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ORTHOGONAL DECOMPOSITION OF DEFINABLE GROUPS

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Abstract. Orthogonality in model theory captures the idea of absence of non-trivial interactions between definable sets. We introduce a somewhat opposite notion of cohesiveness, capturing the idea of interaction among all parts of a given definable set. A cohesive set is indecomposable, in the sense that if it is internal to the product of two orthogonal sets, then it is internal to one of the two. We prove that a definable group in an o-minimal structure is a product of cohesive orthogonal subsets. If the group has dimension one, or it is definably simple, then it is itself cohesive. As an application, we show that an abelian group definable in the disjoint union of finitely many o-minimal structures is a quotient, by a discrete normal subgroup, of a direct product of locally definable groups in the single structures.

Contents

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1. Introduction

Considering a group G interpretable in the disjoint union of finitely many structures $\mathcal{X}_1, \dots, \mathcal{X}_n$ (seen as a multi-sorted structure as in Definition 2.3), one may ask whether G can be understood in terms of groups definable in the individual structures. One may, for instance, ask whether G is definably isomorphic to a quotient, modulo a finite normal subgroup Γ, of a direct product $G_1 \times \dots \times G_n$, where G_i is a group definable in $\mathcal{X}_i$. This, however, is not true in general: a counterexample is provided by [2, Example 1.2] (a torus obtained from two orthogonal copies of $\mathbb{R}$ and a lattice generated by two vectors in generic position). After a talk by the first author at the Oberwolfach workshop "Model Theory: Groups, Geometry, and Combinatorics" (2013), Hrushovski suggested that a result of the above kind would require to pass to the locally definable category. So the natural conjecture would be that, if G is as above, there is a locally definable isomorphism $G \cong G_1 \times \dots \times G_n/\Gamma$, where G_i is a locally definable group in $\mathcal{X}_i$ and Γ is a compatible *discrete* subgroup (i.e. a subgroup which intersects every definable set at a finite set). Here we establish the conjecture under the additional assumption that the structures $\mathcal{X}_i$ are o-minimal and G is abelian. We recall that an o-minimal structure is a structure $\mathcal{M} = (M, <, \dots)$ expanding a dense linear order without end-points such that every definable subset of M is a finite union of intervals and points (see [5]). We prove the following result.

Theorem (9.2)**.** *Let G be a definable abelian group in the disjoint union of finitely many o-minimal structures $\mathcal{X}_1, \dots, \mathcal{X}_n$. Then there is a locally definable homomorphism*

$$G \cong G_1 \times \dots \times G_n/\Gamma,$$

where G_i is a locally definable group in $\mathcal{X}_i$ and Γ is a compatible locally definable discrete subgroup of $G_1 \times \dots \times G_n$.

We will deduce Theorem 9.2 from Theorem 8.4 below, which is interesting in itself and holds for non-abelian groups as well. To state the theorem we need to recall the model-theoretic notion of orthogonality. Given definable sets $X_1, \dots, X_n$ in a structure $\mathcal{M}$, we say that $X_1, \dots, X_n$ are *orthogonal* if, for all $k_1, \dots, k_n \in \mathbb{N}$, any definable subset of $X_1^{k_1} \times \dots \times X_n^{k_n}$ is a finite union of sets of the form $A_1 \times \dots \times A_n$, where A_i is a definable subset of $X_i^{k_i}$. Let us also recall that a definable set X is *internal* to a definable set Y if there is $m \in \mathbb{N}$ and a definable surjective map from Y^m to X.

Theorem (8.4)**.** *Assume $\mathcal{M}$ is an o-minimal structure. Let $X_1, \dots, X_n$ be orthogonal sets definable in $\mathcal{M}$ and G a group definable in $\mathcal{M}$. If G is internal to the product $X_1 \times \cdots \times X_n$, then G is a product*

$$G = A_1 \dots A_n$$

of definable subsets $A_1, \dots, A_n$, where A_i is internal to X_i, for $i = 1, \dots, n$ and $A_1 \dots A_n$ is the set of all products $a_1 \cdot \ldots \cdot a_n \in G$ with $a_i \in A_i$.

To deduce Theorem 9.2 from Theorem 8.4, we take for $\mathcal{M}$ the o-minimal structure consisting of the concatenation of the structures $\mathcal{X}_i$ separated by single points. The sets X_i would then be orthogonal within $\mathcal{M}$, so we can apply Theorem 8.4 and take as G_i an isomorphic copy of the subgroup of G generated by A_i. The only delicate point is to show that G_i is locally definable in $\mathcal{X}_i$, but this is not difficult.

It is worth stressing that Theorem 8.4 holds for an arbitrary o-minimal structure $\mathcal{M}$ and in particular we do not assume that $\mathcal{M}$ has a group structure. This is important for the application to Theorem 9.2 because the concatenation of o-minimal structures does not have a group structure even if the single structures do.

The proof of Theorem 8.4 uses a number of deep results about groups definable in o-minimal structures, such as the solution of Pillay's Conjecture and Compact Domination Conjecture (see [16] and [10] for the definitions). It is, however, conceivable that the theorem could be extended far beyond the o-minimal context: indeed we have no counterexample even if $\mathcal{M}$ is allowed to be a completely arbitrary structure. Theorem 6.1 provides a partial result in this direction, when $\mathcal{M}$ is NIP and G is compactly dominated and abelian.

Before stating our final result, we notice that the sets $A_1, \dots, A_n$ in Theorem 8.4 are orthogonal, so we may ask whether, for a group G definable in an arbitrary o-minimal structure $\mathcal{M}$, there is always a natural way to generate it as a product $G = A_1 \dots A_n$ of orthogonal definable subsets A_i. Of course, there is always the trivial solution with $n = 1$ and $A_1 = G$, but we would like each A_i to be in some sense "minimal". To this aim, we introduce the following model-theoretic notions. Let Z be a set definable in an arbitrary structure $\mathcal{M}$. We say that:

- Z is **indecomposable** if whenever Z is internal to the cartesian product of two orthogonal sets, then Z is internal to one of the two.
- Z is **cohesive** if whenever two definable sets are non-orthogonal to Z, they are non-orthogonal to each other.

A cohesive set Z is indecomposable (Proposition 3.2(2)), but it can be shown that the converse fails, see Example 11.4. We show that cohesive sets have nice model-theoretic properties which we are not able to prove for indecomposable sets. In particular, a set internal to a cohesive set is cohesive and the cartesian product of two cohesive sets is cohesive. The intuition is that the various parts of a cohesive set interact in a non-trivial way, so in particular a cohesive set cannot contain two orthogonal infinite definable subsets. Our final result is the following.

Theorem (8.7)**.** *Let G be a group interpretable in an o-minimal structure $\mathcal{M}$. Then there are cohesive orthogonal definable sets $A_1, \dots, A_n$, such that $G = A_1 \dots A_n$.*

In the setting of Theorem 8.7 we call the tuple $A_1, \dots, A_n$ a **cohesive orthogonal decomposition** of G, while in Theorem 8.4 the tuple $A_1, \dots, A_n$ is called an **orthogonal decomposition** of G with respect to $X_1, \dots, X_n$.

If G is infinite, we can choose each A_i in Theorem 8.7 to be infinite, and in this case the number n is an invariant of G up to definable isomorphism. Indeed, if $G = B_1 \dots B_m$ is another cohesive orthogonal decomposition of G where $B_1, \dots, B_m$ are infinite, then $m = n$ and each B_i is bi-internal to a single A_j. We may call the invariant n the **dimensionality** of G (not to be confused with the dimension of G). In this terminology, the unidimensional groups in the sense of [14, Claim 1.26] have dimensionality 1.

We can show that if G has dimension one, or it is definably simple, then G is itself cohesive, so these groups have dimensionality one.

For the proof of Theorem 8.7, we need both Theorem 8.4 and the results from [8]. In particular, we need that for every group G interpretable in an o-minimal structure, there is an injective definable map f (not necessarily a morphism) from G to the cartesian product of finitely many 1-dimensional definable groups ([8, Theorem 3]).

Related work. Groups definable in the disjoint union of orthogonal structures have already been considered in [2, 20]. The original motivation comes from the model theory of group extensions [3, 21]. For instance, the universal cover of a group definable in an o-minimal expansion of the field $\mathbb{R}$ is definable in the disjoint union $\mathbb{R} \sqcup \mathbb{Z}$, where $\mathbb{Z}$ has only the additive structure [11]. From a model-theoretic point of view, $(\mathbb{Z}, +)$ is an example of a superstable structure of finite and definable Lascar rank. In [2], it is shown that if a group G is definable in the disjoint union of an arbitrary structure $\mathcal{R}$ and a superstable structure $\mathcal{Z}$ of finite and definable Lascar rank, then G is an extension of a group internal to $\mathcal{R}$ by a group internal to $\mathcal{Z}$. In [20], Wagner weakened the superstability assumption to assuming only that $\mathcal{Z}$ is simple. The simplicity assumption cannot be entirely removed, or replaced by an o-minimality one: quotients by a lattice of a product of copies of $\mathbb{R}$ provide examples of definable groups which may not have infinite definable subgroups internal to any of the copies ([2, Example 1.2]). However, as Theorem 8.4 shows, the o-minimality assumption allows for another, in fact more symmetric analysis in terms of generating subsets instead of subgroups. Theorem 8.7 can be seen as a continuation of the work in [8].

