On rectangle intersection graphs with stab number at most two

Dibyayan Chakraborty[1], Sandip Das[1], Mathew C. Francis[2], and Sagnik Sen[3]

[1] Indian Statistical Institute, Kolkata, India.
[2] Indian Statistical Institute, Chennai, India.
[3] Ramakrishna Mission Vivekananda Educational and Research Institute, Kolkata.

Abstract. Rectangle intersection graphs are intersection graphs of axis-parallel rectagles on the plane. A graph G is said to be a *k-stabbable rectangle intersection graph*, or *k-SRIG* for short, if it has a rectangle intersection representation in which k horizontal lines can be chosen such that each rectangle is intersected by at least one of them. The stab number of a graph G, denoted by $stab(G)$, is the minimum integer k such that G is a k-SRIG. In this paper, we introduce "natural" subclasses of 2-SRIG and study the containment relationship among them. We introduce a graph class named $(\mathcal{P},\mathcal{P})$-graph and give a linear time algorithm to recognise triangle-free $(\mathcal{P},\mathcal{P})$-graph. In this paper, we prove that finding CHROMATIC NUMBER is NP-complete even for 2-SRIGs. We also give a linear time algorithm to find the chromatic number of triangle-free 2-SRIGs.

1 Introduction

A *rectangle intersection representation* of a graph is a collection of axis-parallel rectangles on the plane such that each rectangle in the collection represents a vertex of the graph and two rectangles intersect if and only if the vertices they represent are adjacent in the graph. The graphs that have rectangle intersection representation are called *rectangle intersection graphs*. The *boxicity* $box(G)$ of a graph G is the minimum d such that G is representable as an intersection graph of d-dimensional (axis-parallel) hyper-rectangles. A graph G is an interval graph if $box(G) = 1$ and G is a rectangle intersection graph if $box(G) \leq 2$.

A *k-stabbed rectangle intersection representation* is a rectangle intersection representation, along with a collection of k horizontal lines called *stab lines*, such that every rectangle intersects at least one of the stab lines. A graph G is a *k-stabbable rectangle intersection* graph (*k-SRIG*), if there exists a k-stabbed rectangle intersection representation of G. The *stab number* of a rectangle intersection graph, denoted by $stab(G)$, is the minimum integer k such that there exists a k-stabbed rectangle intersection representation of G. In other words $stab(G)$ is the minimum integer k such that G is k-SRIG. A *k-exactly stabbed rectangle intersection representation* is a k-stabbed rectangle intersection representation in which every rectangle intersects exactly one of the stab lines. A graph G is a *k-exactly stabbable rectangle intersection* graph, or *k-ESRIG* for short, if there exists a k-exactly stabbed rectangle intersection representation of G. The *exact stab number* of a rectangle intersection graph, denoted by $estab(G)$, is the minimum integer k such that there exists a k-exactly stabbed rectangle intersection representation of G. In other words, $estab(G)$ is the minimum integer k such that G is k-ESRIG.

Boxicity of a graph has been an active field of research for many decades [1, 9–11, 14]. While recognizing graphs with boxicity at most d is NP-complete for all $d \geq 2$ [21, 26], there are efficient algorithms to recognize interval graphs, i.e. graphs with boxicity at most 1 [12,23]. There seems to be a "jump in the difficulty level" of problems as the boxicity of the input graph increases from 1 to 2. For example, the Maximum Independent Set and Chromatic Number problems, while being linear-time solvable for interval graphs, become NP-complete for rectangle intersection graphs (even with the rectangle intersection representation given as input) [17,22]. To understand the reason of this jump, the concept of stab number and exact stab number was introduced [7].The concept of stab number is a generalization of the idea behind a class of graphs known as "2SIG", which was introduced in an earlier paper [3]. Even though our definitions of 2-SRIG and 2-ESRIG are both slightly different from that of "2SIG", all three classes of graphs turn out to be equivalent [7].

As mentioned earlier, recognizing graphs with boxicity at most 2 is NP-complete. But there are efficient algorithms to recognize interval graphs. Over the years, researchers have defined several subclasses of interval graphs. The class of *proper interval graphs* and *unit interval* graphs are popular examples. It is well known that these two families are in fact equivalent. A short proof for this fact was given by Bogart and West [5]. The class *k-LengthINT* consists intersection graphs of intervals whose lengths are restricted to have at most k different sizes. These classes of interval graphs have been a popular topic of research [6,15,18–20,24]. It is known that $(k-1)$-LengthINT $\subsetneq k$-LengthINT [19], for each $k \geq 2$. Klavik et al. [19] introduced the class of *k-NestedINT*, the class of interval graphs which have representations with no $k+1$ intervals $I_0, I_2, \ldots, I_k$ such that $I_1 \subseteq I_2 \subseteq \ldots I_k$. In this paper, we introduce "natural" subclasses of 2-SRIG and study the containment relationship among them. We shall introduce a graph class named $(\mathcal{P}, \mathcal{P})$-graph (defined later) and give a linear time algorithm to recognise triangle-free $(\mathcal{P}, \mathcal{P})$-graph.

We also study the Chromatic Number of 2-SRIGs. The study of Chromatic Number of rectangle intersection graphs started in 1948, when Bielecki [4] asked whether the ratio ($\sigma(G)$) of Chromatic Number ($\chi(G)$) and Clique Number ($\omega(G)$) is independent of the number of vertices in the graph. This question was answered positively by Asplund and Grunbaum [2] in 1960, when they show that for a rectangle intersection graph G, $\sigma(G) \leq 4\omega(G) - 3$. The best known lower bound result was also obtained by Asplund and Grunbaum [2] which states that $\sigma(G) \geq 3$. Chalermsook [8] established the connection between Independence number and chromatic number of rectangle intersection graphs and also provided improved bounds of $\sigma(G)$ for special classes of rectangle intersection graphs. In this paper, we prove that Chromatic Num-

BER is NP-complete even for rectangle intersection graphs with stab number at most two. This strengthens the result of Imai and Asano [17]. Moreover, we show that the chromatic number of triangle-free 2-SRIGs can be found in linear time.

1.1 Our results

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A set of intervals is *proper* (or $\mathcal{P}$ for short) if no two intervals in the set is a subset of the other. Now we shall introduce the subclasses of 2-SRIG studied in this paper.

For a 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of a graph G let $y = a_1$ and $y = a_2$ be the stab lines with $a_1 < a_2$. Let $\mathcal{R}_t$ and $\mathcal{R}_b$ be the sets of intervals obtained by projecting the rectangles that intersect $y = a_2$ and $y = a_1$ respectively on the $x-$axis. The graph G is an $(\mathcal{I},\mathcal{P})$-graph if there is a 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of G such that $\mathcal{R}_b$ is a set of proper intervals. See Table 1 for an illustration. The graph G is an $(\mathcal{I},\mathcal{U})$-graph if there is a 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of G such that $\mathcal{R}_b$ is a set of unit intervals. The graph G is a $(\mathcal{P},\mathcal{P})$-graph if there is a 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of G such that both $\mathcal{R}_t$ and $\mathcal{R}_b$ are sets of proper intervals. The graph G is a $(\mathcal{P},\mathcal{U})$-graph if there is a 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of G such that $\mathcal{R}_t$ is a set of proper intervals and $\mathcal{R}_b$ is a set of unit intervals. The graph G is a $(\mathcal{U},\mathcal{U})$-graph if there is a 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of G such that both $\mathcal{R}_t$ and $\mathcal{R}_b$ are sets of unit intervals. A graph G is 2-stabbable unit square intersection graph or 2-SUIG, if G has a 2-stabbed rectangle intersection representation $\mathcal{R}$ in which all rectangles are unit squares. Following is the main result of this paper.

Theorem 1. *2-SUIG* $=$ *$(\mathcal{U},\mathcal{U})$-graphs* $\subset$ *$(\mathcal{P},\mathcal{U})$-graphs* $=$ *$(\mathcal{P},\mathcal{P})$-graphs* $\subset$ *$(\mathcal{I},\mathcal{U})$-graphs* $=$ *$(\mathcal{I},\mathcal{P})$-graphs* $\subset$ *2-ESRIG* $=$ *2-SRIG.*

Name of the graph class	$\mathcal{R}_t$	$\mathcal{R}_b$
$(\mathcal{I},\mathcal{P})$-graph	interval $(\mathcal{I})$	proper interval $(\mathcal{P})$
$(\mathcal{I},\mathcal{U})$-graph	interval $(\mathcal{I})$	unit interval $(\mathcal{U})$
$(\mathcal{P},\mathcal{P})$-graph	proper interval $(\mathcal{P})$	proper interval $(\mathcal{P})$
$(\mathcal{P},\mathcal{U})$-graph	proper interval $(\mathcal{P})$	unit interval $(\mathcal{U})$
$(\mathcal{U},\mathcal{U})$-graph	unit interval $(\mathcal{U})$	unit interval $(\mathcal{U})$

Table 1: Different subclasses of 2-SRIG.

