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# On dominating set of some subclasses of string graphs[1]

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**Abstract**

We provide constant factor approximation algorithms for the Minimum Dominating Set (MDS) problem on several subclasses of *string graphs* i.e. intersection graphs of simple curves on the plane. For $k \geq 0$, *unit $B_k$-VPG graphs* are intersection graphs of simple rectilinear curves having at most $k$ cusps (bends) and each segment of the curve being unit length. We give an 18-approximation algorithm for the MDS problem on unit $B_0$-VPG graphs. This partially addresses a question of Katz et al. (Comput. Geom. 2005). We also give an $O(k^4)$-approximation algorithm for the MDS problem on unit $B_k$-VPG graphs. We show that there is an 8-approximation algorithm for the MDS problem on *vertically-stabbed* L*-graphs*. We also give a 656-approximation algorithm for the MDS problem on *stabbed rectangle overlap graphs*. This is the first constant-factor approximation algorithm for the MDS problem on stabbed rectangle overlap graphs and extends a result of Bandyapadhyay et al. (Comput. Geom. 2019). We prove some hardness results to complement the above results.

*Keywords:*
Dominating set, Approximation algorithm, geometric intersection graph, string graph.

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## 1. Introduction and Results

An *intersection representation* $\mathcal{R}$ of a graph $G = (V, E)$ is a family of sets $\{R_u\}_{u \in V}$ such that $uv \in E$ if and only if $R_u \cap R_v \neq \emptyset$. When $\mathcal{R}$ is a collection of geometric objects, it is said to be a *geometric intersection representation* of $G$. When $\mathcal{R}$ is a collection of *simple unbounded curves* on the plane, it is called an *string representation*. A graph $G$ is a *string graph* if $G$ has a string representation. String graphs are important as it contains all intersection graphs of connected sets in $\mathbb{R}^2$. String graphs have been intensively studied both for practical applications and theoretical interest. To the best of our knowledge, Benzer [3] was the first to introduce string graphs in 1959 while exploring the topology of genetic structures. In 1966, Sinden [4] considered the same constructs at Bell Labs. In 1976, Graham [5] introduced string graphs to the mathematics community at the open problem

[1]Preliminary versions of this paper was published in WG 2019 [1] and COCOON 2019 [2].

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session of a conference in Keszthely. Since then, string graphs have become an exciting topic of research.

Many popular graph classes like *planar* graphs, *chordal* graphs, *cocomparability* graph, *disk* graphs, *rectangle intersection* graphs, *segment* graphs, *circular arc* graphs are subclasses of string graphs. In fact, any intersection graph of arc-connected sets on the plane is a string graph [4, 6, 7, 8]. However, not all graphs are string graphs [6] and this motivates further study of computational complexities of various optimisation problems in string graphs and its subclasses [9, 10, 11, 12, 13]. In this paper, we propose constant factor approximation algorithms for the MINIMUM DOMINATING SET (MDS) problem on string graphs.

A *dominating set* of a graph $G = (V, E)$ is a subset $D$ of vertices $V$ such that each vertex in $V \setminus D$ is adjacent to some vertex in $D$. The MINIMUM DOMINATING SET (MDS) problem is to find a minimum cardinality dominating set of a graph $G$. It is not possible to approximate the MDS problem on string graphs with $n$ vertices within $(1 - \alpha) \ln n$ for any $\alpha > 0$ unless $NP \subseteq DTIME(n^{O(\log \log n)})$ [14]. Hence, researchers have developed approximation algorithms for the MDS problem on various subclasses of string graphs. Examples are planar graphs, chordal graphs, disk graphs, *unit disk* graphs, *rectangle intersection* graphs, intersection graphs of homothets of *convex objects* etc [15, 16, 17, 18, 19, 20]. De Berg et al. [21] studied the fixed parameter tractablity of the MDS problem on various classes of geometric intersection graphs. Erlebach and Van Leeuwen [22] provided constant-factor approximation algorithms for intersection graphs of $r$-regular polygons, where $r$ is an arbitrary constant, for pairwise homothetic triangles, and for rectangles with bounded aspect ratio.

Asinowski et al. [23] introduced the concept of *$B_k$-VPG graphs* to initiate a systematic study of string graphs and its subclasses. A *path* is a simple rectilinear curve made of axis-parallel line segments, and a *$k$-bend path* is a path having $k$ bends. The *$B_k$-VPG graphs* are intersection graphs of $k$-bend paths. Any string graph has a $B_k$-VPG representation for some $k$ [23]. Katz et al. [24] proved the NP-hardness for the MDS problem on $B_0$-VPG graphs. However, a sublogarithmic approximation algorithm for the MDS problem on $B_0$-VPG graphs is still unknown. Observe that intersection graphs of orthogonal segments having unit length, i.e. *unit* $B_0$-VPG graphs is a subclass of $B_0$-VPG graphs. In this paper, we show that the MDS problem is NP-hard on unit $B_0$-VPG graphs. This strengthens a result of Katz et al. [24]. We also propose the first constant-factor approximation algorithm for the MDS problem on unit $B_0$-VPG graphs. Specifically, we prove the following theorems.

**Theorem 1.** *It is NP-Hard to solve the* MDS *problem on unit $B_k$-VPG graphs with $k \geq 0$.*

**Theorem 2.** *Given a unit $B_0$-VPG representation of a graph $G$ with $n$ vertices, there is an $O(n^5)$-time* 18*-approximation algorithm to solve the* MDS *problem on $G$.*

We generalise Theorem 2 in the following way. A *unit $k$-bend path* is a $k$-bend path with each segment being of unit length. A *unit $B_k$-VPG representation* of a graph $G = (V, E)$ is a set, $\mathcal{C} = \{C_u\}_{u \in V}$, of unit $k$-bend paths, such that $uv \in E$ if and only if $C_u \cap C_v \neq \emptyset$. A graph is a *unit $B_k$-VPG* graph if it has a unit $B_k$-VPG representation. Observe that, any string graph has a unit $B_{k'}$-VPG representation for some $k'$. We prove the following.

**Theorem 3.** *Given a unit $B_k$-VPG representation of a graph $G$ with $n$ vertices, there is an $O(k^2n^5)$-time $O(k^4)$-approximation algorithm to solve the* MDS *problem on $G$.* [2]

The MDS problem remains difficult in restricted families of string graphs. An L-path is a simple curve consisting of one vertical segment and one vertical segment joined in a point in such a way that it creates the shape 'L'. A set of L-paths is *vertically-stabbed* if all L-paths in the set intersect a common vertical line. A *vertically-stabbed* L*-representation* of a graph $G = (V, E)$ is a set, $\mathcal{C} = \{C_u\}_{u \in V}$, of vertically-stabbed L-paths, such that $uv \in E$ if and only if $C_u \cap C_v \neq \emptyset$. A graph is a *vertically-stabbed* L*-graph* if it has a vertically-stabbed L-representation. The class of vertically-stabbed L-graphs was introduced by McGuinness [25] and it contains many important graph classes like *interval graphs*, *outerplanar graphs*, *permutation graphs*, *interval overlap graphs* as subclasses. Researchers have studied the MDS problem on these classes of graphs ([26, 27, 28, 29, 30]). Bandyapadhyay et al. [31] proved APX-hardness for the MDS problem on vertically-stabbed L-graphs. An $\epsilon$-net based algorithm of Mehrabi [32] gives an $O(1)$-approximation algorithm for the MDS problem on vertically-stabbed L-graphs. The specific value of the constant (which is at least 32) was not reported by the author. We prove the following.

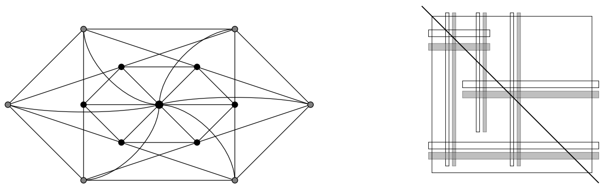
**Theorem 4.** *Given a vertically-stabbed* L*-representation of a graph $G$ with $n$ vertices, there is an $O(n^5)$-time* 8*-approximation algorithm to solve the* MDS *problem on $G$.*

A *rectangle overlap representation* $\mathcal{R}$ of a graph $G = (V, E)$ is a family of axis parallel rectangles $\{R_u\}_{u \in V}$ such that $uv \in E$ if and only if the boundaries of $R_u$ and $R_v$ intersect. A graph $G$ is a *rectangle overlap graph* if $G$ has a rectangle overlap representation. An *interval overlap representation* $\mathcal{R}$ of a graph $G = (V, E)$ is a family of closed intervals $\{I_u\}_{u \in V}$ such that $uv \in E$ if and only if the $I_u \cap I_v \neq \emptyset$ and none of $I_u$ and $I_v$ is contained in the other. A graph $G$ is an *interval overlap graph* if $G$ has an interval overlap representation. Finding a constant-factor approximation algorithm for the MDS problem on rectangle overlap graphs is a challenging open problem. The MDS problem remains APX-hard even on interval overlap graphs [28]. We prove the following assuming Unique Games Conjecture [33] to be true.

**Theorem 5.** *Assuming the* Unique Games Conjecture *to be true, it is not possible to have a polynomial time $(2 - \epsilon)$-approximation algorithm for the* MDS *problem on rectangle overlap graphs for any $\epsilon > 0$.*

Constant-factor approximation algorithms are known only for restricted subclasses of rectangle overlap graphs and rectangle intersection graphs. Damian-Iordache and Pemmaraju [29] gave a $(2 + \epsilon)$-approximation for the MDS problem on interval overlap graphs. Pandit [34] introduced the intersection graph of *diagonally anchored* rectangles which also turns out to be a subclass of rectangle overlap graphs. A set $\mathcal{R}$ of rectangles is a set of *diagonally anchored* rectangles if there is a straight line $l$ with slope $-1$ such that intersection of any $R \in \mathcal{R}$ with $l$ is exactly one corner of $R$. Surprisingly, the MDS problem remains NP-Hard on intersection graphs of diagonally anchored rectangles [34]. Bandyapadhyay et al. [31] gave a $(2 + \epsilon)$-approximation algorithm for the same. Erlebach and Van

[2] Preliminary versions of the proofs of Theorems 3 and 4 appeared in WG 2019 [1].



(a) (b)

Figure 1: A graph which is a stabbed rectangle overlap graphs but neither an interval overlap graph nor an intersection graph of diagonally anchored rectangles.

Leeuwen [22] provided constant-factor approximation algorithms for intersection graphs of rectangles with bounded aspect ratios. The work of Govindarajan et al. [20] implies a PTAS for approximation algorithm for MDS of intersection graphs of unit-height rectangles.

A set $\mathcal{R}$ of axis-parallel rectangles is *stabbed* if there is a straight line that intersects all rectangles in $\mathcal{R}$. A *stabbed rectangle overlap representation* $\mathcal{R}$ of a graph $G = (V, E)$ is a family of stabbed axis parallel rectangles $\{R_u\}_{u \in V}$ such that $uv \in E$ if and only if the boundaries of $R_u$ and $R_v$ intersect. A graph $G$ is a *stabbed rectangle overlap graph* if $G$ has a stabbed rectangle overlap representation.

**Theorem 6.** *Given a stabbed rectangle overlap representation of a graph $G$ with $n$ vertices, there is an $O(n^5)$-time 656-approximation algorithm for the* MDS *problem on $G$.*[3]

We note that interval overlap graphs and intersection graphs of diagonally anchored rectangles are strict subclasses of stabbed rectangle overlap graphs. See Figure 1(a) for a separating example [35]. Note that approximation algorithms for optimisation problems like Maximum Independent Set and Minimum Hitting Set on intersection graphs of "stabbed" geometric objects have been studied [35, 36, 12, 37, 38].

### *1.1. Main lemma*

Proofs of Theorem 2, 3, 4 and 6 use two crucial lemmas. The first one is about the *stabbing segment with rays* (SSR) problem and the second one is about the *stabbing rays with segment* (SRS) problem, both introduced by Katz et al. [24]. Below we provide definitions of both SSR and SRS problems.

*Stabbing segments with rays (*SSR*)*

**Input:** A set $R$ of disjoint leftward-directed horizontal semi-infinite rays and a set of disjoint vertical segments.

**Output:** A minimum cardinality subset of $R$ that intersect all segments in $V$.

*Stabbing rays with segments (*SRS*)*

[3]The authors proposed a 768-approximation algorithm for the MDS problem on stabbed rectangle intersection graphs in COCOON 2019 [2].

**Input:** A set $R$ of disjoint leftward-directed horizontal semi-infinite rays and a set of disjoint vertical segments.

**Output:** A minimum cardinality subset of $V$ that intersect all rays in $R$.

Let $\mathcal{SSR}(R, V)$ (resp. $\mathcal{SRS}(R, V)$) denote an SSR instance (resp. an SRS instance) where $R$ is a given set of disjoint leftward-directed horizontal semi-infinite rays and $V$ is a given set of disjoint vertical segments. Katz et al. [24] gave dynamic programming based polynomial time algorithms for both the SSR problem and SRS problem. However, to prove Theorems 2, 3, 4 and 6, we required an upper bound on the ratio of the cardinality of the optimal solution of an SSR instance (and SRS instance) with the optimal cost of the corresponding relaxed LP formulation(s). Therefore, we proved the following lemmas.

**Lemma 1.** *Let $\mathcal{C}$ be an ILP formulation of an $\mathcal{SSR}(R, V)$ instance. There is an $O((n + m)\log(n + m))$-time algorithm to compute a set $D \subseteq R$ which gives a feasible solution of $\mathcal{C}$ and $|D| \leq 2 \cdot OPT(\mathcal{C}_l)$ where $n = |R|, m = |V|$ and $\mathcal{C}_l$ is the relaxed LP formulation of $\mathcal{C}$.*

**Lemma 2.** *Let $\mathcal{C}$ be an ILP formulation of an $\mathcal{SRS}(R, V)$ instance. There is an $O(n \log n)$ time algorithm to compute a set $D \subseteq V$ which gives a feasible solution of $\mathcal{C}$ and $|D| \leq 2 \cdot OPT(\mathcal{C}_l)$ where $n = |V|$ and $\mathcal{C}_l$ is the relaxed LP formulation of $\mathcal{C}$.*

Note that to prove both the above lemma, we do not need to explicitly solve the LP(s). Moreover, since $OPT(\mathcal{C}_l) \leq OPT(\mathcal{C})$, the algorithm of Lemma 1 provides an approximate solution to the $\mathcal{SSR}(R, V)$ instance with approximation ratio 2. Therefore, the following theorem is a consequence of Lemma 1.

**Theorem 7.** *There is an $O((n + m)\log(n + m))$-time 2-approximation algorithm for* SSR *problem where $n$ and $m$ are the number of rays and segments, respectively.*

*1.2. Organisation of paper*

In Section 2.1 and Section 2.2, we prove the hardness results (Theorem 1 and Theorem 5). In Section 3 and Section 4, we prove Lemma 1 and Lemma 4, respectively. In Section 5, we shall apply both Lemma 1 and Lemma 2 to prove Theorem 4. Then in Section 6, 7 and 8, we prove Theorem 2, 3 and 6, respectively.

