Calculus of Variations

Nonlocal estimates for the volume preserving mean curvature flow and applications

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Abstract

We obtain estimates on nonlocal quantities appearing in the volume preserving mean curvature flow (VPMCF) in the closed, Euclidean setting. As a result we demonstrate that blowups of finite time type I singularities of VPMCF are ancient solutions to mean curvature flow (MCF), prove that monotonicity methods may always be applied at these finite times and obtain information on the asymptotics of the flow. In the case of type II singularities, asymptotic flows corresponding to 'Hamilton's rescaling procedures' are eternal solutions of the MCF.

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1 Introduction

A key question for the study of Volume Preserving Mean Curvature Flow (VPMCF) is the extent to which it has similar properties to the Mean Curvature Flow (MCF). It is now known that the extra nonlocal term in VPMCF causes significant differences. For example, Cabezas-Rivas and Miquel [13] prove that key properties used in many MCF methods do not hold for VPMCF, such as preservation of mean convexity (along with most other curvature conditions bar convexity). This demonstrates the need to a VPMCF specific methods, and in this paper we begin to develop some of these. In this paper, we give evidence that there are still significant links between singularities of these two flows: We show that blowups of type I finite time singularities of VPMCF are indeed ancient solutions to MCF (compare with [13, point (c), p 288]). Finite time type II singularities of the VPMCF look like eternal solutions of MCF after a suitable parabolic blowup procedure (first used by Hamilton [29]). This is

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particularly interesting in the light of the significant progress has been made in recent years in the classification of such singularities – see for example [3, 8, 14, 15].

In order to study these blowup solutions we needed to get control over the (nonlocal) Lagrange parameter in the equation. We found links between L^2-bounds on the (nonlocal) averaged mean curvature and the extrinsic diameter along the flow. These estimates play an important role in the monotonicity formula, and essentially allow initial results on the analysis of singularities to be proven without assuming any further bounds on $\overline{H}$.

The VPMCF of n dimensional hypersurfaces in $\mathbb{R}^{n+1}$ is the L^2-gradient flow of the area functional under the constraint that the $(n + 1)$-dimensional enclosed volume is fixed. This flow was introduced by Gage [25] for curves and by Huisken [30] for hypersurfaces. We consider the closed setting, meaning that M^n is a smooth, orientable, compact n-dimensional manifold without boundary. Then a smooth family of immersions $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is a solution of the VPMCF if

$$\begin{cases} \dfrac{dX}{dt} = -(H - \overline{H})\nu \,, & \text{on } M^n \times [0, T), \\ X(\cdot, t) = X_0 & \text{on } M^n, \end{cases}$$

where $X_0 : M^n \to \mathbb{R}^{n+1}$ is a given immersion. Here, ν is a unit normal along $X(\cdot, t) =: M_t$, $h_{ij} = -\langle \partial_i \partial_j X, \nu \rangle$ are the local coefficients of the second fundamental form A, $g_{ij} = \langle X_i, X_j \rangle$ is the induced metric and $H = \sum g^{ij} h_{ij}$ is the mean curvature. Furthermore, $\overline{H} = \frac{\int H d\mu_t}{|M_t|}$ is the Lagrange parameter of the above mentioned geometric variational problem. Thus, $\frac{d}{dt}|M_t| \leq 0$ and $\frac{d}{dt}V(t) = 0$ is satisfied along the flow (for the precise definition of the enclosed volume $V(t)$ see Sect. 2). Note that the formulation of the VPMCF does not depend on the choice of the unit normal. If it is possible to choose between an "inner" and "outer" unit normal with respect to M_t, then we choose the outer unit normal, so that $H > 0$ for round spheres.

We collect some known results about the VPMCF. Gage (for $n = 1$) and Huisken (for $n \geq 2$) proved that convex solutions of the VPMCF stay convex and do not develop singularities. They converge smoothly to a round sphere enclosing the same amount of volume as the initial hypersurface. In [23], Escher and Simonett proved that hypersurfaces being $h^{1+\beta}$ close to a round sphere converge to (possibly another) round sphere with the same enclosed volume as the initial surface. Li obtained related results in [37]. For example, he proved convergence to a round sphere if the integrated trace-free second fundamental form $\overset{\circ}{A}_{ij} = h_{ij} - \frac{H}{n} g_{ij}$ of the initial surface is small enough in L^2. The needed smallness depends on $\max |A|(0)$, $\overline{H}(0) > 0$ and $V(0) > 0$, see [37, Theorem 1]. Further results about the VPMCF in a non-Euclidean ambient manifold can be found for example in [2, 11, 12, 21, 33, 44].

Athanassenas, later with Kandanaarachchi, studied in [4–6] the VPMCF of a rotational symmetric surface with and without boundary conditions. For example, in [6] it is shown that a height bound on the generating curve for the rotational symmetric surfaces prevents the flow from developing a singularity. The hypersurfaces then converge to a sphere (or a half-sphere in the considered boundary setting). Note that rotational symmetry (additionally to convexity) is one the of few properties that is known to be preserved under the VPMCF.

For curves, we expect stronger results compared to the hypersurface case because the main part of the Lagrange parameter has a geometric interpretation – the term $\int \kappa ds$ is scaling invariant and measures the turn of the angle of a curve. In the closed setting, this is just $2\pi m$ with $m \in \mathbb{N}$ (the index of a closed curve). So an L^∞-in-time bound on $\overline{\kappa} = \frac{\int \kappa ds_t}{L_t}$ is immediate if one controls the length from below (e.g. by an isoperimetric inequality). Inspired by [16],

Escher and Ito proved in [22] the existence of singularities of the VPMCF for curves under some conditions on the initial curve. One scenario of the initial curve leading to a singularity is that the enclosed volume of the initial curve with $\frac{1}{2\pi}\int \kappa \, ds \geq 1$ is negative. Inspired by [22], the second author proved in [40] an analogous result for curves in the (volume preserving) free boundary setting. In this setting, $\int \kappa \, ds$ is not bounded a priori, but still, an L^∞-in-time bound proven earlier in [39] is used in the proof of existence of singularities. In [39], the second author had introduced the (Neumann) free boundary setting for the VPMCF in detail (for curves and surfaces) and proved convergence to a stationary solution for (non-closed) curves under some conditions on the initial curve. The most advanced result about closed curves can be found in [19]. There, Dittberner was able to derive a comparison principle between the intrinsic and extrinsic distance – based on previous work of Huisken [32] for the MCF – to show convergence to a sphere of a closed curve under the condition that the initial curve is embedded and satisfies $\int_x^y \kappa \, ds \geq -\pi$ for all x, y. This can be seen as the analogue of Grayson's theorem – which says that the MCF first makes every embedded, closed curve convex and then shrinks it to a point while becoming more and more round, see [24, 27]. Without assuming $\int_x^y \kappa \, ds \geq -\pi$ for all x, y we cannot expect that such a strong result holds because there is an example, already suggested by Gage [25] and also studied by Dittberner [19] where this condition is violated and a self-intersections develops after starting the VPMCF. Numerical computations in fact indicate that then a singularity appears for this initial curve, see Mayer [42]. For curves, it was also shown recently, that a star-shaped, centrosymmetric set keeps this properties and converges to a round sphere, see [26].

Whether star-shapedness is perserved in general seems to be an open questions. In the work of Kim and Kwon certain approximating solutions of the VPMCF are considered. Using them, it is proven that a strong version of star-shapedness is preserved [34]. In [43], Mugnai, Seis and Spadaro presented a distributional formulation of the VPMCF using the setting of Caccioppoli sets. They show global-in-time existence of their weak flow. A phase field method of the VPMCF was studied for example in the work of Takasao [46, 47], see also the references therein. In which generality these weak flows or limits of approximating sets can be applied to any initially smoothly immersed surfaces in unclear to the authors. An immersed surface does not need to bound a domain, see also the definition of Alexandrov immersions below. Recently, Laux showed in in [36] that a (strong) solution where the hypersurfaces bound a domain agrees with a distributional solution he defines (as long as the smooth one exists).

We now point out the structure of the paper and formulate the main results. Due to the lack of preserved quantities under the VPMCF we need to work in a quite general setting. In Sect. 3, we study the connection between (extrinsic) diameter bounds and L^2-estimates of $\overline{H}(t) = \frac{\int H d\mu_t}{|M_t|}$. One of our results is the following:

If the initial immersion has non-vanishing enclosed volume $V_0 \neq 0$, then there are constants c and C only depending on M_0 such that
$$\int_0^t \overline{H}^2(\tau) d\tau \leq C(1 + t^2)e^{c\sqrt{t}}.$$

We provide an example of a convex, embedded curve where the diameter is growing for a short time (note the contrast to the classical MCF). In Sect. 4, we re-prove a monotonicity formula for the VPMCF already shown in [38] and study consequences for finite time singularities. In the monotonicity formula it is important that the L^2-norm of $\overline{H}$ is appearing in form of $\exp(-\frac{1}{2}\int_0^t \overline{H}^2(\tau)d\tau)$ as a multiplicative factor in front of the usual integrated Gaußian in Huisken's monotonicity formula [31]. Consequently, the L^2-control from Sect. 3 allows us

to get properties of asymptotic flows appearing after suitable parabolic blowups at finite time singularities. One result is:

Parabolic blowups of VPMCF about finite time singularities of type I ($\max |A|^2 \leq \frac{C}{T-t}$)
*produce properly immersed, ancient, non-compact homothetically
shrinking solutions of MCF.
Finite type II blowups are eternal solutions of the MCF after Hamilton's parabolic
rescaling.*

In Sect. 4.3, we illustrate that an L^∞-bound on $\overline{H}$ in fact implies that the diameter of $|M_t|$ along the VPMCF always stays uniformly bounded for all times.

Section 5 contains statements about the VPMCF with infinite life-span and uniform diameter bound. We introduce a new "extended isoperimetric ratio" by

$$\mathcal{I}(M_t) := \frac{n+1}{n} \overline{H} \frac{V(t)}{|M_t|}.$$

Note that $\mathcal{I}$ of the round sphere is equal to one. We motivate this definition with the Alexandrov-Fenchel inequalities proven for k-convex, star-shaped domains in [28]. The main result of Sect. 5 is the following:

*If $V_0 \neq 0$ and M_t satisfies a uniform diameter bound along the VPMCF with $T = \infty$,
then, for each $t_i \to \infty$, there exists a subsequence (not relabeled) such that
either $\mathcal{I}(M_{t_i}) \to 0$ or $\mathcal{I}(M_{t_i}) \to 1$.*

As a corollary of the Alexandrov-Fenchel inequalities [28] we get that

*Any mean convex, star-shaped solution of the VPMCF with uniformly bounded curvature
exists for all times and converges to a round sphere.*

In Sect. 6, we remind the reader of the notion of an *Alexandrov immersion*, which is—roughly speaking—an n-dimensional immersion that bounds an $(n + 1)$-dimensional manifold that is immersed in $\mathbb{R}^{n+1}$ (see Definition 1 for the precise definition). Alexandrov immersions bound an $(n + 1)$-dimensional "domain" that is allowed to have a certain kind of self-overlaps. The definition goes back to Alexandrov [1] in his work about closed surfaces of constant mean curvature in Euclidean space.[1] It has also been used successfully for example by Brendle in his work about minimal tori in $\mathbb{S}^3$, see [7]. Using maximum principle arguments in a "one-sided situation" we prove the following result:

*The property of being Alexandrov immersed is preserved under the VPMCF as long as
$\overline{H} \geq 0$ (using the outer unit normal for the definition of $\overline{H}$).*

Unfortunately, we found an (even embedded) example where $\overline{H} \geq 0$ is lost along the VPMCF. We explain this example in Appendix A.

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[1] Note that being Alexandrov immersed is not called like this in [1], of course. Also, the property that the $(n + 1)$-dimensional domain is *immersed* in $\mathbb{R}^{n+1}$ is missing in this paper (there, only the expression *smooth mapping* is used). But it is meant to be part of the definition as it is used in the proof.

2 Notation

We complement the definitions from the introduction with the formula of the signed volume on the immersion M_t

$$V(t) := \text{Vol}(M_t) = \frac{1}{n+1} \int_{M_t} \langle X, \nu \rangle \, d\mu.$$

The defining property of VPMCF is that it preserves this volume, that is,

$$\text{Vol}^{n+1}(M_t) = \text{Vol}^{n+1}(M_0) =: V_0 .$$

In particular if $\Omega_t \subset \mathbb{R}^{n+1}$ is a domain with smooth boundary $\partial \Omega_t = M_t$, then the divergence theorem implies that $\text{Vol}^{n+1}(M_t) = \mathcal{L}^{n+1}(\Omega_t)$ where $\mathcal{L}^{n+1}$ is the $(n+1)$–dimensional Lebesgue measure.

We will regularly use the *extrinsic* diameter of the immersion M_t defined by

$$\text{diam}(M_t) := \text{diam}_{\text{ext}}(M_t) = \max_{(x,y) \in M^n \times M^n} |X(x,t) - X(y,t)| . \tag{1}$$

This is distinct to the intrinsic metric diameter of M_t, $\text{diam}_{\text{int}}(M_t)$ where we have the inequality $\text{diam}(M_t) \leq \text{diam}_{\text{int}}(M_t)$.