In this paper, we also give an algorithm to decide whether a triangle-free graph is a $(\mathcal{P},\mathcal{P})$-graph or not. Specifically, we prove the following theorem.

Theorem 2. *Let G be a triangle-free graph. There is a $O(|V(G)|)$ time algorithm to decide if G is a $(\mathcal{P},\mathcal{P})$-graph.*

Given a graph G and an integer c, *c-coloring* of G is a mapping $\phi\colon V(G) \to [c]$ such that $\phi(u) \neq \phi(v)$ when $uv \in E(G)$. For an integer c, when there exists a c-coloring of G, we say G is *c-colorable*. The *chromatic number* of G is the minimum integer c for which there is a c-coloring of G. We study the computational complexity of finding the chromatic Number of 2-SRIGs. Specifically, we prove the following theorems.

Theorem 3. *Finding the chromatic Number is NP-hard even for rectangle intersection graphs with stab number at most two.*

Theorem 4. *Let G be a triangle-free 2-SRIG. There is a $O(|V(G)|)$ time algorithm to find the chromatic number of G.*

In Section 2, we give some definitions and notation that will be used throughout the paper. In Section 5 and Section 3 we prove Theorem 3 and Theorem 1, respectively. Finally we draw conclusion in Section 7.

2 Preliminaries

We present some definitions in this section. Let $N(v) = \{u \in V(G) \colon uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ denote the *open neighbourhood* and the *closed neighbourhood* of a vertex v, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced in G by the vertices in S, and by $G - S$ the graph obtained by removing the vertices in S from G. For an edge $e \in E(G)$, we denote by $G - e$ the graph on vertex set $V(G)$ having edge set $E(G) \setminus \{e\}$.

Let G be a rectangle intersection graph with rectangle intersection representation $\mathcal{R}$. A rectangle in $\mathcal{R}$ corresponding to the vertex v is denoted as r_v. All rectangles considered in this article are closed rectangles. Denote by x_v^+ (x_v^-), the $x-$coordinate of the right (left) bottom corner of r_v. Also y_v^+ (y_v^-) is the $y-$coordinate of the left top (bottom) corner of r_v. In other words, $r_v = [x_v^-, x_v^+] \times [y_v^-, y_v^+]$. The *span* of a vertex u, denoted as *span*(u), is the projection of r_u on the $X-$axis, i.e. $span(u) = [x_u^-, x_u^+]$. For two intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$, we write $I_1 < I_2$ to indicate that $b_1 < a_2$. Clearly, $I_1 \cap I_2 = \emptyset$ if and only if $I_1 < I_2$ or $I_2 < I_1$. For an edge $uv \in E(G)$, we define $span(uv) = span(u) \cap span(v)$. For an induced subgraph H of G,

$$span(H) = \bigcup_{u \in V(H)} span(u)$$

Note that when H is connected, $span(H)$ is an interval.

Let G be a k-SRIG with a k-stabbed rectangle intersection representation $\mathcal{R}$ in which the stab lines are $y = a_1$, $y = a_2$, ..., $y = a_k$, where $a_1 < a_2 < \cdots < a_k$. The *top* (resp. *bottom*) stab line of $\mathcal{R}$ is the stab line $y = a_k$ (resp. $y = a_1$). For $1 \leq i < k$, we say that the stab lines $y = a_i$ and $y = a_{i+1}$ are "consecutive". A vertex $u \in V(G)$ is said to be "on" a stab line if r_u intersects that stab line. Two vertices u, v of G "have a common stab" if there is some stab line that intersects both r_u and r_v. Similarly, a set of vertices is said to have a common stab if there is one stab line that intersects the rectangles corresponding to each of them. It is easy to see that if $uv \in E(G)$, then there must be either a stab line such that u and v are on it or two consecutive stab lines such that u is on one of them and v is on the other. We say $uv \in E(G)$ is a *bridge edge* in $\mathcal{R}$ if there are two consecutive stab lines such that u is on one of them and v is on the other. Whenever the k-stabbed rectangle intersection representation of a graph G under consideration is clear from the context, the terms r_u, x_u^-, x_u^+, y_u^-, y_u^+, for every vertex $u \in V(G)$ and usages such as "on a stab line", "have a common stab", "span" etc. are considered to be defined with respect to this representation.

The (h,w)-*grid* is the undirected graph G with $V(G) = \{(x,y) \colon x, y \in \mathbb{Z}, 1 \leq x \leq h, 1 \leq y \leq w\}$ and $E(G) = \{(u,v)(x,y) \colon |u-x| + |v-y| = 1\}$.

3 Proof of Theorem 1

The proof Theorem 1 is divided into seven lemmas. The following lemma was proved previously in an earlier paper, and therefore we only provide the statement here.

Lemma 1 ([7]). *The classes* 2*-SRIG and* 2*-ESRIG are equivalent.*

In Lemma 2, we shall prove that the family of $(\mathcal{I}, \mathcal{P})-$graphs is a proper subset of 2-ESRIG. But before that, we prove some observations and propositions.

Observation A *Let $\mathcal{R}$ be a 2-exactly stabbed representation of a graph G. Let uv be a bridge edge in $\mathcal{R}$ and let $S = \{w \in V(G) \colon span(w) \cap span(uv) \neq \emptyset\}$. Let $a, b \in V(G)$ such that $span(a) < span(uv) < span(b)$. Then a and b are in different connected components of $G - S$.*

Proof. Suppose for the sake of contradiction that a and b are in the same connected component C of $G - S$. As $span(C)$ is an interval that contains both $span(a)$ and $span(b)$, it is clear that $span(C)$ also contains $span(uv)$. But this means that C contains some vertex w such that $span(w) \cap span(uv) \neq \emptyset$, which is a contradiction.

Note that in the above observation, if $a \notin N[u] \cup N[v]$ and $b \notin N[u] \cup N[v]$, then because $S \subseteq N[u] \cup N[v]$, we can conclude that a and b are in different connected components of $G - (N[u] \cup N[v])$. We shall use this form of Observation A in several places.

Observation B *Let $\mathcal{R}$ be a 2-exactly stabbed rectangle intersection representation of a triangle-free graph G. Let $e_1, e_2 \in E(G)$ be two bridge edges in $\mathcal{R}$. Then, $span(e_1) \cap span(e_2) = \emptyset$.*

Proof. Suppose for the sake of contradiction that $I = span(e_1) \cap span(e_2) \neq \emptyset$. Let $e_1 = uv$, $e_2 = ab$ and u, a being distinct vertices. Then, we have $I \subseteq [x_u^-, x_u^+]$, $I \subseteq [x_v^-, x_v^+]$, $I \subseteq [x_a^-, x_a^+]$, and $I \subseteq [x_b^-, x_b^+]$. Let us assume without loss of generality that u and a are on the bottom stab line and that v and b are on the top stab line. Now observe that if $y_u^+ \geq y_a^+$, then u, a, b form a triangle in G ($(I \times [y_b^-, y_a^+]) \subseteq r_u \cap r_a \cap r_b$) and that if $y_u^+ < y_a^+$, then a, u, v form a triangle in G ($(I \times [y_v^-, y_u^+]) \subseteq r_a \cap r_u \cap r_v$).

Proposition 1. *In any 2-exactly stabbed rectangle intersection representation of a cycle of order greater than 3, there are exactly two bridge edges.*

Proof. Let G be a cycle of order greater than 3. Clearly, all the vertices of G cannot have a common stab as G is not an interval graph. This implies that in any 2-exactly stabbed rectangle intersection representation of G, there are at least two bridge edges. Suppose for the sake of contradiction assume that is a 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of G that have more than two bridge edges. As G is triangle-free, we can use Observation B to conclude that the bridge edges in $\mathcal{R}$ can be ordered as $e_1, e_2, \ldots, e_k$, where $k \geq 3$, such that $span(e_1) < span(e_2) < \cdots < span(e_k)$. Let $e_i = u_i v_i$ for all i. As $span(e_1) < span(e_2) < span(e_k)$, it is clear from the definition of $span(e_i)$ that there exists a vertex $w_1 \in \{u_1, v_1\}$ and a vertex $w_2 \in \{u_k, v_k\}$ such that $span(w_1) < span(e_2) < span(w_k)$. We can now apply Observation A to conclude that w_1 and w_k are in different connected components of $G - (N[u_2] \cup N[v_2])$. But this is a contradiction as in any cycle of order greater than 3, it is not possible to remove the closed neighbourhoods of two consecutive vertices to obtain a disconnected non-empty graph.

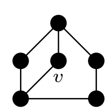


Fig. 1: A graph belonging to the $W_{6,2}$ family and v is the central vertex.

