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# The shadows of quantum gravity on Bell's inequality

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This study delves into the validity of quantum mechanical operators in the context of quantum gravity, recognizing the potential need for their generalization. A primary objective is to investigate the repercussions of these generalizations on the inherent non-locality within quantum mechanics, as exemplified by Bell's inequality. Additionally, the study scrutinizes the consequences of introducing a non-zero minimal length into the established framework of Bell's inequality. The findings contribute significantly to our theoretical comprehension of the intricate interplay between quantum mechanics and gravity. Moreover, this research explores the impact of quantum gravity on Bell's inequality and its practical applications within quantum technologies, notably in the realms of device-independent protocols, quantum key distribution and quantum randomness generation.



## I. INTRODUCTION

The quantum realm is governed by the Heisenberg uncertainty principle (HUP), which mandates that the Hamiltonian be written as the starting point, leading to the Schrodinger equation and, eventually, the eigenvalues and wave function of the quantum system under consideration. In Heisenberg's formulation of quantum mechanics (QM) in the Hilbert space, we encounter states rather than wave functions (although they are connected). In general, QM fails to produce satisfactory solutions for systems featuring the Newtonian gravitational potential in their Hamiltonian. Therefore, in conventional and widely accepted quantum mechanics, gravity is not accounted for in terms of its operators or corresponding Hilbert space (quantum states) carrying gravitational information.

The incompatibility of gravity and quantum mechanics is not limited to Newtonian gravity and persists even when general relativity is considered. On the other hand, the existence of gravity, even in a purely Newtonian regime, leads to a non-zero minimum (of the order of $10^{-35}$m (Planck length) [1]) for the uncertainty in position measurement [1–4]. Consistently, various scenarios of quantum gravity (QG), like String theory, also propose a non-zero minimal for the length measurement [3, 4]. The non-zero minimal length existence may affect the operators, and it leads to the generalization of HUP, called generalized uncertainty principle (GUP), [3, 4].

Operators and system states in QG may differ from those in QM. They are, in fact, functions of ordinary operators that appear in QM [4]. For instance, when considering the first order of the GUP parameter ($\beta$), we find that the momentum operator $\hat{P}$ can be expressed as $\hat{p}(1+\beta\hat{p}^2)$, where $\hat{P}$ and $\hat{p}$ represent momentum operators in QG and QM, respectively. In this representation, $\beta$ is positive, and the position operator remains unchanged [4]. It follows that gravity could impact our understanding of classical physics-based operator sets that have been established by QM [5, 6]. Consequently, it is generally possible to write $\hat{O} = \hat{o} + \beta\hat{o}_p$, where $\hat{O}$ and $\hat{o}$ are operators in QG and QM, respectively, and $\hat{o}_p$ is the first-order correction obtained using perturbation theory [7].

| QM | QG |
|---|---|
| $\Delta\hat{x}\Delta\hat{p} \geq \frac{\hbar}{2}$ (HUP) | $\Delta\hat{x}\Delta\hat{P} \geq \frac{\hbar}{2}[1+\beta(\Delta\hat{P})^2]$ (GUP) |
| $\hat{o}$ | $\hat{O} = \hat{o} + \beta\hat{o}_p$ |
| $\|\psi\rangle$ | $\|\psi_{GUP}\rangle = \|\psi\rangle + \beta\|\psi\rangle_p$ |

TABLE I: A comparison between QM and QG (up to the first order of $\beta$). Here, $|\psi\rangle$ and $|\psi_{GUP}\rangle$ denote the quantum states in QM and QG, respectively, and $|\psi\rangle_p$ is also calculable using the perturbation theory.

Motivated by the correlation between HUP and quantum non-locality (which is easily demonstrated in the square of Bell's inequality) [8–10], as well as the impact of GUP on operators, particularly angular momentum [11, 12], recent studies have revealed that minimal length can alter the square of Bell's operator [13]. Furthermore, GUP can affect the entanglement between energy and time, as evidenced by the results of a Franson experiment (which serves as a testing setup for time-energy entanglement) [14]. Table I clearly displays the generally expected modifications to operators and states resulting from minimal length. The term $|\psi\rangle_p$ indicates an increase in a quantum superposition, which is a probabilistic signal for entanglement enhancement [5, 6] and, therefore, non-locality beyond quantum mechanics [15]. It is apparent that gravity impacts the information bound [7].

