Article:

Babiker, M. orcid.org/0000-0003-0659-5247, Koksal, K. and Lembessis, V. E. (2023) Angular momentum-carrying radially-polarised twisted light. Optics Communications. 129469. ISSN 0030-4018

https://doi.org/10.1016/j.optcom.2023.129469

Angular momentum-carrying radially-polarised twisted light

M. Babiker[1,*], K. Koksal[2], V. E. Lembessis[3]
[1] *School of Physics, Engineering and Technology, University of York, YO10 5DD,UK*
[2] *Physics Department, Bitlis Eren University, Bitlis, Turkey*
[3] *Quantum Technology Group, Department of Physics and Astronomy, College of Science, King Saud University, Riyadh 11451, Saudi Arabia and*
[*] *Corresponding author: m.babiker@york.ac.uk*
(Dated: June 23, 2023)

It is well-known that the fields of pure radially-polarised optical vortex modes do not possess the usual phase function $e^{i\ell\phi}$ of typical twisted light and so they do not carry angular momentum. However, recent work has confirmed that radially-polarised modes with helical wave fronts have been created in the laboratory by several groups and they can now be readily generated using commercially available devices. They are endowed with the phase function $e^{i\ell\phi}$ for arbitrary values of the winding number ℓ. Crucially, the original theoretical treatments disregard the longitudinal electric and magnetic field components. We show here that the longitudinal components must be included, not just for the consistency of the treatment, but because only this can lead to angular momentum. We show that for such created modes the cycle-averaged angular momentum is $\alpha\hbar\ell$ where $\alpha = (\mathcal{P}w_0^2)/(\hbar c^2)$ is a dimensionless constant, with $\mathcal{P}$ the power in the mode and w_0 the mode width at focus.

Vector optical vortex modes are modes of light with spatially-varying polarisation [1] as exemplified by the modes with radial and azimuthal polarisation. Radially-polarised modes, in particular, have so far featured in useful applications, including focusing and microscopy, laser machining and optical trapping, among others [2–5]. A characteristic property of these modes is that their beams can be focused into much smaller waists than in the case of uniformly-polarised modes.

However, conventional radially-polarised optical vortex modes have been known as non-rotating waves as they do not have the usual azimuthal phase function $e^{i\ell\phi}$ and so do not carry angular momentum. Until recently, no rotating radially-polarised mode with helical wave fronts has been created in the laboratory. The theoretical work has consistently predicted that the phase angle of this type of mode is zero [6–9].

However, recently a number of reports [10] confirming that radially and azimuthally-polarised optical vortex modes, typically of the Laguerre-Gaussian type, have been created in the laboratory [11–14] and seem to be routinely producible using commercially available devices in the form of polarisation converters followed by phase plates which adds an azimuthal phase dependence. [15]. These reports indicate that such modes are endowed with the phase function $e^{i\ell\phi}$, with the magnitude of the winding number as large as $|\ell|$ up to 200 [16].

It therefore appears reasonable to suggest that there is a pressing need to emphasise the distinction between the pure vortex modes and the recent radially-polarised modes. The former are devoid of 'twistedness' and lacking vortex properties, including angular momentum, but the latter have twisted wavefronts and are expected to possess angular momentum.

In what follows we describe how the experimental technique involving linearly polarised optical vortex of winding number ℓ entering a polarisation converter simply changes the linear polarisation to a radial polarisation, but leaves the vortex features of the linearly polarised incident mode unchanged. Once the radially-polarised mode is created due to passing a linearly-polarised mode through a polarization converter, it is then made to pass through a device which has a spiral phase plate (or several spiral phase plates, or a stack of these, up to 200). Each spiral phase plate has a step that adds an integral number of 2π per turn. This would keep the radial polarization and add an azimuthal phase and so results in a radially-polarised optical vortex with the usual $e^{i\ell\phi}$ phase function [17]. It is this type of radially-polarised modes that we focus on in this article.

The starting point in the development of the currently available theory of radially-polarised vortex modes is to state the wave equation to be satisfied by the electric field vector, namely

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{E}+\frac{\omega^2}{c^2}\mathbf{E}=0 \tag{1}$$

This equation arises from the two Maxwell curl equations, namely

$$\boldsymbol{\nabla}\times\mathbf{E}=i\omega\mathbf{B};\quad \boldsymbol{\nabla}\times\mathbf{B}=\frac{i\omega}{c^2}\mathbf{E}; \tag{2}$$

In fact Eq.(1) and the curl equations (2) are <u>the same statement</u>. So once we have used the two curl equations we should not then also use Eq.(1).