The possibility of extending the results beyond the o-minimal context raises a number of questions, which are included in Section 10.

Structure of the paper. In Sections 2 – 4, we introduce and study the key notions of this paper. In particular we prove, for definable sets $X_1, \ldots, X_n$ in an arbitrary structure, that $X_1, \ldots, X_n$ are orthogonal if and only if they are pairwise orthogonal. We then study indecomposable and cohesive sets and establish that 1-dimensional groups definable in o-minimal structures are cohesive (Theorem 3.4). The proofs of Theorems 8.4 and 8.7 then proceed in several steps. In Section 5, we recall the basics of compact domination for NIP structures, which we employ in Section 6 to prove Theorem 8.4 for compactly dominated abelian NIP groups that are *contained* in $X_1 \times \cdots \times X_n$ (Theorem 6.1). In Section 7, we specialize in o-minimal structures and prove Theorem 8.4 for definably compact abelian groups *internal* to $X_1 \times \cdots \times X_n$. In Section 8, we employ the rich machinery available for groups definable in o-minimal structures, and conclude the full Theorem 8.4. Together with the results from [8], we then establish Theorem 8.7. In the final part of the paper we prove Theorem 9.2.

Notation. Throughout this paper, we work in a first-order structure $\mathcal{M}$. By "definable" we mean definable in $\mathcal{M}$, with parameters. Unless stated otherwise, X, Y, Z denote definable sets. By convention, $X \times Y^0 = X$.

We assume familiarity with the basics of o-minimality, as in [5]. We also assume familiarity with the definable manifold topology of definable groups [15, Proposition 2.5]. All topological notions for definable groups, such as connectedness and definable compactness, are taken with respect to this group topology.

2. Orthogonality

In this section we work in an arbitrary structure $\mathcal{M}$. We recall the notions of orthogonality and internality, and prove some basic facts.

Definition 2.1. Given definable sets $X_1, \dots, X_n$, a $(X_1, \dots, X_n)$-**box** is a definable set of the form $U_1 \times \cdots \times U_n$ where $U_i \subseteq X_i$ for every $i = 1, \dots, n$. When clear from the context, we omit the prefix $(X_1, \dots, X_n)$- in front of "box".

Definition 2.2. Let $X_1, \dots, X_n$ be definable sets. We say that $X_1, \dots, X_n$ are **orthogonal** if, for every $k_1, \dots, k_n \in \mathbb{N}$, every definable subset S of $X_1^{k_1} \times \dots \times X_n^{k_n}$ is the union of finitely many $(X_1^{k_1}, \dots, X_n^{k_n})$-boxes.

An example of the notion of orthogonal sets is provided by the following definition.

Definition 2.3. Given finitely many structures $\mathcal{X}_1, \dots, \mathcal{X}_n$, their **disjoint union** $\bigsqcup_i \mathcal{X}_i$ is the multi-sorted structure with a sort for each $\mathcal{X}_i$ and whose basic relations are the definable sets in the single structures $\mathcal{X}_i$. Notice that if X_i is the domain of $\mathcal{X}_i$, then $X_1, \dots, X_n$ are orthogonal in $\bigsqcup_i \mathcal{X}_i$.

The definition of orthogonality can be rephrased in terms of types using the following remark of Wagner. We include a proof to facilitate comparison with similar notions of orthogonality in the model-theoretic literature (see [19]), but we will not need this fact.

Remark 2.4 ([20, Remark 3.4])**.** Let $X_1, \dots, X_n$ be definable sets. The following conditions are equivalent:

(1) Every definable subset of $X_1 \times \cdots \times X_n$ is a finite union of $(X_1, \dots, X_n)$-boxes.
(2) For every type $p(x_1, \dots, x_n)$ over $\mathcal{M}$ with $p(x_1, \dots, x_n) \vdash X_1 \times \cdots \times X_n$, we have that $p_1(x_1) \cup \cdots \cup p_n(x_n) \vdash p(x_1, \dots, x_n)$ where p_i is the i-th projection of p.

Proof. To simplify notation we assume $n = 2$ and write X, Y for X_1, X_2.

Assume (1). Let $p(x, y)$ be a type over $\mathcal{M}$ concentrated on $X \times Y$. Let $\varphi(x, y)$ be a formula over $\mathcal{M}$ defining a subset of $X \times Y$. The set Z defined by φ is a finite union of boxes $U_i \times V_i$. It follows that $p_1(x) \cup p_2(y) \vdash p(x, y)$.

Assume (2). Let $Z \subseteq X \times Y$ be definable. We must show that Z is a finite union of boxes $U_i \times V_i$. For any type $p(x, y)$ containing the defining formula of Z, we have that $p_1(x) \cup p_2(y) \vdash (x, y) \in Z$. By compactness, there are $\varphi_1^p(x) \in p_1(x)$ and $\varphi_2^p(y) \in p_2(y)$, such that $\varphi_1^p(x) \wedge \varphi_2^p(y) \vdash (x, y) \in Z$. Again by compactness, $(x, y) \in Z$ is equivalent to a finite disjunction of formulas of the form $\varphi_1^p(x) \wedge \varphi_2^p(y)$ (if not we reach a contradiction considering a type containing the defining formula of Z and the negation of all the formulas $\varphi_1^p(x) \wedge \varphi_2^p(y)$). It follows that Z is a finite union of boxes $U_i \times V_i$ as desired. □

The following fact follows at once from the definition.

Proposition 2.5. *Let X, Y, Z be definable sets. Suppose that, for every positive integer n, all definable subsets of $X^n \times Y$ are a finite union of (X^n, Y)-boxes, and all definable subsets of $X^n \times Z$ are finite unions of (X^n, Z)-boxes. Then, for all n, every definable subset of $X^n \times Y \times Z$ is a finite union of $(X^n, Y \times Z)$-boxes.*

Proof. Let S be a definable subset of $X^n \times Y \times Z$. Given $z \in Z$, consider the fiber $S_z \subseteq X^n \times Y$ consisting of the pairs (x, y) such that $(x, y, z) \in S$. Let $E_z \subseteq X^n \times X^n$ be the following equivalence relation:

$$a \, E_z \, b \quad \iff \quad \forall y \in Y \ \ (a, y) \in S_z \longleftrightarrow (b, y) \in S_z \ .$$

Then $\{E_z \mid z \in Z\}$ is a family of subsets of $X^n \times X^n$ indexed by Z, so by the hypothesis applied to $X^{2n} \times Z$, it is finite. By the hypothesis on $X^n \times Y$, each equivalence relation E_z has finitely many equivalence classes; in fact, S_z is a finite union of (X^n, Y)-boxes and the equivalence classes of E_z are the atoms of the boolean algebra generated by the projections of these boxes on the component X^n. It follows that $E = \bigcap_z E_z \subseteq X^n \times X^n$ is again an equivalence relation with finitely many classes. On the other hand, for $a, b \in X^n$,

$$a\,E\,b \iff \forall y \in Y, z \in Z \;\; (a, y, z) \in S \longleftrightarrow (b, y, z) \in S\ .$$

We have thus proved that there are finitely many subsets of the form $\pi_{Y\times Z}(\pi^{-1}_{X^n}(x) \cap S)$ with $x \in X^n$, which is desired result. □

Corollary 2.6. *Let X, Y, Z be definable sets. If X is orthogonal to both Y and Z, then X is orthogonal to $Y \times Z$.*

Corollary 2.7. *Suppose $X_1, \dots, X_n$ are pairwise orthogonal definable sets. Then $X_1, \dots, X_n$ are orthogonal.*

Proof. If suffices to show that each X_i is orthogonal to the product of the other sets X_j. This follows by 2.6 and induction on n. □

We shall need the following result.

Corollary 2.8. *Let X and Y be definable sets. Then X and Y are orthogonal if and only if for every positive integer n, all definable subsets of $X^n \times Y$ are finite unions of (X^n, Y)-boxes.*

For the main results of this paper we make no saturation assumptions on the ambient structure $\mathcal{M}$. It is however worth mentioning that under a saturation assumption we can strengthen Corollary 2.8 as follows.

