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Finding Geometric Representations of Apex Graphs is NP-Hard[1]

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Abstract

Planar graphs can be represented as the intersection graphs of different types of geometric objects in the plane, *e.g.*, circles (Koebe, 1936), line segments (Chalopin & Gonçalves, SODA 2009), L-shapes (Gonçalves *et al.*, SODA 2018). For general graphs, however, even deciding whether such representations exist is often NP-hard. We consider apex graphs, *i.e.* , graphs that can be made planar by removing one vertex from them. We show, somewhat surprisingly, that deciding whether geometric representations exist for apex graphs is NP-hard as well.

More precisely, we show that for every fixed positive integer g and every graph class $\mathscr{G}$ such that PURE-2-DIR $\subseteq \mathscr{G} \subseteq$ 1-STRING, it is NP-hard to decide whether an input graph belongs to the graph class $\mathscr{G}$, even when the inputs are restricted to apex graphs of girth g. Here, PURE-2-DIR is the class of intersection graphs of axis-parallel line segments (where horizontal segments intersect only vertical segments), and 1-STRING is the class of intersection graphs of simple curves (where two intersecting curves cross each other exactly once) in the plane. This partially answers an open question raised by Kratochvíl & Pergel (COCOON, 2007).

Most known reductions for earlier proofs of NP-hardness for these problems are from variants of 3-SAT (mainly PLANAR 3-CONNECTED 3-SAT). We reduce from the PLANAR HAMILTONIAN PATH COMPLETION problem, which uses the more intuitive notion of planarity. As a result, our proof is much simpler and encapsulates several classes of geometric intersection graphs.

[1]A preliminary version of this paper appeared in the proceedings of WALCOM 2022 [1]

Keywords: Hamiltonian path, planar graph, apex graph, NP-hard, recognition problem, geometric intersection graph, VLSI design, 1-STRING, PURE-2-DIR

1. Introduction

The *recognition* of a graph class is the decision problem of determining whether a given simple, undirected, unweighted graph belongs to the graph class. Recognition of graph classes is a fundamental problem in graph theory with a wide range of applications. In particular, when the graph class relates to intersection patterns of geometric objects, the corresponding recognition problem finds usage in disparate areas like VLSI design [2, 3, 4], map labelling [5], wireless networks [6], and computational biology [7].

The study of graphs that arise out of intersection patterns of geometric objects began with the celebrated circle packing theorem in 1936 [8] (also see [9, 10]), which states that all *planar graphs* can be expressed as intersection graphs[2] of touching disks[3]. Since then, there has been a long line of research on finding representations of planar graphs using other types of geometric objects. In his PhD thesis, Scheinerman [11] conjectured that all planar graphs can be expressed as intersection graphs of line segments. Scheinerman's conjecture has motivated researchers to study representations of planar graphs using many different types of geometric objects, mostly of them culminating in elegant results.

Hartman *et al*. [12] proved that planar bipartite graphs are contained into 2-DIR *i.e*. they are intersection graphs of orthogonal segments on the plane. This result contrasts the fact that deciding whether a bipartite graph is in 2-DIR is NP-hard [13]. Chalopin & Gonçalves [14] proved Scheinerman's conjecture [11] by showing that all planar graphs are in SEGMENTS *i.e*. they are intersection graphs of line segments on the plane. This result contrasts the fact that recognising the class of SEGMENTS is $\exists R$-complete [15]. In fact, there are many more results that imply representing planar graphs as intersection graphs of "certain" geometric is "easy" whereas even deciding if a general graph admits the same representation is difficult [16, 17, 18, 19, 20, 21, 22, 23].

The contrasting phenomenon (as evident from the results mentioned above) motivated us to investigate the following question in this paper: *If an input graph is "almost planar", then is it possible to decide in polynomial time whether the input graph can be represented as the intersection graph of geometric objects on the plane?* A natural notion of " almost planarity" is the following.

[2]For a set of geometric objects $\mathscr{C}$, its intersection graph, $I(\mathscr{C})$, has $\mathscr{C}$ as the vertex set and two vertices are adjacent if and only if the corresponding geometric objects intersect.

[3]Formally, two closed disks are said to touch each other if they share exactly one point.

Definition 1 (Apex number)**.** *The apex number of a graph G is the minimum positive integer k for which G contains a set of k vertices whose removal makes it planar. A graph whose apex number is one is simply called an apex graph.*

Given this definition, we reformulate our earlier question as follows: *For which classes of geometric intersection graphs $\mathscr{G}$ does there exist a function f such that the recognition problem for $\mathscr{G}$ admits an $n^{f(a)}$-time algorithm (here, n and a denote the number of vertices and the apex number of the input graph, respectively)?* Before we proceed, we recall some definitions.

Definition 2 ([24])**.** *A* parameterized problem *is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a fixed, finite alphabet. For an instance $(x,k) \in \Sigma^* \times \mathbb{N}$, k is called the parameter.*

Definition 3 ([24])**.** *A parameterized problem $L \subseteq \Sigma^* \times \mathbb{N}$ is called* slice-wise polynomial (XP) *if there exists an algorithm A and two computable functions $f, g \colon \mathbb{N} \to \mathbb{N}$ such that, given $(x,k) \in \Sigma^* \times \mathbb{N}$, the algorithm A correctly decides whether $(x,k) \in L$ in time bounded by $f(k)|(x,k)|^{g(k)}$. The complexity class containing all slice-wise polynomial problems is called XP.*

Definition 4 ([24])**.** *A parameterized problem $L \subseteq \Sigma^* \times \mathbb{N}$ is called fixed parameter tractable (FPT) if there exists an algorithm A (called a fixed parameter algorithm), a computable function $f \colon \mathbb{N} \to \mathbb{N}$, and a constant c such that, given $(x,k) \in \Sigma^* \times \mathbb{N}$, the algorithm A correctly decides whether $(x,k) \in L$ in time bounded by $f(k) \cdot |(x,k)|^c$. The complexity class containing all fixed-parameter tractable problems is called FPT.*

Given the above definitions, we again reformulate our earlier question as follows: for which geometric intersection graphs $\mathscr{G}$ does the recognition of $\mathscr{G}$ admit an XP-time algorithm or even a FPT algorithm (with respect to the apex number)? As our main contribution, we show that recognizing various classes of geometric intersection graphs remains NP-hard even when the input graphs are both bipartite and apex (Theorem 1). This is surprising, given the fact that an apex graph is simply a planar graph with one additional vertex.

Definition 5. *A parameterised problem is* PARA-NP*-hard if it is already NP-hard for a fixed value of the parameter.*

Hence, our result implies that the recognition problem for various classes of geometric intersection graphs is PARA-NP-hard with respect to apex number.

Our proof technique deviates significantly from that of Kratochvíl [25] and other similar NP-hardness proofs that reduce from PLANAR 3-CONNECTED 3-SAT. We reduce

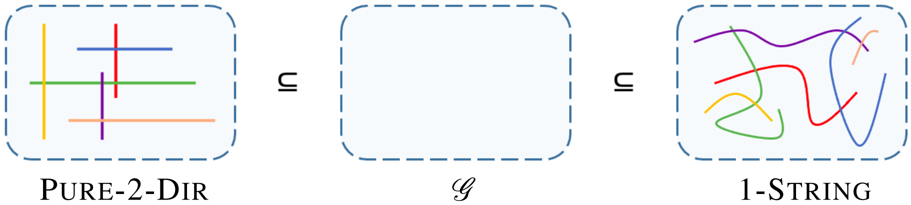

Figure 1: A visual depiction of Theorem 1 (this figure is for representational purposes only)

from a different NP-hard problem called PLANAR HAMILTONIAN PATH COMPLETION, which uses the more intuitive notion of planarity, making our proof easier to understand.

Organisation of the paper: In Section 2, we state our main result and its significance. We describe our proof techniques in Section 3, and prove our main result in Section 4. Finally, we conclude in Section 6.

2. Main Result

For our main result, we are particularly interested in two natural and well-studied classes of geometric intersection graphs called PURE-2-DIR and 1-STRING.

Definition 6. PURE-2-DIR *is the class of all graphs G, such that G is the intersection graph of axis-parallel line segments in the plane, where horizontal segments intersect only vertical segments (Figure 1 (Left)).*

Definition 7. 1-STRING *is the class of all graphs G, such that G is the intersection graph of simple curves in the plane, where two intersecting curves share exactly one point, at which they cross each other (Figure 1 (Right)).*

It is known that recognizing PURE-2-DIR and 1-STRING are both NP-hard [13, 25].

Theorem 1 (Main Result)**.** *Let g be a fixed positive integer and $\mathscr{G}$ be a graph class such that*

$$\text{PURE-2-DIR} \subseteq \mathscr{G} \subseteq \text{1-STRING}.$$

Then it is NP-hard to decide whether an input graph belongs to $\mathscr{G}$, even when the input graphs are restricted to bipartite apex graphs of girth at least g.

