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Path isomorphisms between quiver Hecke and diagrammatic Bott–Samelson endomorphism algebras

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A B S T R A C T

We construct an explicit isomorphism between (truncations of) quiver Hecke algebras and Elias–Williamson's diagrammatic endomorphism algebras of Bott–Samelson bimodules. As a corollary, we deduce that the decomposition numbers of these algebras (including as examples the symmetric groups and generalised blob algebras) are tautologically equal to the associated p-Kazhdan–Lusztig polynomials, provided that the characteristic is greater than the Coxeter number. We hence give an elementary and more explicit proof of the main theorem of Riche–Williamson's recent monograph and extend their categorical equivalence to cyclotomic quiver Hecke algebras, thus solving Libedinsky–Plaza's categorical blob conjecture.

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1. Introduction

The symmetric group lies at the intersection of two great categorical theories. The first is Khovanov–Lauda and Rouquier's categorification of quantum groups and their knot invariants [15,27]; this setting has provided powerful new graded presentations of the symmetric group and its affine Hecke algebra [8]. The second is Elias–Williamson's diagrammatic categorification in terms of endomorphisms of Bott–Samelson bimodules; it was in this setting that the counterexamples to Lusztig's conjecture were first found [28] and that the first general character formulas for decomposition numbers of symmetric groups were discovered [26] (in characteristic $p > h$, the Coxeter number).

The purpose of this paper is to construct an explicit isomorphism between these two diagrammatic worlds. This allows us to provide an elementary algebraic proof of [26, Theorem 1.9] and to vastly generalise this theorem to the quiver Hecke (or KLR) algebras $\mathcal{H}_n$; we hence settle Libedinsky–Plaza's categorical blob conjecture [17]. Understanding its simple modules is equivalent to understanding those of its cyclotomic quotients $\mathcal{H}_n^\sigma$ for $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{\ell-1}) \in \mathbb{Z}^\ell$. We prove that $\mathcal{H}_n^\sigma$ has graded decomposition numbers $d_{\lambda,\mu}(t)$ equal to the p-Kazhdan–Lusztig polynomials of type

$$A_{h_0} \times \ldots \times A_{h_{\ell-1}} \backslash \widehat{A}_{h_0+\cdots+h_{\ell-1}}$$

provided that λ and μ have at most h_m columns in the mth component (where $h_m \leqslant \sigma_{m+1} - \sigma_m$ for $0 \leqslant m < \ell - 1$ and $h_{\ell-1} < e + \sigma_0 - \sigma_{\ell-1}$). We denote the set of such ℓ-multipartitions by $\mathscr{P}_{\underline{h}}(n)$ for $\underline{h} = (h_0, \ldots, h_{\ell-1}) \in \mathbb{Z}^\ell_{\geqslant 0}$ and refer to such an $\underline{h} \in \mathbb{Z}^\ell$ as being (σ, e)-*admissible*. This is the broadest possible generalisation, in the context of the quiver Hecke algebra, of studying the category of tilting modules of the principal block of the general linear group, $\mathrm{GL}_h(\Bbbk)$, in characteristic $p > h$.

Theorem A. *Let $\sigma \in \mathbb{Z}^\ell$ and $e \in \mathbb{Z}_{>1}$ and suppose that $\underline{h} \in \mathbb{Z}^\ell_{\geqslant 0}$ is (σ, e)-admissible. We have a canonical isomorphism of graded $\mathbb{Z}$-algebras between certain subquotients of the quiver Hecke algebra $\mathcal{H}_n^\sigma$ and Elias–Williamson's diagrammatic category under which the simple and standard modules labelled by $\mathscr{P}_{\underline{h}}(n)$ are preserved. The isomorphism is defined in equation* (5.4).

Perhaps most importantly, our isomorphism allows one to pass information back and forth between these two diagrammatic categorifications for the first time. Combining our result with [8] allows one to import Soergel calculus to calculate decomposition numbers directly within the setting of the symmetric group (and more generally, within the cyclotomic quiver Hecke algebras). For instance, the key to the counterexamples of [28] are the mysterious "intersection forms" controlling decompositions of Bott–Samelson bimodules; in light of our isomorphism, these intersection forms can be seen simply as an efficient version of James' classical bilinear form on the Specht modules of $\Bbbk\mathfrak{S}_n$, and the efficiency arises by way of idempotent truncation (in particular, the Gram matrices of these forms are equal). In other words, by virtue of our isomorphism, one can view the current state-of-the-art regarding p-Kazhdan–Lusztig theory (in type A) entirely within the context of the group algebra of the symmetric group, without the need for calculating intersection cohomology groups, or working with parity sheaves, or appealing to the deepest results of 2-categorical Lie theory. In Subsection 7.3 we will explain that the regular decomposition numbers of cyclotomic quiver Hecke algebras are *tautologically equal to* p-Kazhdan–Lusztig polynomials, simply by the categorical definition of these polynomials.

Theorem B. *The isomorphism of Theorem A maps each choice of light leaves cellular basis to a cellular basis element of $\mathcal{H}_n^\sigma$. Thus the Gram matrix of the intersection form associated to the fibre of a Bott–Samelson resolution of a Schubert variety coincides with the Gram matrix of James' bilinear form on the idempotent truncated Specht module for $\lambda \in \mathscr{P}_{\underline{h}}(n)$.*

In the other direction: Soergel diagrammatics is, at present, confined to regular blocks — whereas quiver Hecke diagrammatics is not so restricted — we expect our isomorphism to offer insight toward constructing Soergel diagrammatics for singular blocks. In particular, our isomorphism interpolates between the (well-understood) LLT-style combinatorics of KLR algebras and the (more mysterious) Kazhdan–Lusztig-style combinatorics of diagrammatic Bott–Samelson endomorphism algebras.

Symmetric groups. For $\ell = 1$ our Theorem A has the immediate corollary of reproving the famous result of Riche–Williamson (and later Elias–Losev) which states that regular decomposition numbers of symmetric groups are equal to p-Kazhdan–Lusztig polynomials [26,11]. Our proof is conceptually simpler than both existing proofs, as it does not require any higher categorical Lie theory. Once one has developed the appropriate combinatorial framework, our proof simply verifies that the two diagrammatically defined algebras are isomorphic by checking the relations. In this regard, our proof is akin to the work of Brundan–Kleshchev [8] and extends their ideas to the world of Soergel diagrammatics. We state the simplified version of Theorem A now, for ease of reference.

Corollary A. *For $\Bbbk$ a field of characteristic $p > h$, we have an isomorphism of graded $\Bbbk$-algebras between certain subquotients of $\Bbbk\mathfrak{S}_n$ and Elias–Williamson's diagrammatic category of type $A_{h-1}\backslash\widehat{A}_{h-1}$. The decomposition numbers of symmetric groups labelled by partitions with at most $h < p$ columns are tautologically equal to the p-Kazhdan–Lusztig polynomials of type $A_{h-1}\backslash\widehat{A}_{h-1}$.*

Blob algebras and statistical mechanics. The (generalised) blob algebras first arose as the transfer matrix algebras for the one-boundary Potts models in statistical mechanics. In a series of beautiful and prophetic papers [21–23], Paul Martin and his collaborators conjectured that these algebras would be controlled by non-parabolic affine Kazhdan–Lusztig polynomials and verified this conjecture for level $\ell = 2$. It was the advent of quiver

Hecke and Cherednik algebras that provided the necessary perspective for solving this conjecture [7]. This perspective allowed Libedinsky–Plaza to push these ideas still further (into the modular setting) in the form of a beautiful conjecture which brings together ideas from statistical mechanics, diagrammatic algebra, and p-Kazhdan–Lusztig theory for the first time [17]. For $h = (1^\ell)$ our Theorem A verifies their conjecture, as follows:

Corollary B *(Libedinsky–Plaza's categorical blob conjecture). For $\Bbbk$ a field, we have an isomorphism of graded $\Bbbk$-algebras, between certain subquotients of the generalised blob algebra of level ℓ and Elias–Williamson's diagrammatic category of type $\widehat{A}_{\ell-1}$. In particular the decomposition numbers of generalised blob algebras are tautologically equal to the p-Kazhdan–Lusztig polynomials of type $\widehat{A}_{\ell-1}$.*

Weightings and gradings on cyclotomic quiver Hecke algebras. Recently, Elias–Losev generalised [26, Theorem 1.9] to calculate decomposition numbers of cyclotomic quiver Hecke algebras. However, we emphasise that our Theorem A and Elias–Losev's work intersect only in the case of the symmetric group (providing two independent proofs of [26, Theorem 1.9]). In particular, Elias–Losev's work *does not* imply Libedinsky–Plaza's conjecture (as explained in detail in Libedinsky–Plaza's paper [17]). This lack of overlap arises from different choices of weightings on the cyclotomic quiver Hecke algebra, we refer the reader to [17,7,19] for more details.

The structure and ideas of the paper. The isomorphism of this paper was a surprise to many of the experts in this field. This is because of the fundamental differences in the ways we think of Bott–Samelson endomorphism algebras versus quiver Hecke algebras. The elements of the former algebras are thought of as morphisms between words (in the Coxeter generators of $\widehat{\mathfrak{S}}_h$), their complex representation theory is controlled by Soergel's algorithm, which can be thought of in terms of paths in the Bruhat graph of $\mathfrak{S}_h \leqslant \widehat{\mathfrak{S}}_h$. The elements of the latter algebras are thought of as "graded versions" of permutations, the complex representation theory of these algebras is controlled by the LLT algorithm, which can be thought of in terms of graded standard tableaux [16]. Of course the LLT algorithm and Soergel's algorithm produce the same results, even though the steps involved appear quite different. One can think of this as being because the LLT algorithm has many more "degree zero steps" which simply "pad out" the tableaux. This is a good heuristic for this paper, which we now expound section by section.

Sections 2 and 3 introduce the combinatorics and basic definitions of quiver Hecke and diagrammatic Bott–Samelson endomorphism algebras in tandem. We provide a dictionary for passing between standard tableaux (of the former world) and expressions in cosets of affine Weyl group (of the latter world) by means of coloured paths in our alcove geometries. We subtly tweak the classical perspective for quiver Hecke algebras by recasting each element of the algebra as a morphism between a pair of paths in the alcove geometry. Heuristically, we "equate the combinatorics" of the LLT and Soergel algorithms by writing tableaux/paths as the concatenation of component paths (each of which corresponds to a single reflection hyperplane).

One of the core principles of this paper is that diagrammatic Bott–Samelson endomorphisms are simply a "condensed shorthand" for KLR path-morphisms. Section 4 details the reverse process by which we "dilate" simple elements of the KLR algebra and hence construct these path-morphisms. Section 4 also provides a translation principle by which we can see that a path-morphism depends only on the series of hyperplanes in the path's trajectory, not the individual steps taken within the path. Heuristically, this translation principle says that "the degree zero steps in the LLT algorithm are unimportant".

In Section 5, we recast the generators of the diagrammatic Bott–Samelson endomorphism algebra within the setting of the quiver Hecke algebra; this allows us to explicitly state the isomorphism, Ψ, of Theorem A. In Section 6 we verify that Ψ is a graded $\mathbb{Z}$-algebra homomorphism by recasting the relations of the diagrammatic Bott–Samelson endomorphism within the setting of the quiver Hecke algebra. This involves rewriting products of the path-morphisms in the KLR algebra one step at a time — for the products involving forks and spots there is a single "important step" in this procedure with the others corresponding to "LLT padding".

Finally, in Section 7 we match-up the light leaves bases of these algebras under the map Ψ and hence prove that Ψ is bijective and thus complete the proofs of Theorems A and B.

In Appendix A we provide a coherence theorem for weakly graded monoidal categories which allows us to relate the classical Bott-Samelson endomorphism algebras to certain breadth-enhanced versions which are more convenient for the purposes of this paper. The reader can think of this as inserting "extra monoidal identity padding" into the diagrammatic Bott–Samelson endomorphisms algebras which corresponds (on the KLR side of the isomorphism) to the *steps of degree zero* in paths/tableaux.

Finally we emphasise that the LLT/Soergel analogy above is motivated by the situation over $\mathbb{C}$. This is merely a heuristic and our results work over a field of arbitrary characteristic (indeed, the isomorphism is actually proven to hold over the integers).

For the convenience of the reader we provide three tables summarising the notation used throughout the paper in Appendix B.

2. Parabolic and non-parabolic alcove geometries and path combinatorics

Without loss of generality, we assume that $\sigma \in \mathbb{Z}^\ell$ is weakly increasing and $e > h \in \mathbb{Z}_{\geqslant 1}$. We say that $\underline{h} = (h_0, \dots, h_{\ell-1}) \in \mathbb{Z}^\ell_{\geqslant 0}$ with $h_0 + h_1 + \dots + h_{\ell-1} = h$ is (σ, e)-admissible if $h_m \leqslant \sigma_{m+1} - \sigma_m$ for $0 \leqslant m < \ell - 1$ and $h_{\ell-1} < e + \sigma_0 - \sigma_{\ell-1}$. (This condition on $\underline{h} = (h_0, \dots, h_{\ell-1}) \in \mathbb{Z}^\ell_{\geqslant 0}$ is equivalent to the empty partition not lying on any hyperplane of our alcove geometry, so that the resulting Kazhdan–Lusztig theory is "non-singular".)

2.1. Multipartitions, residues and tableaux

We define a composition, λ, of n to be a finite sequence of non-negative integers $(\lambda_1, \lambda_2, \dots)$ whose sum, $|\lambda| = \lambda_1 + \lambda_2 + \dots$, equals n. We say that λ is a partition if, in addition, this sequence is weakly decreasing. An ℓ-multicomposition (respectively ℓ-multipartition) $\lambda = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ of n is an ℓ-tuple of compositions (respectively of partitions) such that $|\lambda^{(0)}| + \dots + |\lambda^{(\ell-1)}| = n$. We will denote the set of ℓ-multicompositions (respectively ℓ-multipartitions) of n by $\mathscr{C}_\ell(n)$ (respectively by $\mathscr{P}_\ell(n)$). Given $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell-1)}) \in \mathscr{P}_\ell(n)$, the (dual) Young diagram of λ is defined to be the set of nodes,

$$[\lambda] = \{(r, c, m) \mid 1 \leqslant r \leqslant (\lambda^{(m)})_c, 0 \leqslant m < \ell\}.$$

Notice that we have taken the transpose-dual of the usual conventions so that the multipartitions are the sequences whose columns are weakly decreasing (this is a trivial, if unfortunate, relabelling inherited from our earlier work [3,4]). We do not distinguish between the multipartition and its (dual) Young diagram. We refer to a node (r, c, m) as being in the rth row and cth column of the mth component of λ. Given a node, (r, c, m), we define the content of this node to be $\mathrm{ct}(r, c, m) = \sigma_m + c - r$ and we define its residue to be $\mathrm{res}(i, j, m) = \mathrm{ct}(i, j, m) \pmod e$. We refer to a node of residue $i \in \mathbb{Z}/e\mathbb{Z}$ as an i-node. Given $\lambda \in \mathscr{C}_\ell(n)$ or $\mathscr{P}_\ell(n)$, we let $\mathrm{Rem}(\lambda)$ (respectively $\mathrm{Add}(\lambda)$) denote the set of all removable (respectively addable) nodes of the Young diagram of λ so that the resulting diagram is the Young diagram of an ℓ-composition or an ℓ-partition.

Given $\lambda \in \mathscr{C}_\ell(n)$, we define a tableau of shape λ to be a filling of the nodes of λ with the numbers $\{1, \dots, n\}$. We define a standard tableau of shape λ to be a tableau of shape λ such that entries increase along the rows and down the columns of each component. We let $\mathrm{Std}(\lambda)$ denote the set of all standard tableaux of shape $\lambda \in \mathscr{P}_\ell(n)$. We let $\varnothing$ denote the empty multipartition.

Definition 2.1. Given a pair of i-nodes $(r, c, m), (r', c', m')$, we write $(r, c, m) \lhd (r', c', m')$ if either $\mathrm{ct}(r, c, m) < \mathrm{ct}(r', c', m')$ or $\mathrm{ct}(r, c, m) = \mathrm{ct}(r', c', m')$ and $m > m'$. For $\lambda, \mu \in \mathscr{P}_\ell(n)$, we write $\mu \trianglelefteq \lambda$ if there is a bijective map $\mathsf{A} : [\lambda] \to [\mu]$ such that either $\mathsf{A}(r, c, m) \lhd (r, c, m)$ or $\mathsf{A}(r, c, m) = (r, c, m)$ for all $(r, c, m) \in \lambda$.

Given $\mathsf{S} \in \mathrm{Std}(\lambda)$ a, we write $\mathsf{S}{\downarrow}_{\leqslant k}$ or $\mathsf{S}{\downarrow}_{\{1,\dots,k\}}$ (respectively $\mathsf{S}{\downarrow}_{\geqslant k}$) for the subtableau of S consisting solely of the entries 1 through k (respectively of the entries k through n). Given $\lambda \in \mathscr{P}_\ell(n)$, we let T_λ denote the λ-tableau in which we place the entry n in the minimal (under the $\rhd$-ordering) removable node of λ, then continue in this fashion inductively. Given $1 \leqslant k \leqslant n$, we let $(r_k, c_k, m_k) \in \lambda$ be the node such that $\mathsf{T}(r_k, c_k, m_k) = k$. We let $\mathcal{A}_\mathsf{T}(k)$ (respectively $\mathcal{R}_\mathsf{T}(k)$) denote the set of all addable (respectively removable) $\mathrm{res}(r_k, c_k, m_k)$-nodes of the multipartition $\mathrm{Shape}(\mathsf{T}{\downarrow}_{\{1,\dots,k\}})$ which are less than (r_k, c_k, m_k) in the $\rhd$-order. We define the ($\rhd$)-degree of $\mathsf{T} \in \mathrm{Std}(\lambda)$ for $\lambda \in \mathscr{P}_\ell(n)$ as follows,

$$\deg(\mathsf{T}) = \sum_{k=1}^{n} \left(|\mathcal{A}_\mathsf{T}(k)| - |\mathcal{R}_\mathsf{T}(k)|\right).$$

Definition 2.2. Given $\underline{h} \in \mathbb{Z}^{\ell}_{\geqslant 0}$, we let $\mathscr{P}_{\underline{h}}(n) \subseteq \mathscr{C}_{\underline{h}}(n)$ denote the subsets of ℓ-multipartitions and ℓ-multicompositions with at most h_m columns in the mth component for $0 \leqslant m < \ell$.