3 Diameter bounds and L^2-bounds of the Lagrange parameter

Lemma 1 *Let* $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ *be a closed VPMCF for* $t \in [0, T)$. *Then the extrinsic diameter (as in Eq. 1) is Lipschitz continuous and therefore differentiable almost everywhere in* $[0, T)$. *For a time of differentiability* t *it satisfies*

$$\frac{d}{dt} \text{diam}(M_t) = \langle -(H(x,t) - \overline{H}(t))\nu(x,t) + (H(y,t) - \overline{H}(t))\nu(y,t), \frac{X(x,t) - X(y,t)}{|X(x,t) - X(y,t)|} \rangle,$$

where $(x, y) \in M^n \times M^n$ *are points where the maximum of the diameter is attained.*

Proof First we note that the map $(x, y, t) \mapsto |X(x,t) - X(y,t)|^2$ is smooth and $M^n \times M^n$ is compact. Thus, Mantegazza's version of 'Hamilton's trick'[2] [41, Lemma 2.1.3] tells us that $t \mapsto \max_{(x,y) \in M^n \times M^n} |X(x,t) - X(y,t)|^2$ is Lipschitz and therefore differentiable almost everywhere in $[0, T)$. For such a time t, the derivative can be computed as

$$\frac{d}{dt} \max_{(x,y) \in M^n \times M^n} |X(x,t) - X(y,t)|^2 = \frac{\partial}{\partial t} |X(x,t) - X(y,t)|^2 \big|_{(x,y)=(x^*,y^*)},$$

where (x^*, y^*) is a pair such that the maximum in $\max_{(x,y) \in M^n \times M^n} |X(x,t) - X(y,t)|^2$ is attained. Note that $|X(x^*,t) - X(y^*,t)|^2 > 0$, otherwise $X(\cdot, t)$ is just a point which contradicts the fact that $X(\cdot, t)$ is an immersion. So, when t is such a time of differentiability of $u(t) := \max_{(x,y) \in M^n \times M^n} |X(x,t) - X(y,t)|^2$, then also $\text{diam}(M_t)$ is differentiable and $\frac{d}{dt} \text{diam}(M_t) = \frac{\partial_t u(t)}{2 \text{diam}(M_t)}$. This shows that $t \mapsto \text{diam}(M_t)$ is differentiable almost everywhere in $[0, T)$ and

$$\frac{d}{dt} \text{diam}(M_t) = \frac{1}{|X(x,t) - X(y,t)|} \langle \partial_t X(x,t) - \partial_t X(y,t), X(x^*,t) - X(y,t) \rangle$$

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[2] An alternative to working with the smooth map $(x, y, t) \mapsto |X(x,t) - X(y,t)|^2$ is to differentiate the diameter directly and use the results in [17, Section 2] or [18, Section 2] to see that $t \mapsto \max |X(x,t) - X(y,t)|$ is in $W^{1,1}$ and that one can compute $\frac{d}{dt} \text{diam}$ by evaluation $\partial_t |X(x,t) - X(y,t)|$ in any point of maximality.

$$= \langle -(H(x,t) - \overline{H}(t))\nu(x,t) + (H(y,t) - \bar{H}(t))\nu(y,t), \tfrac{X(x,t)-X(y,t)}{|X(x,t)-X(y,t)|}\rangle,$$

where $(x,y) \in M^n \times M^n$ are points where the maximum in $u(t)$ und thus also in the diameter is attained.   □

Lemma 2 *Let* $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ *be a closed VPMCF for* $t \in [0,T)$. *Then the extrinsic diameter satisfies for almost every time* $t \in [0,T)$

$$\frac{d}{dt} \operatorname{diam}(M_t) \leq \overline{H}\left[\langle \nu^x, \tfrac{X(x,t)-X(y,t)}{|X(x,t)-X(y,t)|}\rangle - \langle \nu^y, \tfrac{X(x,t)-X(y,t)}{|X(x,t)-X(y,t)|}\rangle\right] - \frac{4n}{\operatorname{diam}(M_t)}$$

$$\leq 2|\overline{H}| - \frac{4n}{\operatorname{diam}(M_t)}.$$

Proof We work again with the smooth squared distance function $\varphi : M^n \times M^n \times [0,T) \to \mathbb{R}$, $\varphi(x,y,t) := |X(x,t) - X(y,t)|^2$ as in the proof of Lemma 1. We write $(z_1, \ldots, z_{2n}) := (x_1, \ldots, x_n, y_1, \ldots, y_n)$ if needed. We leave out the time dependence in the notation from now on. We calculate the second derivatives of φ. But we only want to use them for x and y such that the maximum in $\operatorname{diam}(M_t)^2 = \max_{(x,y)\in M^n \times M^n} |X(x,t) - X(y,t)|^2$ is attained. Thus, we compute

$$\partial_{x_i}\varphi = 2\langle \partial_{x_i}X(x), X(x) - X(y)\rangle, \qquad \partial_{y_i}\varphi = -2\langle \partial_{y_i}X(y), X(x) - X(y)\rangle$$

$$\nabla^2_{x^i x^j}\varphi = -2h_{ij}(x)\langle \nu(x), X(x) - X(y)\rangle + 2g_{ij}(x)$$
$$\nabla^2_{x^i y^j}\varphi = -2\langle \partial_{x_i}X(x), \partial_{y_j}X(y)\rangle$$
$$\nabla^2_{y^i y^j}\varphi = 2h_{ij}(y)\langle \nu(y), X(x) - X(y)\rangle + 2g_{ij}(y)$$

At almost every time t, $\operatorname{diam}(M_t)^2$ is differentiable, and by Hamilton's trick [41, Lemma 2.1.3], we know that

$$\frac{d}{dt}\operatorname{diam}(M_t)^2 = 2\langle -(H(x) - \overline{H})\nu(x) + (H(y) - \bar{H})\nu(y), X(x) - X(y)\rangle$$

for x and y attaining the maximum. At such points we have that $0 = \partial_{x_i}\varphi = \partial_{y_i}\varphi$ and $0 \geq \nabla^2_{z^i z^j}\varphi$. This particularly implies that $X(x) - X(y)$ is normal to the surface in x and y, so the tangent spaces at x and y are parallel and we may take local orthonormal coordinates so that $\partial_{x_i}X(x) = \partial_{y_i}X(y) = e_i$. Furthermore,

$$0 \geq \sum_i \nabla^2_{e_i^x - e_i^y, e_i^x - e_i^y}\varphi = -2\langle \nu(x), X(x) - X(y)\rangle H(x) + 2\langle \nu(y), X(x) - X(y)\rangle H(y) + 8n$$

which implies

$$\frac{d}{dt}\operatorname{diam}^2(M_t) \leq 2\overline{H}[\langle \nu(x), X(x) - X(y)\rangle - \langle \nu(y), X(x) - X(y)\rangle] - 8n.$$

Since $\operatorname{diam}(M_t) = |X(x,t) - X(y,t)| > 0$, we get

$$\frac{d}{dt}\operatorname{diam}(M_t) \leq \overline{H}\left[\langle \nu(x), \tfrac{X(x)-X(y)}{|X(x)-X(y)|}\rangle - \langle \nu(y), \tfrac{X(x)-X(y)}{|X(x)-X(y)|}\rangle\right] - \frac{4n}{\operatorname{diam}(M_t)}$$

$$=: 2\left(\overline{H}\sigma - \frac{2n}{\operatorname{diam}(M_t)}\right)$$

$$\leq 2\left(|\overline{H}| - \frac{2n}{\operatorname{diam}(M_t)}\right)$$

(2)

as stated in the lemma. Here, $\sigma \in \{1, 0, -1\}$ depends on the configuration of normals and positions.   □

Remark 1 Lemma 2 implies that the diameter along the VPMCF is bounded on every bounded time interval $[0, T)$, $T < \infty$, if one is able to show a bound $\sup_{t \in [0,T)} |\overline{H}(t)| \leq c$. Unfortunately, such a bound is hard to get in a very general setting. See Proposition 15 for a weakening of this condition.

Remark 2 The above calculation is to some extent optimal in bounding diameter, even in the case of convex curves. Indeed, even in the convex setting, the nonlocal term can force the diameter to increase initially, before converging back to a circle. See Example 1 for details.

Definition 1 An immersion $X : M^n \to \mathbb{R}^{n+1}$ is *Alexandrov immersed* if there exists an $(n + 1)$-dimensional manifold $\overline{\Omega}$ with $\partial \overline{\Omega} = M^n$ and an immersion $G : \overline{\Omega} \to \mathbb{R}^{n+1}$ such that $G|_{\partial \overline{\Omega}}$ also parametrises $\mathrm{Im}(X)$.

Remark 3 For an Alexandrov immersion, there is obviously a natural notion of an an *inner unit normal* $\tilde{\nu}$ (the one where $G^*(\tilde{\nu})$ shows into Ω) and an *outer unit normal* $-\tilde{\nu}$.

Corollary 3 *Let* $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ *be a closed VPMCF for* $t \in [0, T)$ *that is an Alexandrov immersion for all* $t \in [0, T)$. *Then we choose* ν *to be the outer unit normal. In this case*

$$\left\langle \nu(x), \frac{X(x) - X(y)}{|X(x) - X(y)|} \right\rangle - \left\langle \nu(y), \frac{X(x) - X(y)}{|X(x) - X(y)|} \right\rangle = 2$$

for $x(t)$ *and* $y(t)$ *such that the maximum in the definition of the diameter is attained. As a consequence, we have that*

$$\frac{d}{dt} \mathrm{diam}(M_t) \leq 2 \left(\overline{H} - \frac{2n}{\mathrm{diam}(M_t)} \right)$$

for almost every $t \in [0, T)$.

Proof Since there are no points $X(\tilde{x})$ that are further apart from $X(y)$ in $\mathbb{R}^{n+1}$ than $X(x)$, the outer unit normal $\nu(x)$ must agree with $\frac{X(x)-X(y)}{|X(x)-X(y)|}$. For $X(y)$, an analogous argument works.   □

Corollary 4 *Let* $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ *be a closed VPMCF that is Alexandrov-immersed for all* $t \in [0, T)$ *and* $\overline{H} \leq 0$ *on* $t \in [0, T)$, *where* $\overline{H}$ *is computed with respect to the outer unit normal. Then* $T \leq \frac{\mathrm{diam}(M_0)^2}{8n}$.

Proof The assumption $\overline{H} \leq 0$ and Corollary 3 imply

$$\frac{d}{dt} \mathrm{diam}(M_t) \leq -\frac{4n}{\mathrm{diam}(M_t)}, \quad t \in [0, T)$$

which is equivalent to

$$\mathrm{diam}(M_t)^2 \leq \mathrm{diam}(M_0)^2 - 8nt \ \text{ for } t \in [0, T)$$

via integration. So we have that

$$\limsup_{t \to T} \mathrm{diam}(M_t)^2 \leq \mathrm{diam}(M_0)^2 - 8nT$$

which leads to a contradiction for $T > \frac{\mathrm{diam}(M_0)^2}{8n}$.   □

Remark 4 This above statement should be interpreted in the following way: If we know the flow exists for quite some time *and stays Alexandrov immersed*, then $\overline{H} \leq 0$ is not preserved. For more on this see also Theorem 16.

Lemma 5 *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a VPMCF with enclosed volume $V_0 \neq 0$. Then we can express $\overline{H}$ as*

$$\overline{H} = \frac{\int_M \langle \partial_t X, X - x \rangle \, d\mu + n|M_t|}{(n+1)V_0} \tag{3}$$

for any $x = x(t) \in \mathbb{R}^{n+1}$.

Proof We have that $\Delta |X - x|^2 = -2H \langle \nu, X - x \rangle + 2n$ and so by the divergence theorem

$$|M_t| = \frac{1}{n} \int_M H \langle \nu, X - x \rangle \, d\mu \ . \tag{4}$$

The formula for the signed volume enclosed by M_t reads $V(t) = \frac{1}{n+1} \int_{M_t} \langle X - x, \nu \rangle \, d\mu$ for any $x = x(t) \in \mathbb{R}^{n+1}$. We compute

$$\int_M \langle X_t, X - x \rangle \, d\mu = \overline{H} \int_M \langle \nu, X - x \rangle \, d\mu - \int_M H \langle \nu, X - x \rangle \, d\mu$$

and so

$$\overline{H} = \frac{\int_M \langle \partial_t X, X - x \rangle \, d\mu + \int_M H \langle \nu, X - x \rangle \, d\mu}{\int_M \langle \nu, X - x \rangle \, d\mu} = \frac{\int_M \langle \partial_t X, X - x \rangle \, d\mu + n|M_t|}{(n+1)V_0}$$

using the volume preserving property. $\qquad\square$

Proposition 6 *Suppose that $(M_t)_{t \in [0,T)}$ satisfies $V_0 \neq 0$ and $diam(M_t) < R(t)$ with $R(s) \leq R(t)$ for $s < t \leq T$. Then we have for $t \in [0, T)$,*

$$\int_0^t \overline{H}^2(\tau) d\tau \leq \frac{2|M_0|^2}{V_0^2(n+1)^2}(R(t)^2 + n^2 t)$$

Proof First we choose $x(t) \in \mathbb{R}^{n+1}$ to be any point such that $M_t \subset B_R(x(t))$. We have that

$$\frac{d}{dt}|M_t| = -\int_M H(H - \overline{H})d\mu = -\int_M (H - \overline{H})^2 d\mu = -\int_M |\partial_t X|^2 d\mu.$$

