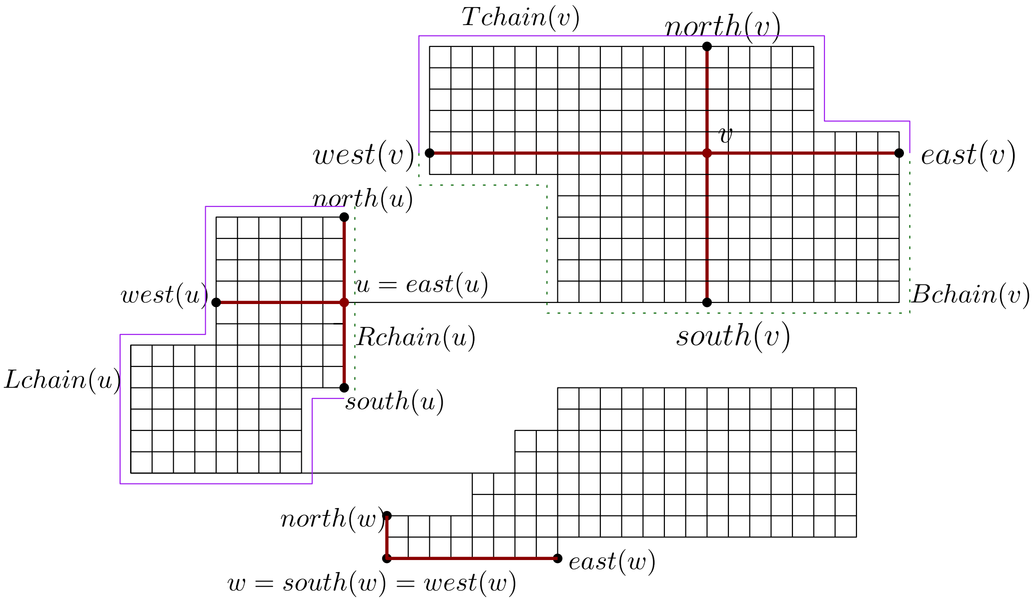

Figure 9: For vertex v, we outline $Tchain(v)$ with a purple line and $Bchain(v)$ with a green dotted line. Similarly, for vertex u, we outline $Lchain(u)$ with a purple line and $Rchain(u)$ with a dotted green line.

in a counter-clockwise traversal of the boundary of $\mathcal{R}_H$ from $south\,(v)$ to $north\,(v)$. See Figure 9 for an illustration.

A biconnected component H of G is *trivial* if H is either just an edge or a single vertex. For all non-trivial biconnected components H of G, we have the following observation.

Observation 14. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let H be a non-trivial biconnected component of G. For a vertex v of H, all the following hold.*

(a) The vertices $north\,(v)\,, south\,(v)$ are distinct and the vertices $east\,(v)\,, west\,(v)$ are distinct.

(b) Let $a \in \{north\,(v)\,, south\,(v)\}$ and $b \in \{east\,(v)\,, west\,(v)\}$ be two vertices such that $a = b$. Then, $a = b = v$.

(c) The subgraphs $Tchain\,(v)$, $Bchain\,(v)\,, Lchain\,(v)\,, Rchain\,(v)$ are paths in G.

(d) The path $Tchain\,(v)$ contains $north\,(v)$ and the path $Bchain\,(v)$ contains $south\,(v)$.

(e) The union of the edges of $Lchain\,(v)$ and $Rchain\,(v)$ creates a cycle C in G and $\mathcal{R}_C$ is the boundary of $\mathcal{R}_H$.

(f) The union of the edges of $Tchain\,(v)$ and $Bchain\,(v)$ creates a cycle C in G and $\mathcal{R}_C$ is the boundary of $\mathcal{R}_H$.

(g) Let P be a path such that $V(P) \subseteq V(H)$. Then $\mathcal{R}_P \subset I\left(\mathcal{R}_C\right) \cup \mathcal{R}_C$ where C is the cycle corresponding to the boundary of $\mathcal{R}_H$.