If $\underline{h} \in \mathbb{Z}^{\ell}_{\geqslant 0}$ is (σ, e)-admissible, then $\deg(\mathsf{T}_\lambda) = 0$ for $\lambda \in \mathscr{P}_{\underline{h}}(n)$.

Example 2.3. Let $\sigma = (0, 3, 8) \in \mathbb{Z}^3$ and $e = 13$. We note that $\underline{h} = (3, 5, 4)$ is (σ, e)-admissible. We depict $\lambda = ((5, 4, 2), (5, 4, 3, 2^2), (5, 3^2, 2)) \in \mathscr{P}_{\underline{h}}(n)$ along with the residues of this multipartition as follows:

$$\left(\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\\hline 12 & 0 & 1 \\\hline 11 & 12 \\\cline{1-2} 10 & 11 \\\cline{1-2} 9 \\\cline{1-1}\end{array} \; , \; \begin{array}{|c|c|c|c|c|}\hline 3 & 4 & 5 & 6 & 7 \\\hline 2 & 3 & 4 & 5 & 6 \\\hline 1 & 2 & 3 \\\cline{1-3} 0 & 1 \\\cline{1-2} 12 \\\cline{1-1}\end{array} \; , \; \begin{array}{|c|c|c|c|}\hline 8 & 9 & 10 & 11 \\\hline 7 & 8 & 9 & 10 \\\hline 6 & 7 & 8 \\\cline{1-3} 5 \\\cline{1-1} 4 \\\cline{1-1}\end{array} \right).$$

Notice that any given residue $i \in \mathbb{Z}/e\mathbb{Z}$ appears at most once in a fixed row of the multipartition.

2.2. Alcove geometry

For ease of notation, we set $H_m = h_0 + \cdots + h_m$ for $0 \leqslant m < \ell$, and $h = h_0 + \cdots + h_{\ell-1}$. For each $1 \leqslant i \leqslant h_m$ and $0 \leqslant m < \ell$ we let $\varepsilon_{i,m} := \varepsilon_{(h_0+\cdots+h_{m-1})+i}$ denote a formal symbol, and define an h-dimensional real vector space

$$\mathbb{E}_{\underline{h}} = \bigoplus_{\substack{0 \leqslant m < \ell \\ 1 \leqslant i \leqslant h_m}} \mathbb{R}\varepsilon_{i,m}$$

and $\overline{\mathbb{E}}_{\underline{h}}$ to be the quotient of this space by the one-dimensional subspace spanned by

$$\sum_{\substack{0 \leqslant m < \ell \\ 1 \leqslant i \leqslant h_m}} \varepsilon_{i,m}.$$

We have an inner product $\langle\ ,\ \rangle$ on $\mathbb{E}_{\underline{h}}$ given by extending linearly the relations

$$\langle \varepsilon_{i,p}, \varepsilon_{j,q} \rangle = \delta_{i,j}\delta_{p,q}$$

for all $1 \leqslant i, j \leqslant n$ and $0 \leqslant p, q < \ell$, where $\delta_{i,j}$ is the Kronecker delta. We identify $\lambda \in \mathscr{P}_{\underline{h}}(n)$ with an element of the integer lattice inside $\mathbb{E}_{\underline{h}}$ via the map

$$\lambda \longmapsto \sum_{\substack{0 \leqslant m < \ell \\ 1 \leqslant i \leqslant h_m}} \lambda_i^{(m)} \varepsilon_{i,m}. \tag{2.1}$$

We let Φ denote the root system of type A_{h-1} consisting of the roots

$$\{\varepsilon_{i,p} - \varepsilon_{j,q} : \ 0 \leqslant p, q < \ell, \ 1 \leqslant i \leqslant h_p, 1 \leqslant j \leqslant h_q, \text{with } (i,p) \neq (j,q)\}$$

and Φ_0 denote the root system of type $A_{h_0-1} \times \cdots \times A_{h_{\ell-1}-1}$ consisting of the roots $\{\varepsilon_{i,m} - \varepsilon_{j,m} : 0 \leqslant m < \ell, 1 \leqslant i \neq j \leqslant h_m\}$. We choose Δ (respectively Δ_0) to be the set of simple roots inside Φ (respectively Φ_0) of the form $\varepsilon_t - \varepsilon_{t+1}$ for some $1 \leqslant t \leqslant h$, and write Φ^+ (respectively Φ_0^+) for the set of positive roots with respect to this choice of simple roots. Given $r \in \mathbb{Z}$ and $\alpha \in \Phi$ we define $s_{\alpha,re}$ to be the reflection which acts on $\mathbb{E}_{\underline{h}}$ by

$$s_{\alpha,re}x = x - (\langle x, \alpha \rangle - re)\alpha.$$

The group generated by the $s_{\alpha,0}$ with $\alpha \in \Phi$ (respectively $\alpha \in \Phi_0$) is isomorphic to the symmetric group $\mathfrak{S}_h$ (respectively to $\mathfrak{S}_f := \mathfrak{S}_{h_0} \times \cdots \times \mathfrak{S}_{h_{\ell-1}}$), while the group generated by the $s_{\alpha,re}$ with $\alpha \in \Phi$ and $r \in \mathbb{Z}$ is isomorphic to $\widehat{\mathfrak{S}}_h$, the affine Weyl group of type A_{h-1}. We set $\alpha_0 = \varepsilon_h - \varepsilon_1$ and $\Pi = \Delta \cup \{\alpha_0\}$. The elements $S = \{s_{\alpha,0} : \alpha \in \Delta\} \cup \{s_{\alpha_0,-e}\}$ generate $\widehat{\mathfrak{S}}_h$. We have chosen $\alpha_0 = \varepsilon_h - \varepsilon_1$ (rather than $\alpha_0 = \varepsilon_1 - \varepsilon_h$) as this is compatible with our path combinatorics.

Notation 2.4. We shall frequently find it convenient to refer to the generators in S in terms of the elements of Π, and will abuse notation in two different ways. First, we will write s_α for $s_{\alpha,0}$ when $\alpha \in \Delta$ and s_{α_0} for $s_{\alpha_0,-e}$. This is unambiguous except in the case of the affine reflection $s_{\alpha_0,-e}$, where this notation has previously been used for the element $s_{\alpha,0}$. As the element $s_{\alpha_0,0}$ will not be referred to hereafter this should not cause confusion. Second, we will write $\alpha = \varepsilon_i - \varepsilon_{i+1}$ in all cases; if $i = h$ then all occurrences of $i + 1$ should be interpreted modulo h to refer to the index 1.

We shall consider a shifted action of the affine Weyl group $\widehat{\mathfrak{S}}_h$ on $\mathbb{E}_{\underline{h}}$ by the element $\rho := (\rho_0, \rho_2, \ldots, \rho_{\ell-1}) \in \mathbb{Z}^h$ where $\rho_m := (\sigma_m + h_m - 1, \sigma_m + h_m - 2, \ldots, \sigma_m) \in \mathbb{Z}^{h_m}$, that is, given an element $w \in \widehat{\mathfrak{S}}_h$, we set $w \cdot x = w(x + \rho) - \rho$. This shifted action induces a well-defined action on $\overline{\mathbb{E}}_{\underline{h}}$; we will define various geometric objects in $\mathbb{E}_{\underline{h}}$ in terms of this action, and denote the corresponding objects in the quotient with a bar without further comment. We let $\mathbb{E}(\alpha, re)$ denote the affine hyperplane consisting of the points

$$\mathbb{E}(\alpha, re) = \{x \in \mathbb{E}_{\underline{h}} \mid s_{\alpha,re} \cdot x = x\}.$$

Note that our assumption that $\underline{h} \in \mathbb{Z}^{\ell}_{\geqslant 0}$ is (σ, e)-admissible implies that the origin does not lie on any hyperplane. Given a hyperplane $\mathbb{E}(\alpha, re)$ we remove the hyperplane from $\mathbb{E}_{\underline{h}}$ to obtain two distinct subsets $\mathbb{E}^{>}(\alpha, re)$ and $\mathbb{E}^{<}(\alpha, re)$ where the origin lies in $\mathbb{E}^{<}(\alpha, re)$. The connected components of

$$\overline{\mathbb{E}}_{\underline{h}} \setminus (\cup_{\alpha \in \Phi_0} \overline{\mathbb{E}}(\alpha, 0))$$

are called chambers. The dominant chamber, denoted $\overline{\mathbb{E}}_{\underline{h}}^{+}$, is defined to be

$$\overline{\mathbb{E}}_{\underline{h}}^{+} = \bigcap_{\alpha \in \Phi_0} \overline{\mathbb{E}}^{<}(\alpha, 0).$$

The connected components of

$$\overline{\mathbb{E}}_{\underline{h}} \setminus (\cup_{\alpha \in \Phi, r \in \mathbb{Z}} \overline{\mathbb{E}}(\alpha, re))$$

are called alcoves, and any such alcove is a fundamental domain for the action of the group $\widehat{\mathfrak{S}}_h$ on the set **Alc** of all such alcoves. We define the fundamental alcove A_0 to be the alcove containing the origin (which is inside the dominant chamber). We note that the map $\mathscr{P}_{\underline{h}}(n) \to \mathbb{E}_{\underline{h}} \cap \mathbb{Z}_{\geqslant 0}\{\varepsilon_1, \dots, \varepsilon_h\}$ restricts to be surjective when we restrict the codomain to the dominant Weyl chamber.

We have a bijection from $\widehat{\mathfrak{S}}_h$ to **Alc** given by $w \longmapsto wA_0$. Under this identification **Alc** inherits a right action from the right action of $\widehat{\mathfrak{S}}_h$ on itself. Consider the subgroup

$$\mathfrak{S}_f := \mathfrak{S}_{h_0} \times \cdots \times \mathfrak{S}_{h_{\ell-1}} \leqslant \widehat{\mathfrak{S}}_h.$$

The dominant chamber is a fundamental domain for the action of $\mathfrak{S}_f$ on the set of chambers in $\overline{\mathbb{E}}_{\underline{h}}$. We let $\mathfrak{S}^f$ denote the set of minimal length representatives for right cosets $\mathfrak{S}_f \backslash \widehat{\mathfrak{S}}_h$. So multiplication gives a bijection $\mathfrak{S}_f \times \mathfrak{S}^f \to \widehat{\mathfrak{S}}_h$. This induces a bijection between right cosets and the alcoves in our dominant chamber. Under this identification, the alcoves are partially ordered by the Bruhat-ordering on $\mathfrak{S}^f$. (This is the opposite of the ordering, $\trianglelefteq$, on multipartitions belonging to these alcoves.)

If the intersection of a hyperplane $\overline{\mathbb{E}}(\alpha, re)$ with the closure of an alcove A is generically of codimension one in $\overline{\mathbb{E}}_{\underline{h}}$ then we call this intersection a wall of A. The fundamental alcove A_0 has walls corresponding to $\overline{\mathbb{E}}(\alpha, 0)$ with $\alpha \in \Delta$ together with an affine wall $\overline{\mathbb{E}}(\alpha_0, e)$. We will usually just write $\overline{\mathbb{E}}(\alpha)$ for the walls $\overline{\mathbb{E}}(\alpha, 0)$ (when $\alpha \in \Delta$) and $\overline{\mathbb{E}}(\alpha, e)$ (when $\alpha = \alpha_0$). We regard each of these walls as being labelled by a distinct colour (and assign the same colour to the corresponding element of S). Under the action of $\widehat{\mathfrak{S}}_h$ each wall of a given alcove A is in the orbit of a unique wall of A_0, and thus inherits a colour from that wall. We will sometimes use the right action of $\widehat{\mathfrak{S}}_h$ on **Alc**. Given an alcove A and an element $s \in S$ we have that $A = wA_0$ for some w under the identification above (that is, $\widehat{\mathfrak{S}}_h$ to **Alc** given by $w \longmapsto wA_0$). Thus the right action of s on A gives the element wsA_0 in **Alc**, and this can easily be seen to be obtained by reflecting A in the wall of A with colour corresponding to the colour of s. With this observation it is now easy to see that if $w = s_1 \dots s_t$ where the s_i are in S then wA_0 is the alcove obtained from A_0 by successively reflecting through the walls corresponding to s_1 up to s_t.

If a wall of an alcove A corresponds to $\overline{\mathbb{E}}(\alpha, re)$ and $A \subset \overline{\mathbb{E}}^{>}(\alpha, re)$ then we call this a lower alcove wall of A; otherwise we call it an upper alcove wall of A. We will call a multipartition σ-regular (or just regular) if its image in $\overline{\mathbb{E}}_{\underline{h}}$ lies in some alcove; the multipartitions whose images lies on one or more walls will be called σ-singular.

Let $\lambda \in \overline{\mathbb{E}}_{\underline{h}}$. There are only finitely many hyperplanes $\mathbb{E}(\alpha, re)$ for $\alpha \in \Pi$ and $r \in \mathbb{Z}$ lying between the points $\lambda \in \mathbb{E}_{\underline{h}}$ and the point $\varnothing \in \overline{\mathbb{E}}_{\underline{h}}$. We let $\ell_\alpha(\lambda)$ denote the total number of these hyperplanes for a given $\alpha \in \Pi$ (including any hyperplane upon which λ lies).

2.3. *Paths in the geometry*

We now develop the combinatorics of paths inside our geometry. Given a map $p : \{1, ..., n\} \to \{1, ..., h\}$ we define points $\mathsf{P}(k) \in \mathbb{E}_{\underline{h}}$ by

$$\mathsf{P}(k) = \sum_{1 \leqslant i \leqslant k} \varepsilon_{p(i)}$$

for $1 \leqslant i \leqslant n$. We define the associated path of length n by

$$\mathsf{P} = (\varnothing = \mathsf{P}(0), \mathsf{P}(1), \mathsf{P}(2), \dots, \mathsf{P}(n))$$

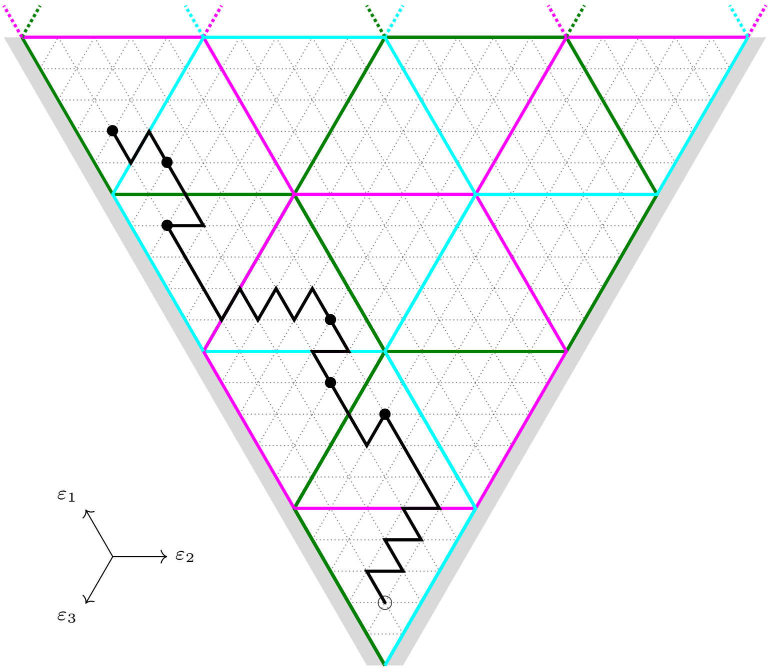
and we say that the path has shape $\pi = \mathsf{P}(n) \in \mathbb{E}_{\underline{h}}$. We also denote this path by $\mathsf{P} = (\varepsilon_{p(1)}, \dots, \varepsilon_{p(n)})$ and call $\varepsilon_{p(i)}$ the ith step in this path. Given $\lambda \in \mathbb{E}_{\underline{h}} \cap \mathbb{Z}_{\geqslant 0}\{\varepsilon_1, \dots, \varepsilon_h\}$ we let $\mathrm{Path}(\lambda)$ denote the set of paths of length n with shape λ. We define $\mathrm{Path}_{\underline{h}}(\lambda)$ to be the subset of $\mathrm{Path}(\lambda)$ consisting of those paths lying entirely inside the dominant chamber, $\mathbb{E}^+_{\underline{h}}$; in other words, those P such that $\mathsf{P}(i)$ is dominant for all $0 \leqslant i \leqslant n$.

Given a path P defined by such a map p of length n and shape λ we can write each $p(j)$ uniquely in the form $\varepsilon_{p(j)} = \varepsilon_{c_j, m_j}$ where $0 \leqslant m_j < \ell$ and $1 \leqslant c_j \leqslant h_j$. We record these elements in a tableau of shape λ^T by induction on j, where we place the positive integer j in the first empty node in the c_jth column of component m_j. By definition, such a tableau will have entries increasing down columns; if λ is a multipartition then the entries also increase along rows if and only if the given path is in $\mathrm{Path}_{\underline{h}}(\lambda)$, and hence there is a bijection between $\mathrm{Path}_{\underline{h}}(\lambda)$ and $\mathrm{Std}(\lambda)$. For this reason we will sometimes refer to paths as tableaux, to emphasise that what we are doing is generalising the classical tableaux combinatorics for the symmetric group.

Notation 2.5. Given a path P we will let $\mathsf{P}^{-1}(r, \varepsilon_{c,m})$ with $0 \leqslant m < \ell$ and $1 \leqslant c \leqslant h_m$ denote the (r, c)-entry of the mth component of the tableau corresponding to P. In terms of our path this is the point at which the rth step of the form $+\varepsilon_{c,m}$ occurs in P. Given a path P we define

$$\mathrm{res}(\mathsf{P}) = (\mathrm{res}_{\mathsf{P}}(1), \dots, \mathrm{res}_{\mathsf{P}}(n))$$

where $\mathrm{res}_{\mathsf{P}}(i)$ denotes the residue of the node labelled by i in the tableau corresponding to P.



