Isometric Path Complexity of Graphs

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Abstract

A set S of isometric paths of a graph G is "v-rooted", where v is a vertex of G, if v is one of the end-vertices of all the isometric paths in S. The *isometric path complexity* of a graph G, denoted by $ipco(G)$, is the minimum integer k such that there exists a vertex $v \in V(G)$ satisfying the following property: the vertices of any isometric path P of G can be covered by k many v-rooted isometric paths.

First, we provide an $O(n^2m)$-time algorithm to compute the isometric path complexity of a graph with n vertices and m edges. Then we show that the isometric path complexity remains bounded for graphs in three seemingly unrelated graph classes, namely, *hyperbolic graphs*, *(theta, prism, pyramid)-free graphs*, and *outerstring graphs*. Hyperbolic graphs are extensively studied in *Metric Graph Theory*. The class of (theta, prism, pyramid)-free graphs are extensively studied in *Structural Graph Theory*, e.g. in the context of the *Strong Perfect Graph Theorem*. The class of outerstring graphs is studied in *Geometric Graph Theory* and *Computational Geometry*. Our results also show that the distance functions of these (structurally) different graph classes are more similar than previously thought.

There is a direct algorithmic consequence of having small isometric path complexity. Specifically, using a result of Chakraborty et al. [ISAAC 2022], we show that if the isometric path complexity of a graph G is bounded by a constant k, then there exists a k-factor approximation algorithm for Isometric Path Cover, whose objective is to cover all vertices of a graph with a minimum number of isometric paths.

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1 Introduction

A path is *isometric* if it is a shortest path between its endpoints. An *isometric path cover* of a graph G is a set of isometric paths such that each vertex of G belongs to at least one of the paths. The *isometric path number* of G is the smallest size of an isometric path cover of G. Given a graph G and an integer k, the objective of the algorithmic problem Isometric Path Cover is to decide if there exists an isometric path cover of cardinality at most k. Isometric Path Cover has been introduced and studied in the context of pursuit-evasion games [2, 3]. However, until recently the algorithmic aspects of Isometric Path Cover remained unexplored. After proving that Isometric Path Cover remains NP-hard on *chordal graphs* (graphs without any induced cycle of length at least 4), Chakraborty et al. [7] provided constant-factor approximation algorithms for many graph classes, including *interval graphs*, chordal graphs, and more generally, graphs with bounded *treelength*. To prove the approximation ratio of their algorithm, the authors introduced a parameter called *isometric path antichain cover number* of a graph G, denoted as $ipacc(G)$ (see Definition 6), and proved (i) when $ipacc(G)$ is bounded by a constant, Isometric Path Cover admits a constant-factor approximation algorithm on G; and (ii) the isometric path antichain cover number of graphs with bounded *treelength* is bounded.

The objectives of this paper are three fold: **(A)** provide a more intuitive definition of isometric path antichain cover number; **(B)** provide a polynomial-time algorithm to compute $ipacc(G)$; and **(C)** prove that it remains bounded for seemingly unrelated graph classes. Along the way, we also extend the horizon of approximability of Isometric Path Cover. To achieve **(A)** we introduce the following new metric graph parameter, that we will show to be always equal to the isometric path antichain cover number, and whose definition is simpler.

▶ **Definition 1.** *Given a graph G and a vertex v of G, a set S of isometric paths of G is v-*rooted *if v is one of the end-vertices of all the isometric paths in S. The* isometric path complexity *of a graph G, denoted by $ipco(G)$, is the minimum integer k such that there exists a vertex $v \in V(G)$ satisfying the following property: the vertices of any isometric path P of G can be covered by k many v-rooted isometric paths.*

A consequence of Dilworth's theorem is that for any graph G, $ipacc(G) = ipco(G)$ (see Lemma 7). We will give a polynomial-time algorithm to compute $ipco(G)$, and therefore $ipacc(G)$ for an arbitrary undirected graph G. This achieves **(B)**. Finally, to achieve **(C)**, we consider the following three seemingly unrelated graph classes, namely, *δ-hyperbolic graphs*, *(theta, prism, pyramid)-free graphs* and *outerstring* graphs, and show that their isometric path complexity is bounded by a constant.

δ-hyperbolic graphs. A graph G is said to be *δ-hyperbolic* [20] if for any four vertices u, v, x, y, the two larger of the three distance sums $\mathsf{d}(u,v) + \mathsf{d}(x,y)$, $\mathsf{d}(u,x) + \mathsf{d}(v,y)$ and $\mathsf{d}(u,y) + \mathsf{d}(v,x)$ differ by at most 2δ. A graph class $\mathcal{G}$ is *hyperbolic* if there exists a constant δ such that every graph $G \in \mathcal{G}$ is δ-hyperbolic. This parameter comes from geometric group theory and was first introduced by Gromov [20] in order to study groups via their *Cayley graphs*. The hyperbolicity of a tree is 0, and in general, the hyperbolicity measures how much the distance function of a graph deviates from a tree metric. Many structurally defined graph classes like chordal graphs, *cocomparability* graphs [13], *asteroidal-triple free* graphs [14], graphs with bounded *chordality* or *treelength* are hyperbolic [8, 22]. Moreover, hyperbolicity has been found to capture important properties of several large practical graphs such as the Internet graph [25] or database relation graphs [30]. Due to its importance in discrete

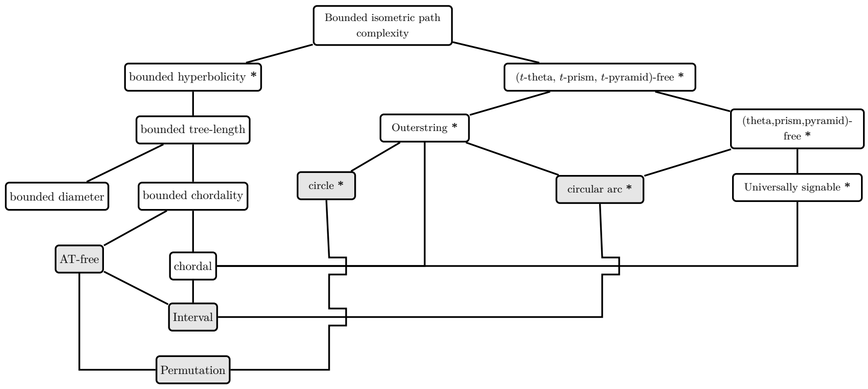


Figure 1 Inclusion diagram for graph classes. If a class A has an upward path to class B, then A is included in B. Constant bounds for the isometric path complexity on graph classes marked with * are contributions of this paper.

mathematics, algorithms, *metric graph theory*, researchers have studied various algorithmic aspects of hyperbolic graphs [8, 15, 10, 16]. Note that graphs with diameter 2 are hyperbolic, which may contain any graph as an induced subgraph.