Proposition 2.9. *If $\mathcal{M}$ is $\aleph_0$-saturated, then X and Y are orthogonal if and only if all definable subsets of $X \times Y$ are finite unions of (X, Y)-boxes. The same conclusion holds without saturation provided $\mathcal{M}$ is o-minimal, or more generally if $\mathcal{M}$ eliminates the quantifier $\exists^\infty$.*

Proof. Suppose that all definable subsets of $X \times Y$ are finite unions of (X, Y)-boxes. Let S be a definable subset of $X^n \times Y$. It suffices to show that S is a finite union of (X^n, Y)-boxes (by Corollary 2.8). We reason by induction on n. The case $n = 1$ holds by the assumptions. Assume $n > 1$. For $t \in X$, consider the set

$$S_t = \{(x, y) \mid (t, x, y) \in S\} \subseteq X^{n-1} \times Y.$$

By induction, S_t is a finite union of (X^{n-1}, Y)-boxes. Let $R_t \subseteq Y \times Y$ be the equivalence relation defined by $uR_tv \iff (\forall x \in X^{n-1})\ (x, u) \in S_t \leftrightarrow (x, v) \in S_t$. Note that R_t has finitely many equivalence classes. By saturation (or elimination of $\exists^\infty$), there is a uniform bound $k \in \mathbb{N}$, such that for all $t \in X$, there are at most k equivalence classes modulo R_t.

We claim that there is a finite subset A of Y such that for all $t \in X$ each equivalence class of R_t intersects A. To this aim we prove, by induction on $j \leq k$, that there is a finite subset A_j of Y such that for all $t \in Y$ there are at most $k - j$ equivalence classes of R_t which do not intersect A_j. Clearly we can take A_0 to be the empty set. Let us construct A_{j+1}. Consider the definable family $(Q_t)_{t\in X}$ where $Q_t \subseteq Y$ is the union of the R_t-equivalence classes which intersect A_j. By the assumptions the family $(Q_t)_{t\in X}$ contains finitely many distinct subsets $Q_1, \dots, Q_l$ of Y. We obtain A_{j+1} adding to A_j a point in $Y \setminus Q_i$ for each $i = 1, \dots, l$ such that $Q_i \neq Y$. The claim is thus proved taking $A = A_k$.

Now we claim that the equivalence relation $R = \bigcap_{t\in X} R_t$ has finitely many equivalence classes. Indeed, given $a \in A$ consider the family $(P_{t,a})_{t\in X}$ where $P_{t,a} \subseteq Y$ is the R_t-equivalence class of a. For each $a \in A$, by the assumption there are finitely many sets of the form $P_{t,a}$ for $t \in X$. Each R_t-equivalence class is of the form $P_{t,a}$. Hence each R-equivalence class belongs to the finite boolean algebra generated by the sets $P_{t,a}$. The claim is thus proved.

Now for each R-equivalence class $B \subseteq Y$ and for each $y_1, y_2 \in B$, we have $\forall t \in X, \forall x \in X^{n-1}$ $(t, x, y_1) \in S \iff (t, x, y_2) \in S$. Thus $S \cap \pi_Y^{-1}B$ is a box where $\pi_Y : X^n \times Y \to Y$ is the projection. Therefore S is a finite union of (X^n, Y)-boxes. □

We now turn to the notion of internality.

Definition 2.10 ([19, Lemma 10.1.4])**.** Let $\mathcal{M}$ be a structure. Given two definable sets X and V, we say that X is **internal to** V, or V**-internal**, if there is a definable surjection from V^n to X, for some n.

Fact 2.11 ([2, Lemma 2.2])**.** *Let X and Y be orthogonal definable sets and let $(T_s \mid s \in S)$ be a definable family of subsets of a Y-internal set T indexed by an X-internal set S. Then the family contains only finitely many distinct sets $T_s \subseteq T$.*

Fact 2.12 ([2, Proposition 2.3])**.** *Let X and Y be orthogonal definable sets with $|X| \geq 2$, and S a definable subset of $X \times Y$. Then S is X-internal if and only if its projection onto Y is finite.*

It is clear that if X and Y are infinite and one of them is internal to the other, then the two sets are non-orthogonal. Also, if one of X and Y is finite, then the two sets are orthogonal.

Proposition 2.13. *Let X and Y be orthogonal definable sets. If U is internal to X and V is internal to Y, then U and V are orthogonal.*

Proof. Let $S \subseteq U^k \times V^l$ be definable. By the assumption there are definable surjective maps $f : X^m \to U^k$ and $g : Y^n \to V^l$.Then, by orthogonality, $(f \times g)^{-1}(S)$ is a finite union $\bigcup_i A_i \times B_i$ of (X^m, Y^n)-boxes. So $S = \bigcup_i f(A_i) \times g(B_i)$ is a finite union of (U^k, V^l)-boxes. □

3. Cohesive sets

In this section we work in an arbitrary structure $\mathcal{M}$, except for Theorem 3.4 where we assume o-minimality. We introduce and study the following two key notions, also mentioned in the introduction.

Definition 3.1. Let Z be a definable set.

- Z is **indecomposable** if for every orthogonal definable sets X, Y, if Z is internal to $X \times Y$, then Z is either internal to X or internal to Y.
- Z is **cohesive** if all definable sets X and Y not orthogonal to Z are not orthogonal to each other.

Proposition 3.2.

(1) *A definable set internal to a cohesive set is cohesive.*
(2) *Any cohesive set is indecomposable.*

Proof. (1) Let X be internal to Y. If A and B are non-orthogonal to X, then by Proposition 2.13 they are non-orthogonal to Y. Thus if Y is cohesive, so is X.

We prove (2). Let Z be a cohesive set and suppose that Z is internal to $X \times Y$ where X and Y are orthogonal. By definition there is $m \in \mathbb{N}$ and a surjective definable map $f : X^m \times Y^m \to Z$. Since X^m and Y^m are orthogonal, for the sake of our argument we can assume $m = 1$. So we have a surjective definable map $f : X \times Y \to Z$ and we need to show that Z is X-internal or Y-internal. For $x \in X$ and $y \in Y$, let $f_x(y) = f(x, y) = f^y(x)$. The image of f_x is Y-internal and the image of f^y is X-internal. These two images are then orthogonal (Proposition 2.13), so they cannot be both infinite by the hypothesis on Z. It follows that either $\mathrm{Im}(f_x)$ is finite for all $x \in X$ or $\mathrm{Im}(f^y)$ is finite for all $y \in Y$. By symmetry let us assume that $\mathrm{Im}(f^y) \subseteq Z$ is finite for all y. Let $E_y \subseteq X^2$ be the equivalence relation defined by $xE_yx' \iff f(x, y) = f(x', y)$. Then E_y is a definable equivalence relation on X of finite index. Since $(E_y \subseteq X \times X \mid y \in Y)$ is a definable family of subsets of an X-internal set indexed by a Y-internal set, by orthogonality there are finitely many sets of the form E_y for $y \in Y$ (Fact 2.11). The intersection $E = \bigcap_{y \in Y} E_y$ is then again a definable equivalence relation of finite index on X. Let $x_1, \dots, x_k$ be representatives for the equivalence classes of E. Then Z is the image of the restriction of f to $\bigcup_{i \leq k}\{x_i\} \times Y$, and therefore it is Y-internal. □

Proposition 3.3. *If X and Y are cohesive and non-orthogonal, then $X \times Y$ is cohesive.*

Proof. Let A and B be non-orthogonal to $X \times Y$. We need to prove that A and B are non-orthogonal. By Corollary 2.6 either X or Y is non-orthogonal to A. Similarly, either X or Y is non-orthogonal to B. There are four cases to consider, but by symmetry we can consider the following two cases.

Case 1. X is non-orthogonal to both A and B.

Case 2. X is non-orthogonal to A, and Y is non-ortogonal to B.

In the first case by the cohesiveness of X the sets A and B are non-orthogonal. In the second case, since X and Y are non-orthogonal and Y is non-orthogonal to B, by the cohesiveness of Y we conclude that X is non-orthogonal to B, so we have a reduction to the first case. □

We now turn to groups definable in o-minimal structures.

Theorem 3.4. *Let G be a definable group of dimension* 1 *in an o-minimal structure $\mathcal{M}$. Then G is cohesive.*

Proof. Let A and B be definable sets non-orthogonal to G. We need to prove that A and B are not orthogonal. Let us first concentrate on A. By Corollary 2.8 there are $n \in \mathbb{N}$ and a definable relation $R \subseteq A^n \times G$ which is not a finite union of boxes. Let $\mathcal{P}(G)$ be the power set of G and let $f \colon A^n \to \mathcal{P}(G)$ be defined by $f(a) = \{g \in G \mid (a, g) \in R\}$. Since $f(a)$ has dimension ≤ 1, its boundary $\delta f(a)$, closure minus interior, is finite. By the assumption on R, the image of f is infinite, thus also the image of $\delta f \colon A^n \to \mathcal{P}(G)$ is infinite, since in a group of dimension 1 there can only be finitely many disjoint definable subsets with a given boundary (because, given a cell decomposition compatible with the boundary any definable subset with the given boundary is a union of some cells of the decomposition). Now recall that in an o-minimal structure a definable family of finite sets is uniformly finite. So there is $k \in \mathbb{N}$ such that $\delta f(a)$ has at most k elements for every $a \in A^n$. By ordering the points of $\delta f(a)$ lexicographically, we have a map $h\colon A^n \to G^k$ with infinite image. It follows that there is $i \leq k$ such that $\pi_i \circ h\colon A^n \to G$ has infinite image, where $\pi_i\colon G^k \to G$ is the projection onto the i-th component. We have thus proved that there is a definable map $f_A\colon A^n \to G$ with infinite image. Similarly, there is $l \in \mathbb{N}$ and a definable map $f_B\colon B^l \to G$ with infinite image. Infinite definable subsets of a one-dimensional group have non-empty interior, thus, composing with a group translation, we can assume that the two images have an infinite intersection. The relation $xQy :\iff f_A(x) = f_B(y)$ then witnesses the non-orthogonality of A and B. $\square$

Remark 3.5. Notice that, by Theorem 3.4, any o-minimal expansion $\mathcal{M}$ of a group is cohesive, hence all sets definable in $\mathcal{M}$ are cohesive by 3.2. For Theorems 8.4 and 8.7 it is, therefore, important to work in an arbitrary o-minimal structure.