1	2	10
3	4	17
5	6	19
7	14	20
8	25	29
9		
11		
12		
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15		
16		
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30		

Fig. 1. An alcove path in $\mathrm{Path}_{(3)}(20, 5^2)$ and the corresponding tableau in $\mathrm{Std}(20, 5^2)$. The black vertices denote vertices on the path in the orbit of the origin. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

Example 2.6. We will illustrate our various definitions with an example in $\overline{\mathbb{E}}^+_{3,1}$ with $e = 5$. This space is the projection of $\mathbb{R}^3$ in two dimensions, which we shall represent as shown in Fig. 1. Notice in particular that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ in this projection, as required. Only the dominant chamber is illustrated, with the origin marked in the fundamental alcove A_0.

The affine Weyl group $\widehat{\mathfrak{S}}_3$ has generating set S corresponding to the green and blue (non-affine) reflections $s_{\varepsilon_2-\varepsilon_3,0}$ and $s_{\varepsilon_1-\varepsilon_2,0}$ about the lower walls of the fundamental alcove, together with the (affine) reflection $s_{\varepsilon_3-\varepsilon_1,-5}$ about the red wall of that alcove. Recall that we will abuse notation, and refer to these simply as $s_{\varepsilon_2-\varepsilon_3}$, $s_{\varepsilon_1-\varepsilon_2}$, and $s_{\varepsilon_3-\varepsilon_1}$. The associated colours for the remaining alcove walls are as shown.

Given $\lambda = (3^5, 1^{15})$ we have illustrated a path P from the origin to λ with a black line. Recall that we embed partitions via the transpose map (as in equation (2.1)) and so the final point in the path corresponds to the point $(20, 5, 5) \in E_{3,1}$. The corresponding steps in the path are recorded in the standard tableau at the bottom of the figure, where an entry i in column j of the tableau (again, note the transpose) corresponds to the ith step of the path being in the direction ε_j. This is an element of $\mathrm{Path}_{\underline{h}}(\lambda)$ as it never leaves the dominant region.

The path passes through the sequence of alcoves obtained from the fundamental alcove by reflecting through the walls labelled R then G then B then R then G then B, and so the final alcove corresponds to the element $s_{\varepsilon_3-\varepsilon_1}s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_3-\varepsilon_2}s_{\varepsilon_3-\varepsilon_1}s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_3-\varepsilon_1}A_0$. If $\sigma = (0)$ then we have

$$\mathrm{res}(\mathsf{P}) = (0, 1, 4, 0, 3, 4, 2, 1, 0, 2, 4, \dots, 1).$$

Example 2.7. Further examples of paths and tableaux are given in Figs. 2 to 4.

Given paths $\mathsf{P} = (\varepsilon_{p(1)}, \dots, \varepsilon_{p(n)})$ and $\mathsf{Q} = (\varepsilon_{q(1)}, \dots, \varepsilon_{q(n)})$ we say that $\mathsf{P} \sim \mathsf{Q}$ if there exists an $\alpha \in \Phi$ and $r \in \mathbb{Z}$ and $s \leqslant n$ such that

$$\mathsf{P}(s) \in \mathbb{E}(\alpha, re) \qquad \text{and} \qquad \varepsilon_{q(t)} = \begin{cases} \varepsilon_{p(t)} & \text{for } 1 \leqslant t \leqslant s \\ s_\alpha \varepsilon_{p(t)} & \text{for } s+1 \leqslant t \leqslant n. \end{cases}$$

In other words the paths P and Q agree up to some point $\mathsf{P}(s) = \mathsf{Q}(s)$ which lies on $\mathbb{E}(\alpha, re)$, after which each $\mathsf{Q}(t)$ is obtained from $\mathsf{P}(t)$ by reflection in $\mathbb{E}(\alpha, re)$. We extend $\sim$ by transitivity to give an equivalence relation on paths, and say that two paths in the same equivalence class are related by a series of wall reflections of paths. We say that $\mathsf{P} = (\varepsilon_{p(1)}, \dots, \varepsilon_{p(n)})$ is a reduced path if $\ell_\alpha(\mathsf{P}(s+1)) \geqslant \ell_\alpha \mathsf{P}(s))$ for all $1 \leqslant s < n$ and $\alpha \in \Pi$. There exists a unique reduced path in each $\sim$-equivalence class.

Lemma 2.8. *We have* $\mathsf{P} \sim \mathsf{Q}$ *if and only if* $\mathrm{res}(\mathsf{P}) = \mathrm{res}(\mathsf{Q})$.

Proof. Let $\alpha = \varepsilon_{i,a} - \varepsilon_{j,b}$. We first note that a path of shape λ lies on $\mathbb{E}(\alpha, re)$ if and only if the addable nodes in λ in the ith column of the ath component and in the jth column of the bth component have the same residue. (This is straightforward from the definition of the inner product, see for example [3, Lemma 6.19].) Also $s_\alpha \varepsilon_t = \varepsilon_t$ for all $t \notin \{H_{a-1} + i, H_{b-1} + j\}$ and s_α permutes the elements of this set. So if two paths coincide up to some point and then diverge, but have the same sequence of residues, then the point where they diverge must lie on some $\mathbb{E}(\alpha, re)$ and the divergence must initially be by a reflection in this hyperplane. From this the result easily follows by induction on the number of hyperplanes which the two paths cross. □

We recast the degree of a tableau in the path-theoretic setting as follows.

Definition 2.9. Given a path $\mathsf{S} = (\mathsf{S}(0), \mathsf{S}(1), \mathsf{S}(2), \dots, \mathsf{S}(n))$ we set $\deg(\mathsf{S}(0)) = 0$ and define

$$\deg(\mathsf{S}) = \sum_{1 \leqslant k \leqslant n} d(\mathsf{S}(k), \mathsf{S}(k-1)),$$

for $d(\mathsf{S}(k), \mathsf{S}(k-1))$ defined as follows. For $\alpha \in \Phi^+$ we set $d_\alpha(\mathsf{S}(k), \mathsf{S}(k-1))$ to be

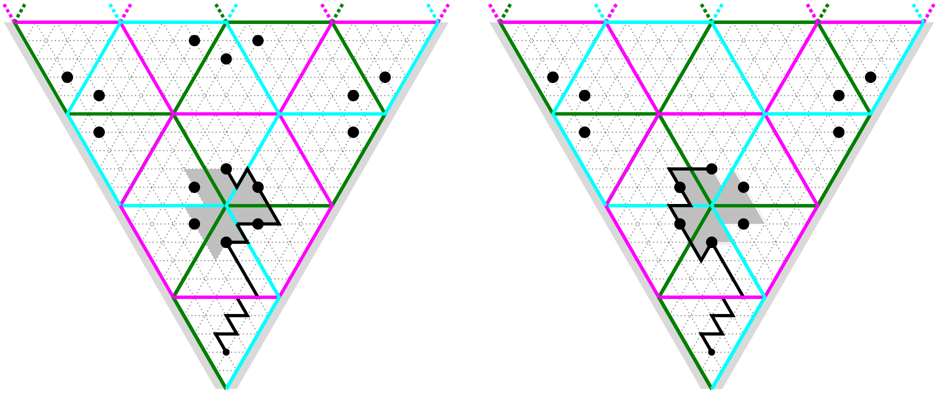

Fig. 2. Two paths S and T in an alcove geometry. These paths are used in Example 2.30.

1	2	17
3	4	
5	6	
7	10	
8	12	
9	13	
11		
14		
15		
16		
18		

1	2	10
3	4	
5	6	
7	14	
8	17	
9	18	
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16		

Fig. 3. The two tableaux S and T corresponding to the paths in Fig. 2. These paths are used in Example 2.30.

- $+1$ if $\mathsf{S}(k-1) \in \mathbb{E}(\alpha, re)$ and $\mathsf{S}(k) \in \mathbb{E}^{<}(\alpha, re)$ for some $r \in \mathbb{Z}$;
- -1 if $\mathsf{S}(k-1) \in \mathbb{E}^{>}(\alpha, re)$ and $\mathsf{S}(k) \in \mathbb{E}(\alpha, re)$ for some $r \in \mathbb{Z}$;
- 0 otherwise.

We let

$$\deg(\mathsf{S}) = \sum_{1 \leqslant k \leqslant n} \sum_{\alpha \in \Phi^+} d_\alpha(\mathsf{S}(k-1), \mathsf{S}(k)).$$

2.4. Alcove paths

When passing from multicompositions to our geometry $\overline{\mathbb{E}}_{\underline{h}}$, many non-trivial elements map to the origin. One such element is $\delta = ((h_0), \ldots, (h_{\ell-1})) \in \mathscr{P}_{\underline{h}}(h)$. (Recall our transpose convention for embedding multipartitions into our geometry, as in equation (2.1).) We will sometimes refer to this as the determinant as (for $\ell = 1$) it corresponds to the determinant representation of the associated general linear group. We will also need to consider elements corresponding to powers of the determinant, namely $\delta_n = ((h_0^n), \ldots, (h_{\ell-1}^n)) \in \mathscr{P}_\ell(nh)$.

We now restrict our attention to paths between points in the principal linkage class, in other words to paths between points in $\widehat{\mathfrak{S}}_h \cdot 0$. Such points can be represented by the μ in the orbit $\widehat{\mathfrak{S}}_h \cdot \delta_n$ for some choice of n.

Definition 2.10. We will associate alcove paths to certain words in the alphabet

$$S \cup \{1\} = \{s_\alpha \mid \alpha \in \Pi \cup \{\emptyset\}\}$$

where $s_\emptyset = 1$. That is, we will consider words in the generators of the affine Weyl group, but enriched with explicit occurrences of the identity in these expressions. When we wish to consider a particular expression for an element $w \in \widehat{\mathfrak{S}}_h$ in terms of our alphabet we will denote this by $\underline{w}$.

Our aim is to define certain distinguished paths from the origin to multipartitions in the principal linkage class; for this we will need to proceed in stages. In order to construct our path we want to proceed inductively. There are two ways in which we shall do this.

Definition 2.11. Given two paths

$$\mathsf{P} = (\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_p}) \in \mathrm{Path}(\mu) \quad \text{and} \quad \mathsf{Q} = (\varepsilon_{j_1}, \varepsilon_{j_2}, \ldots, \varepsilon_{j_q}) \in \mathrm{Path}(\nu)$$

we define the naive concatenated path

$$\mathsf{P} \boxtimes \mathsf{Q} = (\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_p}, \varepsilon_{j_1}, \varepsilon_{j_2}, \ldots, \varepsilon_{j_q}) \in \mathrm{Path}(\mu + \nu).$$

There are several problems with naive concatenation. Most seriously, the naive concatenation of two paths between points in the principal linkage class will not in general itself connect points in that class. Also, if we want to associate to our path the coloured sequence of walls through which it passes, then this is not compatible with naive concatentation. To remedy these failings, we will also need to define a contextualised concatenation.

Given a path P between points in the principal linkage class, the end point lies in the interior of an alcove of the form wA_0 for some $w \in \widehat{\mathfrak{S}}_h$. If we write w as a word in our alphabet, and then replace each element s_α by the corresponding non-affine reflection

s_α in $\mathfrak{S}_h$ to form the element $\overline{w} \in \mathfrak{S}_h$ then the basis vectors ε_i are permuted by the corresponding action of $\overline{w}$ to give $\varepsilon_{\overline{w}(i)}$, and there is an isomorphism from $\overline{\mathbb{E}}_{\underline{h}}$ to itself which maps A_0 to wA_0 such that 0 maps to $w \cdot 0$, coloured walls map to walls of the same colour, and each basis element ε_i map to $\varepsilon_{\overline{w}(i)}$. Under this map we can transform a path Q starting at the origin to a path starting at $w \cdot 0$ which passes through the same sequence of coloured walls as Q does.

More generally, the end point of a path P may lie on one or more walls. In this case, we can choose a distinct transformation as above for each alcove wA_0 whose closure contains the endpoint. We can now use this to define our contextualised concatenation.

Definition 2.12. Given two paths $\mathsf{P} = (\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_p}) \in \mathrm{Path}(\mu)$ and $\mathsf{Q} = (\varepsilon_{j_1}, \varepsilon_{j_2}, \dots, \varepsilon_{j_q}) \in \mathrm{Path}(\nu)$ with the endpoint of P lying in the closure of some alcove wA_0 we define the contextualised concatenated path

$$\mathsf{P} \otimes_w \mathsf{Q} = (\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_p}) \boxtimes (\varepsilon_{\overline{w}(j_1)}, \varepsilon_{\overline{w}(j_2)}, \dots, \varepsilon_{\overline{w}(j_q)}) \in \mathrm{Path}(\mu + (w \cdot \nu)).$$

If there is a unique such w then we may simply write $\mathsf{P} \otimes \mathsf{Q}$. If $w = s_{\boldsymbol{\alpha}}$ we will simply write $\mathsf{P} \otimes_{\boldsymbol{\alpha}} \mathsf{Q}$.

It is not difficult to understand contextualised concatenation in terms of tableaux. Each symbol ε_i for $1 \leqslant i \leqslant h$ labels a column of a partition. Contextualised concatenation is then given by permuting the columns (according to the rule in Definition 2.12) and then vertically stacking the tableaux (and shifting the entries), see Fig. 5.

Our next aim is to define the building blocks from which all of our distinguished paths will be constructed. We begin by defining certain integers that describe the position of the origin in our fundamental alcove.

Definition 2.13. Given $\boldsymbol{\alpha} \in \Pi$ we define $b_{\boldsymbol{\alpha}}$ to be the distance from the origin to the wall corresponding to $\boldsymbol{\alpha}$, and let $b_\emptyset = 1$. Given our earlier conventions this corresponds to setting

$$b_{\varepsilon_{i,p} - \varepsilon_{j,q}} = \sigma_q - \sigma_p + j - i \qquad b_{\varepsilon_h - \varepsilon_1} = e - \sigma_0 + \sigma_{\ell-1} + h_{\ell-1} - 1$$

for $0 \leqslant p \leqslant q < \ell$ and $1 \leqslant i \leqslant h_p$, $1 \leqslant j \leqslant h_q$. We sometimes write $\delta_{\boldsymbol{\alpha}}$ for the element $\delta_{b_{\boldsymbol{\alpha}}}$. Given $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Pi$ we set $b_{\boldsymbol{\alpha}\boldsymbol{\beta}} = b_{\boldsymbol{\alpha}} + b_{\boldsymbol{\beta}}$.

Example 2.14. Let $e = 5$, $h = 3$ and $\ell = 1$ as in Fig. 1. Then $b_{\varepsilon_2 - \varepsilon_3}$ and $b_{\varepsilon_1 - \varepsilon_2}$ both equal 1, while $b_{\varepsilon_3 - \varepsilon_1} = 3$ and $b_\emptyset = 1$.

Example 2.15. Let $e = 7$, $h = 2$ and $\ell = 2$ and $\sigma = (0, 3) \in \mathbb{Z}^2$. Then $b_{\varepsilon_1 - \varepsilon_2}$ and $b_{\varepsilon_3 - \varepsilon_4}$ both equal 1, while $b_{\varepsilon_4 - \varepsilon_1} = 3$, $b_{\varepsilon_2 - \varepsilon_3} = 2$, and $b_\emptyset = 1$.

We can now define our basic building blocks for paths.

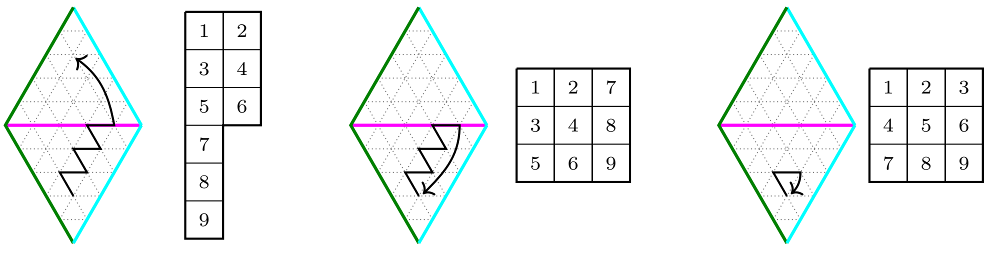


Fig. 4. Three paths and their corresponding tableaux. The leftmost two paths are the path P_{α} which walks through an α-hyperplane in $\overline{\mathbb{E}}^{+}_{1,3}$, and the path $\mathsf{P}^{\flat}_{\alpha}$ which reflects the former path through the same α-hyperplane. The rightmost path is $\mathsf{P}_{\emptyset}$ (which we repeat thrice).

Definition 2.16. Given $\alpha = \varepsilon_i - \varepsilon_{i+1} \in \Pi$, we consider the multicomposition $s_{\alpha} \cdot \delta_{\alpha}$ with all columns of length b_{α}, with the exception of the ith and $(i+1)$st columns, which are of length 0 and $2b_{\alpha}$, respectively. We set

$$\mathsf{M}_i = (\varepsilon_1, \ldots, \varepsilon_{i-1}, \widehat{\varepsilon_i}, \varepsilon_{i+1}, \ldots, \varepsilon_h) \quad \text{and} \quad \mathsf{P}_i = (+\varepsilon_i)$$

where $\widehat{\cdot}$ denotes omission of a coordinate. Then our distinguished path corresponding to s_{α} is given by

$$\mathsf{P}_{\alpha} = \mathsf{M}_i^{b_{\alpha}} \boxtimes \mathsf{P}_{i+1}^{b_{\alpha}} \in \mathrm{Path}(s_{\alpha} \cdot \delta_{\alpha}).$$

The distinguished path corresponding to $\emptyset$ is given by

$$\mathsf{P}_{\emptyset} = (\varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_h) \in \mathrm{Path}(\delta) = \mathrm{Path}(s_{\emptyset} \cdot \delta)$$

and set $\mathsf{P}_{\varnothing} = (\mathsf{P}_{\emptyset})^{b_{\alpha}}$. We will also find it useful to have the following variant of M_i. We set

$$\mathsf{M}_{i,j} = (\varepsilon_1, \ldots, \varepsilon_{i-1}, \widehat{\varepsilon_i}, \varepsilon_{i+1}, \ldots, \varepsilon_{j-1}, \widehat{\varepsilon_j}, \varepsilon_{j+1}, \ldots, \varepsilon_h).$$

Example 2.17. The paths/tableaux S and T from Figs. 2 and 3 are equal to $\mathsf{P}_{\alpha} \otimes_{\alpha} \mathsf{P}_{\beta} \otimes_{\beta} \mathsf{P}_{\gamma} \otimes_{\gamma} \mathsf{P}_{\beta}$ and $\mathsf{P}_{\alpha} \otimes_{\alpha} \mathsf{P}_{\gamma} \otimes_{\gamma} \mathsf{P}_{\beta} \otimes_{\beta} \mathsf{P}_{\gamma}$ respectively for $\alpha = \varepsilon_1 - \varepsilon_3$, $\beta = \varepsilon_1 - \varepsilon_2$, $\gamma = \varepsilon_2 - \varepsilon_3$.