## 2. Hardness results

In this section, we prove the two hardness results of this paper.

*2.1. Proof of Theorem 1*

We shall reduce the NP-complete MDS problem on grid graphs [39] to the MDS problem on unit $B_0$-VPG graphs. The $(h, w)$-*grid* is the undirected graph $G$ with vertex set $\{(x, y) : x, y \in \mathbb{Z}, 1 \leq x \leq h, 1 \leq y \leq w\}$ and edge set $\{(u, v)(x, y) : |u - x| + |v - y| = 1\}$. A graph $G$ is a *grid* graph if $G$ is an induced subgraph of $(h, w)$-grid for some positive integers $h, w$.

We shall show that any grid-graph is a unit-$B_0$-VPG graph and thus prove Theorem 1. Observe that it is sufficient to show that for any positive even integer $n$, the $(n, n)$-grid has

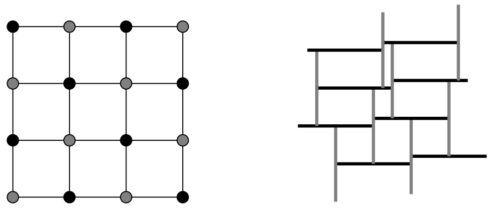


Figure 2: (a) A $(4,4)$-grid. In this case, $X$ consists of the gray vertices and $Y$ consists of black vertices. (b) A unit $B_0$-VPG representation of (a).

a unit $B_0$-VPG representation. Let $n$ be a fixed positive even integer and $H = (V', E')$ be a $(n, n)$-grid. Let $X = \{(i, j) \in V' : i, j \text{ have same parity}\}$ and $Y = V' \setminus X$. See Figure 2(a) for an example. We have the following lemma.

**Lemma 3.** *The graph $H$ has a unit $B_0$-VPG representation $\mathcal{R}$ where the vertical segments represent the pairs in $X$ and the horizontal segments represent the pairs in $Y$.*

The proof of Lemma 3 is not difficult but requires involved calculation. For the sake of completion, we provide detailed proof of Lemma 3 in Section 9.

*2.2. Proof of Theorem 5*

A *vertex cover* of a graph $G = (V, E)$ is a subset $C$ of $V$ such that each edge in $E$ has an endvertex which lies in $C$. The Minimum Vertex Cover problem is to find a minimum cardinality vertex cover of a graph. Assuming *Unique Games Conjecture* to be true, the Minimum Vertex Cover has no polynomial time $(2-\epsilon)$-approximation algorithm for any $\epsilon > 0$ [33]. We shall reduce the Minimum Vertex Cover problem to the MDS problem on rectangle overlap graphs.

Given a graph $G = (V, E)$, construct another graph $G' = (V', E')$ as follows. Define $V' = V \cup E$. Define $E' = \{uv : u, v \in V\} \cup \{ue : u \in V, e \in E \text{ and } u \text{ is an endvertex of } e \text{ in } G\}$. We have the following observation

**Observation A.** *The graph $G$ has a vertex cover of size $k$ if and only if $G'$ has a dominating set of size $k$.*

*Proof.* Let $C$ be a vertex cover of $G$. Then at least one endpoint of every edge of $G$ belongs to $C$. From construction of $G'$, it follows that $C$ is a dominating set of $G'$. Now let $D$ be a dominating set of $G'$ and $e \in E$ be a vertex of $D$. Let $v_e$ be a neighbour of $e$ in $G'$. Observe that if a vertex $w$ of $G'$ is adjacent to $e$, it must be adjacent to $v_e$ also. Hence, $D' = (D \setminus \{e\}) \cup \{v_e\}$ is also a dominating set of $G'$ with $|D'| \leq |D|$. Arguing in similar way for all vertices in $D \cap E$, we have a dominating set $D^*$ of $G'$ which is a subset of $V$. Therefore, $D^*$ is a vertex cover of $G$. □

We will be done by showing that $G'$ is a rectangle overlap graph. Let $V = \{v_1, v_2, \ldots, v_n\}$ and for each $v_i \in V$ define $R_{v_i} = [i, n+1] \times [-i, 0]$ (See Figure 3(c) for illustration).

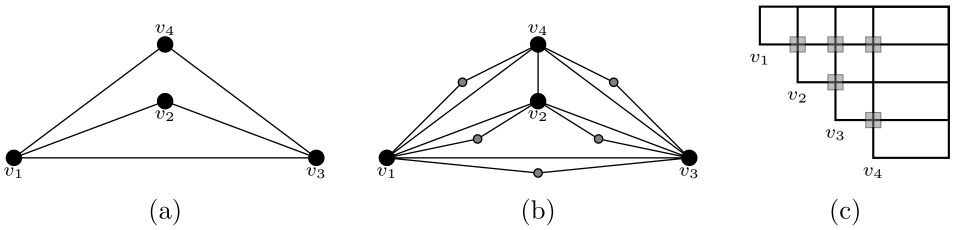



Figure 3: Reduction procedure for Theorem 5. (a) Input graph $G$, (b) The graph $G'$ and (c) rectangle overlap representation of $G'$.

Notice that, each vertex $u \in V' \setminus V$, has degree two and is adjacent to exactly two vertices of $V$. For each vertex $u \in V' \setminus V$, introduce a rectangle $R_u$ which overlaps only with $R_{v_i}$ and $R_{v_j}$ where $\{v_i, v_j\}$ is the set of vertices adjacent to $u$ with $i < j$. This is possible as $R_u$ can be kept around the unique intersection point of the bottom boundary of $R_{v_i}$ and the left boundary of $R_{v_j}$ (see Figure 3(c) for illustration). Formally, for each $u \in V' \setminus V$, define $R_u = [p - \epsilon, p + \epsilon] \times [q - \epsilon, q + \epsilon]$ where $\epsilon = \frac{1}{|V|}$ and $(p, q)$ is the intersection point of the bottom boundary of $R_{v_i}$ and the left boundary of $R_{v_j}$. Observe that the set of rectangles $\mathcal{R}' = \{R_{v_i} : v_i \in V\} \cup \{R_u : u \in V' \setminus V\}$ is a rectangle overlap representation of $G'$. This completes the proof.

**Remark B.** For a graph $G$, the graph $G'$ is also an intersection graph of line segments on the plane, i.e., a *segment graph*. Hence, unless the Unique Games Conjecture is false, it is not possible to have a polynomial-time $(2 - \epsilon)$-approximation algorithm for the MDS problem on segment graphs, for any $\epsilon > 0$.

## 3. Proof of Lemma 1

In this section we shall prove Lemma 1 and Theorem 7. Recall that in the SSR problem, the inputs are a set of disjoint leftward-directed horizontal rays and a set of disjoint vertical segments. The objective is to select a minimum number of leftward-directed horizontal rays that intersect all vertical segments.

In this section, we call a *leftward-directed horizontal semi-infinite ray* by simply a *ray* and a *vertical segment* by a *segment* in short. Let $R$ be a set of disjoint rays and $V$ be a set of disjoint vertical segments.

To prove Lemma 1, first, we present an iterative algorithm consisting of three main steps. The first step is to include all rays $r \in R$ in solution $S$ whenever some segments in $V$ intersect a single ray $r$ in that iterative step. Next, delete all segments intersecting any ray in $S$ from $V$. In the final step, find a ray in $R \setminus S$ whose $x$-coordinate of the right endpoint is the smallest among all rays in $R \setminus S$ and delete it from $R$ (when there are multiple such rays, choose one arbitrarily). We repeat the above three steps until $V$ is empty. The above algorithm takes $O((|R| + |V|) \log(|R| + |V|))$ time (using segment trees [40]) and outputs a set $S$ of rays such that all segments in $V$ intersect at least one ray in $S$.

We describe the above algorithm formally in Algorithm 1. Below we introduce some notations used to describe the algorithm. We assign *token* $T_r = \{r\}$ for each $r \in R$ initially.

**Algorithm 1** The SSR-Algorithm

**Input:** A set $R$ of leftward-directed rays and a set $V$ of vertical segments.
**Output:** A subset of $R$ that intersects all segments in $V$.

$T_r = \{r\}$ for each $r \in R$ and $i \leftarrow 1, V_0 \leftarrow V, R_0 \leftarrow R, S \leftarrow \emptyset, S_0 \leftarrow \emptyset$ ▷ Initialisation.
**while** $V_{i-1} \neq \emptyset$ **do**
  $S \leftarrow S \cup \{r : r \in R_{i-1}, r$ is critical after $(i-1)^{th}$ iteration$\}$ and $S_i \leftarrow S$. ▷ Critical ray collection.
  $V_i \leftarrow$ the set obtained by deleting all segments from $V_{i-1}$ that intersect a ray in $S_i$.
  Find a $r \in R_{i-1} \setminus S_i$ whose $x$-coordinate of the right endpoint is the smallest.
  $r$ discharges the token to its neighbours.
  $R_i \leftarrow$ The set obtained by deleting $\{r\} \cup S_i$ from $R_{i-1}$. ▷ Discharging token step.
  $i \leftarrow i + 1$;
**end while**
**return** $S$

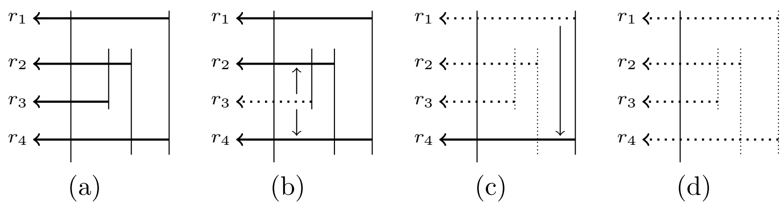



Figure 4: (a) An input SSR instance, (b) $1^{st}$ iteration, (c) $2^{nd}$ iteration and (d) $3^{rd}$ iteration of the SSR-Algorithm with (a) as input. A dotted ray (or segment) indicates that it is deleted.

For $i \geq 1$, let $R_i, V_i, S_i$ be the set of rays, the set of segments and the solution constructed by Algorithm 1, respectively at the *end* of $i^{th}$ iteration. A ray $r \in R_i$ is *critical* if there is a segment $v \in V_i$ such that $r$ is the only ray in $R_i$ that intersects $v$. We describe a *discharging technique* below.

Let $D$ be a subset of $R$. A ray $r \in D$ lies *between* two rays $r', r'' \in D$ if the $y$-coordinate of $r$ lies between those of $r', r''$. A ray $r \in D$ lies *just above* (resp. *just below*) a ray $r' \in D$ if $y$-coordinate of $r$ is greater (resp. smaller) than that of $r'$ and no other ray lies between $r, r'$ in $D$. Two rays $r, r' \in D$ are *neighbours* of each other if $r$ lies just above or below $r'$.

**Discharging Method:** Let $r \in R_{i-1} \setminus S_i$ be a ray whose $x$-coordinate of the right endpoint is the smallest. The phrase "$r$ discharges the token to its neighbours" in the $i^{th}$ iteration means the following operations in the given order.

(i) Let $r'$ lie just above $r$ and $r''$ lie just below $r$ in $R_{i-1} \setminus S_i$. For all $x \in T_r$ ($x$ and $r$ not necessarily distinct) do the following. If there is a segment in $V_i$ that intersects $x, r'$ and $r$ then assign $T_{r'} = T_{r'} \cup \{x\}$ and if there is a segment in $V_i$ that intersects $x, r''$ and $r$ then $T_{r''} = T_{r''} \cup \{x\}$.

(ii) Make $T_r = \emptyset$ after performing the above step.

For an illustration, consider the input instance shown in Figure 4(a). At the first iteration of Algorithm 1, $r_3$ passes the token to its neighbours $(r_2, r_4)$ and gets deleted. After the

$1^{st}$ iteration, notice that $r_2$ has become critical. So, at the begining of the $2^{nd}$ iteration Algorithm 1 put $r_2$ in the solution. Then all segment intersecting $r_2$ is deleted and $r_2$ itself is also deleted. Also in the second iteration $r_1$ passes the token to its neighbour ($r_4$) and gets deleted. Finally in the third iteration $r_4$ is put in the solution. We have the following observation.

**Observation C.** *For some $v \in V_k$, $k \geq 1$, if some ray $r \in R_0$ intersects $v$, then either $r \in R_k$ or there exists some ray $r' \in R_k$ such that $r \in T_{r'}$.*

*Proof.* Assume $r \notin R_k$. Let $< r_1, r_2, \ldots, r_k >$ be a sorted order of the rays such that for $i < j$, $r_i$ discharged the token to the neighbours before $r_j$. Due to step 5 of the SSR-algorithm, $X =< r_1, r_2, \ldots, r_k >$ is an increasing sequence based on the $x$-coordinate of their right endpoint. Observe that, whenever a ray $r_i \in X$ discharged its token to its neighbours in the $i^{th}$ iteration, all the vertical segments in $V_i$ intersected by $r_i$ also intersects one of the immediate neighbours of $r_i$. Again as $v \in V_k$, $v$ is not intersected by critical ray within $k$ iteration. Hence the result follows. □

**Lemma 4.** *For a ray $r$, there are at most two tokens containing $r$.*

*Proof.* If $r$ never discharged its token to its neighbours, the statement is true. Let $r$ discharge the token to its neighbours at iteration $i$. Note that $r$ discharged tokens to at most two of its neighbours. Since $r$ gets deleted after the discharging step, the rays whose tokens contain $r$ become neighbours.

Let $j$ be the minimum integer with $i < j$ such that at the end of $(j-1)^{th}$ iteration, there is a ray $p \in R_{j-1}$ which is critical and $r \in T_p$. Note that iteration of the SSR-Algorithm may stop before encountering such events. However, within iteration $i$ to $j-1$, there may exist some rays which discharged their tokens containing $r$ due to step 5 of the SSR-Algorithm.

To prove the lemma, we use induction to show that there are at most two tokens containing $r$ in any iteration from $i$ to $j-1$, and if there are indeed two tokens containing $r$, then the corresponding rays are neighbours.

Consider some $k$, $i < k < j$, such that $x_1$, $x_2 \in R_{k-1}$ be only two rays where $r \in T_{x_1}$ and $r \in T_{x_2}$. Notice that, $x_1$ and $x_2$ are neighbours of each other and without loss of generality assume $x_1$ lies just above $x_2$ in $V_{k-1}$. Assume $x_1$ discharged its token at $k^{th}$ iteration. If there exists a neighbour of $x_1$ (say $x_3$) which is different from $x_2$, then due to the discharging step of $k^{th}$ iteration, $x_1$ passes the token to its neighbours (i.e $x_2$ and $x_3$) and gets deleted from $R_{k-1}$ to create $R_k$. If $x_3$ does not exist, then $x_1$ shall pass the token only to $x_2$. Therefore $x_2$ becomes the top-most ray among those rays in $R_k$ which intersect some segment intersecting $r$.

Moreover, if $x$ was the only ray in $R_{k-1}$ such that $r \in T_x$, then $x$ was the top-most (or bottom-most) ray among those rays in $R_{k-1}$ which intersect some segment intersecting $r$. Therefore, at the end of $k^{th}$ iteration there is exactly one ray $x' \in R_k$ such that $r \in T_{x'}$ and $x'$ must be the top-most (resp. bottom-most) ray among those rays in $R_k$ which intersect some segment intersecting $r$.