The graph family $W_{n+1,d}$ with $n \geq 4, d \geq 2, n \geq d$ consists of triangle free graphs that are isomorphic to a cycle of order n with d vertices adjacent to a new *central vertex*. For example, Figure 1 shows a graph belonging to the $W_{6,2}$ family.

Proposition 2. *Let $n \geq 4, d \geq 2$ be two integers and $\mathcal{R}$ be any 2-exactly stabbed rectangle intersection representation of a graph $G \in W_{n+1,d}$ with central vertex v. Then the number of bridge edges incident on v is $|N(v)| - 1$. Moreover, if $d \geq 3$ and uv be an edge such that u, v have a common stab, then $span(v) \subset span(u)$.*

Proof. Consider an arbitrary 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of G. Let C be the cycle obtained by removing the central vertex v

from G and let $u_1, u_2, \ldots, u_d$ be the neighbours of v on C in the cyclic order. Let P_i denote the subpath of C from u_i to u_{i+1} (where $u_{d+1} = u_1$) that does not contain any neighbour of v as an internal vertex. Notice that each of C, $C_1 = G[V(P_1) \cup \{v\}]$, $C_2 = G[V(P_2) \cup \{v\}]$, $\ldots$, $C_d = G[V(P_d) \cup \{v\}]$ are induced cycles of G that are of order greater than 3. Applying Observation 1 to each of $C, C_1, C_2, \ldots, C_d$, we can conclude that each of them contain exactly two bridge edges. Let us first consider C_i, for some $i \in \{1, 2, \ldots, d\}$. Suppose that the two bridge edges of C_i are also in P_i. Then these two edges are exactly the two bridge edges of C, implying that none of the paths in $P_1, P_2, \ldots, P_d$ other than P_i contain any bridge edges. This means that the two bridge edges of C_{i+1} (where again, $C_{d+1} = C_1$) are vu_i and vu_{i+1}. But then vu_i is a third bridge edge in C_i other than the two bridge edges on P_i, which is a contradiction. So we can assume that there is at most one bridge edge in P_i, for each i. As C has exactly two bridge edges, it follows that there are exactly two values in $\{1, 2, \ldots, d\}$, say t and t', such that P_t and $P_{t'}$ contain a bridge edge each. This tells us that for $i \in \{1, 2, \ldots, d\} \setminus \{t, t'\}$, the edges vu_i and vu_{i+1} are the two bridge edges in C_i. Now if t and t' are not consecutive (i.e., $t - 1 \neq t' \neq t + 1$), then by our previous observation, both vu_t and vu_{t+1} are bridge edges, which is a contradiction as we would then have three bridge edges in C_t. Therefore, we can conclude that t and t' are consecutive. Let us assume without loss of generality that $t' = t + 1$. By our earlier observation, we know that every edge in vu_i, where $i \in \{1, 2, \ldots, d\} \setminus \{t + 1\}$ is a bridge edge, as it belongs to some C_j, where $j \in \{1, 2, \ldots, d\} \setminus \{t, t'\}$. Also, we can see that vu_{t+1} is not a bridge edge as otherwise, the cycles C_t and $C_{t'}$ will have more than two bridge edges. Therefore, the set of bridge edges incident on v is $\{u_1, u_2, \ldots, u_d\} \setminus \{u_{t+1}\}$.

We will now show that $span(v) \subset span(u_{t+1})$. For ease of notation, let $a = u_t$, $b = u_{t+1}$ and $c = u_{t+2}$. Let us assume without loss of generality that v is on the bottom stab line. Then, we know that a and c are both on the top stab line as va and vc are bridge edges. Since a and c are nonadjacent, it follows that $span(a) \cap span(c) = \emptyset$. Let us assume by symmetry that $span(a) < span(c)$. Since v is a neighbour of both a and c, it follows that $span(v)$ intersects both $span(a)$ and $span(c)$, or in other words, $[x_a^+, x_c^-] \subseteq span(v)$. Notice that there are no bridge edges in the path $P = P_{t+2} \cup P_{t+3} \cup \cdots \cup P_d \cup P_1 \cup P_2 \cup \cdots P_{t-1}$. Therefore, all the vertices in P are have a common stab line, in particular, have a common stab line with a and c. As $a, c \in P$, we have that $span(P)$ contains both $span(a)$ and $span(c)$, which implies that $[x_a^+, x_c^-] \subseteq span(P)$. Let ww' and zz' be the bridge edges on P_t and P_{t+1} respectively, where w, z, a, c have a common stab line and w', z', b, v have a common stab line. Let P'_t be the path $P_t - \{a, b\}$ and P'_{t+1} the path $P_{t+1} - \{b, c\}$. As no vertex of P'_t is adjacent to any vertex of P or to v, we can conclude that $span(P'_t) \cap [x_a^+, x_c^-] = \emptyset$. As there is a neighbour of a on P'_t, $span(P'_t)$ intersects $span(a)$, leading us to the conclusion that $span(P'_t) < [x_a^+, x_c^-]$. Since at least one of w, w' is on P'_t, this means that $span(ww') < [x_a^+, x_c^-]$. With the same kind of arguments, we can also deduce that $[x_a^+, x_c^-] < span(P'_{t+1})$ and that $[x_a^+, x_c^-] < span(zz')$. By Observation B, we know that the spans of any two bridge edges of G are disjoint. Since it is clear that $span(va) \cap [x_a^+, x_c^-] \neq \emptyset$ and $span(vc) \cap [x_a^+, x_c^-] \neq \emptyset$, we now have $span(ww') < span(va) < span(vc) < span(zz')$ (recall that $span(a) < span(c)$). As $span(ww') < span(va)$, there exists a vertex $w'' \in \{w, w'\}$ such that $span(w'') < span(va)$. Let $S = \{u \in V(G) \colon span(u) \cap span(va) \neq \emptyset\}$. It is easy to see that $S \subseteq N[v] \cup N[a]$. Now, by Observation A, w'' and c are in two connected components of $G - S$ (note that $c \notin S$). This implies that $b \in S$, or in other words, $span(b) \cap span(va) \neq \emptyset$. Using the same kind of reasoning for the bridge edges vc and zz', we can conclude that $span(b) \cap span(vc) \neq \emptyset$. Together, we get $[x_a^+, x_c^-] \subseteq span(b)$. Recall that $[x_a^+, x_c^-] \subseteq span(v)$. As b and v have a common stab line and because $a, c \in N(v) \setminus N(b)$, we can conclude that $y_b^+ < y_v^+$. Now suppose that $x_v^- \leq x_b^-$. Let b' be the neighbour of b on P'_t. As $span(P'_t) < [x_a^+, x_c^-]$, we have $span(b') < [x_a^+, x_c^-]$. But then, $r_{b'}$ cannot intersect r_b without intersecting r_v. This contradiction lets us conclude that $x_b^- < x_v^-$. Arguing symmetrically, we can also derive $x_b^+ > x_v^+$. This shows that $span(v) \subset span(b)$.

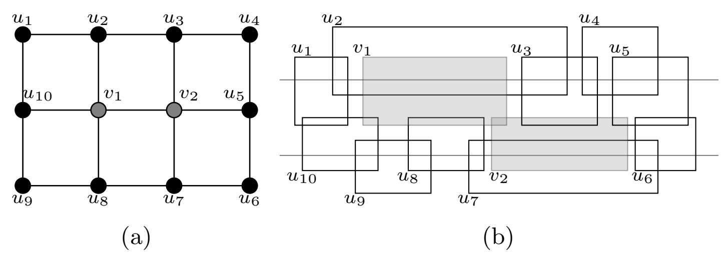


Fig. 2: 2-exactly stabbed rectangle intersection representation of $(3, 4)$-grid graph.

Observation C *Consider the* $(3, 4)$*-grid graph* H *as shown in Figure 2(a). In any 2-exactly stabbed rectangle intersection representation of* H*, the edge* v_1v_2 *is a bridge edge.*

Proof. Suppose for the sake of contradiction that there is a 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of H such that v_1 and v_2 have a common stab. In H, both the subsets $\{v_1, u_1, u_2, u_3, v_2, u_7, u_8, u_9, u_{10}\}$ and $\{v_2, u_2, u_3, u_4, u_5, u_6, u_7, u_8, v_1\}$ induce subgraphs belonging to the $W_{9,4}$ family. Hence, by Proposition 2, for each $i \in \{1, 2\}$, there is exactly one vertex $w_i \in N(v_i)$ that have a common stab line as v_i and $span(v_i) \subset span(w_i)$ in $\mathcal{R}$. Then by definition of w_1 and w_2, we have $w_1 = v_2$ and $w_2 = v_1$. Now we can use our earlier observation to infer that $span(v_1) \subset span(v_2) \subset span(v_1)$, which is a contradiction.