Formula (3) implies

$$|\overline{H}| \leq \frac{1}{(n+1)|V_0|}\left(\sqrt{\int_M |\partial_t X|^2 d\mu}\sqrt{\int_M |X - x|^2 d\mu} + n|M_0|\right).$$

and thus

$$\overline{H}^2 \leq \frac{2}{(n+1)^2 V_0^2}\left(-\frac{d}{dt}|M_t| \int_M |X - x|^2 d\mu + n^2|M_0|^2\right) \ .$$

Estimating $|X - x|^2|_s \leq \text{diam}^2(M_s) \leq R^2(s) \leq R^2(t)$ for $s < t$ and using $|M_s| \leq |M_0|$ again, we know now

$$\int_0^t \overline{H}^2 ds \leq \frac{2}{V_0^2(n+1)^2}|M_0|^2(R^2(t) + n^2 t) \ .$$

$\qquad\square$

Proposition 7 *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a closed VPMCF for $t \in [0, T)$ with enclosed volume $V_0 \neq 0$. Then there are constants c, C only depending on M_0 such that*

$$\operatorname{diam}(M_t) \leq C(1 + t)e^{c\sqrt{t}}$$

for $t \in [0, T)$

Proof We have

$$\frac{d}{dt}\operatorname{diam}(M_t) \leq 2|\overline{H}| - \frac{4n}{\operatorname{diam}(M_t)} \leq 2|\overline{H}|$$

and from (3)

$$|\overline{H}| \leq \frac{1}{(n+1)|V_0|}\left(\sqrt{-\frac{d}{dt}|M_t|}\sqrt{|M_0|}\operatorname{diam}(M_t) + n|M_0|\right)$$

$$:= c_1 + c_2\operatorname{diam}(M_t)\sqrt{-\frac{d}{dt}|M_t|}.$$

We pick

$$\log\psi = -2c_2\int_0^t\sqrt{-\frac{d}{dt}|M_t|} \geq -2c_2\sqrt{t}(|M_0| - |M_t|)^{\frac{1}{2}} \geq -2c_2\sqrt{t}|M_0|^{\frac{1}{2}} \qquad (5)$$

so that

$$\frac{1}{\psi}\frac{d}{dt}(\operatorname{diam}(M_t)\psi) = \frac{d}{dt}\operatorname{diam}(M_t) - 2c_2\operatorname{diam}(M_t)\sqrt{-\frac{d}{dt}|M_t|} \leq 2c_1$$

and thus (using that $\psi \leq 1$ and (5))

$$\operatorname{diam}(M_t) \leq \frac{1}{\psi(t)}\operatorname{diam}(M_0) + 2c_1\frac{1}{\psi(t)}\int_0^t\psi(\tau)d\tau$$

$$\leq (\operatorname{diam}(M_0) + 2c_1t)\frac{1}{\psi(t)}$$

$$\leq C(1 + t)e^{2c_2|M_0|^{\frac{1}{2}}\sqrt{t}}.$$

$\square$

Corollary 8 *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a closed VPMCF for $t \in [0, T)$ with enclosed volume $V_0 \neq 0$. Then there are constants $c, C > 0$ only depending on n, V_0 and M_0 such that*

$$\int_0^t\overline{H}^2(s)ds \leq C(1 + t^2)e^{c\sqrt{t}}.$$

Proof Put together Propositions 6 and 7. $\square$

Remark 5 (1) In general, we can not expect to have a bound of the form $\int_0^t\overline{H}^2(s)ds \leq C$, where C is independent of t because round spheres are (stationary) solutions of the VPMCF, and they satisfy $\int_0^t\overline{H}^2(s)ds = c^2t$. We do not know whether the $e^{\sqrt{t}}$-growth can be improved (e.g. to a linear growth in t) in this very general setting.

(2) Integrability conditions of a Lagrange multiplier was successfully used earlier, see for example the work of Rupp [45] on the volume preserving Willmore flow. But there, the fixed-enclosed-volume constraint is of lower order compared to the order of the flow, therefore Rupp was able to get strong results with the integrability.

3.1 Optimality of diameter bounds

In this subsection we consider a curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$, where ∂_s is differentiation with respect to arclength and $\nu = -J\partial_s\gamma$ is the rotation of the tangent $\partial_s\gamma$ by $\frac{\pi}{2}$ in negative direction. In order to be consistent with the definitions for surfaces, we use the definition $\kappa = -\langle \partial_s^2\gamma, \nu \rangle$. The *area preserving curve shortening flow* reads

$$\partial_t\gamma = -(\kappa - \overline{\kappa})\nu$$

for $\overline{\kappa} := \frac{\int \kappa ds}{L(\gamma_t)}$. Here, $ds = |\partial_x\gamma|dx$ denotes integration with respect to arclength and $L(\gamma_t)$ is the length of $\gamma_t := \gamma(\cdot, t)$ for the family $\gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$. Of course, as γ is closed and $\int \kappa ds$ is the turn of the tangents, we have that $\overline{\kappa} = \frac{2\pi m}{L(\gamma_t)}$. If γ_0 is embedded, then $m = 1$.

On a general curve we have that

$$\kappa_0(x) = \kappa_0(y), \quad \frac{\gamma_0(x) - \gamma_0(y)}{|\gamma_0(x) - \gamma_0(y)|} = \nu_0(x), \quad \frac{\gamma_0(x) - \gamma_0(y)}{|\gamma_0(x) - \gamma_0(y)|} = -\nu_0(y), \qquad (6)$$

where $x, y \in \mathbb{S}^1$ are such that the maximum in $\mathrm{diam}(\gamma_0)$ is attained. Note that for a positively oriented curve γ_0, ν_0 is the outer unit normal. The proof of Lemma 2 shows that

$$0 \geq -2\langle \nu_0(x), \gamma_0(x) - \gamma_0(y)\rangle\kappa_0(x) + 2\langle \nu_0(y), \gamma_0(x) - \gamma_0(y)\rangle\kappa_0(y) + 8.$$

Then (6) implies that

$$\langle \nu_0(x), \gamma_0(x) - \gamma_0(y)\rangle = |\gamma_0(x) - \gamma_0(y)| = \mathrm{diam}(\gamma_0)$$
$$\langle \nu_0(y), \gamma_0(x) - \gamma_0(y)\rangle = -|\gamma_0(x) - \gamma_0(y)| = -\mathrm{diam}(\gamma_0)$$

and thus

$$\kappa_0(x) \geq \frac{2}{\mathrm{diam}(\gamma_0)}. \qquad (7)$$

This inequality can be interpreted geometrically: The curvature at the points where the maximum is attained cannot be smaller than $\frac{2}{\mathrm{diam}(\gamma_0)}$ – otherwise the straight line realising the maximal distance would be attained at other points (tilting that line would increase the distance). Bearing this is mind, we construct a somehow sharp example.

We now provide an example demonstrating, somewhat counter-intuitively, that even on a convex curve, (which we know will converge to a round circle in infinite time [30]), the diameter will initially increase while the curvature at extremal points initially decreases.

Example 1 In this example we show that $\frac{d}{dt}\mathrm{diam}(\gamma_t)\big|_{t=0} > 0$ can occur, even on closed convex curves. We construct an explicit example as in Figure 1 as follows: First we set $\mathrm{diam}(\gamma_0) = 1$. We put a circle with radius $r := \frac{1}{2}$ at the midpoint of the line realising the maximal distance (w.l.o.g. $\{0\} \times [0, 1] \subset \mathbb{R}^2$). So we use very short arcs of the circle at the points $(0, 1) = \gamma_0(x)$ and $(0, 0) = \gamma_0(y)$. Thus, in x and y, the curvature is precisely equal to 2. We close this curve (smoothly) by straight lines (going from south to north and the other way round at the end of the arcs). The diameter is clearly realised by the points x and y.

In Example 1 we have equality in Eq. (7). With Lemma 1, we know that

$$\frac{d}{dt}\mathrm{diam}(\gamma_t)\big|_{t=0} = \langle -2(\kappa_0(x) - \overline{\kappa}_0)\nu_0(x), \frac{\gamma_0(x) - \gamma_0(y)}{|\gamma_0(x) - \gamma_0(y)|}\rangle = -4(1 - \frac{\pi}{L(\gamma_0)})$$
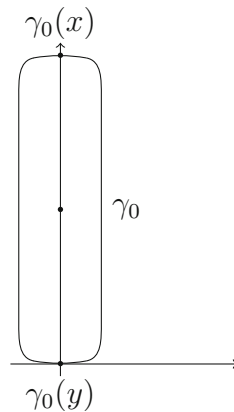
Fig. 1 An embedded, convex initial curve for which the diameter increases

taking into account our situation. The length of γ_0 can be chosen to be $L(\gamma_0) = 2 + \epsilon$. The curve only needs to satisfy $L(\gamma_0) < \pi$ to get an example where $\frac{d}{dt} \operatorname{diam}(\gamma_t)\big|_{t=0} > 0$, in particular $\kappa_0(x) - \overline{\kappa}(0) < 0$.

There is something else worth noticing about the above example: The evolution equation of the curvature along the area preserving curve shortening flow is

$$\partial_t \kappa = \partial_s^2 \kappa + (\kappa - \overline{\kappa})\kappa^2.$$

We use this for our initial curve γ_0 at the points x and y (where the maximal distance is attained). Since $\kappa \equiv 2$ on a small neighborhood around x and y, we know $\partial_s^2 \kappa(x) = 0 = \partial_s^2 \kappa(y)$ and hence

$$\partial_t \kappa(x)\big|_{t=0} = \partial_t \kappa(y)\big|_{t=0} = (\kappa_0(x) - \overline{\kappa}(0))\kappa^2(x) < 0.$$

This means that, at the points $\gamma_0(x)$ and $\gamma_0(y)$, the curve becomes even a bit flatter, and the diameter grows a bit. This is surprising, because eventually, this curve converges to a round circle. This example shows that the short-time behavior of the flow can be somehow counterintuitive.

4 Finite time singularities and monotonicity results

4.1 Monotonicity formula

We derive a monotonicity formula for VPMCF and demonstrate various corollaries following a similar course to [20]. This has been calculated by the second author in [38] already, inspired by [5]. Note that also [46] contains a version of a monotonicity formula.

The evolution equation of the area measure is

$$\frac{d}{dt} d\mu = -(H - \overline{H})H d\mu.$$

We now set $\Phi = \Phi_{(x_0,t_0)}$ the standard (appropriately rescaled version of the) "ambient backward heat kernel",

$$\Phi_{(x_0,t_0)}(x,t) := \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}, \quad x \in \mathbb{R}^{n+1}, t < t_0,$$

where in the following calculation we will suppress the subscript for simplicity of notation.

Proposition 9 *(Monotonicity formula) Suppose that M_t satisfies smooth VPMCF. We define*

$$\psi(t) := e^{-\frac{1}{2}\int_0^t \overline{H}^2(s)ds}$$

and let ϕ be a C^2-function that is either defined on $M^n \times [0,T)$ or on a neighborhood of M_t in $\mathbb{R}^{n+1}$. Then we have that

$$\frac{d}{dt}\left(\psi \int_{M_t} \phi\Phi d\mu\right) = \psi \int_{M_t} \Phi\left(\frac{d}{dt} - \Delta\right)\phi d\mu + \psi \int_{\partial M_t} \Phi\nabla_\mu\phi - \phi\nabla_\mu\Phi d\mu^\partial$$

$$- \frac{\psi}{2}\int_{M_t}\left[\left|(H-\overline{H}) - \frac{\langle v, x-x_0\rangle}{2(t_0-t)}\right|^2 + \left|H - \frac{\langle v, x-x_0\rangle}{2(t_0-t)}\right|^2\right]$$

$$\times \phi\Phi d\mu.$$

Proof The proof can be found in [38], see also [39, Remark before Proposition 4.10]. For the convenience of the reader we repeat it here. For general functions ϕ and $\Phi : \mathbb{R}^{n+1} \times [0,T) \to \mathbb{R}$ we have that

$$\frac{d}{dt}\int_{M_t}\phi\Phi d\mu = \int_{M_t}\Phi\left(\frac{d}{dt} - \Delta\right)\phi + \mathrm{div}(\Phi\nabla\phi - \phi\nabla\Phi)$$

$$+ \phi\left(\frac{d\Phi}{dt} + \Delta\Phi\right) - H(H-\overline{H})\phi\Phi d\mu$$

$$= \int_{M_t}\Phi\left(\frac{d}{dt} - \Delta\right)\phi + \mathrm{div}(\Phi\nabla\phi - \phi\nabla\Phi)d\mu$$

$$+ \int_{M_t}\left[\frac{1}{\Phi}\left(\left(\frac{\partial}{\partial t} + \overline{\Delta}\right)\Phi - \overline{\nabla}^2_{vv}\Phi\right.\right.$$

$$\left.\left. - (H-\overline{H})\overline{\nabla}_v\Phi - H\overline{\nabla}_v\Phi\right) - H(H-\overline{H})\right]\phi\Phi d\mu.$$

For our choice of Φ, we get that

$$\frac{\partial\Phi}{\partial t} = \left(\frac{n}{2(t_0-t)} - \frac{|x-x_0|^2}{4(t_0-t)^2}\right)\Phi, \qquad \overline{\nabla}_V\Phi = -\frac{\langle V, x-x_0\rangle}{2(t_0-t)}\Phi,$$

$$\overline{\nabla}^2_{VW}\Phi = \left(\frac{\langle V, x-x_0\rangle\langle W, x-x_0\rangle}{4(t_0-t)^2} - \frac{\langle V, W\rangle}{2(t_0-t)}\right)\Phi.$$