Hence we conclude that for each $k$ with $i \leq k < j$, there is at most two rays $r', r'' \in R_k$ such that $r \in T_{r'} \cap T_{r''}$ and they are neighbours. If there is exactly one ray $r''' \in R_k$ such that $r \in T_{r'''}$ then $r'''$ must be the top-most or bottom-most ray among those rays in $R_k$ which intersect some segment intersecting $r$.

In iteration $j$, ray $p$ is critical and $r \in T_p$ and $p$ is put in the solution. If $p$ is the only ray whose token contained $r$, only $T_p$ will contain $r$ after the termination of Algorithm 1. Let $r', p \in R_{j-1}$ be the rays whose token contained $r$. They must be neighbours. Without loss of generality, assume that $p$ lies just above $r'$. If both $r', p$ are selected in $S_j$, there is nothing to prove. Now consider the set $A$ of segments in $V_j$ that intersects $r$ but not $p$. Note that no ray above $p$ intersects any segment in $A$. Hence $r'$ becomes the only ray in the next iterative step whose token contains $r$ and $r'$ turns to be the bottom-most ray among those rays in $R_{j-1}$ which intersect some segment intersecting $r$. Now consider any iteration $k > j$. By similar arguments as above, there would be at most one ray in $R_k$ that contains the token $r$. Hence the lemma follows. □

For a segment $v \in V$, let $N(v) \subseteq R$ be the set of rays that intersect $v$. Let $r \in S$ be a ray, $i$ be the minimum integer such that $r \in S_i$. There must exist a segment $\nu_r \in V_{i-1}$ such that $r$ is the only ray in $R_{i-1}$ that intersects $\nu_r$ and all rays in $N(\nu_r) \setminus \{r\}$ must have passed the token to its neighbours. So, for each ray $r \in S$, there exists a segment $\nu_r$ such that for all $x \in N(\nu_r) \setminus \{r\}$ we have $T_x = \emptyset$. We call $\nu_r$ a *critical segment with respect to $r$*.

**Observation D.** *For a ray $r \in S$ let $\nu_r$ be a critical segment with respect to $r$. Then $N(\nu_r) \subseteq T_r$.*

*Proof.* Consider any arbitrary but fixed deleted ray $y \in N(\nu_r) \setminus \{r\}$ which was deleted at some $j^{th}$ iteration. By Observation C, there exists a ray $y' \in R_j$ such that $y'$ intersects $v$ and $y \in T_{y'}$. Applying the above argument for all rays in $N(\nu_r) \setminus \{r\}$, we have the proof. □

**Lemma 5.** *If $S$ is the set returned by the* SSR*-algorithm with rays $R$ and segments $V$, then $|S| \leq 2|OPT|$, where $OPT$ is an optimum solution of $\mathcal{SSR}(R, V)$.*

*Proof.* Let $R$ be the set of rays and $V$ be the set of segments with $|R| = n, |V| = m$. Consider the ILP formulation $Q$ of $\mathcal{SSR}(R, V)$. For each ray $r \in R$, let $x_r \in \{0, 1\}$ denote the variable corresponding to $r$. Objective is to minimize $\sum_{r \in R} x_r$ with constraints $\sum_{r \in N(v)} x_r \geq 1$ for all $v \in V$. Let the corresponding relaxed LP formulation be $Q_l$.

Let $\mathbf{Q}_l = \{x_r\}_{r \in R}$ be an optimal solution of $Q_l$. Consider the SSR-algorithm. Here, define $y_r = 1$ if $r \in S$, $y_r = 0$ if $r \notin S$ and $\mathbf{Q}' = \{y_r\}_{r \in R}$, obtained by the algorithm. This is a feasible solution of $Q$ as the SSR-algorithm terminates only when no segments are left in $V_i$. Now we fix any arbitrary $r \in S$ and $\nu_r$ be a critical segment with respect to $r$. Then due to Observation D, we know that for all $z \in N(\nu_r) \setminus \{r\}$ we have $T_z = \emptyset$ and $N(\nu_r) \subseteq T_r$. Therefore, for the constraint corresponding to $\nu_r$ in $Q_l$, we have that

$$\sum_{z \in N(\nu_r)} y_z = 1 \leq \sum_{z \in N(\nu_r)} x_z \leq \sum_{z \in T_r} x_z \qquad \text{[since } N(\nu_r) \subseteq T_r \text{ by Observation D]}$$

Therefore, from above argument and from Lemma 4 we conclude that

$$|S| = \sum_{r \in S} y_r = \sum_{r \in S} \sum_{z \in N(\nu_r)} y_z \leq \sum_{r \in S} \sum_{z \in T_r} x_z \leq 2 \sum_{z \in R} x_z \leq 2|OPT|.$$

Hence we have the proof. □

The proofs of Lemma 1 and Theorem 7 follows directly from the proof of Lemma 5.

## 4. Proof of Lemma 2

In this section, we shall prove Lemma 2. Recall that in the SRS problem, the input is a set of disjoint leftward-directed horizontal rays and a set of disjoint vertical segments. The objective is to select a minimum number of vertical segments intersecting all leftward-directed horizontal rays.

**2-approximation algorithm for the SRS problem:** With each segment $v \in V$, we associate a token $T_v$ which is a subset of $V$. Initialise $T_v = \emptyset$ for each $v \in V$. Let $r_i$ be a ray whose right-endpoint, $(x_i, y_i)$, has the smallest $x$-coordinate. Assuming that there is a feasible solution to the SRS instance, there must exist a segment of $V$ that intersects $r_i$. Let $N(r_i) \subseteq V$ be the set of segments that intersect $r_i$. Let $v_{top}$ (resp. $v_{bot}$) be a segment in $N(r_i)$ whose top endpoint is top-most (resp., bottom endpoint is bottom-most); it may be that $v_{top} = v_{bot}$. We add both $v_{top}$ and $v_{bot}$ to our heuristic solution set $S$. Also we set $T_{v_{top}} = T_{v_{bot}} = N(r_i)$. We remove from $R$ all of the rays that intersect $v_{top}$ or $v_{bot}$, delete all segments in $N(r_i)$ and then repeat the above steps untill $R = \emptyset$. Observe that for each ray $r$, there is a segment $v \in S$ that intersects $r$. Also observe that for each segment $v \in V$, there are at most two tokens such that both of them contains $v$. Ob serve that, the running time of the above algorithm is $O(n \log n)$ where $n = |V|$.

**Lemma 6.** *Let $Q$ be the ILP of the SRS instance with a set of rays $R$ and set of segments $V$ as input and $Q_l$ be the corresponding relaxed LP. Then $OPT(Q) \leq 2 \cdot OPT(Q_l)$.*

*Proof.* Let $\mathbf{X} = \{x_v\}_{v \in V}$ be an optimal solution of $Q_l$ where $x_v$ denotes the value of the variable in $Q_l$ corresponding to $v \in V$. Let $S$ be the solution returned by the above algorithm with $R, V$ as input. Now define for each $v \in V$, $y_v = 1$ if $v \in S$, $y_v = 0$ if $v \notin S$ and let $\mathbf{Y} = \{y_v\}_{v \in V}$. Observe that $\mathbf{Y}$ is a feasible solution of $Q$. For each $z \in S$, there is a ray $r_i$ such that $T_z = N(r_i)$. Therefore, $y_z = 1 \leq \sum_{v \in N(r_i)} x_v = \sum_{v \in T_z} x_v$

As a segment $v$ is contained in at most two tokens, using the above inequality we have

$$|S| = \sum_{v \in S} y_v \leq \sum_{v \in S} \sum_{v' \in T_v} x_{v'} \leq 2 \sum_{v' \in V} x_{v'} = 2 \cdot OPT(Q_l)$$

Hence the result follows. □

## 5. Algorithm for vertically-stabbed L-graphs

Given a vertically-stabbed L-representation of a graph $G$ with $n$ vertices, we shall give an $O(n^5)$-time 8-approximation algorithm to solve the MDS problem on $G$. In the rest of the paper, $OPT(Q)$ and $OPT(Q_l)$ denote the cost of the optimum solution of an ILP formulation $Q$ and LP formulation $Q_l$, respectively.

**Overview of the algorithm:** First, we solve the relaxed LP formulation of the ILP formulation of the MDS problem on the input vertically-stabbed L-graph $G$ and create two subproblems. We shall show that one of those two subproblems is equivalent to the SSR problem, and the other is equivalent to the SRS problem. Using Lemma 1 and 2 we shall give a performance guarantee of our algorithm. The running time of the algorithm becomes $O(n^5)$ where $n$ is the number of vertices in the input graph [41]. We note that such techniques have been previously used to design approximation algorithms [42, 43, 44].

Now we describe our approximation algorithm for the MDS problem on vertically-stabbed L-graphs. Let $\mathcal{R} = \{\mathrm{L}_u\}_{u\in V}$ be a vertically-stabbed L-representation of a graph $G = (V, E)$. We assume that (i) the vertical line $x = 0$ intersects all the L-paths in $\mathcal{R}$ and the $x$-coordinate of the corner point of each L-path in $\mathcal{R}$ is strictly less than 0, and (ii) whenever two distinct L-paths intersect in $\mathcal{R}$, they intersect at exactly one point (otherwise we can apply small perturbation to the L-paths so that whenever two distinct L-paths intersect in $\mathcal{R}$, they intersect at exactly one point [45]).

For a vertex $u \in V$, let $N[u]$ denote the closed neighbourhood of $u$ in $G$, $H_u = \{c \in N[u] \colon \mathrm{L}_c$ intersects the horizontal segment of $\mathrm{L}_u\}$ and let $V_u$ denote the set $N(u) \setminus H_u$. Based on these, we have the following ILP (say $Q$) of the problem of finding a minimum dominating set of $G$.

$$\begin{array}{ll} \text{minimize} & \sum_{v\in V} x_v \\ \text{subject to} & \sum_{v\in H_u} x_v + \sum_{v\in V_u} x_v \geq 1, \forall u \in V \\ x_v \in \{0,1\}, & \forall v \in V \end{array}$$
$$Q$$

Let $Q_l$ be the the relaxed LP formulation of $Q$ and $\mathbf{Q}_l = \{x_v \colon v \in V\}$ be an optimal solution of $Q_l$. Now we define the following sets.

$$A_1 = \left\{u \in V \colon \sum_{v\in H_u} x_v \geq \frac{1}{2}\right\}, A_2 = \left\{u \in V \colon \sum_{v\in V_u} x_v \geq \frac{1}{2}\right\}$$

$$H = \bigcup_{u\in A_1} H_u, V = \bigcup_{u\in A_2} V_u$$

Based on these, we consider the following two integer programs $Q'$ and $Q''$.

| $Q'$ | $Q''$ |
|---|---|
| minimize $\sum_{v\in H} x'_v$<br>subject to $\sum_{v\in H_u} x'_v \geq 1, \forall u \in A_1$<br>$x'_v \in \{0,1\}$, $v \in H$ | minimize $\sum_{v\in V} x''_v$<br>subject to $\sum_{v\in V_u} x''_v \geq 1, \forall u \in A_2$<br>$x''_v \in \{0,1\}$, $v \in V$ |

Let $Q'_l$ and $Q''_l$ be the relaxed LP of $Q'$ and $Q''$ respectively. Clearly, the solutions of $Q'$ and $Q''$ gives a feasible solution for $Q$. Hence $OPT(Q) \leq OPT(Q') + OPT(Q'')$. For each $x_v \in \mathbf{Q}_l$, define $y_v = \min\{1, 2x_v\}$ and define $\mathbf{Y}_l = \{y_v\}_{x_v\in\mathbf{Q}_l}$. Notice that $\mathbf{Y}_l$ gives a solution to $Q'_l$ and $Q''_l$. Therefore, $OPT(Q'_l) + OPT(Q''_l) \leq 4 \cdot OPT(Q_l)$. We have the following lemma.

**Lemma 7.** *$Q'$ and $Q''$ are* SRS *and* SSR *instances, respectively. Therefore, $OPT(Q') \leq 2 \cdot OPT(Q'_l)$ and $OPT(Q'') \leq 2 \cdot OPT(Q''_l)$.*

*Proof.* Note that for each vertex $u \in A_1$, $H_u$ is non-empty and for each $v \in H_u$, $L_v$ intersects the horizontal segment of $L_u$. Let $R$ be the set of horizontal segments of the L-paths representing the vertices in $A_1$ and $S$ be the set of vertical segments of the L-paths representing the vertices in $H$. Since all horizontal segments in $R$ intersect the vertical line $x = 0$ and the $x$-coordinates of the vertical segments in $S$ is strictly less than 0, we can consider the horizontal segments in $R$ as rightward directed rays. Hence, solving $Q'$ is equivalent to solving the ILP, say $\mathcal{E}$, of the problem of finding a minimum cardinality subset of vertical segments $S$ that intersects all rays in the set $R$ of rightward-directed rays. Hence solving $\mathcal{E}$ is equivalent to solving an SRS instance with $R$ and $S$ as input. By Lemma 6, we have that

$$OPT(Q') = OPT(\mathcal{E}) \leq 2 \cdot OPT(\mathcal{E}_l) \leq 2 \cdot OPT(Q'_l)$$

where $\mathcal{E}_l$ is the relaxed LP of $\mathcal{E}$. Hence we have proof of the first part.

For the second part, using similar arguments as above, we can show that solving $Q''$ is equivalent to solving an SSR instance. Hence, by Lemma 1, we have that $OPT(Q'') \leq 2 \cdot OPT(Q''_l)$. Hence the proof follows. □

**Proof of Theorem 4:** Lemma 7 implies that solving $Q'$ (resp. $Q''$) is equivalent to solving the SRS (resp. SSR) problem instance. Let $A$ be the union of the solutions returned by 2-approximation algorithm for SRS problem and the SSR-algorithm, used to solve $Q'$ and $Q''$ respectively. Hence,

$$|A| \leq 2(OPT(Q'_l) + OPT(Q''_l)) \leq 8 \cdot OPT(Q_l) \leq 8 \cdot OPT(Q)$$

Since $Q_l$ consists of $n$ variables where $n = |V|$, solving $Q_l$ takes $O(n^5)$ time [41]. Solving both the SSR and the SRS instances takes a total of $O(n \log n)$ time and the total running time of the algorithm is $O(n^5)$.

## 6. Algorithm for unit $B_0$-VPG graphs

Given a unit $B_0$ representation of a graph $G$ with $n$ vertices we shall give an 18-approximation algorithm for the MDS problem on $G$. We shall prove the following stronger theorem.

**Theorem 8.** *Let $S_1$ and $S_2$ be sets of orthogonal unit length segments. Let $\mathcal{C}$ be the ILP of the problem of finding a minimum cardinality subset $D$ of $S_2$ such that every segment in $S_1$ intersects some segment in $D$. There is an $O(n^5)$-time algorithm to compute a set $D' \subseteq S_2$ which gives a feasible solution of $\mathcal{C}$ and $|D'| \leq 18 \cdot OPT(\mathcal{C}_l)$ where $n = |S_1 \cup S_2|$ and $\mathcal{C}_l$ is the relaxed LP of $\mathcal{C}$.*

Theorem 2 follows from Theorem 8. Moreover, we shall use Theorem 8 to prove Theorem 3. In the next section, we give an overview of our algorithm.