Let $\mathcal{R}$ be a 2-exactly stabbed rectangle intersection representation of a graph G along with the stab lines $y = a_1$ and $y = a_2$ where $a_1 < a_2$. Recall that $\mathcal{R}_t$ and $\mathcal{R}_b$ are the sets of intervals obtained by projecting the rectangles that intersect $y = a_2$ and $y = a_1$ respectively on the $x-$axis.

Lemma 2. *The family of $(\mathcal{I},\mathcal{P})-$graphs is a proper subset of 2-ESRIG.*

Proof. By definition, an $(\mathcal{I},\mathcal{P})-$graph is a 2-ESRIG. We show that there is a graph which is a 2-ESRIG but not an $(\mathcal{I},\mathcal{P})-$graph. Let H be the $(3,4)$-grid as shown in Figure 2(a). Clearly, there is a 2-exactly stabbed rectangle intersection representation of H (Figure 2(b)).

In H, both the subsets $\{v_1,u_1,u_2,u_3,v_2,u_7,u_8,u_9,u_{10}\}$ and $\{v_2,u_2,u_3,u_4,u_5,u_6,u_7,u_8,v_1\}$ induce subgraphs belonging to the $W_{9,4}$ family. Hence, by Proposition 2, in any 2-exactly stabbed rectangle intersection representation of H, for each $i \in \{1,2\}$, there is exactly one vertex $w_i \in N(v_i)$ that have a common stab line as v_i and $span(v_i) \subset span(w_i)$. Moreover, by Observation C, in any 2-exactly stabbed rectangle intersection representation of H, the edge v_1v_2 is a bridge edge. This implies that, in any 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of H, none of the sets $\mathcal{R}_t$ and $\mathcal{R}_b$ is a set of proper interval. Hence, H is not an $(\mathcal{I},\mathcal{P})-$graph.

Lemma 3. *The family of $(\mathcal{I},\mathcal{P})-$graphs is equivalent to the family of $(\mathcal{I},\mathcal{U})-$graphs.*

Proof. By definition, an $(\mathcal{I},\mathcal{U})-$graph is a $(\mathcal{I},\mathcal{P})-$graph. Let G be an $(\mathcal{I},\mathcal{P})-$graph and $\mathcal{R}$ be a 2-exactly stabbed rectangle intersection representation of G such that at least one of $\mathcal{R}_t$ and $\mathcal{R}_b$ is a set of proper intervals. We shall assume without loss of generality that $\mathcal{R}_b$ is a set of proper intervals and for any two vertices $u,v \in V(G)$ we have $\{x_v^-,x_v^+\} \cap \{x_u^-,x_u^+\} = \emptyset$. We shall use V_1 and V_2 to denote the sets of vertices that are on the top and bottom stab lines respectively. Note that $\mathcal{R}_b$ is a proper interval representation of $G[V_2]$. Let $p_1,p_2,\ldots,p_{2|V_2|}$ be the endpoints of the intervals in $\mathcal{R}_b$ written in ascending order. We now use the fact that every graph that has a proper interval representation also has a unit interval representation in which the endpoints of the intervals are in the same order [5]. Let $\mathcal{U}$ be the unit interval representation corresponding to $\mathcal{R}_b$ and let $p'_1,p'_2,\ldots,p'_{2|V_2|}$ be the endpoints of the intervals in $\mathcal{U}$ written in ascending order. We now construct a 2-exactly stabbed rectangle intersection representation $\mathcal{R}' = \{r'_u = [x_u'^-,x_u'^+] \times [y_u'^-,y_u'^+]\}_{u\in V(G)}$ of G such that at least one of $\mathcal{R}'_t$ and $\mathcal{R}'_b$ is a set of unit intervals as follows.

In the representation $\mathcal{R}'$, the rectangle corresponding to a particular vertex intersect the same stab line as it intersects in $\mathcal{R}$. We define $[y_u'^-,y_u'^+] = [y_u^-,y_u^+]$ for every vertex $u \in V(G)$. For each vertex $u \in V_2$, we let $x_u'^+ = p'_i$ and $x_u'^- = p'_j$, where $x_u^+ = p_i$ and $x_u^- = p_j$. Define $f : \bigcup_{u\in V_1}\{x_u^+,x_u^-\} \to \mathbb{R}$ as follows: $f(p) = p'_i + \frac{j}{t+1}(p'_{i+1} - p'_i)$, where $p_i < p < p_{i+1}$, and $q_1,q_2,\ldots,q_{k-1},(q_k = p),q_{k+1},\ldots,q_t$ are the points in $\{x_u^+ \colon u \in V_1, p_i < x_u^+ < p_j\} \cup \{x_u^- \colon u \in V_1, p_i < x_u^- < p_j\}$, in ascending order. For each vertex $u \in V_1$, we let $x_u'^+ = f(x_u^+)$ and $x_u'^- = f(x_u^-)$. It is not difficult to verify that $\mathcal{R}'$ is a 2-exactly stabbed rectangle intersection representation of G such that $\mathcal{R}'_b$ is a set of unit intervals.

Remark 1. From the proof of Lemma 3, it is clear that the left and right edges of the rectangles of $\mathcal{R}'$ are in the same order as they are in $\mathcal{R}$.

Let G is a $(\mathcal{P},\mathcal{P})$-graph and $\mathcal{R}$ be a 2-exactly stabbed rectangle intersection representation of G such that both $\mathcal{R}_t$ and $\mathcal{R}_b$ are sets of proper intervals. By Remark 1, the construction procedure described in Lemma 3 when applied on $\mathcal{R}$ gives us a 2-exactly stabbed rectangle intersection representation $\mathcal{R}'$ of G such that one of $\mathcal{R}'_t$ and $\mathcal{R}'_b$ is a set of unit intervals and the other is a set of proper intervals. This gives us the following lemma.

Lemma 4. *The family of $(\mathcal{P},\mathcal{P})-$graphs is equivalent to the family of $(\mathcal{P},\mathcal{U})-$graphs.*

Now we show that there is a graph which is an $(\mathcal{I},\mathcal{P})-$graph but not a $(\mathcal{P},\mathcal{P})-$graph.

Lemma 5. *The family of $(\mathcal{P},\mathcal{P})-$graphs is a proper subset of the family of $(\mathcal{I},\mathcal{P})-$graphs.*

Proof. By definition, a $(\mathcal{P},\mathcal{P})-$graph is an $(\mathcal{I},\mathcal{P})-$graph. We show that there is a graph which is a $(\mathcal{I},\mathcal{P})-$graph but not a $(\mathcal{P},\mathcal{P})-$graph. Consider the labelled $(3,3)$-grid graph H shown in Figure 3(a). Clearly, there is a 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of H such that at least one of $\mathcal{R}_t$ and $\mathcal{R}_b$ is a set of proper intervals (Figure 3(b)). Note that H is a graph belonging to the $W_{9,4}$ family. By Proposition 2, in any 2-exactly stabbed rectangle intersection representation of H, there is a vertex $w \in N(v)$ such that w, v have a common stab and $span(v) \subset span(w)$. Hence, H is not a $(\mathcal{P},\mathcal{P})$-graph.

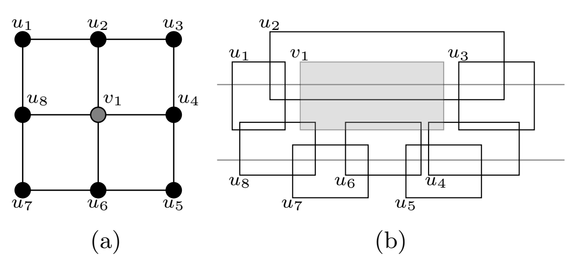


Fig. 3: 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of $(3,3)$-grid graph such that $\mathcal{R}_b$ is a set of proper intervals.

Observation D *Let $\mathcal{R}$ be a 2-exactly stabbed rectangle intersection representation of a triangle free graph G such that both $\mathcal{R}_t$ and $\mathcal{R}_b$ are sets of proper intervals. Let $e = uv$ be a bridge edge and $a, b \in V(G)$ such that $span(a) < span(uv) < span(b)$. Then a and b are in different connected components of $G - \{u, v\}$.*

Proof. Assume for the sake of contradiction that a and b are in the same connected component C of $G - \{u, v\}$. Then there exists a path $P = \{u_1 =$

$a, u_2 \ldots, u_t = b\}$ between a and b in C. As $span(P)$ is an interval that contains both $span(a)$ and $span(b)$, it is clear that $span(P)$ also contains $span(uv)$ in $\mathcal{R}$. Let $j \in \{2, \ldots, t-1\}$ be the minimum value such that $span(u_j) \cap span(uv) \neq \emptyset$. As G is trinagle free, u_j intersects exactly one of u, v. Without loss of generality, assume u_j intersects u. Since G is triangle free, u_j and u must have a common stab in $\mathcal{R}$. Without loss of generality assume that both u_j and u are on the bottom stab line. Therefore, we must have $y^+_{u_j} < y^-_v$. As G is triangle free, the vertex u_{j-1} does not intersect u and from our definition of j, $span(u_{j-1}) < span(uv)$. If $x^-_{u_j} > x^-_u$, then $r_{u_{j-1}}, r_u, r_{u_j}$ intersects each other contradicting the fact that $\mathcal{R}$ is a valid 2-exactly stabbed rectangle intersection representation of G. Therefore, we must have $x^-_{u_j} < x^-_u$ in $\mathcal{R}$. Similarly, we can show that $x^+_{u_j} > x^+_u$. This implies that $span(u) \subset span(u_j)$. But this contradicts the fact $\mathcal{R}_b$ is a set of proper intervals.