Given all of the above, we can finally define our distinguished paths for general words in our alphabet. There will be one such path for each word in our alphabet, and they will be defined by induction on the length of the word, as follows.

Definition 2.18. We now define a distinguished path $\mathsf{P}_{\underline{w}}$ for each word $\underline{w}$ in our alphabet $S \cup \{1\}$ by induction on the length of $\underline{w}$. If $\underline{w}$ is $s_{\emptyset}$ or a simple reflection s_{α} we have already defined the distinguished path in Definition 2.16. Otherwise if $\underline{w} = s_{\alpha}\underline{w}'$ then we define

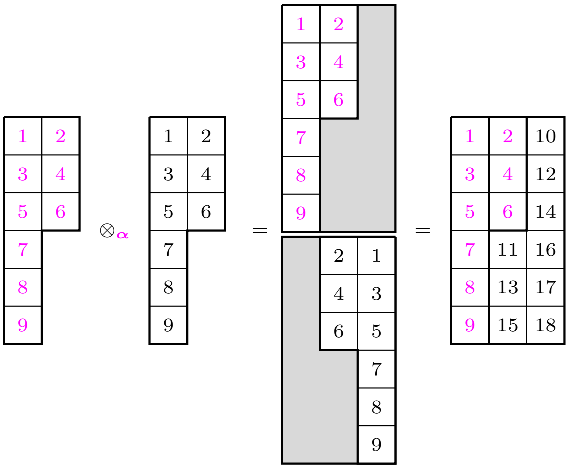


Fig. 5. The tableau $\mathsf{P}_{\alpha} \otimes_{\alpha} \mathsf{P}_{\alpha}$ obtained by contextualised concatenation from the path/tableau P_{α} in Fig. 4. The reflection s_{α} for $\alpha = \varepsilon_1 - \varepsilon_3$ permutes the first and third columns of P_{α}. The entries of tableaux are coloured to facilitate comparison. The reader is invited to draw the corresponding path.

$$\mathsf{P}_{\underline{w}} := \mathsf{P}_{\alpha} \ \otimes_{\alpha} \ \mathsf{P}_{\underline{w}'}.$$

If $\underline{w}$ is a reduced word in $\widehat{\mathfrak{S}}_h$, then the path $\mathsf{P}_{\underline{w}}$ is a reduced path.

Remark 2.19. Contextualised concatenation is not associative (if we wish to decorate the tensor products with the corresponding elements w). As we will typically be constructing paths as in Definition 2.18 we will adopt the convention that an unbracketed concatenation of n terms corresponds to bracketing from the right:

$$\mathsf{Q}_1 \otimes \mathsf{Q}_2 \otimes \mathsf{Q}_3 \otimes \cdots \mathsf{Q}_n = \mathsf{Q}_1 \otimes (\mathsf{Q}_2 \otimes (\mathsf{Q}_3 \otimes (\cdots \otimes \mathsf{Q}_n) \cdots)).$$

We will also need certain reflections of our distinguished paths corresponding to elements of Π.

Definition 2.20. Given $\alpha \in \Pi$ we set

$$\mathsf{P}^{\flat}_{\alpha} = \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_i^{b_\alpha} = \mathsf{M}_i^{b_\alpha} \otimes_{\alpha} \mathsf{P}_{i+1}^{b_\alpha} = (+\varepsilon_1, \dots, +\varepsilon_{i-1}, \widehat{+\varepsilon_i}, +\varepsilon_{i+1}, \dots, +\varepsilon_h)^{b_\alpha} \boxtimes (\varepsilon_i)^{b_\alpha}$$

the path obtained by reflecting the second part of P_{α} in the wall through which it passes.

Example 2.21. We illustrate these various constructions in a series of examples. In the first two diagrams of Fig. 4, we illustrate the basic path P_{α} and the path $\mathsf{P}^{\flat}_{\alpha}$ and in the rightmost diagram of Fig. 4, we illustrate the path $\mathsf{P}_{\emptyset}$. A more complicated example is illustrated in Fig. 1, where we show the distinguished path $\mathsf{P}_{\underline{w}}$ for $\underline{w} = s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_1-\varepsilon_2} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_1-\varepsilon_2} s_{\varepsilon_2-\varepsilon_3}$ as in Example 2.6. The components of the path between consecutive black nodes correspond to individual P_{α}s.

Remark 2.22. There are plenty of other paths we could have chosen. For example, we could replace the leftmost path in Fig. 4 with the path

$$(\varepsilon_1, \varepsilon_1, \varepsilon_1, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_1, \varepsilon_1, \varepsilon_1) \in \mathrm{Path}(6, 3).$$

In Proposition 4.4 we will see that it does not matter which path we pick, providing it "does not hit any extra hyperplanes". Our "zig zagging" paths are merely the easiest to define such general paths.

Remark 2.23. By Lemma 2.8 we have $\mathrm{res}(\mathsf{P}_{\boldsymbol{\alpha}}) = \mathrm{res}(\mathsf{P}^{\flat}_{\boldsymbol{\alpha}})$. This fact is key to our construction of the KLR versions of the diagrammatic Bott–Samelson generators using step-preserving permutations.

Definition 2.24. We say that a word $\underline{w} = s_{\alpha^{(1)}} \dots s_{\alpha^{(p)}}$ in either of the alphabets S or $S \cup \{1\}$ has breadth

$$\mathsf{breadth}_\sigma(\underline{w}) = \sum_{1 \leqslant i \leqslant p} b_{\alpha^{(i)}}$$

which we denote simply by $b_{\underline{w}}$ when the context is clear. We let $\Lambda(n, \sigma)$ (respectively $\Lambda^+(n, \sigma)$) denote the set of words $\underline{w}$ in the alphabet $S \cup \{1\}$ (respectively the alphabet S) such that $\mathsf{breadth}_\sigma(\underline{w}) = n$. We define

$$\mathscr{P}_{\underline{h}}(n, \sigma) = \{\lambda \in \mathscr{P}_{\underline{h}}(n) \mid \text{there exists } \mathsf{P}_{\underline{w}} \in \mathrm{Std}(\lambda), \underline{w} \in \Lambda(n, \sigma)\}.$$

Example 2.25. We can insert the path $\mathsf{P}_\emptyset = (+\varepsilon_1, +\varepsilon_2, +\varepsilon_3)$ into the path in Fig. 1 at seven distinct points to obtain a new alcove path. For example, we can insert two copies of this path (in two distinct ways) to obtain $\mathsf{P}_{\underline{w}}$ and $\mathsf{P}_{\underline{w}'}$ for $\underline{w} = s_\emptyset s_\emptyset s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2} s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2}$ and $\underline{w}' = s_{\varepsilon_3-\varepsilon_1} s_\emptyset s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2} s_\emptyset s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2}$ respectively. Then $\mathsf{res}(\mathsf{P}_{\underline{w}})$ and $\mathsf{res}(\mathsf{P}_{\underline{w}'})$ are equal to

$$(0, 1, 2, 4, 0, 1, 3, 4, 2, 3, 1, 2, 0, 4, 3, 0, 2, 1, 0, 1, 4, 3, 4, 2, 3, 1, 2, 0, 4, 3, 0, 2, 1, 0, 1, 4),$$
$$(0, 1, 4, 0, 3, 4, 2, 1, 0, 2, 3, 4, 4, 1, 0, 4, 0, 3, 2, 3, 4, 3, 4, 2, 3, 1, 2, 0, 4, 3, 0, 2, 1, 0, 1, 4).$$

For any $\lambda \in \mathscr{P}_{\underline{h}}(n)$, we define the set of alcove-tableaux, $\mathrm{Std}_{n,\sigma}(\lambda)$, to consist of all standard tableaux which can be obtained by contextualised concatenation of paths from the set

$$\{\mathsf{P}_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \Pi\} \cup \{\mathsf{P}^{\flat}_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \Pi\} \cup \{\mathsf{P}_\emptyset\}.$$

We let $\mathrm{Std}^+_{n,\sigma}(\lambda) \subseteq \mathrm{Std}_{n,\sigma}(\lambda)$ denote the subset of strict alcove-tableaux of the form $(\mathsf{P}_\emptyset)^{\otimes p} \otimes \mathsf{Q}$ for Q obtained by contextualised concatenation of paths from the set $\{\mathsf{P}_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \Pi\} \cup \{\mathsf{P}^{\flat}_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \Pi\}$ and some $p \geqslant 0$.

Example 2.26. The tableau of shape $(20, 5^2)$ in Fig. 1 is the strict alcove tableau given by $\mathsf{P}_{\alpha} \otimes_{\alpha} \mathsf{P}_{\gamma} \otimes_{\gamma} \mathsf{P}_{\beta} \otimes_{\beta} \mathsf{P}_{\alpha} \otimes_{\alpha} \mathsf{P}_{\gamma} \otimes_{\gamma} \mathsf{P}_{\beta}$.

Clearly any such (strict) alcove tableau terminates at a regular partition in the principal linkage class of the algebra. By definition, we have that there is precisely one alcove-tableau $\mathsf{P}_{\underline{w}}$ for each expression $\underline{w}$ in the simple reflections (and the emptyset). Similarly, we have that there is precisely one strict alcove-tableau $\mathsf{P}_{\underline{w}}$ for each expression $\underline{w}$ in the simple reflections.

Example 2.27. Let $h = 3$ and $\ell = 1$ and $e = 5$ and $\alpha = \varepsilon_3 - \varepsilon_1$. We have that $b_{\alpha} = 3$. We have that

$$\begin{aligned}\mathsf{P}_{\alpha\varnothing} &= (\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_1, \varepsilon_1) \otimes (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3)\\ &= (\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_1, \varepsilon_1, \varepsilon_3, \varepsilon_2, \varepsilon_1, \varepsilon_3, \varepsilon_2, \varepsilon_1, \varepsilon_3, \varepsilon_2, \varepsilon_1)\\ \mathsf{P}_{\varnothing\alpha} &= (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_1, \varepsilon_1)\end{aligned}$$

are both dominant paths of shape $(3^3, 2^3, 1^3)$.

2.4.1. Permutations as morphisms between paths

We now discuss how one can think of a permutation as a morphism between pairs of paths in the alcove geometries of Section 3. This shift in perspective, from permutations acting on tableaux (the usual combinatorics of $\mathfrak{S}_n$) to "morphisms between paths" is a central idea of this paper.

Definition 2.28. Let $\lambda \in \mathbb{Z}_{\geqslant 0}\{\varepsilon_1, \dots, \varepsilon_h\}$. Given a pair of paths $\mathsf{S}, \mathsf{T} \in \mathrm{Path}(\lambda)$ we write the steps ε_i in S and T in sequence along the top and bottom edges of a frame, respectively. We define $w^{\mathsf{S}}_{\mathsf{T}} \in \mathfrak{S}_n$ to be the unique step-preserving permutation with the minimal number of crossings.

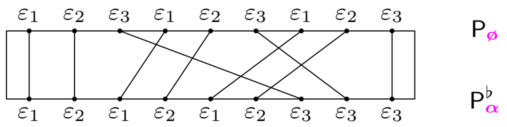
Recall that a step ε_i in a path corresponds to adding a node in the ith column (indexed from left to right) of the multi-partition tableau. Thus one can rewrite the above for pairs of *column standard* tableaux as follows: $w^{\mathsf{S}}_{\mathsf{T}}$ is the unique element such that $w^{\mathsf{S}}_{\mathsf{T}}(\mathsf{S}) = \mathsf{T}$ (under the usual action of the symmetric group on tableaux). An example is given in Example 2.30.

Example 2.29. We consider $\Bbbk\mathfrak{S}_9$ in the case of $p = 5$. We set $\alpha = \varepsilon_3 - \varepsilon_1 \in \Pi$. Here we have

$$\mathsf{P}_{\varnothing} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3) \quad \text{and} \quad \mathsf{P}^{\flat}_{\alpha} = (\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_3, \varepsilon_3)$$

(the corresponding tableaux are given in Fig. 4). The unique step-preserving permutation of minimal length is given by

$w^{\mathsf{P}_{\varnothing}}_{\mathsf{P}^{\flat}_{\alpha}} =$

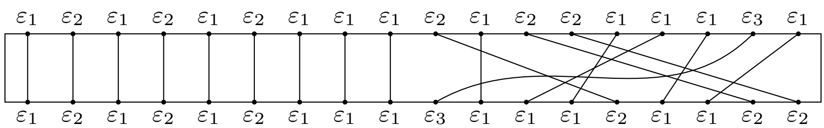


(2.2)

Notice that if two strands have the same step-label, then they do not cross. This is, of course, exactly what it means for a step-preserving permutation to be of minimal length.

Example 2.30. We depict two paths $\mathsf{S}, \mathsf{T} \in \text{Path}(11, 6, 1)$ in Fig. 2 and the corresponding tableaux in Fig. 3. The path-morphism $w^{\mathsf{S}}_{\mathsf{T}}$ is as follows

$w^{\mathsf{S}}_{\mathsf{T}} =$



.

Notice that the sequence of ε_i along the top (bottom) of the word simply record the columns of the entries of the tableaux S, T read in order according to the entries $1 \leqslant i \leqslant 18$. We always use ε_i as our labels of strand (dropping the epsilons would cause confusion later on, when we further attach KLR residues to these strands).

When we wish to explicitly write down a specific reduced expression for $w^{\mathsf{S}}_{\mathsf{T}}$ for concreteness, we will find the following notation incredibly useful.

Definition 2.31. Given t an integer, we let $r_h(t)$ denote the remainder of t modulo h. Given $p, q \geqslant 1$ such that $r_h(p) \neq i$ and $\alpha = \varepsilon_i - \varepsilon_{i+1} \in \Pi$, we set

$$\alpha(p) = \mathsf{P}_{\alpha}^{-1}(1, r_h(p)) \qquad \text{and} \qquad \emptyset(q) = \mathsf{P}_{\emptyset}^{-1}(1, r_h(q))$$

This notation allows us to implicitly use the cyclic ordering on the labels of roots without further ado.

Convention 2.32. Throughout the paper, we let $\alpha = \varepsilon_i - \varepsilon_{i+1}$, $\beta = \varepsilon_j - \varepsilon_{j+1}$, $\gamma = \varepsilon_k - \varepsilon_{k+1}$, $\delta = \varepsilon_m - \varepsilon_{m+1}$. We will assume that β, γ, δ label distinct commuting reflections. We will assume throughout that β and α label non-commuting reflections. Here we read these subscripts in the obvious cyclotomic ordering, without further ado (in other words, we read occurrences of $h + 1$ simply as 1).

3. The diagrammatic algebras

We now introduce the two protagonists of this paper: the diagrammatic Bott–Samelson endomorphism algebras and the quiver Hecke algebras — these can be defined either as monoidal (tensor) categories or as finite-dimensional diagrammatic algebras. We favour the latter perspective for aesthetic reasons, but we borrow the notation from the former world by letting $\otimes$ denote horizontal concatenation of diagrams — in the quiver Hecke case, we must first "contextualise" before concatenating as we shall explain in Subsection 3.3.2. (We refer to [9] for a more detailed discussion of the interchangeability of these two languages.) The relations for both algebras are entirely local (here a local relation means one that can be specified by its effect on a sufficiently small region of the wider diagram). We then consider the cyclotomic quotients of these algebras: these can be viewed as quotients by right-tensor-ideals, or equivalently (as we do in this paper) as quotients by a *non-local* diagrammatic relation concerning the leftmost strand in the ambient concatenated diagram. (We remark that the cyclotomic relations break the monoidal structure of both categories.) We continue with the notation of Convention 2.32.

Remark 3.1. The cyclotomic quotients of (anti-spherical) Hecke categories are small categories with finite-dimensional morphism spaces given by the light leaves basis of [13,18]. Working with such a category is equivalent working to with a locally unital algebra, as defined in [9, Section 2.2], see [9, Remark 2.3]. Throughout this paper we will work in the latter setting. The reader who prefers to think of categories can equivalently phrase everything in this paper in terms of categories and representations of categories.

3.1. The diagrammatic Bott–Samelson endomorphism algebras

These algebras were defined by Elias–Williamson in [13]. In this section, all our words will be in the alphabet S.

Definition 3.2. Given $\boldsymbol{\alpha} = \varepsilon_i - \varepsilon_{i+1}$ we define the corresponding Soergel idempotent, $1_{\boldsymbol{\alpha}}$ to be a frame of width 1 unit, containing a single vertical strand coloured with $\boldsymbol{\alpha} \in \Pi$. For $\underline{w} = s_{\alpha^{(1)}} \dots s_{\alpha^{(p)}}$ an expression with $\alpha^{(i)} \in \Pi$ simple roots, we set

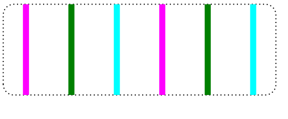
$$1_{\underline{w}} = 1_{\alpha^{(1)}} \otimes 1_{\alpha^{(2)}} \otimes \cdots \otimes 1_{\alpha^{(p)}}$$

to be the diagram obtained by horizontal concatenation.