Let $TLchain\,(v)$ be the subpath of $Tchain\,(v)$ between $north\,(v)$ and $west\,(v)$. Similarly, let $TRchain\,(v)$ be the subpath of $Tchain\,(v)$ between $north\,(v)$ and $east\,(v)$. Let $BLchain\,(v)$ be the subpath of $Bchain\,(v)$

between $south\,(v)$ and $west\,(v)$. Similarly, let $BRchain\,(v)$ be the subpath of $Bchain\,(v)$ between $south\,(v)$ and $east\,(v)$. Due to Observation 14, the above terminologies are well defined. Observe that $Lchain\,(v)$ is the concatenation of $TLchain\,(v)$ with $BLchain\,(v)$ and $Rchain\,(v)$ is the concatenation of $BRchain\,(v)$ with $TRchain\,(v)$. We have the following lemma.

Lemma 15. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let H be a non-trivial biconnected component of G. For a vertex v of H, we have $V(TRchain\,(v)) \cap V(BLchain\,(v)) \subseteq \{v\}$ and $V(TLchain\,(v)) \cap V(BRchain\,(v)) \subseteq \{v\}$.*

Proof. Due to symmetry, it is enough to prove that $V(TRchain\,(v)) \cap V(BLchain\,(v)) \subseteq \{v\}$. If each path in $TLchain\,(v), TRchain\,(v), BLchain\,(v), BRchain\,(v)$ has at least one edge, then it follows from the definitions that $V(TRchain\,(v)) \cap V(BLchain\,(v)) = \emptyset$. Moreover, since $north\,(v) \neq south\,(v)$ and $east\,(v) \neq west\,(v)$ (due to Observation 14(a)), we have to consider only the following four exhaustive cases.

First, suppose $south\,(v) = west\,(v)$. Due to Observation 14(b) we have that $south\,(v) = west\,(v) = v$, and therefore, $V(BLchain\,(v)) = \{v\}$. This concludes the proof of the lemma for this case.

Second, suppose $south\,(v) = east\,(v)$. Due to Observation 14(b) we have that $south\,(v) = east\,(v) = v$, and therefore, $BLchain\,(v) = Bchain\,(v)$. From the definitions it is clear that $Tchain\,(v) \cap Bchain\,(v) = \{east\,(v), west\,(v)\}$. Using Observation 14(a), we infer that $north\,(v) \neq v$ and therefore, $TRchain\,(v)$ is proper subpath of the path induced by $Tchain\,(v)$ and does not contain $west\,(v)$. This implies $V(TRchain\,(v)) \cap V(BLchain\,(v)) \subseteq \{east\,(v)\} = \{v\}$.

As the cases $north\,(v) = east\,(v)$ and $north\,(v) = west\,(v)$ are symmetrical to the above cases, we have the proof of the Lemma. □

Lemma 16. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let H be a non-trivial biconnected component of G. Let v be any vertex of H and u, w be two vertices such that $u \in V(TRchain\,(v))$ and $w \in V(BLchain\,(v))$. Let P be any path between u and w in H. Then $V(P) \cap V(X_v) \neq \emptyset$ and $V(P) \cap V(Y_v) \neq \emptyset$.*