### *6.1. Overview*

First, we solve the relaxed LP formulation $\mathcal{C}_l$ of $\mathcal{C}$ and create two subproblems. Since $\mathcal{C}$ consists of $n$ variables where $n = |S_2|$, solving $Q_l$ takes $O(n^5)$ time [41]. We shall show that these subproblems are equivalent to one of the following optimisation problems.

1. ***The Subset Unit Interval Domination*** **(SUID)** problem: In this problem, the input is (i) a set $X$ of horizontal unit length segments, (ii) a set $Y$ of vertical unit-length segments, and (iii) two sets $X', Y'$ such that $X' \subseteq X$ and $Y' \subseteq Y$. The objective is to find a minimum cardinality subset $D$ of $X \cup Y$ such that every horizontal (resp. vertical) segment in $X'$ (resp. $Y'$) intersects at least one horizontal (resp. vertical) segment in $D \cap X$ (resp. $D \cap Y$). Through out this article, $\mathcal{SUD}(X', X, Y', Y)$ shall denote an SUID instance.
2. ***The Unit Orthogonal Segment Stabbing*** **(UOSS)** problem: In this problem, the input is (i) two sets $X_1, X_2$ containing horizontal unit length segments and (ii) two sets $Y_1, Y_2$ containing vertical unit length segments. The objective is to find a minimum cardinality subset $D$ of $X_2 \cup Y_2$ such that every horizontal (resp. vertical) segment in $X_1$ (resp. $Y_1$) intersect at least one vertical (resp. horizontal) segment in $D \cap Y_2$ (resp. $D \cap X_2$). Through out this article, $\mathcal{US}(X_1, Y_1, X_2, Y_2)$ shall denote a UOSS instance.

We shall prove the following lemmas.

**Lemma 8.** *Let $X$ (resp. $Y$) be a set of horizontal (resp. vertical) unit length segments. For $X' \subseteq X$ and $Y' \subseteq Y$, let $\mathcal{A}$ be the ILP formulation of the $\mathcal{SUD}(X', X, Y', Y)$ instance. Then $OPT(\mathcal{A}) = OPT(\mathcal{A}_l)$ where $\mathcal{A}_l$ is the relaxed LP of $\mathcal{A}$. Moreover, $OPT(\mathcal{A})$ can be computed in $O(n \log n)$ time where $n = |X| + |Y|$.*

**Lemma 9.** *Let $X_1, X_2$ (resp. $Y_1, Y_2$) be sets of horizontal (resp. vertical) unit length segments. Let $\mathcal{B}$ be the ILP formulation of the $\mathcal{US}(X_1, Y_1, X_2, Y_2)$ instance. Then there is an $O(n^5)$-time algorithm to compute a set $D' \subseteq X_2 \cup Y_2$ which gives a feasible solution of $\mathcal{B}$ with $|D'| \leq 8 \cdot OPT(\mathcal{B}_l)$ where $n = |X_1 \cup X_2 \cup Y_1 \cup Y_2|$ and $\mathcal{B}_l$ is the relaxed LP of $\mathcal{B}$.*

In Section 6.2, we prove Lemma 8. Then in Section 6.3, we shall prove Lemma 9 using Lemma 1. Using the above lemmas we shall complete the proof of Theorem 8 in Section 6.4.

*6.2. Proof of Lemma 8*

Recall that $X$ is a set of horizontal unit length segments, $Y$ is a set of vertical unit length segments, $X' \subseteq X, Y' \subseteq Y$ and $\mathcal{A}$ is the ILP formulation of the $\mathcal{SUD}(X', X, Y', Y)$ instance.

Let $\mathcal{A}'$ be the ILP formulation of the problem of finding a subset $D_1$ of $X$ with minimum cardinality such that any segment in $X'$ intersects a segment in $D_1$. Let $\mathcal{A}''$ be the ILP formulation of the problem of finding a subset $D_2$ of $Y$ with minimum cardinality such that any segment in $Y'$ intersects a segment in $D_2$. Observe that, $OPT(\mathcal{A}) = OPT(\mathcal{A}') + OPT(\mathcal{A}'')$ and $OPT(\mathcal{A}_l) = OPT(\mathcal{A}'_l) + OPT(\mathcal{A}''_l)$ where $\mathcal{A}'_l$ and $\mathcal{A}''_l$ are the relaxed LP formulations of $\mathcal{A}'$ and $\mathcal{A}''$, respectively. Now we have the following observation.

**Observation E.** *$OPT(\mathcal{A}') = \mathcal{A}'_l$ and $OPT(\mathcal{A}'') = OPT(\mathcal{A}''_l)$.*

*Proof.* We shall only prove the observation for $OPT(\mathcal{A}')$ as similar arguments will suffice for the other case. Let $X'_i \subseteq X'$ be the set of all horizontal segments whose $y$-coordinate is $i$. Similarly let $X_i \subseteq X$ be the set of all horizontal segments whose $y$-coordinate is $i$. Let $\mathcal{A}'_i$ be the ILP formulation of the problem of finding a subset $D'_i$ of $X$ with minimum cardinality such that any segment in $X'_i$ intersects a segment in $D'$. Since for $i \neq j$, $X_i \cap X_j = \emptyset$ and

$X'_i \cap X'_j = \emptyset$, observe that, $OPT(\mathcal{A}') = \sum_i OPT(\mathcal{A}'_i)$ and $OPT(\mathcal{A}'_l) = \sum_i OPT(\mathcal{A}'_{i,l})$ where $\mathcal{A}'_{i,l}$ is the relaxed LP formulation of $\mathcal{A}'_i$. Now we prove the following claim.

*Claim 1. For each $i$, $OPT(\mathcal{A}'_i) = OPT(\mathcal{A}'_{i,l})$.*

To prove the claim first define for each horizontal segment $h \in X_i$, let $l(h)$ denote the left endpoints of $h$. Let $h_1, h_2, \ldots, h_k$ be the segments in $X_i$ sorted in the ascending order of the $x$-coordinates of $l(h)$. For a segment $h' \in X'_i$, let $N(h')$ denote the set of intervals in $X_i$ that intersect $h'$. Let $\mathcal{M}$ be the coefficient matrix of $\mathcal{A}'_i$ such that the $i^{th}$ column of $\mathcal{M}$ corresponds to the variable corresponding to $h_i \in X_i$. Observe that in each row of $\mathcal{M}$, the set of 1's are consecutivel. Therefore, $\mathcal{M}$ is a totally unimodular matrix [46]. Thus any optimal solution of $\mathcal{A}'_{i,l}$ is integral. Thus we have the proof.

Hence $OPT(\mathcal{A}') = \sum_i OPT(\mathcal{A}'_i) = \sum_i OPT(\mathcal{A}'_{i,l}) = OPT(\mathcal{A}'_l)$. This completes the proof. □

Using the above observation, we have that $OPT(\mathcal{A}) = OPT(\mathcal{A}') + OPT(\mathcal{A}'') = OPT(\mathcal{A}'_l) + OPT(\mathcal{A}''_l) = OPT(\mathcal{A}_l)$. This completes the proof of the lemma.

### *6.3. Proof of Lemma 9*

Recall that $X_1, X_2$ are sets of horizontal unit length segments, $Y_1, Y_2$ are sets of vertical unit length segments and $\mathcal{B}$ is the ILP formulation of the $\mathcal{US}(X_1, Y_1, X_2, Y_2)$ instance.

Let $\mathcal{B}'$ be the ILP formulation of the problem of finding a subset $D'$ of $Y_2$ with minimum cardinality such that any segment in $X_1$ intersects a segment in $D'$. Let $\mathcal{B}''$ be the ILP formulation of the problem of finding a subset $D''$ of $X_2$ with minimum cardinality such that any segment in $Y_1$ intersects a segment in $D''$. Observe that, $OPT(\mathcal{B}) = OPT(\mathcal{B}') + OPT(\mathcal{B}'')$ and $OPT(\mathcal{B}_l) = OPT(\mathcal{B}'_l) + OPT(\mathcal{B}''_l)$ where $\mathcal{B}'_l$ and $\mathcal{B}''_l$ are the relaxed LP formulations of $\mathcal{B}'$ and $\mathcal{B}''$, respectively. Now we prove the following proposition.

**Proposition 9.** *$OPT(\mathcal{B}') \leq 8 \cdot OPT(\mathcal{B}'_l)$ and $OPT(\mathcal{B}'') \leq 8 \cdot OPT(\mathcal{B}''_l)$.*

*Proof.* We shall only prove the proposition for $OPT(\mathcal{B}'')$ as similar arguments suffice for the other case. Let $X_2 = S$ and $Y_1 = T$ and let $\mathcal{I}_S$ be the set of intervals obtained by projecting the horizontal segments in $S$ onto the $x$-axis. Observe that $\mathcal{I}_S$ is a set of unit intervals.

We assume that (i) no two interval in $\mathcal{I}_S$ contain each other, and (ii) $x$-coordinate of any vertical segment in $T$ is distinct from the left and right endpoints of any interval in $\mathcal{I}_S$. Since no two interval in $\mathcal{I}_S$ contain each other, there exists a set $P$ of real numbers such that each interval in $\mathcal{I}_S$ contains exactly one real number from $P$. (To see this, consider the right endpoints of the intervals in the maximum cardinality subset of $\mathcal{I}_S$ with pairwise non-intersecting intervals which is obtained using the greedy algorithm [47]). Add in $P$ two more dummy values $q, q'$ which are not contained in any interval in $\mathcal{I}_S$ and $q$ (resp. $q'$) is less than (resp. greater than) that of all values in $P$. Let $p_1, p_2, \ldots, p_t$ be the values in $P$ sorted in the ascending order (notice that $p_1 = q$ and $p_t = q'$). For each $i \in \{1, 2, \ldots, t-1\}$, let $T_i$ denote the vertical segments of $T$ that lies inside the strip bounded by the lines $y = p_i$ and $y = p_{i+1}$. Due to our general position assumption for any $i \neq j$, $T_i$ and $T_j$ are disjoint. For each $i \in \{1, 2, \ldots, t-1\}$, and each vertical segment $v \in T_i$, let $S_v^{left}$ (resp. $S_v^{right}$) be the subset of $S$ that intersects $v$ and the line $y = p_i$ (resp. $y = p_{i+1}$). Since any interval in $\mathcal{I}_S$

contains exactly one value from $P$ and therefore from $\{p_i, p_{i+1}\}$, $S_v^{left} \cap S_v^{right} = \emptyset$, for each vertical segment $v \in T$. Based on these we have the following equivalent ILP formulation (say $W$) of $\mathcal{B}''$.

$$\begin{array}{ll} \text{minimize} & \sum_{v \in S} x_v \\ \text{subject to} & \sum_{v \in S_u^{left}} x_v + \sum_{v \in S_u^{right}} x_v \geq 1, \forall u \in T \\ & x_v \in \{0,1\}, \qquad \forall v \in S \\ & \qquad W \end{array}$$

Let $\mathbf{W}_l = \{x_v : v \in S\}$ be an optimal solution of the relaxed LP formulation (say $W_l$) of $W$. Consider the following sets.

$$A_1 = \left\{ u \in T \colon \sum_{v \in S_u^{left}} x_v \geq \frac{1}{2} \right\}, A_2 = \left\{ u \in T \colon \sum_{v \in S_u^{right}} x_v \geq \frac{1}{2} \right\}$$

$$L = \bigcup_{v \in A_1} S_v^{left}, R = \bigcup_{v \in A_2} S_v^{right}$$

Based on these, we consider the following two integer programs $W'$ and $W''$.

| | |
|---|---|
| $\begin{array}{ll} \text{minimize} & \sum_{v \in L} x'_v \\ \text{subject to} & \sum_{v \in S_u^{left}} x'_v \geq 1, \forall u \in A_1 \\ & x'_v \in \{0,1\}, \quad v \in L \\ & \quad W' \end{array}$ | $\begin{array}{ll} \text{minimize} & \sum_{v \in R} x''_v \\ \text{subject to} & \sum_{v \in S_u^{right}} x''_v \geq 1, \forall u \in A_2 \\ & x''_v \in \{0,1\}, \quad v \in R \\ & \quad W'' \end{array}$ |

Let $W'_l$ and $W''_l$ be the corresponding relaxed LPs of $W'$ and $W''$ respectively. The union of the solutions of $W'$ and $W''$ is a solution for $W$ implying $OPT(W) \leq OPT(W') + OPT(W'')$. For each $x_v \in \mathbf{W}_l$, define $y_v = \min\{1, 2x_v\}$ and define $\mathbf{Y}_l = \{y_v\}_{x_v \in \mathbf{W}_l}$. Notice that $\mathbf{Y}_l$ gives a solution to $W'_l$ (and $W''_l$). Hence, $OPT(W'_l) \leq 2 \cdot OPT(W_l)$ and $OPT(W''_l) \leq 2 \cdot OPT(W_l)$. Therefore, $OPT(W'_l) + OPT(W''_l) \leq 4 \cdot OPT(W_l)$. Notice that, solving $W'$ (resp. $W''$) is equivalent to the problem of finding a minimum cardinality subset of the horizontal segments in $L$ (resp. $R$) to intersect all vertical segments in $A_1$ (resp. $A_2$). Now we have the following claim.

*Claim 2.* $OPT(W') \leq 2 \cdot OPT(W'_l)$ *and* $OPT(W'') \leq 2 \cdot OPT(W''_l)$.

We shall prove the above claim only for $W'$ as proof for the other case is similar. Recall that solving $W'$ is equivalent to the problem of finding a minimum cardinality subset of the horizontal segments in the set $L$ (defined earlier) to intersect all vertical segments in $A_1$. For each $i \in \{1, 2, \ldots, (t-1)\}$ let $T_{1,i} = A_1 \cap T_i$ and $L_i$ be the set of horizontal segments in $L$ that intersect some vertical segment in $T_{1,i}$. Formally, $L_i = \bigcup_{v \in T_{1,i}} S_v^{left}$. For any $i \neq j$, $T_{1,i} \cap T_{1,j} = \emptyset$ and $L_i \cap L_j = \emptyset$ (this follows from the fact no horizontal segment in $S$ intersects both $y = p_i$ and $y = p_j$). For each $i \in \{1, 2, \ldots, (t-1)\}$, let $\mathcal{D}_i$ (resp, $\mathcal{D}_{i,l}$) denote the ILP (resp. relaxed LP) of the problem of selecting minimum subset $D_i$ horizontal segments in $L_i$ such that all vertical segments in $T_{1,i}$ intersect at least one horizontal segment in $D_i$. Clearly, $OPT(W') = \sum_{i=1}^{t-1} OPT(\mathcal{D}_i)$ and $OPT(W'_l) = \sum_{i=1}^{t-1} OPT(\mathcal{D}_{i,l})$. For each $i \in \{1, 2, \ldots, (t-1)\}$ notice that, all horizontal segments intersect the vertical line $y = p_i$ and all vertical segments in $T_{1,i}$ lies to the left of the vertical line $y = p_i$. For each $i \in \{1, 2, \ldots, (t-1)\}$ if we consider the segments in $L_i$ to be leftward-directed rays then solving $\mathcal{D}_i$ is equivalent to solving an SSR instance with $T_{1,i}$ and $L_i$ as input. Due to Lemma 1, for each $i \in \{1, 2, \ldots, (t-1)\}$, $OPT(\mathcal{D}_i) \leq 2 \cdot OPT(\mathcal{D}_{i,l})$. Hence,

$$OPT(W') = \sum_{i=1}^{t-1} OPT(\mathcal{D}_i) \leq 2 \cdot \sum_{i=1}^{t-1} OPT(\mathcal{D}_{i,l}) = 2 \cdot OPT(W'_l)$$

This completes the proof of the claim.