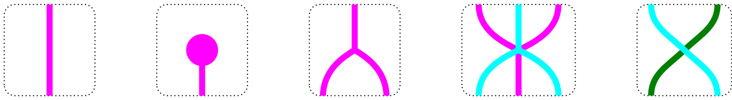
Example 3.3. Consider the colour-word from the path in Fig. 1. Namely,

$$\underline{w} = s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2} s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2} \in \widehat{\mathfrak{S}}_3.$$

The corresponding Soergel idempotent is as follows

$1_{\underline{w}} =$ 

Definition 3.4. Given $\underline{w} = s_{\alpha^{(1)}} \dots s_{\alpha^{(p)}}, \underline{w}' = s_{\beta^{(1)}} \dots s_{\beta^{(q)}} \in \mathfrak{S}_h$, a $(\underline{w}, \underline{w}')$-Soergel diagram D is defined to be any diagram obtained by horizontal and vertical concatenation of the following diagrams

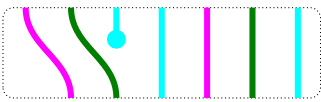
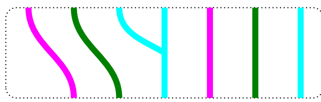


their flips through the horizontal axis and their isotypic deformations such that the top and bottom edges of the graph are given by the idempotents $1_{\underline{w}}$ and $1_{\underline{w}'}$ respectively. Here the vertical concatenation of a $(\underline{w}, \underline{w}')$-Soergel diagram on top of a $(\underline{v}, \underline{v}')$-Soergel diagram is zero if $\underline{v} \neq \underline{w}'$. We define the degree of these generators (and their flips) to be $0, 1, -1, 0$, and 0 respectively.

Example 3.5. Examples of $(\underline{w}, \underline{w}')$-Soergel diagrams, for

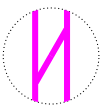
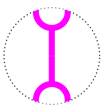
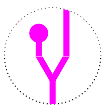
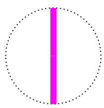
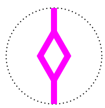
$$\underline{w} = s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2} s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2} s_{\varepsilon_1-\varepsilon_2},$$
$$\underline{w}' = s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2} s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2}$$

are as follows

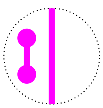
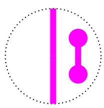
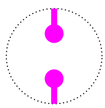
 

We let $*$ denote the map which flips a diagram through its horizontal axis.

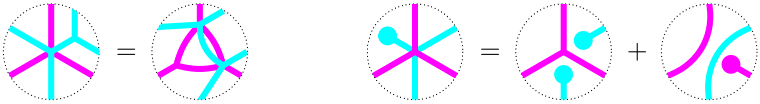
Definition 3.6. Let $\Bbbk$ be an arbitrary commutative ring. We define the diagrammatic Bott–Samelson endomorphism algebra, $\mathscr{S}(n, \sigma)$ to be the span of all $(\underline{w}, \underline{w}')$-Soergel diagrams for $\underline{w}, \underline{w}' \in \Lambda(n, \sigma)$, with $\Bbbk$-associative multiplication given by vertical concatenation and subject to isotypic deformation and the following local relations: For each colour (i.e. each generator $s_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in \Pi$) we have

 =   =   = 0 (S1)

along with their horizontal and vertical flips and the Demazure relation

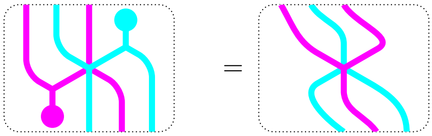
 +  = 2  (S2)

We now picture the two-colour relations for non-commuting reflections $s_{\boldsymbol{\alpha}}, s_{\boldsymbol{\beta}} \in \widehat{\mathfrak{S}}_h$. We have



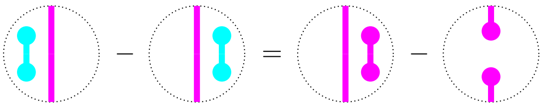
(S3)

along with their flips through the horizontal and vertical axes. We also have the cyclicity relation



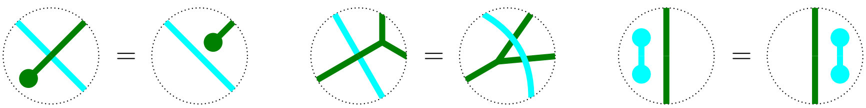
(S4)

and the two-colour barbell relations



(S5)

for Φ of rank greater than 1 (or double the righthand-side if Φ has rank 1). For commuting reflections $s_\beta, s_\gamma \in \widehat{\mathfrak{S}}_h$ we have the following relations

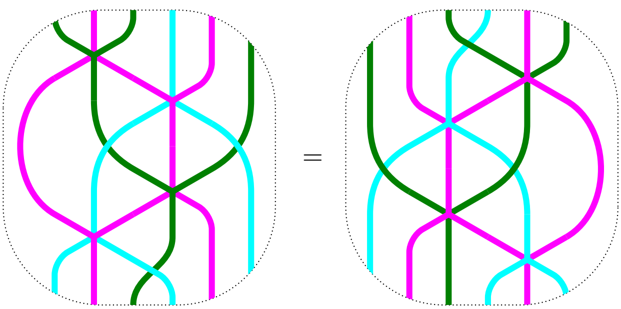


(S6)

along with their flips through the horizontal and vertical axes. In order to picture the three-colour commuting relations we require a fourth root $s_\delta \in \widehat{\mathfrak{S}}_h$ which commutes with all other roots (such that $s_\delta s_\alpha = s_\alpha s_\delta$, $s_\delta s_\beta = s_\beta s_\delta$, $s_\delta s_\gamma = s_\gamma s_\delta$) and we have the following,

(S7)

Finally, we require the tetrahedron relation for which we make the additional assumption on γ that it does not commute with α. This relation is as follows,



(S8)

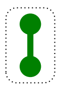
Remark 3.7. The diagrammatic Bott–Samelson category of $\widehat{\mathfrak{S}}_h$ is normally defined using an underlying reflection representation $\mathfrak{h} = (V, \{\alpha^\vee_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in S\}, \{\alpha_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in S\})$ of $\widehat{\mathfrak{S}}_h$ called a realisation. Our construction of the diagrammatic Bott–Samelson endomorphism algebra implicitly assumes that the roots $\{\alpha_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in S\} \subset V^*$ form a basis, and that the pairing between roots and coroots is given by the usual Cartan matrix of type $\widehat{A}_{h-1}$. These two conditions uniquely determine the realisation, which we call the universal realisation of $\widehat{\mathfrak{S}}_h$ with respect to this Cartan matrix [5]. It coincides with the modular reduction of the "dual geometric" realisation of $\widehat{\mathfrak{S}}_h$ (which can be defined over $\mathbb{Z}$ as $\widehat{\mathfrak{S}}_h$ is simply laced) [18].

Remark 3.8. We do not include "isotopy" as an explicit relation here (unlike in [13]) as it follows from the one-colour relations and cyclicity of the braid generator (see [12, Proposition 8.6]). This is the more modern definition, see for example [25, Section 2.3]

Definition 3.9. We define the cyclotomic diagrammatic Bott–Samelson endomorphism algebra,

$$\mathscr{S}_{\underline{h}}(n,\sigma) := \mathrm{End}_{\mathcal{D}^{\mathrm{asph},\oplus}_{\mathrm{BS}}(A_{h-1}\times\ldots\times A_{h-1}\backslash\widehat{A}_{h-1})}\left(\oplus_{\underline{w}\in\Lambda(n,\sigma)} B_{\underline{w}}\right)$$

to be the quotient of $\mathscr{S}(n,\sigma)$ by the relations

$$1_{\boldsymbol{\alpha}} \otimes 1_{\underline{w}} = 0 \qquad \text{and}$$  $$\otimes 1_{\underline{w}} = 0 \tag{S9}$$

for $\boldsymbol{\gamma} \in \Pi$ arbitrary, $\boldsymbol{\alpha} \in \Pi$ corresponding to a wall of the dominant chamber, and $\underline{w}$ any word in the alphabet S.

3.2. The breadth-enhanced diagrammatic Bott–Samelson endomorphism algebra

We now use the notion of a weakly graded monoidal category (see Appendix A) to introduce the breadth-enhanced diagrammatic Bott–Samelson endomorphism algebra. On one level this definition and construction is utterly superficial. It merely allows us to keep track of occurrences of the identity of $\widehat{\mathfrak{S}}_h$ in a given expression. The occurrences of $s_\emptyset = 1$ are usually ignored in the world of Soergel diagrammatics and so this will seem very foreign to some. We ask these readers to be patient as this extra "blank space" will be very important in this paper: each occurrence of $s_\emptyset$ corresponds to adding h additional strands in the quiver Hecke algebra or, if you prefer, corresponds to "tensoring with the determinant". For this reason, in this section all our words will be in the alphabet $S \cup \{1\}$.

Definition 3.10. Given $\boldsymbol{\alpha} = \varepsilon_i - \varepsilon_{i+1}$ we define the breadth-enhanced Soergel idempotent, $1_{\boldsymbol{\alpha}}$, to be a frame of width $2b_{\boldsymbol{\alpha}}$ with a single vertical strand coloured with $\boldsymbol{\alpha} \in \Pi$ placed in the centre. We define the breadth-enhanced Soergel idempotent $1_\emptyset$ to be an empty frame of width 2. For $\underline{w} = s_{\alpha^{(1)}} \ldots s_{\alpha^{(p)}}$ an expression with $\alpha^{(i)} \in \Pi \cup \{\emptyset\}$, we set

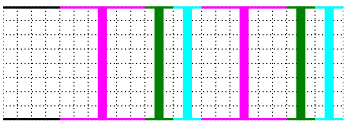
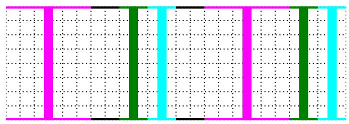
$$1_{\underline{w}} = 1_{\alpha^{(1)}} \otimes 1_{\alpha^{(2)}} \otimes \cdots \otimes 1_{\alpha^{(p)}}$$

to be the diagram obtained by horizontal concatenation. In order that we better illustrate this idea, we colour the top and bottom edges of a frame with the corresponding element of $\Pi \cup \{\emptyset\}$.

Example 3.11. Continuing with Fig. 1 and Example 2.14, we let

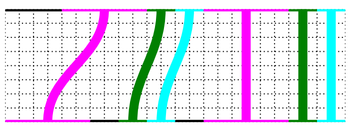
$$\underline{w} = s_\emptyset s_\emptyset s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2} s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2}$$
$$\underline{w}' = s_{\varepsilon_3-\varepsilon_1} s_\emptyset s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2} s_\emptyset s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2}.$$

The breadth-enhanced Soergel idempotents are as follows

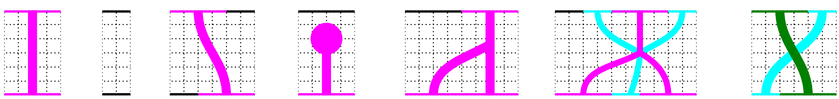
$1_{\underline{w}} =$  $1_{\underline{w}'} =$  (3.1)

Definition 3.12. Let $w \in \mathfrak{S}_h$ and suppose $\underline{w} = s_{\alpha^{(1)}} \dots s_{\alpha^{(p)}}$ and $\underline{w}' = s_{\beta^{(1)}} \dots s_{\beta^{(p)}}$ for $\alpha^{(i)}, \beta^{(j)} \in \Pi \cup \{\emptyset\}$ are two expressions which differ only by occurrences of $s_\emptyset$ within the word. We define the corresponding Soergel adjustment $1^{\underline{w}}_{\underline{w}'}$, to be the diagram with $1_{\underline{w}}$ along the top and $1_{\underline{w}'}$ along the bottom and no crossing strands.

Example 3.13. Continuing with Example 3.11, we have that

$1^{\underline{w}}_{\underline{w}'} =$ 

Definition 3.14. Given $\underline{w} = s_{\alpha^{(1)}} \dots s_{\alpha^{(p)}}, \underline{w}' = s_{\beta^{(1)}} \dots s_{\alpha^{(q)}}$ for $\alpha^{(i)}, \beta^{(j)} \in \Pi \cup \{\emptyset\}$, a breadth-enhanced $(\underline{w}, \underline{w}')$-Soergel diagram D is defined to be any diagram obtained by horizontal and vertical concatenation of the following generators

 (3.2)

and their flips through the horizontal axes such that the top edge of the graph is given by the breadth-enhanced idempotent $1_{\underline{w}}$ and the bottom edge given by the breadth-enhanced idempotent $1_{\underline{w}'}$. Here the vertical concatenation of a $(\underline{w}, \underline{w}')$-diagram on top of a $(\underline{v}, \underline{v}')$-diagram is zero if $\underline{v} \neq \underline{w}'$. The degree of these generators (and their flips) are $0, 0, 0, 1, -1, 0$, and 0 respectively. When we wish to avoid drawing diagrams, we will denote the above diagrams by

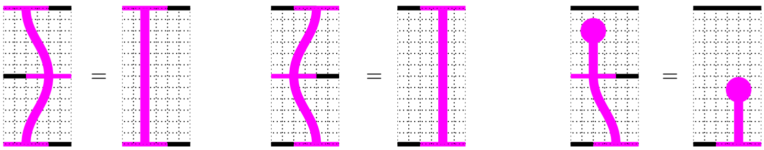


Fig. 6. The adjustment-inversion relations and the naturality relation for the spot diagram (we also require their flips through horizontal axis).

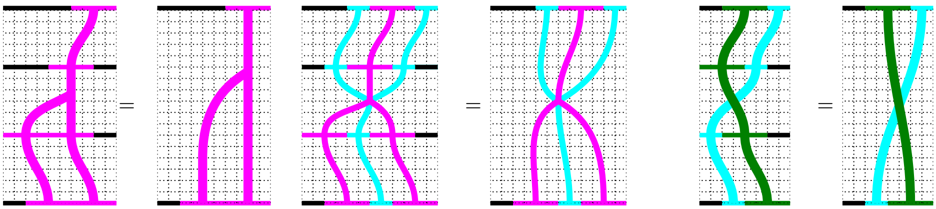


Fig. 7. The remaining naturality relations (we also require their flips through horizontal axis).

$$1_{\alpha} \quad 1_{\emptyset} \quad 1^{\alpha\emptyset}_{\emptyset\alpha} \quad \mathsf{SPOT}^{\emptyset}_{\alpha} \quad \mathsf{FORK}^{\emptyset\alpha}_{\alpha\alpha} \quad \mathsf{HEX}^{\beta\alpha\beta}_{\alpha\beta\alpha} \quad \text{and} \quad \mathsf{COMM}^{\gamma\beta}_{\beta\gamma}.$$

These diagrams are known as "single strand", "blank space", "single adjustment", "spot", "fork", "hexagon" (in order to distinguish from the symmetric group braid) and "commutator".

Definition 3.15. We define the breadth-enhanced diagrammatic Bott–Samelson endomorphism algebra, $\mathscr{S}^{\rm br}(n,\sigma)$ (respectively, its cyclotomic quotient $\mathscr{S}^{\rm br}_{\underline{h}}(n,\sigma)$) to be the span of all $(\underline{w},\underline{w}')$-breadth enhanced Soergel diagrams for $\underline{w},\underline{w}' \in \Lambda(n,\sigma)$, with multiplication given by vertical concatenation, subject to the breadth-enhanced analogues of the relations (S1) to (S8) (plus the additional cyclotomic relation (S9), respectively) which are explicitly pictured in Section 6, the adjustment inversion and naturality relations pictured in Figs. 6 and 7 and their flips through the horizontal axis.

We are free to use the breadth-enhanced form of the diagrammatic Bott–Samelson endomorphism algebra instead of the usual one because of the following result. We let $\phi : \cup_{0\leqslant m\leqslant n}\Lambda^+(m,\sigma) \hookrightarrow \Lambda(n,\sigma)$ denote the map which takes $\underline{w} \in \Lambda^+(m,\sigma)$ to $(s_{\emptyset})^{n-m}\underline{w} \in \Lambda(n,\sigma)$. We refer to the image, $\mathrm{im}(\phi) = \Lambda^+(\leqslant n,\sigma)$, as the subset of left-adjusted words in $\Lambda(n,\sigma)$ and we define an associated idempotent

$$1^+_{n,\sigma} = \sum_{\underline{w}\in\Lambda^+(\leqslant n,\sigma)} 1_{\underline{w}}.$$

Proposition 3.16. *We have the following isomorphisms of graded* $\Bbbk$*-algebras,*

$$\mathscr{S}(n,\sigma) \cong 1^+_{n,\sigma}\mathscr{S}^{\rm br}(n,\sigma)1^+_{n,\sigma} \qquad \mathscr{S}_{\underline{h}}(n,\sigma) \cong 1^+_{n,\sigma}\mathscr{S}^{\rm br}_{\underline{h}}(n,\sigma)1^+_{n,\sigma}.$$

Proof. This is the one point in the paper in which we require the notions from Appendix A and all references within this proof are to the appendix. Thus for this proof only, we briefly switch perspectives and think of the algebras above as *categories* $\mathscr{S}$ and $\mathscr{S}^{\rm br}$ and use the notation in Appendix A. The category $\mathscr{S}$ (resp. $\mathscr{S}^{\rm br}$) has objects given by expression in the alphabet S (resp. $S \cup \{1\}$) and Hom-spaces given by $1_{\underline{w}}\mathscr{S}(n,\sigma)1_{\underline{w}'}$ (resp. $1_{\underline{w}}\mathscr{S}(n,\sigma)1_{\underline{w}'}$) for some sufficiently large n (resp. for some n).