The inquiry into the influence of special and general relativity (SR and GR) on Bell's inequality (quantum non-locality) has been extensively studied over the years [16–20]. The existing research on the effects of SR on Bell's inequality can be classified into three general categories, depending on the method of applying Lorentz transformations: (i) the operators change while the states remain unchanged, (ii) only the states undergo the Lorentz transformation while the operators remain unaltered (the reverse of the previous one), and (iii) both the operators and states are affected by the Lorentz transformation [21–32]. Furthermore, certain implications of GR and non-inertial observers have also been addressed in Refs. [33–36]. Given the ongoing effort to bridge QG with QM [37], exploring the effects of QG on quantum non-locality is deemed inevitable and advantageous.

Bell's theorem suggests that certain experimental outcomes are constrained if the universe adheres to local realism. However, quantum entanglement, which seemingly allows distant particles to interact instantaneously, can breach these constraints [45]. This led to cryptographic solutions like quantum key distribution (QKD) [50] and quantum random number generation (QRNG) [43, 46]. However, classical noise can enter QKDs and QRNGs during implementation, which hackers can exploit to gain partial information. A device-independent (DI) method was developed to address this, ensuring security when a particular correlation is detected, irrespective of device noise. DI protocols often hinge on non-local game violations, like the CHSH inequality [39]. Section IV delves into the impacts of QG on these applications.

In this study, our primary goal is to explore the ramifications of QG on Bell's inequality, specifically by investigating the implications of minimal length (up to the first order of $\beta$). To address this objective, we adopt a methodology analogous to the three scenarios previously examined concerning the effects of Special Relativity (SR) on quantum non-locality. To facilitate this exploration, we categorize the existing cases into three distinct groups, which we elaborate on in the following section. The paper concludes by providing a comprehensive summary of our research findings, shedding light on the intricate interplay between quantum mechanics and gravity, elucidating the impact of QG on Bell's inequality, and exploring potential applications within various quantum-based systems.

## II. BELL'S INEQUALITY AND THE IMPLICATIONS OF QG

In the framework of QM, assume two particles and four operators $\hat{A},\hat{A}',\hat{B},\hat{B}'$ with eigenvalues $\lambda^J$ ($J \in \{\hat{A},\hat{A}',\hat{B},\hat{B}'\}$), while the first (second) two operators act on the first (second) particle. Now, operators $\hat{j}=\frac{\hat{J}}{|\lambda^J|}\in\{\hat{a},\hat{a}',\hat{b},\hat{b}'\}$ have eigenvalues $\pm1$, and Bell's inequality is defined as

$$\langle\hat{B}\rangle\equiv\langle\hat{a}(\hat{b}+\hat{b}')+\hat{a}'(\hat{b}-\hat{b}')\rangle\leq2. \tag{1}$$

Taking into account the effects of QG (up to the first order), the operators are corrected as $\hat{J}_{GUP}=\hat{J}+\beta\hat{J}_p$ and $\hat{j}_{GUP}=\frac{\hat{J}+\beta\hat{J}_p}{|\lambda^J_{GUP}|}$ where $\lambda^J_{GUP}$ represents the eigenvalue of $\hat{J}_{GUP}$. Since QM should be recovered at the limit $\beta\to0$, one may expect $\lambda^J_{GUP}\simeq\lambda^J+\beta\lambda^J_p$. Moreover, as the $\beta\lambda^J_p$ term is perturbative, it is reasonable to expect $|\beta\frac{\lambda^J_p}{\lambda^J}|<<1$ leading to $|\lambda^J+\beta\lambda^J_p|=|\lambda^J|(1+\beta\frac{\lambda^J_p}{\lambda^J})$.

Applying modifications to the states, operators, or both in quantum gravity can result in three distinct situations. Similar studies conducted on the effects of SR on Bell's inequality have also revealed three cases [21–27, 32]. Therefore, it is necessary to consider the possibilities arising from these situations to understand the implications of quantum gravitational modifications. In the following paragraphs, we will examine these possibilities in depth.