In particular we have

$$\left(\frac{\partial}{\partial t} + \overline{\Delta}\right)\Phi = \frac{-1}{2(t_0-t)}\Phi, \qquad \overline{\nabla}^2_{vv}\Phi = \left(\left(\frac{\langle v, x-x_0\rangle}{2(t_0-t)}\right)^2 - \frac{1}{2(t_0-t)}\right)\Phi$$

and so the contents of the square brackets in the last integrand above are

$$
-\left(\frac{\langle \nu, x - x_0 \rangle}{2(t_0 - t)}\right)^2 + (H - \overline{H})\frac{\langle \nu, x - x_0 \rangle}{2(t_0 - t)} + H\frac{\langle \nu, x - x_0 \rangle}{2(t_0 - t)} - H(H - \overline{H})
$$

$$
= -\frac{1}{2}\left|(H - \overline{H}) - \frac{\langle \nu, x - x_0 \rangle}{2(t_0 - t)}\right|^2 - \frac{1}{2}\left|H - \frac{\langle \nu, x - x_0 \rangle}{2(t_0 - t)}\right|^2 + \frac{1}{2}\overline{H}^2 .
$$

We have chosen $\psi(t) = e^{-\frac{1}{2}\int_0^t \overline{H}^2(s)ds}$ so that $\frac{d\psi}{dt} = -\frac{1}{2}\overline{H}^2$ which yields

$$
\frac{d}{dt}\int_{M_t}\psi\phi\Phi d\mu = \int_{M_t}\psi\Phi\left(\frac{d}{dt} - \Delta\right)\phi d\mu + \psi\int_{\partial M_t}\Phi\nabla_\mu\phi - \phi\nabla_\mu\Phi d\mu^\partial
$$

$$
- \frac{\psi}{2}\int_{M_t}\left[\left|(H - \overline{H}) - \frac{\langle \nu, x - x_0 \rangle}{2(t_0 - t)}\right|^2 + \left|H - \frac{\langle \nu, x - x_0 \rangle}{2(t_0 - t)}\right|^2\right]\phi\Phi d\mu .
$$

$\square$

Remark 6 We have that

$$
\frac{d}{dt}\log|M_t| = -\fint_{M_t}H^2 d\mu + \overline{H}^2
$$

and so we may rewrite

$$
\psi(t) = \sqrt{\frac{|M_0|}{|M_t|}}e^{\int_0^t \fint_{M_s} H^2 d\mu ds}
$$

Remark 7 Consider a parabolic rescaling of M_t, $\widetilde{M}_s$ where $t = \lambda^2 s + t_0$, and $M_t = \lambda\widetilde{M}_s + x_0$. Then $\overline{H}(\widetilde{M}_s) = \lambda\overline{H}(M_t)$ and we have that

$$
\log\widetilde{\psi}(s) = \int_0^s \overline{H}(\widetilde{M}_{\hat{s}})^2 d\hat{s} = \int_0^s \lambda^2\overline{H}(M_{t(\hat{s})})^2 d\hat{s} = \int_{t_0}^t \overline{H}(M_{\hat{t}})^2 d\hat{t} = \log\psi(t) ,
$$

which implies that our object to differentiate is invariant under parabolic rescaling.

4.2 VPMCF blowups are ancient MCFs

We will use the following MCF terminology.

Definition 2 Given any $X(x, t)$ satisfying VPMCF the parabolic rescaling about a point $p \in \mathbb{R}^{n+1}$ at time T given by

$$
X_p^\lambda(x, \tau) := \lambda(X(x, \lambda^{-2}\tau + T) - p)
$$

for $\tau \le 0$. Equivalently we may write this as

$$
M_\tau^{\lambda, p} = \lambda(M_{\lambda^{-2}\tau + T} - p) .
$$

A parabolic rescaling of VPMCF gives another solution of VPMCF (although with a different fixed volume).

Definition 3 Let M_t be a mean curvature flow on the time interval $[0, T)$, and suppose that the second fundamental form blows up at time T that is, $\limsup_{t \to T}\sup_{M_t}|A| = \infty$. Then we say that this is a singularity of VPMCF where

(i) the singularity is of type I if there exists a $C > 0$ such that

$$\sup_{M_t} |A|^2 \leq \frac{C}{T - t}$$

for all $t < T$ and
(ii) the singularity is of type II otherwise.

Definition 4 Let M_t be a mean curvature flow on the time interval $[0, T)$. Then

(i) Suppose we are given a point $p \in \mathbb{R}^{n+1}$ and a sequence $\lambda_i \to \infty$. Then a type I (or centred) blow up sequence is the sequence of solutions to VPMCF given by

$$M_\tau^i = \lambda_i (M_{\lambda_i^{-2}\tau + T} - p)$$

for $\tau \in [-\lambda_i^2 T, 0)$.
(ii) Assuming that the second fundamental form blows up as $t \to T$, then we follow Hamilton [29] and define a type II blow up sequence by a sequence times t_i and points $x_i \in M_{t_i}$ such that

$$\lambda_i := \lambda_i^2 (T - i^{-1} - t_i) = \max_{(x,t) \in M^n \times [0, T - i^{-1}]} |A(x,t)|^2 (T - i^{-1} - t)$$

Then the type II blow up along (x_i, t_i) is given by

$$M_\tau^i = \lambda_i (M_{t_i + \lambda_i^{-2}\tau} - x_i)$$

for $\tau \in [-\lambda_i^2 t_i, \lambda_i^2 (T - i^{-1} - t_i))$.

Remark 8 Once we have uniformly bounded curvature, $\overline{H}$ is bounded and Shi-type estimates hold to all orders, as in [20, Proposition 3.22]. As a result, if a type I singularity occurs, then any type I blowup sequence will converge locally smoothly to a smooth flow for $\tau \in (-\infty, 0)$ (which is possibly empty depending on the p). This will be referred to as an ancient flow. Later in Lemma 11 we will see that this is a self similar solution of MCF. If we have a type II singularity then the type II blowup sequence converges smoothly to a flow for $\tau \in (-\infty, \infty)$ [29].

We are now in a position to prove the following:

Theorem 10 *Suppose that a singularity of VPMCF occurs at a finite time. Then*

(a) If the singularity is type I then any locally smoothly converging subsequence of the type I blow up sequence is an ancient solution to MCF.
(b) If the singularity if type II then any locally smoothly converging subsequence of a type II blow up sequence yields an eternal solution to MCF.

Proof We first consider the type I blow up case: By dilating we may assume wlog that $|A|^2(x,t) < \frac{1}{T-t}$. By Corollary 8, we know that for any finite T

$$\int_0^T \overline{H}^2(t)dt < C .$$

By continuous dependence of an integral on it's domain of definition (Lebesgue's differentiation theorem) we have that there exist $T_i \to T$ such that

$$\int_{T_j}^T \overline{H}^2 dt = j^{-1} \tag{8}$$

and therefore, writing $t = \lambda_i^{-2}(\tau + T)$ then

$$j^{-1} = \int_{T_j}^{T} \overline{H}^2(t)dt = \int_{-\lambda^2(T-T_j)}^{0} \lambda^{-2}\overline{H}^2(\tau)d\tau = \int_{-\lambda_i^2(T-T_j)}^{0} (\overline{H}^i)^2(\tau)d\tau .$$

where $\overline{H}^i$ is the average mean curvature of M_τ^i of the type I blowup. Due to the type I singularity hypothesis, the second fundamental form of M_τ^i satisfies

$$|A^i(x,\tau)|^2 \leq \frac{1}{|\tau|} .$$

Furthermore, we have that

$$\left| \frac{d}{dt}\overline{H}^i \right| = \left| \int (H^i - \overline{H}^i)|A^i|^2 - H^i(H^i - \overline{H}^i)^2 d\mu \right| \leq \frac{4}{|\tau|^3} \qquad (9)$$

Given any $a < b < 0$, if λ_i is so large that $[a,b] \subset [-\max\{\lambda_i^2(T-T_j), \lambda_i^2 T\}, 0)$ then we have that there is a constant $C_1 = C_1(a,b)$ such that

$$\overline{H}^i < C_1(a,b)j^{-1}$$

where we used the estimate (9). Therefore, for any smoothly converging subsequence, $\overline{H}^i \to 0$ uniformly on $[a,b]$ and so the flow converges smoothly on $[a,b]$ to a solution to MCF,

$$\frac{dX^\infty}{dt} = -H^\infty v^\infty .$$

Note that the above argument implies that for *any* subsequence such that the blow up sequence converges in C^2, we still get a mean curvature flow.

The argument for the type II blow up is similar: Suppose that we have a converging subsequence of a type II blow up. Then, defining T_j as in (8) then substituting $t = t_i + \lambda_i^{-2}\tau$

$$j^{-1} = \int_{T_j}^{T} \overline{H}^2(t)dt \geq \int_{\lambda_i^2(T_j-t_i)}^{\lambda_i^2(T-t_i-i^{-1})} \lambda_i^{-2}\overline{H}^2(\tau)d\tau = \int_{\lambda_i^2(T_j-t_i)}^{\lambda_i^2(T-t_i-i^{-1})} \overline{H}_i^2(\tau)d\tau .$$

By the type II hypothesis we know that $\lambda_i^2(T-t_i-i^{-1}) \to \infty$, see for example [41, p. 89], while we may assume that for i large enough $T_j < t_i$ and so $\lambda_i^2(T_j-t_i) \to -\infty$. This time, by our choice of blow up sequence, $|A^i(x,t)| \leq 1$ on the time interval $[-\lambda_i^2 t_i, \lambda_i^2(T-i^{-1}-t_i))$ and so, estimating similarly to (9), on this time interval $|\frac{d}{dt}\overline{H}| \leq 4$. As a result, given any finite time interval $[a,b] \subset \mathbb{R}$, if i is large enough so that $[a,b] \subset [\lambda_i^2(T_j-t_i), \lambda_i^2(T-t_i-i^{-1})]$ then there exists a $C_2 = C_2(b-a)$ such that $\overline{H}^i < C_2 j^{-1}$. As we know that the type II sequence converges locally smoothly, we see that on $[a,b]$ we $\overline{H}^i \to 0$ uniformly and again the limit is a solution to MCF. $\qquad \square$

Lemma 11 *Suppose that a singularity of VPMCF occurs at a finite time T which is of type I. Then the asymptotic limit flow from the above theorem is a (possibly trivial) properly immersed, non-compact homothetically shrinking solution of the Mean Curvature Flow.*

Proof The proof works as in the case of type I singularities of the Mean Curvature Flow. We sketch the proof for the convenience of the reader. Note that the Monotonicity formula implies

$$\frac{d}{dt}\left(\psi(t) \int_{M_t} \Phi_{(x_0,T)}(x,t) d\mu_t^n \right) \leq 0,$$

thus the limit $\lim_{t \to T} \Lambda_{(x_0, T)}(t)$ exists, where

$$\Lambda_{(x_0, T)}(t) := \psi(t) \int_{M_t} \Phi_{(x_0, T)}(x, t) \, d\mu_t^n \, .$$

By parabolic rescaling we have that

$$\Lambda_{(x_0, T)}(t) = \psi(t) \int_{M_t} \Phi_{(x_0, T)}(x, t) \, d\mu_t = \psi_i(\tau) \int_{M_\tau^i} \Phi_{(0,0)}(y, \tau) \, d\mu_\tau^i =: \Lambda_{(0,0)}^i(\tau) \, ,$$

where $t = T + \frac{\tau}{\lambda_i^2}$. We apply the rescaled monotonicity formula and estimate for $\tau_1 < \tau_2$

$$0 \leq \frac{1}{2} \int_{\tau_1}^{\tau_2} \psi_i(\tau) \int_{M_\tau^i} \left| H^i + \frac{\langle y, \nu \rangle}{2\tau} \right|^2 \Phi_{0,0} \, d\mu_\tau^n d\tau$$

$$\leq \frac{1}{2} \int_{\tau_1}^{\tau_2} \psi_i(\tau) \int_{M_\tau^i} \left\{ \left| H^i - \overline{H}^i + \frac{\langle y, \nu \rangle}{2\tau} \right|^2 + \left| H^i + \frac{\langle y, \nu \rangle}{2\tau} \right|^2 \right\} \times \Phi_{0,0} \, d\mu_\tau^n d\tau$$

$$\leq \Lambda_{(0,0)}^i(\tau_1) - \Lambda_{(0,0)}^i(\tau_2)$$

$$= \Lambda_{(x_0, T)} \left(T + \frac{\tau_1}{\lambda_i^2} \right) - \Lambda_{(x_0, T)} \left(T + \frac{\tau_2}{\lambda_i^2} \right)$$

for all i. Since

$$T + \frac{\tau_l}{\lambda_i^2} \to T$$

for $i \to \infty$ and $l = 1, 2$, the right-hand side of the above converges to 0 for $i \to \infty$. The flows $\left((M_\tau^i)_{\tau \in [\tau_1, \tau_2]} \right)_{i \in \mathbb{N}}$ converge smoothly along a subsequence and on compact subsets of $\mathbb{R}^{n+1}$ to a smooth flow $(M_\tau^\infty)_{\tau \in [\tau_1, \tau_2]}$. Let $R > 0$. There exists a $i_0 \in \mathbb{N}$ so that for all $i \geq i_0$, $M_\tau^i \cap B_R(0)$ can be parametrised over $M_\tau^\infty \cap B_R(0)$. That is, there exist immersions $Y_i : M_\tau^\infty \cap B_R(0) \to \mathbb{R}^{n+1}$ with

$$M_\tau^i \cap B_R(0) = Y_i(M_\tau^\infty \cap B_R(0))$$

and $Y_i \to \mathrm{id}$ for $i \to \infty$. From the above we have that

$$\Lambda_{(x_0, T)} \left(T + \frac{\tau_1}{\lambda_i^2} \right) - \Lambda_{(x_0, T)} \left(T + \frac{\tau_2}{\lambda_i^2} \right)$$

$$\geq \frac{1}{2} \int_{\tau_1}^{\tau_2} \psi_i(\tau) \int_{M_\tau^i \cap B_R(0)} \left| H^i + \frac{\langle y, \nu \rangle}{2\tau} \right|^2 \Phi_{0,0} \, d\mu_\tau^n d\tau \, .$$