The graph family $C_{n,d}$ with $n \geq 4, d \geq 1, n \geq d$ consists of triangle free graphs G isomorphic to a cycle of order n and d new vertices each adjacent to a unique vertex of the cycle. For example, Figure 4 shows a graph belonging to the $C_{10,1}$ family.

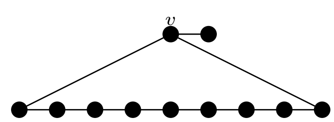


Fig. 4: A graph belonging to the $C_{10,1}$ family.

Proposition 3. *Let $n \geq 4$ be a positive integer and G be a graph isomorphic to a graph belonging to $C_{n,1}$. Let v be the vertex in G with degree 3 and $\mathcal{R}$ be any 2-exactly stabbed rectangle intersection representation of G such that both $\mathcal{R}_t$ and $\mathcal{R}_b$ are sets of proper intervals. Then at least one of the two edges which are in the cycle and incident on v is a bridge edge.*

Proof. Let v_1, v_2, w be the vertices adjacent to v, C be the induced cycle in G and w is the vertex with degree 1. For the sake of contradiction, assume that none of vv_1 and vv_2 is a bridge edge. We shall use V_1 and V_2 to denote the sets of vertices that are on the top and bottom stab lines respectively. Note that $G[V_1]$ and $G[V_2]$ are proper interval graphs. We know that a proper interval graph cannot have a $K_{1,3}$ as an induced subgraph. Since the graph induced by $\{v, w, v_1, v_2\}$ is isomorphic to $K_{1,3}$, therefore it must be the case that $e = vw$ is a bridge edge. Moreover, by proposition 1, C has exactly two bridge edges. Let the bridge edges in C be $e_1 = z_1w_1$ and $e_2 = z_2w_2$ with z_1, z_2, w_1, w_2 being distinct from v. By Observation B, we know that $span(e_1) \cap span(e_2) = span(e_1) \cap span(e) = span(e_2) \cap span(e) = \emptyset$. Without loss of generality assume that $span(e_1) < span(e_2)$. If $span(e_1) < span(e) < span(e_2)$ then there are vertices $a \in \{z_1, w_1\}$ and $b \in \{z_2, w_2\}$ such that $span(a) < span(e) < span(b)$. By Observation D, we can infer that a and b lies in two different components of $G - \{v, w\}$. This is a contradiction as there will be only one component in $G - \{v, w\}$. If $span(e) < span(e_1) < span(e_2)$ then there are vertices $a \in \{w, v\}$ and $b \in \{z_2, w_2\}$ such that $span(a) < span(e_1) < span(b)$. Observation D, we can infer that a and b lies in two different components of $G - \{z_1, w_1\}$. But due to our initial assumption that $v \notin \{w_1, z_1\}$, this is impossible. By similar type of reason we can show that $span(e) < span(e_1) < span(e_2)$ is impossible. Therefore, at least one of vv_1 or vv_2 must be a bridge edge.

For a graph G, let $\alpha(G)$ denote the cardinality of the maximum independent set of G.

Observation E *Let $\mathcal{R}$ be a 2-exactly stabbed rectangle intersection representation of a graph G such that both $\mathcal{R}_t$ and $\mathcal{R}_b$ are sets of unit intervals. Let v be a vertex of G and G' is the graph induced by the set $\{w \in V(G)\colon span(w) \cap span(v) \neq \emptyset$ and w, v are on different stab lines$\}$. Then $\alpha(G') \leq 2$.*

Proof. Suppose for the sake of contradiction $\alpha(G') > 2$ and S be the maximum independent set of G'. There are at least three vertices $\{w_1, w_2, w_3\} \subseteq S$ such that $span(w_1) < span(w_2) < span(w_3)$ in $\mathcal{R}$. Hence, we have $x^-_{w_3} - x^+_{w_1} > 1$. Since $span(v)$ intersects with both $span(w_1)$ and $span(w_3)$, we must have $x^+_v - x^-_v > 1$ which is a contradiction.

Observation F *Let $\mathcal{R}$ be a 2-exactly stabbed rectangle intersection representation of a graph G such that both $\mathcal{R}_t$ and $\mathcal{R}_b$ are sets of unit intervals. Let uv is an edge such that both u, v have a common same stab and G' is the graph induced by the set $\{w \in V(G)\colon span(w) \cap (span(v) \cup span(u) \neq \emptyset$ and w, v are on different stab lines$\}$. Then $\alpha(G') \leq 3$.*

Proof. Without loss of generality assume $x^-_u < x^-_v$. Since u, v are adjacent we must have $x^+_v - x^-_u \leq 2$. Suppose for the sake of contradiction $\alpha(G') > 3$ and S be the maximum independent set of G'. There are at least four vertices $\{w_1, w_2, w_3, w_4\} \subseteq S$ such that $span(w_1) < span(w_2) < span(w_3) < span(w_4)$ in $\mathcal{R}$. Hence, we have $x^-_{w_4} - x^+_{w_1} > 2$. Since $span(u) \cup span(v)$ intersects with both $span(w_1)$ and $span(w_4)$, we must have $x^+_v - x^-_u > 2$ which is a contradiction.

Lemma 6. *The family of $(\mathcal{U},\mathcal{U})-$graphs is a proper subset of the family of $(\mathcal{P},\mathcal{U})-$graph.*

Proof. By definition, a $(\mathcal{U},\mathcal{U})-$graph is a $(\mathcal{P},\mathcal{U})-$graph. We show that there is a graph G which is a $(\mathcal{P},\mathcal{U})-$graph but not a $(\mathcal{U},\mathcal{U})-$graph. Let H be a graph isomorphic to the graph shown in Figure 5(a). As shown in Figure 5(b), there is a 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of H such that one of $\mathcal{R}_t$ and $\mathcal{R}_b$ is a set of proper intervals and the other is a set of unit intervals. We prove that H is not an $(\mathcal{U},\mathcal{U})-$graph.

For the sake of contradiction, assume there is a 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of H such that both $\mathcal{R}_t$ and $\mathcal{R}_b$ are sets of unit

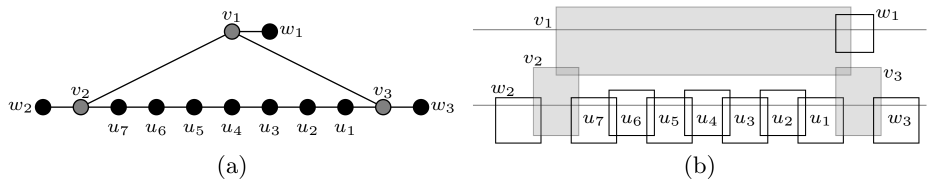


Fig. 5: A 2-exactly stabbed rectangle intersection representation $\mathcal{R}$ of the graph shown in (a) such that one of $\mathcal{R}_t$ and $\mathcal{R}_b$ is a set of proper intervals and the other is a set of unit intervals.

intervals. Without loss of generality, assume that v_1 is on the top stab line and let C denote the induced cycle in H. Notice that, the subgraph of H induced by $V(C) \cup \{w_1\}$ is isomorphic to a graph in the $C_{10,1}$ family. By Proposition 3, at least one of v_1v_2 and v_1v_3 is a bridge edge. By proposition 1, C will have exactly two bridge edges in C. Hence, if both v_1v_2 and v_1v_3 are bridge edges, then all vertices of the path $P = u_1u_2 \ldots u_7$ will be on the bottom stab line and for each $i \in \{1, 2, \ldots, 7\}$, $span(u_i) \cap span(v_1) \neq \emptyset$. But $\alpha(P) = 4$, which by Observation E is a contradiction.

Therefore, exactly one of v_2 and v_3 must be on the top stab line. Without loss of generality, assume that v_3 is on the top stab line. Notice that, the subgraph of H induced by $V(C) \cup \{w_3\}$ is isomorphic to a graph in the $C_{10,1}$ family. Since v_3v_1 is not a bridge edge, therefore by Proposition 3 the edge v_3u_1 must be a bridge edge. Since both v_1v_2 and v_3u_1 are bridge edges of C, all vertices of the path $P = u_1u_2 \ldots u_7$ will be on the bottom stab line. Now for each $i \in \{1, 2, \ldots, 7\}$, $span(u_i) \cap (span(v_1) \cup span(v_3)) \neq \emptyset$. But $\alpha(P) = 4$, which by Observation F is a contradiction.