(theta, prism, pyramid)-free graphs. A *theta* is a graph made of three vertex-disjoint induced paths $P_1 = a \dots b$, $P_2 = a \dots b$, $P_3 = a \dots b$ of lengths at least 2, and such that no edges exist between the paths except the three edges incident to a and the three edges incident to b. A *pyramid* is a graph made of three induced paths $P_1 = a \dots b_1$, $P_2 = a \dots b_2$, $P_3 = a \dots b_3$, two of which have lengths at least 2, vertex-disjoint except at a, and such that $b_1 b_2 b_3$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident to a. A *prism* is a graph made of three vertex-disjoint induced paths $P_1 = a_1 \dots b_1$, $P_2 = a_2 \dots b_2$, $P_3 = a_3 \dots b_3$ of lengths at least 1, such that $a_1 a_2 a_3$ and $b_1 b_2 b_3$ are triangles and no edges exist between the paths except those of the two triangles. A graph G is *(theta, pyramid, prism)*-free if G does not contain any induced subgraph isomorphic to a theta, pyramid or prism. A graph is a 3-*path configuration* if it is a theta, pyramid or prism. The study of 3-path configurations dates back to the works of Watkins and Meisner [31] in 1967 and plays "special roles" in the proof of the celebrated *Strong Perfect Graph Theorem* [11, 18, 27, 29]. Important graph classes like chordal graphs, *circular arc* graphs, *universally-signable* graphs [12] exclude all 3-path configurations. Popular graph classes like *perfect* graphs, *even hole*-free graphs exclude some of the 3-path configurations. Note that, (theta, prism, pyramid)-free graphs are not hyperbolic. To see this, consider a cycle C of order n. Clearly, C excludes all 3-path configurations and has hyperbolicity $\Omega(n)$.

Outerstring graphs. A set S of simple curves on the plane is *grounded* if there exists a horizontal line containing one endpoint of each of the curves in S. A graph G is an *outerstring* graph if there is a collection C of grounded simple curves and a bijection between $V(G)$ and C such that two curves in S intersect if and only if the corresponding vertices are adjacent in G. The term "outerstring graph" was first used in the early 90's [23] in the context of studying intersection graphs of simple curves on the plane. Many well-known graph classes like chordal graphs, *circular arc* graphs [19], *circle* graphs (intersection graphs of chords of a circle [17]), or cocomparability graphs [13] are also outerstring graphs and thus, motivated researchers from the *geometric graph theory* and *computational geometry* communities to study algorithmic and structural aspects of outerstring graphs and its subclasses [4, 5, 6, 21, 24]. Note that, in general, outerstring graphs may contain a prism, pyramid or theta as an induced subgraph. Moreover, cycles of arbitrary order are outerstring graphs, implying that outerstring graphs are not hyperbolic.

It is clear from the above discussion that the classes of hyperbolic graphs, (theta, prism, pyramid)-free graphs, and outerstring graphs are pairwise incomparable (with respect to the containment relationship). We show that the isometric path complexities of all the above graph classes are small.

1.1 Our contributions

The main technical contributions of this paper are as follows. First we prove that the isometric path complexity can be computed in polynomial time.

▶ **Theorem 2.** *Given a graph G with n vertices and m edges, it is possible to compute $ipco(G)$ in $O(n^2m)$ time.*

Recall that, the above theorem and Lemma 7 imply that for any undirected graph G, $ipacc(G)$ can be computed in polynomial time. Then we show that the isometric path complexity remains bounded on hyperbolic graphs, (theta, pyramid, prism)-free graphs, and outerstring graphs. Specifically, we prove the following theorem.

▶ **Theorem 3.** *Let G be a graph.*
(a) *If the hyperbolictiy of G is at most δ, then $ipco(G) \leq 4\delta + 3$.*
(b) *If G is a (theta, pyramid, prism)-free graph, then $ipco(G) \leq 71$.*
(c) *If G is an outerstring graph, then $ipco(G) \leq 95$.*

To the best of our knowledge, the isometric path complexity being bounded (by constant(s)) is the only known non-trivial property shared by any two or all three of these graph classes. Theorem 3 shows that isometric path complexity (equivalently isometric path antichain cover number), as recently introduced graph parameters, are general enough to unite these three graph classes by their metric properties. We hope that this definition will be useful for the field of metric graph theory, for example by enabling us to study (theta,prism,pyramid)-free graphs and outerstring graphs from the perspective of metric graph theory.

We provide a unified proof for Theorem 3b and 3c by proving that the isometric path complexity of *(t-theta, t-pyramid, t-prism)*-free graphs [28] (see Section 4 for a definition) is bounded by a linear function of t. Due to the above theorems, we also have as corollaries that there is a polynomial-time approximation algorithm for Isometric Path Cover with approximation ratio
(a) $4\delta + 3$ on δ-hyperbolic graphs,
(b) 73 on (theta, prism, pyramid)-free graphs,
(c) 95 on outerstring graphs, and
(d) $8t + 63$ on (t-theta, t-pyramid, t-prism)-free graphs.

To contrast with Theorem 3, we construct highly structured graphs with small *treewidth* and large isometric path complexity. A *wheel* consists of an induced cycle C of order at least 4 and a vertex $w \notin V(C)$ adjacent to at least three vertices of C. The three path configurations introduced earlier and the wheel together are called *Truemper configurations* [29] and they are important objects of study in structural and algorithmic graph theory [1, 18].

▶ **Theorem 4.** *For every $k \geq 1$,*

(a) *there exists a (pyramid, prism, wheel)-free graph G with tree-width 2, hyperbolicity at least $\lceil \frac{k}{2} \rceil - 1$ and $ipco(G) \geq k$.*

(b) *there exists a (theta, prism, wheel)-free graph G with tree-width at most 3, hyperbolicity at least $\lceil \frac{k}{2} \rceil - 1$ and $ipco(G) \geq k$.*

(c) *there exists a (theta, pyramid, wheel)-free graph G with hyperbolicity at least $\lceil \frac{k}{2} \rceil - 1$ and $ipco(G) \geq k$.*

Organisation. In Section 2, we recall some definitions and some results. In Section 3, we present an algorithm to compute the isometric path complexity of a graph and prove Theorem 2. In Section 4, we prove Theorem 3. In Section 5, we prove Theorem 4. We conclude in Section 6. Proofs of lemma and observations marked with (*) are provided in the main version of the paper.