We will establish the first isomorphism; the second isomorphism is similar. Let b : ${\rm Ob}(\mathscr{S}) \to \mathbb{Z}_{\geqslant 0}$ be a monoidal homomorphism given by $b(s_{\boldsymbol{\alpha}}) = b_{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in \Pi$. We now apply Theorem A.3 to show that $\mathscr{S}^{\rm br}(n,\sigma)$ is isomorphic to the weak grading of $\mathscr{S}(n,\sigma)$ concentrated in breadth b. Most of the hypotheses of this result follow by design. For example, since $\mathscr{S}$ is already defined by generators and relations, it's enough to add breadth-enhanced versions of the relations to ensure the composition and tensor product axioms in the theorem. Also, adjustments on objects are defined monoidally, so the weak grading axioms (WG2) and (WG3) automatically hold. Finally (WG1) follows from the adjustment inversion and naturality relations above. $\square$

3.3. The quiver Hecke algebra

We now introduce the second star of the paper, the quiver Hecke or KLR algebras. Given $\underline{i} = (i_1, \dots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n$ and $s_r = (r, r+1) \in \mathfrak{S}_n$ we set $s_r(\underline{i}) = (i_1, \dots, i_{r-1}, i_{r+1}, i_r, i_{r+2}, \dots, i_n)$.

Definition 3.17 *([8,15,27]).* Fix $e > 2$. The quiver Hecke algebra (or KLR algebra), $\mathcal{H}_n$, is defined to be the unital, associative $\mathbb{Z}$-algebra with generators

$$\{e_{\underline{i}} \mid \underline{i} = (i_1, \dots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\},$$

subject to the following relations, for all $r, s, \underline{i}, \underline{j}$ we have that

$$\sum e_{\underline{i}} = 1_{\mathcal{H}_n}; \quad e_{\underline{i}}e_{\underline{j}} = \delta_{\underline{i},\underline{j}}e_{\underline{i}} \quad y_r e_{\underline{i}} = e_{\underline{i}}y_r \quad \psi_r e_{\underline{i}} = e_{s_r\underline{i}}\psi_r \quad y_r y_s = y_s y_r \tag{R1}$$

where the sum is over all $\underline{i} \in (\mathbb{Z}/e\mathbb{Z})^n$ and

$$\psi_r y_s = y_s \psi_r \ \text{ for } s \neq r, r+1 \qquad \psi_r\psi_s = \psi_s\psi_r \ \text{ for } |r-s| > 1 \tag{R2}$$

$$y_r\psi_r e_{\underline{i}} = (\psi_r y_{r+1} - \delta_{i_r,i_{r+1}})e_{\underline{i}} \qquad y_{r+1}\psi_r e_{\underline{i}} = (\psi_r y_r + \delta_{i_r,i_{r+1}})e_{\underline{i}} \tag{R3}$$

$$\psi_r\psi_r e_{\underline{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ e_{\underline{i}} & \text{if } i_{r+1} \neq i_r, i_r \pm 1, \\ (y_{r+1} - y_r)e_{\underline{i}} & \text{if } i_{r+1} = i_r + 1, \\ (y_r - y_{r+1})e_{\underline{i}} & \text{if } i_{r+1} = i_r - 1 \end{cases} \tag{R4}$$

$$\psi_r\psi_{r+1}\psi_r = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} - 1)e_{\underline{i}} & \text{if } i_r = i_{r+2} = i_{r+1} + 1, \\ (\psi_{r+1}\psi_r\psi_{r+1} + 1)e_{\underline{i}} & \text{if } i_r = i_{r+2} = i_{r+1} - 1 \\ \psi_{r+1}\psi_r\psi_{r+1}e_{\underline{i}} & \text{otherwise} \end{cases} \tag{R5}$$

for all permitted r, s, i, j. We identify such elements with decorated permutations and the multiplication with vertical concatenation, $\circ$, of these diagrams in the standard fashion of [8, Section 1]. We let $*$ denote the anti-involution which fixes the generators (this can be visualised as a flip through the horizontal axis of the diagram).

We identify an undecorated single strand with the sum over all possible residues on that strand, as in $\sum_{\underline{i}\in(\mathbb{Z}/e\mathbb{Z})^n} e_{\underline{i}} = 1_{\mathcal{H}_1}$. We freely identify an element $d \in \mathcal{H}_n$ with an element of $\mathcal{H}_{n+1}$ by adding such an undecorated vertical strand to the right; we extend this to all $\mathcal{H}_m$ with $m > n$. The y_k elements are visualised as dots on strands; we hence refer to them as KLR dots. Given $\mathsf{T} \in \mathrm{Std}(\lambda)$, we set $e_\mathsf{T} := e_{\mathrm{res}(\mathsf{T})} \in \mathcal{H}_n$. Using the notation of Subsection 2.1, we define

$$y_\mathsf{T} = \prod_{k=1}^{n} y_k^{|\mathcal{A}_\mathsf{T}(k)|} e_\mathsf{T}, \tag{3.3}$$

such elements should be familiar to those working in KLR algebras, see for example [14, Section 4.3]. Given $p < q$ we set

$$w^p_q = s_p s_{p+1} \dots s_{q-1} \qquad w^q_p = s_{q-1} \dots s_{p+1} s_p$$
$$\psi^p_q = \psi_p\psi_{p+1}\dots\psi_{q-1} \qquad \psi^q_p = \psi_{q-1}\dots\psi_{p+1}\psi_p,$$

and given an expression $\underline{w} = s_{i_1} \dots s_{i_p} \in \mathfrak{S}_n$ we set $\psi_{\underline{w}} = \psi_{i_1} \dots \psi_{i_p} \in \mathcal{H}_n$.

Definition 3.18. Fix $e > 2$ and $\sigma \in \mathbb{Z}^\ell$. The cyclotomic quiver Hecke algebra, $\mathcal{H}^\sigma_n$, is defined to be the quotient of $\mathcal{H}_n$ by the relation

$$y_1^{\sharp\{\sigma_m | \sigma_m = i_1, 1 \leqslant m \leqslant \ell\}} e_{\underline{i}} = 0 \quad \text{ for } \underline{i} \in (\mathbb{Z}/e\mathbb{Z})^n. \tag{3.4}$$

Definition 3.19. We define the degree on the generators as follows,

$$\deg(e_{\underline{i}}) = 0 \quad \deg(y_r) = 2 \quad \deg(\psi_r e_{\underline{i}}) = \begin{cases} -2 & \text{if } i_r = i_{r+1} \\ 1 & \text{if } i_r = i_{r+1} \pm 1 \\ 0 & \text{otherwise} \end{cases}.$$

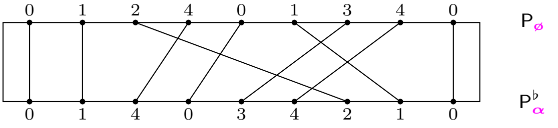


Fig. 8. The element $\psi^{\mathsf{P}_{\varnothing}}_{\mathsf{P}^{\flat}_{\alpha}}$ for $\Bbbk\mathfrak{S}_9$ in the case $p = 5$ and $\alpha = \varepsilon_3 - \varepsilon_1 \in \Pi$ (see also Example 2.29).

Definition 3.20. Given a pair of paths $\mathsf{S}, \mathsf{T} \in \mathrm{Path}(\lambda)$, and a fixed choice of reduced expression for $w^{\mathsf{S}}_{\mathsf{T}} = s_{i_1} s_{i_2} \dots s_{i_k}$ we define $\psi^{\mathsf{S}}_{\mathsf{T}} = e_{\mathsf{S}} \psi_{i_1} \psi_{i_2} \dots \psi_{i_k} e_{\mathsf{T}}$.

Remark 3.21. The quiver Hecke algebra and its cyclotomic quotients are isomorphic (over a field) to the classical affine Hecke algebra and its cyclotomic quotients (at a root of unity) by [8, Main Theorem]. Setting $e = p$ and $\sigma = (0) \in \mathbb{Z}^1$ we have that $\Bbbk\mathfrak{S}_n$ is isomorphic to $\mathcal{H}^{\sigma}_n$ and we freely identify these algebras without further mention. (See Fig. 8.)

3.3.1. Our quotient algebra and regular blocks

A long-standing belief in modular Lie theory is that we should (first) restrict our attention to fields whose characteristic, p, is greater than the Coxeter number, h, of the algebraic group we are studying. This allows one to consider a "regular" or "principal block" of the algebraic group in question. For example, the diagrammatic Bott–Samelson endomorphism algebras categorify the endomorphisms between tilting modules for the *principal* block of the algebraic group, $\mathrm{GL}_h(\Bbbk)$, and this is the crux of the proof of [26, Theorem 1.9]. Extending this "Soergel diagram calculus" to singular blocks is a difficult problem. As such, all results in [26,1] and this paper are restricted to *regular* blocks. In the language of [26,1] this restricts the study of the algebraic group in question to primes $p > h$.

What does this mean on the other side of the Schur–Weyl duality relating $\mathrm{GL}_h(\Bbbk)$ and $\Bbbk\mathfrak{S}_n$? By the second fundamental theorem of invariant theory, the kernel of the group algebra of the symmetric group acting on n-fold h-dimensional tensor space is the element $\sum_{g \in \mathfrak{S}_{h+1} \leqslant \mathfrak{S}_n} \mathrm{sgn}(g) g \in \Bbbk\mathfrak{S}_n$. Modulo "more dominant terms" this element is equal to $\mathsf{y}_{\mathsf{T}^{(h+1)}}$ (the element introduced in equation (3.3)). The module category of $\Bbbk\mathfrak{S}_n / \Bbbk\mathfrak{S}_n \mathsf{y}_{\mathsf{T}^{(h+1)}} \Bbbk\mathfrak{S}_n$ is the Serre subcategory of $\Bbbk\mathfrak{S}_n$-mod whose simple modules are indexed by partitions with at most h columns. For $p > h$, the algebra $\Bbbk\mathfrak{S}_n / \Bbbk\mathfrak{S}_n \mathsf{y}_{\mathsf{T}^{(h+1)}} \Bbbk\mathfrak{S}_n$ is the largest quotient of $\Bbbk\mathfrak{S}_n$ controlled by the diagrammatic Bott–Samelson endomorphism algebra with h distinct colours. Combinatorially, the condition that $p > h$ ensures that $\varnothing$ does not lie on any hyperplane in the alcove geometry (and so the p-Kazhdan–Lusztig theory is "regular" not "singular"). The importance of this Serre subcategory and the condition $p > h$ can also be explained in the context of calibrated/unitary modules [6, Introduction]. The main theorem of [26] calculates decomposition numbers of $\Bbbk\mathfrak{S}_n / \Bbbk\mathfrak{S}_n \mathsf{y}_{\mathsf{T}^{(h+1)}} \Bbbk\mathfrak{S}_n$.

There is a canonical manner in which the above situation generalises to cyclotomic Hecke algebras. For a given $e > h$, one can ask *"what is the largest quotient of $\mathcal{H}_n^\sigma$ controlled by the diagrammatic Bott–Samelson endomorphism algebra with h distinct colours?"* Assuming that $\underline{h} \in \mathbb{Z}_{>0}^\ell$ is (σ, e)-admissible, we define

$$\mathsf{y}_{\underline{h}} = \sum_{\substack{\alpha=(\varnothing,\dots,\varnothing,(h_a+1),\varnothing,\dots,\varnothing)\\ 0\leqslant a<\ell}} y_{\mathsf{T}^\alpha}$$

and we claim that the answer to the question is provided by the quotient algebras $\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{y}_{\underline{h}} \mathcal{H}_n^\sigma$ for (σ, e)-admissible $\underline{h} \in \mathbb{Z}_{\geqslant 0}^\ell$. Our claim is justified as follows: for $e > h$ the condition that $\underline{h} \in \mathbb{Z}_{>0}^\ell$ is (σ, e)-admissible is equivalent to requiring that $\varnothing$ does not lie on any hyperplane in the alcove geometry (so that our p-Kazhdan–Lusztig theory is "regular" not "singular" as required). We further remark that the importance of the Serre subquotient with regards to calibrated/unitary modules goes through verbatim to our setting, see [6, Introduction].

Example 3.22. Let $e = 3$ and $h = 3 \in \mathbb{Z}$ (and let $\sigma = (0) \in \mathbb{Z}$). We have that $\mathsf{y}_{\underline{h}} = y_{\mathsf{T}^{(3)}} = y_4 e(0, 1, 2, 3)$.

Example 3.23. Continuing with Example 2.3, we let $\sigma = (0, 3, 8) \in \mathbb{Z}^3$ and $e = 13$. We have that $\mathsf{y}_{\underline{h}} = y_4 e(0, 1, 2, 3) + y_6 e(3, 4, 5, 6, 7, 8) + e(8, 9, 10, 11, 12)$. The reader should compare these residue sequences with the residues appearing in the first row of the tableau in Example 2.3.

Remark 3.24. The tableaux T^α for $0 \leqslant a < \ell$ all have different residue sequences, in particular the corresponding e_{T^α} are pairwise orthogonal idempotents. For $h_a < \sigma_{a+1} - \sigma_a$ and $0 \leqslant a \leqslant \ell - 2$, we have that $y_{\mathsf{T}^\alpha} = e_{\mathsf{T}^\alpha}$. Similarly, for $a = \ell - 1$ and $h_a < e + \sigma_0 - \sigma_{a-1} - 1$, we have that $y_{\mathsf{T}^\alpha} = e_{\mathsf{T}^\alpha}$. If we replace either of the strict inequalities above with an equality, then we obtain $y_{\mathsf{T}^\alpha} = y_{h_a+1} e_{\mathsf{T}^\alpha}$. Thus the element $\mathsf{y}_{\underline{h}}$ need not be homogenous, however each element y_{T^α} is homogeneous in the grading (of degree 0 or 1). We have that the ideal generated by $\mathsf{y}_{\underline{h}}$ is the same as the ideal generated by the set of homogeneous elements $\{y_{\mathsf{T}^\alpha} \mid 0 \leqslant a < \ell\}$ and therefore the quotient is a graded algebra.

Remark 3.25. In [14, 4.1 Lemma] it is proven that relation (3.4) is equivalent to $e_{\underline{i}} = 0$ for any $\underline{i} \neq \operatorname{res}(\mathsf{S})$ for some $\mathsf{S} \in \operatorname{Std}(\lambda)$ with $\lambda \in \mathscr{P}_\ell(n)$. In $\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{y}_{\underline{h}} \mathcal{H}_n^\sigma$ we have that $e_{\underline{i}} = 0$ for any $\underline{i} \neq \operatorname{res}(\mathsf{S})$ for some $\mathsf{S} \in \operatorname{Std}(\lambda)$ with $\lambda \in \mathscr{P}_{\underline{h}}(n)$. For more details, see [4, Theorem 1.19(a)].

3.3.2. The Bott–Samelson truncation

In the previous section, we defined the Bott–Samelson endomorphism algebra and its breadth-enhanced counterpart. The idempotents in the former (respectively latter)

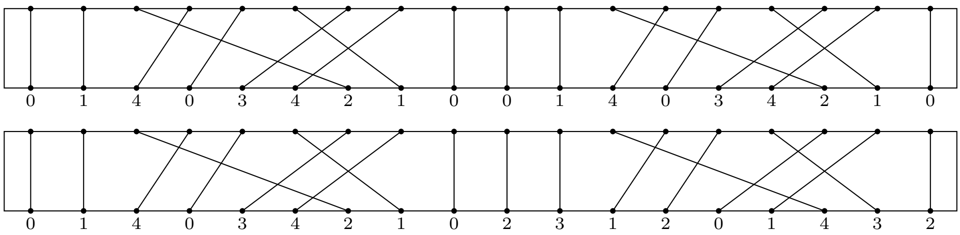


Fig. 9. Continuing Example 2.29, we depict $\psi^{\mathsf{P}_{\varnothing}}_{\mathsf{P}^{\flat}_{\alpha}} \boxtimes \psi^{\mathsf{P}_{\varnothing}}_{\mathsf{P}^{\flat}_{\alpha}}$ and $\psi^{\mathsf{P}_{\varnothing}}_{\mathsf{P}^{\flat}_{\alpha}} \otimes \psi^{\mathsf{P}_{\varnothing}}_{\mathsf{P}^{\flat}_{\alpha}}$ respectively.

algebra were indexed by expressions $\underline{w}$ in the simple reflections (respectively, in the simple reflections *and* the empty set). We define

$$\mathsf{f}^{+}_{n,\sigma} = \sum_{\substack{\mathsf{S}\in \mathrm{Std}^{+}_{n,\sigma}(\lambda) \\ \lambda \in \mathscr{P}_{\underline{h}}(n)}} e_{\mathsf{S}} \qquad \mathsf{f}_{n,\sigma} = \sum_{\substack{\mathsf{S}\in \mathrm{Std}_{n,\sigma}(\lambda) \\ \lambda \in \mathscr{P}_{\underline{h}}(n)}} e_{\mathsf{S}}$$

and the bulk of this paper will be dedicated to proving that

$$\mathsf{f}^{+}_{n,\sigma}(\mathcal{H}^{\sigma}_{n}/\mathcal{H}^{\sigma}_{n}\mathsf{y}_{\underline{h}}\mathcal{H}^{\sigma}_{n})\mathsf{f}^{+}_{n,\sigma} \quad \text{and} \quad \mathsf{f}_{n,\sigma}(\mathcal{H}^{\sigma}_{n}/\mathcal{H}^{\sigma}_{n}\mathsf{y}_{\underline{h}}\mathcal{H}^{\sigma}_{n})\mathsf{f}_{n,\sigma}$$

are isomorphic to the cyclotomic Bott–Samelson endomorphism algebra and its breadth-enhanced counterpart, respectively. For the most part, we work in the breadth-enhanced Bott–Samelson endomorphism algebra where the isomorphism is more natural (and we then finally truncate at the end of the paper to deduce our Theorem A).