Lemma 7. *The family of 2-SUIG is equialent to the family of $(\mathcal{U},\mathcal{U})$-graphs.*

Proof. By definition, a 2-SUIG is a $(\mathcal{U},\mathcal{U})-$graph. Let G be a $(\mathcal{U},\mathcal{U})-$graph and $\mathcal{R}$ be a 2-exactly stabbed rectangle intersection representation of G such that both $\mathcal{R}_t$ and $\mathcal{R}_b$ are sets of unit intervals. Let $\mathcal{I}$ be the set of intervals $\{[y_v^+, y_v^-]\}_{v \in V(G)}$. We now use the fact that every graph that has a proper interval representation also has a unit interval representation in which the endpoints of the intervals are in the same order [5]. Let $\mathcal{U} = \{[y_v'^-, y_v'^+]_{v \in V(G)}\}$ be the unit interval representation corresponding to $\mathcal{I}$. Consider the set of rectangles $\mathcal{R}' = \{[x_v^+, x_v^-] \times [y_v'^-, y_v'^+]\}_{v \in V(G)}$. Clearly, $\mathcal{R}'$ is a 2-exactly stabbed rectangle intersection representation of G and all rectangles are unit squares. Hence, G is a 2-SUIG.

Combining Lemma 1,2,3,4,5,6,7, we have the proof of Theorem 1.

4 Proof of Theorem 2

A planar graph G is said to have an *LL-drawing* if G has a straight line embedding on the plane such that the point corresponding to a vertex of G lies on one of two given horizontal lines. A planar graph G is an *LL-graph* if G has an LL-drawing. Cornelsen et al. [] proved the following result which we state as a theorem.

Theorem 5 ([]). *Given a graph G, there is a $O(|V(G)|)$ time algorithm to decide if G is an LL-graph.*

Let $\mathcal{L}$ be an LL-drawing of a graph G and $y = a_1$,$y = a_2$ be the horizontal lines in $\mathcal{L}$ with $a_1 < a_2$. For a vertex $v \in V(G)$ let p_v denote the point corresponding to v in $\mathcal{L}$. Let $\mathcal{A}_{\mathcal{L}} = \{w \in V(G)\colon p_w \text{ lies on } y = a_1\}$ and $\mathcal{B}_{\mathcal{L}} = \{w \in V(G)\colon p_w \text{ lies on } y = a_2\}$. For two vertices $u, v \in V(G)$, we say $u <_{\mathcal{L}} v$ if both u, v lies on $y = a_i$ for some $i \in \{1, 2\}$ and x-coordinate of p_u is less than that of p_v. Let $C_{\mathcal{L}}$ denote the subgraph of G such that $E(C_{\mathcal{L}}) = \{uv \in E(G)\colon p_u \text{ lies on } y = a_1 \text{ and } p_v \text{ lies on } y = a_2\}$. When the LL-drawing of G under consideration is clear from the context, the sets $A_{\mathcal{L}}$ and $B_{\mathcal{L}}$, the relation $<_{\mathcal{L}}$ and the graph $C_{\mathcal{L}}$ are considered to be defined with respect to this drawing.

Observation G *Let $\mathcal{L}$ be an LL-drawing of a graph G. Then the subgraph $C_{\mathcal{L}}$ of G is a caterpillar graph.*

Lemma 8. *If a graph G is a LL-graph then G is also a $(\mathcal{P}, \mathcal{P})$-graph.*

Proof. Let $\mathcal{L}$ be an LL-drawing of a graph G and $y = a_1$,$y = a_2$ be the horizontal lines in $\mathcal{L}$. We shall give a 2-exactly stabbed rectangle intersection representation of G where the stab lines are $y = a_1$ and $y = a_2$.

First take a 2-exactly stabbed rectangle intersection representation $\mathcal{C}$ of the subgraph $C_{\mathcal{L}}$ of G such that $\mathcal{C}$ satisfies (i) $y = a_1$ and $y = a_2$ are the stab lines in $\mathcal{C}$, (ii) $A_{\mathcal{L}} \cap V(C_{\mathcal{L}})$ is the set of vertices that are on the bottom stab line in $\mathcal{C}$, (iii) $B_{\mathcal{L}} \cap V(C_{\mathcal{L}})$ is the set of vertices that are on the top stab line in $\mathcal{C}$, and (iv) if $u <_{\mathcal{L}} v$ then $span(u) < span(v)$ in $\mathcal{C}$.

Now for each vertex $v \in A_{\mathcal{L}} \setminus V(C_{\mathcal{L}})$ (resp. $v \in B_{\mathcal{L}} \setminus V(C_{\mathcal{L}})$), introduce a rectangle such that $y_v^+ = y_v^- = a_1$ (resp. $y_v^+ = y_v^- = a_2$). Moreover, we can ensure that after introducing all rectangles corresponding to the vertices in $(A_{\mathcal{L}} \cup B_{\mathcal{L}}) \setminus V(C_{\mathcal{L}})$ we can ensure that whenever $u <_{\mathcal{L}} v$ we have $span(u) < span(v)$ (resp. $span(v) < span(u)$). Let $\mathcal{C}'$ denote the set of these new rectangles. We can further modify the rectangles in $\mathcal{C}'$ to get $\mathcal{C}''$ such that (i) whenever there is an edge uv with $u, v \in (A_{\mathcal{L}} \cup B_{\mathcal{L}}) \setminus V(C_{\mathcal{L}})$, the rectangles r_u and r_v intersects and (ii) whenever there is an edge $u'v'$ with $u' \in V(C_{\mathcal{L}})$ and $v' \in (A_{\mathcal{L}} \cup B_{\mathcal{L}}) \setminus V(C_{\mathcal{L}})$, $span(u')$ in $\mathcal{C}$ and $span(v')$ in $\mathcal{C}''$ intersects.

Let $\mathcal{R} = \mathcal{C} \cup \mathcal{C}''$. It is not difficult to verify that $\mathcal{R}$ is a 2-exactly stabbed rectangle intersection representation of G. This completes the proof.

Lemma 9. *If a triangle-free graph G is a $(\mathcal{P}, \mathcal{P})$-graph then G a LL-graph.*

Proof. Let $\mathcal{R}$ be a 2-stabbed rectangle intersection representation of a triangle-free $(\mathcal{P}, \mathcal{P})$-graph G such that both $\mathcal{R}_t$ and $\mathcal{R}_b$ are sets of proper interval. Without loss of generality, we can assume that $\mathcal{R}$ is a 2-exactly stabbed rectangle intersection representation of G and for any two vertices $u, v \in V(G)$ we have that $\{x_u^+, x_u^-, y_u^+, y_u^-\} \cap \{x_v^+, x_v^-, y_v^+, y_v^-\} = \emptyset$. For each $v \in V(G)$, consider the point $p_v = (a_v, b_v)$ such that $a_v = x_v^+$, $b_v = 0$ if v is on the bottom stab line and $b_v = 1$ if v is on the top stab line. Now for each eadge $uv \in E(G)$, let s_{uv} be the line segment joining p_u and p_v. It is not difficult to verify that $\{p_v\}_{v \in V(G)}$ and $\{s_{uv}\}_{uv \in E(G)}$ gives an LL-drawing of G.

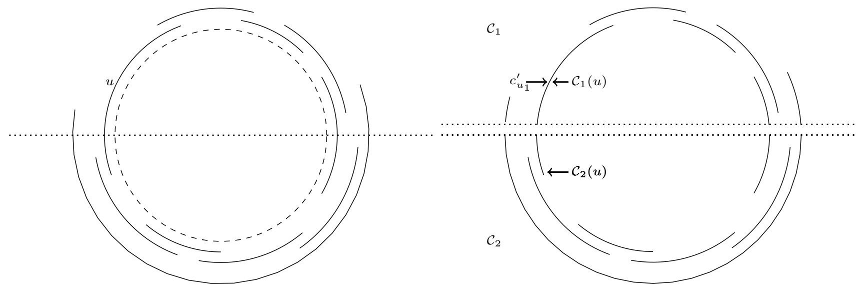
Combining Theorem 5, Lemma 8 and Lemma 9 we have the proof of Theorem 2.