Using the above claim and previous observations, we can infer that

$$OPT(W) \leq OPT(W') + OPT(W'') \leq 2(OPT(W'_l) + OPT(W''_l)) \leq 8 \cdot OPT(W_l)$$

This completes the proof of the proposition. □

Hence, Observe that, $OPT(\mathcal{B}) = OPT(\mathcal{B}') + OPT(\mathcal{B}'') \leq 8(OPT(\mathcal{B}'_l) + OPT(\mathcal{B}''_l)) = 8 \cdot OPT(\mathcal{B}_l)$. This completes the proof of the lemma.

### *6.4. Completion of proof of Theorem 8*

Recall that $S_1$ and $S_2$ are sets of orthogonal unit length segments, $\mathcal{C}$ is an ILP formulation of the problem of finding a minimum cardinality subset $D$ of $S_2$ such that every segment in $S_1$ intersects some segment in $D$. We shall give an $O(n^5)$-time algorithm to compute a set $D' \subseteq S_2$ which gives a feasible solution of $\mathcal{C}$ and $|D'| \leq 18 \cdot OPT(\mathcal{C}_l)$ where $n = |S_1 \cup S_2|$ and $\mathcal{C}_l$ is the relaxed LP formulation of $\mathcal{C}$.

Let $V_1$ and $H_1$ are the sets of vertical and horizontal segments in $S_1$, respectively. Similarly, let $V_2$ and $H_2$ are the sets of vertical and horizontal segments in $S_2$, respectively. For $v \in V_1 \cup H_1$, let $N(v) \subseteq V_2 \cup H_2$ denote the set of segments that intersects $v$. For $w \in H_1$, let $N_o(w) = N(w) \cap H_2$. For $w \in V_1$, let $N_o(w) = N(w) \cap V_2$. Based on these we have the following equivalent ILP formulation (say $Z$) of $\mathcal{C}$.

$$\begin{aligned} \text{minimize} \quad & \sum_{w \in V_2 \cup H_2} x_w \\ \text{subject to} \quad & \sum_{w \in N_o(u)} x_w + \sum_{w \in N(u) \setminus N_o(u)} x_w \geq 1, \forall u \in V_1 \cup H_1 \\ & x_w \in \{0, 1\}, \qquad \forall w \in V_2 \cup H_2 \end{aligned}$$

$Z$

The first step of our algorithm is to solve the relaxed LP formulation (say $Z_l$) of $Z$. Let $\mathbf{Z}_l = \{x_w : w \in V_2 \cup H_2\}$ be an optimal solution of $Z_l$. Let

$$A_1 = \left\{ u \in V_1 \cup H_1 : \sum_{w \in N_o(u)} x_w \geq \frac{1}{2} \right\}$$

$$A_2 = \left\{ u \in V_1 \cup H_1 \colon \sum_{w \in N(u) \setminus N_o(u)} x_w \geq \frac{1}{2} \right\}$$

,

$$B_1 = \bigcup_{u \in A_1} N_o(u), \;\; B_2 = \bigcup_{u \in A_2} N(u) \setminus N_o(u)$$

Based on these, we consider the following two integer programs $Z'$ and $Z''$.

| $Z'$ | $Z''$ |
|---|---|
| minimize $\sum_{w \in B_1} x'_w$<br>subject to $\sum_{w \in N_o(v)} x'_w \geq 1, \forall v \in A_1$<br>$x'_w \in \{0,1\}, \quad w \in B_1$ | minimize $\sum_{w \in B_2} x''_w$<br>subject to $\sum_{w \in N(v) \setminus N_o(v)} x''_w \geq 1, \forall v \in A_2$<br>$x''_w \in \{0,1\}, \quad w \in B_2$ |

Let $Z'_l$ and $Z''_l$ be the corresponding relaxed LPs of $Z'$ and $Z''$ respectively. Clearly, the union of the solutions of $Z'$ and $Z''$ is a solution for $Z$. Hence, $OPT(Z) \leq OPT(Z') + OPT(Z'')$. For each $x_v \in \mathbf{Z}_l$, define $y_v = \min\{1, 2x_v\}$ and define $\mathbf{Y}_l = \{y_v\}_{x_v \in \mathbf{Z}_l}$. Notice that $\mathbf{Y}_l$ gives a solution for $Z'_l$ and $Z''_l$. Hence, $OPT(Z'_l) \leq 2 \cdot OPT(Z_l)$ and $OPT(Z''_l) \leq 2 \cdot OPT(Z_l)$. Now we prove the following lemma.

**Lemma 10.** *$OPT(Z') = OPT(Z'_l)$ and $OPT(Z'') \leq 8 \cdot OPT(Z''_l)$.*

*Proof.* To prove the first part, let $X$ (resp. $Y$) be the set of horizontal (resp. vertical) segments in $B_1$ and $X'$ (resp. $Y'$) be the set of horizontal (resp. vertical) segments in $A_1$. Notice that $X' \subseteq X$ and $Y' \subseteq Y$. Hence, $Z'$ is the ILP formulation of finding minimum cardinality subset $D$ of $X \cup Y$ such that every horizontal (resp. vertical) segment in $X'$ (resp. $Y'$) intersects at least one horizontal (resp. vertical) segment in $D \cap X$ (resp. $D \cap Y$). By Lemma 8, we have that $OPT(Z') = OPT(Z'_l)$.

To prove the second part, let $X_1$ and $X_2$ (resp. $Y_1$ and $Y_2$) be the sets of horizontal (resp. vertical) segments in $A_2$ and $B_2$, respectively. Notice that $Z''$ is the ILP formulation of finding minimum cardinality subset $D$ of $X_2 \cup Y_2$ such that every horizontal (resp. vertical) segment in $X_1$ (resp. $Y_1$) intersects at least one vertical (resp. horizontal) segment in $D \cap Y_2$ (resp. $D \cap X_2$). By Lemma 9, we have that $OPT(Z'') \leq 8 \cdot OPT(Z''_l)$. □

Using Lemma 10 and previous arguments, we can conclude that in $O(n^5)$ time it is possible to compute a set $D' \subseteq S_2$ which gives a feasible solution of $Z$ where $n = |S_1 \cup S_2|$. Moreover, $|D'| \leq OPT(Z') + OPT(Z'') \leq OPT(Z'_l) + 8 \cdot OPT(Z''_l) \leq 18 \cdot OPT(Z_l) \leq 18 \cdot OPT(\mathcal{C}_l)$. This completes the proof of the theorem.

## 7. Algorithm for unit $B_k$-VPG graphs

Let $\mathcal{R}$ be a unit $B_k$-VPG representation of a unit $B_k$-VPG graph $G = (V, E)$. Throughout this section, we assume that the segments of each path $P \in \mathcal{R}$ are numbered consecutively, starting from a segment containing one of the endpoints of $P$, by $1, 2, \ldots, t$ where $t (\leq k+1)$ is the number of segments in $P$. For a path $P \in \mathcal{R}$, let $N(P)$ denote the set of paths in $\mathcal{R}$ that intersect $P$.

Define $\Phi\colon \mathcal{R}\times\mathcal{R}\to\mathbb{N}\times\mathbb{N}$ such that for two paths $P,Q\in\mathcal{R}$, $\Phi(P,Q)=(i,j)$ if and only if the $i^{th}$ segment of $P$ intersects the $j^{th}$ segment of $Q$, and for all $1\le a<i$, the $a^{th}$ segment of $P$ does not intersect any segment of $Q$.

For a path $P\in\mathcal{R}$, let $\mathcal{X}_P(i,j)=\{Q\in N[P]\colon \Phi(P,Q)=(i,j)\}$. For distinct pairs $(i,j)$ and $(i',j')$ the sets $\mathcal{X}_P(i,j)$ and $\mathcal{X}_P(i',j')$ are disjoint. Let $\mathcal{K}$ denote the set $\{1,2,\ldots,k+1\}\times\{1,2,\ldots,k+1\}$. Based on these we have the following ILP formulation of the MDS problem on $G$.

$$\begin{array}{ll} \text{minimize} & \sum\limits_{Q\in\mathcal{R}} x_Q \\ \text{subject to} & \sum\limits_{(i,j)\in\mathcal{K}}\ \sum\limits_{Q\in\mathcal{X}_P(i,j)} x_Q\ge 1,\ \forall P\in\mathcal{R} \\ & x_Q\in\{0,1\},\qquad \forall P\in\mathcal{R} \end{array}$$
$$Z$$

First step of our algorithm is to solve the relaxed LP formulation (say $Z_l$) of $Z$. Let $\mathbf{Z}_l=\{x_Q\colon Q\in\mathcal{R}\}$ be an optimal solution of $Z_l$. For each path $P\in\mathcal{R}$, there is a pair $(i,j)\in\mathcal{K}$ such that $\sum\limits_{Q\in\mathcal{X}_P(i,j)} x_Q\ge\frac{1}{(k+1)^2}$. For each pair $(i,j)\in\mathcal{K}$, define

$$\mathcal{A}(i,j)=\left\{P\in\mathcal{R}\colon \sum_{Q\in\mathcal{X}_P(i,j)} x_Q\ge\frac{1}{(k+1)^2}\right\},\mathcal{B}(i,j)=\bigcup_{P\in\mathcal{A}(i,j)}\mathcal{X}_P(i,j)$$

Based on these, we have the following ILP formulation for each pair $(i,j)\in\mathcal{K}$.

$$\begin{array}{ll} \text{minimize} & \sum\limits_{Q\in\mathcal{B}(i,j)} x'_Q \\ \text{subject to} & \sum\limits_{Q\in\mathcal{X}_P(i,j)} x'_Q\ge 1,\ \forall P\in\mathcal{A}(i,j) \\ & x'_Q\in\{0,1\},\qquad \forall Q\in\mathcal{B}(i,j) \end{array}$$
$$Z(i,j)$$

For each pair $(i,j)\in\mathcal{K}$, let $Z_l(i,j)$ be the relaxed LP formulation of $Z(i,j)$. We have the following

$$OPT(Z)\le\sum_{(i,j)\in\mathcal{K}} OPT(Z(i,j))$$

For each $x_P\in\mathbf{Z}_l$, define $y_P=\min\{1,x_P(k+1)^2\}$ and define $\mathbf{Y}_l=\{y_P\}_{x_P\in\mathbf{Z}_l}$. Clearly, $\mathbf{Y}_l$ gives a solution to $Z_l(i,j)$ for each $(i,j)\in\mathcal{K}$. Moreover,

$$\sum_{(i,j)\in\mathcal{K}} OPT(Z_l(i,j))\le(k+1)^4\cdot OPT(Z_l)$$

Now we have the following lemma.

**Lemma 11.** *For each pair $(i,j)\in\mathcal{K}$, there is a solution $D(i,j)$ for $Z(i,j)$ such that $|D(i,j)|\le 18\cdot OPT(Z_l(i,j))$. Moreover, $D(i,j)$ can be found in $O(n^5)$ time.*

*Proof.* For any $(i,j)\in\mathcal{K}$, solving $Z(i,j)$ is equivalent to finding a minimum cardinality subset $D$ of $\mathcal{B}(i,j)$ such that each path $P\in\mathcal{A}(i,j)$ intersects at least one path is $D\cap\mathcal{X}_P(i,j)$.

Notice that, for each $P \in \mathcal{A}(i,j)$ the set $\mathcal{X}_u(i,j)$ is non-empty and for each $Q \in \mathcal{X}_P(i,j)$, the $i^{th}$ segment of $P$ intersects the $j^{th}$ segment of $Q$. Let $S_1 = \{i^{th}$ segment of $P \colon P \in \mathcal{A}(i,j)\}$, $S_2 = \{j^{th}$ segment of $Q \colon Q \in \mathcal{B}(i,j)\}$.

Solving $Q(i,j)$ is equivalent to the problem finding a minimum cardinality subset $D$ of $S_2$ such that every segment in $S_1$ intersect at least one segment in $D$. Moreover, every segment in $S_1 \cup S_2$ have unit length. Hence by Theorem 8, we have a solution (say $D(i,j)$) for $Z(i,j)$ such that $|D(i,j)| \leq 18 \cdot OPT(Z_l(i,j))$. The running time also follows from Theorem 8. □

For each pair $(i,j) \in \mathcal{K}$, due to Lemma 11, we have a solution $D(i,j)$ of $Z(i,j)$ such that $|D(i,j)| \leq 18 \cdot OPT(Z_l(i,j))$. Let $D$ be the union of $D(i,j)$'s for all $(i,j) \in \mathcal{K}$. We have that

$$\begin{aligned} |D| &= \sum_{(i,j)\in\mathcal{K}} |D(i,j)| \\ &\leq \sum_{(i,j)\in\mathcal{K}} 18 \cdot OPT(Z_l(i,j)) \\ &\leq 18 \cdot (k+1)^4 \cdot OPT(Z_l) \leq 18 \cdot (k+1)^4 \cdot OPT(Z) \end{aligned}$$

Since $|\mathcal{K}|$ is $O(k^2)$ and due to Lemma 11, in $O(k^2 n^5)$ time it is possible to construct the set $D$. This completes the proof of Theorem 3.

## 8. Algorithm for stabbed rectangle overlap graphs

Given a stabbed rectangle overlap representation of a graph $G$ with $n$ vertices, we shall give a 656-approximation algorithm for the MDS problem on $G$. Below we give an overview of the algorithm.

### *8.1. Overview*

First, we solve the relaxed LP formulation of the ILP formulation of the MDS problem on the input graph $G$ and create eight subproblems. We shall show that these subproblems are equivalent to one of the following optimisation problems.

1. **The *local vertical segment covering* (LVSC) problem:** In this problem, the input is a set $H$ of disjoint horizontal segments intersecting a common straight line $l$ and a set $V$ containing disjoint vertical segments none of which intersects $l$. The objective is to select a minimum number of horizontal segments that intersect all vertical segments. Throughout this article, we let $\mathcal{LVSC}(V,H)$ denote an LVSC instance.
2. **The *local horizontal segment covering* (LHSC) problem:** In this problem, the input is a set $H$ of disjoint horizontal segments all intersecting a common straight line and a set $V$ of disjoint vertical segments. The objective is to select a minimum number of vertical segments that intersect all horizontal segments. Throughout this article, we let $\mathcal{LHSC}(V,H)$ denote an LHSC instance.