4. Splitting and decomposition

In this section we work in an arbitrary structure $\mathcal{M}$. Fix definable orthogonal sets $X_1, \dots, X_n$. We introduce the notions of splitting and decomposition for definable functions and groups, respectively, contained in products of the sets X_i, as follows.

Definition 4.1. We let $\mathrm{Def}(\Pi_i X_i)$ be the collection of all definable sets in $\mathcal{M}$ that are contained in some cartesian product $\prod_{j=1}^{k} X_{t(j)}$ with $t(j) \in \{1, \dots, n\}$, for all $j = 1, \dots, k$. We say that a function f is in $\mathrm{Def}(\Pi_i X_i)$ if its graph belongs to $\mathrm{Def}(\Pi_i X_i)$.

Notation 4.2. If $S \subseteq \prod_{j=1}^{k} X_{t(j)}$ we define

$$\pi_i : S \to X_i^{k_i}$$

as the projection of S onto the X_i-components, where k_i is the number of indexes j with $t(j) = i$. For instance, if $S \subseteq X_1 \times X_2 \times X_1$, then $\pi_1 : S \to X_1^2$ maps (a, b, c) to (a, c).

We now introduce the notion of splitting, which will be a crucial tool for our proofs.

Definition 4.3. Let $f : A \to B$ be a function in $\mathrm{Def}(\Pi_i X_i)$. We say that f **splits** (with regard to $X_1, \dots, X_n$) if for every $x, y \in \mathrm{dom}(f)$ and every $i \leq n$,

$$\pi_i(x) = \pi_i(y) \ \Rightarrow \ \pi_i(f(x)) = \pi_i(f(y)).$$

That is, $\pi_i(f(x))$ depends only on $\pi_i(x)$. Note that if f splits, then up to a permutation of the indexes we can write $f = f_1 \times \cdots \times f_n | A$ where $f_i : \pi_i(A) \to \pi_i(B)$.

We will be using the following two facts without specific mentioning.

Fact 4.4. *Let $f : A \to B$ be a function in* $\mathrm{Def}(\Pi_i X_i)$. *Then there is a finite partition $\mathcal{D}$ of* $\mathrm{dom}(f)$ *into definable sets, such that for each $D \in \mathcal{D}$, the restriction of f to D splits.*

Proof. By orthogonality of $X_1, \dots, X_n$, up to a permutation of the variables, f is a finite disjoint union $\bigcup_j f_j$, where $f_j = U_{1,j} \times \dots \times U_{n,j}$ and $U_{i,j} \subseteq \pi_i(A) \times \pi_i(B)$. We conclude observing that each f_j splits. ☐

Splitting is preserved under composition in the following sense.

Fact 4.5. *Let* $h, f_1, \dots, f_m$ *be maps in* $\mathrm{Def}(\Pi_i X_i)$*, and suppose that the map* $h \circ (f_1 \times \dots \times f_m)$ *is defined (in the sense that the range of* $f_1 \times \ldots \times f_m$ *is contained in the domain of* h*). If* $h, f_1, \dots, f_m$ *split, so does* $h \circ (f_1 \times \dots \times f_m)$*.*

Proof. Straightforward. ☐

We now turn to definable groups. Recall that $X_1, \dots, X_n$ are orthogonal.

Definition 4.6. A definable group G **admits an orthogonal decomposition** (or decomposition for short) with respect to $X_1, \dots, X_n$, if there are definable subsets $A_1, \dots, A_n$ of G, such that $G = A_1 \dots A_n$, and A_i is internal to X_i, for all i.

Note that, since a set internal to a cohesive set is cohesive (Proposition 3.2(1)), a definable group admits a (cohesive) orthogonal decomposition as in the introduction if and only if there are definable (cohesive) orthogonal sets $X_1, \dots, X_n$, such that G admits a decomposition with respect to $X_1, \dots, X_n$.

Observation 4.7. Let H be a finite index subgroup of G. If H admits an orthogonal decomposition with respect to $X_1, \dots, X_n$, then so does G.

Our goal in the next sections is to prove that if G is definable in an o-minimal structure and internal to $X_1 \times \ldots \times X_n$, then it admits a decomposition in the sense above. A relevant case is when G is contained in – as opposed to being internal to – $X_1 \times \ldots \times X_n$. In this situation, by Remark 4.8 below, if the group operation of G splits, then G admits a decomposition. The converse, however, is not true (Example 4.9). On the other hand, if G is a direct product of groups $G_1, \dots, G_n$ with $G_i \subseteq X_i$ for all i, then the group operation obviously splits. In the appendix, we will give an example in which the group operation splits, yet the group is not even definably isomorphic to a direct product.

The rest of this section contains some remarks that help to demonstrate the newly defined notions.

Remark 4.8. Let (G, μ, e) be a definable group with $G \subseteq X_1 \times \ldots \times X_n$. Then the following are equivalent:

(1) the group operation μ splits (with respect to $X_1, \dots, X_n$);
(2) there are definable groups $H_1, \dots, H_n$, contained in $X_1, \dots, X_n$, respectively, such that $G < H_1 \times \ldots \times H_n$;
(3) there are definable groups $H_1, \dots, H_n$, contained in $X_1, \dots, X_n$, respectively, such that G is a finite index subgroup of $H_1 \times \ldots \times H_n$;

and, if either of these holds, then G admits a decomposition.

Proof. Assume (1). Consider the projections $H_1, \dots, H_n$ of G on $X_1, \dots, X_n$, respectively. The group operation μ, by the splitting hypothesis, takes the form

$$\mu((x_1, \dots, x_n), (y_1, \dots, y_n)) = (\mu_1(x_1, y_1), \dots, \mu_n(x_n, y_n))$$

It is straightforward to check that μ_i is a group operation on H_i, and (2) is thus established.

Now assume (2). Let $\pi_i : \prod_i X_i \to X_i$ be the projection onto the i-th component. Replacing H_i with $\pi_i(G)$ we can assume, without loss of generality, that $\pi_i(G) = H_i$ for each i. We prove that G has finite index in $H_1 \times \ldots \times H_n$. Let

$$p_i : \prod_j X_j \to \prod_{j \neq i} X_j$$

be the projection that omits the i-th coordinate. Fix an index i and consider an element k of $\prod_{j \neq i} H_j$. Let $H_i(k) = \pi_i(G \cap p_i^{-1}(k))$. Observe that $L_i := H_i(p_i(e))$ is a subgroup of H_i. We claim that $L := L_1 \times \ldots \times L_n$ has finite index in $H_1 \times \ldots \times H_n$. Assuming the claim, since $L < G$, also G must have finite index. It suffices to prove that L_i has finite index in H_i. The coset hL_i, for $h \in H_i$, coincides with $H_i(p_i(g))$ for any $g \in G \cap \pi_i^{-1}(h)$. Thus, in particular, the cosets of L_i belong to the family $H_i(-)$ indexed over $\prod_{j \neq i} X_j$. This family is finite by Fact 2.11, and this concludes the proof of (3).

It is immediate that (2), hence also (3), implies (1). It remains to show that G admits a decomposition. To this aim, observe that L admits a decomposition: it is, in fact, the product of the subgroups $G \cap p_i^{-1}(p_i(e)) \cong L_i$. We conclude by Observation 4.7. $\square$

Example 4.9. We give an example of a group whose operation does not split, even up to definable isomorphism, but the group admits a decomposition. Let $\mathcal{M} = \mathcal{R}_1 \sqcup \mathcal{R}_2$ where $\mathcal{R}_1$ and $\mathcal{R}_2$ are isomorphic copies of $(\mathbb{R}, <, +)$ and note that their domains R_1 and R_2 are orthogonal in $\mathcal{M}$. Let $(R_1 \times R_2, +, 0)$ be the product group. Fix $a \in [0, 1)$. Consider the lattice $\Lambda = \mathbb{Z} \cdot (0, 1) + \mathbb{Z} \cdot (1, a) \subseteq R_1 \times R_2$ and the definable set $[0, 1)^2 \subseteq R_1 \times R_2$. Define $G = ([0, 1)^2, \mu, 0)$, where $\mu = + \mod \Lambda$. Clearly, $G = A_1 + A_2$, where $A_1 = [0, 1) \times \{0\}$ and $A_2 = \{0\} \times [0, 1)$, and hence it admits a decomposition with respect to R_1, R_2. It is easy to see that if $a \neq 0$, then μ does not split. Moreover, $a \in \mathbb{Q}$, if and only if G is definably isomorphic to a group whose operation splits, if and only if G is definably isomorphic to a direct product of groups definable in R_1 and R_2, respectively.