2 Definitions and preliminary observations

In this section, we recall some definitions and some related observations. A sequence of distinct vertices forms a *path* P if any two consecutive vertices are adjacent. Whenever we fix a path P of G, we shall refer to the subgraph formed by the edges between the consecutive vertices of P. The *length* of a path P, denoted by $|P|$, is the number of its vertices minus one. A path is *induced* if there are no graph edges joining non-consecutive vertices. A path is *isometric* if it is a shortest path between its endpoints. For two vertices u, v of a graph G, $\mathsf{d}(u,v)$ denotes the length of an isometric path between u and v.

In a directed graph, a *directed path* is a path in which all arcs are oriented in the same direction. For a path P of a graph G between two vertices u and v, the vertices $V(P) \setminus \{u,v\}$ are *internal vertices* of P. A path between two vertices u and v is called a (u,v)-path. Similarly, we have the notions of *isometric (u,v)-path* and *induced (u,v)-path*. The interval $I(u,v)$ between two vertices u and v consists of all vertices that belong to an isometric (u,v)-path. For a vertex r of G and a set S of vertices of G, the *distance of S from r*, denoted as $\mathsf{d}(r,S)$, is the minimum of the distance between any vertex of S and r. For a subgraph H of G, the *distance of H w.r.t. r* is $\mathsf{d}(r, V(H))$. Formally, we have $\mathsf{d}(r,S) = \min\{\mathsf{d}(r,v) : v \in S\}$ and $\mathsf{d}(r,H) = \mathsf{d}(r,V(H))$.

For a graph G and a vertex $r \in V(G)$, consider the following operations on G. First, remove all edges xy from G such that $\mathsf{d}(r,x) = \mathsf{d}(r,y)$. Let G'_r be the resulting graph. Then, for each edge $e = xy \in E(G'_r)$ with $\mathsf{d}(r,x) = \mathsf{d}(r,y) - 1$, orient e from y to x. Let $\overrightarrow{G_r}$ be the directed acyclic graph formed after applying the above operation on G'. Note that this digraph can easily be computed in linear time using a Breadth-First Search (BFS) traversal with starting vertex r. The following definition is inspired by the terminology of posets (as the graph $\overrightarrow{G_r}$ can be seen as the Hasse diagram of a poset).

▶ **Definition 5.** *For a graph G and a vertex $r \in V(G)$, two vertices $x, y \in V(G)$ are* antichain vertices *if there are no directed paths from x to y or from y to x in $\overrightarrow{G_r}$. A set X of vertices of G is an* antichain set *if any two vertices in X are antichain vertices.*

▶ **Definition 6** ([7])**.** *Let r be a vertex of a graph G. For a subgraph H, $A_r(H)$ shall denote the maximum antichain set of H in $\overrightarrow{G_r}$. The* isometric path antichain cover number *of $\overrightarrow{G_r}$, denoted by $ipacc\left(\overrightarrow{G_r}\right)$, is defined as follows:*

$$ipacc\left(\overrightarrow{G_r}\right) = \max\left\{|A_r(P)| : \ P \text{ is an isometric path}\right\}.$$

The isometric path antichain cover number *of graph* G*, denoted as* $ipacc(G)$*, is defined as the minimum over all possible antichain covers of its associated directed acyclic graphs:*

$$ipacc(G) = \min \left\{ ipacc\left(\overrightarrow{G_r}\right) : r \in V(G) \right\}.$$

For technical purposes, we also introduce the following definition. For a graph G and a vertex r of G, let $ipco\left(\overrightarrow{G_r}\right)$ denote the minimum integer k such that any isometric path P of G can be covered by k r-rooted isometric paths (The notation reflects that it is a dual notion of $ipacc\left(\overrightarrow{G_r}\right)$). Using Dilworth's Theorem we prove the following important lemma.

▶ **Lemma 7.** *For any graph* G *and vertex* r*,* $ipco\left(\overrightarrow{G_r}\right) = ipacc\left(\overrightarrow{G_r}\right)$*. Therefore,* $ipco(G) = ipacc(G)$*.*

Proof. Let r be a vertex of G such that any isometric path of G can be covered by $ipco\left(\overrightarrow{G_r}\right)$ r-rooted isometric paths. Let P be an arbitrary isometric path of G. Since two vertices of an antichain of $\overrightarrow{G_r}$ cannot be covered by a single r-rooted path and P is covered by $ipco\left(\overrightarrow{G_r}\right)$ r-rooted path, we deduce $|A_r(P)| \leq ipco\left(\overrightarrow{G_r}\right)$. This is true for any isometric path P of G. Hence, $ipacc\left(\overrightarrow{G_r}\right) \leq ipco\left(\overrightarrow{G_r}\right)$. Conversely, consider a vertex $r \in V(G)$. By definition of $ipco\left(\overrightarrow{G_r}\right)$, there is an isometric path P that cannot be covered by $(ipco\left(\overrightarrow{G_r}\right) - 1)$ r-rooted isometric paths. By Dilworth theorem, P contains an antichain of $\overrightarrow{G_r}$ of size $ipco\left(\overrightarrow{G_r}\right)$. Hence $|A_r(P)| \geq ipco\left(\overrightarrow{G_r}\right)$ and $ipacc\left(\overrightarrow{G_r}\right) \geq ipco\left(\overrightarrow{G_r}\right)$. The second part of the lemma follows immediately. ◀

We also recall the following theorem and proposition from [7].