The standard procedure demands that the electric field be radially polarised and it simply multiplies the general amplitude function Ψ by the polarisation unit vector $\hat{\boldsymbol{\rho}}$

$$\mathbf{E}=\hat{\boldsymbol{\rho}}\Psi(\rho,\phi,z) \tag{3}$$

where the function $\Psi(\rho,\phi,z)$ has the dimensions of the electric field and is to be determined subject to the paraxial limit. It is easy to show that the unit vector $\hat{\boldsymbol{\rho}}$ can be written in terms of Cartesian unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ so that Eq.(3) becomes

$$\mathbf{E} \;=\; \{\hat{\boldsymbol{x}}\cos(\phi) + \hat{\boldsymbol{y}}\sin(\phi)\}\,\Psi \tag{4}$$

and it can also be written as

$$\mathbf{E} = \frac{1}{2}\left\{(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})e^{i\phi} + (\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})e^{-i\phi}\right\}\Psi \tag{5}$$

Note that the radial polarisation in Eq.(5) is represented by a point on the surface of the first order Poincare unit sphere [18]. However, the form of field represented by Eq.(5) may be the source of the suggestion that radially-polarised modes do not carry angular momentum. The argument leading to this conclusion is as follows. The radially-polarised beam can be regarded as a superposition of two beams: one with $\ell = +1$ (OAM$= +\hbar$) and spin $\sigma = -1$ (left-circularly polarized, so spin angular momentum $-\hbar$), and the other has $\ell = -1$ (OAM$= -\hbar$) and spin $\sigma = +1$ (right-circularly polarized, so spin angular momentum $+\hbar$). It follows that the total spin and the total orbital angular momenta of each superposition are zero [19]. This is assumed to hold true irrespective of the form the vortex properties of the mode function Ψ. At this stage the only feature that Ψ has is that it has the space-time phase $\exp i(k_z z - \omega t)$ where k_z is the axial wavenumber along the propagation direction.

The conventional theory based on the above equations is reviewed in the article by Zhan [6] in which the main step was to demand that since $\hat{\boldsymbol{\rho}}$ is not a constant vector, the transverse electric vector field on the right-hand side of Eq.(3) satisfies Eq.(1) in the paraxial regime. This led to a paraxial equation where there is no phase function $\exp(i\ell\phi)$ and as a consequence, the mode has no angular momentum.

We now argue that a rigorous and a consistent treatment can be developed which focuses on the manner of production as described recently for radially-polarised modes endowed with the $\exp(i\ell\phi)$ phase. We show how this treatment leads to the field equations as well as the equation for the amplitude function and is based on electromagnetic potentials with a gauge condition [20, 21].

The standard electromagnetic theory in terms of the vector potential $\mathbf{A}$ and scalar potential Φ are such that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ and $\mathbf{E} = i\omega\mathbf{A} - \boldsymbol{\nabla}\Phi$. Substituting in the first curl equation $\boldsymbol{\nabla}\times\mathbf{B} = -(i\omega/c)\mathbf{E}$, we have

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A} = \frac{\omega^2}{c^2}\mathbf{A} + \frac{i\omega}{c^2}\boldsymbol{\nabla}\Phi \tag{6}$$

In the Lorenz gauge we have $\frac{i\omega}{c^2}\Phi = \boldsymbol{\nabla}\cdot\mathbf{A}$ and we therefore find

$$\nabla^2\mathbf{A} + \frac{\omega^2}{c^2}\mathbf{A} = 0 \tag{7}$$

Each of the three Cartesian components of $\mathbf{A} = (A_x, A_y, A_z)$ satisfy $\nabla^2 A_i + \frac{\omega^2}{c^2}A_i = 0$. So, in the paraxial regime we write

$$A_i = \mathcal{U}_i(x, y, z)e^{ik_z z - i\omega t} \tag{8}$$

Substituting A_i as a component of Eq.(7) we have for the functions $\mathcal{U}_i,\ i = x, y$

$$\nabla_\perp^2\mathcal{U}_i + \frac{\partial^2\mathcal{U}_i}{\partial z^2} + 2ik_z\frac{\partial\mathcal{U}_i}{\partial z} = 0 \quad (i = 1, 2, 3) \tag{9}$$

where $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. A version of wave equation for the amplitude function component $\mathcal{U}$ emerges on dropping the term $\partial^2\mathcal{U}/\partial z^2$ which is regarded as small relative to the term $k_z\partial\mathcal{U}/\partial z$. Thus we obtain from Eq.(9) on dropping the second term

$$i\frac{\partial\mathcal{U}}{\partial z} = -\frac{1}{2k_z}\nabla_\perp^2\mathcal{U} \tag{10}$$

This is the paraxial approximation which is widely used and identified as providing a reasonable description of optical beam propagation along the z-axis. Physically it emphasises the fact that the beam profile for most laser beams changes slowly with axial position z relative to its profile at the focal plane.