5. Compact domination

Let again $\mathcal{M}$ be an arbitrary structure and G a definable group. We call a definable set $S \subseteq G$ *left-generic* (*right-generic*) if finitely many left-translates (respectively right-translates) of S cover G. We call S *generic* if it both left-generic and right-generic (which are equivalent when G has *finitely satisfiable generics* (*fsg*) [10, Proposition 4.2]). We are mainly interested in o-minimal structures, but in this section we consider the larger NIP class. We recall that NIP structures include both the o-minimal structures (e.g. the real field) and the stable structures (e.g. the complex field). If $\mathcal{M}$ is a NIP structure and G is *compactly dominated* (see [17] for the definitions), then G has *fsg* [17, Theorem 8.37], [18, Proposition 3.23], [17, Proposition 8.33]. Compact domination is a model-theoretic form of compactness: for instance a semialgebraic linear group $G < GL(n, \mathbb{R})$ is compactly dominated if and only if it is compact. The following proposition subsumes all we need about these notions.

Proposition 5.1. *Let $\mathcal{M}$ be a NIP structure and G a compactly dominated group definable in $\mathcal{M}$. Then:*

1. *If the union of two definable subsets of G is generic, then one of the two is generic.*
2. *Suppose $G = HK$, where H and K are definable subgroups and K is normal. Then $S \subseteq G$ is generic if and only if it contains a set of the form AB where A is a generic subset of H and B is a generic subset of K.*
3. *If G and H are compactly dominated, then $G \times H$ is compactly dominated.*

Proof. Point (1) holds for all groups with *fsg* [10, Proposition 4.2].

To prove point (2) we may assume $\mathcal{M}$ is κ-saturated for some sufficiently big cardinal κ. We make use of the *infinitesimal subgroup* G^{00} (see [17] for the definition). Specifically, we need to observe that in a compactly dominated group G, a subset S is generic if and only if some translate of S contains G^{00} [1, Proposition 2.1]. If $G = HK$ with $H < G$ and $K \lhd G$, then $G^{00} = (HK)^{00} = H^{00}K^{00}$ [4, Theorem 4.2.5] (this holds without the hypothesis that G is compactly dominated). Now suppose $S \subseteq G$ is generic. So there is $g \in G$ such that $gG^{00} \subseteq S$. Now, K^{00} is a normal subgroup of G (being a definably characteristic subgroup of K), and so, writing $g = kh$ for $h \in H$ and $k \in K$, we get $gG^{00} = khK^{00}H^{00} = (kK^{00})(hH^{00}) \subseteq S$. By κ-saturation there are definable sets U, V with $K^{00} \subseteq U \subseteq K$ and $H^{00} \subseteq V \subseteq H$ such that $(kU)(hV) \subseteq S$. By construction, kU and hV are generic in K and H, respectively.

Point (3) follows from [18, Corollary 3.17] (the product of smooth measures is smooth) and the fact that a group is compactly dominated if and only if it has a smooth left-invariant measure [17, Theorem 8.37]. □

Fact 5.2. *Definably compact groups in an o-minimal structure are compactly dominated.*

Proof. This was first proved for o-minimal expansions of a field in [9]. We give some bibliographical pointers to obtain the result for arbitrary o-minimal structures. First one shows that a definably compact group in an o-minimal structures has *fsg* [12, Theorem 8.6]. From this one deduces that G admits a *generically stable left-invariant measure* [17, Proposition 8.32]. In an o-minimal structure (and more generally in a *distal* structure), a generically stable measure is *smooth* [17, Proposition 9.26]. Finally, a NIP group with a smooth left-invariant measure is compactly dominated [17, Theorem 8.37]. □

Proposition 5.3. *Let G be a compactly dominated group. Given two generic sets $A \subseteq G$ and $B \subseteq G$, there is $h \in G$ such that $A \cap hB$ is generic.*

Proof. There is a finite subset $I \subseteq G$ such that $A \subseteq \bigcup_{h \in I} hB$. By Proposition 5.1(1), there is $h \in I$ such that $A \cap hB$ is generic. □

6. Decomposition of compactly dominated abelian groups (NIP)

In this section $\mathcal{M}$ is a NIP structure and $X_1, \dots, X_n$ are orthogonal definable sets.

Theorem 6.1. *Let G be a compactly dominated abelian group contained in $X_1 \times \cdots \times X_n$. Then G admits a decomposition with respect to $X_1, \dots, X_n$.*

Proof. Let $P_n : G^n \to G$ be the function sending $(x_1, \dots, x_n)$ to $\prod_{i=1}^n x_i$. Then $P_n \in \mathrm{Def}(\Pi_i X_i)$. Since G is compactly dominated, so is G^n (Proposition 5.1(3)). By Fact 4.4 and Proposition 5.1(1), P_n splits on a generic definable set $S \subseteq G^n$. By Proposition 5.1(2) we find definable generic sets $A_1, \dots, A_n \subseteq G$ such that $A_1 \times \dots \times A_n \subseteq S$. By Proposition 5.3 we find $a_1, \dots, a_n \in G$ such that the set

$$U = \bigcap_{i=1}^n a_i A_i$$

is generic in G. Again by Fact 4.4, there is a generic set $D \subseteq U$ such that for every $i = 1, \dots, n$ the function

$$f_i : x \in D \mapsto a_i^{-1} x$$

splits on D. Since $D \subseteq a_i A_i$, the image of f_i is contained in A_i. It follows that the function

$$f_1 \times \dots \times f_n : D^n \to A_1 \times \dots \times A_n$$

splits. By Fact 4.5, since P_n splits on $A_1 \times \dots \times A_n$, we deduce that the function

$$f = P_n \circ (f_1 \times \dots \times f_n)$$

splits on D^n. Since G is abelian,

$$f(x_1, \dots, x_n) = a \prod_{i=1}^n x_i \tag{6.1}$$

where $a = \prod_{i=1}^n a_i^{-1}$. By the orthogonality assumption, D is a finite union of sets of the form $U_1 \times \dots \times U_n$ with $U_i \subseteq X_i$. By Proposition 5.1(1) one of these sets is generic, so by replacing D with a smaller set we can assume that

$$D = U_1 \times \dots \times U_n.$$

Let $\pi_i : \prod_i X_i \to X_i$ be the projection onto the i-th component and let

$$p_i : \prod_j X_j \to \prod_{j \neq i} X_j$$

be the projection that omits the i-th coordinate. Now fix $k \in D$ and let

$$D_i = p_i^{-1} p_i(k) \cap D = \{x \in D \mid \forall j \neq i \ \pi_j(x) = \pi_j(k)\}.$$

Notice that D_i is U_i-internal, hence *a fortiori* it is X_i-internal. We claim that

$$k^{n-1} D \subseteq D_1 \dots D_n \tag{6.2}$$

To prove the claim, let $g \in D$ and let $x_i \in G$ be such that

$$\pi_i(x_i) = \pi_i(g) \ \& \ \pi_j(x_i) = \pi_j(k) \text{ for } j \neq i.$$

Notice that $x_i \in D_i$. Since f splits on D^n, the value of $\pi_i f(x_1, \dots, x_n)$ does not change if we replace x_i with g and x_j with k for $j \neq i$. By Equation (6.1) we then obtain

$$\pi_i f(x_1, \dots, x_n) = \pi_i(a k^{n-1} g).$$

Since this holds for every i, we deduce that

$$f(x_1, \dots, x_n) = a k^{n-1} g$$

and since $f(x_1, \dots, x_n) = a \prod_{i=1}^n x_i$ we obtain Equation 6.2.

We have thus shown that a translate of D is contained in a product of X_i-internal sets. Since D is generic, the same holds for G, so G is a product of X_i-internal sets. $\square$

7. Decomposition of definably compact abelian groups (o-minimal)

In this section $\mathcal{M}$ is an o-minimal structure and $X_1, \dots, X_n$ are orthogonal definable sets. We prove the following variant of Theorem 6.1, where the NIP hypothesis is replaced by o-minimality, but the group is only assumed to be internal to $X_1 \times \dots \times X_n$.

Theorem 7.1. *Let G be a definably compact abelian group internal to $X_1 \times \dots \times X_n$. Then G admits a decomposition with respect to $X_1, \dots, X_n$.*

We need the following lemma.