### *1. Purely quantum mechanical entangled states in the presence of operators modified by QG*

Firstly, let us contemplate the scenario in which an entangled state ($|\xi\rangle$) has been prepared away from the QG influences. This implies that the objective has been accomplished using purely quantum mechanical procedures. Furthermore, it is assumed that an observer utilizes Bell measurements that are constructed through the incorporation of operators containing the QG corrections ($\hat{j}_{GUP}$). In the framework of QM, the violation amount of inequality (1) depends on the directions of Bell's measurements. Here, we have $\hat{j}=\hat{j}_{GUP}+\beta(\frac{\lambda^J_p}{\lambda^J}\hat{j}_{GUP}-\frac{\hat{J}_p}{|\lambda^J|})$ inserted into Eq. (1) to reach

$$\begin{aligned}&\langle\hat{B}_{GUP}\rangle\equiv \\ &\langle\hat{a}_{GUP}(\hat{b}_{GUP}+\hat{b}'_{GUP})+\hat{a}'_{GUP}(\hat{b}_{GUP}-\hat{b}'_{GUP})\rangle\leq2 \\ &-\langle\beta'_a\hat{a}_{GUP}(\hat{b}_{GUP}+\hat{b}'_{GUP})+\beta'_{a'}\hat{a}'_{GUP}(\hat{b}_{GUP}-\hat{b}'_{GUP})\rangle- \\ &\langle\hat{a}_{GUP}(\beta'_b\hat{b}_{GUP}+\beta'_{b'}\hat{b}'_{GUP})+\hat{a}'_{GUP}(\beta'_b\hat{b}_{GUP}-\beta'_{b'}\hat{b}'_{GUP})\rangle \\ &+\beta''_a\langle\hat{A}_{GUP}(\hat{b}_{GUP}+\hat{b}'_{GUP})+\hat{A}'_{GUP}(\hat{b}_{GUP}-\hat{b}'_{GUP})\rangle+ \\ &\beta''_b\langle\hat{a}_{GUP}(\hat{B}_{GUP}+\hat{B}'_{GUP})+\hat{a}'_{GUP}(\hat{B}_{GUP}-\hat{B}'_{GUP})\rangle,\end{aligned} \tag{2}$$

where $\beta'_j=\beta\frac{\lambda^J_p}{\lambda_J}$, $\beta''_j=\beta|\lambda_J|^{-1}$ and the last two expressions have been written using $\beta''_a=\beta''_{a'}$ and $\beta''_b=\beta''_{b'}$. In this manner, it is clearly seen that although the state is unchanged, in general, $\langle\hat{B}_{GUP}\rangle\neq\langle\hat{B}\rangle$ as the operators are affected by quantum features of gravity [12–14]. In studying the effects of SR on Bell's inequality, whenever the states remain unchanged, and Lorentz transformations only affect Bell's operator, a similar situation is also obtained [21–27, 32].

### *2. Purely quantum mechanical measurements and quantum gravitational states*

Now, let us consider the situation in which the Bell apparatus is built using purely quantum mechanical operators $j$, and the primary entangled state carries the Planck scale information, i.e., the quantum features of gravity. It means that the entangled state is made using the $j_{GUP}$ operators. A similar case in studies related to the effects of SR on Bell's inequality is the case where the Bell measurement does not go under the

Lorentz transformation while the system state undergoes the Lorentz transformation [21–27, 32]. In this setup, we have $|\xi_{GUP}\rangle = |\xi\rangle + \beta|\xi\rangle_p$ and thus

$$\begin{aligned} &\langle \xi_{GUP}|\hat{B}|\xi_{GUP}\rangle \equiv \langle \hat{B}\rangle_{GUP} = \langle \hat{B}\rangle + 2\beta\langle \xi|\hat{B}|\xi\rangle_p \\ &\Rightarrow \langle \hat{B}\rangle_{GUP} \leq 2\big(1 + \beta\langle \xi|\hat{B}|\xi\rangle_p\big). \end{aligned} \tag{3}$$

Correspondingly, if one considers a Bell measurement apparatus that yields $\langle \hat{B}\rangle = 2\sqrt{2}$, then such an apparatus cannot lead $\langle \hat{B}\rangle_{GUP}$ to its maximum possible value whenever Lorentz symmetry is broken [38].