▶ **Theorem 8** ([7])**.** *For a graph* G*, if* $ipacc(G) \leq c$*, then* Isometric Path Cover *admits a polynomial-time* c*-approximation algorithm on* G*.*

▶ **Proposition 9** ([7])**.** *Let* G *be a graph and* r*, an arbitrary vertex of* G*. Consider the directed acyclic graph* $\overrightarrow{G_r}$*, and let* P *be an isometric path between two vertices* x *and* y *in* G*. Then* $|P| \geq |\mathsf{d}(r,x) - \mathsf{d}(r,y)| + |A_r(P)| - 1$*.*

Proof. Orient the edges of P from y to x in G. First, observe that P must contain a set E_1 of oriented edges such that $|E_1| = |\mathsf{d}(r,y) - \mathsf{d}(r,x)|$ and for any $\overrightarrow{ab} \in E_1$, $\mathsf{d}(r,a) = \mathsf{d}(r,b) + 1$. Let the vertices of the largest antichain set of P in $\overrightarrow{G_r}$, i.e., $A_r(P)$, be ordered as $a_1, a_2, \ldots, a_t$ according to their occurrence while traversing P from y to x. For $i \in [2,t]$, let P_i be the subpath of P between a_{i-1} and a_i. Observe that for any $i \in [2,t]$, since a_i and a_{i-1} are antichain vertices, there must exist an oriented edge $\overrightarrow{b_ic_i} \in E(P_i)$ such that either $\mathsf{d}(r,b_i) = \mathsf{d}(r,c_i)$ or $\mathsf{d}(r,b_i) = \mathsf{d}(r,c_i) - 1$. Let $E_2 = \{b_ic_i\}_{i\in[2,t]}$. Observe that $E_1 \cap E_2 = \emptyset$ and therefore $|P| \geq |E_1| + |E_2| = |\mathsf{d}(r,y) - \mathsf{d}(r,x)| + |A_r(P)| - 1$. ◀

3 Proof of Theorem 2

In this section we provide a polynomial-time algorithm to compute the isometric path complexity of a graph. Let G be a graph. In the following lemma, we provide a necessary and sufficient condition for two vertices of an isometric path to be covered by the same isometric r-rooted path in $\overrightarrow{G_r}$ for some vertex $r \in V(G)$.

▶ **Lemma 10.** *Let r be a vertex of G. If $P = (u = v_0, \ldots, v_k = v)$ is an isometric (u,v)-path with $\mathsf{d}(r,u) \leq \mathsf{d}(r,v)$ then there exists an isometric r-rooted path containing u, v in $\overrightarrow{G_r}(P)$ if and only if $\mathsf{d}(v_{i+1}, r) = \mathsf{d}(v_i, r) + 1$ for all $i \in \{0, \ldots, k-1\}$.*

Proof. If $\mathsf{d}(v_{i+1}, r) = \mathsf{d}(v_i, r) + 1$ for every $i \in \{0, \ldots, k-1\}$ then the path obtained by concatenating an isometric (r,u)-path and the path P is an isometric r-rooted (r,v)-path containing u, v in $\overrightarrow{G_r}(P)$. Now suppose that there exists an isometric r-rooted path containing u, v in $\overrightarrow{G_r}(P)$, i.e., $\mathsf{d}(r,v) - \mathsf{d}(r,u) = \mathsf{d}(u,v)$. Then, along any path from u to v, we need to traverse at least $\mathsf{d}(u,v)$ edges increasing the distance to r. Since P is an isometric (u,v)-path, it contains exactly $\mathsf{d}(u,v)$ edges. Hence, $\mathsf{d}(r, v_{i+1}) = \mathsf{d}(r, v_i) + 1$ for every $i \in \{0, \ldots, k-1\}$. ◀

3.1 Notations and preliminary observations

We now introduce some notations that will be used to describe the algorithm and prove its correctness. Consider three vertices r, x, v of G such that $x \neq v$. Let $\mathcal{P}^r_{\searrow}(x,v)$ denote the set of all isometric (x,v)-paths P containing a vertex u that is adjacent to v and satisfies $\mathsf{d}(r,u) = \mathsf{d}(r,v) - 1$. Analogously, let $\mathcal{P}^r_{\rightarrow}(x,v)$ denote the set of all isometric (x,v)-paths P containing a vertex u that is adjacent to v and satisfies $\mathsf{d}(r,u) = \mathsf{d}(r,v)$ and let $\mathcal{P}^r_{\nearrow}(x,v)$ denote the set of all isometric (x,v)-paths P containing a vertex u that is adjacent to v and satisfies $\mathsf{d}(r,u) = \mathsf{d}(r,v) + 1$. Observe that the set of isometric (x,v)-paths is precisely $\mathcal{P}^r_{\searrow}(x,v) \cup \mathcal{P}^r_{\rightarrow}(x,v) \cup \mathcal{P}^r_{\nearrow}(x,v)$ and that some of these sets may be empty.

Given a path P, we denote by $|S_r(P)|$ the minimum size of a set of isometric r-rooted paths covering the vertices of P. We denote by $\gamma^r_{\searrow}(x,v)$ and $\beta^r_{\searrow}(x,v)$ respectively the minimum of $|S_r(P)|$ and $|S_r(P - \{v\})|$ over all paths $P \in \mathcal{P}^r_{\searrow}(x,v)$. More formally,

$$\gamma^r_{\searrow}(x,v) = \max\left\{|S_r(P)| : P \in \mathcal{P}^r_{\searrow}(x,v)\right\},$$
$$\beta^r_{\searrow}(x,v) = \max\left\{|S_r(P - \{v\})| : P \in \mathcal{P}^r_{\searrow}(x,v)\right\}.$$

Note that if $\mathcal{P}^r_{\searrow}(x,v)$ is empty, we have $\gamma^r_{\searrow}(x,v) = \beta^r_{\searrow}(x,v) = 0$. We define similarly $\gamma^r_{\nearrow}(x,v)$, $\beta^r_{\nearrow}(x,v)$, and $\gamma^r_{\rightarrow}(x,v)$:

$$\gamma^r_{\nearrow}(x,v) = \max\left\{|S_r(P)| : P \in \mathcal{P}^r_{\nearrow}(x,v)\right\},$$
$$\beta^r_{\nearrow}(x,v) = \max\left\{|S_r(P - \{v\})| : P \in \mathcal{P}^r_{\nearrow}(x,v)\right\},$$
$$\gamma^r_{\rightarrow}(x,v) = \max\left\{|S_r(P)| : P \in \mathcal{P}^r_{\rightarrow}(x,v)\right\}.$$

Finally, let $\gamma^r(x,v) = \max\left\{\gamma^r_{\searrow}(x,v), \gamma^r_{\rightarrow}(x,v), \gamma^r_{\nearrow}(x,v)\right\}$ be the maximum of $|S_r(P)|$ over all isometric (x,v)-paths P. In our algorithm, we will need also to consider the case where $v = x$ as an initial case. For practical reasons, we let $\gamma^r(x,x) = \gamma^r_{\searrow}(x,x) = \gamma^r_{\rightarrow}(x,x) = \gamma^r_{\nearrow}(x,x) = 1$ and $\beta^r_{\searrow}(x,x) = \beta^r_{\nearrow}(x,x) = 0$. Based on the above notations and Lemma 7, we have the following observation.