2.1. Significance of the Main Result

Our main result has several corollaries, obtained by substituting different values for the graph class $\mathscr{G}$. Recall that the recognition of a graph class $\mathscr{G}$ asks if a given graph G is a member of $\mathscr{G}$.

STRING is the class of intersection graphs of simple curves in the plane. Kratochvíl & Pergel [26] posed the question of determining the complexity of recognizing STRING when the inputs are restricted to graphs of large girth. The above question was answered by Mustaţă & Pergel [27], where they showed that recognizing STRING is NP-hard, even when the inputs are restricted to graphs of arbitrarily large girth. However, the graphs they constructed were far from planar. Since 1-STRING $\subsetneq$ STRING, the following corollary of our main result partially answers Kratochvíl & Pergel's [26] question.

Corollary 1. *For every positive integer g, recognizing* 1-STRING *is* NP*-hard, even for bipartite apex graphs with girth at least g.*

Chalopin & Gonçalves [14] showed that every planar graph can be represented as an intersection graph of line segments in polynomial time. The following corollary shows that a similar result does not hold for apex graphs.

Corollary 2. *For every positive integer g, recognizing intersection graphs of line segments is* NP*-hard, even for bipartite apex graphs with girth at least g.*

Gonçalves, Isenmann & Pennarun [17] showed that every planar graph can be represented as an intersection graph of L-shapes in polynomial time. The following corollary shows that a similar result does not hold for apex graphs.

Corollary 3. *For every positive integer g, recognizing intersection graphs of* L*-shapes is* NP*-hard, even for bipartite apex graphs with girth at least g.*

Our main result also has a connection to a graph invariant called *boxicity*. The boxicity of a graph is the minimum integer d such that the graph can be represented as an intersection graph of d-dimensional axis-parallel boxes. Thomassen showed three decades ago that the boxicity of every planar graph is either one, two or three [28]. It is easy to check if the boxicity of a planar graph is one [29]. However, the complexity of determining whether a planar graph has boxicity two or three is not yet known. A result of Hartman, Newman & Ziv [12] states that the class of bipartite graphs with boxicity 2 is precisely PURE-2-DIR. Combined with our main result, this implies that determining the boxicity of apex graphs is NP-hard.

Corollary 4. *For every positive integer g, recognizing graphs with boxicity* 2 *is* NP*-hard, even for bipartite apex graphs with girth at least g.*

CONV is the class of intersection graphs of convex objects in the plane. Kratochvíl & Pergel [26] asked if recognizing CONV remains NP-hard when the inputs are restricted to graphs of large girth. Note that the class of graphs with boxicity 2 (alternatively, intersection graphs of rectangles) is a subclass of CONV. Similarly, intersection graphs of line segments on the plane is also a subclass of CONV. Hence, Corollary 2 and Corollary 4 also partially address the aforementioned open question of Kratochvíl & Pergel [26].

Our main result implies that no graph class $\mathscr{G}$ satisfying PURE-2-DIR $\subseteq \mathscr{G} \subseteq$ 1-STRING can be recognized in $n^{f(c)}$ time, where c is the apex number and f is a computable function depending only on c. This means recognizing $\mathscr{G}$ is PARA-NP-hard and most likely, not fixed-parameter tractable, with respect to the apex number.

Corollary 5. *Let g be a positive integer and $\mathscr{G}$ be a graph class such that*

$$\text{PURE-2-DIR} \subseteq \mathscr{G} \subseteq \text{1-STRING}.$$

Then assuming $\mathsf{P} \neq \mathsf{NP}$, it is PARA-NP*-hard with respect to the apex number even when the inputs are bipartite graphs with girth at least g.*

Owing to a long line of work involving Robertson & Seymour [30], several graph classes can be characterized by a finite set of forbidden minors. For example, planar graphs are $\{K_5, K_{3,3}\}$-minor free graphs. Interestingly, the set of forbidden minors is not known for apex graphs, although it is known that the set is finite [31]. However, it is easy to see that apex graphs are K_6-minor free, which means that our main result has the following implication.

Corollary 6. *Let g be a positive integer and $\mathscr{G}$ be a graph class such that*

$$\text{PURE-2-DIR} \subseteq \mathscr{G} \subseteq \text{1-STRING}.$$

Then it is NP*-hard to decide whether an input graph belongs to $\mathscr{G}$, even for bipartite K_6-minor free graphs with girth at least g.*

Finally, using techniques different from ours, Kratochvíl & Matoušek [32] had shown that recognizing PURE-2-DIR is NP-hard, and so is the recognition of line segment intersection graphs. Theorem 1 and Corollary 2 show that these recognition problems remain NP-hard even when the inputs are restricted to bipartite apex graphs of arbitrarily large girth, thereby strengthening their results.

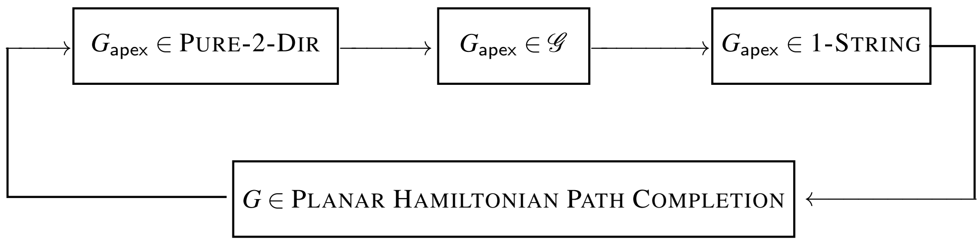

Figure 2: An illustration of our proof strategy of Theorem 3

3. Proof Techniques

We reduce from the NP-complete PLANAR HAMILTONIAN PATH COMPLETION problem [33], which in turn was inspired by another NP-complete problem known as the PLANAR HAMILTONIAN CYCLE COMPLETION problem [34]. Let us now describe the PLANAR HAMILTONIAN PATH COMPLETION problem [33]. A Hamiltonian path in a graph is a path that visits each vertex of the graph exactly once.

Definition 8. PLANAR HAMILTONIAN PATH COMPLETION *is the following decision problem.*
Input: *A planar graph G.*
Output: *Yes, if G is a subgraph of a planar graph with a Hamiltonian path; no, otherwise.*

Theorem 2 (Auer & Gleißner [33])**.** PLANAR HAMILTONIAN PATH COMPLETION *is* NP*-hard.*

We will use Theorem 2 to show Theorem 1. Similar to Mustaţă & Pergel [27], we show NP-hardness for graph classes "sandwiched" between two classes of geometric intersection graphs. A more technical formulation of Theorem 1 is as follows.

Theorem 3. *For every planar graph G and positive integer g, there exists a bipartite apex graph G_{apex} of girth at least g, which can be obtained in polynomial time from G, satisfying the following properties.*

(a) If G_{apex} is in 1-STRING*, then G is a yes-instance of* PLANAR HAMILTONIAN PATH COMPLETION*.*

(b) If G is a yes-instance of PLANAR HAMILTONIAN PATH COMPLETION*, then G_{apex} is in* PURE-2-DIR*.*

Proof of Theorem 1 from Theorem 3. See Figure 2 for an outline of our proof strategy. Let $\mathscr{G}$ be a graph class such that PURE-2-DIR $\subseteq \mathscr{G} \subseteq$ 1-STRING, and let G be a planar graph. If G is a yes-instance of PLANAR HAMILTONIAN PATH COMPLETION, then using Theorem 3 (b), we obtain that $G_{\mathsf{apex}} \in$ PURE-2-DIR $\subseteq \mathscr{G}$. And if $G_{\mathsf{apex}} \in \mathscr{G} \subseteq$ 1-STRING, then by Theorem 3 (a), G is a yes-instance of PLANAR HAMILTONIAN PATH COMPLETION.

Thus, $G_{\mathsf{apex}} \in \mathscr{G}$ if and only if G is a yes-instance of PLANAR HAMILTONIAN PATH COMPLETION. Since PLANAR HAMILTONIAN PATH COMPLETION is NP-hard (Theorem 2) and G_{apex} can be obtained in polynomial time from G, this implies that deciding whether the bipartite apex graph G_{apex} belongs to $\mathscr{G}$ is NP-hard. □

Therefore, as Theorem 3 implies our main result (Theorem 1), the rest of this paper is devoted to the proof of Theorem 3.

4. Proof of the Main Result

In this section, we will prove our main result (Theorem 3). First, given a planar graph G on n vertices, we will construct a bipartite apex graph G_{apex} in poly(n) time (Subsection 4.1). Then, we will show that $G_{\mathsf{apex}} \in$ 1-STRING $\Rightarrow$ $G \in$ PLANAR HAMILTONIAN PATH COMPLETION, proving Theorem 3 (a) (Subsection 4.2). Finally, we will show that $G \in$ PLANAR HAMILTONIAN PATH COMPLETION $\Rightarrow$ $G_{\mathsf{apex}} \in$ PURE-2-DIR, proving Theorem 3 (b) (Subsection 4.3).