*3. Bell's inequality in a purely quantum gravitational regime*

In deriving Bell's inequality, it is a significant step to ensure that the operators' eigenvalues are only either $\pm 1$, regardless of their origin, whether it be from QM or QG. If both the Bell measurement and the entangled state were prepared using the quantum gravitational operators, then it is evident that $\langle \xi_{GUP}|\hat{B}_{GUP}|\xi_{GUP}\rangle \leq 2$. This result indicates that, when considering the effects of QG on both the state and the operators, Bell's inequality and the classical regime's limit (which is 2 in the inequality) remain unchanged compared to the previous setups. The same outcome is also achieved when it comes to the relationship between SR and Bell's inequality, provided that both the system state and Bell's measurement undergo a Lorentz transformation [26].

## III. RESULTS

This section studies QG's implications on Bell's inequality, specifically within the contexts delineated earlier. The CHSH inequality, a specific form of Bell's inequality, provides a quantifiable limit on the correlations predicted by local hidden-variable theories [51]. A violation of the CHSH inequality underscores the inability of such approaches to account for the observed correlations in specific experiments with entangled quantum systems, as predicted by quantum mechanics [47].

Now, we define the scenario where there are two parties where an entangled pair is shared between them. The entangled state of two qubits can be represented by the Bell state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{4}$$

Alice and Bob each measure their respective states. They can choose between two measurement settings: $\hat{a}, \hat{a}'$ for Alice and $\hat{b}, \hat{b}'$ for Bob. The measurement results can be either $+1$ or $-1$. The expected value of the CHSH game using the above quantum strategy and the Bell state is given in Eq. 1. Classically, the maximum value of $\langle \hat{B}\rangle$ is 2. However, this value can reach $2\sqrt{2}$ with the quantum strategy, violating the CHSH inequality.

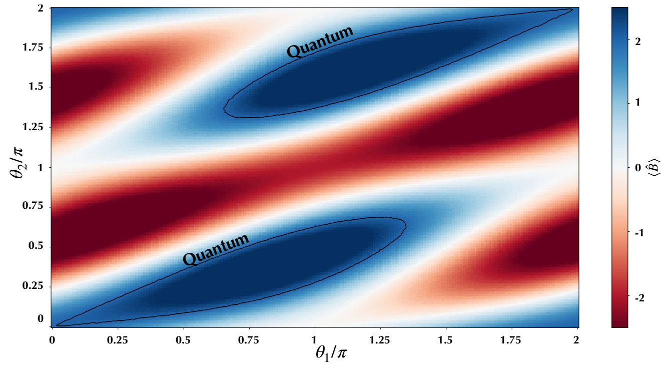


FIG. 1: The 2D plot of the CHSH inequality values as functions of detection angles $\theta_1/\pi$ and $\theta_2/\pi$. Different colors indicate different $\langle \hat{B}\rangle$ values, with a contour distinguishing the classical and quantum regions.

Fig. 1 illustrates that the CHSH inequality can be surpassed by judiciously selecting the appropriate detection angles, denoted as $\theta_1$ and $\theta_2$. The color bar quantitatively represents the value of the inequality, highlighting two distinct regions where the value exceeds the classical limit of 2. In Fig. 1, the simulation of Bell's inequality is conducted solely based on QM representations without incorporating QG impact.

Next, we consider the QG impact on Bell's inequality for various cases; better to say, we extend the well-known Bell inequality to account for the effects of QG. Equations 2 and 3 introduce new terms that are parameterized by $\beta$, a constant that quantifies the strength of quantum gravitational effects. These equations represent the modified Bell inequalities in the presence of QG. To explore the implications of these modifications, we plot, see Fig. 2, the degree of Bell inequality violation, denoted as $\langle \hat{B}\rangle$, as a function of $\theta$ for various angles $\beta$. Each sub-figure of Fig. 2 features three curves: the blue curve represents the Bell inequality in the framework of QM, while the orange and green curves correspond to the modified Bell inequalities given by Equations 2 and 3, respectively, which incorporate the effects of QG.

The results notably indicate an escalating violation of the Bell inequality with the introduction of QG. As the parameter $\beta$ increases, the violation surpasses the quantum mechanical limit of $\sqrt{8}$, signifying a more pronounced breach of the inequality. This implies that the presence of quantum gravitational effects could lead to a more pronounced violation of the Bell inequality than what is predicted by standard quantum mechanics.