Lemma 7.2. *Let X be an infinite set definable in an o-minimal structure $\mathcal{M}$. Then there is a definable set $Y = [X]^{o\text{-}min}$ such that:*

1. *X and Y are internal to each other;*
2. *there is $k \in \mathbb{N}$ and a definable injective map from X to Y^k;*
3. *Y has a definable linear order $\prec$ such that $(Y, \prec)$ with the induced structure from $\mathcal{M}$ is o-minimal.[1] We call the resulting structure $\mathcal{Y}$ an* **o-minimal envelope** *of X.*

Moreover we have:

4. *if $X_1, \dots, X_n$ are definable infinite sets, there there are definable sets $X'_1, \dots, X'_n$ such that X'_i is bi-internal to X_i for each i and $[X_1 \times \dots \times X_n]^{o\text{-}min}$ can be definably embedded in $X'_1 \times \dots \times X'_n$.*

Proof. Suppose $X \subseteq M^m$ and let $\pi_i : M^m \to M$ be the projection onto the i-th coordinate ($i = 1, \dots, m$). Fix parameters $a_1 < \dots < a_m$ in M, let $Z_i = \{a_i\} \times \pi_i(X)$ and let $Z = \bigcup_i Z_i \subseteq M \times M$. Then Z has dimension 1 and is bi-internal to X. Each $\pi_i(X) \subseteq M$ is a finite union of points and intervals with the induced order from $\mathcal{M}$, and we have an induced order on Z_i via the obvious bijection. We then order Z lexicographically by stipulating that all the elements of Z_i preceed all the elements of Z_{i+1}. We thus obtain a definable linear order $<_Z$ on Z, but notice that Z need not be dense, and even if it is, Z_i need not be a finite union of points and intervals in the order $(Z, <_Z)$ (e.g. Z_i may be bounded but with no supremum in Z). We can remedy this by adding and removing from Z finitely many points, thus obtaining a definable set Y with a dense linear order $\prec$ without end-points which agrees with $<_Z$ on $Y \cap Z$ (for example if $\mathcal{M} = \mathbb{R}$ and $Z = [1,2) \cup (3,4] \cup [5,6) \subset M$, we can add the point 2 and remove 1 and 4). The set Y can be constructed as a disjoint union $\bigcup_{i=1}^m Y_i$ where, for each i, the symmetric difference $Y_i \triangle Z_i$ is finite and Y_i is a finite union of intervals and points of $(Y, \prec)$, so in particular it is definable in $(Y, \prec)$.

To prove (3) it suffices to observe that any definable subset D of Y is the union of its traces $D \cap Y_i$ on the various Y_i, so it is a finite union of points and intervals of $(Y, \prec)$.

To prove the remaining points we make some preliminary observations (where all the relevant sets are assumed to be definable and n is arbitary):

---

[1] The induced structure contains a predicate for each $\mathcal{M}$-definable subset of Y^n for $n \in \mathbb{N}$.

i) $A_1 \times \ldots \times A_n$ can be definably embedded in $(A_1 \cup \ldots \cup A_n)^n$.

ii) If A is infinite and F is finite, $A \cup F$ can be embedded in $A \times A$.

iii) $A_1 \cup \ldots \cup A_n$ can be definably embedded in $A_1 \times \ldots \times A_n \times B$ where B is any infinite set. If, for some i, the set A_i is infinite, we can take $B = A_i$ and embed $A_1 \cup \ldots \cup A_n$ in $A_1 \times \ldots \times A_n \times A_i$, hence *a fortiori* in $A_1^2 \times \ldots \times A_n^2$.

By i) and iii) $A_1 \times \ldots \times A_n$ is bi-internal to $A_1 \cup \ldots \cup A_n$ provided at least one of the sets A_i is infinite.

The set X is included in the cartesian product of its projections $\pi_i(X)$, so it can be embedded in the cartesian product of the sets $Z_i = \{a_i\} \times \pi_i(X)$. On the other hand, by i), $Z_1 \times \ldots \times Z_n$ can be embedded in $(Z_1 \cup \ldots \cup Z_m)^m$. Moreover, $Z_1 \cup \ldots \cup Z_m$ differs from the o-minimal envelope Y by a finite set, so by ii) it can be embedded into Y^2 (note that Y must be infinite as X is assumed to be infinite). It follows that X can be embedded into Y^{2m}, thus proving (2).

By a similar argument, using ii), Y can be embedded into $(Z_1 \cup \ldots \cup Z_n)^2$, which in turn can be embedded, by iii), into the cartesian product $\prod_i \pi_i(X)^4$ of the projections of X to the 4-th power. In particular X and Y are bi-internal and we get (1).

To prove point (4), let $X = X_1 \times \ldots \times X_n$ where each $X_i \subseteq M^{n_i}$ is infinite. Let X_i' be the product of the projections of X_i to the 4-th power. Then the o-minimal envelope of X can be embedded in $X_1' \times \ldots \times X_n'$ and we get (4). □

Definition 7.3 ([8, Def. 1.1])**.** Let X, Y be definable sets, E_1, E_2 two definable equivalence relations on X and Y respectively. A function $f : X/E_1 \to Y/E_2$ is called definable if the set $\{(x, y) \in X \times Y \mid f([x]_{E_1}) = [y]_{E_2}\}$ is definable.

Proposition 7.4. *Let G be a definable group in an o-minimal structure $\mathcal{M}$ and let $X_1, \ldots, X_n$ be definable sets in $\mathcal{M}$. If G is internal to $X_1 \times \cdots \times X_n$, then there is an injective definable map $f : G \to X_1' \times \cdots \times X_n'$, where each X_i' is bi-internal to X_i ($i = 1, \ldots, n$).*

Proof. We can assume that G is infinite as otherwise the result is clear. We can then also assume that each X_i is infinite, because if X_n is finite, say, then G is internal to $X_1 \times \ldots \times X_{n-1}$.

By Lemma 7.2, G is internal to $Y = [X_1 \times \cdots \times X_n]^{\text{o-min}}$, so it can be considered as an interpretable group in the o-minimal structure $\mathcal{Y}$. By [8] interpretable groups in an o-minimal structure are definably isomorphic to definable groups (in the sense of Definition 7.3). It follows that there is a definable injective map $f : G \to Y^k$ for some $k \in \mathbb{N}$. By the construction of the o-minimal envelope, Y can be embedded into a product of sets Y_i, where Y_i is bi-internal to X_i (Lemma 7.2(4)). Now it suffices to take $X_i' = Y_i^k$. □

We are now ready to finish the proof of the theorem.

Proof of Theorem 7.1. By Proposition 7.4 there are definable sets $X_1', \ldots, X_n'$ with X_i' bi-internal to X_i such that G is definably isomorphic to a group G' contained in $X_1' \times \cdots \times X_n'$. Since G is definably compact, G' also is. By Theorem 6.1 G' admits a decomposition with respect to $X_1', \ldots, X_n'$, hence also with respect to $X_1, \ldots, X_n$. Hence so does G. □

8. Decomposition: general case (o-minimal)

In this section, we prove our main theorems (8.4 and 8.7). Fix an o-minimal structure $\mathcal{M}$ and definable orthogonal sets $X_1, \dots, X_n$. We first prove that the existence of decompositions is preserved under taking central extensions.

Lemma 8.1. *Let $1 \to N \to G \xrightarrow{f} H \to 1$ be a definable exact sequence of definable groups internal to $X_1 \times \dots \times X_n$ with $N < Z(G)$. If N and H admit decompositions with respect to $X_1, \dots, X_n$, then G too admits a decomposition with respect to the same orthogonal sets.*

Proof. By assumption, we can write $H = H_1 \dots H_n$ and $N = N_1 \dots N_n$, where H_i and N_i are X_i-internal definable sets (not necessarily subgroups). We have

$$G = f^{-1}(H_1) \dots f^{-1}(H_n).$$

By [7, Theorem 2.5], there is a definable section $\sigma : H \to G$. Since $f^{-1}(H_i) = \sigma(H_i)N$, we have

$$G = \sigma(H_1)N \dots \sigma(H_n)N.$$

Since $N = N_1 \dots N_n$ is contained in the center of G, it follows that $G = U_1 \dots U_n$, where U_i is the X_i-internal set $\sigma(H_i)N_i \dots N_i$ (n occurrences of N_i). ☐

We can now handle the abelian case.

Proposition 8.2. *Let G be a definable group internal to $X_1 \times \dots \times X_n$. If G is abelian, then G admits a decomposition with respect to $X_1, \dots, X_n$.*

Proof. We reason by induction on dimension. For $\dim(G) = 1$, G is cohesive by Theorem 3.4, so it is indecomposable (Proposition 3.2), hence it is internal to one of the X_i. Let $\dim(G) > 1$. If G is definably compact, then we conclude by 7.1. If not, then by [13, Theorem 1.2], G has a 1-dimensional torsion-free definable subgroup $H < G$. Since $\dim(H) = 1$, H admits a decomposition. By induction on dimension, so does G/H. Therefore, since G is abelian, we can apply Lemma 8.1. ☐

The following fact must be well-known, but we include a proof for completeness.

Fact 8.3. *Let G be a connected group definable in an o-minimal structure. If $Z(G)$ is finite, then $G/Z(G)$ is centerless.*

Proof. Let $a \in G$, such that $aZ(G)$ is in the center of $G/Z(G)$. We want to prove that $a \in Z(G)$. We have that for all $b \in G$, $a^{-1}b^{-1}ab \in Z(G)$. Since G is connected, the image of the map $f : G \to Z(G)$ sending $b \in G$ to $a^{-1}b^{-1}ab \in Z(G)$ is connected. Since $Z(G)$ is finite, f must be constant. Since f maps the identity $e \in G$ to e, we have $a^{-1}b^{-1}ab = e$. Thus $a \in Z(G)$, as needed. ☐

We can now prove our first main result.