4.1. Construction of the Apex Graph

We begin our proof of Theorem 3 by describing the construction of G_{apex}. Let G be a planar graph. G_{apex} is constructed in two steps.

$$G \to G_{k\text{-div}} \to G_{\mathsf{apex}}.$$

Let $g \geq 6$ be a positive integer constant, and $k \geq 3$ be the minimum odd integer greater or equal to $g-3$. Let $G_{k\text{-div}}$ be the full k-subdivision of G, *i.e.* , $G_{k\text{-div}}$ is the graph obtained by replacing each edge of G by a path with $k+1$ edges. Figure 3 (a) shows an example graph G, and Figure 3 (b) shows the full 3-subdivision of G. Formally, we replace each $e=(x,y) \in E(G)$ by the path $(x, u_e^1, u_e^2, u_e^3, \ldots, u_e^k, y)$.

$$V(G_{k\text{-div}}) = V(G) \cup \{u_e^1, u_e^2, u_e^3, \ldots, u_e^k \mid e \in E(G)\};$$
$$E(G_{k\text{-div}}) = \{xu_e^1, u_e^1u_e^2, u_e^2u_e^3, \ldots, u_e^{k-1}u_e^k, u_e^ky \mid e = xy \in E(G)\}.$$

We call the vertices of $V(G) \subseteq V(G_{k\text{-div}})$ as the *original vertices* of $G_{k\text{-div}}$ and the remaining vertices as the *subdivision vertices* of $G_{k\text{-div}}$. Finally, we construct G_{apex} by

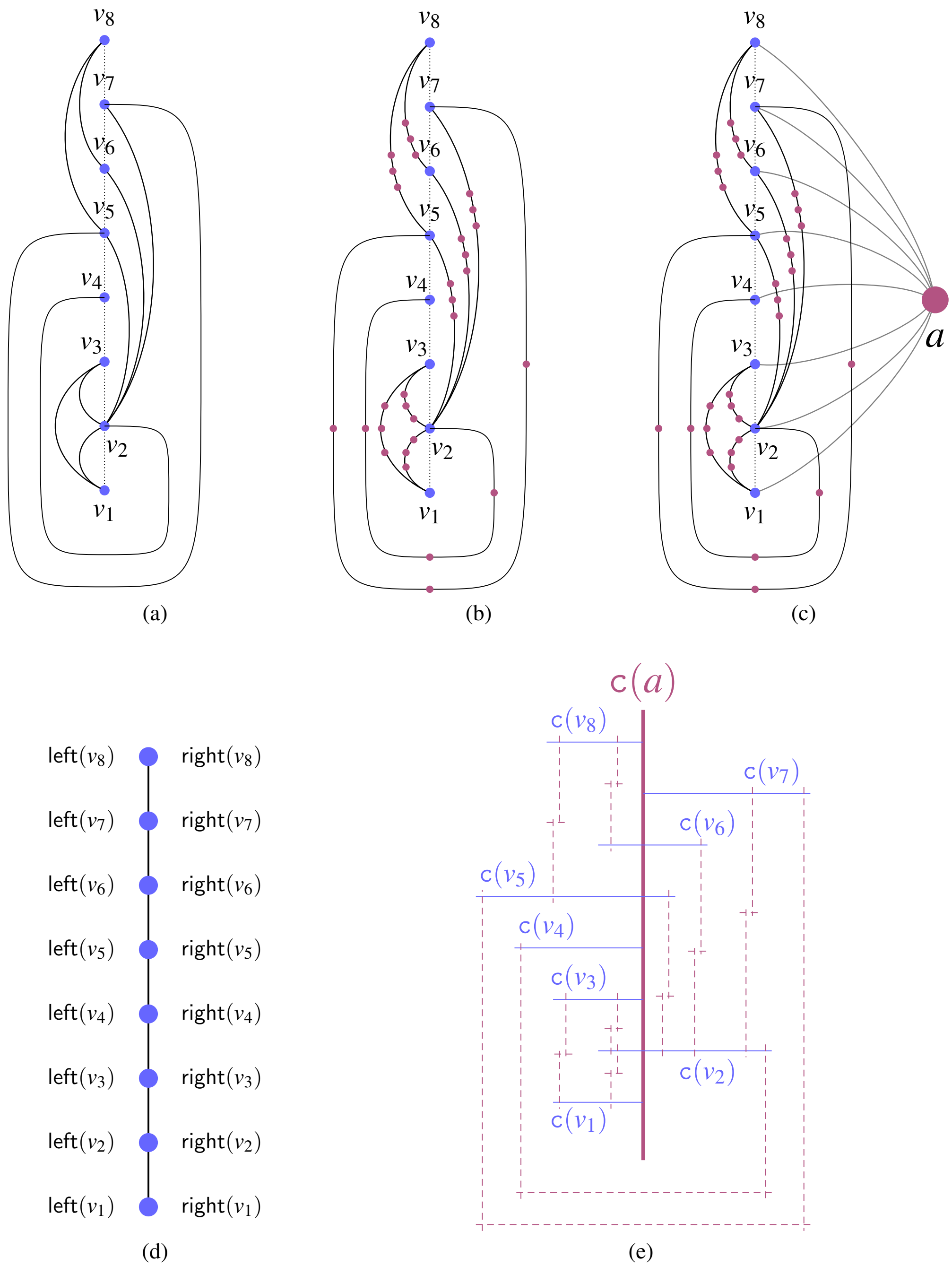

Figure 3: (a) G, a yes-instance of PLANAR HAMILTONIAN PATH COMPLETION*; (b)* $G_{k\text{-div}}$ *for* $k = 3$*; (c)* G_{apex}*; (d)* left *and* right *semi-disks representing the vertices of G; (e) C, a* PURE-2-DIR *representation of* G_{apex}*. (See Subsection 4.1 for detailed explanations of (a), (b) and (c).)*

adding a new vertex a to $G_{k\text{-div}}$ and making it adjacent to all the original vertices of $G_{k\text{-div}}$ (Figure 3 (c)). Formally, G_{apex} is defined as follows.

$$V(G_{\mathsf{apex}}) = V(G_{k\text{-div}}) \cup \{a\};$$
$$E(G_{\mathsf{apex}}) = E(G_{k\text{-div}}) \cup \{av \mid v \in V(G)\}.$$

Observation A. *If G is planar, then G_{apex} is a bipartite apex graph of girth at least g.*

Proof. G is a planar graph and subdivision does not affect planarity, so $G_{k\text{-div}}$ is also planar, implying that G_{apex} is an apex graph. The vertex set of G_{apex} can be expressed as the disjoint union of two sets A and B, where

$$A = \{x \mid x \in V(G)\} \cup \{u_e^i \mid e \in E(G), i \text{ is even}\};$$
$$B = \{a\} \cup \{u_e^i \mid e \in E(G), i \text{ is odd}\}.$$

Note that A induces an independent set in G_{apex}, and so does B. Thus, G_{apex} is a bipartite apex graph. As for the girth, note that every cycle in G_{apex} contains at least $k+2$ vertices $x, u_e^1, u_e^2, u_e^3, \ldots, u_e^k, y$, for some $e = (x,y) \in E(G)$. At least one more vertex is needed to complete the cycle, implying that the girth of G_{apex} is at least $k+3 \geq g$. □

It is easy to see that this entire construction of G_{apex} from G can be carried out in polynomial time.

4.2. Proof of Theorem 3 (a)

In this section, we will show that if G_{apex} is in 1-STRING, then G is a yes-instance of PLANAR HAMILTONIAN PATH COMPLETION. In other words, if G_{apex} has a 1-STRING representation, then G is a subgraph of a planar graph with a Hamiltonian path.

In our proofs, we will demonstrate the planarity of our graphs by embedding them in the plane. Typically, a planar graph is defined as a graph whose vertices are *points* in the plane and edges are *strings* connecting pairs of points such that no two strings intersect (except possibly at their end points). The same definition holds in more generality, *i.e.* , if the vertices are also allowed to be strings (see Figure 4). Let us now state this formally.