We note that Bandyapadhyay and Mehrabi [48] considered restricted cases of LVSC and LHSC problem. They proved that LVSC problem remains NP-hard even if all horizontal segments in the input instance intersect a common vertical line. We also note that PTAS are known for both LVSC and LHSC problems [49]. However, to prove Theorem 6, we need to prove the following lemmas.

**Lemma 12.** *Let $\mathcal{C}$ be an ILP formulation of an $\mathcal{LVSC}(V, H)$ instance. There is an $O(n^5)$ time algorithm to compute a set $D \subseteq H$ which gives a feasible solution of $\mathcal{C}$ and $|D| \leq 4 \cdot OPT(\mathcal{C}_l)$ where $n = |V \cup H|$ and $\mathcal{C}_l$ is the relaxed LP formulation of $\mathcal{C}$.*

**Lemma 13.** *Let $\mathcal{C}$ be an ILP formulation of an $\mathcal{LHSC}(V, H)$ instance. There is an $O(n^5)$ time algorithm to compute a set $D \subseteq V$ which gives a feasible solution of $\mathcal{C}$ and $|D| \leq 8 \cdot OPT(\mathcal{C}_l)$ where $n = |V \cup H|$ and $\mathcal{C}_l$ is the relaxed LP formulation of $\mathcal{C}$.*

In Section 8.2 and Lemma 8.3, we shall use Lemma 1 and Lemma 2 to prove Lemma 12 and Lemma 13, respectively. In Section 8.4 we complete the proof of Theorem 6.

*8.2. Proof of Lemma 12*

Let $l$ be the straight line intersecting all horizontal segments in $H$. We assume that $l$ passes through the origin at an angle in $[\frac{\pi}{2}, \pi)$. (Otherwise, first, we translate all segments to the first quadrant and reflect the segments w.r.t the $y$-axis). For a vertical segment $v \in V$, let $N(v)$ denote the set of horizontal segments intersecting $v$. Let $V_1 \subseteq V$ be the set of vertical segments that lie above $l$ and $V_2 = V \setminus V_1$. Based on these, consider the following equivalent ILP formulation of $\mathcal{C}$.

$$\begin{array}{lll} \text{minimize} & \sum_{h \in H} x_h & \\ \text{subject to} & \sum_{h \in N(v)} x_h \geq 1, \ \forall v \in V_1 & \\ & \sum_{h \in N(v)} x_h \geq 1, \ \forall v \in V_2 & \\ & x_h \in \{0,1\}, & \forall h \in H \end{array}$$

$Q$

Let $Q_l$ be the relaxed LP formulation of $Q$. Now consider the following two ILPs.

| | |
|---|---|
| $\begin{array}{ll} \text{minimize} & \sum_{h \in H} x'_h \\ \text{subject to} & \sum_{h \in N(v)} x'_h \geq 1, \forall v \in V_1 \\ & x'_h \in \{0,1\}, \quad h \in H \end{array}$ <br> $Q'$ | $\begin{array}{ll} \text{minimize} & \sum_{h \in H} x''_h \\ \text{subject to} & \sum_{h \in N(v)} x''_h \geq 1, \forall v \in V_2 \\ & x''_h \in \{0,1\}, \quad h \in H \end{array}$ <br> $Q''$ |

Observe that $OPT(Q'_l) + OPT(Q''_l) \leq 2 \cdot OPT(Q_l)$ where $Q'_l$ and $Q''_l$ are the relaxed LP formulation of $Q'$ and $Q''$ respectively. We have the following claim.

*Claim 3.* $OPT(Q') \leq 2 \cdot OPT(Q'_l)$ *and* $OPT(Q'') \leq 2 \cdot OPT(Q''_l)$.

We shall only prove the first part as similar arguments suffice for the latter. Since all segments in $H$ intersect the straight line $l$, we can consider the horizontal segments in $H$

as leftward-directed rays and all vertical segments in $V_1$ lie above $l$. Hence, solving $Q'$ is equivalent to solving an ILP formulation, say $\mathcal{E}$, of the problem of finding a minimum cardinality subset of leftward-directed rays in $H$ that intersects all vertical segments in the set $V_1$. Hence solving $\mathcal{E}$ is equivalent to solving an SSR instance with $H$ and $V_1$ as input. By Lemma 1, we have that

$$OPT(Q') = OPT(\mathcal{E}) \leq 2 \cdot OPT(\mathcal{E}_l) \leq 2 \cdot OPT(Q'_l)$$

where $\mathcal{E}_l$ is the relaxed LP formulation of $\mathcal{E}$. Hence we have proof of the claim.

By Lemma 1, we can solve both $Q'$ and $Q''$ in polynomial time. Let $D'$ and $D''$ be solutions of $Q'$ and $Q''$, respectively. Observe that, $D' \cup D''$ is a feasible solution to the $\mathcal{LVSC}(V, H)$ instance. Hence,

$$|D' \cup D''| \leq 2(OPT(Q'_l) + OPT(Q''_l) \leq 4 \cdot OPT(Q_l)$$

This completes the proof.

*8.3. Proof of Lemma 13*

Let $l$ be the straight line that intersects all horizontal segment in $H$. We assume that $l$ passes through the origin at an angle in $[\frac{\pi}{2}, \pi)$. (Otherwise, first, we translate all segments to the first quadrant and reflect the segments w.r.t the $y$-axis). For a horizontal segment $h \in H$, let $N(h)$ denote the set of vertical segments intersecting $h$, $A(h)$ be the set of vertical segments that intersect $h$ *above* $l$ and $B(h) = N(h) \setminus A(h)$. Observe that for a horizontal segment $h$ and a vertical segment $v \in B(h)$, $v$ intersects $h$ *on* or *below* $l$.

Based on these, we have the following ILP formulation of the $\mathcal{LHSC}(V, H)$ instance.

$$\begin{array}{ll} \text{minimize} & \sum_{v \in V} x_v \\ \text{subject to} & \sum_{v \in A(h)} x_v + \sum_{v \in B(h)} x_v \geq 1, \ \forall h \in H \\ & x_v \in \{0, 1\}, \qquad \forall v \in V \end{array}$$
$$Q$$

Let $Q_l$ be the the relaxed LP formulation of $Q$ and $\mathbf{Q}_l = \{x_v : v \in V\}$ be an optimal solution of $Q_l$. Now we define the following sets.

$$H_1 = \left\{ h \in H : \sum_{v \in A(h)} x_v \geq \frac{1}{2} \right\}, H_2 = \left\{ h \in H : \sum_{v \in B(h)} x_v \geq \frac{1}{2} \right\}$$

$$V_1 = \bigcup_{h \in H_1} A(h), V_2 = \bigcup_{h \in H_2} B(h)$$

Based on these, we consider the following two integer programs $Q'$ and $Q''$.

| | |
|---|---|
| $$\begin{array}{ll} \text{minimize} & \sum_{v \in V_1} x'_v \\ \text{subject to} & \sum_{v \in A(h)} x'_v \geq 1, \forall h \in H_1 \\ & x'_v \in \{0,1\}, \quad v \in V_1 \end{array}$$ $Q'$ | $$\begin{array}{ll} \text{minimize} & \sum_{v \in V_2} x''_v \\ \text{subject to} & \sum_{v \in B(h)} x''_v \geq 1, \forall h \in H_2 \\ & x''_v \in \{0,1\}, \quad v \in V_2 \end{array}$$ $Q''$ |

Let $Q'_l$ and $Q''_l$ be the relaxed LP formulation of $Q'$ and $Q''$ respectively. Clearly, the solutions of $Q'$ and $Q''$ gives a feasible solution for $Q$. Hence $OPT(Q) \leq OPT(Q') + OPT(Q'')$. For each $x_v \in \mathbf{Q}_l$, define $y_v = \min\{1, 2x_v\}$ and define $\mathbf{Y}_l = \{y_v\}_{x_v \in \mathbf{Q}_l}$. Notice that $\mathbf{Y}_l$ gives a feasible solution to $Q'_l$ and $Q''_l$. Therefore, $OPT(Q'_l) + OPT(Q''_l) \leq 4 \cdot OPT(Q_l)$. We have the following claim.

*Claim 4.* $OPT(Q') \leq 2 \cdot OPT(Q'_l)$ *and* $OPT(Q'') \leq 2 \cdot OPT(Q''_l)$.

To prove the first part, note that for each vertex $h \in H_1$, $A(h)$ is non-empty and for each $v \in A(h)$, $v$ intersects $h$ above the line $l$ (the straight line which intersects all segments in $H$). Since all segments in $H_1$ intersect the straight line $l$ we can consider the horizontal segments in $H_1$ as leftward-directed rays and all vertical segments in $V_1$ lie above $l$. Hence, solving $Q'$ is equivalent to solving an ILP formulation, say $\mathcal{E}$, of the problem of finding a minimum cardinality subset of vertical segments in $V_1$ that intersects all leftward-directed rays in the set $H_1$. Hence solving $\mathcal{E}$ is equivalent to solving an SRS instance with $V_1$ and $H_1$ as input. By Lemma 2, we have that

$$OPT(Q') = OPT(\mathcal{E}) \leq 2 \cdot OPT(\mathcal{E}_l) \leq 2 \cdot OPT(Q'_l)$$

where $\mathcal{E}_l$ is the relaxed LP formulation of $\mathcal{E}$. Hence we have proof of the first part. For the second part, using similar arguments as above, we can show that solving $Q''$ is equivalent to solving an SRS instance and therefore by Lemma 2, we have that $OPT(Q'') \leq 2 \cdot OPT(Q''_l)$. Hence the proof of the claim follows.

By Lemma 2, we can solve both $Q'$ and $Q''$ in polynomial time. Let $D'$ and $D''$ be solutions of $Q'$ and $Q''$, respectively. Clearly, $D' \cup D''$ is a feasible solution to the $\mathcal{LHSC}(V, H)$ instance. Hence,

$$|D' \cup D''| \leq 2(OPT(Q'_l) + OPT(Q''_l) \leq 8 \cdot OPT(Q_l)$$

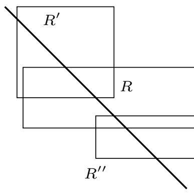
Hence we have the proof of Lemma 13.

### *8.4. Completion of proof of Theorem 6*

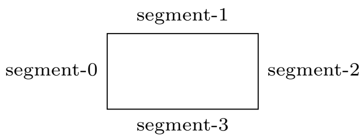
Let $\mathcal{R}$ be a stabbed rectangle overlap representation of a graph $G = (V, E)$ and $l$ be the line that intersects all rectangles in $\mathcal{R}$. We shall also refer to $l$ as the *cutting line.*

For a vertex $u \in V$, let $R_u$ denote the rectangle corresponding to $u$ in $\mathcal{R}$. We assume that the coordinates of all corner points of all the rectangles in $\mathcal{R}$ are distinct and that the cutting line passes through the origin at an angle in $[\frac{\pi}{2}, \pi)$ with the positive $x$-axis.

Each rectangle $R_u$ consists of four *boundary segments* i.e. *left segment*, *top segment*, *right segment* and *bottom segment.* We assume that the cutting line intersects exactly two boundary segments of each rectangle in $\mathcal{R}$. (Otherwise we can perturb the rectangles



(a)



(b)

Figure 5: (a) In this example $R' \in N'(R)$ and $R'' \in N''(R)$. (b) Nomenclature for the four boundadry segments of a rectangle.

without changing the corresponding overlap graph so that the cutting line intersects exactly two boundary segments of each rectangle). For a rectangle $R \in \mathcal{R}$, let $B(R)$ denote the set of boundary segments of $R$ that intersect the cutting line. Similarly, let $\overline{B(R)}$ denote the set of boundary segments of $R$ that do not intersect the cutting line. For a rectangle $R \in \mathcal{R}$, let $N(R)$ denote the set of rectangles which overlap with $R$. Let

$$N'(R) = \{X \in N(R) \colon \; \exists \, a \in B(X), \exists \, b \in \overline{B(R)}, \; a \cap b \neq \emptyset\}$$

See Figure 5(a) for an example. Now define $N''(R) = N(R) \setminus N'(R)$. We have the following observation.

**Observation F.** *For a rectangle $R \in \mathcal{R}$ and a rectangle $X \in N''(R)$, there is a segment of $B(R)$ that intersects some boundary segment of $X$.*

*Proof.* Suppose $X$ has a segment in $B(X)$ that intersects some boundary segment $s$ of $R$. In this case, $s$ must be in $B(R)$, and we are done. Suppose, there is a segment $s \in \overline{B(X)}$ such that no boundary segment of $R$ intersects $s$. Since $R$ intersects at two different boundary segments of $X$, in this case, there exists one segment $s' \in B(X)$ that intersect some boundary segment of $R$. Then again, $s' \in B(R)$ and we are done.

Otherwise, observe that $X$ contains two boundary segments $s_1, s_2 \in \overline{B(X)}$ such that $R$ intersects both of them. If $s_1$ and $s_2$ belong to opposite sides of the cutting line, then both $s_1$ and $s_2$ are horizontal or both of them are vertical. In either case, $R$ must have a boundary segment $t \in B(R)$ that intersect both $s_1, s_2$. Consider the case when both $s_1$ and $s_2$ lie below the cutting line. Then without loss of generality, we can assume that $s_1$ is a vertical segment and $s_2$ is a horizontal segment. Hence, $R$ must have a horizontal boundary segment $w$ that intersects $s_1$ and a vertical boundary segment $z$ that intersects $s_2$. If neither $w$ nor $z$ intersects the cutting line, then observe that the top-right corner of $R$ must lie below the cutting line, implying that $R$ does not intersect the cutting line. This is a contradiction. Similarly, the case when both $s_1, s_2$ lie above the cutting line also leads to a contradiction. □

We shall denote the left segment of a rectangle $R \in \mathcal{R}$ also as the *segment*-0 of $R$. Similarly *segment*-1, *segment*-2 and *segment*-3 of $R_u$ shall refer to the top segment, the right segment and the bottom segment of $R$, respectively. See Figure 5(b) for an illustration. Let $\mathcal{S} = \{(0,1), (0,3), (1,0), (1,2), (2,1), (2,3), (3,0), (3,2)\}$. Since no two horizontal segments or two vertical segments intersect, we have the following observation.

**Observation G.** *If two rectangles $R, R' \in \mathcal{R}$ overlap there must be a pair $(i,j) \in \mathcal{S}$ such that segment-i of $R$ intersects segment-j of $R'$.*

Based on the above observation, we partition the sets $N'(R)$ and $N''(R)$ in the following way. For each rectangle $R \in \mathcal{R}$ and $(i,j) \in \mathcal{S}$, a rectangle $X \in N'(R)$ belongs to the set $Z'_R(i,j)$ if and only if $(i,j)$ is the smallest pair in the lexicographic order such that *(a)* segment-$i$ of $R$ intersects the segment-$j$ of $X$ and *(b)* segment-$j$ of $X$ intersects the cutting line.

Similarly, for each rectangle $R \in \mathcal{R}$ and $(i,j) \in \mathcal{S}$, a rectangle $X \in N''(R)$ belongs to the set $Z''_R(i,j)$ if and only if $(i,j)$ is the smallest pair in the lexicographic order such that *(a)* segment-$i$ of $R$ intersects the segment-$j$ of $X$ and *(b)* segment-$i$ of $R$ intersects the cutting line. The next observation follows from the above definitions.