Proof. First, we show that $V(P) \cap V(X_v) \neq \emptyset$. If $\{u, w\} \cap V(X_v) \neq \emptyset$, then we are done. Let C denote the subgraph induced by $V(X_v) \cup V(Tchain\,(v))$. Observe that if no vertex of $V(X_v) \setminus \{west\,(v), east\,(v)\}$ lie on the boundary of $\mathcal{R}_H$, then C is a cycle. Otherwise, C can be decomposed into cycles $C_1, C_2, \ldots, C_t$ and paths $Q_1, Q_2, \ldots, Q_{t'}$. Since $TRchain\,(v) \subseteq Tchain\,(v)$ and $BLchain\,(v) \subseteq Bchain\,(v)$, $u \in Tchain\,(v)$ and $w \in Bchain\,(v)$. Hence $w \notin \bigcup_{i=1}^{t} I\left(\mathcal{R}_{C_i}\right) \bigcup_{i=1}^{t'} I\left(\mathcal{R}_{Q_i}\right)$ and therefore, $\mathcal{R}_P \not\subseteq \bigcup_{i=1}^{t} I\left(\mathcal{R}_{C_i}\right) \bigcup_{i=1}^{t'} I\left(\mathcal{R}_{Q_i}\right)$. Let p be the point closest to w (in $\mathcal{R}_P$) that also lies on $\mathcal{R}_C$. If $p \in Tchain\,(v)$, then it follows that $\mathcal{R}_P \not\subseteq I\left(\mathcal{R}_{C'}\right) \cup \mathcal{R}_{C'}$ where C' is the boundary of H. But this would contradict Observation 14(g). Hence we have that, $p \notin Tchain\,(v)$ and it must be the case that $p \in \mathcal{R}_{X_v}$ and therefore $V(P) \cap V(X_v) \neq \emptyset$. Through symmetric arguments, it can be shown that $V(P) \cap V(Y_v) \neq \emptyset$. □

3.3.2 Algorithm 1 returns a geodetic set

The reader may observe that Algorithm 1 never picks a cut vertex in its solution. To prove the rationale behind it, we shall use the following result of Ekim and Erey [15].

Proposition 17 ([15])**.** *Let F be a graph and $F_1, \ldots, F_k$ its biconnected components. Let C be the set of cut vertices of G. If $X_i \subseteq V(F_i)$ is a minimum set such that $X_i \cup (V(F_i) \cap C)$ is a minimum geodetic set of F_i, then $\cup_{i=1}^{k} X_i$ is a minimum geodetic set of F.*

The next observation follows from Proposition 17.

Observation 18. *Let $C(G)$ be the set of cut-vertices of G and let $\{H_1, H_2, \ldots, H_k\}$ be the set of biconnected components of G. A set D is a geodetic set of G if and only if $(D \cup C(G)) \cap V(H_i)$ is a geodetic set of H_i for all $1 \leq i \leq k$.*

Lemma 19. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let $C(G)$ be the set of cut-vertices of G and D be the set returned by Algorithm 1. Let H be a non-trivial biconnected component of G and v be any vertex of H. Then, for each $Z \in \{Lchain\,(v)\,, Rchain\,(v)\,, Tchain\,(v)\,, Bchain\,(v)\}$ we have that $Z \cap (D \cup C(G)) \neq \emptyset$.*

Proof. Let D_H denote the set $(D \cup C(G)) \cap V(H)$. Due to symmetry, it is sufficient to prove the lemma for $Z = Tchain\,(v)$. Let m be a vertex of Z whose y-coordinate is maximum among all vertices of Z, and X_m be the maximal horizontal path of H containing m. Observe that X_m is a subpath of $Tchain\,(v)$. We shall show that either X_m contains a cut vertex or it is a corner path.

Let X_m do not contain any cut vertex and let a and b be the end-vertices of X_m. Suppose there exists a vertex $u \in \{a, b\}$ that has degree 3 in G. Then, there exists a vertex $u' \in N[u]$, which is not a vertex of H. Hence u is a cut-vertex, a contradiction. Therefore we have the following.

(+) Degrees of both a and b in G are two.

Similarly, if any vertex w of X_m has a neighbour w' whose y-coordinate is greater than that of w, then w is also a cut-vertex which is a contradiction. Now, consider any vertex c of X_m distinct from both a and b. Let the coordinate of c be (ℓ, ℓ'). Consider the unit square U that has (ℓ, ℓ') and $(\ell - 1, \ell' - 1)$ as diagonally opposite points. Also consider the subgraph F formed by the vertices of $V(Tchain\,(v)) \cup V(Bchain\,(v))$. Due to Observation 14(e), F is a cycle of G and observe that $I\,(\mathcal{R}_F)$ contains U. Due to Observation 11, the four corner points of U induce a cycle in G. Therefore, we have the following.