3.3.3. Concatenation

We now discuss horizontal concatenation of diagrams in (our truncation of) the quiver Hecke algebra. First we let $\boxtimes$ denote the "naive concatenation" of KLR diagrams side-by-side as illustrated in Fig. 9. Now, given two quiver Hecke diagrams $\psi^{\mathsf{P}}_{\mathsf{Q}}$ and $\psi^{\mathsf{P}'}_{\mathsf{Q}'}$ we define

$$\psi^{\mathsf{P}}_{\mathsf{Q}} \otimes \psi^{\mathsf{P}'}_{\mathsf{Q}'} = e_{\mathsf{P}'\otimes\mathsf{Q}'} \circ \psi^{\mathsf{P}\otimes\mathsf{Q}}_{\mathsf{P}'\otimes\mathsf{Q}'} \circ e_{\mathsf{P}'\otimes\mathsf{Q}'}.$$

We refer to this as the contextualised concatenation of diagrams (as the residue sequences appearing along the bottom of the diagram are not obtained by simple concatenation, but rather from considering the residue sequence of the concatenated path).

4. Translation and dilation

In this section we prove some technical results for KLR elements which will appear repeatedly in what follows. The reader should feel free to skip this section on first reading. We continue with the notation of Convention 2.32.

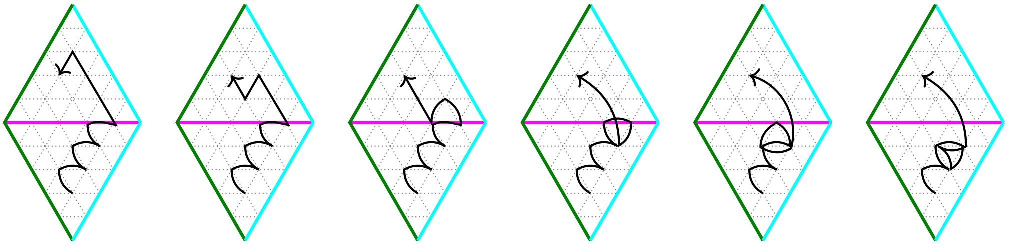


Fig. 10. A series of paths P, Q, R, S, T and U. The paths P, Q, U are α-crossing paths.

4.1. The translation principle for paths

Our first result of this section says that our choice of distinguished path $\mathsf{P}_{\underline{w}}$ in Definition 2.18 for $\underline{w} = \alpha_1\alpha_2\dots\alpha_p$ was entirely arbitrary (the only thing that matters is that the path crosses the hyperplanes $\alpha_1, \alpha_2, \dots \alpha_p$ in sequence).

Lemma 4.1. *Let* P *denote any path which terminates at a regular point and let* $r \in \mathbb{Z}/e\mathbb{Z}$. *Then*

$$e_{\mathsf{P}} \boxtimes e_{r,r} = 0.$$

Proof. The result follows from Remark 3.25 in light of the proof of Lemma 2.8. □

For $\boldsymbol{\alpha} \in \Pi$, we say that a path P of length n is an $\boldsymbol{\alpha}$-crossing path if (i) there exists $1 < p_1 \leqslant p_2 < n$ such that $\mathsf{P}(k) \in \mathbb{E}(\boldsymbol{\alpha})$ if and only if $k \in [p_1, p_2]$ and (ii) $\mathsf{P}(k) \notin \mathbb{E}(\boldsymbol{\beta}, se) \neq \mathbb{E}(\boldsymbol{\alpha})$ for any $1 \leqslant k \leqslant n$. We say that P is an $\emptyset$-crossing path if $\mathsf{P}(k)$ is a regular point for all $1 \leqslant k \leqslant n$. We say a path is $\boldsymbol{\alpha}$-bouncing if it is obtained from an $\boldsymbol{\alpha}$-crossing path by reflection through the $\boldsymbol{\alpha}$-hyperplane.

Example 4.2. Let $e = 5$, $\ell = 1$, $h = 3$, and $\boldsymbol{\alpha} = \varepsilon_3 - \varepsilon_1$. For the paths in Fig. 10, we have that $\mathrm{res}(\mathsf{P}) = (0,1,4,0,3,4,2,1,0,2)$, $\mathrm{res}(\mathsf{Q}) = (0,1,4,0,3,4,2,1,2,0)$, $\mathrm{res}(\mathsf{R}) = (0,1,4,0,3,4,2,2,1,0)$, $\mathrm{res}(\mathsf{S}) = (0,1,4,0,3,4,2,2,1,0)$, $\mathrm{res}(\mathsf{T}) = (0,1,4,0,3,2,4,2,1,0)$, and $\mathrm{res}(\mathsf{U}) = (0,1,4,0,2,3,4,2,1,0)$ and we have that

$$\mathrm{res}_{\mathsf{P}}(\mathsf{P}^{-1}(1,\varepsilon_3)) = 2 \quad \mathrm{res}_{\mathsf{P}}(\mathsf{P}^{-1}(3,\varepsilon_1)) = 3 \quad \mathrm{res}_{\mathsf{P}}(\mathsf{P}^{-1}(4,\varepsilon_1)) = 2 \quad \mathrm{res}_{\mathsf{P}}(\mathsf{P}^{-1}(5,\varepsilon_1)) = 1.$$

It is not difficult to see that the elements $\psi^{\mathsf{P}}_{\mathsf{Q}}$, $\psi^{\mathsf{P}}_{\mathsf{R}}$, $\psi^{\mathsf{P}}_{\mathsf{S}}$, $\psi^{\mathsf{P}}_{\mathsf{T}}$, and $\psi^{\mathsf{P}}_{\mathsf{U}}$ have 0, 1, 2, 3, 3, crossings of non-zero degree respectively. We will see that $\mathsf{e}_{\mathsf{P}} = \psi^{\mathsf{P}}_{\mathsf{Q}}\psi^{\mathsf{Q}}_{\mathsf{P}} = \psi^{\mathsf{P}}_{\mathsf{T}}\psi^{\mathsf{T}}_{\mathsf{P}} = \psi^{\mathsf{P}}_{\mathsf{U}}\psi^{\mathsf{U}}_{\mathsf{P}}$.

Remark 4.3. Given P and U two ($\boldsymbol{\alpha}$-crossing) paths, we can pass between them inductively, this lifts to a factorisation of $w^{\mathsf{P}}_{\mathsf{U}}$ as a product of Coxeter generators. An example is given by the sequence of paths P, Q, R, S, T and U in Fig. 10 (for example $w^{\mathsf{S}}_{\mathsf{T}} = (6,7)$).

The degree of each of these crossings is determined by whether we are stepping onto or off-of a wall. For example, the elements $\psi^{\mathsf{R}}_{\mathsf{Q}} = e_{\mathsf{R}}\psi_8 e_{\mathsf{Q}}$, $\psi^{\mathsf{S}}_{\mathsf{R}} = e_{\mathsf{S}}\psi_7 e_{\mathsf{R}}$, and $\psi^{\mathsf{T}}_{\mathsf{S}} = e_{\mathsf{T}}\psi_6 e_{\mathsf{S}}$ have degrees $1, -2$, and 1 respectively.

Proposition 4.4. *Fix $\alpha \in \Pi \cup \{\emptyset\}$. Let P, Q be a pair of α-crossing/bouncing paths of length n from $\varnothing \in A_0$ to $\lambda \in s_\alpha A_0$. We have that*

$$\psi^{\mathsf{P}}_{\mathsf{Q}}\psi^{\mathsf{Q}}_{\mathsf{P}} = e_{\mathsf{P}} \quad \textit{and} \quad \psi^{\mathsf{Q}}_{\mathsf{P}}\psi^{\mathsf{P}}_{\mathsf{Q}} = e_{\mathsf{Q}}. \tag{4.1}$$

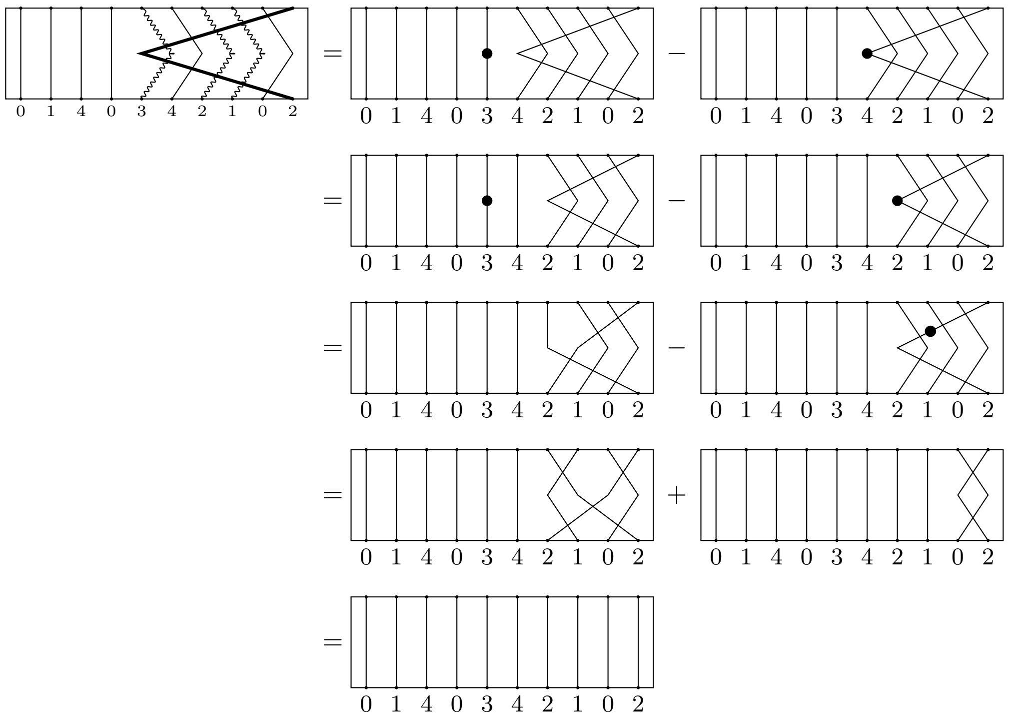
Proof. The $\alpha = \emptyset$ case is trivial, and so we set $\alpha = \varepsilon_i - \varepsilon_{i+1}$. We fix $\mathsf{P} = (\varepsilon_{j_1}, \dots, \varepsilon_{j_n})$ and $\mathsf{Q} = (\varepsilon_{k_1}, \dots, \varepsilon_{k_n})$. Recall that $w^{\mathsf{P}}_{\mathsf{Q}}$ is minimal and step-preserving and that the paths P and Q only cross the hyperplane $\alpha \in \Pi$. This implies, for any pair of strands from $1 \leqslant x < y \leqslant n$ to $1 \leqslant w^{\mathsf{P}}_{\mathsf{Q}}(y) < w^{\mathsf{P}}_{\mathsf{Q}}(x) \leqslant n$ whose crossing has *non-zero degree*, that $\varepsilon_{j_x} = \varepsilon_{i+1}$ and $\varepsilon_{j_y} = \varepsilon_i$ and $\mathsf{P}(y) \in s_\alpha A_0$ and $\mathsf{Q}(w^{\mathsf{P}}_{\mathsf{Q}}(y)) \in A_0$ (one can swap P and Q and hence reorder so that $1 \leqslant y < x \leqslant n$). We let $1 \leqslant y \leqslant n$ be minimal such that $\mathsf{P}(y) \in s_\alpha A_0$ and we suppose that $\mathrm{res}_{\mathsf{P}}(y) = r \in \mathbb{Z}/e\mathbb{Z}$. We let Y denote this r-strand from y to $w^{\mathsf{P}}_{\mathsf{Q}}(y)$.

We recall our assumption that P and Q cross the α-hyperplane precisely once. This implies that there exists a unique $1 \leqslant z \leqslant n$ such that $\mathsf{P}^{-1}(z, \varepsilon_{i+1}) \in [p_1, p_2]$. We have that $\mathsf{res}_{\mathsf{P}}(\mathsf{P}^{-1}(z, \varepsilon_{i+1})) = r + 1$, $\mathsf{res}_{\mathsf{P}}(\mathsf{P}^{-1}(z+1, \varepsilon_{i+1})) = r$, and $\mathsf{res}_{\mathsf{P}}(\mathsf{P}^{-1}(z+2, \varepsilon_{i+1})) = r - 1$. The Y strand crosses each of the strands connecting the points $\mathsf{P}^{-1}(z, \varepsilon_{i+1})$, $\mathsf{P}^{-1}(z+1, \varepsilon_{i+1})$, and $\mathsf{P}^{-1}(z+2, \varepsilon_{i+1})$ to the points $\mathsf{Q}^{-1}(z, \varepsilon_{i+1})$, $\mathsf{Q}^{-1}(z+1, \varepsilon_{i+1})$, and $\mathsf{Q}^{-1}(z+2, \varepsilon_{i+1})$ and these are all the crossings involving the Y-strand which are of non-zero degree. We refer to these strands as Z_{+1}, Z_0, Z_{-1}.

We are ready to consider the product $\psi^{\mathsf{P}}_{\mathsf{Q}}\psi^{\mathsf{Q}}_{\mathsf{P}}$. We use case 4 of relation (R4) to resolve the double-crossing of the Y and Z_{+1} strands, which yields two terms with KLR-dots on these strands. The term with a KLR-dot on the Z_{+1} strand vanishes after applying case 1 of (R4) to the like-labelled double-crossing r-strands Y and Z_0. The remaining term has a KLR-dot on the Y strand. We next use (R3) to pull this KLR-dot through one of the like-labelled crossings of Y and Z_0. Again we obtain the difference of two terms, one of which vanishes by applying case 1 of (R4). This remaining term has the r-strands Y and Z_0 crossing only once. We then pull the Z_{-1}-strand through this crossing using the second case of relation (R5), to obtain another sum of two terms. The term with more crossings is zero by Lemma 4.1, while the remaining term has no non-trivial double-crossings involving the Y strand. As the Y strand was chosen to be minimal, we now repeat the above argument with the next such strand; we proceed until all double-crossings of non-zero degree have been undone. □

Remark 4.5. More generally, given P and Q two α- and β-crossing/bouncing paths, we can apply Proposition 4.4 to any local regions $\mathsf{S} \otimes \mathsf{P} \otimes \mathsf{T}$ and $\mathsf{S} \otimes \mathsf{Q} \otimes \mathsf{T}$ of a wider pair of paths. The proof again follows simply by applying the same sequence of relations as in the proof of Proposition 4.4. Indeed, P and Q can be said to be "translation-equivalent" if the non-zero double-crossings in $\psi^{\mathsf{P}}_{\mathsf{Q}}\psi^{\mathsf{Q}}_{\mathsf{P}}$ are precisely those detailed in the proof of Proposition 4.4 (and so are in bijection with the crossings of non-zero degree in Example 4.6).

Example 4.6. We now go through the steps of the above proof for the product $\psi_{\mathsf{U}}^{\mathsf{P}}\psi_{\mathsf{P}}^{\mathsf{U}} = e_{(0,1,4,0,3,4,2,1,0,2)}$ from Example 4.2.



The first and second equalities hold by case 4 and case 3 of relation (R4). The first term in the second line and the second term in the third line are both zero by case 1 of relation (R4). Thus the third equality follows by relation (R3) and the fourth equality follows from case 1 or relation (R5). The first term in the fourth line is zero by Lemma 4.1 (the partition (2^3) does not have an addable node of residue 1). The second term in the fourth line is equal the term in the fifth line by case 2 of relation (R4).

4.2. Good and bad braids

Given $w \in \mathfrak{S}_n$, we define a w-braid to be any triple $1 \leqslant p < q < r \leqslant n$ such that $w(p) > w(q) > w(r)$. We recall that an element $w \in \mathfrak{S}_n$ is said to be fully-commutative if there do not exist any w-braid triples. We define a bad w-braid to be a triple $1 \leqslant p < q < r \leqslant n$ with $i_p = i_r = i_q \pm 1$ such that $w(p) > w(q) > w(r)$. We say that a w-braid which is not bad is good. We say that w is residue-commutative if there do not exist any bad-braid triples.

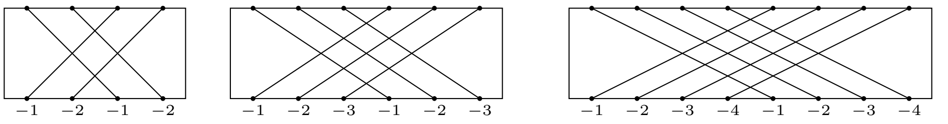


Fig. 11. The 2- 3- and 4- dilated elements $e_B\psi_{(1,2)_b}e_B$ for $b = 2, 3, 4$.

Lemma 4.7. *Suppose that w is residue-commutative and let $\underline{w}$ be a reduced expression for w. Then $\psi_{\underline{w}}$ is independent of the choice of reduced expression and we denote this element simply by ψ_w.*

Proof. If w is fully-commutative then any two reduced expressions differ only by the commuting Coxeter relations see [2, Theorem 2.1] (in particular, one need not use the braid relation). Thus the claim follows by the second equality of (R2). An identical argument shows that if w is residue-commutative, then any two reduced expressions differ only by the commuting Coxeter relations and good braid relations. The condition for a braid to be good is precisely the commuting case of relation (R5). Thus the claim follows by relation (R2) and (R5). □

4.3. Breadth dilation of permutations

We will see later on in the paper that the commutator and hexagonal generators of equation (3.2) roughly correspond to “dilated” copies of transpositions and braids in the KLR algebra. Similarly, the tetrahedron relation roughly corresponds to the equality between two expressions for a “dilated” copy of $(1, 4)(2, 3)$. In this section, we provide the necessary background results which will allow us to make these ideas more precise in Sections 5 and 6. Given $b > 1$, we define the b-dilated transpositions to be the elements

$$(i, i+1)_b = s_{bi}(s_{bi-1}s_{bi+1})\dots(s_{bi-b+1}s_{bi-b+3}\dots s_{bi-b-3}s_{bi+b-1})\dots(s_{bi-1}s_{bi+1})s_{bi}$$

for $1 \leqslant i < n$. (The examples in Fig. 11 should make this definition clear.) Now, we note that $\mathfrak{S}_n \cong \langle (i, i+1)_b \mid 1 \leqslant i < n\rangle \leqslant \mathfrak{S}_{bn}$. We remark that $(i, i+1)_b$ is fully commutative. Given any permutation $w \in \mathfrak{S}_n$ and $\underline{w}$ an expression for $w \in \mathfrak{S}_n$, we let $\underline{w}_b$ denote the corresponding expression in the generators $(i, i+1)_b$ of this b-dilated copy of $\mathfrak{S}_n \leqslant \mathfrak{S}_{bn}$. We set $B = (-1, -2, \dots, -b)^n \in (\mathbb{Z}/e\mathbb{Z})^{bn}$ and we let $\psi_{\underline{w}_b}e_B$ denote the corresponding element in $\langle e_B\psi_{(i,i+1)_b}e_B \mid 1 \leqslant i < n\rangle \subseteq \mathcal{H}_n^\sigma$.