Taking a lim inf and applying Fatou's lemma,

$$0 = \liminf_{i \to \infty} \int_{\tau_1}^{\tau_2} \int_{M_\tau^i \cap B_R(0)} \frac{\psi_i(\tau)}{2} \left| H^i + \frac{\langle y, \nu^i \rangle}{2\tau} \right|^2 \Phi_{0,0} \, d\mu_\tau^i d\tau$$

$$= \liminf_{i \to \infty} \int_{\tau_1}^{\tau_2} \int_{M_\tau^i \cap B_R(0)} \frac{\psi_i(\tau)}{2} \left| H^i + \frac{\langle y, \nu^i \rangle}{2\tau} \right|^2 \Phi_{0,0} \sqrt{\det(DY_i)} \, dx d\tau$$

$$\geq \int_{\tau_1}^{\tau_2} \int_{M_\tau^\infty \cap B_R(0)} \liminf_{i \to \infty} \left(\frac{\psi_i(\tau)}{2} \left| H^i + \frac{\langle Y_i, \nu^i \rangle}{2\tau} \right|^2 \Phi_{0,0} \sqrt{\det(DY_i)} \right) dx d\tau$$

$$= \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_{M_\tau^\infty \cap B_R(0)} \left| H^\infty + \frac{\langle y, \nu^\infty \rangle}{2\tau} \right|^2 \Phi_{0,0} \, d\mu_\tau^\infty d\tau \, ,$$

where we have used that $\overline{H}^i \to 0$ uniformly as shown in the proof of Theorem 10, and thus $\psi_i(\tau) \to 1$. Since $R > 0$ was arbitrary, we deduce

$$\int_{\tau_1}^{\tau_2} \int_{M_\tau^\infty} \left| H^\infty + \frac{\langle y, \nu^\infty \rangle}{2\tau} \right|^2 \Phi_{0,0} \, d\mu_\tau d\tau = 0 \, .$$

Since the convergence is smooth, and sending $\tau_1 \to -\infty$ and $\tau_2 \to 0$ yields

$$\left| H^\infty + \frac{\langle y, \nu^\infty \rangle}{2\tau} \right|^2 = 0$$

for every $\tau \in (-\infty, 0)$ and every $y \in M_\tau^\infty$, so M_τ^∞ is a homothetically shrinking (possibly trivial) solution of the Mean Curvature Flow.

It remains to show that the limit flow is proper and non-compact. The rescaled monotonicity formula and the L^2-bound on $\overline{H}$ implies

$$\int_{M_\tau^i} \Phi_{(0,0)}(y, \tau) d\mu_\tau^n \leq C$$

for all $\tau \in (-\infty, 0)$ and for all i. Fatou's lemma on ambient balls yields

$$C \geq \int_{M_\tau^\infty \cap B_R(0)} \Phi_{(0,0)} d\mu_\tau^\infty$$

$$\geq \frac{1}{(-4\pi\tau)^{\frac{n}{2}}} \exp\left(-\frac{R^2}{-4\tau} \right) |M_\tau^\infty \cap B_R(0)|,$$

which implies that M_τ^∞ is proper. The enclosed volume of the rescaled flows is $V^i = \lambda_i^{n+1} V_0$. Since $V_0 \neq 0$, this clearly blows up as $\lambda_i \to \infty$. If the limit flow were compact, it would be contained in a ball with finite radius (for each τ). But because of the properness of the limiting hypersurface M_τ^∞ there can only be finite volume contained in that ball which is a contradiction. Thus, the limiting ancient flow is non-compact. □

Remark 9 See [10] for examples of non-compact self-shrinkers of arbitrary genus.

4.3 Density estimates and the clearing out lemma

In this section we demonstrate that the monotonicity arguments in [20, Chapter 4] may be modified to the case of VPMCF with essentially minor additional assumptions depending only on the L^2 norm of $\overline{H}$. Throughout this section we will assume that our flow is smooth and properly immersed up to the final, possibly singular, time.

From the Monotonicity formula we have that

$$\frac{d}{dt} \left(\psi \int_{M_t} \phi \Phi_{(x_0, t_0)} d\mu \right) \leq \psi \int_{M_t} \Phi_{(x_0, t_0)} \left(\frac{d}{dt} - \Delta \right) \phi d\mu \, .$$

and from Corollary 8 we have that $1 = \psi(0) \geq \psi(t) \geq \epsilon(t) > 0$ for some strictly positive function $\epsilon(t)$, and by the dominated convergence theorem, $\psi(t)$ is continuous.

Definition 5 The space-time track of a flow is defined by

$$\mathcal{M} := \cup_{t \in [0,T)} M_t \times \{t\} \subset \mathbb{R}^{n+1} \times [0, T) .$$

The Gaussian density on $\mathcal{M}$ is determined by

$$\Theta(\mathcal{M}, x_0, t_0) := \lim_{t \nearrow t_0} \int_{M_t} \Phi_{(x_0, t_0)} d\mu_t$$

where, as usual, integration also counts multiplicities in $\mathcal{M}$.

Indeed, by the monotonicity formula (with $\phi = 1$) we see that

$$\int_{M_t} \Phi_{(x_0, t_0)} d\mu_t \leq \frac{1}{\psi(t)} \int_{M_0} \Phi_{(x_0, t_0)} d\mu_0 =: \frac{C_0(M_0)}{\psi(t)}$$

and so by dominated convergence theorem, the density converges everywhere. It is also useful to define a localised density, and so we consider the localisation function

$$\phi_{(x_0, t_0), \rho}(x, t) := \left(1 - \frac{|x - x_0|^2 + 2n(t - t_0)}{\rho^2} \right)_+^3$$

then we have that on the support of $\phi_{(x_0, t_0), \rho}(x, t)$,

$$\left(\frac{d}{dt} - \Delta \right) \phi_{(x_0, t_0), \rho}(x, t) = -\frac{6}{\rho^2} \overline{H} \langle x - x_0, \nu \rangle \phi_{(x_0, t_0), \rho}^{\frac{2}{3}} \leq \frac{6}{\rho} |\overline{H}|$$

and so by the monotonicity formula

$$\frac{d}{dt} \left(\psi(t) \int_{M_t} \phi_{(x_0, t_0), \rho} \Phi_{(x_0, t_0)} d\mu \right) \leq \frac{6}{\rho} |\overline{H}| \int_{M_t} \psi(t) \Phi_{(x_0, t_0)} d\mu \leq \frac{6}{\rho} |\overline{H}| C_0(M_0)$$

where we apply the monotonicity formula to estimate the integral in terms of M_0. In particular, we may estimate that for any $t_2 < t_3$,

$$\psi(t_3) \int_{M_{t_3}} \phi_{(x_0, t_0), \rho} \Phi_{(x_0, t_0)} d\mu$$

$$\leq \psi(t_2) \int_{M_{t_2}} \phi_{(x_0, t_0), \rho} \Phi_{(x_0, t_0)} d\mu + 6 C_0(M_0) \sqrt{\frac{t_3 - t_2}{\rho^2}} \sqrt{\int_{t_2}^{t_3} \overline{H}^2 dt} . \quad (10)$$

With some abuse of notation, we define localised monotonicity to be

$$\Theta(\mathcal{M}, x_0, t_0) := \lim_{t \nearrow t_0} \int_{M_t} \phi_{(x_0, t_0), \rho} \Phi_{(x_0, t_0)} d\mu_t ,$$

and note that, by (10), this limit always exists and we may estimate that for $t \in (t_0 - \rho^2, t_0)$

$$\Theta(\mathcal{M}, x_0, t_0) \leq \frac{\psi(t)}{\psi(t_0)} \int_{M_t} \phi_{(x_0, t_0), \rho} \Phi_{(x_0, t_0)} d\mu + 6 \frac{C_0(M_0)}{\psi(t_0)} \sqrt{\frac{t_0 - t}{\rho^2}} \sqrt{\int_t^{t_0} \overline{H}^2 dt}. \quad (11)$$

Lemma 12 (*Upper semicontinuity of* Θ) $\Theta(\mathcal{M}, x_0, t_0)$ *is upper semi continuous in time and space. Explicitly, if* $x_j \to x_0$, $t_j \nearrow t_0$ *then*

$$\limsup_{j \to \infty} \Theta(\mathcal{M}, x_j, t_j) \leq \Theta(\mathcal{M}, x_0, t_0) .$$

Proof We have that for any $t \in (t_0 - \rho^2, t_0)$, if j is large enough then

$$\Theta(\mathcal{M}, x_j, t_j) \leq \frac{\psi(t)}{\psi(t_j)} \int_{M_t} \phi_{(x_j, t_j), \rho} \Phi_{(x_j, t_j)} d\mu + 6 \frac{C_0(M_0)}{\psi(t_j)} \sqrt{\frac{t_j - t}{\rho^2}} \sqrt{\int_t^{t_j} \overline{H}^2 dt} \ .$$

so, taking a limsup,

$$\limsup_{j \to \infty} \Theta(\mathcal{M}, x_j, t_j) \leq \frac{\psi(t)}{\psi(t_0)} \int_{M_t} \phi_{(x_0, t_0), \rho} \Phi_{(x_0, t_0)} d\mu + +6 \frac{C_0(M_0)}{\psi(t_0)} \sqrt{\frac{t_0 - t}{\rho^2}} \sqrt{\int_t^{t_0} \overline{H}^2 dt}$$

and limiting $t \to t_0$ yields the required inequality (as ψ is continuous). □

Corollary 13 *For any point* (x_0, t_0) *reached by a smooth properly immersed flow,*

$$\Theta(\mathcal{M}, x_0, t_0) \geq 1$$

Proof On any smooth manifold with $x_j \in M_{t_j}$, $\Theta(\mathcal{M}, x_j, t_j) = 1$, so the statement follows from upper semicontinuity. □

Proposition 14 *(The clearing out lemma) Suppose that M_t is smooth and properly immersed for $t \in [t_0 - \rho_0^2, t_0)$. We suppose that for positive constants C_0 and L*

$$\int_{t_0 - \rho_0^2}^{t_0} \overline{H}^2 dt \leq L \qquad \psi(t_0 - \rho_0^2) \int_{M_{t_0 - \rho_0^2}} \Phi_{(x_0, t_0)} d\mu \leq C_0 \ .$$

Then there exists a constant $\beta_0 = \beta_0(n, C_0, L)$ such that for any $\beta \in (0, \beta_0)$ there exists a constant $\theta(\beta, n)$ such that for any $\rho \in (0, \rho_0)$

$$\frac{\mathcal{H}^n(M_{t_0 - \beta \rho^2} \cap B_\rho)}{\rho^n} \geq \theta e^{-\frac{L}{2}} \ .$$

Proof By Corollary 13 and (11) we have that for $t \in (t_0 - \sigma^2, t_0)$ where $\sigma \in (0, \rho_0)$,

$$1 \leq \frac{\psi(t)}{\psi(t_0)} \int_{M_t} \phi_{(x_0, t_0), \sigma} \Phi_{(x_0, t_0)} d\mu + 6 \frac{C_0(M_0)}{\psi(t_0)} \sqrt{\frac{t_0 - t}{\sigma^2}} \sqrt{\int_t^{t_0} \overline{H}^2 dt} \ .$$

Rewriting, and setting $t_0 - t = \alpha \sigma^2$ for some $\alpha \in (0, 1]$,

$$1 - 6 \frac{C_0(M_0)}{\psi(t_0)} \sqrt{\frac{t_0 - t}{\sigma^2}} \sqrt{\int_t^{t_0} \overline{H}^2 dt}$$

$$\leq \frac{\psi(t)}{\psi(t_0)} \int_{M_t} \left(1 - \frac{|x - x_0|^2 + 2n(t - t_0)}{\sigma^2}\right)_+^3 \frac{1}{(4\pi(t_0 - t))^{\frac{n}{2}}} e^{-\frac{|x - x_0|^2}{4(t_0 - t)}} d\mu$$

$$\leq \frac{\psi(t)}{\psi(t_0)} \frac{(1 + 2n\alpha)^3}{(4\pi\alpha\sigma^2)^{\frac{n}{2}}} \mathcal{H}^n(M_{t_0 - \alpha\sigma^2} \cap B_{\sqrt{1 + 2n\alpha}\sigma}(x_0)) \ .$$

We wish to change constants from (α, σ) to (β, ρ) by setting $\rho = \sqrt{1 + 2n\alpha}\sigma$ and $\beta\rho^2 = \alpha\sigma^2$. For this to be possible, we need $\beta < \frac{1}{2n}$ at which point we have that

$$\beta = \frac{\alpha}{1 + 2n\alpha} \qquad \alpha = \frac{\beta}{1 - 2n\beta} \qquad \sigma = \sqrt{1 - 2n\beta}\rho$$

Therefore,

$$\frac{(4\pi\beta)^{\frac{n}{2}}}{(1+2n\alpha)^3}\left(1-6\sqrt{\alpha}C_0\sqrt{L}e^{\frac{L}{2}}\right) \le e^{\frac{L}{2}}\frac{\mathcal{H}^n(M_{t_0-\beta\rho^2}\cap B_\rho)}{\rho^n}\ .$$

To get a positive density we want $\alpha = \min\{\frac{e^{-L}}{144C_0^2L}, 1\}$ which corresponds to

$$\beta \le \min\left\{\frac{1}{144C_0^2Le^L+2n}, \frac{1}{1+2n}\right\} =: \beta_0\ .$$

Finally we obtain,

$$\theta(n,\beta)e^{-\frac{L}{2}} := \frac{1}{2}(4\pi\beta)^{\frac{n}{2}}(1-2n\beta)^3e^{-\frac{L}{2}} \le \frac{\mathcal{H}^n(M_{t_0-\beta\rho^2}\cap B_\rho(x_0))}{\rho^n}\ .$$

□

4.4 A condition for a uniform diameter bound

In this section we apply the clearing out lemma to obtain conditions under which we may ensure a uniform curvature bound.