## IV. APPLICATIONS

QKD and QRNG represent two extensively researched and commercially implemented areas where the applications of quantum mechanics come to life. While quantum

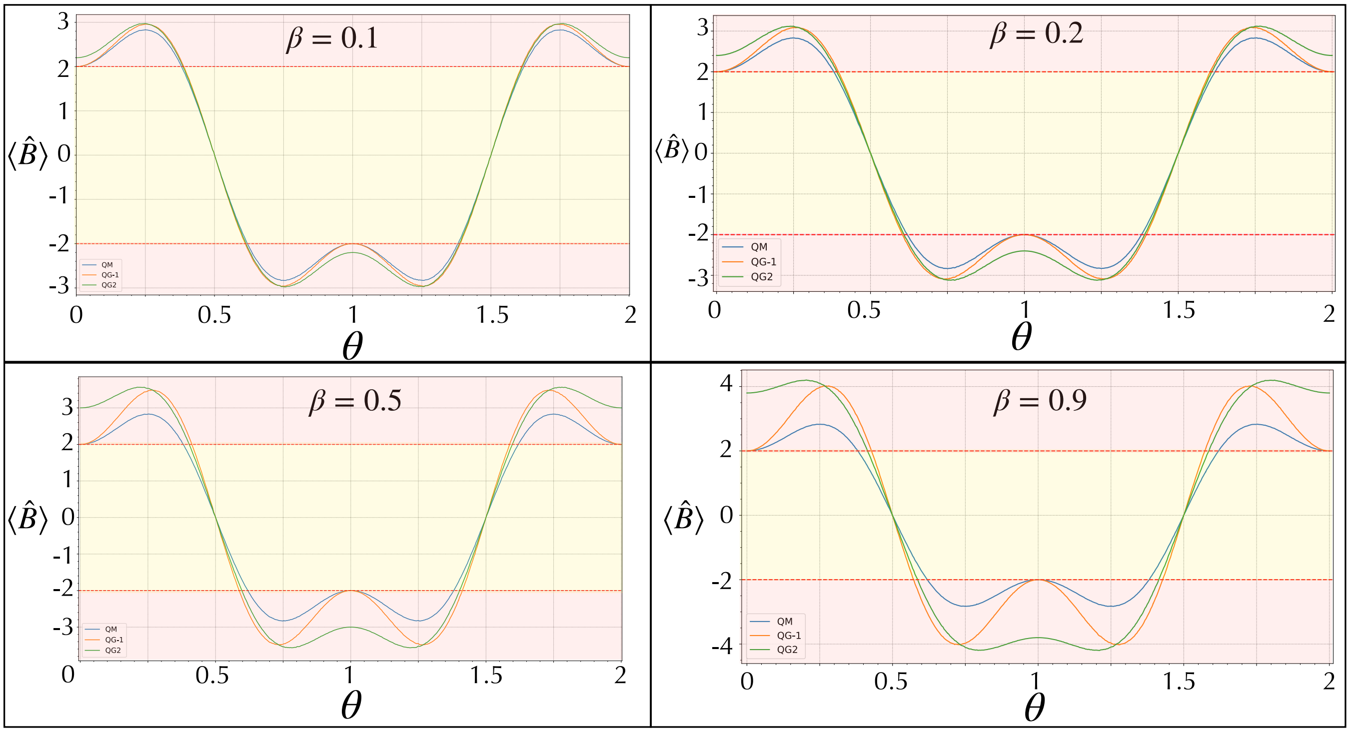


FIG. 2: The figure depicts the Bell inequality values as a function of $\theta$, the rotation angle modulating the measurement basis. Results are stratified by distinct $\beta$ values: 0.1, 0.2, 0.5, and 0.9. Each subplot features three curves—blue, orange, and green—corresponding to quantum mechanical (QM, Eq. 1), first quantum gravitational (QG-1, Eq. 2), and second quantum gravitational (QG-2, Eq. 3) formulations of the Bell inequality, respectively. The dual horizontal lines at $-2$ and 2 (colored in faded yellow) demarcate the classical regime, while the regions above and below these lines (highlighted in faded red) signify violations of the classical limit, thereby entering the quantum domain.

mechanics underpins the security of these systems, experimental imperfections can introduce vulnerabilities. To address this, DI protocols have been developed. These protocols harness the non-local correlations inherent in quantum entanglement. Importantly, they don't rely on an intricate understanding of the devices in use; their security is grounded solely in the observed violation of non-local correlations, such as the Bell inequalities. This approach offers a robust solution to the security challenges posed by device imperfections [40, 42].