Definition 9 (Planarizable representation of a graph)**.** *A graph G on n vertices and m edges is said to admit a planarizable representation if there are two mutually disjoint sets of strings V and E (with $|V| = n$ and $|E| = m$) in the plane such that*

- *the strings of V correspond to the vertices of G, and those of E correspond to the edges of G;*

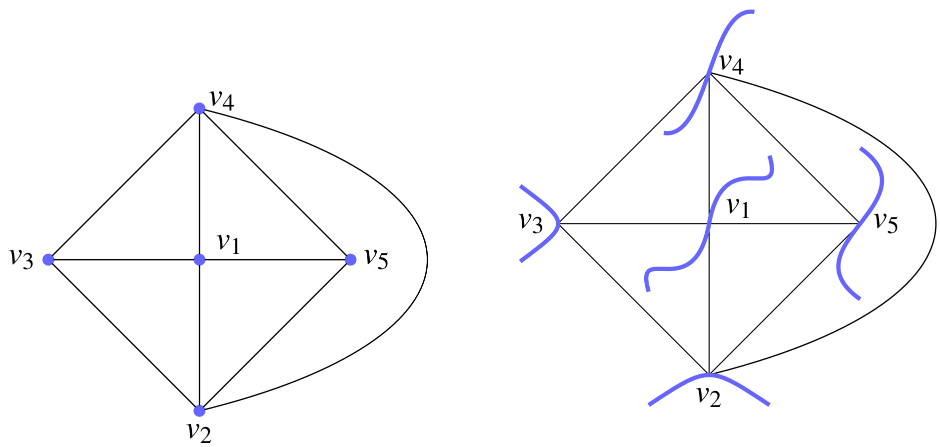

Figure 4: (Left) A standard representation of a planar graph with $n = 5$ vertices and $m = 9$ edges, where the vertices are points and the edges are strings. (Right) A planarizable representation (Definition 9) of the same graph, where the vertices as well as the edges are strings.

- *no two strings of V intersect;*
- *no two strings of E intersect, except possibly at their end points;*
- *apart from its two end points, a string of E does not intersect any string of V;*
- *for every vertex v and every edge $e = (x,y)$ of G, an end point of the string corresponding to e intersects the string corresponding to v if and only if $v = x$ or $v = y$.*

Figure 4 illustrates a planar graph and a planarizable representation of it.

Lemma 4. *A graph admits a planarizable representation if and only if it is planar.*

Lemma 4 may seem obvious. For completeness, we provide a formal proof of it in Section 5. We now use this lemma to prove Theorem 3 (a).

Proof of Theorem 3 (a). Given $G_{\mathsf{apex}} \in$ 1-STRING, we will show that the planar graph G is a yes-instance of the PLANAR HAMILTONIAN PATH COMPLETION problem. Let C be a 1-STRING representation of G_{apex} in the plane. It is helpful to follow Figure 5 while reading this proof. We will use C to construct a graph G_{pl} with the following properties.

(a) G_{pl} is a supergraph of G on the same vertex set as G.

(b) G_{pl} is planar.

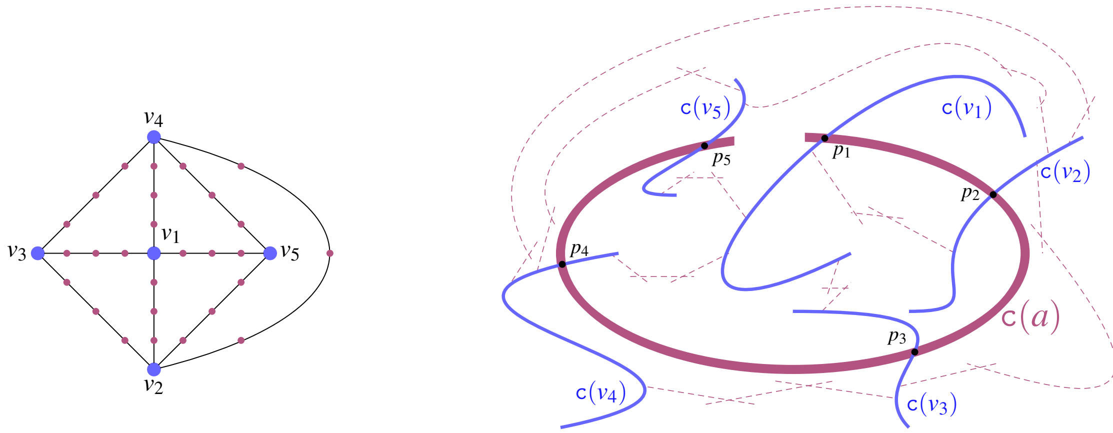

Figure 5: (Left) $G_{k\text{-div}}$ ($k = 3$) for a planar graph G on $n = 5$ vertices. (Right) C, a planarizable representation of G_{apex}. The thickest string denotes $\mathtt{c}(a)$, the apex vertex of G_{apex}. The bold strings denote the original vertices of G. The thin dashed strings denote $\mathtt{c}(u_e^1)$, $\mathtt{c}(u_e^2)$ and $\mathtt{c}(u_e^3)$.

(c) G_{pl} has a Hamiltonian path.

Note that (a), (b), (c) together imply that G is a subgraph of a planar graph with a Hamiltonian path (*i.e.* , G is a yes-instance of PLANAR HAMILTONIAN PATH COMPLETION). Let $n = |V(G)|$ and assume that $n \geq 4$. Along with our construction of G_{pl}, we will also describe DRAW(G_{pl}), a planarizable representation (Definition 9) of G_{pl} in the plane.

In C, consider the strings corresponding to the n original vertices (the large vertices in Figure 5 (Left)) of G. Since the original vertices form an independent set in G_{apex}, the bold strings are pairwise disjoint. We add these n strings to DRAW(G_{pl}), which correspond to the n vertices of G_{pl}.

Proof of (c): So far, G_{pl} has no edge. We will now add $n-1$ edges to G_{pl} to connect these vertices via a Hamiltonian path. Recall that all n original vertices are adjacent to the apex vertex a in G_{apex}, implying that each of the n bold strings intersects $\mathtt{c}(a)$ at exactly one point (as C is a 1-STRING representation). Starting from one end point of $\mathtt{c}(a)$ and travelling along the curve $\mathtt{c}(a)$ until we reach its other end point, we encounter these n points one-by-one. Let $(v_1, v_2, \ldots, v_n)$ be the order in which they are encountered.

For each $i \in [n]$, let p_i be the point at which $\mathtt{c}(v_i)$ intersects $\mathtt{c}(a)$. For each $i \in [n-1]$, let $\mathtt{s_i}$ be the substring of $\mathtt{c}(a)$ between p_i and p_{i+1}. Add the strings $\mathtt{s_1}, \mathtt{s_2}, \ldots, \mathtt{s_{n-1}}$ as edges to DRAW(G_{pl}), where $\mathtt{s_i}$ represents the edge between v_i and v_{i+1}. Thus the edges corresponding to the $n-1$ strings $\mathtt{s_1}, \mathtt{s_2}, \ldots, \mathtt{s_{n-1}}$ constitute a Hamiltonian path

$(v_1, v_2, \dots, v_n)$ in G_{pl}. This shows (c).

Proof of (a): To show (a), we need to add all the edges of G to G_{pl} (other than those already added by the previous step), so that G_{pl} becomes a supergraph of G. For each edge $e = v_i v_j \in E(G)$, there are k strings $\mathtt{c}(u_e^1), \mathtt{c}(u_e^2), \dots, \mathtt{c}(u_e^k)$ (corresponding to the subdivision vertices $u_e^1, u_e^2, \dots, u_e^k$ in G_{apex}) in C. Note that for each $t \in \{1, 2, \dots, k\}$, the string $\mathtt{c}(u_e^t)$ intersects exactly two other strings. Let $\mathtt{s}(u_e^t)$ be the substring of $\mathtt{c}(u_e^t)$ between those two intersection points. Let $\mathtt{s_e}$ be the string obtained by concatenating the k substrings thus obtained.

$$\mathtt{s_e} \triangleq \bigcup_{t=1}^{k} \mathtt{s}(u_e^t). \tag{1}$$

If the edge $e = v_i v_j$ is not already present in G_{pl}, then add the string $\mathtt{s_e}$ to $\textsc{Draw}(G_{\mathsf{pl}})$, where $\mathtt{s_e}$ represents the edge between v_i and v_j (one end point of $\mathtt{s_e}$ lies on $\mathtt{c}(v_i)$ and the other on $\mathtt{c}(v_j)$). This completes the construction of $\textsc{Draw}(G_{\mathsf{pl}})$, and shows (a).

Proof of (b): To show (b), it is enough to show that $\textsc{Draw}(G_{\mathsf{pl}})$ is a planarizable representation of G_{pl} (Lemma 4). Note that there are three types of strings in $\textsc{Draw}(G_{\mathsf{pl}})$: (i) substrings of $\mathtt{c}(a)$, (ii) strings of the type $\mathtt{s_e}$, for some $e = v_i v_j \in E(G)$, and (iii) n strings corresponding to the original vertices of G.