Proposition 15 *Suppose that there exists constants $h, C > 0$ such that for all $t \in [0, T-h)$*

$$\int_t^{t+h} \overline{H}^2 dt < C\ . \tag{12}$$

Then there exists a constant $R = R(C, n, M_0)$ such that for all $t \in [0, T)$

$$\mathrm{diam}(M_t) < R\ .$$

Proof We apply the clearing out lemma (Proposition 14) with $\rho_0 = \sqrt{h}$ on any time interval of the form $[t, t+h)$. On such an interval by assumption we have the L^2-bound on $\overline{H}$, and the second assumption is automatically fulfilled for some $C_0 = C_0(M_0)$ by the monotonicity formula. Therefore we have: for $\beta = \frac{\beta_0(n,C,M_0)}{2}$ there is a $\tilde{\theta} = \theta(\beta, n)e^{-\frac{C}{2}}$ such that $\mathcal{H}^n(M_{t_0-\beta\rho^2}\cap B_\rho) \ge \tilde{\theta}\rho^n$.

Suppose that $t > h\beta$. We have that $|M_t| \le |M_0|$ and there exists $\frac{\rho_0}{4} \le \rho < \rho_0$ such that $\mathrm{diam}(M_t) = 2N\rho$ for $N \in \mathbb{N}$. Then there must be at least N disjoint ambient balls of radius ρ whose centers lie on M_t. We use the area of M_t in these balls as lower bounds. We see that (as $t > \beta\rho^2$)

$$|M_0| > |M_{t-\beta\rho^2}| > N\tilde{\theta}\rho^n = \frac{1}{2}\tilde{\theta}\rho^{n-1}\mathrm{diam}(M_t).$$

As $\rho \ge \frac{\rho_0}{4}$ and $\tilde{\theta}$ is fixed, we see that $\mathrm{diam}(M_t) \le 2\tilde{\theta}^{-1}\rho_0^{1-n}4^{n-1}|M_0|$.
If $t < h\beta$, we may instead apply Proposition 7 to see that $\mathrm{diam}(M_t) \le C(M_0, h, \beta)$. □

Remark 10 The condition (12) in the above Proposition may also be replaced by $\int_t^{t+h}\int_{M_t} H^2 d\mu dt < C$ or equivalently $|\frac{d}{dt}|M_t|| < C$. Clearly a uniform bound on $\overline{H}$ also implies this condition.

5 Long time behaviour of VPMCF

In this section we consider the flow under the assumptions that

a) the flow exists for all time, and
b) the diameter is uniformly bounded, that is, there exists an $R_0 > 0$ such that

$$\text{diam}(M_t) \leq R_0 \tag{13}$$

for all $t \in [0, \infty)$.

By Proposition 15 we can replace (13) with (12). Both of the above assumptions are therefore implied by a curvature bound on the flow which is uniform in time.

Definition 6 We define a "Newton inequality-like" extended isoperimetric ratio by the scale invariant quantity

$$\mathcal{I}(M_t) = \frac{n+1}{n} \overline{H} \frac{\text{Vol}^{n+1}(M_t)}{|M_t|} .$$

Remark 11 This isoperimetric ratio appears to be a natural one to consider: If we think of the Alexandrov–Fenchel inequalities as an "averaged" version of the Mclaurin inequalities for symmetric polynomials, then the above would be a ratio similar to an "averaged" Newton inequalities. Importantly, the Newton inequalities hold without any assumptions on the curvature cone.

Remark 12 Denoting ω_n for the area of the n-sphere, if we write $v_{n+1} = \frac{n+1}{\omega_n} \text{Vol}^{n+1}(M_t)$, $v_n = |M_t|$ and $v_{n-1} = \frac{1}{n} \int_{M_t} H d\mu$ then, while M_t is mean convex and starshaped, then it bounds some domain Ω_t and we know that, by the Alexandrov–Fenchel inequalities, $v_{n+1}^{\frac{1}{n+1}} \leq v_n^{\frac{1}{n}} \leq v_{n-1}^{\frac{1}{n-1}}$ with equalities if and only if M_t is a sphere (see [28] – here we are writing, $v_{n+1-k} = \frac{V_{n+1-k}(\Omega_t)}{V_{n+1-k}(B)}$ in that paper). Applying these we have that

$$1 \leq \frac{v_{n+1}}{v_n^{\frac{n+1}{n}}} \leq \mathcal{I}(M_t) = \frac{v_{n-1} v_{n+1}}{v_n^2} \leq \frac{v_{n-1}}{v_n^{\frac{n-1}{n}}} = \frac{1}{\mathcal{I}_1(\Omega_t)^{n-1}} ,$$

where $\mathcal{I}_1(\Omega_t)$ is the first Quermass integral [28, equation (6)]. Equalities hold in the above if and only if M_t is a sphere. Therefore if M_t is starshaped and mean convex with $\mathcal{I}(M_t) = 1$, M_t is a round sphere.

Under the above assumptions we may improve the assumptions of Sect. 3 to get the following:

Theorem 16 *Suppose that M_t satisfies VPMCF for all $t > 0$ with $V_0 \neq 0$ and has uniformly bounded diameter as in (13). Then for any sequence $t_i \to \infty$ there exists a subsequence (not relabeled) such that either*

$$\mathcal{I}(M_{t_i}) \to 1 \quad or \quad \mathcal{I}(M_{t_i}) \to 0$$

as $i \to \infty$. Equivalently, there exists a subsequence such that the extended isoperimetric ratio goes to that of a constant mean curvature surface or of a minimal surface.

Corollary 17 *Let X be a mean convex embedded starshaped solution of the VPMCF with uniformly bounded curvature for all times. Then $X(\cdot, t)$ converges smoothly to a round sphere with enclosed volume $V(X(\cdot, 0))$ for $t \to \infty$.*

Proof of Corollary 17 We first note that the uniform curvature bound implies that the diameter is bounded by Proposition 15. The area remains also bounded due to the gradient flow property. This allows us to use Shi-type estimates as sketched in Remark 8 to get uniform estimates on all derivatives of curvature.

We now pick a sequence of points and times (p_i, t_i) such that $M_{t_i} =: M_i$ is starshaped over p_i. We denote $X_i : S^n \to \mathbb{R}^{n+1}$ to be the immersion for M_i translated so that p_i is moved to the origin. We may now apply the compactness theorem of Breuning [9] and get a limiting immersion $\tilde{X} : \tilde{M} \to \mathbb{R}^{n+1}$ and smooth subconvergence of X_i to $\tilde{X}$ on compact subsets of $\tilde{M}^n$ with the following properties: a) $\tilde{M}^n$ is connected and a topological sphere because the domain enclosed by X_i is starshaped around 0 for all i – so $\tilde{M}^n$ can be assumed to be $M^n = \mathbb{S}^n$ (we compose with appropriate diffeomorphisms if necessary, see [9]); b) in fact, the convergence is uniform because M^n is compact.

For the convenience of the reader, we sketch the above compactness result under our stronger assumptions (which include having a sequence of immersions on a fixed manifold): Due to the curvature bound there exists an $r > 0$ such that for any i, M_i may be locally written as graphs over a ball $B_{3r} \subset T_p M_i$ for any $p \in M_i$. As a result we claim that there exists an N such that for all i, M_i may be covered by N graphs over B_r tangent balls. Take all such balls of radius $B_{\frac{r}{5}}$ then by the Vitali covering lemma there exists a countable subcollection of balls of radius r with centres $p_1^i, p_2^i, \ldots$ such that $B_r \subset T_{p_i} M_i$ cover M_i and the graphs over $B_{\frac{r}{5}}$ are disjoint. Each of the $B_{\frac{r}{5}}$ graphs contribute a fixed amount of area and so there can only be a finite number N given by our (uniform-in-i) area bound. Writing $p_1^i, \ldots, p_N^i$ for the tangent space centres on M_i, using that M_i stays in a bounded region given by the diameter bound, and by compactness of $S^n = Gr(\mathbb{R}^n; \mathbb{R}^{n+1})$, we may choose a subsequence (also labelled with i) so that these centres and the tangent spaces converge to a sequence of points $p_1^\infty, \ldots, p_N^\infty$, so that graphs over B_{2r} cover a smooth manifold M_∞. After taking further subsequences, Arzelà–Ascoli implies the convergence is smooth and that for i large enough, images of the X_i may be written as smooth local graphs over the p_i^∞ converging to M_∞. However we immediately see that for i large enough M_∞ has the same topology as M_i, which due to starshapedness must be a topological sphere.

As $\left\langle \frac{x}{|x|}, \nu \right\rangle \geq 0$ on the M_i, we get $\mathcal{I}(M_i) \geq 1$, which implies $\mathcal{I}(M_\infty) \geq 1$. By applying Theorem 16, we conclude $\mathcal{I}(M_\infty) = 1$. Therefore, as in Remark 12, we know that M_∞ is a round sphere. Finally, picking a t_{i_0} so that M_{i_0} is (smoothly) close enough to M_∞, and restarting the flow from t_{i_0} we may now apply [23] to see that the flow converges exponentially and smoothly to a round sphere. But now by smooth convergence it must be M_∞, which is a round sphere at finite distance from p_{i_0}. So the translating procedure can be reversed and the original immersions $X(\cdot, t)$ converge to a round sphere for $t \to \infty$.

Remark 13 • If we have a VPMCF that stays Alexandrov immersed for all times and $|A| \leq C$ uniformly, then for each $t_j \to \infty$, the flow subconverges smoothly to a closed limiting surface with constant mean curvature ($H(\cdot, \tilde{t}_j)$ must converge to $\overline{H}_\infty$ see e.g. [30]). In this Alexandrov immersed case the limiting object must be a sphere – and thus $\mathcal{I}(M_\infty) = 1$. This follows from Alexandrov's theorem: A closed Alexandrov immersion having constant mean curvature must be a sphere [1]. We can also improve subconvergence to convergence by [23] again.

• In general, if no singularity appears, i.e. $T = \infty$ and $|A| \leq C$ (uniformly in t), then $\mathcal{I}(M_{t_j}) \to 0$ cannot happen because any limiting surface satisfies $|M_\infty| > 0$, so $\mathcal{I}(M_{\tilde{t}_j}) \to 0$ would imply we found a closed minimal surface as limiting object which

is clearly impossible. At this point we do not know whether $\mathcal{I}(M_\infty) = 1$ implies that M_∞ is a sphere. We only know this for the starshaped case and for Alexandrov immersions.

In order to show Theorem 16 we first improve our estimates on $\overline{H}^2$ in the two cases of $\mathcal{I}(M_t) \leq 1$ (where $\mathcal{I}(\mathbb{S}^n) = 1$) and $\mathcal{I}(M_t) \geq 1$.

Proposition 18 *Suppose that M_t as in (13) and $V_0 \neq 0$. Let $\alpha \in (0, 1)$, and suppose that on the time interval $[t_1, t_2]$ we may estimate*

$$\overline{H}(n+1)V_0 \leq \alpha n |M_t| .$$

Then we have that

$$\int_{t_1}^{t_2} |\overline{H}|^2 dt \leq \max\left(\frac{\alpha^2}{(1-\alpha)^2}, 1\right) \frac{R_0^2}{2(n+1)^2 V_0^2} |M_{t_1}|^2 .$$

Note that $\frac{\alpha^2}{(1-\alpha)^2} \geq 1$ if and only if $\alpha \geq \frac{1}{2}$.

Proof In Proposition 6, we estimated $\overline{H}$ using Young's inequality. Here we estimate more carefully using the above assumption.