▶ **Observation 11.** *For any graph G and any vertex r of G, we have $ipco\left(\overrightarrow{G_r}\right) = ipacc\left(\overrightarrow{G_r}\right) = \max_{x,v} \gamma^r(x,v)$ and $ipco(G) = ipacc(G) = \min_r \max_{x,v} \gamma^r(x,v)$.*

Observation 11 implies that to compute the isometric path complexity of a graph it is enough to compute the parameter $\gamma^r(x,v)$ for all $r, x, v \in V(G)$ in polynomial time. In the next section, we focus on achieving this goal without computing explicitly any of the sets $\mathcal{P}^r_{\searrow}(x,v)$, $\mathcal{P}^r_{\rightarrow}(x,v)$ or $\mathcal{P}^r_{\nearrow}(x,v)$. (Note that the size of these sets could be exponential in the number of vertices of the graph).

3.2 An algorithm to compute $\gamma^r(x, v)$

Throughout this section, let r and x be two fixed vertices of G. We shall call r as the "root" and x as the "source" vertex. The objective of this section is to compute the parameter $\gamma^r(x, v)$ for all vertices $v \in V(G)$.

In the sequel, since we always refer to a fixed root r and source x, we omit r and x and use the shorthand $\gamma(v)$ for $\gamma^r(x, v)$. We do the same with the notations $\gamma_{\nearrow}(v)$, $\gamma_{\rightarrow}(v)$, $\gamma_{\searrow}(v)$, $\beta_{\nearrow}(v)$, and $\beta_{\searrow}(v)$ that also refer to fixed vertices r and x In the following lemmas, we shall provide explicit (recursive) formulas to compute $\gamma_{\nearrow}(v)$, $\gamma_{\rightarrow}(v)$, $\gamma_{\searrow}(v)$, $\beta_{\nearrow}(v)$, and $\beta_{\searrow}(v)$. Using these formulas, we will show how to compute $\gamma(v)$ for all $v \in V(G)$ in a total of $O(|E(G)|)$-time.

▶ **Observation 12.** *If r is the root vertex, x the source vertex, and v is distinct from x, then*

$$\beta_{\searrow}(v) = \max\{\gamma(u) : u \in I(x, v) \cap N(v);\ \mathsf{d}\,(r, u) = \mathsf{d}\,(r, v) - 1\},$$
$$\beta_{\nearrow}(v) = \max\{\gamma(u) : u \in I(x, v) \cap N(v);\ \mathsf{d}\,(r, u) = \mathsf{d}\,(r, v) + 1\}.$$

▶ **Lemma 13 (*).** *If r is the root vertex, x the source vertex, and v is distinct from x, then $\gamma_{\rightarrow}(v) = \max\{1 + \gamma(u) : u \in I(x, v) \cap N(v);\ \mathsf{d}\,(r, u) = \mathsf{d}\,(r, v)\}$.*

▶ **Lemma 14 (*).** *If r is the root vertex, x the source vertex, and v is a vertex distinct from x, then $\gamma_{\searrow}(v) = \max\{\max\{\gamma_{\searrow}(u), \gamma_{\rightarrow}(u), \beta_{\nearrow}(u)+1\} : u \in I(x, v) \cap N(v);\ \mathsf{d}\,(r, u) = \mathsf{d}\,(r, v) - 1\}$*

▶ **Lemma 15 (*).** *If r is the root vertex, x the source vertex, and v is a vertex distinct from x, then $\gamma_{\nearrow}(v) = \max\{\max\{\gamma_{\nearrow}(u), \gamma_{\rightarrow}(u), \beta_{\searrow}(u)+1\} : u \in I(x, v) \cap N(v);\ \mathsf{d}\,(r, u) = \mathsf{d}\,(r, v) + 1\}$.*

Now we provide a BFS based algorithm to compute the above parameters. Let r and x be fixed root and source vertices of G, respectively. For a vertex $u \in V(G)$, let $\mathcal{D}(u) = \{\gamma(u), \gamma_{\nearrow}(u), \gamma_{\rightarrow}(u), \gamma_{\searrow}(u), \beta_{\nearrow}(u), \beta_{\searrow}(u)\}$. Clearly, the set $\mathcal{D}(x)$ can be computed in constant time. Now let X_i be the set of vertices at distance i from x. Clearly, the sets X_i can be computed in $O(|E(G)|)$-time (using a BFS) and $X_0 = \{x\}$. Let $i \geq 1$ be an integer and assume that for all vertices $u \in \bigcup_{j=0}^{i-1} X_j$, the set $\mathcal{D}(u)$ is already computed. Let $v \in X_i$ be a vertex. Then due to the formulas given in Observation 12 and Lemmas 13–15, the set $\mathcal{D}(v)$ can be computed by observing only the sets $\mathcal{D}(u)$, $u \in N(v) \cap X_{i-1}$. Hence, for all vertices $v \in V(G)$, the sets $\mathcal{D}(v)$ can be computed in a total of $O(|E(G)|)$ time. Hence, we have the following lemma.

▶ **Lemma 16.** *For a root vertex r and source vertex x, for all vertices $v \in V(G)$, the value $\gamma^r(x, v)$ can be computed in $O(|E(G)|)$ time.*

We can now finish the proof of Theorem 2. Let G be a graph with n vertices and m edges. For a root vertex r, by applying Lemma 16, for every source $x \in V(G)$, it is possible to compute $ipco\left(\overrightarrow{G_r}\right) = \max_{x,v} \gamma^r(x, v)$ in $O(nm)$ time. By repeating this for every root $r \in V(G)$, it is possible to compute $ipco\,(G) = \min_r ipco\left(\overrightarrow{G_r}\right)$ in $O(n^2 m)$ time.

4 Proof of Theorem 3

First we prove Theorem 3a. We recall the definition of Gromov products [20] and its relation with hyperbolicity. For three vertices r, x, y of a graph G, the Gromov product of x, y with respect to r is defined as $(x|y)_r = \frac{1}{2}\left(\mathsf{d}\,(x, r) + \mathsf{d}\,(y, r) - \mathsf{d}\,(x, y)\right)$. Then, a graph G is δ-hyperbolic [9, 20] if and only if for any four vertices x, y, z, r, we have $(x|y)_r \geq \min\{(x|z)_r, (y|z)_r\} - \delta$.