Theorem 8.4. *Let G be internal to $X_1 \times \dots \times X_n$ where $X_1, \dots, X_n$ are orthogonal definable sets. Then G admits a decomposition with respect to $X_1, \dots, X_n$.*

Proof. We observe that, since the connected component of the identity G^0 has finite index in G, by Observation 4.7, it suffices to find a decomposition of G^0. We may thus assume that G is connected.

We prove, now, the theorem, by induction on $\dim(G)$. For $\dim(G) = 0$, it is obvious. Assume $\dim(G) > 0$.

Case 1. Suppose that the centre $Z(G)$ is finite. Then $Z(G)$ admits an orthogonal decomposition, and by Lemma 8.1 it suffices to prove that $H = G/Z(G)$ has a decomposition. Since $Z(G)$ is finite, H is centerless (Fact 8.3).

By [8], H is definably isomorphic to a definable group. By [14, Theorems 3.1 and 3.2], it follows that there are definable real closed fields $R_1, \dots, R_k$ and definable linear groups $H_i < GL(n, R_i)$ such that H is definably isomorphic to $H_1 \times \dots \times H_k$. A definable real closed field in an o-minimal structure has dimension 1 [13, Theorem 4.1]. We can arrange so that R_i is internal to $X_1 \times \dots \times X_n$ because we can apply the cited results inside the o-minimal structure $[X_1 \times \dots \times X_n]^{\text{o-min}}$ (as defined in Lemma 7.2). By Theorem 3.4 R_i is cohesive, so it is internal to some X_j by Proposition 3.2(2). It follows that each H_i is internal to some X_j. We have thus proved that H admits a decomposition with respect to the orthogonal sets $X_1, \dots, X_n$. Therefore we can conclude by Lemma 8.1.

Case 2. Suppose $Z(G)$ is infinite. By the abelian case (Proposition 8.2), $Z(G)$ has a decomposition with respect to $X_1, \dots, X_n$. By induction on the dimension, $G/Z(G)$ has a decomposition too. Therefore we can again conclude by Lemma 8.1. ☐

As a by-product of the proof we obtain:

Proposition 8.5. *If G is definably simple, then G is cohesive.*

Proof. Let G be definably simple. By the proof of Theorem 8.4 there is a definable real closed field R such that G is definably isomorphic to a definable subgroup of $GL(n, R)$. Since $\dim(R) = 1$, by Theorem 3.4 R is cohesive. But $GL(n, R)$ is internal to R, thus all its definable subsets are cohesive. ☐

We now proceed towards our second main result.

Lemma 8.6. *Let G be an interpretable group. Then there are cohesive orthogonal definable sets $X_1, \dots, X_n$ and an injective definable map $h : G \to \prod_{i=1}^n X_i$.*

Proof. By [8, Theorem 3], there is a definable injective map $f : G \to \prod_{j=1}^k G_j$ where each G_j is a 1-dimensional definable group. Define a relation R on $\{1, \dots, k\}$ by iRj if G_i and G_j are not orthogonal. By Theorem 3.4, the groups G_i are cohesive, hence R is an equivalence relation. Suppose there are n equivalence classes. Let X_i be the product of the groups G_j, with j in the i-th class. Using f, it is easy to define (by a permutation of the coordinates on the image) the injective map $h : G \to \prod_{i=1}^n X_i$.

The sets X_i are cohesive by Proposition 3.3 and mutually orthogonal by Corollary 2.6. ☐

Recall the notion of cohesive orthogonal decomposition from the introduction.

Theorem 8.7. *If G is a group interpretable in an o-minimal structure $\mathcal{M}$, then G admits a cohesive orthogonal decomposition $G = A_1 \dots A_n$.*

Proof. Let $X_1, \dots, X_n$ be the sets provided by Lemma 8.6. By Theorem 8.4, there are X_i-internal sets $A_i \subseteq G$, $i = 1, \dots, n$, such that

$$G = A_1 \dots A_n.$$

Clearly, the A_i's are orthogonal and cohesive, as they inherit those properties from the X_i's. $\square$

Corollary 8.8. *If G is infinite, in Theorem 8.7 we can choose each A_i to be infinite. In this case, the number n is an invariant of G up to definable isomorphism. Indeed, if $G = B_1 \dots B_m$ is another cohesive orthogonal decomposition of G with $B_1, \dots, B_m$ infinite, then $m = n$ and each B_i is bi-internal to a unique A_j.*

Proof. Suppose that G is infinite and fix a cohesive orthogonal decomposition $G = A_1 \dots A_n$. Then at least one A_i is infinite, say A_1. If some A_i is finite we may replace A_1 with $A_1 A_i$ and omit A_i obtaining another cohesive orthogonal decomposition. So we may assume that $A_1, \dots, A_n$ are all infinite. Now consider another cohesive orthogonal decomposition $G = B_1 \dots B_m$ into infinite sets. Fix B_i and observe that B_i is internal to G, which is internal to the cartesian product $A_1 \times \dots \times A_n$. Since B_i is indecomposable, it must be internal to some A_j. Moreover, j must be unique, because if B_i is internal to both A_j and A_h, with $j \neq h$, then it is non-orthogonal to both, so by cohesiveness of B_i, A_i and A_h are non-orthogonal, a contradiction. The argument also shows that $m = n$ and B_i is in fact bi-internal to the corresponding A_j. $\square$

9. Locally definable groups

In this section, we fix again an o-minimal structure $\mathcal{M}$, and prove Theorem 9.2. Let us first recall a few definitions concerning locally definable sets and groups (which can, in fact, be given for arbitrary structures).

Definition 9.1. A **locally definable set** X is a countable union of definable sets together with a given presentation as such a countable union. A subset of a locally definable set X is said to be **definable** if it is definable in $\mathcal{M}$ and is contained in the union of finitely many sets of the presentation of X (this last condition is automatically satisfied if $\mathcal{M}$ is $\aleph_0$-saturated). A **compatible** subset of a locally definable set X is a subset which intersects every definable subset of X at a definable set. A compatible subset is **discrete** if it intersects every definable set into a finite set. A **locally definable function** is a function between locally definable sets whose restriction to each definable set is definable. Similar definitions apply to groups. A locally definable group is a locally definable set with a locally definable group operation. We can then speak of compatible and discrete subgroups and locally definable homomorphisms. A locally definable group is **definably generated** if it is generated by a definable subset.

Theorem 9.2. *Let G be an abelian group definable in the disjoint union $\bigsqcup_i \mathcal{X}_i$ of finitely many o-minimal structures $\mathcal{X}_1, \dots, \mathcal{X}_n$. Then there is a locally definable isomorphism*

$$G \cong G_1 \times \dots \times G_n / \Gamma,$$

where G_i is a locally definable and definably generated group in $\mathcal{X}_i$, and Γ is a compatible locally definable discrete subgroup of $G_1 \times \dots \times G_n$.

Proof. The structure $\bigsqcup_i \mathcal{X}_i$ is bi-interpretable with the o-minimal structure $\mathcal{M}$ obtained by concatenating $\mathcal{X}_1, \dots, \mathcal{X}_n$ in the given order and adding $n - 1$ points to separate $\mathcal{X}_i$ from $\mathcal{X}_{i+1}$ for $i < n$. We can therefore apply to $\mathcal{M} = \bigsqcup_i \mathcal{X}_i$ the various results concerning o-minimal structures. By Theorem 8.4, the group G admits a decomposition $G = A_1 \dots A_n$ with respect to $X_1, \dots, X_n$, where X_i is the domain of $\mathcal{X}_i$. Let $\langle A_i \rangle$ be the locally definable subgroup of G generated by A_i.

We claim that $\langle A_i \rangle$ is locally definably isomorphic to a definably generated group G_i in the structure $\mathcal{X}_i$. To this aim, let $A_i^{(m)} \subseteq G$ consist of the m-fold products $a_1 \dots a_m$, where each a_i is either an element of A_i or is the group-inverse of an element of A_i. Without loss of generality, after permuting coordinates, we can choose $k_1, \dots, k_n \in \mathbb{N}$ such that G is included in $X_1^{k_1} \times \dots \times X_n^{k_n}$. Since $A_i^{(m)}$ is X_i-internal and included in G, by Fact 2.12 it must have a finite projection on the factors different from X_i. Thus we can write $A_i^{(m)} \subseteq L_m \times X_i^{k_i} \times F_m$ where L_m and F_m are finite sets. It follows that the subgroup $\langle A_i \rangle$ of G generated by A_i is included in $L \times X_i^{k_i} \times F$ where $L = \bigcup_m L_m$ and $F = \bigcup_m F_m$ are countable sets. Consider a bijection sending $L \cup F$ to a countable subset of X_i. This induces a locally definable bijection between $\langle A_i \rangle$ and a locally definable subset $G_i \subseteq X_i^{k_i+1}$. We can endow G_i with a group operation via the bijection. The resulting group G_i will then be locally definable, and in fact definably generated, in the structure $\mathcal{X}_i$. There is a locally definable group homomorphism $f : G_1 \times \dots \times G_n \to G$ induced by the composition $G_1 \times \dots \times G_n \cong \langle A_1 \rangle \times \dots \times \langle A_n \rangle \to G$, where the last map sends $(x_1, \dots, x_n)$ to their product in G. Note that f is a homomorphism since G is abelian.