(++) For any vertex of $c \in V(X_m) \setminus \{a, b\}$, degree of c in G is three.

Due to (+) and (++), we have that X_m is a corner path. Now recall the definition of maximal corner sequence and observe that there exists a maximal corner sequence S where a and b are consecutive. From the definition of D, we have that at least one vertex among a and b lies in D and, therefore, in D_H. This completes the proof. ☐

Lemma 20. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let $C(G)$ be the set of cut-vertices of G and D be the set returned by Algorithm 1. Let H be a non-trivial biconnected component of G, and v be any vertex of H. Then, at least one of the following is true.*

(a) There exist two vertices $\{u, w\} \subseteq (D \cup C(G)) \cap V(H)$ such that $u \in V(TRchain\,(v))$ and $w \in V(BLchain\,(v))$.

(b) There exist two vertices $\{u, w\} \subseteq (D \cup C(G)) \cap V(H)$ such that $u \in V(TLchain\,(v))$ and $w \in V(BRchain\,(v))$.

Proof. Let $D_H = (D \cup C(G)) \cap V(H)$. Due to Lemma 19, we know that there exists at least one vertex $u \in D_H$ such that $u \in V(Bchain\,(v))$. Due to symmetry, we assume that $u \in BLchain\,(v)$. Now suppose (a) is false. Then $D_H \cap TRchain\,(v) = \emptyset$. Applying Lemma 19, we infer that there exists a vertex $u' \in D_H \cap Tchain\,(v)$ and therefore $u' \in TLchain\,(v)$. We apply Lemma 19 again to infer that there exists a vertex $w' \in D_H \cap Rchain\,(v)$. Since $Rchain\,(v)$ is the concatenation of $BRchain\,(v)$ with $TRchain\,(v)$, we infer that v' must lie in $BRchain\,(v)$. Hence (b) is true. ☐

Lemma 21. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let H be a non-trivial biconnected component of G. Let v be any vertex of H and u, w be two vertices such that $u \in V(TRchain\,(v))$ and $w \in V(BLchain\,(v))$. Then there exists a shortest path between u and w that contains v.*

Proof. Let P be a shortest path between u and w. If $v \in V(P)$, we are done. Recall that X_v (resp. Y_v) is the maximal horizontal (resp. vertical) path of H containing v. Lemma 16 implies that $V(P) \cap X_v \neq \emptyset$ and $V(P) \cap Y_v \neq \emptyset$. Let P be be written as $u = z_0\ z_1\ z_2\ \ldots z_t = w$. Let i be the minimum index such that $z_i \in V(X_v) \cup V(Y_v)$ and $Q \in \{X_v, Y_v\}$ such that $z_i \in V(Q)$. Since $z_i \neq v$, there exists a maximum index j, distinct from i, and a path $Q' \in \{X_v, Y_v\}$ distinct from Q such that $z_j \in V(Q')$. Now, consider the path P' between z_i and z_j such that $V(P') \subseteq V(X_v) \cup V(Y_v)$. Observation 12 implies that P' is an isometric path. Hence, the path $Q = z_0 \ldots z_{i-1}\ P'\ z_{j+1} \ldots z_t$ is a also an isometric path that contains v. ☐

Due to symmetry, we also have the following lemma.