In cylindrical coordinates $\mathbf{r} = (\rho, \phi, z)$ the paraxial equation (10) is as follows

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\mathcal{U}}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\mathcal{U}}{\partial\phi^2} + 2ik_z\frac{\partial}{\partial z}\mathcal{U} = 0 \tag{11}$$

A Laguerre-Gaussian mode of winding number ℓ that is linearly-polarised along $\hat{\boldsymbol{x}}$ is a solution of this Paraxial equation such that

$$\mathbf{A}_x(\rho, \phi, z) = \hat{\boldsymbol{x}}\mathcal{U}(\rho, \phi, z) \tag{12}$$

where $\mathcal{U}$ is given by

$$\mathcal{U}(\rho, \phi, z) = \mathcal{F}(\rho, \phi, z) = \tilde{\mathcal{F}}_{\ell,p}(\rho)e^{i(\ell\phi + k_z z)} \tag{13}$$

with the amplitude function defined as

$$\tilde{\mathcal{F}}_{\ell,p}(\rho) = \mathcal{E}_0\sqrt{\frac{p!}{(p+|\ell|)!}}e^{-\frac{\rho^2}{w_0^2}}\left(\frac{\sqrt{2}\rho}{w_0}\right)^{|\ell|}L_p^{|\ell|}\left(\frac{2\rho^2}{w_0^2}\right) \tag{14}$$

This describes the amplitude function of a paraxial Laguerre-Gaussian mode of winding number ℓ, radial number p and waist w_0. The factor $\mathcal{E}_0$ can be determined in terms of the applied power $\mathcal{P}$.

The generation of a radially-polarised mode involves passing the above linearly polarised mode through a polarisation converter, which simply changes the polarisation from $\hat{\boldsymbol{x}}$ to a radial polarisation $\hat{\boldsymbol{\rho}}$ and this radially-polarised mode emerging from the polarisation converter keeps the Laguerre-Gaussian profile of the incident mode, changing its polarisation to $\hat{\boldsymbol{\rho}}$ and moves on to acquire a phase function .

As a result we may now write the vector potential of the mode emerging from the polarisation converter as

$$\mathbf{A}_{\rho,\phi,z} = \hat{\boldsymbol{\rho}}\mathcal{U}(\rho,\phi,z) \tag{15}$$

Note that the mode function in Eq.(15) is the same $\mathcal{U}$ carried by the linearly-polarised mode. From this radially-polarised vector potential we can evaluate the fields. First the magnetic field follows as $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. With $\mathbf{A}$ given by Eq.(15), we obtain

$$\mathbf{B}(\rho,\phi,z) = \frac{\partial\mathcal{U}}{\partial z}\hat{\boldsymbol{\phi}} - \frac{1}{\rho}\frac{\partial\mathcal{U}}{\partial\phi}\hat{\boldsymbol{z}} \tag{16}$$

For the electric field we make use of the Maxwell's equation

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{17}$$

to obtain, to the same leading order as for the magnetic field [20, 22]

$$\mathbf{E}(\rho,\phi,z) = c\frac{\partial\mathcal{U}}{\partial z}\hat{\boldsymbol{\rho}} - c\frac{1}{\rho}\frac{\partial(\rho\mathcal{U})}{\partial\rho}\hat{\boldsymbol{z}} \tag{18}$$

In both the electric and the magnetic fields we have retained the zero-order transverse terms and the longitudinal field only to first order. In so doing, we have followed Lax et al [22] who emphasised this as an essential feature of paraxial wave optics. These authors showed that 'the first-order field is a longitudinal field, obtained explicitly in terms of the zeroth-order field which is transverse.'

As pointed out earlier, the radially-polarised mode of interest here is created as a result of a linearly polarised optical vortex of winding number ℓ entering a polarisation converter which changes the linear polarisation to a radial polarisation, but leaves the vortex features of the linearly polarised incident mode unchanged. It is then made to pass through a device which has a spiral phase plate (or several spiral phase plates, or a stack of these, up to 200). Each spiral phase plate has a step that adds an integral number of 2π per turn. This would keep the radial polarization and the beam profile and simply adds an azimuthal phase [23]. This results in a radially-polarised optical vortex with the usual $e^{i\ell\phi}$ phase function. It is this type of radially-polarised modes that we focus on in this article.