It remains to prove that the kernel Γ of the above f is discrete. To this aim fix, for $i = 1, \dots, n$, a definable subset U_i of $\langle A_i \rangle$ and let S be the set of all tuples $(a_1, \dots, a_n) \in U_1 \times \dots \times U_n$ such that $a_1 \dots a_n = 1_G$. It suffices to show that S is finite. The sets $U_1, \dots, U_n$ are orthogonal by Proposition 2.13, since they are internal to $A_1, \dots, A_n$ respectively. It follows that S is a finite union of sets of the form $B_1 \times \dots \times B_n$ with $B_i \subseteq U_i$. However, each B_i can only be a singleton because any choice of $n-1$ elements from $a_1, \dots, a_n$ determines the last one via the equation $a_1 \dots a_n = 1_G$. □

10. Questions

(1) Does Theorem 8.4 extend to the case when $\mathcal{M}$ is an arbitrary structure?
(2) Can the abelianity hypothesis in Theorem 9.2 be removed?
(3) Does Proposition 2.9 hold without the saturation hypothesis?

11. Appendix

We construct an example of a definable group whose group operation splits, but the group is not a product of two infinite groups, hence in particular two orthogonal groups. First we need the following observation.

Example 11.1. There is a real Lie group G, definable in the pure real field structure, which is not a semidirect product of the connected component of the identity G^0 and a finite non-trivial group.

Proof. Let p be an odd prime. The Heisenberg group mod p is the semialgebraic group H of matrices of the form $\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ where $c \in \mathbb{R}/p\mathbb{Z}$ and $a, b \in \mathbb{Z}/p\mathbb{Z}$. We claim that H is not Lie isomorphic to a semidirect product of a connected real Lie group and a discrete group. Taking $a, b = 0$ we obtain the center $Z(H)$ of H, which coincides with H^0 and it is isomorphic to the circle group $\mathbb{R}/p\mathbb{Z}$. The quotient H/H^0 is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Since H is not abelian, H is not isomorphic to the direct product $H^0 \times (\mathbb{Z}/p\mathbb{Z})^2$. Moreover the direct product is the only possible semidirect product in the Lie category because $(\mathbb{Z}/p\mathbb{Z})^2$ has no non-trivial continuous action on $\mathbb{R}/p\mathbb{Z}$ (since the only non-trivial definable automorphism of $\mathbb{R}/p\mathbb{Z}$ is the inverse, and p is odd). ☐

Example 11.2. Let R_1 and R_2 be two orthogonal copies of the field $\mathbb{R}$ and work in the o-minimal structure $M = R_1 \sqcup R_2$ obtained by concatenation of R_1 and R_2 with a separating element between them. Let H_i be the Heisenberg group mod 3 over R_i. Consider the definable group $H_1 \times H_2$. Now consider the definable subgroup $G < H_1 \times H_2$ consisting of the pairs of matrices

$$\left\langle \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a & c' \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$$

with $a, b \in \mathbb{Z}/3\mathbb{Z}$, $c, c' \in \mathbb{R}/3\mathbb{Z}$. Note that G is definably isomophic to an extension of $(\mathbb{R}/3\mathbb{Z})^2$ by $(\mathbb{Z}/3\mathbb{Z})^2$. In Proposition 11.3 below we prove that G is not a direct product of two infinite definable subgroups. This shows that, although the group operation splits with respect to R_1 and R_2 (because it is induced by the direct product $H_1 \times H_2$), G is not a direct product of orthogonal subgroups.

Proposition 11.3. *The group G in Example 11.2 is not a direct product of two infinite definable subgroups.*

Proof. Assume that G is the direct product of two definable infinite subgroups G_1 and G_2. Then $\dim(G_1) = \dim(G_2) = 1$. Since definable groups of dimension 1 are cohesive (Theorem 3.4), we may assume that G_1 is R_1-internal and G_2 is R_2-internal. Consider the natural (surjective) projections $\pi_i : G \to H_i$.

We claim that $G_1^0 = \pi_2^{-1}(1_G)$ and $G_2^0 = \pi_1^{-1}(1_G)$ where G_i^0 is the connected component of the identity of G_i. Consider for instance G_1. Clearly $\pi_2(G_1)$ is finite by orthogonality. Hence $\pi_2^{-1}(1_G) \cap G_1$ has finite index in G_1, so it is infinite. Moreover $\pi_2^{-1}(1_G)$ is $H_1^0 \times \{1_G\}$, hence it is connected and, since two definably connected one dimensional groups having infinite intersection coincide, it must coincide with G_1^0. The claim is thus proved.

Observe that $Z(G) = G^0 = G_1^0 \times G_2^0$, thus $[G_1 : G_1^0][G_2 : G_2^0] = [G : G^0] = 9$. So, there are three cases for the possible values of the indexes of G_1^0 and G_2^0: $(1, 9), (3, 3), (9, 1)$.

First case: $[G_1 : G_1^0] = 1$ and $[G_2 : G_2^0] = 9$ (observe that the third case is symmetric). In this case G_1 is connected, thus $G_1 = G_1^0$, and $|\pi_1(G_2)| = 9$ because $G_2^0 = \pi_1^{-1}(1_G)$ has index 9 in G_2. On the other hand

$$H_1 = \pi_1(G) = \pi_1(G_1)\pi_1(G_2) = H_1^0\pi_1(G_2) = Z(H_1)\pi_1(G_2)$$

and since $\pi_1(G_2)$ has 9 elements and $Z(H_1)$ has index 9 in H_1, it follows that H_1 must be the direct product of $Z(H_1)$ and $\pi_1(G_2)$, contradicting the claim in Example 11.1.

Second case: $[G_1 : G_1^0] = 3 = [G_2 : G_2^0]$. In this case we show that G_1 and G_2 are abelian and we reach a contradiction since G is not abelian. By symmetry it suffices to show that G_1 is abelian. First recall that $G_1^0 < Z(G)$, so in particular G_1^0 is central in G_1, and definably isomorphic to $\mathbb{R}/3\mathbb{Z}$. It follows that G_1 is definably isomorphic to a central extension of $\mathbb{R}/3\mathbb{Z}$ by $\mathbb{Z}/3\mathbb{Z}$. We claim that such a group is necessarily abelian. To this aim we show that there is a copy of $\mathbb{Z}/3\mathbb{Z}$ which is a complement of G_1^0. Let G_1^0, aG_1^0, bG_1^0 be the three connected components of G_1. Note that the map $x \mapsto x^3$ has image contained in G_1^0 and its restriction to G_1^0 is onto. Consider the map sending $x \in G_1^0$ to $(ax)^3 \in G_1^0$. Since G_1^0 is central, $(ax)^3 = a^3x^3$. Now let $y \in G_1^0$ be such that $y^3 = a^3$. Then ay^{-1} has order three and generates a complement C of G_1^0 in G_1. Since G_1^0 is central we conclude that G_1 is the direct product of C and G_1^0, hence it is abelian. ☐

Example 11.4. Let $\mathcal{M} = (K, k, \Gamma; \pi, v)$ be a three sorted structure consisting of an algebraically closed valued field K with residue field k of characteristic zero, value group Γ, and the two projections of the field on the other sorts. Then K is indecomposable in $\mathcal{M}$ but not cohesive.

Proof. We need the following facts:

1. k and Γ are internal to K.
2. k and Γ are orthogonal.
3. K is internal to any infinite definable subset of K.
4. K is not internal to $k \times \Gamma$.

Point (1) is obvious. The proof of (2) is in [6, Corollary 5.25]. (3) follows from the fact that any definable subset of K is a boolean combination of valuation balls [6, Corollary 3.32], so in particular if it is infinite it contains a valuation ball, and K itself is internal to any valuation ball. Point (4) is clear if k and Γ have cardinality less than K and we can reduce to this case going to an elementary equivalent model (for instance, by the theorem of Ax-Kochen and Ershov (see [6, Theorem 5.1]) we can take $K = \mathbb{Q}^{ac}((t^{\mathbb{Q}}))$, $\Gamma = \mathbb{Q}$, $k = \mathbb{Q}^{ac}$ where $\mathbb{Q}^{ac}$ is the algebraic closure of $\mathbb{Q}$).

Granted (1)–(4) we continue as follows. By (1) and (2) K is not cohesive. We claim that K is indecomposable. So let K be internal to a product $X \times Y$ of orthogonal definable sets. We must show that K is internal to either X or Y. If this is not the case, then by (3) neither X nor Y can have an infinite projection on K. But then both X and Y are internal to $k \times \Gamma$. So K itself is internal to $k \times \Gamma$, contradicting (4). ☐

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