Two strings of type (i) are either disjoint or intersect at their end points, since $\mathtt{c}(a)$ is non-self-intersecting. More precisely, for each $i \in [n-1]$, the point p_{i+1} (the unique intersection point of $\mathtt{s_i}$ and $\mathtt{s_{i+1}}$) lies on $\mathtt{c}(v_{i+1})$, which denotes a vertex in $\textsc{Draw}(G_{\mathsf{pl}})$. A string of type (ii) intersects exactly two strings, $\mathtt{c}(v_i)$ and $\mathtt{c}(v_j)$, which denote vertices in $\textsc{Draw}(G_{\mathsf{pl}})$. Finally, strings of type (iii) are mutually disjoint. This shows (b). $\square$

4.3. Proof of Theorem 3 (b)

In this section, we will show that if G is a yes-instance of Planar Hamiltonian Path Completion, then G_{apex} is in Pure-2-Dir. In other words, if G is a subgraph of a planar graph with a Hamiltonian path, then G_{apex} has a Pure-2-Dir representation.

Before we jump in to our proof, let us elucidate the main idea behind it. We are given a plane drawing of G in which its vertices are placed in a collinear fashion on a vertical line, respecting their ordering on the Hamiltonian path (Figure 3 (a)). Our construction, in essence, makes use of this plane drawing and tweaks it to obtain a Pure-2-Dir representation of G_{apex} (Figure 3 (e)).

The apex segment $\mathtt{c}(a)$ is placed on the vertical line. The vertices of the original graph within G_{apex} are represented using horizontal segments (of appropriate length). The edges of G, which were strings in the plane drawing, are now replaced by rectilinear piecewise linear curves where each individual orthogonal segment represents a subdivided vertex of G_{apex}. If we were allowed a large (unbounded) number of rectilinear pieces for each edge, then this construction is trivial, since every curve can be viewed as a series of infinitesimally small vertical and horizontal segments. Our proof formally justifies that this can always be done even when the number of allowed rectilinear pieces is a fixed odd integer greater than or equal to three.

We achieve this through a slightly modified version of a folklore observation concerning *book embeddings* of graphs [35]: if a graph is embedded in a book and a,b,c,d are four vertices on the spine of the book arranged in the order $a<b<c<d$, then (a,c) and (b,d) cannot both be edges on the same page of the book.[4] Also note that Figure 3 (a) is for representational purposes only. Owing to the observation above, our construction does not rely on the topology of the strings representing the edges of G in Figure 3 (a). Now we provide a formal proof below.

Proof of Theorem 3 (b). Given a graph G, a yes-instance of PLANAR HAMILTONIAN PATH COMPLETION, we will construct C, a PURE-2-DIR representation of G_{apex}. Recall that the construction of G_{apex} from G uses an intermediate graph $G_{k\text{-}\mathsf{div}}$, where $k \geq 3$ is an odd integer.

$$G \to G_{k\text{-}\mathsf{div}} \to G_{\mathsf{apex}}.$$

Our proof is by induction on k. For almost the entirety of this proof, we will work with $k=3$. At the end, we will show that if the proof works for k, then it also works for $k+2$, and therefore for all odd integers k.

Base case $(k=3)$. G is a subgraph of a planar graph with a Hamiltonian path (say G_{pl}), as G is a yes-instance of PLANAR HAMILTONIAN PATH COMPLETION (Definition 8).

Let $n=|V(G)|$ and assume that $n \geq 4$. Let $(v_1,v_2,\ldots,v_n)$ be a Hamiltonian path in G_{pl}. Consider a plane drawing $D(G_{\mathsf{pl}})$ of G_{pl} in which its vertices are represented by points on a vertical line, ordered $(v_1,v_2,\ldots,v_n)$ from bottom to top, and the edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ (called Hamiltonian edges) are represented by vertical segments (Figure 3 (a)). All other edges of G_{pl} are called non-Hamiltonian edges. In $D(G_{\mathsf{pl}})$, we

[4] In order to see how book embeddings are applicable to our proof, it is helpful to think of the regular |-shaped spine generally used in book embeddings a bit differently. In Figure 3 (d), imagine that the |-shaped spine has a minuscule but non-zero thickness. "Cut" the |-shaped spine vertically down its middle from the top, stopping just before the bottom of the |-shape. Open up and spread out the two parts of the spine. Since the two parts are only connected to each other at the bottom, the new shape thus obtained looks like a $\vee$-shape. Now, the vertices, represented by tiny disks in the standard |-shaped spine become semi-disks in the $\vee$-shaped spine, each left semi-disk on the left arm of the $\vee$-shape, and each right semi-disk on its right arm. This now resembles a standard book embedding on one side of a $\vee$-shaped spine instead of on a straight-line spine.

retain all Hamiltonian edges of G_{pl}, but only those non-Hamiltonian edges of G_{pl} that are also present in G. The set of non-Hamiltonian edges of G is denoted by E_{non}.

The points representing the vertices of G (or G_{pl}) are slightly expanded (similar to the proof of Lemma 4) so that they are tiny circular disks. Each Hamiltonian edge $v_i v_{i+1}$ is represented by a vertical line segment connecting the top of the lower disk v_i to the bottom of the upper disk v_{i+1}. Thus, the line segments $v_{i-1}v_i$ and $v_i v_{i+1}$ divide the disk v_i into two: a left semi-disk ($\mathsf{left}(v_i)$) and a right semi-disk ($\mathsf{right}(v_i)$) (Figure 3 (d)).

Now, each $v_i v_j \in E_{\mathsf{non}}$ is a string that connects a semi-disk of v_i to a semi-disk of v_j. The edge is relabelled accordingly. In Figure 3 (a), $v_1 v_3$ becomes $(\mathsf{left}(v_1), \mathsf{left}(v_3))$, $v_2 v_5$ becomes $(\mathsf{right}(v_2), \mathsf{right}(v_5))$, $v_5 v_7$ becomes $(\mathsf{left}(v_5), \mathsf{right}(v_7))$, and $v_2 v_4$ becomes $(\mathsf{right}(v_2), \mathsf{left}(v_4))$. For simplicity of exposition, we denote each Hamiltonian edge $v_i v_{i+1}$ of G_{pl} that is also present in G as $(\mathsf{right}(v_i), \mathsf{right}(v_{i+1}))$.

We refer to this new updated plane drawing as $D(G)$. Note that the edges of $E(D(G))$ have $2n$ possible end points. We define a relation "$<$" on these end points, as follows.

$$\mathsf{left}(v_n) < \mathsf{left}(v_{n-1}) < \cdots < \mathsf{left}(v_1) < \mathsf{right}(v_1) < \mathsf{right}(v_2) < \cdots < \mathsf{right}(v_n). \quad (2)$$

For two end points a and b, we say that $a \le b$ if and only if $a < b$ or $a = b$. The following observation is easy to see, as it is a direct consequence of the planarity of $D(G)$.

Observation B. *Let (a,c) and (b,d) be two edges in the plane drawing $D(G)$. Then, under the ordering given by Equation 2, the following is not possible.*

$$\mathsf{left}(v_n) \le a < b < c < d \le \mathsf{right}(v_n).$$

We define a partial order "$\subseteq$" on $E(D(G))$. Let $e_1 = (b,c), e_2 = (a,d) \in E(D(G))$ such that $b < c$ and $a < d$ according to the ordering given by Equation 3. Then,

$$e_1 \subseteq e_2 \iff \mathsf{left}(v_n) \le a \le b < c \le d \le \mathsf{right}(v_n). \quad (3)$$

For example, $(\mathsf{right}(v_2), \mathsf{right}(v_5)) \subseteq (\mathsf{right}(v_2), \mathsf{right}(v_7))$ and $(\mathsf{left}(v_4), \mathsf{right}(v_2)) \subseteq (\mathsf{left}(v_5), \mathsf{right}(v_7))$ in Figure 3 (a). It is easy to see that $\subseteq$ is reflexive, anti-symmetric and transitive. Thus $(E(D(G)), \subseteq)$ is a poset. Consider the Hasse diagram of this poset, where the minimal elements are placed at the bottom and the maximal elements at the top. For an edge $e \in E(D(G))$, let $\mathsf{rank}(e)$ be the number of elements (including e) on the longest downward chain starting from e. For example, $\mathsf{rank}((\mathsf{right}(v_2), \mathsf{right}(v_7))) = 3$, as

$$(\mathsf{right}(v_2), \mathsf{right}(v_5)) \subseteq (\mathsf{right}(v_2), \mathsf{right}(v_6)) \subseteq (\mathsf{right}(v_2), \mathsf{right}(v_7))$$

is the longest downward chain starting from $(\mathsf{right}(v_2), \mathsf{right}(v_7))$ in Figure 3 (a). All Hamiltonian edges of G are minimal elements in this poset, and thus their rank is one.

We partition the edge set of $E(D(G))$ into three: an edge of $E(D(G))$ belongs to E_{left} if both its end points are $\mathsf{left}()$, to E_{right} if both its end points are $\mathsf{right}()$, and to E_{cross} if one of its end points is $\mathsf{left}()$ and the other is $\mathsf{right}()$. Therefore,

$$E(D(G)) = E_{\mathsf{left}} \cup E_{\mathsf{right}} \cup E_{\mathsf{cross}}.$$

We are now set to construct C (Figure 3 (e)), our PURE-2-DIR representation of G_{apex} for $k = 3$ (see next page for details).