Lemma 22. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let H be a non-trivial biconnected component of G. Let v be any vertex of H and u, w be two vertices such that $u \in V(TLchain\,(v))$ and $w \in V(BRchain\,(v))$. Then there exists a shortest path P between u and w such that $v \in V(P)$.*

Lemma 23. *Algorithm 1 returns a geodetic set of the input solid grid graph.*

Proof. Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let D be the set returned by Algorithm 1 with G as input. Let H be any biconnected component of G. If H is an edge, then either both vertices of H are cut vertices, or at least one of the vertices is degree one. In either case, both vertices lie in some shortest path between some pair of vertices in D. So, we assume H is a non-trivial biconnected component and v to be any vertex of H. Then, due to Lemma 20, there exists two vertices $\{u, w\} \subseteq (D \cup C(G)) \cap V(H)$ such that either (*i*) $u \in V(TRchain\,(v))$ and $w \in V(BLchain\,(v))$, or (*ii*) $u \in V(TLchain\,(v))$ and $w \in V(BRchain\,(v))$. In either case, if $u = w$ then due to Lemma 15, we have that $u = w = v$. Otherwise, due to Lemma 21 and Lemma 22, we have that v lies in a shortest path between u and w. Hence, $(D \cup C(G)) \cap V(H)$ is a geodetic set of H. Now arguing for all biconnected components of G as above, and due to Observation 18, we have that D is a geodetic set of G. ☐

Lemma 24. *Algorithm 1 returns an edge geodetic set of the input solid grid graph.*

Proof. Targeting contradiction, assume that the set S returned by Algorithm 1 is not an edge geodetic set of the input solid grid graph. Then, there is an edge uv that is not covered by S. Let (x_u, y_u) and (x_v, y_v) be the coordinates of u and v in the grid embedding $\mathcal{R}$. Without loss of generality, assume $y_u = y_v$ and that $x_v = x_u + 1$.

Recall that S is a geodetic set (due to Lemma 23). Hence, if uv is a cut edge, then S covers uv, and we are done. Now assume uv belongs to some non-trivial biconnected component H. Due to Lemma 20 and without loss of generality assume, there exist two vertices $\{w, w'\} \subseteq S \cap V(H)$ such that $w \in V(TLchain\,(u))$ and $w' \in V(BRchain\,(u))$. Observe that if $w' \in V(BRchain\,(v))$ as well, then the edge uv lies in some shortest path between w and w'. Otherwise, w' must be the bottom-most endpoint of the maximal vertical path passing through u, i.e., Y_u.

Again applying Lemma 20, we know that either there exists a vertex $\{w''\} \subseteq S \cap V(H)$ such that $w'' \in V(TRchain\,(v))$ or there exists some vertices $\{z, z'\} \subseteq S \cap V(H)$ such that $z \in V(TLchain\,(u))$ and $z' \in V(BRchain\,(u))$. Now observe that if w'' exists, then the edge uv lies in some shortest path between w' and w''. Otherwise, the edge uv lies in some shortest path between w and z'. $\square$

3.4 Optimality

In this section, we shall prove that Algorithm 1 indeed returns a minimum geodetic set. In the next section, we shall prove some properties of the corner paths. In Section 3.4.2 we shall establish a lower bound on the geodetic number of a solid grid.

3.4.1 Properties of corner paths

In the following lemma, we prove a structural property of the subgraph induced by the neighbours of the vertices of a corner path P that lies outside P.

Lemma 25. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let P be a vertical (resp. horizontal) corner path of G such that the x-coordinates (resp. y-coordinates) of all vertices of P is ℓ. Consider the set $Q = \{v \in V(G) \colon v \notin V(P), N(v) \cap P \neq \emptyset\}$. Then the following holds:*

(a) for a vertex $u \in P$, there exists a unique neighbour $u' \in Q$ and y-coordinates (resp. x-coordinates) of both u, u' are same;

(b) x-coordinates (resp. y-coordinates) of all vertices of Q are the same and are equal to either $\ell - 1$ or $\ell + 1$;

(c) Q induces a path in G.

Proof. We shall prove the observation only for the case when P is a vertical corner path, as the other case is symmetric.

First, we prove (a). Let u be a vertex of P. Note that every internal vertex of P must have degree three, and both end vertices of P have degree two in G. Therefore, u has exactly one neighbour in Q. As P is a vertical path, the neighbour of u that is not in the path must have the same y-coordinate as u. Moreover, since any two distinct vertices of P have distinct y coordinates, they have distinct neighbours in Q, as well.