Using (4), we note that,

$$\begin{aligned}
\int_{M_t} \langle X_t, X - x \rangle \, d\mu &= \overline{H} \int_M \langle \nu, X - x \rangle \, d\mu - \int_M H \langle \nu, X - x \rangle \, d\mu \\
&= \overline{H}(n+1)V_0 - n|M_t| \\
&\leq -n(1-\alpha)|M_t|
\end{aligned} \tag{14}$$

We set $b = n|M_t|$ and $a = -\int_{M_t} \langle X_t, X - x \rangle \, d\mu \geq 0$. Then we consider two cases. First, we assume that $a < b$. We use $a = \alpha a + (1-\alpha)a \geq (1-\alpha)b$ which implies $|b - a| = b - a \leq \frac{\alpha}{1-\alpha}a = \frac{\alpha}{1-\alpha}|a|$. On the other hand, if $a > b$, then we know that $|b - a| = a - b < a = |a|$ because $b > 0$. Putting this together, we get that $|b - a| \leq \max(1, \frac{\alpha}{1-\alpha})|a|$ and thus

$$\left| \int_{M_t} \langle X_t, X - x \rangle \, d\mu + n|M_t| \right| \leq \max\left(\frac{\alpha}{1-\alpha}, 1\right) \left| \int_M \langle X_t, X - x \rangle \, d\mu \right|$$

and

$$|\overline{H}| \leq \max\left(\frac{\alpha}{1-\alpha}, 1\right) \frac{\left|\int_M \langle X_t, X - x \rangle \, d\mu\right|}{(n+1)|V_0|} .$$

Estimating as before we have

$$\begin{aligned}
|\overline{H}|^2 &\leq \max\left(\frac{\alpha^2}{(1-\alpha)^2}, 1\right) \frac{\int_M |H - \overline{H}|^2 d\mu \int_M |X|^2 d\mu}{(n+1)^2 V_0^2} \\
&\leq -\max\left(\frac{\alpha^2}{(1-\alpha)^2}, 1\right) \frac{R_0^2}{2(n+1)^2 V_0^2} \frac{d}{dt} |M_t|^2 .
\end{aligned}$$

And so

$$\int_{t_1}^{t_2} |\overline{H}|^2 dt \leq \max\left(\frac{\alpha^2}{(1-\alpha)^2}, 1\right) \frac{R_0^2}{2(n+1)^2 V_0^2} [|M_{t_1}|^2 - |M_{t_2}|^2] .$$

$\square$

We may now repeat this but for the opposite inequality of the (extended) isoperimetric (or Alexandrov-Fenchel) inequality.

Proposition 19 *Suppose that M_t has* $\mathrm{diam}(M_t) \leq R$ *and* $V_0 \neq 0$. *If on the time interval* $[t_1, t_2]$ *we may estimate*

$$\overline{H}(n+1)V_0 \geq (1+\beta)n|M_t| ,$$

for some $\beta > 0$, *then we have that*

$$\int_{t_1}^{t_2} |\overline{H}|^2 dt \leq \frac{R^2(1+\beta^{-1})^2}{2(n+1)^2 V_0^2} |M_{t_1}|^2 .$$

Proof We note that from the assumption,

$$\int_{M_t} \langle X_t, X - x \rangle \, d\mu = \overline{H}(n+1)V_0 - n|M_t|$$

$$\geq \beta n |M_t| .$$

This time, may directly estimate using the triangle inequality that

$$|\overline{H}| \leq \frac{(1+\beta^{-1}) \left| \int_M \langle X_t, X - x \rangle \, d\mu \right|}{(n+1)|V_0|} .$$

so, as above

$$|\overline{H}|^2 \leq -\frac{R_0^2(1+\beta^{-1})^2}{2(n+1)^2 V_0^2} \frac{d}{dt} |M_t|^2 ,$$

and so

$$\int_{t_1}^{t_2} |\overline{H}|^2 dt \leq \frac{R_0^2(1+\beta^{-1})^2}{2(n+1)^2 V_0^2} [|M_{t_1}|^2 - |M_{t_2}|^2] .$$

$\square$

Proof of Theorem 16 Suppose not. Then there exists a $T, \epsilon > 0$ such that for all $t > T$, $\mathcal{I}(M_t) \in (-\infty, -\epsilon) \cup (\epsilon, 1 - \epsilon) \cup (1 + \epsilon, \infty)$. By continuity we have that $\mathcal{I}(M_t)$ is in one of the intervals $(-\infty, -\epsilon)$, $(\epsilon, 1 - \epsilon)$ or $(1 + \epsilon, \infty)$ for all $t > T$. We particularly have that $\overline{H}(n+1)V_0 = n|M_t|\mathcal{I}(M_t)$ is either not larger than $(1-\epsilon)n|M_t|$ or not less than $(1+\epsilon)n|M_t|$. As a result, by applying Propositions 18 and 19 we have one of

$$\int_T^\infty \overline{H}^2 dt \leq \max\left(\frac{\epsilon^2}{(1-\epsilon)^2}, 1\right) \frac{R_0^2}{2(n+1)^2 V_0^2} |M_{t_1}|^2 .$$

or

$$\int_T^\infty \overline{H}^2 dt \leq \frac{R_0^2(1+\epsilon^{-1})^2}{2(n+1)^2 V_0^2} |M_{t_1}|^2 .$$

Using the isoperimetric inequality, we have that $|\overline{H}| = \frac{n|M_t|}{(n+1)|V_0|}|\mathcal{I}(M_t)| \geq c > 0$ in both cases. This is clearly impossible.

Corollary 20 *If a solution exists for all $t \geq 0$ and for all this time $\overline{H}(n+1)V_0 \leq \alpha n|M_t|$ for some $\alpha \in (0,1)$ then there exist t_i such that $\overline{H}(t_i) \to 0$ as $i \to \infty$.*

6 Barriers and preservation of Alexandrov immersions

In Corollary 4 we already saw that the property of being Alexandrov immersed and having $\overline{H} \leq 0$ is not preserved for long under the VPMCF. Now we will study under which conditions the property of being Alexandrov immersed is preserved for all times. We will also describe how barriers for the VPMCF can be constructed.

We start by introducing some notation. We will use descriptive superscripts to distinguish between manifold, e.g. M_t^{outer}, M^{Inner} and so on. For example, we let $f^* : [0, T) \to \mathbb{R}$ be some smooth function.

Definition 7 We will say a flow M_t^* satisfies *Forced Mean Curvature Flow* with forcing term f^* (and write this as FMCF(f^*)) if it has a time dependent parametrisation $X^* : N^n \times [0, T) \to \mathbb{R}^{n+1}$ (and chosen normal ν^*) so that

$$\left\langle \frac{dX^*}{dt} + H^* \nu^*, \nu^* \right\rangle = f^* .$$

We will use this to produce barriers for the flow.

Definition 8 Suppose that M_t^1 satisfies FMCF(f^1) and M_t^2 satisfies FMCF(f^2). Suppose that there is an open ball $B_r(x)$ so that M_t^1 has non-trivial intersection with $B_r(x)$ so that $B_r(x) \setminus M_t^1$ is made up of two connected components.

We define the component which the normal points away from to be the *local interior* of M_t^1, and the component the normal points into to be the *local exterior* of M_t^1.

We will say that M_t^2 is *locally inside* M_t^1 if it is contained in the local interior of M_t^1. We will say that M_t^2 is *locally outside* M_t^1 if it is contained in the local exterior of M_t^1

Lemma 21 *Suppose that M_t^{out} satisfies FMCF(f^{out}) and M_t^{in} satisfies FMCF(f^{in}). Suppose that in open some ball $B_{2r}(x)$, M_0^{in} is locally inside M_0^{out}. Given $\sigma \in \{1, -1\}$, suppose that*

$$f^{out} - \sigma f^{in} \geq 0 .$$

Then $M_t^{in} \cap B_r(x)$ cannot intersect $M_t^{out} \cap B_r(x)$ for the first time with $\langle \nu^{in}, \nu^{out} \rangle = \sigma$ at an interior point.

Proof Step 1: The graph equation. We describe the evolution of a graphical piece of a surface moving by FMCF(f^{gr}):
We consider a solution M_t^{gr} to FMCF(f^{gr}) locally as a graph of the function graph function u^{gr} in direction e_{n+1} (possibly after rotation). Then we calculate with respect to unit normal μ^{gr} showing upwards in the graphical setting (the graphical normal μ^{gr} may not be the same normal as the a priori chosen ν^{gr})

$$v^{gr} = \sqrt{1 + |Du^{gr}|^2}, \qquad \mu^{gr} = v_{gr}^{-1}(-Du^{gr} + e_{n+1}),$$
$$g_{ij} = \delta_{ij} + D_i u^{gr} D_j u^{gr}, \qquad g^{ij} = \delta^{ij} - v_{gr}^{-2} D^i u^{gr} D^j u^{gr},$$

and

$$h_{ij}^{gr} = -v_{gr}^{-1} D_{ij}^2 u^{gr}, \qquad H = -v_{gr}^{-1}(\delta^{ij} - v_{gr}^{-2} D^i u^{gr} D^j u^{gr}) D_{ij}^2 u^{gr}$$

and so

$$\left\langle u_t^{gr} e_{n+1} - v_{gr}^{-1}(\delta^{ij} - v^{-2} D^i u^{gr} D^j u^{gr}) D_{ij}^2 u^{gr} \nu^{gr}, \nu^{gr} \right\rangle = f^{gr} \langle \nu^{gr}, \mu^{gr} \rangle$$

or

$$u_t^{\text{gr}} = (\delta^{ij} - v_{\text{gr}}^{-2} D^i u^{\text{gr}} D^j u^{\text{gr}}) D_{ij}^2 u^{\text{gr}} + v^{\text{gr}} f^{\text{gr}} \left\langle v^{\text{gr}}, \mu^{\text{gr}} \right\rangle .$$

Here (while hypersurface is graphical), $\langle v^{\text{gr}}, \mu^{\text{gr}} \rangle \in \{1, -1\}$.

Step 2: Proof of the statement.

Suppose not. Suppose that the first point of intersection is (p_0, t_0) for some $t_0 > 0$. Then after rotation we may assume that $v^{\text{out}}(p_0, t_0) = e_{n+1}$ and so $v^{\text{in}}(p_0, t_0) = \pm e_{n+1}$. For some $\epsilon > 0$ we can write both M_t^{out} and M_t^{in} graphically on $B_\epsilon(p_0) \times (t_0 - \epsilon, t_0]$ where

$$u_t^{\text{out}} = (\delta^{ij} - v_{\text{out}}^{-2} D^i u^{\text{out}} D^j u^{\text{out}}) D_{ij}^2 u^{\text{out}} + v^{\text{out}} f^{\text{out}} ,$$

and

$$u_t^{\text{in}} = (\delta^{ij} - v_{\text{in}}^{-2} D^i u^{\text{in}} D^j u^{\text{in}}) D_{ij}^2 u^{\text{in}} + v^{\text{in}} f^{\text{in}} \left\langle v^{\text{in}}, \mu^{\text{in}} \right\rangle .$$

Due to our definitions we have that $w := u^{\text{out}} - u^{\text{in}} > 0$ for $t < t_0$ and

$$
\begin{aligned}
w_t &= a^{ij} D_{ij}^2 w + \hat{b}^i D_i w + \sqrt{1 + |Du^{\text{out}}|^2} f^{\text{out}} - \sqrt{1 + |Du^{\text{in}}|^2} \left\langle v^{\text{in}}, \mu^{\text{in}} \right\rangle f^{\text{in}} \\
&= a^{ij} D_{ij}^2 w + \hat{b}^i D_i w + \sqrt{1 + |Du^{\text{out}}|^2} (f^{\text{out}} - \left\langle v^N, \mu^N \right\rangle f) \\
&\quad + (\sqrt{1 + |Du^{\text{out}}|^2} - \sqrt{1 + |Du^{\text{in}}|^2}) \left\langle v^{\text{in}}, \mu^{\text{in}} \right\rangle f^{\text{in}} \\
&= a^{ij} D_{ij}^2 w + \hat{b}^i D_i w + \sqrt{1 + |Du^{\text{out}}|^2} (f^{\text{out}} - \left\langle v^{\text{in}}, \mu^{\text{in}} \right\rangle f) \\
&\quad + \frac{(D^i u^{\text{out}} + D^i u^{\text{in}})(D^i u^{\text{out}} - D^i u^{\text{in}})}{\sqrt{1 + |Du^{\text{in}}|^2} + \sqrt{1 + |Du^{\text{in}}|^2}} \left\langle v^{\text{in}}, \mu^{\text{in}} \right\rangle f^{\text{in}} \\
&= a^{ij} D_{ij}^2 w + b^i D_i w + \sqrt{1 + |Du^{\text{out}}|^2} (f^{\text{out}} - \left\langle v^{\text{in}}, \mu^{\text{in}} \right\rangle f^{\text{in}})
\end{aligned}
$$

where a^{ij} and b^i are the usual terms and we have computed the additional terms to needless accuracy. We know that $\left\langle v^{\text{in}}, \mu^{\text{in}} \right\rangle = \left\langle v^{\text{in}}(p_0, t_0), v^{\text{out}}(p_0, t_0) \right\rangle = \sigma$. Therefore as $f^{\text{out}} - \left\langle v^{\text{in}}, \mu^{\text{in}} \right\rangle f^{\text{in}} \geq 0$, we have a contradiction to the strong maximum principle.   □

Lemma 22 *Suppose that M_t^1 satisfies FMCF(f^1) and M_t^{out} satisfies FMCF(f^{out}). Suppose that in a ball $B_r(x)$ we have that M_0^1 is locally outside M_0^{in}. Suppose that for some $\sigma \in \{1, -1\}$*

$$f^1 - \sigma f^{out} \leq 0$$

Then $M_t^1 \cap B_r(x)$ cannot intersect $M_t^{out} \cap B_r(x)$ for the first time with $\left\langle v^1, v^{out} \right\rangle = \sigma$ at an interior point.