The Construction of C (Figure 3 (e))

The apex vertex a: Let $\mathsf{c}(a)$ be the vertical segment $((0,0.5),(0,n+0.5))$.

The vertices $\{v_1,v_2,\ldots,v_n\}$: For each $i\in[n]$, let $\mathsf{c}(v_i)$ be the horizontal segment $((-a_i-0.1,i),(b_i+0.1,i))$, where a_i and b_i are defined as follows.

$$a_i = \mathsf{max}(\{0\}\cup\{\mathsf{rank}(e)\mid e \text{ is incident to } \mathsf{left}(v_i)\}); \tag{4}$$

$$b_i = \mathsf{max}(\{0\}\cup\{\mathsf{rank}(e)\mid e \text{ is incident to } \mathsf{right}(v_i)\}). \tag{5}$$

The $\{0\}$ set is included to ensure that the argument for max is not an empty set.
The vertices $\{u_e^1,u_e^2,u_e^3\}$: For each edge $e\in E(G)$, we define a set of four points $\ell_e=(\alpha_e,\beta_e,\gamma_e,\delta_e)$, such that

$\mathsf{c}(u_e^1)$ is a vertical line segment connecting α_e and β_e;
$\mathsf{c}(u_e^2)$ is a horizontal line segment connecting β_e and γ_e;
$\mathsf{c}(u_e^3)$ is a vertical line segment connecting γ_e and δ_e.

We may think of ℓ_e as a piecewise linear curve with three pieces. Let $\zeta: E\to\{1,2,\ldots,n^2\}$ be an injective function[a], *i.e.* , ζ maps each edge e of G to a distinct number from the set $\{1,2,\ldots,n^2\}$. For each edge $e\in E(G)$, let

$$\mathsf{xpos}(e)=\mathsf{rank}(e)+\frac{\zeta(e)}{n^4}. \tag{6}$$

Let $\varepsilon_k=1/(k^2n^5)$ (for this construction, $k=3$). For each $e=v_iv_j\in E(G)$ (where $1\le i<j\le n$), we define ℓ_e as follows.

$$\ell_e=\begin{cases}\left((-\mathsf{xpos}(e),i),\left(-\mathsf{xpos}(e),\frac{i+j}{2}\right),\left(-\mathsf{xpos}(e)-\varepsilon_k,\frac{i+j}{2}\right),(-\mathsf{xpos}(e)-\varepsilon_k,j)\right) & \text{if } e\in E_{\mathsf{left}};\\ \left((\mathsf{xpos}(e),i),\left(\mathsf{xpos}(e),\frac{i+j}{2}\right),\left(\mathsf{xpos}(e)+\varepsilon_k,\frac{i+j}{2}\right),(\mathsf{xpos}(e)+\varepsilon_k,j)\right) & \text{if } e\in E_{\mathsf{right}};\\ \left((-\mathsf{xpos}(e),i),(-\mathsf{xpos}(e),-\mathsf{rank}(e)),(\mathsf{xpos}(e),-\mathsf{rank}(e)),(\mathsf{xpos}(e),j)\right) & \text{if } e\in E_{\mathsf{cross}}.\end{cases}$$

[a]One such function is $\zeta(e) = ni + j$ for each $e = v_i v_j \in E(G)$, where $1 \le i < j \le n$.

Let G_{rep} be the intersection graph of C. To complete our proof, we need to show that $G_{\mathsf{apex}} = G_{\mathsf{rep}}$. First, let us understand the idea behind our construction of C.

Think of each ℓ_e as a single (piecewise linear) segment. Note that ℓ_e always consists of two vertical segments $\left(\mathsf{c}(u_e^1), \mathsf{c}(u_e^3)\right)$ and one horizontal segment $\left(\mathsf{c}(u_e^2)\right)$. Also, the x-coordinate of the vertical segments of ℓ_e is essentially the rank (or the negation of the rank) of e (Equation 6). The $\zeta(e)/n^4$ term (and also the ε_k term) is simply a tiny perturbation added to its x-coordinate to ensure that the vertical parts of ℓ_e do not intersect the vertical parts of any other $\ell_{e'}$. (In fact, ζ was chosen to be an injection for precisely this reason.)

Since G_{apex} and G_{rep} are graphs on the same vertex set, it is sufficient to show that $e \in E(G_{\mathsf{apex}}) \Leftrightarrow e \in E(G_{\mathsf{rep}})$ in order to demonstrate their equality.

Proof of $e \in E(G_{\mathsf{apex}}) \Rightarrow e \in E(G_{\mathsf{rep}})$: The $\mathsf{c}(v_i)$'s are horizontal segments, all intersecting the vertical apex segment $\mathsf{c}(a)$. Further, the $\mathsf{c}(v_i)$'s are made to extend as far (to the left and right of $\mathsf{c}(a)$) as the maximum rank (plus an additional ± 0.1) of their incident edges (Equation 4, Equation 5). This ensures that they intersect the vertical segments of all their corresponding ℓ_e's. The fact that the $\mathsf{c}(u_e^1)$'s and $\mathsf{c}(u_e^3)$'s intersect their corresponding $\mathsf{c}(u_e^2)$'s is implicit from the definition of ℓ_e.

Proof of $e \notin E(G_{\mathsf{apex}}) \Rightarrow e \notin E(G_{\mathsf{rep}})$: Note that C has three types of segments, namely

(i) the apex segment $\mathsf{c}(a)$;

(ii) the horizontal segments $\mathsf{c}(v_i)$;

(iii) the piecewise linear segments ℓ_e.

We will consider all pairs of non-adjacent vertices (p,q) of G_{apex}, and show that their respective segments $\mathsf{c}(p)$ and $\mathsf{c}(q)$ do not intersect in C. We have three cases.

Case 1: one of $\mathsf{c}(p)$ or $\mathsf{c}(q)$ is of type (i). Let us say $\mathsf{c}(p)$ is of type (i), *i.e.* , p is the apex vertex a. Then $\mathsf{c}(p) = ((0,0.5),(0,n+0.5))$, and $\mathsf{c}(q)$ must be of type (iii) (since all type (ii) vertices are adjacent to a, and we are only considering q's such that $(p,q) \notin E(G_{\mathsf{apex}})$). Note that the x-coordinates of the vertical pieces of all the ℓ_e's in $E_{\mathsf{left}} \cup E_{\mathsf{right}}$ are either less than -0.1 or greater than 0.1, and the y-coordinates of the horizontal pieces of all the ℓ_e's in E_{cross} are less than 0. Therefore, $\mathsf{c}(p)$ intersects none of the ℓ_e's.

Case 2: one of $\mathsf{c}(p)$ *or* $\mathsf{c}(q)$ *is of type (ii).* Let us say $\mathsf{c}(p)$ is of type (ii). If $\mathsf{c}(q)$ is also of type (ii), then we are done, since all the $\mathsf{c}(v_i)$'s are mutually disjoint (Equation 4, Equation 5). If $\mathsf{c}(q)$ is of type (iii), then let $\mathsf{c}(q)$ be a piece of ℓ_{e_q} for some edge e_q, and let $p = v_{i_p}$ for some $i_p \in [n]$ such that e_q is not incident to v_{i_p}.

Note that all the horizontal pieces of ℓ_{e_q} of non-ε_k length belong to edges of E_{cross}, which lie below $\mathsf{c}(a)$, and none of them can intersect $\mathsf{c}(v_{i_p})$.

Now, assume to contrary that $\mathsf{c}(p)$ and $\mathsf{c}(q)$ do intersect. Then $\mathsf{c}(q)$ must be a vertical segment. Let $x(\mathsf{c}(p),\mathsf{c}(q))$ be the x-coordinate of their point of intersection. If $x(\mathsf{c}(p),\mathsf{c}(q)) < 0$, then let e_p be an edge of maximum rank incident to $\mathsf{left}(v_{i_p})$. If $x(\mathsf{c}(p),\mathsf{c}(q)) > 0$, then let e_p be an edge of maximum rank incident to $\mathsf{right}(v_{i_p})$. (If no such edge exists, then the $\{0\}$ set (Equation 4, Equation 5) comes into play, and we are done, as $\mathsf{c}(v_{i_p})$ falls short of ℓ_{e_q}.)