We now seek to examine whether the radially-polarised modes whose fields are as specified above carry angular momentum. We therefore consider the cycle-averaged angular momentum density, which is defined as follows

$$\bar{\boldsymbol{j}} = \frac{1}{2c^2\mu_0}\mathbf{r}\times\Re[\mathbf{E}^*\times\mathbf{B}] \tag{19}$$

where $\Re\{...\}$ indicates taking the real part of $\{...\}$. To proceed we need the electric field and magnetic field for the general paraxial form of a linearly-polarised mode. We find

$$\mathbf{B}(\rho,\phi,z) = ik_z\hat{\boldsymbol{\phi}}\mathcal{F} - \hat{\boldsymbol{z}}\frac{1}{\rho}\frac{\partial\mathcal{F}}{\partial\phi} \tag{20}$$

and

$$\mathbf{E}(\rho,\phi,z) = ick_z\hat{\boldsymbol{\rho}}\mathcal{F} - \hat{\boldsymbol{z}}c\frac{1}{\rho}\frac{\partial(\rho\mathcal{F})}{\partial\rho} \tag{21}$$

Substituting for the fields using Eqs.(20) and (21), we find straightforwardly

$$\bar{\boldsymbol{j}} = \left(\frac{k_z}{2c\mu_0}\right)\{\rho\hat{\boldsymbol{\rho}}\}\times\left\{\ell\frac{|\tilde{\mathcal{F}}|^2}{\rho}\hat{\boldsymbol{\phi}} + k_z|\tilde{\mathcal{F}}|^2\hat{\boldsymbol{z}}\right\} \tag{22}$$

Direct evaluations lead us to the angular momentum density vector

$$\bar{\boldsymbol{j}} = \left(\frac{k_z}{2c\mu_0}\right)\left\{\ell|\tilde{\mathcal{F}}|^2\hat{\boldsymbol{z}} - k_z|\tilde{\mathcal{F}}|^2\rho\hat{\boldsymbol{\phi}}\right\} \tag{23}$$

Since we have $\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi$ the angular momentum density vector has all three Cartesian components. It is also easy to verify that the first term of the angular momentum density would not have existed if the longitudinal components were not included

However, the transverse ($x-$ and $y-$) components are $\phi-$ dependent and, as we point out shortly, will result in zero on angular integration.

Finally, we evaluate the total angular momentum as the space integral of the angular momentum density.

$$\bar{\boldsymbol{\mathcal{J}}} = \int_0^{2\pi}d\phi\int_0^{\infty}\rho\, d\rho\,\bar{\boldsymbol{j}} \tag{24}$$

The $x-$ and $y-$ components give zero each due to vanishing angular integration. We are left only with the z-component, so we have

$$\bar{\boldsymbol{\mathcal{J}}} = \hat{\boldsymbol{z}}\ell\left(\frac{k_z\pi}{c\mu_0}\right)I_P \tag{25}$$

where the integral I_P is related to the applied power $\mathcal{P}$ of the mode, evaluated as the space integral over the beam cross-section of the z-component of the Poynting vector. We have

$$\mathcal{P} = \frac{1}{2\mu_0}\int_0^{2\pi}d\phi\int_0^{\infty}|(\mathbf{E}^*\times\mathbf{B})_z|\rho d\rho \tag{26}$$

with

$$(\mathbf{E}^*\times\mathbf{B})_z = ck_z^2|\tilde{\mathcal{F}}|^2 \tag{27}$$

We can then write for I_P

$$I_P = \int_0^{\infty}|\tilde{\mathcal{F}}|^2\rho\, d\rho \tag{28}$$

Thus we obtain for the power $\mathcal{P}$

$$\mathcal{P} = \left(\frac{\pi c k_z^2}{\mu_0}\right) I_P \tag{29}$$

Substituting for I_P, we have for the total angular momentum per unit length

$$\bar{\boldsymbol{\mathcal{J}}} = \ell\left(\frac{\mathcal{P}}{k_z c^2}\right)\hat{\boldsymbol{z}} \tag{30}$$

Thus we find that $\bar{\boldsymbol{\mathcal{J}}}$ is axial and proportional to ℓ. Note that we have determined the angular momentum without specifying the type of mode. If we now assume that the axial extent of the radial mode is of the order of twice the Rayleigh range $2z_R = w_0^2 k_z$, we can evaluate the angular momentum carried by the radially-polarised mode, so we can write

$$\begin{aligned}\mathcal{J}_T &= 2z_R\bar{\mathcal{J}} = \hbar\ell\left(\frac{\mathcal{P}}{\hbar c^2}\right)w_0^2 \\ &= \alpha\hbar\ell \end{aligned}\tag{31}$$

where α is a dimensionless constant

$$\alpha = \frac{\mathcal{P}w_0^2}{\hbar c^2} \tag{32}$$

For orientation as to orders of magnitude we set the mode waist $w_0 = \lambda = 10^2$nm and the power $\mathcal{P} \approx 1$mW, we then have $\alpha \approx 1$. This means that a radially polarised mode with these parameters carries one unit $\hbar\ell$ of angular momentum.