5 Proof of Theorem 3

Given a 2-SRIG H, its 2-stabbed rectangle intersection representation $\mathcal{R}$ and an integer c, the 2-SRIG COLORING problem is to decide whether the chromatic number of H is at most c. A *circular arc representation* of a graph is a collection of circular arcs of a circle such that each circular arc in the collection represents a vertex of the graph and two circular arc intersect if and only if the vertices they represent are adjacent in the graph. The graphs that have circular arc representation are called *circular arc graphs*. Given a circular arc graph G, its circular arc representation $\mathcal{C}$ and an integer c, the CIRCULAR ARC COLORING problem is to decide whether the chromatic number of G is at most c. To prove our theorem we shall reduce the NP-complete CIRCULAR ARC COLORING [16] problem to 2-SRIG COLORING problem. First we introduce some definition and notations. When traversing around the circle in the clockwise direction, we first encounter the *left endpoint* of a circular arc, then its *interior*, and then its *right endpoint*. For points p, q, r on the circle, we write $p < q < r$ to indicate that the points p, q, r appear in this order in a full traversal of the circle in the clockwise direction starting from the point p. Similarly, we write $p_1 < p_2 < \ldots < p_n$ for the clockwise order of points $p_1, p_2, \ldots, p_k$. We write $[p, q]$ for the clockwise arc from p to q. Let G be a connected circular arc graph with a circular arc representation $\mathcal{C}$. For a vertex $v \in V(G)$, let $c_v = [a_v, b_v]$ denote the circular arc corresponding to v in $\mathcal{C}$ where a_v and b_v are the left and right endpoints of c_v, respectively.

Given a connected circular arc graph G with a circular arc representation $\mathcal{C}$ and an integer c. We shall construct a 2-SRIG H such that G is c-colorable if and only if H is c-colorable. Without loss of generality we can assume that all circular arcs in $\mathcal{C}$ are part of the unit circle with centre on the origin, all circular arcs in $\mathcal{C}$ have distinct endpoints, no circular arc have an endpoint on the x-axis. Below we describe the reduction procedure which consitutes of two steps viz. *partition step* and *joining step*.

Partition step: Let H_1 and H_2 be the two closed half spaces induced by the x-axis. For each $i \in \{1, 2\}$ and $v \in V(G)$, let $\mathcal{C}_i(v) = c_v \cap H_i$. Notice that, for each $i \in \{1, 2\}$ and $v \in V(G)$, the number of circular arcs in $\mathcal{C}_i(v)$ is at most two and total number of circular arcs in $\mathcal{C}_1(v) \cup \mathcal{C}_2(v)$ is at most three (see Figure 6). For each $i \in \{1, 2\}$ let $\mathcal{C}_i$ be the set of circular arcs in $\{\mathcal{C}_i(v)\}_{v \in V(G)}$ and G_i



(a) (b)

Fig. 6: Illustration of the partition step.

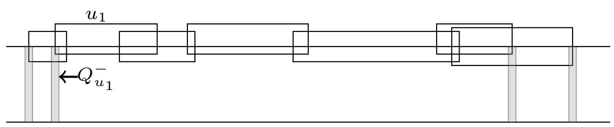
be the intersection graph induced by the arcs in $\mathcal{C}_i$. For a vertex $w_1 \in V(G_1)$, $c'_{w_1} = [a'_{w_1}, b'_{w_1}]$ shall denote the corresponding circular arc in $\mathcal{C}_1$. Similarly, for a vertex $w_2 \in V(G_2)$, $c''_{w_2} = [a''_{w_2}, b''_{w_2}]$ shall denote the corresponding circular arc in $\mathcal{C}_2$. We shall use the following observation in our proof.

Observation H *Both G_1 and G_2 are interval graphs and* $|V(G_1) \cup V(G_2)| \leq 3|V(G)|$.

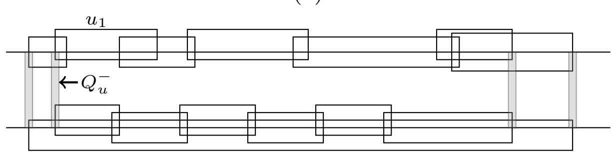
Joining step: Let $\mathcal{R}$ be a 2-stabbed rectangle intersection representation of G_1 such that (i) $y = 1$ and $y = 0$ are the two stab lines in $\mathcal{R}_1$, (ii) for all vertex $u \in V(G_1)$ we have $y_u^- = y_u^+ = 1$ and for two distinct vertices $u, v \in V(G_1)$ we have $\{x_u^-, x_u^+\} \cap \{x_v^+, x_v^+\} = \emptyset$, (iii) for two vertices $u, v \in V(G_1)$ if $a'_u < a'_v$ in $\mathcal{C}_1$ then $x_u^- < x_v^-$ in $\mathcal{R}$, and (iv) for two vertices $u, v \in V(G_1)$ if $b'_u < b'_v$ in $\mathcal{C}_1$ then $x_u^+ < x_v^+$ in $\mathcal{R}$ (see Figure 7(a)). For each vertex $u \in V(G_1)$, define $f^-(u)$ and $f^+(u)$ to be cardinality of the sets $\{v \in V(G_1) \colon x_v^- < x_u^-\}$ and $\{v \in V(G_1) \colon x_v^+ > x_u^+\}$, respectively. Now we do the following.

(i) For each vertex $u \in V(G_1)$ with $a'_u = (-0.5, 0)$ in $\mathcal{C}_1$, let $z = x_u^-$ (with respect to $\mathcal{R}$) and consider a new complete graph Q_u^- consisting of $(c - f^-(u) - 1)$ vertices. Take a 2-stabbed rectangle intersection representation $\mathcal{R}' = \{r'_v\}_{v \in V(Q_u^-)}$ of Q_u^- such that $x_v'^- = x_v'^+ = z, y_v^- = 0, y_v^+ = 1$ in $\mathcal{R}'$ for all $v \in V(Q_u^-)$ and define $\mathcal{R} = \mathcal{R} \cup \mathcal{R}'$ (see Figure 7(a)).
(ii) For each vertex $u \in V(G_1)$ with $b'_u = (0.5, 0)$ in $\mathcal{C}_1$, let $z' = x_u^+$ (with respect to $\mathcal{R}$) and consider a new complete graph Q_u^+ consisting of $(c - f^+(u) - 1)$ vertices. Take a 2-stabbed rectangle intersection representation $\mathcal{R}' = \{r'_v\}_{v \in V(Q_u^+)}$ of Q_u^+ such that $x_v'^- = x_v'^+ = z', y_v^- = 0, y_v^+ = 1$ in $\mathcal{R}'$ and define $\mathcal{R} = \mathcal{R} \cup \mathcal{R}'$ (see Figure 7(a)).
(iii) Take a 2-stabbed rectangle intersection representation $\mathcal{R}'' = \{r''_u\}_{u \in V(G_2)}$ of G_2 such that (i) $y = 1$ and $y = 0$ are the two stab lines in $\mathcal{R}''$, (ii) for all vertex $u \in V(G_2)$ we have $y_u^- = y_u^+ = 0$ and for two distinct vertices $u, v \in V(G_2)$ we have $\{x_u''^-, x_u''^+\} \cap \{x_v''^+, x_v''^+\} = \emptyset$, (iii) for a vertex $u_2 \in V(G_2)$ if there is a vertex $u_1 \in V(G_1)$ and a vertex $v \in V(G)$ satisfying the following properties: (a) $c'_{u_1} \in \mathcal{C}_1(v)$, (b) $c''_{u_2} \in \mathcal{C}_2(v)$ and (c) $c'_{u_1} \cup c''_{u_2}$ cuts

the negative x-axis (resp. positive x-axis), then $x''^{-}_{u_2} = x^{-}_{u_1}$ (resp. $x''^{+}_{u_2} = x^{+}_{u_1}$). See Figure 7(b). Define $\mathcal{R} = \mathcal{R} \cup \mathcal{R}''$ and H be the 2-SRIG induced by the rectangles in $\mathcal{R}$.



(a)



(b)

Fig. 7: Illustration of the joining step.

Observe that, H has at most $3nc$ vertices and construction of H clearly takes polynomial time. Also, note that G_1 and G_2 are induced subgraphs of H and there is no edge between a vertex of G_1 and a vertex of G_2. We shall show that G is c-colorable if and only if H is c-colorable. Given a c-coloring ϕ of G, we can define a c-coloring ϕ' of H as follows.

(i) For each vertex $u \in V(H) \cap V(G_1)$ we define $\phi'(u) = \phi(v)$ where v is the vertex of G such that $c'_u \in \mathcal{C}_1(v)$.
(ii) For each vertex $u \in V(H) \cap V(G_2)$ we define $\phi'(u) = \phi(v)$ where v is the vertex of G such that $c''_u \in \mathcal{C}_2(v)$.
(iii) After performing the above two steps, for each vertex $u \in V(H) \setminus V(G_1) \cup V(G_2)$, we assign $\phi'(u)$ to be the smallest available color.

From construction it is not difficult to verify that ϕ' is a valid c-coloring of H. Now we shall show if H is c-colorable then G is c-colorable. First we have the following observation.