Now we prove (b). Refer to Figure 10 for an illustration of the notations used in this proof. For a vertex u of P, let u' denote its unique neighbour in Q. Suppose the statement is false, then we deduce that there must exist two vertices u_1, u_2 of P such that they are consecutive in P, the x-coordinate of u_1' is $\ell - 1$ and the x-coordinate of u_2' is $\ell + 1$. Without loss of generality, we assume that the y-coordinate of u_1 is larger than that of u_2 and equals to, say, ℓ_y. Hence, $u_1 = (\ell, \ell_y)$, $u_2 = (\ell, \ell_y - 1)$, $u_1' = (\ell - 1, \ell_y)$, and $u_2' = (\ell + 1, \ell_y - 1)$. Let U_1 denote the unit square formed by the grid points $tl(U_1) = u_1$, $bl(U_1) = u_2$, $br(U_1) = u_2'$ and $tr(U_1) = (\ell + 1, \ell_y)$. Let U_2 denote the unit square formed by the grid points $tl(U_2) = u_1'$, $bl(U_2) = (\ell - 1, \ell_y - 1)$, $br(U_2) = u_2$ and $tr(U_2) = u_1$. Since u_2 is not a cut-vertex, there must exist a path P' between u_1' and u_2' that does not contain u_2. Then the subgraph H formed by $E(P') \cup \{u_1u_1', u_1u_2, u_2u_2'\}$ contains a cycle C such that there exists a $Z \in \{U_1, U_2\}$ with $Z \subseteq I\,(\mathcal{R}_C)$. Due to Observation 11, the four corner points of Z induce a cycle of order four in G. If $Z = U_1$, then u_1 must have two distinct neighbours

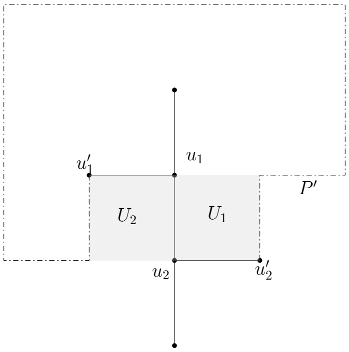

Figure 10: Illustration of the notations used in proof of Lemma 25(b).

in Q, which is a contradiction. Similarly, if $Z = U_2$, then u_2 must have two distinct neighbours in Q, which is also a contradiction.

Finally we prove (c). Let $P = u_1\ u_2 \dots\ u_k$ and $Q = \{u_1', u_2', \dots, u_k'\}$ such that u_i' is the unique neighbour of u_i in $N(u_i) \setminus V(P)$. Due to (b), the x-coordinates of Q are all same. Without loss of generality, we assume that the x-coordinates of Q are $\ell + 1$. Let $i \in [k-1]$ and consider the vertices u_i' and u_{i+1}'. Let X denote the unit square formed by the grid points $u_{i+1}, u_i, u_i', u_{i+1}'$. Since u_{i+1} is not a cut vertex there exists a path P' from u_{i+1}' to u_i that does not contain u_{i+1}. Now consider the subgraph H formed by the edges $E(P') \cup \{u_{i+1}'u_{i+1}, u_{i+1}u_i, u_iu_i'\}$. The subgraph H contains a cycle C such that $X \subseteq I(\mathcal{R}_C)$. Then due to Observation 11 we have that $u_{i+1}, u_i, u_i', u_{i+1}'$ induces a cycle of length four in G and therefore $u_i'u_{i+1}' \in E(G)$. Now for any $i, j \in [k]$ with $|i - j| > 1$, the vertices u_i' and u_j' are non-adjacent. Hence Q induces a path in G. □

Lemma 26. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Two distinct corner paths P and Q have the following properties.*

- *(a) $E(P) \cap E(Q) = \emptyset$.*
- *(b) $|V(P) \cap V(Q)| \leq 1$.*
- *(c) If there exists a vertex $z \in V(P) \cap V(Q)$ then z must be an end-vertex of both P and Q.*

Proof. Without loss of generality, assume that P is a vertical corner path.