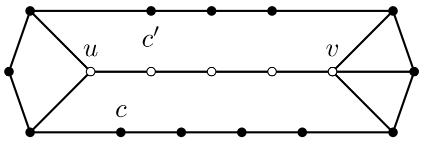


Figure 2 An example of a 4-fat turtle. Let C be the cycle induced by the black vertices, P be the path induced by the white vertices. Then the tuple $(4, C, P, c, c')$ defines a 4-fat turtle.

Let G be a graph with hyperbolicity at most δ. Due to Lemma 7, in order to prove Theorem 3a, it is enough to show that $ipacc(G) \leq 4\delta + 3$. Aiming for a contradiction, let r be a vertex of G and P be an isometric path such that $|A_r(P)| \geq 4\delta + 4$. Let $a_1, a_2, \ldots, a_{2\delta+2}, \ldots, a_{4\delta+4}$ be the vertices of $A_r(P)$ ordered as they are encountered while traversing P from one end-vertex to the other. Let $x = a_1, z = a_{2\delta+2}, y = a_{4\delta+4}$. Let Q denote the (y,z)-subpath of P. Observe that, $|A_r(Q)| \geq 2\delta + 2$. Then we have $(x|y)_r \geq \min\{(x|z)_r, (y|z)_r\} - \delta$. Without loss of generality, assume that $(x|z)_r \leq (y|z)_r$. Hence,

$$\begin{aligned}
(x|y)_r &\geq (x|z)_r - \delta \\
\mathsf{d}(x,r) + \mathsf{d}(y,r) - \mathsf{d}(x,y) &\geq \mathsf{d}(x,r) + \mathsf{d}(z,r) - \mathsf{d}(x,z) - 2\delta \\
\mathsf{d}(y,r) - \mathsf{d}(x,y) &\geq \mathsf{d}(z,r) - \mathsf{d}(x,z) - 2\delta \\
\mathsf{d}(y,r) - \mathsf{d}(z,r) + 2\delta &\geq \mathsf{d}(x,y) - \mathsf{d}(x,z) \\
\mathsf{d}(y,r) - \mathsf{d}(z,r) + 2\delta &\geq \mathsf{d}(y,z) \\
\mathsf{d}(y,z) &\leq |\mathsf{d}(y,r) - \mathsf{d}(z,r)| + 2\delta.
\end{aligned}$$

But this directly contradicts Proposition 9, which implies that $\mathsf{d}(y,z) \geq |\mathsf{d}(y,r) - \mathsf{d}(z,r)| + |A_r(Q)| - 1 \geq |\mathsf{d}(y,r) - \mathsf{d}(z,r)| + 2\delta + 1$. This completes the proof of Theorem 3a.

Now, we shall prove Theorems 3b and 3c. First, we shall define the notions of *t-theta*, *t-prism*, and *t-pyramid* [28]. For an integer $t \geq 1$, a *t-prism* is a graph made of three vertex-disjoint induced paths $P_1 = a_1 \ldots b_1$, $P_2 = a_2 \ldots b_2$, $P_3 = a_3 \ldots b_3$ of lengths at least t, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles and no edges exist between the paths except those of the two triangles. For an integer $t \geq 1$, a *t-pyramid* is a graph made of three induced paths $P_1 = a \ldots b_1$, $P_2 = a \ldots b_2$, $P_3 = a \ldots b_3$ of lengths at least t, two of which have lengths at least $t+1$, they are pairwise vertex-disjoint except at a, such that $b_1b_2b_3$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident to a. For an integer $t \geq 1$, a *t-theta* is a graph made of three internally vertex-disjoint induced paths $P_1 = a \ldots b$, $P_2 = a \ldots b$, $P_3 = a \ldots b$ of lengths at least $t+1$, and such that no edges exist between the paths except the three edges incident to a and the three edges incident to b. A graph G is *(t-theta, t-pyramid, t-prism)*-free if G does not contain any induced subgraph isomorphic to a t-theta, t-pyramid or t-prism. When $t = 1$, (t-theta, t-pyramid, t-prism)-free graphs are exactly (theta, prism, pyramid)-free graphs.

Now, we shall show that the isometric path antichain cover number of (t-theta, t-pyramid, t-prism)-free graphs are bounded above by a linear function on t. We shall show that, when the isometric path antichain cover number of a graph is large, the existence of a structure called "t-fat turtle" (defined below) as an induced subgraph is forced, which cannot be present in a (($t-1$)-theta, ($t-1$)-pyramid, ($t-1$)-prism)-free graph.

▶ **Definition 17.** *For an integer $t \geq 1$, a "t-fat turtle" consists of a cycle C and an induced (u,v)-path P of length at least t such that all of the following hold:*
(a) $V(P) \cap V(C) = \emptyset$,

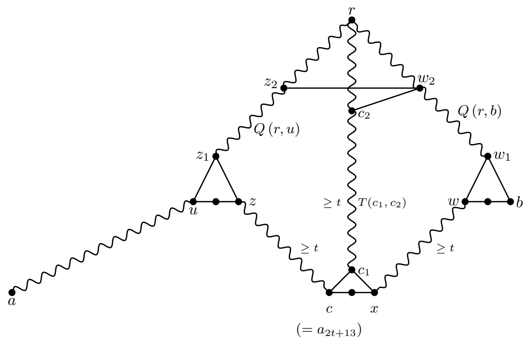


Figure 3 Illustration of the notations used in the proof of Lemma 20.

(b) *For any vertex* $w \in (V(P) \setminus \{u,v\})$, $N(w) \cap V(C) = \emptyset$ *and both* u *and* v *have at least one neighbour in* C,

(c) *For any vertex* $w \in N(u) \cap V(C)$ *and* $w' \in N(v) \cap V(C)$, *the distance between* w *and* w' *in* C *is at least* t,

(d) *There exist two vertices* $\{c, c'\} \subset V(C)$ *and two distinct components* C_u, C_v *of* $C - \{c, c'\}$ *such that* $N(u) \cap V(C) \subseteq V(C_u)$ *and* $N(v) \cap V(C) \subseteq V(C_v)$.

The tuple (t, C, P, c, c') *defines the* t*-fat turtle. See Figure 2 for an example.*

In the following observation, we show that any (t-theta, t-pyramid,t-prism)-free graph cannot contain a $(t+1)$-fat turtle as an induced subgraph.