Let $e_q = (a,c)$ and $e_p = (b,d)$ such that $a < c$. Furthermore, let b be the end point of e_p that is incident to v_{i_p} (*i.e.* , $b = \mathsf{left}(v_{i_p})$ or $b = \mathsf{right}(v_{i_p})$), where the relation "$<$" is defined by Equation 2. Then, it is easy to see that $a < b < c$ (otherwise $\mathsf{c}(p)$ and $\mathsf{c}(q)$ can never intersect). Applying Observation B to this, we get $a < d < c$. This implies that $e_p \subseteq e_q$ (Equation 3). Thus (since $e_p \neq e_q$),

$$e_p \subseteq e_q \;\Rightarrow\; \mathsf{rank}(e_p) < \mathsf{rank}(e_q) \;\Rightarrow\; \mathsf{rank}(e_p) + 1 \leq \mathsf{rank}(e_q).$$

Note that the vertical pieces of ℓ_{e_q} are at least $\mathsf{rank}(e_q)$ units away from the apex segment $\mathsf{c}(a)$. However, the horizontal segment $\mathsf{c}(v_{i_p})$ only reaches as far as $\mathsf{rank}(e_p) + 0.1 < \mathsf{rank}(e_q)$ units away from $\mathsf{c}(a)$ on the same side (left/right) of $\mathsf{c}(a)$ as ℓ_{e_q}. Hence, $\mathsf{c}(v_{i_p}) = \mathsf{c}(p)$ and $\mathsf{c}(q)$ (which is a piece of ℓ_{e_q}) do not intersect, contradicting our assumption.

Case 3: both $\mathsf{c}(p)$ *and* $\mathsf{c}(q)$ *are of type (iii).* Let $\mathsf{c}(p)$ be a piece of ℓ_{e_p} and $\mathsf{c}(q)$ be a piece of ℓ_{e_q}, for some edges e_p and e_q of G. If they are both vertical pieces or one of them is a horizontal piece of length ε_k, then the ζ function guarantees that they do not intersect.

The only remaining case is if one of them (say $\mathsf{c}(p)$) is a horizontal piece of non-ε_k length. Let $e_p \in E_{\mathsf{cross}}$. Then, the y-coordinate of $\mathsf{c}(p)$ is less than 0. If $e_q \in E_{\mathsf{left}} \cup E_{\mathsf{right}}$, then the y-coordinate of each end point of each segment of ℓ_{e_q} is greater than 0, and we are done. If $e_q \in E_{\mathsf{cross}}$, then note that the all the edges contained in E_{cross} constitute a total order (or chain) in the poset $(E(D(G)),\subseteq)$ (Equation 3). Thus either $\mathsf{rank}(e_p) < \mathsf{rank}(e_q)$ or $\mathsf{rank}(e_q) < \mathsf{rank}(e_p)$. Let $\mathsf{rank}(e_p) < \mathsf{rank}(e_q)$ (the proof for $\mathsf{rank}(e_q) < \mathsf{rank}(e_p)$ is similar). Then, $\mathsf{rank}(e_p) + 1 \leq \mathsf{rank}(e_q)$. All the pieces of ℓ_{e_q} are at least $\mathsf{rank}(e_p) + 0.5$ units away from the apex segment $\mathsf{c}(a)$, and all the pieces of ℓ_{e_q} reach less than $\mathsf{rank}(e_p)+0.1$ units away from $\mathsf{c}(a)$. Hence, ℓ_{e_p} and ℓ_{e_q} do not intersect.

This completes the proof of the base case ($k=3$) of our induction. A crucial feature of our construction, which we will use in our proof of the inductive case, is that for all edges $e \in E(D(G))$, the segment $\mathsf{c}(u_e^3)$ is a vertical segment.

Induction hypothesis. Let $k \geq 3$ be an odd integer. Then there exists a PURE-2-DIR representation of G_{apex} in which $\mathsf{c}(u_e^k)$ is a vertical segment for all edges e of G.

Induction step. Given a PURE-2-DIR representation of G_{apex} where $\mathsf{c}(u_e^k)$ is a vertical segment, we will slightly modify it to include two new segments $\mathsf{c}(u_e^{k+1})$ and $\mathsf{c}(u_e^{k+2})$ for each e, such that $\mathsf{c}(u_e^{k+2})$ is a vertical segment. Let

$$\mathsf{c}(u_e^k) = ((\lambda_e^k, \mu_e^k), (\lambda_e^k, \pi_e^k));$$

$$\sigma_e^{k+2} = \begin{cases} -1 & \text{if } \lambda_e^k < 0; \\ +1 & \text{if } \lambda_e^k > 0. \end{cases}$$

Recall that $\varepsilon_k = 1/(k^2 n^5)$. Now for each edge e of G, we replace the segment $\mathsf{c}(u_e^k)$ by the following three segments.

$$\mathsf{c}(u_e^k) = \left(\left(\lambda_e^k, \mu_e^k\right), \left(\lambda_e^k, \frac{\mu_e^k + \pi_e^k}{2}\right)\right);$$

$$\mathsf{c}(u_e^{k+1}) = \left(\left(\lambda_e^k, \frac{\mu_e^k + \pi_e^k}{2}\right), \left(\lambda_e^k + \sigma_e^{k+2}\varepsilon_{k+2}, \frac{\mu_e^k + \pi_e^k}{2}\right)\right);$$

$$\mathsf{c}(u_e^{k+2}) = \left(\left(\lambda_e^k + \sigma_e^{k+2}\varepsilon_{k+2}, \frac{\mu_e^k + \pi_e^k}{2}\right), \left(\lambda_e^k + \sigma_e^{k+2}\varepsilon_{k+2}, \pi_e^k\right)\right).$$

Note that these three new segments roughly coincide with the segment that they replaced, with a tiny perturbation of $\sigma_e^{k+2}\varepsilon_{k+2}$ made to the x-coordinates of $\mathsf{c}(u_e^{k+1})$ and $\mathsf{c}(u_e^{k+2})$. Using the induction hypothesis, it is easy to see that $\mathsf{c}(u_e^k)$, $\mathsf{c}(u_e^{k+1})$ and $\mathsf{c}(u_e^{k+2})$ intersect the segments that they are adjacent to in G_{apex}. The following calculation shows that the $\sigma_e^{k+2}\varepsilon_{k+2}$ perturbation is so minuscule that $\mathsf{c}(u_e^{k+1})$ and $\mathsf{c}(u_e^{k+2})$ do not intersect any additional segments.

$$\left|\sum_{i=3}^{k+2} \sigma_e^i \varepsilon_i\right| \leq \frac{1}{n^5}\left(\sum_{i=3}^{k+2}\frac{1}{i^2}\right) < \frac{1}{n^5}.$$

Note that for every $e' \neq e$ and every odd k' such that $1 \leq k' \leq k+2$, the x-coordinates of $\mathsf{c}(u_{e'}^{k'})$ and $\mathsf{c}(u_e^k)$ differ by roughly $1/n^4$, which is much larger than $1/n^5$. Finally, note that $\mathsf{c}(u_e^{k+2})$ is a vertical segment, as promised. This completes the proof. □

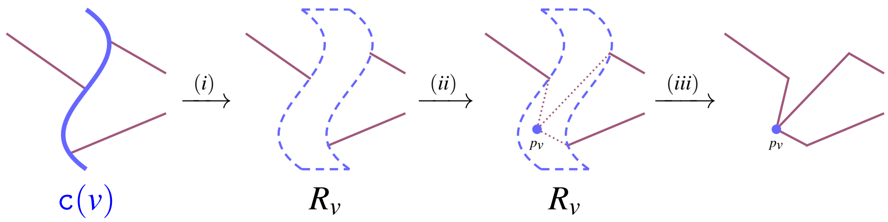

Figure 6: Initially, the vertex v is denoted by the bold string $\mathsf{c}(v)$, and the edges incident to it are the thinner strings. (i) $\mathsf{c}(v)$ is "thickened" to form a region R_v around it. (ii) The end points of the edges on the boundary of R_v are connected to a single point p_v in the interior of R_v. (iii) Strings sharing end points on the boundary of R_v are fused, and the region R_v is "shrunk" to the point p_v.

5. Planarizable Representations of Planar Graphs

In this section, we will show Lemma 4, *i.e.* , a graph admits a planarizable representation (Definition 9) if and only if the graph is planar.

Proof of Lemma 4. It is easy to see that every planar graph admits a planarizable representation. We will show the other direction: every graph that admits a planarizable representation is planar. Let G be a graph with a planarizable representation. Let v be a vertex of G, and $\mathsf{c}(v)$ be its corresponding string. Figure 6 illustrates our proof for a given $\mathsf{c}(v)$. Let

$$R_v = \{p \mid p \in \mathbb{R}^2, d(p, \mathsf{c}(v)) \leq \varepsilon\}$$

be the set of points within a closed ε-neighbourhood of $\mathsf{c}(v)$, choosing ε small enough so that $\mathsf{c}(v)$ does not intersect any additional strings. Delete all substrings lying in the interior of R_v. Thus all strings that intersected $\mathsf{c}(v)$ now have one end point on the boundary of R_v. Connect all these boundary end points to a common point (say p_v) in the interior of R_v via pairwise disjoint substrings (intersecting only at p_v) in the interior of R_v, effectively "shrinking" the region R_v to a single point p_v. (This last step is possible because R_v is a simply connected region.) Now the point p_v corresponds to the vertex v.