In DI QKD, two distant parties share an entangled quantum state. They perform measurements on their respective parts of the state, and due to the non-local nature of entanglement, the outcomes of these measurements are correlated in a way that disobeys classical explanation. These correlations serve as the foundation for key generation, with the security of the key guaranteed by the violation of Bell inequalities. Basically, any eavesdropper attempting to intercept or tamper with the quantum states would disrupt these correlations, making their presence detectable.

The security and randomness of DI QRNG don't depend on trusting the intrinsic workings of the devices. Traditional QRNGs require detailed models and assumptions about the device, but in DI QRNGs, as long as observed outcomes violate Bell inequalities, one can be assured of the randomness. With the rise of quantum computers, many cryptographic methods are at risk. Nevertheless, the unpredictability in DI QRNG is more than just computationally hard for quantum computers; it's theoretically impossible to predict due to the inherent randomness of quantum processes [43, 46].

Incorporating the effects of QG in quantum information science and technology becomes an intellectual exercise and a practical necessity. Given the results in the previous section—significantly that QG effects can enhance the violation of Bell inequalities—let's consider its implications for quantum information science and technology and its applications.

The security of QKD is guaranteed by the quantum mechanical violation of Bell inequalities; increasing the violation value of Bell's inequality makes QKD even more secure against attacks. This disturbance changes the quantum correlations between Alice's and Bob's measurements. In other words, if the eavesdropper is listening in, the observed violations of Bell's inequalities at Alice's and Bob's ends will reduce, moving closer to what would be expected classically. Thus, if you start with a higher violation of Bell's inequalities (thanks to QG effects), you're raising the "quantumness" of your initial state. The higher this initial level, the more sensitive your system becomes to any eavesdropping activities. A significant drop in the observed Bell inequality violation from this higher baseline would more quickly and definitively signal the presence of eavesdropping, thus enabling quicker and more reliable detection of any security breaches.

DI protocols prevent the need for trust in the hardware by utilizing Bell inequality violations—the greater the violation, the higher the level of security. The introduction of QG effects adds an additional layer of robustness to DI protocols, fortifying them through quantum mechanical principles and integrating fundamental theories of nature. Similarly, for QRNGs, a heightened violation signifies a more quantum-coherent system, enhancing the quality of randomness, which comprises not merely an incremental advancement but a paradigmatic leap in the entropy of the generated random numbers. Consequently, this reduces the computational time required to achieve a given level of randomness and unpredictability, analogous to transitioning from conventional vehicular propulsion to advanced warp drives, all while adhering to the fundamental constraints of space-time.

More importantly, quantum gravity could offer richer quantum correlations in multipartite systems. Imagine a quantum network secured by quantum gravity effects—each additional party would enhance not just the computational power but the security, generating what could be termed "quantum gravity-secured entanglement." Enabling a brand-new platform for multiparty quantum computations and secret sharing protocols.

In summary, enhanced violations of Bell inequalities render QKD virtually impregnable, elevate QRNGs to sources of high-entropy randomness, and establish DI protocols as the epitome of trust-free security mechanisms. Dismissing QG as a purely academic endeavor could overlook its potential as a critical element in safeguarding quantum data against even the most advanced computational threats. If quantum mechanics is considered the apex of security and efficiency, the advent of QG compels a reevaluation. It promises to redefine the boundaries of what is secure, efficient, and trustworthy in quantum technologies.

## V. CONCLUSION

The study can be summarized by its two main components: *i*) the origin of entangled states and *ii*) Bell's measurement. Furthermore, the study has introduced the possibility of three outcomes depending on which cornerstone carries the quantum gravitational modifications. The first two scenarios suggest that if only one of the foundations stores the effects of QG, then a precise Bell measurement (depending on the value of $\beta$) could detect the effects of QG. This is due to the differences between $\langle\hat{B}\rangle$, $\langle\hat{B}_{GUP}\rangle$, and $\langle\hat{B}\rangle_{GUP}$. In the third case, Bell's inequality remains invariant if we consider the quantum aspects of gravity on both the states and the operators. Moreover, the results demonstrate that the presence of QG enhances Bell's inequality violation, thereby offering avenues for improving the security and performance of DI QRNG and QKD protocols.

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