Observation I *Let ϕ be any c-coloring of H. For each $i \in \{1,2\}$, let $u_i \in V(H) \cap V(G_i)$. If there is a vertex $v \in V(G)$ such that $\{c'_{u_1}, c''_{u_2}\} \subseteq \mathcal{C}_1(v) \cup \mathcal{C}_2(v)$, then $\phi(u_1) = \phi(u_2)$.*

Now given a c-coloring ϕ of H we define a coloring ϕ' of G as follows. For each vertex $v \in V(G)$, there is a vertex $u \in V(H)$ such that $c'_u \in \mathcal{C}_1(v)$ or $c''_u \in \mathcal{C}_2(v)$ and define $\phi'(v) = \phi(u)$. Using Observation I, we can infer that ϕ' is indeed a c-coloring of G. This completes the proof of Theorem 3.

6 Proof of Theorem 4

Let H be a triangle-free 2-SRIG. In Lemma 10, we shall show that H is a planar graph. Then due to Grötscz theorem, we can infer that H is 3-colorable. Hence, in $O(|V(H)|)$ time we can decide if H is 2-colorable and due to Dvořák et al. [13], we can have a 3-coloring of H in $O(|V(H)|)$ time. Now we state a result of Perepelitsa [25].

Theorem 6 ([25]). *Let G be the triangle-free intersection graph of finite number of compact connected sets Ai with boundaries that are piecewise differentiable Jordan curves. For every i and j, let $A_i \setminus A_j$ be nonempty and arc-connected. Then G is a planar graph.*

A rectangle intersection representation $\mathcal{R}$ of a graph G is *crossing-free* if for any two rectangles r_u and r_v in $\mathcal{R}$, the regions $r_u \setminus r_v$ and $r_v \setminus r_u$ are both arc-connected.

Lemma 10. *Let H be a triangle-free 2-SRIG. Then H is a planar graph.*

Proof. Due to Theorem 6, if we can show that H has a 2-stabbed rectangle intersection representation $\mathcal{R}$ such that $\mathcal{R}$ is crossing-free, then H is planar. Now assume any 2-stabbed rectangle intersection representation $\mathcal{R}$ of H and I be the interval graph induced by the set of intervals $\{[y_v^-, y_v^+]\}_{v \in V(H)}$. Notice that, size of the maximum independent set of H is at most two. Therefore, I must be an unit interval graph and $\mathcal{I} = \{y_v'^-, y_v'^+\}_{v \in V(H)}$ be an unit interval representation of I. Now consider the set of rectangles $\mathcal{R}' = \{[x_v^-, x_v^+] \times [y_v'^-, y_v'^+]\}$. Clearly, $\mathcal{R}'$ is a 2-stabbed rectangle intersection representation of H where each rectangle has unit height and therefore $\mathcal{R}'$ is crossing-free. Hence, H is a planar graph.

7 Conclusions

In this paper, we focus our study on graphs that have stab number at most 2 and its subclasses. We prove that 2-SUIG = $(\mathcal{U},\mathcal{U})$-graphs $\subset$ $(\mathcal{P},\mathcal{U})$-graphs = $(\mathcal{P},\mathcal{P})$-graphs $\subset$ $(\mathcal{I},\mathcal{U})$-graphs = $(\mathcal{I},\mathcal{P})$-graphs $\subset$ 2-ESRIG = 2-SRIG. While proving the theorem, we showed that exact stab number of $(3,4)$-grid is exactly 2. In fact using the observations made in this paper, it is possible to prove that for any $n \geq 1$, the exact stab number of $(3,n)$-grid is at most 2. Moreover, the above observation can be used to show that $estab((h,w) - \text{GRID}) \leq 2\lceil \frac{t}{3} \rceil$ where $t = \min\{h,w\}$.

Since $(\mathcal{P},\mathcal{P})$-graphs are proper subsets of 2-SRIG, a direction of further research could be to investigate the class of $(\mathcal{P},\mathcal{P})$-graphs and try to characterize this class of graphs.

Question 1. Develop a forbidden structure characterization and/or a polynomial-time recognition algorithm for $(\mathcal{P},\mathcal{P})$-graphs.

For a 2-SRIG H, notice that $\chi(H) \leq 2\omega(H)$. Moreover, we proved that the chromatic number of any triangle-free 2-SRIG is at most three. In other words, when a 2-SRIG H is triangle-free, we have $\chi(H) \leq \omega(H) + 1$. Therefore, the following is a natural question in this direction.

Question 2. Is there a constant c such that for any 2-SRIG H we have $\chi(H) \leq \omega(H) + c$?

References

1. Adiga, A., Chandran, L., Sivadasan, N.: Lower bounds for boxicity. Combinatorica **34**(6), 631–655 (2014)
2. Asplund, E., Grünbaum, B.: On a coloring problem. Mathematica Scandinavica **8**(1), 181–188 (1960)
3. Bhore, S., Chakraborty, D., Das, S., Sen, S.: On a special class of boxicity 2 graphs. In: CALDAM. pp. 157–168 (2015)
4. Bielecki, A.: Problem 56. Colloq. Math. **1**, 333 (1948)
5. Bogart, K.P., West, D.B.: A short proof that proper= unit. Discrete Mathematics **201**(1-3), 21–23 (1999)
6. Cerioli, M., Oliveira, F., Szwarcfiter, J.: On counting interval lengths of interval graphs. Discrete Applied Mathematics **159**(7), 532–543 (2011)
7. Chakraborty, D., Francis, M.C.: On the stab number of rectangle intersection graphs. CoRR **abs/1804.06571** (2018), http://arxiv.org/abs/1804.06571
8. Chalermsook, P.: Coloring and maximum independent set of rectangles. In: Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, pp. 123–134. Springer (2011)
9. Chandran, L., Francis, M., Sivadasan, N.: Boxicity and maximum degree. J. Comb. Theory, Ser. B **98**(2), 443–445 (2008)
10. Chandran, L., Mathew, R., Rajendraprasad, D.: Upper bound on cubicity in terms of boxicity for graphs of low chromatic number. Discrete Mathematics **339**(2), 443 – 446 (2016)
11. Chandran, L., Sivadasan, N.: Boxicity and treewidth. Journal of Combinatorial Theory, Series B **97**(5), 733 – 744 (2007)
12. Corneil, D., Olariu, S., Stewart, L.: The LBFS structure and recognition of interval graphs. SIAM Journal on Discrete Mathematics **23**(4), 1905–1953 (2009)
13. Dvorak, Z., Kawarabayashi, K., Thomas, R.: Three-coloring triangle-free planar graphs in linear time. ACM Trans. Algorithms **7**(4), 41:1–41:14 (2011)
14. Esperet, L., Joret, G.: Boxicity of graphs on surfaces. Graphs and Combinatorics pp. 1–11 (2013)
15. Fishburn, P., Graham, R.: Classes of interval graphs under expanding length restrictions. Journal of graph theory **9**(4), 459–472 (1985)
16. Garey, M.R., Johnson, D.S., Miller, G.L., Papadimitriou, C.H.: The complexity of coloring circular arcs and chords. SIAM Journal on Algebraic Discrete Methods **1**(2), 216–227 (1980)
17. Imai, H., Asano, T.: Finding the connected components and a maximum clique of an intersection graph of rectangles in the plane. Journal of Algorithms **4**(4), 310–323 (1983)
18. Joos, F., Löwenstein, C., Oliveira, F., Rautenbach, D., Szwarcfiter, J.: Graphs of interval count two with a given partition. Information Processing Letters **114**(10), 542–546 (2014)

19. Klavík, P., Otachi, Y., Šejnoha, J.: On the Classes of Interval Graphs of Limited Nesting and Count of Lengths. In: ISAAC 2016. LIPIcs, vol. 64, pp. 45:1–45:13 (2016)
20. Köbler, J., Kuhnert, S., Watanabe, O.: Interval graph representation with given interval and intersection lengths. Journal of Discrete Algorithms **34**, 108–117 (2015)
21. Kratochvíl, J.: A special planar satisfiability problem and a consequence of its NP-completeness. Discrete Applied Mathematics **52**(3), 233–252 (1994)
22. Kratochvíl, J., Nešetřil, J.: Independent set and clique problems in intersection-defined classes of graphs. Commentationes Mathematicae Universitatis Carolinae **31**(1), 85–93 (1990)
23. Lekkerkerker, C., Boland, J.: Representation of a finite graph by a set of intervals on the real line. Fundamenta Mathematicae **51**(1), 45–64 (1962)
24. Pe'er, I., Shamir, R.: Realizing interval graphs with size and distance constraints. SIAM Journal on Discrete Mathematics **10**(4), 662–687 (1997)
25. Perepelitsa, I.G.: Bounds on the chromatic number of intersection graphs of sets in the plane. Discrete Mathematics **262**(1-3), 221–227 (2003)
26. Yannakakis, M.: The complexity of the partial order dimension problem. SIAM Journal on Algebraic Discrete Methods **3**(3), 351–358 (1982)