▶ **Lemma 18 (*).** *For some integer* $t \geq 1$, *let* G *be a graph containing a* $(t+1)$*-fat turtle as an induced subgraph. Then* G *is not (*t*-theta,* t*-pyramid,* t*-prism)-free.*

In the remainder of this section, we shall prove that there exists a linear function $f(t)$ such that if the isometric path antichain cover number of a graph is more than $f(t)$, then G is forced to contain a $(t+1)$-fat turtle as an induced subgraph, and therefore is not (t-theta, t-pyramid,t-prism)-free. We shall use the following observation.

▶ **Observation 19 (*).** *Let* G *be a graph,* r *be an arbitrary vertex,* P *be an isometric* (u,v)*-path in* G *and* Q *be a subpath of an isometric* (v,r)*-path in* G *such that one endpoint of* Q *is* v. *Let* P' *be the maximum* (u,w)*-subpath of* P *such that no internal vertex of* P' *is a neighbour of some vertex of* Q. *We have that* $|A_r(P')| \geq |A_r(P)| - 3$.

▶ **Lemma 20.** *For an integer* $t \geq 1$, *let* G *be a graph with* $ipacc(G) \geq 8t + 64$. *Then* G *has a* $(t+1)$*-fat turtle as an induced subgraph.*

Proof. Let r be a vertex of G such that $ipacc\left(\overrightarrow{G_r}\right)$ is at least $8t+64$. Then there exists an isometric path P such that $|A_r(P)| \geq 8t + 64$. Let the two endpoints of P be a and b. (See Figure 3.) Let u be a vertex of P such that $\mathsf{d}(r,u) = \mathsf{d}(r,P)$. Let $P(a,u)$ be the (a,u)-subpath of P and $P(b,u)$ be the (b,u)-subpath of P. Both $P(a,u)$ and $P(b,u)$ are isometric paths and observe that either $|A_r(P(a,u))| \geq 4t + 32$ or $|A_r(P(b,u))| \geq 4t + 32$. Without loss of generality, assume that $|A_r(P(b,u))| \geq 4t + 32$. Let $Q(r,b)$ be an isometric (b,r)-path in G. First observe that u is not adjacent to any vertex of $Q(r,b)$. Otherwise, $\mathsf{d}(u,b) \leq 2 + \mathsf{d}(r,b) - \mathsf{d}(r,u)$, which contradicts Proposition 9. Let $P(u,w)$ be the maximum (u,w)-subpath of $P(b,u)$ such that no internal vertex of $P(u,w)$ is a neighbour of $Q(r,b)$. Note that $P(u,w)$ is an isometric path and w has a neighbour in $Q(r,b)$. Applying Observation 19, we have the following:

▷ Claim 21. $|A_r(P(u,w))| \geq 4t + 29$.

Let $Q(r,u)$ be any isometric (u,r)-path of G. Observe that w is not adjacent to any vertex of $Q(r,u)$. Otherwise, $\mathsf{d}(u,w) \leq 2 + \mathsf{d}(r,u) - \mathsf{d}(r,w)$, which contradicts Proposition 9. Let $P(z,w)$ be the maximum (z,w)-subpath of $P(u,w)$ such that no internal vertex of $P(z,w)$ has a neighbour in $Q(r,u)$. Observe that $P(z,w)$ is an isometric path, and z has a neighbour in $Q(r,u)$. Again applying Observation 19, we have the following:

▷ Claim 22. $|A_r(P(z,w))| \geq 4t + 26$.

Let $a_1, a_2, \ldots, a_k$ be the vertices of $A_r(P(z,w))$ ordered according to their appearance while traversing $P(z,w)$ from z to w. Due to Claim 22, we have that $k \geq 4t + 26$. Let $c = a_{2t+13}$ and $Q(r,c)$ denote an isometric (c,r)-path. Let $T(r,c_1)$ be the maximum subpath of $Q(r,c)$ such that no internal vertex of $T(r,c_1)$ is adjacent to any vertex of $P(z,w)$. Observe that neither z nor w can be adjacent to c_1 (due to Proposition 9). Morevoer, if c_1 is a vertex of $P(z,w)$ then we must have $c_1 = c$.

▷ Claim 23 (*). Let x be a neighbour of c_1 in $P(z,w)$, X be the (x,b)-subpath of $P(u,b)$ and Y be the (x,u)-subpath of $P(u,b)$. Then $|A_r(X)| \geq 2t + 11$ and $|A_r(Y)| \geq 2t + 11$.

Let $T(c_1,c_2)$ be the maximum (c_1,c_2)-subpath of $T(c_1,r)$ such that no internal vertex of $T(c_1,c_2)$ is adjacent to a vertex of $Q(r,b)$ or $Q(r,u)$. We have the following claim.

▷ Claim 24 (*). The length of $T(c_1,c_2)$ is at least $t+3$.

The path $T(c_1,c_2)$ forms the first ingredient to extract a $(t+1)$-fat turtle. Let z_1 be the neighbour of z in $Q(r,u)$ and w_1 be the neighbour of w in $Q(r,b)$. We have the following claim.

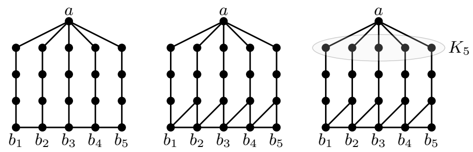
▷ Claim 25 (*). The vertices w_1 and z_1 are non adjacent.

Now we shall construct a (w_1,z_1)-path as follows: Consider the maximum (w_1,w_2)-subpath, say $T(w_1,w_2)$, of $Q(r,b)$ such that no internal vertex of $T(w_1,w_2)$ has a neighbour in $Q(r,u)$. Similarly, consider the maximum (z_1,z_2)-subpath, say $T(z_1,z_2)$, of $Q(r,u)$ such that no internal vertex of $T(z_1,z_2)$ is a neighbour of w_2. (Note that it is possible that $z_2 = w_2 = r$.) Let T be the path obtained by taking the union of $T(w_1,w_2)$ and $T(z_1,z_2)$. Observe that z_2 must be a neighbour of w_2 and T is an induced (w_1,z_1)-path. The definitions of T and $P(z,w)$ imply that their union induces a cycle Z. Here we have the second and final ingredient to extract the $(t+1)$-fat turtle.