**Observation H.** *For each $R \in \mathcal{R}$, $\{Z'_R(i,j)\}_{(i,j)\in\mathcal{S}}$ is a partition of $N'(R)$ and $\{Z''_R(i,j)\}_{(i,j)\in\mathcal{S}}$ is a partition of $N''(R)$.*

For each $R \in \mathcal{R}$, define the sets $\mathcal{S}'_R = \{(i,j) \in \mathcal{S} \colon Z'_R(i,j) \neq \emptyset\}$ and $\mathcal{S}''_R = \{(i,j) \in \mathcal{S} \colon Z''_R(i,j) \neq \emptyset\}$. Recall that according to our assumption, each rectangle intersect the cutting line exactly two times. Since the boundary segment of a retangle intersect exactly two boundary segments of another rectangle, we have the following observation.

**Observation I.** *For each $R \in \mathcal{R}$, $|\mathcal{S}'_R| \leq 4$ and $|\mathcal{S}''_R| \leq 4$.*

*Proof.* Observe that if there is a rectangle $X \in Z'_R(i,j)$ for some $(i,j) \in \mathcal{S}'_R$ then $X$ intersects a boundary segment of $\overline{B(R)}$. There are exactly two segments in $\overline{B(R)}$. Let $\overline{B(R)}$ contains segment-$i$ and segment-$j$ of $R$. Hence $\mathcal{S}'_R$ is a subset of $\{(i,i-1),(i,i+1),(j,j-1),(j,j+1)\}$ where all addition operations are modulo 4. Therefore $|\mathcal{S}'_R| \leq 4$. To prove the second part, we use Observation F to infer that if a rectangle $X \in Z''_R(i,j)$ for some $(i,j) \in \mathcal{S}''_R$ then $X$ intersects a boundary segment of $B(R)$. Now using similar arguments as above we have that $|\mathcal{S}''_R| \leq 4$. □

Let $Q$ denote the following ILP formulation of the MDS problem on $G$ and $Q_l$ be the corresponding relaxed LP formulation.

$$
\begin{array}{lll}
\text{minimize} & \sum_{R\in\mathcal{R}} x_R & \\
\text{subject to} & \sum_{(i,j)\in\mathcal{S}'_R} \sum_{R'\in Z'_R(i,j)} x_{R'} + \sum_{(i,j)\in\mathcal{S}''_R} \sum_{R''\in Z''_R(i,j)} x_{R''} \geq 1, \ \forall R \in \mathcal{R} & \\
& x_R \in \{0,1\}, & \forall R \in \mathcal{R}
\end{array}
$$

$Q$

Let $\mathbf{Q}_l = \{x_R \colon R \in \mathcal{R}\}$ be an optimal solution of $Q_l$. By Observation I, for each rectangle $R \in \mathcal{R}$, we have $|\mathcal{S}'_R| + |\mathcal{S}''_R| \leq 8$. Hence, there is a pair $(i,j) \in \mathcal{S}'_R \cup \mathcal{S}''_R$ such that either $\sum_{R'\in Z'_R(i,j)} x_{R'} \geq \frac{1}{8}$ or $\sum_{R'\in Z''_R(i,j)} x_{R'} \geq \frac{1}{8}$. For each pair $(i,j) \in \mathcal{S}$, define

$$
A'(i,j) = \left\{ R \in \mathcal{R} \colon (i,j) \in \mathcal{S}'_R, \sum_{R'\in Z'_R(i,j)} x_{R'} \geq \frac{1}{8} \right\}
$$

$$B'(i,j) = \bigcup_{R \in A'(i,j)} Z'_R(i,j)$$

$$A''(i,j) = \left\{ R \in \mathcal{R} \colon (i,j) \in \mathcal{S}''_R, \sum_{R' \in Z''_R(i,j)} x_{R'} \geq \frac{1}{8} \right\}$$

$$B''(i,j) = \bigcup_{R \in A''(i,j)} Z''_R(i,j)$$

Based on these, we have the following two ILP formulations for each pair $(i,j) \in \mathcal{S}$.

| | |
|---|---|
| minimize $\sum_{R' \in B'(i,j)} x'_{R'}$<br>subject to $\sum_{R' \in Z'_R(i,j)} x'_{R'} \geq 1, \forall R \in A'(i,j)$<br>$x'_{R'} \in \{0,1\}, \quad R' \in B'(i,j)$<br>$Q'(i,j)$ | minimize $\sum_{R'' \in B''(i,j)} x''_{R''}$<br>subject to $\sum_{R'' \in Z''_R(i,j)} x''_{R''} \geq 1, \forall R \in A''(i,j)$<br>$x''_{R''} \in \{0,1\}, \quad R'' \in B''(i,j)$<br>$Q''(i,j)$ |

For each pair $(i,j) \in \mathcal{S}$, let $Q'_l(i,j)$ and $Q''_l(i,j)$ be the relaxed LP formulation of $Q'(i,j)$ and $Q''(i,j)$, respectively. Observe that

$$OPT(Q) \leq \sum_{(i,j) \in \mathcal{S}} (OPT(Q'(i,j)) + OPT(Q''(i,j)))$$

For each $x_R \in \mathbf{Q}_l$, define $y_R = \min\{1, 8x_R\}$ and $\mathbf{Y}_l = \{y_R\}_{x_R \in \mathbf{Q}_l}$. Due to Observation H and I, $\mathbf{Y}_l$ gives a feasible solution to $Q'_l(i,j)$ and $Q''_l(i,j)$ for all $(i,j) \in \mathcal{S}$. Therefore, $OPT(Q'_l(i,j)) \leq 8 \cdot OPT(Q_l)$ and $OPT(Q''_l(i,j)) \leq 8 \cdot OPT(Q_l)$ for all $(i,j) \in \mathcal{S}$. Now we have the following lemma.

**Lemma 14.** *For each $(i,j) \in \mathcal{S}$ there is a set $D'(i,j) \subseteq B'(i,j)$ such that $D'(i,j)$ gives a feasible solution of $Q'(i,j)$ and $|D'(i,j)| \leq 4 \cdot OPT(Q'_l(i,j))$.*

*Proof.* For any $(i,j) \in \mathcal{S}$, solving $Q'(i,j)$ is equivalent to finding a minimum cardinality subset $D$ of $B'(i,j)$ such that each rectangle $R \in A'(i,j)$ overlaps a rectangle in $D \cap Z'_R(i,j)$. For each $R \in A'(i,j)$ the set $Z'_R(i,j)$ is non-empty. Moreover for each $R' \in Z'_R(i,j)$, the segment-$j$ of $R'$ intersects the cutting line and segment-$i$ of $R$. Moreover, the segment-$i$ of $R$ does not intersect the cutting line. Let $S = \{\text{segment-}i \text{ of } R \colon R \in A'(i,j)\}$, $T = \{\text{segment-}j \text{ of } R' \colon R' \in B'(i,j)\}$.

Solving $Q'(i,j)$ is equivalent to the problem finding a minimum cardinality subset $D$ of $T$ such that every segment in $S$ intersect at least one segment in $D$. Every segment in $T$ intersects the cutting line and no segment in $S$ intersects the cutting line. Without loss of generality we can assume that $S$ consists of vertical segments. Therefore $T$ consists of horizontal segments. Hence solving $Q'(i,j)$ is equivalent to solving the $\mathcal{LVSC}(S,T)$ instance. Hence by Lemma 12, we have a feasible solution (say $D'(i,j)$) for $Q'(i,j)$ such that $|D'(i,j)| \leq 4 \cdot OPT(Q'_l(i,j))$. □

**Lemma 15.** *For each $(i,j) \in \mathcal{S}$ there is a set $D''(i,j) \subseteq B''(i,j)$ such that $D''(i,j)$ gives a feasible solution of $Q''(i,j)$ and $|D''(i,j)| \leq 8 \cdot OPT(Q''_l(i,j))$.*

*Proof.* For any $(i,j) \in \mathcal{S}$, solving $Q''(i,j)$ is equivalent to finding a minimum cardinality subset $D$ of $B''(i,j)$ such that each rectangle $R \in A''(i,j)$ overlaps a rectangle in $D \cap X''_R(i,j)$. Notice that, for each $R \in A''(i,j)$ the set $X''_R(i,j)$ is non-empty. Moreover for each $R'' \in X''_R(i,j)$, the segment-$i$ of $R$ intersects the cutting line and segment-$j$ of $R''$. Let $S = \{\text{segment-}i \text{ of } R\colon R \in A''(i,j)\}$, $T = \{\text{segment-}j \text{ of } R''\colon R'' \in B''(i,j)\}$.

Solving $Q''(i,j)$ is equivalent to the problem finding a minimum cardinality subset $D$ of $T$ such that every segment in $S$ intersect at least one segment in $D$. Moreover, every segment in $S$ intersects the cutting line. Without loss of generality we can assume that $S$ consists of horizontal segments. Therefore $T$ consists of vertical segments. Hence solving $Q(i,j)$ is equivalent to solving the $\mathcal{LHSC}(S,T)$ instance. Hence by Lemma 13, we have a feasible solution (say $D''(i,j)$) for $Q''(i,j)$ such that $|D''(i,j)| \leq 8 \cdot OPT(Q''_l(i,j))$. □

For each $R \in \mathcal{R}$, let $\mathcal{T}'_R = \{(i,j) \in \mathcal{S}\colon R \in B'(i,j)$ and $\mathcal{T}''_R = \{(i,j) \in \mathcal{S}\colon R \in B''(i,j)\}$. The following observation follows from the definitions of $B'(i,j)$ and $B''(i,j)$.

**Observation J.** *For each $R \in \mathcal{R}$, we have that $|\mathcal{T}'_R| \leq 4$ and $|\mathcal{T}''_R| \leq 8$.*

*Proof.* Let $i,j$ be two integers such that $(i,j) \in \mathcal{T}'_R$. Then segment-$j$ must be in $B(R)$. As $B(R)$ contains exactly two segments of $R$, we have that $|\mathcal{T}'_R| \leq 4$. The second part follows from the fact that $|\mathcal{S}| \leq 8$. □

For each pair $(i,j) \in \mathcal{S}$, due to Lemma 14 and Lemma 15, we have a feasible solution $D'(i,j)$ of $Q'(i,j)$ and a feasible solution $D''(i,j)$ such that $|D'(i,j)| \leq 4 \cdot OPT(Q'_l(i,j))$ and $|D''(i,j)| \leq 8 \cdot OPT(Q''_l(i,j))$. Let $D$ be the union of $D'(i,j)$'s and $D''(i,j)$ for all $(i,j) \in \mathcal{S}$. We have that

$$\begin{aligned}
|D| &= \sum_{(i,j)\in\mathcal{S}} |D'(i,j)| + \sum_{(i,j)\in\mathcal{S}} |D''(i,j)| \\
&\leq 4 \cdot \sum_{(i,j)\in\mathcal{S}} OPT(Q'_l(i,j)) + 8 \cdot \sum_{(i,j)\in\mathcal{S}} OPT(Q''_l(i,j)) \\
&\leq 144 \cdot OPT(Q_l) + 512 \cdot OPT(Q_l) && \text{[Due to Observation J, Lemma 14 and 15]} \\
&= 656 \cdot OPT(Q_l) \leq 656 \cdot OPT(Q)
\end{aligned}$$

This completes the proof of Theorem 6.

## 9. Proof of Lemma 3

In this section, we prove Lemma 3. Recall that, $H = (V', E')$ is a $(n,n)$-grid for some even integer $n$. The set $X$ is $\{(i,j) \in V'\colon i,j \text{ have same parity}\}$ and $Y = V' \setminus X$. First, we note the following.

**Observation K.** *For any edge $e \in E'$, one of the endpoints of $e$ belongs to $X$ and the other endpoint belongs to $Y$.*

Let $\epsilon = \frac{1}{n^2}$. For each $(i,j) \in Y$, we define two real values $x_{i,j}$ and $y_{i,j}$ as follows.

$$x_{i,j} = \begin{cases} \lceil \frac{j}{2} \rceil & \text{when } i = 1 \\ \lceil \frac{j}{2} - \epsilon \rceil & \text{when } i = 2 \\ x_{i-1,j+1} + \frac{x_{i-2,j} - x_{i-1,j+1}}{2} & \text{when } i \geq 3, i \equiv 0 \mod 2 \\ x_{i-1,j-1} + \frac{x_{i-2,j} - x_{i-1,j-1}}{2} & \text{when } i \geq 3, i \equiv 1 \mod 2 \end{cases}$$

$$y_{i,j} = \frac{i}{2} + \left\lceil \frac{j}{2} \right\rceil \epsilon$$

Notice that for $i \geq 3$, if $(i,j) \in Y$, then $(i-2,j) \in Y$. Moreover, if $i$ is even then $(i-1,j+1) \in Y$ and if $i$ is odd then $(i-1,j-1) \in Y$. Therefore, the values $x_{i,j}$ for all $(i,j) \in Y$ are well-defined. We have the following observation.

**Observation L.** *Let for some pair $(i,j)$ we have $\{(i,j-1),(i,j+1),(i+1,j),(i-1,j)\} \subseteq Y$. Then*

(i) $x_{i,j-1} + 1 = x_{i,j+1}$ *and* $y_{i,j-1} = y_{i,j+1} - \epsilon$*;*
(ii) $x_{i+1,j} < x_{i,j+1} < (x_{i+1,j}) + 1$ *and* $x_{i-1,j} < x_{i,j+1} < x_{i-1,j} + 1$*;*
(iii) *when* $i \equiv 1 \mod 2$*,* $y_{i-1,j} = y_{i,j+1} - 0.5$ *and* $y_{i+1,j} = y_{i,j+1} + 0.5$*; and*
(iv) *when* $i \equiv 0 \mod 2$*,* $y_{i-1,j} = y_{i,j-1} - 0.5$ *and* $y_{i+1,j} = y_{i,j-1} + 0.5$

Now for each $(i,j) \in Y$, we define a horizontal line segment $s_{i,j}$ as follows.

$$s_{i,j} = [x_{i,j}, x_{i,j} + 1] \times [y_{i,j}, y_{i,j}]$$

Let $S = \{s_{i,j}\}_{(i,j) \in Y}$. Observe that no two segment in $S$ intersect each other and length of every segment in $S$ is one. Now for each $(i,j) \in X$, we define the real values $x'_{i,j}$ and $y'_{i,j}$ as follows.

$$x'_{i,j} = \begin{cases} x_{i,j+1} & \text{when } i \equiv 1 \mod 2 \\ x_{i,j-1} + 1 & \text{when } i \equiv 0 \mod 2 \end{cases} \qquad y'_{i,j} = \begin{cases} y_{i,j+1} - 0.5 & \text{when } i \equiv 1 \mod 2 \\ y_{i,j-1} - 0.5 & \text{when } i \equiv 0 \mod 2 \end{cases}$$

Notice that, for each $(i,j) \in X$ if $i$ is odd then $(i,j+1) \in Y$ and if $i$ is even then $(i,j-1) \in Y$. Therefore, the values $x'_{i,j}$ are well defined. Now for each $(i,j) \in X$, we define a vertical segment $t_{i,j}$ as follows.

$$t_{i,j} = [x'_{i,j}, x'_{i,j}] \times [y'_{i,j}, y'_{i,j} + 1]$$

Let $T = \{t_{i,j}\}_{(i,j) \in X}$. Observe that no two segment in $T$ intersect each other and length of every segment in $T$ is one. Moreover we have the following observation about $T$.