Now we shall prove (a). For the sake of contradiction, suppose P and Q share an edge e. Since P is a vertical corner path, Q must be a vertical corner path containing the edge e. This implies Q must have the same end-vertices as P, and all intermediate vertices of Q must lie in P as well. Therefore $P = Q$, which is a contradiction.

Now we shall prove (b). Suppose P and Q share two vertices z_1, z_2, and without loss of generality, y-coordinate of z_1 is greater than that of z_2. Observe that Q must be a vertical path containing the subpath of P between z_1, z_2. This would imply that P and Q share an edge contradicting (a).

Now we shall prove (c). Suppose P and Q share exactly one vertex z. Then, observe that Q must be a horizontal corner path. Now we have the following cases.

1. Suppose z is neither an end-vertex of P nor of Q. Then z must have two neighbours, say a, a', whose y-coordinate is the same as that of z. Similarly, z must have two neighbours, distinct from both a, a'

whose x-coordinate is the same as that of z. Therefore, z must have degree four in G, a contradiction to the fact that z is a vertex of some corner path.

2. Suppose z is an end-vertex of one of the corner paths but not of the other. But this would directly contradict the definition of a corner vertex (recall that a corner vertex must have degree two in G and other vertices of a corner path must have degree three in G).

Hence, z is an end-vertex of both P and Q. □

Definition 2. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. For an arbitrary maximal corner sequence $S = (u_1, u_2, \ldots, u_{|S|})$ and $1 \leq i \leq |S| - 1$, let T_i denote the corner path between u_i and u_{i+1} and let* $\Phi(S) = \bigcup_{i=1}^{|S|-1} V(T_i)$.

Lemma 27. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let S and S' be two distinct maximal corner sequences. Then $\Phi(S) \cap \Phi(S') = \emptyset$.*

Proof. Let $S = (u_1, u_2, \ldots, u_{|S|})$ and $1 \leq i \leq |S| - 1$, let T_i denote the corner path between u_i and u_{i+1}. Similarly, let $S' = (u'_1, u'_2, \ldots, u'_{|S'|})$ and $1 \leq i \leq |S'| - 1$, let T'_i denote the corner path between u'_i and u'_{i+1}. Now suppose $\Phi(S) \cap \Phi(S') \neq \emptyset$. Then there exist i and j such that the two corner paths T_i and T'_j has a vertex z in common. Due to Lemma 26(c), we know that z is an end-vertex of both T_i and T'_j. Since z has degree 2 in G, we conclude that $z \in \{u_1, u_{|S|}\}$ and $z \in \{u'_1, u'_{|S|}\}$. But this contradicts the maximality of S and S'. □

3.4.2 Lower bound

Chakraborty et al. [7] proved the following observations and lemma using a slightly different terminologies. For completion we provide the proofs.

Lemma 28 ([7])**.** *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Any geodetic set of G contains at least one vertex from each corner path.*

Proof. Let there be a corner path P and a geodetic set X of G such that $V(P) \cap X = \emptyset$. Without loss of generality, we assume P to be a vertical corner path. Let u be the end-vertex of P with the larger y-coordinate. Now, consider two vertices $a, b \in X$ such that there is an isometric path P' between a and b that contains u. Observe that P' can be expressed as $P' = a\ c_1\ c_2 \ldots c_t\ d\ f_1\ f_2 \ldots f_{t'}\ u\ g\ h_1\ h_2 \ldots h_{t''}\ b$ such that $\{f_1, f_2, \ldots, f_{t'}\} \subseteq V(P)$, $d \in N(f_1) \setminus V(P)$, $g \in N(u) \setminus V(P)$.