Do this for all the vertices of G. Since the vertices are now points, and the edges are strings connecting them, the representation thus obtained is a planar drawing of G. □

6. Conclusion

Corollary 4 states that recognizing rectangle intersection graphs is NP-hard, even when the inputs are bipartite apex graphs. This raises the following question. Can we recognize planar rectangle intersection graphs in polynomial time? Corollary 5 states that for all graph classes $\mathscr{G}$ such that PURE-2-DIR $\subseteq \mathscr{G} \subseteq$ 1-STRING, the recognition of $\mathscr{G}$ is not even in XP, when parameterized by the apex number of the graph. As our construction produces graphs of large degree, the sum of maximum degree and apex number of the graph might be a parameter for which the recognition of $\mathscr{G}$ is XP or even FPT. Corollary 6 states that recognizing several geometric intersection graph classes is NP-hard, even when the inputs are restricted to K_6-minor free graphs. On the other hand, the complexity of finding geometric representations of K_5-minor free graphs is unknown. Is it possible to use Wagner's Theorem [36] to decide in polynomial time whether a K_5-minor free graph is in 1-STRING (or STRING)? It would also be interesting to study the complexity of recognizing STRING when the inputs are restricted to apex graphs.

The crossing number of a graph is the minimum number of edge crossings possible in a plane drawing of the graph. Planar graphs are precisely the graphs with crossing number zero. Schaefer showed that apex graphs can have arbitrarily high crossing number, and also exhibited several graphs with crossing number one [37]. Graph classes with a small crossing number, like k-planar graphs [38], have also been studied. Therefore, the complexity of recognizing 1-STRING (or STRING) when the inputs are restricted to graphs with a small crossing number is another potential direction of research.

Finally, it would be interesting to see if our techniques can be used to prove NP-hardness of recognizing other classes of geometric intersection graphs, like outerstring graphs [39] and intersection graphs of grounded L-shapes [40]. Also, the graph classes we study in this paper are for objects embedded in the plane. The complexity of finding geometric intersection representations of apex graphs (appropriately defined) using curves on other surfaces (*e.g.*, torus, projective plane) is another avenue open for exploration.

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References

[1] D. Chakraborty, K. Gajjar, Finding Geometric Representations of Apex Graphs is NP-Hard, in: WALCOM: Algorithms and Computation - 16th International Conference and Workshops, Jember, Indonesia, March 24-26, 2022, Proceedings, Vol. 13174 of Lecture Notes in Computer Science, Springer, 2022, pp. 161–174.

[2] F. Chung, F. T. Leighton, A. Rosenberg, Diogenes: A methodology for designing fault-tolerant VLSI processor arrays, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Microsystems Program Office, 1983.

[3] F. Chung, F. Leighton, A. Rosenberg, A graph layout problem with applications to VLSI design (1984).

[4] N. A. Sherwani, Algorithms for VLSI Physical Design Automation, Springer US, 2007.

[5] P. K. Agarwal, M. J. Van Kreveld, Label placement by maximum independent set in rectangles, Elsevier, 1998.

[6] F. Kuhn, R. Wattenhofer, A. Zollinger, Ad hoc networks beyond unit disk graphs, Wireless Networks 14 (5) (2008) 715–729.

[7] J. Xu, B. Berger, Fast and accurate algorithms for protein side-chain packing, Journal of the ACM (JACM) 53 (4) (2006) 533–557.

[8] P. Koebe, Kontaktprobleme der konformen Abbildung, Hirzel, 1936.

[9] E. Andreev, On convex polyhedra in lobacevskii spaces, Mathematics of the USSR-Sbornik 10 (3) (1970) 413.

[10] W. Thurston, Hyperbolic geometry and 3-manifolds, Low-dimensional topology (Bangor, 1979) 48 (1982) 9–25.

[11] E. R. Scheinerman, Intersection classes and multiple intersection parameters of graphs, Princeton University, 1984.

[12] I. B. Hartman, I. Newman, R. Ziv, On grid intersection graphs, Discrete Mathematics 87 (1) (1991) 41–52.

[13] J. Kratochvíl, A special planar satisfiability problem and a consequence of its NP-completeness, Discrete Applied Mathematics 52 (3) (1994) 233–252.

[14] J. Chalopin, D. Gonçalves, Every planar graph is the intersection graph of segments in the plane, in: STOC, 2009, pp. 631–638.

[15] J. Kratochvíl, J. Matousek, Intersection graphs of segments, Journal of Combinatorial Theory, Series B 62 (2) (1994) 289–315.

[16] N. De Castro, F. J. Cobos, J. C. Dana, A. Márquez, M. Noy, Triangle-free planar graphs as segments intersection graphs, in: International Symposium on Graph Drawing, Springer, 1999, pp. 341–350.

[17] D. Gonçalves, L. Isenmann, C. Pennarun, Planar graphs as L-intersection or L-contact graphs, in: SODA, SIAM, 2018, pp. 172–184.

[18] J. Chalopin, D. Gonçalves, P. Ochem, Planar graphs have 1-string representations, Discrete & Computational Geometry 43 (3) (2010) 626–647.

[19] D. Gonçalves, 3-colorable planar graphs have an intersection segment representation using 3 slopes, in: WG, Springer, 2019, pp. 351–363.

[20] D. Gonçalves, B. Lévêque, A. Pinlou, Homothetic triangle representations of planar graphs, Journal of Graph Algorithms and Applications 23 (4) (2019) 745–753.

[21] D. Gonçalves, Not all planar graphs are in PURE-4-DIR, Journal of Graph Algorithms and Applications 24 (3) (2020) 293–301.

[22] P. Chmel, Algorithmic aspects of intersection representations, Bachelor's Thesis.

[23] S. Chaplick, V. Jelínek, J. Kratochvíl, T. Vyskočil, Bend-bounded path intersection graphs: Sausages, noodles, and waffles on a grill, in: WG, Springer, 2012, pp. 274–285.

[24] M. Cygan, F. V. Fomin, Ł. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, S. Saurabh, Parameterized algorithms, Vol. 5, Springer, 2015.

[25] J. Kratochvíl, String graphs. II. recognizing string graphs is NP-hard, Journal of Combinatorial Theory, Series B 52 (1) (1991) 67–78.

[26] J. Kratochvíl, M. Pergel, Geometric intersection graphs: do short cycles help?, in: International Computing and Combinatorics Conference, Springer, 2007, pp. 118–128.

[27] I. Mustaţă, M. Pergel, On unit grid intersection graphs and several other intersection graph classes, Acta Mathematica Universitatis Comenianae 88 (3) (2019) 967–972.

[28] C. Thomassen, Interval representations of planar graphs, Journal of Combinatorial Theory, Series B 40 (1) (1986) 9–20.

[29] K. S. Booth, G. S. Lueker, Testing for the consecutive ones property, interval graphs, and graph planarity using pq-tree algorithms, Journal of Computer and System Sciences 13 (3) (1976) 335–379.

[30] N. Robertson, P. D. Seymour, Graph minors. XX. Wagner's conjecture, Journal of Combinatorial Theory, Series B 92 (2) (2004) 325–357.

[31] A. Gupta, R. Impagliazzo, Computing planar intertwines, in: FOCS, Citeseer, 1991, pp. 802–811.

[32] J. Kratochvíl, J. Matoušek, NP-hardness results for intersection graphs, Commentationes Mathematicae Universitatis Carolinae 30 (4) (1989) 761–773.

[33] C. Auer, A. Gleißner, Characterizations of deque and queue graphs, in: WG, Springer, 2011, pp. 35–46.

[34] A. Wigderson, The complexity of the hamiltonian circuit problem for maximal planar graphs, Tech. rep., Tech. Rep. EECS 198, Princeton University, USA (1982).

[35] F. Bernhart, P. C. Kainen, The book thickness of a graph, Journal of Combinatorial Theory, Series B 27 (3) (1979) 320–331.

[36] K. Wagner, Über eine eigenschaft der ebenen komplexe, Mathematische Annalen 114 (1) (1937) 570–590.

[37] M. Schaefer, Crossing numbers of graphs, CRC Press, 2018.

[38] P. Angelini, M. A. Bekos, F. J. Brandenburg, G. Da Lozzo, G. Di Battista, W. Didimo, M. Hoffmann, G. Liotta, F. Montecchiani, I. Rutter, C. D. T/'oth, Simple k-planar graphs are simple (k+ 1)-quasiplanar, Journal of Combinatorial Theory, Series B 142 (2020) 1–35.

[39] T. Biedl, A. Biniaz, M. Derka, On the size of outer-string representations, in: SWAT, Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.

[40] S. McGuinness, On bounding the chromatic number of L-graphs, Discrete Mathematics 154 (1-3) (1996) 179–187.

[41] L. Gąsieniec, R. Klasing, T. Radzik, IWOCA 2020, Vol. 12126, Springer Nature, 2020.