**Observation M.** *For a pair $(i,j) \in X$, let $S_{i,j}$ be the set of segments in $S$ that intersect*

$t_{i,j}$. *Then*

$$S_{i,j} = \begin{cases} \{s_{i+1,j}, s_{i,j+1}\} & \text{when } i=1, j=1 \\ \{s_{i+1,j}, s_{i,j+1}, s_{i,j-1}\} & \text{when } i=1, 2 \leq j \leq n-1 \\ \{s_{i-1,j}, s_{i,j+1}, s_{i,j-1}\} & \text{when } i=n, 2 \leq j \leq n-1 \\ \{s_{i+1,j}, s_{i-1,j}, s_{i,j+1}\} & \text{when } 2 \leq i \leq n, j=1 \\ \{s_{i+1,j}, s_{i-1,j}, s_{i,j-1}\} & \text{when } 2 \leq i \leq n, j=n \\ \{s_{i+1,j}, s_{i-1,j}, s_{i,j+1}, s_{i,j-1}\} & \text{when } 2 \leq i \leq n-1, 2 \leq j \leq n-1 \end{cases}$$

*Proof.* We shall prove the observation only for the case when $2 \leq i \leq n-1, 2 \leq j \leq n-1$ and $i$ is odd. For the remaining cases similar arguments will suffice. Notice that when $(i,j) \in X$, we have $\{(i+1,j), (i-1,j), (i,j+1), (i,j-1)\} \subsetneq Y$ and therefore $s_{i+1,j}, s_{i-1,j}, s_{i,j+1}, s_{i,j-1}$ exists.

Since $i$ is odd, the bottom and top endpoints of $t_{i,j}$ are $(x_{i,j+1}, y_{i,j+1} - 0.5)$ and $(x_{i,j+1}, y_{i,j+1}+0.5)$, respectively. Recall that the left endpoint of $s_{i,j+1}$ is $(x_{i,j+1}, y_{i,j+1})$ and using Observation L(i) we can infer that the right endpoint of $s_{i,j-1}$ is $(x_{i,j+1}, y_{i,j+1} - \epsilon)$. These facts imply that the segment $t_{i,j} \cap s_{i,j-1}$ is the right endpoint of $s_{i,j-1}$ and $t_{i,j} \cap s_{i,j-1}$ is the left endpoint of $s_{i,j+1}$. Due to Observation L(ii) and L(iii), the bottom endpoint of $t_{i,j}$ lies between the left and right endpoints of $s_{i-1,j}$ and has the same $y$-coordinate as that of $s_{i-1,j}$. Hence $t_{i,j} \cap s_{i-1,j} = \{(x_{i,j+1}, y_{i,j+1} - 0.5)\} = \{(x'_{i,j}, y'_{i,j})\}$. Similarly, we can show that $s_{i+1,j} = \{(x_{i,j+1}, y_{i,j+1} + 0.5)\} = \{(x'_{i,j}, y'_{i,j} + 1)\}$. This completes the proof. □

Using Observation K and Observation M we can infer that $S \cup T$ is a valid unit $B_0$-VPG representation of $H$.

## 10. Conclusion

In this paper, we studied the *SSR* problem and the *SRS*. Improvements on the lemmas regarding the SSR and the SRS problems will immediately imply better approximation ratios for several optimisation problems, including a few studied by Bandyapadhyay and Meharbi [31]. Therefore the following question might be interesting.

**Question 1.** *What are the integrality gaps of the* SSR *and the* SRS *problems?*

We gave the first constant-factor approximation algorithm for the MDS problem on unit $B_0$-VPG graphs. However, we believe that obtained approximation ratio of 18 to be far from being tight. This motivates the following question.

**Question 2.** *Is there a c-approximation algorithm for the* MDS *problem on unit $B_0$-VPG graphs with $c < 18$?*

Using our results on the SSR and the SRS problems, we gave an $O(k^4)$-approximation algorithm for the MDS problem on unit $B_k$-VPG graphs. On the other hand, it is unlikely that the MDS problem admit a $o(\log k)$-approximation algorithm on $B_k$-VPG graphs. The reason is as follows. It is known that unless $P = NP$, the MDS problem does not admit a $o(\log n)$-approximation algorithm on split graphs (graphs whose vertex set can be partitioned into a clique and an independent set) of order $n$ [14]. On the other hand, given any split graph

$G$ of order $n$, it is not hard to construct a $B_{O(n)}$-VPG representation of $G$ in polynomial time as follows.

Let $G$ be a split graph whose vertex set can be partitioned into a clique $C$ and an independent set $I$. Let the vertices of $I$ are $v_1, v_2, \ldots, v_t$. For each $i \in \{1, 2, \ldots, t\}$, let $s_i$ denote the vertical segment whose bottom point is at $(i, i - 0.1)$ and the top point is at $(i, i + 0.1)$. For each vertex $u \in C$, let $N_I[u]$ denote the set of vertices of $I$ adjacent to $u$. For each vertex $u \in C$, we shall define a path $P_u$ with $O(N_I[u])$ bends. For a vertex $u \in C$, let $N_I[u] = v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ with $i_1 < i_2 < \ldots < i_k$. Then let $P_u$ be the rectilinear path joining the points $(0, 0), (0, i_1), (i_1, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k), (i_k, i_k)$ in the same order. Observe that $P_u$ has at most $2k + 1$ bends and $P_u$ intersect a vertical segment $s_i$ if and only if $i \in \{i_1, i_2, \ldots, i_k\}$.

The above construction imply that unless $P = NP$, the MDS problem does not admit a $o(\log n)$-approximation algorithm on $B_k$-VPG graphs. This motivates the following question.

**Question 3.** *Is there a constant-factor approximation algorithm for the* MDS *problem on $B_0$-VPG graphs? Is there an $O(\log k)$-approximation algorithm for the* MDS *problem on $B_k$-VPG graphs?*

In this paper, we introduce the class of stabbed rectangle overlap graphs and study the MDS problem. Using our results on the SSR and the SRS problems, we gave a 656-approximation algorithm for the MDS problem on stabbed rectangle overlap graphs. As a corollary to Theorem 6, we have the following.

**Corollary 1.** *Let $\mathcal{R}$ be a stabbed rectangle intersection representation of a graph $G = (V, E)$ such that no two rectangles in $\mathcal{R}$ contain each other. There is an $O(|V|^5)$-time* 656-*approximation algorithm for the* MDS *problem on $G$.*

Since the approximation ratio of 656 seems to be far from being tight, the following question is interesting.

**Question 4.** *Is there a $c$-approximation algorithm for the* MDS *problem on stabbed rectangle overlap graphs with $c < 656$?*

## References


[1] D. Chakraborty, S. Das, J. Mukherjee, Approximating minimum dominating set on string graphs, in: WG, Springer, 2019, pp. 232–243.

[2] D. Chakraborty, S. Das, J. Mukherjee, Dominating set on overlap graphs of rectangles intersecting a line, in: COCOON, Springer, 2019, pp. 65–77.

[3] S. Benzer, On the topology of the genetic fine structure, Proceedings of the national Academy of Sciences 45 (11) (1959) 1607–1620.

[4] F. W. Sinden, Topology of thin film rc circuits, Bell System Technical Journal 45 (9) (1966) 1639–1662.

[5] R. Graham, Problem, in: Combinatorics, Vol. II, 1978, p. 1195.

[6] G. Ehrlich, S. Even, R. E. Tarjan, Intersection graphs of curves in the plane, Journal of Combinatorial Theory, Series B 21 (1) (1976) 8–20.

[7] J. Kratochvíl, String graphs. ii. recognizing string graphs is np-hard, Journal of Combinatorial Theory, Series B 52 (1) (1991) 67–78.

[8] J. Kratochvíl, Intersection graphs of noncrossing arc-connected sets in the plane, in: International Symposium on Graph Drawing, Springer, 1996, pp. 257–270.

[9] E. Bonnet, P. Rzazewski, Optimality program in segment and string graphs, in: WG, 2018, pp. 79–90.

[10] J. Fox, J. Pach, Coloring $K_k$-free intersection graphs of geometric objects in the plane, European Journal of Combinatorics 33 (5) (2012) 853–866.

[11] A. Rok, B. Walczak, Coloring curves that cross a fixed curve, Discrete & Computational Geometry 61 (4) (2019) 830–851.

[12] J. Keil, J. Mitchell, D. Pradhan, M. Vatshelle, An algorithm for the maximum weight independent set problem on outerstring graphs, Computational Geometry 60 (2017) 19–25.

[13] R. Bar-Yehuda, D. Hermelin, D. Rawitz, Minimum vertex cover in rectangle graphs, Computational Geometry 44 (6-7) (2011) 356–364.

[14] M. Chlebík, J. Chlebíková, Approximation hardness of dominating set problems in bounded degree graphs, Information and Computation 206 (11) (2008) 1264–1275.

[15] B. S. Baker, Approximation algorithms for np-complete problems on planar graphs, Journal of the ACM (JACM) 41 (1) (1994) 153–180.

[16] M. V. Marathe, H. Breu, H. B. Hunt III, S. S. Ravi, D. J. Rosenkrantz, Simple heuristics for unit disk graphs, Networks 25 (2) (1995) 59–68.

[17] T. Nieberg, J. Hurink, A PTAS for the minimum dominating set problem in unit disk graphs, in: WAOA, 2005, pp. 296–306.

[18] P. Carmi, G. K. Das, R. K. Jallu, S. C. Nandy, P. R. Prasad, Y. Stein, Minimum dominating set problem for unit disks revisited, International Journal of Computational Geometry & Applications 25 (03) (2015) 227–244.

[19] M. Gibson, I. Pirwani, Algorithms for dominating set in disk graphs: breaking the logn barrier, in: ESA, 2010, pp. 243–254.

[20] S. Govindarajan, R. Raman, S. Ray, A. Roy, Packing and covering with non-piercing regions, in: ESA, 2016.

[21] M. de Berg, S. Kisfaludi-Bak, G. Woeginger, The complexity of dominating set in geometric intersection graphs, Theoretical Computer Science 769 (2019) 18–31.

[22] T. Erlebach, E. Van Leeuwen, Domination in geometric intersection graphs, in: LATIN, Springer, 2008, pp. 747–758.

[23] A. Asinowski, E. Cohen, M. Golumbic, V. Limouzy, M. Lipshteyn, M. Stern, Vertex intersection graphs of paths on a grid., Journal of Graph Algorithms and Applications 16 (2) (2012) 129–150.

[24] M. J. Katz, J. Mitchell, Y. Nir, Orthogonal segment stabbing, Computational Geometry: Theory and Applications 30 (2) (2005) 197–205.

[25] S. McGuinness, On bounding the chromatic number of L-graphs, Discrete Mathematics 154 (1-3) (1996) 179–187.

[26] N. Bousquet, D. Gonçalves, G. Mertzios, C. Paul, I. Sau, S. Thomassé, Parameterized domination in circle graphs, in: WG, 2012, pp. 308–319.

[27] C. Colbourn, L. Stewart, Permutation graphs: connected domination and steiner trees, Discrete Mathematics 86 (1-3) (1990) 179–189.

[28] M. Damian, S. Pemmaraju, APX-hardness of domination problems in circle graphs, Information processing letters 97 (6) (2006) 231–237.

[29] M. Damian-Iordache, S. Pemmaraju, A (2+ $\varepsilon$)-approximation scheme for minimum domination on circle graphs, Journal of Algorithms 42 (2) (2002) 255–276.

[30] M. Farber, J. Keil, Domination in permutation graphs, Journal of algorithms 6 (3) (1985) 309–321.

[31] S. Bandyapadhyay, A. Maheshwari, S. Mehrabi, S. Suri, Approximating dominating set on intersection graphs of rectangles and l-frames, Computational Geometry 82 (2019) 32–44.

[32] S. Mehrabi, Approximating domination on intersection graphs of paths on a grid, in: WAOA, Springer, 2017, pp. 76–89.

[33] S. Khot, O. Regev, Vertex cover might be hard to approximate to within 2- $\varepsilon$, Journal of Computer and System Sciences 74 (3) (2008) 335–349.

[34] S. Pandit, Dominating set of rectangles intersecting a straight line., in: CCCG, 2017, pp. 144–149.

[35] J. Correa, L. Feuilloley, P. Pérez-Lantero, J. Soto, Independent and hitting sets of rectangles intersecting a diagonal line: algorithms and complexity, Discrete & Computational Geometry 53 (2) (2015) 344–365.

[36] D. Catanzaro, S. Chaplick, S. Felsner, B. Halldórsson, M. Halldórsson, T. Hixon, J. Stacho, Max point-tolerance graphs, Discrete Applied Mathematics 216 (2017) 84–97.

[37] V. Chepoi, S. Felsner, Approximating hitting sets of axis-parallel rectangles intersecting a monotone curve, Computational Geometry 46 (9) (2013) 1036–1041.

[38] A. Mudgal, S. Pandit, Covering, hitting, piercing and packing rectangles intersecting an inclined line, in: Combinatorial Optimization and Applications, Springer, 2015, pp. 126–137.

[39] B. Clark, C. Colbourn, D. Johnson, Unit disk graphs, Discrete mathematics 86 (1-3) (1990) 165–177.

[40] M. d. Berg, O. Cheong, M. v. Kreveld, M. Overmars, Computational geometry: algorithms and applications, 2008.

[41] E. Tardos, A strongly polynomial algorithm to solve combinatorial linear programs, Operations Research 34 (2) (1986) 250–256.

[42] D. R. Gaur, T. Ibaraki, R. Krishnamurti, Constant ratio approximation algorithms for the rectangle stabbing problem and the rectilinear partitioning problem, Journal of Algorithms 43 (1) (2002) 138–152.

[43] A. Butman, D. Hermelin, M. Lewenstein, D. Rawitz, Optimization problems in multiple-interval graphs, ACM Transactions on Algorithms (TALG) 6 (2) (2010) 40.

[44] A. Acharyya, S. C. Nandy, S. Pandit, S. Roy, Covering segments with unit squares, Computational Geometry 79 (2019) 1–13.

[45] S. Felsner, K. Knauer, G. B. Mertzios, T. Ueckerdt, Intersection graphs of l-shapes and segments in the plane, Discrete Applied Mathematics 206 (2016) 48–55.

[46] A. Schrijver, Theory of linear and integer programming, John Wiley & Sons, 1998.

[47] J. Kleinberg, E. Tardos, Algorithm design, Pearson Education India, 2006.

[48] S. Bandyapadhyay, S. Mehrabi, Constrained orthogonal segment stabbing, CCCG (2019).

[49] S. Bandyapadhyay, A. Roy, Effectiveness of local search for art gallery problems, in: WADS, 2017, pp. 49–60.