Suppose that c_2 has a neighbour in T. Let T' be the maximum subpath of $T(c_1,c_2)$ which is vertex-disjoint from Z. (Note that if $c_1 = c$ or $c_2 \in \{w_2, z_2\}$ (e.g. when $c_2 = w_2 = z_2 = r$), $T(c_1,c_2)$ may share vertices with Z.) Due to Claim 24, the length of T' is at least $t+1$. Let e_1 and e_2 be the end-vertices of T'. Observe the following.

- Each of e_1 and e_2 has at least one neighbour in Z.
- $Z - \{z,w\}$ contains two distinct components C_1, C_2 such that for $i \in \{1,2\}$, $N(e_i) \cap V(Z) \subseteq V(C_i)$.
- For a vertex $e'_1 \in N(e_1) \cap V(Z)$ and $e'_2 \in N(e_2) \cap V(Z)$, the distance between e'_1 and e'_2 is at least $t+1$. This statement follows from Claim 23.

Hence, we have that the tuple $(t+1, Z, T', z, w)$ defines a $(t+1)$-fat turtle. Now consider the case when c_2 does not have a neighbour in T. By definition, c_2 has at least one neighbour in $Q(r,u)$ or $Q(r,b)$. Without loss of generality, assume that c_2 has a neighbour c_3 in $Q(r,u)$ such that the (z_2,c_3)-subpath, say, T'' of $Q(r,u)$ has no neighbour of c_2 other than c_3. Observe that the path $T^* = (T' \cup (T'' - \{z_2\}))$ is vertex-disjoint from Z and has length at least $t+1$. Let e_1, e_2 be the two end-vertices of T^*. Observe the following.



(a) (b) (c)

Figure 4 (a) X_4 (b) Y_4 (c) Z_4.

- Each of e_1 and e_2 has at least one neighbour in Z.
- $Z - \{z, w\}$ contains two distinct components C_1, C_2 such that for $i \in \{1, 2\}$, $N(e_i) \cap V(Z) \subseteq V(C_i)$.
- For a vertex $e'_1 \in N(e_1) \cap V(Z)$ and $e'_2 \in N(e_2) \cap V(Z)$, the distance between e'_1 and e'_2 is at least $t + 1$. This statement follows from Claim 23.

Hence, $(t + 1, Z, T^*, z, w)$ is a $(t + 1)$-fat turtle. ◀

Proof of Theorem 3b. Lemma 7, 18 and 20 together imply Theorem 3b.

▶ **Lemma 26 (*).** *Any outerstring graph is (4-theta, 4-prism, 4-pyramid)-free.*

Proof of Theorem 3c. Lemma 7, 18, 20, and 26 together imply Theorem 3c.

5 Proof of Theorem 4

We shall provide a construction for every $k \geq 4$, this implies the statement of Theorem 4 for any $k \geq 1$. First we shall prove Theorem 4a. For a fixed integer $k \geq 4$, first we describe the construction of a graph X_k as follows. Consider $k + 1$ paths $P_1, P_2, \ldots, P_{k+1}$ each of length k and having a common endvertex a. For $i \in [k + 1]$, let the other endvertex of P_i be denoted as b_i. Moreover, for $i \in [k + 1]$, let the neighbours of a and b_i in P_i be denoted as a'_i and b'_i, respectively. For $i \in [k]$, introduce an edge between b_i and b_{i+1}. The resulting graph is denoted X_k and the special vertex a is the *apex* of X_k. See Figure 4(a). For a fixed integer $k \geq 4$, consider the graph X_k and for each $i \in [k]$, introduce an edge between b_i and b'_{i+1}. Let Y_k denote the resulting graph and the special vertex a is the *apex* of Y_k. See Figure 4(b). For a fixed integer $k \geq 4$, consider the graph Y_k and for each $\{i, j\} \subseteq [k]$, introduce an edge between a'_i and a'_j. Let Z_k denote the resulting graph and the special vertex a is the *apex* of Z_k. See Figure 4(c). We prove the following lemmas.

▶ **Lemma 27 (*).** *For $k \geq 4$, let G be the graph constructed by taking two distinct copies of X_k and identifying the two apex vertices. Then G is a wheel-free, (pyramid, prism)-free graph with treewidth 2, hyperbolicity at least $\lceil \frac{k}{2} \rceil - 1$ and ipacc$(G) \geq k$.*

▶ **Lemma 28 (*).** *For $k \geq 4$, let G be the graph constructed by taking two distinct copies of Y_k and identifying the two apex vertices. Then G is a wheel-free, (theta, prism)-free graph with treewidth 3, hyperbolicity at least $\lceil \frac{k}{2} \rceil - 1$, and ipacc$(G) \geq k$.*

▶ **Lemma 29 (*).** *For $k \geq 4$, let G be the graph constructed by taking two distinct copies of Z_k and identifying the two apex vertices. Then G is a wheel-free, (theta, pyramid)-free graph with hyperbolicity at least $\lceil \frac{k}{2} \rceil - 1$ and ipacc$(G) \geq k$.*

Lemma 7, 27, 28, 29 imply Theorem 4.

6 Conclusion

In this paper, we have introduced the new graph parameter *isometric path complexity*. We have shown that the isometric path complexity of a graph with n vertices and m edges can be computed in $O(n^2m)$-time. It would be interesting to provide a faster algorithm to compute the isometric path complexity of a graph. We have derived upper bounds on the isometric path complexity of three seemingly (structurally) different classes of graphs, namely hyperbolic graphs, (theta,pyramid,prism)-free graphs and outerstring graphs. An interesting direction of research is to generalise the properties of hyperbolic graphs or (theta,pyramid,prism)-free graphs to graphs with bounded isometric path complexity.

Note that, in our proofs we essentially show that, for any graph G that belongs to one of the above graph classes, any vertex v of G, and any isometric path P of G, the path P can be covered by a small number of v-rooted isometric paths. This implies our "choice of the root" is arbitrary. This motivates the following definition. The *strong isometric path complexity* of a graph $G$is the minimum integer k such that for each vertex $v \in V(G)$ we have that the vertices of any isometric path P of G can be covered by k many v-rooted isometric paths. Our proofs imply that the strong isometric path complexity of graphs from all the graph classes addressed in this paper are bounded. We also wonder whether one can find other interesting graph classes with small (strong) isometric path complexity.

Our results imply a constant-factor approximation algorithm for Isometric Path Cover on hyperbolic graphs, (theta, pyramid, prism)-free graphs and outerstring graphs. However, the existence of a constant-factor approximation algorithm for Isometric Path Cover on general graphs is not known (an $O(\log n)$-factor approximation algorithm is designed in [26]).

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