Proof Suppose not. This time, we may take $e_{n+1} = v^1(p_0, t_0) = \sigma v^{\text{out}}(p_0, t_0)$ (as otherwise one would have been locally inside the other). This time $w = u^1 - u^{\text{out}} < 0$ and

$$w_t = a^{ij} D_{ij}^2 w + b^i D_i w + \sqrt{1 + |Du^1|^2} (f^1 - \sigma f^{\text{out}}) .$$

Again we get a contradiction to the strong maximum principle.   □

Proposition 23 *(Outer Barriers) Suppose that M_t^{out} is a solution for FMCF(f^{out}) which is embedded for all time and bounds the compact region $\Lambda(t)$, and v^N points out of $\Lambda(t)$. Suppose that M_t satisfies VPMCF and $M_0 \subset \overset{\circ}{\Lambda}(0)$ and for all $t \in [0, T)$*

$$f^{out} \geq |\overline{H}|.$$

Then for all $t \in [0, T)$, $M_t \subset \Lambda(t)$.

Proof Suppose not. We have that for $\sigma \in \{1, -1\}$,

$$f^{\text{out}} - \sigma \overline{H} \geq f^{\text{out}} - |\overline{H}| \geq 0$$

and so by Lemma 21, at a first time of intersection the two normals cannot be multiples of each other, a contradiction. □

Proposition 24 (*Inner Barriers*) *Suppose that N_t is a solution for FMCF(f) which is embedded for all time and bounds the compact region $\Lambda(t)$, and ν^N points out of $\Lambda(t)$. Suppose that $M_0 \cap \overline{\Lambda(0)} = \emptyset$, and for all $t \in [0, T)$*

$$f \leq -|\overline{H}|.$$

Then for all $t \in [0, T)$, $M_t \cap \overline{\Lambda(0)} = \emptyset$.

Proof Suppose not. For any $\sigma \in \{1, -1\}$

$$f^1 - \sigma \overline{H} \leq f^1 + |\overline{H}| \leq 0$$

and so by Lemma 22, at a first time of intersection the two normals cannot be multiples of each other, a contradiction. □

We need the following corollary from our previous paper:

Corollary 25 ([35, Corollary 5]) *A compact flowing manifold with bounded curvature may only loose the property of being Alexandrov immersed at time T if there exist points p and $q(t) \in M^n$ so that $|X(p, t) - X(q(t), t)|$ goes to zero with $\langle \nu(p, t), \nu(q(t), t) \rangle = -1$ and where $\langle \nu(p), X(q(t), t) - X(p, t) \rangle < 0$ and $\langle \nu(q, t), X(p, t) - X(q(t), t) \rangle < 0$ for $T - \delta < t < T$.*

Theorem 26 *Let M_0 be Alexandrov immersed M_t the VPMCF starting from M_0. As long as $\overline{H} \geq 0$, the flow stays Alexandrov immersed.*

Proof From Corollary 25 we know that at a time where Alexandrov immersion property is lost two disjoint pieces of the flow must intersect, with the normals are in opposite directions. In Lemma 21 this corresponds to $\sigma = -1$ while $f^{\text{out}} = f^{\text{in}} = \overline{H}$, and so we immediately see that this cannot happen. □

A natural question to ask is when $\overline{H} \geq 0$ is preserved if we know that M_t is Alexandrov immersed. This is not always true without further assumptions – see the Example 2 below.

Appendix A: The Trilobite - an example in which $\overline{H} \geq 0$ is lost

A natural hope, given the previous section, is to attempt to show that under Alexandrov immersed VPMCF, $\overline{H}(t) \geq 0$ is preserved. The following example demonstrates that this is not the case without further assumptions on the flow.

The evolution equation for $\overline{H}$ is given by

$$\frac{d\overline{H}}{dt} = \fint (H - \overline{H})|A|^2 + (H - \overline{H})(-H^2 + \overline{H}H)d\mu = \fint (H - \overline{H})|A|^2 - (H - \overline{H})^2 H d\mu.$$

So, for surfaces ($n = 2$), if $\overline{H}(0) = 0$, then

$$\frac{d\overline{H}}{dt}\Big|_{t=0} = \fint H(|A|^2 - H^2)d\mu\big|_{t=0} = -2\fint HK d\mu\big|_{t=0}$$

using the Gauss equations. We now describe initial data such that $\int_M H d\mu = 0$ and $\int_M K H d\mu > 0$, meaning that $\overline{H} \geq 0$ is not preserved.

Our example is constructed from rotationally symmetric pieces. If γ is the profile curve (parametrised by arclength) of a rotationally symmetric surface (rotated about the y-axis) then we have that the curvatures are given by

$$\kappa_1 = \kappa, \qquad \kappa_2 = \frac{\langle \dot{\gamma}, e_y \rangle}{x} = \frac{\dot{y}}{x}, \qquad H = \frac{\dot{y}}{x} + \kappa, \qquad K = \frac{\dot{y}\kappa}{x}$$

and we may see that $d\mu = 2\pi x ds$.
Therefore we have that (for $I = [a, b]$)

$$\int_M H d\mu = 2\pi \int_\gamma x\kappa ds + 2\pi(y(b) - y(a)), \qquad \int_M HK d\mu = 2\pi \int_\gamma \dot{y}\kappa^2 + \kappa\frac{\dot{y}^2}{x}ds .$$
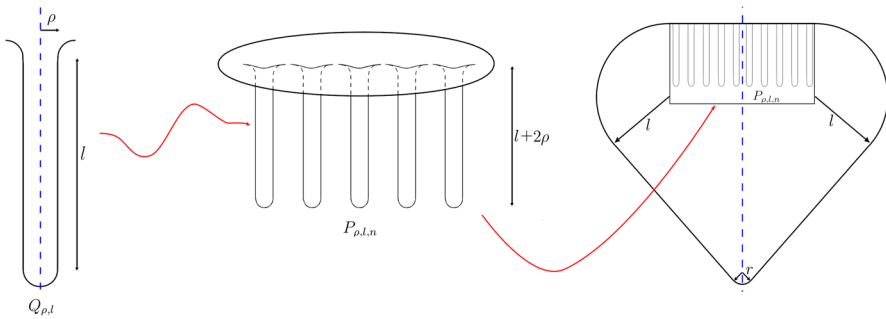
$$(15)$$

We calculate the above for various profiles.
On a hemisphere E_ρ of radius ρ: We have that $\kappa_1 = \kappa_2 = \rho^{-1}$ and so in this case $H = 2\rho^{-1}$, $K = \rho^{-2}$,

$$\int_{E_\rho} H = 4\pi\rho, \qquad \int_{E_\rho} HK d\mu = 4\pi\rho^{-1} .$$

On a cylinder $C_{\rho,l}$ of radius ρ and length l: We have that $\kappa_1 = 0$, $\kappa_2 = \rho^{-1}$ and so in this case $H = \rho^{-1}$, $K = 0$,

$$\int_{C_{\rho,l}} H = 2\pi l, \qquad \int_{C_{\rho,l}} HK d\mu = 0 .$$

Fig. 2 The construction of the Trilobite example in pictorial form

On a rotated arc $I_{(\tilde{x},\tilde{y}),\rho}^{\theta_1,\theta_2} \subset \mathbb{R}_+$ centred at $(\tilde{x}, \tilde{y})$ through angles $[\theta_1, \theta_2]$ from the horizontal: Parametrising as $\gamma(s) = (\rho \cos(\rho^{-1}s) + \tilde{x}, \rho \sin(\rho^{-1}s) + \tilde{y})$ for $s \in [\rho\theta_1, \rho\theta_2]$, we have that $\kappa = \rho^{-1}$ so

$$\int_{I_{(\tilde{x},\tilde{y}),\rho}^{\theta_1,\theta_2}} H = 2\pi\rho^{-1} \int_{\rho\theta_1}^{\rho\theta_2} \rho \cos(\rho^{-1}s) + \tilde{x}\, ds + 2\pi\rho(\sin(\theta_2) - \sin(\theta_1))$$

$$= 4\pi\rho[\sin(\theta_2) - \sin(\theta_1)] + 2\pi\tilde{x}(\theta_2 - \theta_1)$$

while

$$\int_{I_{(\tilde{x},\tilde{y}),\rho}^{\theta_1,\theta_2}} HK d\mu = 2\pi\rho^{-1} \int_{\rho\theta_1}^{\rho\theta_2} \cos(\rho^{-1}s)\rho^{-1} + \frac{\cos^2(\rho^{-1}s)}{\rho \cos(\rho^{-1}s) + \tilde{x}} ds$$

$$= 2\pi \int_{\theta_1}^{\theta_2} \frac{\cos^2(t)}{\rho \cos(t) + \tilde{x}} dt + 2\pi\rho^{-1}[\sin(\theta_2) - \sin(\theta_1)] .$$

As $\int_{\theta_1}^{\theta_2} \cos^2(t) dt = \frac{1}{4}[\sin(2\theta_2) - \sin(2\theta_1)] + \frac{1}{2}[\theta_2 - \theta_1]$ we may estimate the integral as

$$\pi \frac{\frac{1}{2}[\sin(2\theta_2) - \sin(2\theta_1)] + [\theta_2 - \theta_1]}{\tilde{x} + \rho} \leq 2\pi \int_{\theta_1}^{\theta_2} \frac{\cos^2(t)}{\rho \cos(t) + \tilde{x}} dt$$

$$\leq \pi \frac{\frac{1}{2}[\sin(2\theta_2) - \sin(2\theta_1)] + [\theta_2 - \theta_1]}{\tilde{x} - \rho}$$

Example 2 (The Trilobite) We construct a family of embedded surfaces in $\mathbb{R}^3$ as follows. We start with a construction that is C^1 everywhere and smooth away from 1 dimensional glueing points (Fig. 2).

Step 1 - Capped cylinders, $Q_{\rho,l}$ We start with a hemisphere of radius ρ, E_ρ, attach this to a long cylinder $C_{\rho,l}$ and then attach this to the plane using a quarter circle $I_{(2\rho,-\rho),\rho}^{(\frac{\pi}{2},\pi)}$ of radius ρ, as shown. We will call this entire union $Q_{\rho,l}$. In our constuction, we will choose out normals to point into the cylinders meaning that they contribute negatively to both integrals. We calculate

We get that

$$\int_{Q_{\rho,l}} H d\mu = 2\pi[-(4 - \pi)\rho - l], \qquad \int_{Q_{\rho,l}} K H d\mu \geq -6\pi\rho^{-1}$$

	$\int H d\mu$	$\int H K d\mu$
E_ρ^1	$-4\pi\rho$	$-4\pi\rho^{-1}$
$C_{\rho,l}$	$-2\pi l$	0
$I_{(2\rho,-\rho),\rho}^{(\frac{\pi}{2},\pi)}$	$2\pi\rho(\pi-2)$	$\geq \frac{\pi^2}{6\rho} - 2\pi\rho^{-1}$

Step 2 - Attach n **capped cylinders to a disk** We now attach n of these to a flat disk of radius $2n\rho$ which we may always do (e.g. in a line). We will refer to this modified disk as $P_{\rho,l,n}$ where we note that the integrals over H and HK on this disk are just n times those on $Q_{\rho,l}$.

Step 3 - Glue into a final rotationally symmetric surface We now attach this to the rotationally symmetric surface given by taking another third of a circle $I_{(2n\rho,-l),l}^{-\frac{\pi}{6},\frac{\pi}{2}}$ before closing the surface with a cone of slope $\sqrt{3}$ which we will denote $O_{\sqrt{3}}$. We round off the point of the cone using a spherical cap of radius r denoted A_r.

Step 4 - A careful choice of constants and smoothing In the above, altering l in the cylinders $Q_{\rho,l}$ changes only $\int H$, and by increasing n sufficiently, we will see below see that this may be used to ensure that in the above we have $\int H = 0$. Unfortunately, the caps and joins add a significant negative quantity to $\int H K$. However A_r has only negligable $\int H$ contribution while adding an arbitrarily large amount to $\int H K$. Balancing these (see calculations below) mean we can ensure that $\int H = 0$ while $\int H K$ can be made arbitrarily large. However, we would like a smooth manifold - to do this we smooth only locally to the joins, perturbing $\int H$, $\int H K$ by an arbitrarily small amount. Away from the joins, the interior of the cylinders are still cylinders and so shortening or lengthening one of these slightly will again restore $\int H = 0$.

We now do the accountancy and calculate the following contributions to the integrals:

	$\int H d\mu$	$\int H K d\mu$
$P_{\rho,l,n}$	$2n\pi[-(4-\pi)\rho - l]$	$\geq -6n\pi\rho^{-1}$
$I_{(2n\rho,-l),l}^{-\frac{\pi}{6},\frac{\pi}{2}}$	$6\pi l + \frac{8}{3}n\pi^2\rho$	positive
$O_{\sqrt{3}}$	$2n\rho + l - r$	0
A_r	$2\pi(2-\sqrt{3})r$	$2\pi(2-\sqrt{3})r^{-1}$

so

$$\int H d\mu = 2\pi[(3 + (2\pi)^{-1} - n)l + (\frac{11}{3}\pi + \pi^{-1} - 4)n\rho + (2 - \sqrt{3} - (2\pi)^{-1})r]$$

$$\int H K d\mu \geq 2\pi(2-\sqrt{3})r^{-1} - 6n\pi\rho^{-1}.$$

Therefore we may pick (for example) $\rho = 1$, $n \geq 4$ (e.g. $n = 7$), and choose

$$l(r) = \frac{(\frac{11}{3}\pi + \pi^{-1} - 4)n\rho + (2 - \sqrt{3} - (2\pi)^{-1})r}{n - 3 - (2\pi)^{-1}}.$$

Then, by setting r to sufficiently small (compared to $n^{-1}\rho$) we may make $\int H K d\mu$ arbitrarily large.

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