In conclusion, we have shown how a theory focused on the recently generated phase-bearing radially-polarised optical vortex mode can be constructed, based on electromagnetic potentials. This provides the means of including the longitudinal components of both the electric and magnetic fields in a manner consistent with the Maxwell curl equations and a path towards the paraxial regime.

We then followed a treatment based on the vector and scalar potentials in the Lorenz gauge [20, 21] and showed that it leads to the electric and magnetic field equations which satisfy Eqs. (2) and (1). We then discussed how the radially-polarised vortex mode which is endowed with the phase factor $e^{i\ell\phi}$ is created using a linearly-polarised optical vortex mode passing through a polarisation converter followed by phase plates of the desired number, resulting in the radially-polarised mode being endowed with a phase function $e^{i\ell\phi}$. We then proceeded to evaluate the angular momentum density carried by the radially-polarised mode as Eq.(23) which has both azimuthal and radial components. Finally, we evaluated the integrated angular momentum density leading to the total angular momentum carried by the radially-polarised mode which is given by Eq.(31).

As far as we know, there has been no experiments which have attempted to measure the angular momentum of the radially-polarised modes in question. In view of the prospects of radially-polarised modes in a number of applications, most notably quantum communications [12], we propose that experimental work be carried out with the aim of measuring the angular momentum of these modes.

ACKNOWLEDGEMENTS

The authors wish to thank Professors S. Franke-Arnold, K.D. Dholakia, M. Mansuripur, E.J. Galvez, Q. Zhan, G. Milione and P. Banzer for useful correspondence.

[1] B. M. Holmes and E. J. Galvez, Journal of Optics **21**, 104001 (2019).
[2] Q. Zhan, Opt. Express **12**, 3377 (2004).
[3] O. J. Allegre, W. Perrie, S. P. Edwardson, G. Dearden, and K. G. Watkins, Journal of Optics **14**, 085601 (2012).
[4] R. Dorn, S. Quabis, and G. Leuchs, Physical Review Letters **91**, 1 (2003).
[5] U. Levy, Y. Silberberg, and N. Davidson, Adv. Opt. Photon. **11**, 828 (ts).
[6] Q. Zhan, Advances in Optics and Photonics **1**, 1 (2009).
[7] A. V. Nesterov and V. G. Niziev, Journal of Physics D: Applied Physics **33**, 1817 (2000).
[8] K. Volke-Sepulveda, V. Garcés-Chávez, S. Chávez-Cerda, J. Arlt, and K. Dholakia, Journal of Optics B: Quantum and Semiclassical Optics **4**, S82 (2002).
[9] M. Chekhova and P. Banzer, *Polarization of Light* (De Gruyter, Berlin, Boston, 2021).
[10] Porfirev, A.P., Ustinov, A.V. Khonina, S.N. Polarization conversion when focusing cylindrically-polarized vortex beams. Sci Rep 6, 6 (2016). https://doi.org/10.1038/s41598-016-0015-2.
[11] M. Neugebauer, T. Bauer, A. Aiello, and P. Banzer, Phys. Rev. Lett. **114**, 063901 (2015).
[12] F. Bouchard, N. H. Valencia, F. Brandt, R. Fickler, M. Huber, and M. Malik, Opt. Express **26**, 31925 (2018).
[13] J. Hu, S. Kim, C. Schneider, S. Höfling, and H. Deng, Phys. Rev. Appl. **14**, 044001 (2020).
[14] "Orbital angular momentum and polarization characteristics of vector vortex beam passing through aperture-lens system,".
[15] Https://www.thorlabs.com, http://www.arcoptix.com.
[16] S. Frank-Arnold; private correspondence.
[17] E. J. Galves:private communication.
[18] G. Milione, H. I. Sztul, D. A. Nolan, and R. R. Alfano, Phys. Rev. Lett. **107**, 053601 (2011).
[19] K. Dholakia: private communication.
[20] H. A. Haus, *Waves and Fields in Optoelectronics* (Prentice Hall, 1984).
[21] L. Allen, M. Padgett, and M. Babiker, Progress in optics **39**, 291 (1999).
[22] M. Lax, W. H. Louisell, and W. B. McKnight, Phys. Rev. A **11**, 1365 (1975).
[23] E. J. Galvez: private communication.