Now consider the set $Q = \{v \in V(G) \colon v \notin V(P), N(v) \cap P \neq \emptyset\}$. Observe that $\{d, g\} \subseteq Q$ and therefore, due to Lemma 25(b), the x-coordinate of both d and g is the same. Let the y-coordinate of d and g be ℓ and ℓ', respectively. Due to Lemma 25(c) and Observation 12, we have that there is a isometric path between d and g of length $|\ell - \ell'|$. But the subpath of P' between d and g is of length $|\ell - \ell'| + 2$, which is not isometric, and, therefore, cannot be a subpath of an isometric path. Hence, this contradicts the fact that P' is an isometric path. □

We have the following lemma.

Lemma 29. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let $\mathcal{S}$ be the set of all distinct maximal corner sequences of G, and let t be the number of vertices of G with degree* 1*. Then,* $gn(G) \geq t + \sum_{S \in \mathcal{S}} \left\lfloor \frac{|S|}{2} \right\rfloor$.

Proof. Let X be a minimum geodetic set of G and V_1 be the set of all vertices with degree one. Observe that $V_1 \subseteq X$ and therefore $gn(G) \geq t$. Recall that for an arbitrary maximal corner sequence $S = u_1, u_2, \ldots, u_{|S|} \in \mathcal{S}$ and for $1 \leq i \leq |S|-1$, T_i denotes the corner path between u_i and u_{i+1} and $\Phi(S) = \bigcup_{i=1}^{|S|-1} V(T_i)$. Lemma 28 implies that for each $1 \leq i < |S|$, at least one vertex of the corner path T_i must belong to X. Lemma 26(c) implies that no vertex of G lies in more than two corner paths. Therefore, X must contain at least $\left\lfloor \frac{|S|}{2} \right\rfloor$ vertices from $\Phi(S)$.

Now, let $S_1, S_2, \ldots S_t$ be the maximal corner sequences. Observe that for any $\{i, j\} \subseteq \{1, 2, \ldots, |\mathcal{S}|\}$, due to Lemma 27, $\Phi(S_i) \cap \Phi(S_j) = \emptyset$. Now, previous arguments imply that X must contain a set V_2 whose cardinality is at least $\sum_{S \in \mathcal{S}} \left\lfloor \frac{|S|}{2} \right\rfloor$. Moreover, $V_2 \subseteq \bigcup_{i=1}^{t} \Phi(S_i)$. Since no vertex of V_1 is a vertex of any corner path, $V_1 \cap V_2 = \emptyset$. Therefore, $gn(G) = |X| \geq t + \sum_{S \in \mathcal{S}} \left\lfloor \frac{|S|}{2} \right\rfloor$. ☐

3.5 Proof of Theorem 2

Now, proof of Theorem 2 follows from Observation 13, Lemma 23 and Lemma 29. Moreover, we have the following structural lemma, as a consequence of Observation 13, Lemma 23, and Lemma 29, that provides a tight bound on the geodetic number of a solid grid in terms of the cardinality of its corner sequences.

Lemma 30. *Let G be a solid grid graph and $\mathcal{R}$ be a grid embedding of G. Let $\mathcal{S}$ be the set of all maximal corner sequences of G, and let t be the number of vertices of G with degree* 1*. Then,* $gn(G) = t + \sum_{S \in \mathcal{S}} \left\lfloor \frac{|S|}{2} \right\rfloor$.

The above (structural) lemma could be of independent interest.

Proof of Corollary 2 The proof follows from Observation 13, Lemma 24, and Lemma 29.

4 Conclusion

We proved that Minimum Geodetic Set is NP-hard on partial grids and that Minimum Geodetic Set admits linear time algorithm on solid grids. An interesting question is whether Minimum Geodetic Set is FPT on partial grids (or planar graphs) parameterized by the solution size? Another interesting question is the existence of constant factor approximation algorithms for Minimum Geodetic Set on partial grids or planar graphs.

Finally, we think that studying the approximability and parameterized (in)tractability of related problems like Isometric Path Cover [5], Strong Geodetic Set [24], and Geodetic Hull [20] on planar graphs is another interesting direction of research.

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