We fix $\underline{w}$ a reduced word for $w \in \mathfrak{S}_n$. We say that $D \in \mathcal{H}^\sigma_{bn}$ is a quasi-b-dilated expression for $\underline{w}$ if for each $1 \leqslant r < b$, the subexpression consisting solely of the $-r$-strands and $-(r+1)$-strands from D forms the 2-dilated element $\psi_{\underline{w}_2}e_{(-r,-r-1)^n}$. It is easy to see that a quasi-b-dilated element for $\underline{w}$ differs from $\psi_{\underline{w}_b}$ simply by undoing some crossings of *degree zero*. In particular, all quasi-b-dilated expressions for $\underline{w}$ (including

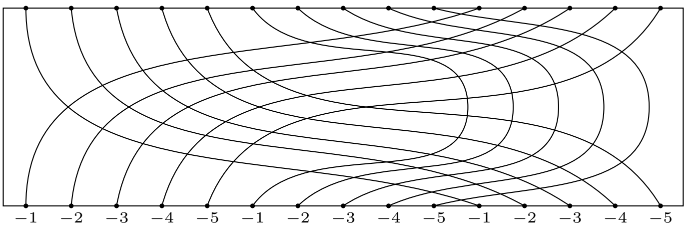


Fig. 12. The 5-dilated element $e_B\psi_{(2,3)_5(1,2)_5(2,3)_5}e_B$ for $B = 5$.

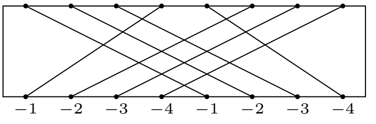


Fig. 13. A quasi-4-dilated expression for $(1, 2)$. This diagram is obtained from the final diagram of Fig. 11 by undoing a degree zero crossing.

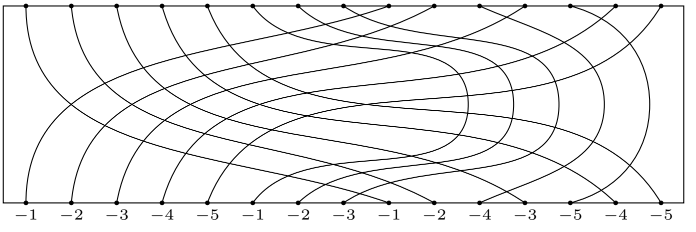


Fig. 14. A quasi-5-expression element for $\underline{w} = (23)(12)(23)$. Conjugating this diagram by the invertible element $(\psi_{10}\psi_{12}\psi_9\psi_{11}\psi_{10})e_{(-1,-2,-3,-4,-5)^3}$ we obtain the diagram in Fig. 12.

$\psi_{\underline{w}_b}$ itself) have the same bad braids (in the same order, modulo the commutativity relations). (See Fig. 13.)

Finally, we define the nibs of a permutation w to be the nodes 1 and n and $w^{-1}(1)$ and $w^{-1}(n)$ from the top edge and the nodes 1 and n and $w(1)$ and $w(n)$ from the bottom edge. We define the nib-truncation of $\underline{w}$ to be the expression, $\mathrm{nib}(\underline{w})$, obtained by deleting the 4 pairs of nibs of w and then deleting the (four) strands connecting these vertices. Similarly, we define $\mathrm{nib}(\psi_{\underline{w}}e_{\underline{i}}) = \psi_{\mathrm{nib}(\underline{w})}e_{\mathrm{nib}(\underline{i})}$ where the residue sequence $\mathrm{nib}(\underline{i}) \in (\mathbb{Z}/e\mathbb{Z})^{bn-4}$ is inherited by deleting the 1st, nth, $w(1)$th and $w(n)$th entries of $\underline{i} \in (\mathbb{Z}/e\mathbb{Z})^n$. See Figs. 14 and 15 for examples.

4.4. Freedom of expression

We now prove that the quasi-dilated elements and their nib-truncations are independent of the choice of reduced expressions. For $0 \leqslant q \leqslant b$, we define the element $\psi_{[b,q]}$

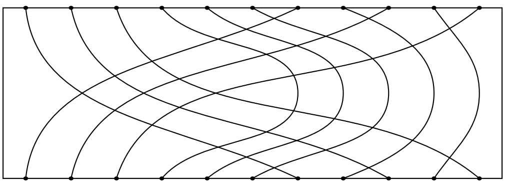


Fig. 15. A diagram obtained by nib-truncation from that in Fig. 14. This diagram is a subdiagram of the hexagonal generator in Fig. 23.

which breaks the strands into two groups (left and right) according to their residues as follows

$$\psi_{[b,q]} = \prod_{0\leqslant p<n} \Big(\prod_{1\leqslant i\leqslant q} \psi^{pb+i}_{pq+i} \Big)$$
$$\text{where} \quad e_B\psi_{[b,q]} \in e_{(-1,\dots,-b)^n}\mathcal{H}^\sigma_n e_{(-1,\dots,-q)^n\boxtimes(-q-1,\dots,-b)^n}.$$

We remark that $\psi_{[b,0]} = \psi_{[b,b]} = 1 \in \mathfrak{S}_{bn}$.

Lemma 4.8. *We have that $e_B\psi_{(1,2)_b}\psi_{(1,2)_b}e_B = 0$ for $b \geqslant 1$.*

Proof. For $b = 1$ the result is immediate by case 1 of relation (R4). Now let $b > 1$. We pull the strand connecting the strand connecting the 1st top and bottom vertices to the right through the strand connecting the $(b+2)$th top and bottom vertices using case 4 of relation (R4) and hence obtain

$$e_B\psi_{[b,b-1]}\Big(\big(\psi_{(1,2)_{b-1}}y_{2b-2}\psi_{(1,2)_{b-1}} \boxtimes \psi_{(1,2)}\psi_{(1,2)}\big)$$
$$-\big(\psi_{(1,2)_{b-1}}\psi_{(1,2)_{b-1}} \boxtimes \psi_{(1,2)}y_1\psi_{(1,2)}\big)\Big)\psi^*_{[b,b-1]}e_B$$

and the first (respectively second) term is zero by the $(b-1)$th (respectively 1st) inductive step. □

Proposition 4.9. *Let $1 \leqslant b < e$. The elements $e_B\psi_{(i,i+2)_b}e_B$ and $\mathrm{nib}(e_B\psi_{(i,i+2)_b}e_B)$ are independent of the choice of reduced expression of $(i, i+2)_b \in \mathfrak{S}_{bn}$.*

Proof. For ease of notation we consider the $i = 1$ case, the general case is identical up to relabelling of strands. We first consider $e_B\psi_{(1,3)_b}e_B$, as the enumeration of strands is easier. We will refer to two reduced expressions in the KLR algebra as *distinct* if they are not trivially equal by the commuting relations (namely, the latter case of (R2), case 2 of relation (R4) and case 3 of relation (R5)). There are precisely $b+1$ distinct expressions, Ω_q, of $e_B\psi_{(1,3)_b}e_B$ as follows

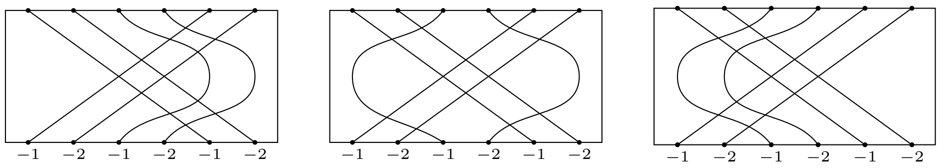


Fig. 16. The 3 distinct expressions, Ω_0, Ω_1, and Ω_2 for $\psi_{(1,3)_2}$. The $b+1$ distinct expressions for $\psi_{(1,3)_b}$ are determined by where the central "fat strand" is broken into "left" and "right" parts.

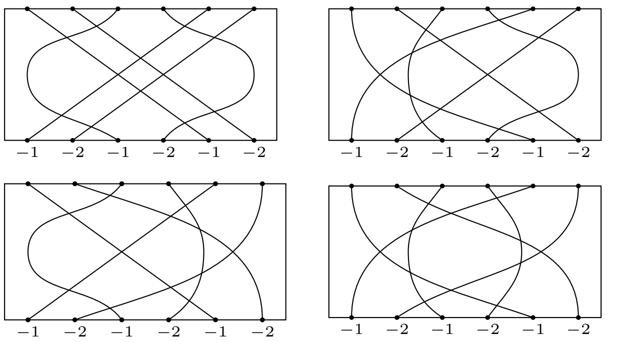


Fig. 17. The 4 equivalent expressions for Ω_1 of Fig. 16. These differ only by applications of case 3 of relation (R5) (and so the bad braids are all the same).

$$\Omega_q = e_B\psi_{[b,q]}\big(\psi_{(12)_q}\psi_{(23)_q}\psi_{(12)_q} \boxtimes \psi_{(23)_{b-q}}\psi_{(12)_{b-q}}\psi_{(23)_{b-q}}\big)\psi^*_{[b,q]}e_B \qquad (4.2)$$

for $0 \leqslant q \leqslant b$. See Figs. 16 and 17 for examples. We remark that $\Omega_0 = e_B\psi_{(23)_b}\psi_{(12)_b}\psi_{(23)_b}e_B$ and $\Omega_b = e_B\psi_{(12)_b}\psi_{(23)_b}\psi_{(12)_b}e_B$. We will show that $\Omega_q = \Omega_{q+1}$ for $1 \leqslant q < b$ and hence deduce the result.

Step 1. If $q = 0$ proceed to Step 2, otherwise we pull the $(-q)$-strand connecting the $(b+q)$th northern and southern nodes of Ω_q to the right. We first use relation (R5) to pull $(-q)$-strand through the crossing of $(1-q)$-strands connecting the $(q-1)$th and $(2b+q-1)$th top and bottom vertices. We obtain two terms: the first is equal to

$$\begin{aligned} &e_B\psi_{[b,q]}\big(\psi_{[q,q-1]}\big(\psi_{(12)_{q-1}}\psi_{(23)_{q-1}}\psi_{(12)_{q-1}} \boxtimes \psi_{(12)(23)(12)}\big)\psi^*_{[q,q-1]} \\ &\quad\boxtimes\psi_{(23)_{b-q}}\psi_{(12)_{b-q}}\psi_{(23)_{b-q}}\big)\psi^*_{[b,q]}e_B \end{aligned} \qquad (4.3)$$

and an error term of strictly smaller length (in which we undo the crossing pair of $(1-q)$-strands). If $q = 1$, the error term contains a double-crossing of $(r-q)$-strands and so is zero by case 1 of relation (R4). If $q > 1$, then we apply relation (R5) to the error term to obtain two distinct terms; one of which is zero by Lemma 4.8 and the other is zero by case 2 or relation (R4) and the commutativity relations.

Step 2. The output from Step 1 has a subexpression $\psi_{(12)(23)(12)}$ which we rewrite as $\psi_{(23)(12)(23)}$ using case 3 of relation (R5) (as the three strands are all of the same residue, $-q \in \mathbb{Z}/e\mathbb{Z}$). We also have that $\psi_{[b,q]}\psi_{[q,q-1]} = \psi_{[b,q-1]}(1_{\mathcal{H}^{\sigma}_{3b-3}} \boxtimes \psi_{[b-q+1,1]})$. Thus (4.3) is equal to

$$\psi_{[b,q]}\big(\psi_{(12)_{q-1}}\psi_{(23)_{q-1}}\psi_{(12)_{q-1}}$$
$$\boxtimes\psi_{[b-q+1,1]}\big(\psi_{(23)(12)(23)} \boxtimes \psi_{(23)_{b-q}}\psi_{(12)_{b-q}}\psi_{(23)_{b-q}}\big)\psi^*_{[b-q+1,1]}\big)\psi^*_{[b,q]}$$

Now, by the mirror argument to that used in Step 1, we have that this equals

$$\psi_{[b,q-1]}\big(\psi_{(12)_{q-1}}\psi_{(23)_{q-1}}\psi_{(12)_{q-1}} \boxtimes \psi_{(23)_{b-q+1}}\psi_{(12)_{b-q+1}}\psi_{(23)_{b-q+1}}\big)\psi^*_{[b,q-1]}$$

as required. The argument for $\mathrm{nib}(e_B\psi_{(1,3)_b}e_B)$ is identical (up to relabelling of strands) except that the $q = 0$ and $q = b$ cases do not appear. □

Corollary 4.10. *Let $\underline{x}$ be any expression in the Coxeter generators of $\mathfrak{S}_n$. Any quasi-b-dilated expression of $\underline{x}$ is independent of the choice of expression $\underline{x}$. Similarly, the nib truncations of these elements are independent of the choice of expression $\underline{x}$.*

Proof. By Lemma 4.7 it is enough to consider the bad braids in $\psi_{\underline{x}}$. If $\underline{x} = \underline{w}_b$ for some $w \in \mathfrak{S}_n$, then we can resolve each bad braid in $\psi_{\underline{x}}$ and $\mathrm{nib}(\psi_{\underline{x}})$ using Proposition 4.9. Now, if $\psi_{\underline{x}}$ is quasi-b-dilated then $\psi_{\underline{x}}$ and $\mathrm{nib}(\psi_{\underline{x}})$ are obtained from $\psi_{\underline{w}_b}$ and $\mathrm{nib}(\psi_{\underline{w}_b})$ by undoing some degree zero crossings (thus introducing no new bad braids) and the result follows. □

5. Recasting the diagrammatic Bott–Samelson generators in the quiver Hecke algebra

We continue with the notation of Convention 2.32. The elements of the (breadth-enhanced) diagrammatic Bott–Samelson endomorphism algebras can be thought of as morphisms relating pairs of expressions from $\widehat{\mathfrak{S}}_h$. We have also seen that one can think of an element of the quiver Hecke algebra as a morphism between pairs of paths in the alcove geometries of Section 3. This will allow us, through the relationship between paths and their colourings described in Section 3, to define the isomorphism behind Theorem A. In what follows we will define generators

$$\mathsf{adj}^{\varnothing\alpha}_{\alpha\varnothing} \quad \mathsf{spot}^{\varnothing}_{\alpha} \quad \mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} \quad \mathsf{com}^{\beta\gamma}_{\gamma\beta} \quad \mathsf{hex}^{\beta\alpha\beta}_{\alpha\beta\alpha}$$

for $\alpha, \beta, \gamma \in \Pi$ and their duals. The hyperplane labelled by α (respectively β) is a wall of the dominant chamber if and only if P_{α} (respectively P_{β}) leaves the dominant chamber. By the cyclotomic KLR relation, one of the above generators is zero if (and only if) one of its indexing roots labels a path which leaves the dominant chamber. However, one should think of these as generators in the sense of a right tensor quotient of a monoidal category. In other words, we still require every generator for every simple root (even if they are zero) as the *left concatenates* of these generators will not be zero, in general.

In order to construct our isomorphism, we must first "sign-twist" the elements of the KLR algebra. This twist counts the number of degree -2 crossings (heuristically, these are the crossings which "intersect an alcove wall"). For $\underline{w}$ an arbitrary reduced expression, we set

$$\Upsilon_{\underline{w}} = (-1)^{\sharp\{1\leqslant p<q\leqslant n \mid w(p)>w(q), i_p=i_q\}} e_{\underline{i}}\psi_{\underline{w}} e_{w(\underline{i})}.$$

While KLR diagrams are usually only defined up to a choice of expression, we emphasise that each of the generators we define is independent of this choice. Thus there is no ambiguity in defining the elements $\Upsilon^{\mathsf{P}}_{\mathsf{Q}}$ for $w^{\mathsf{P}}_{\mathsf{Q}}$ without reference to the underlying expression. In other words: these generators are *canonical elements* of $\mathcal{H}^{\sigma}_n$. Examples of concrete choices of expression can be found in [4, Section 2.3]. In various proofs it will be convenient to denote by T and B the top and bottom paths of certain diagrams (which we define case-by-case).

5.1. Idempotents in diagrammatic algebras

We consider an element of the quiver Hecke or diagrammatic Bott–Samelson endomorphism algebra to be a morphism between paths, lifting the ideas of Subsection 2.4.1. The easiest elements to construct are the idempotents corresponding to the trivial morphism from a path to itself. Given $\boldsymbol{\alpha}$ a simple reflection, we have an associated path $\mathsf{P}_{\boldsymbol{\alpha}}$, a trivial bijection $w^{\mathsf{P}_{\boldsymbol{\alpha}}}_{\mathsf{P}_{\boldsymbol{\alpha}}} = 1 \in \mathfrak{S}_{b_{\boldsymbol{\alpha}}}$, and an idempotent element of the quiver Hecke algebra

$$e_{\mathsf{P}_{\boldsymbol{\alpha}}} := e_{\mathrm{res}(\mathsf{P}_{\boldsymbol{\alpha}})} \in \mathcal{H}^{\sigma}_{b_{\boldsymbol{\alpha}}}$$

where we reemphasise that $e_{\mathrm{res}(\mathsf{P}_{\boldsymbol{\alpha}})} = e_{\mathrm{res}(\mathsf{P}^{\flat}_{\boldsymbol{\alpha}})}$ (see Remark 2.23). Given $\boldsymbol{\alpha}$ a simple reflection, we also have a Soergel diagram $1_{\boldsymbol{\alpha}}$ given by a single vertical strand coloured by $\boldsymbol{\alpha}$. We define

$$\Psi(1_{\boldsymbol{\alpha}}) = e_{\mathsf{P}_{\boldsymbol{\alpha}}}. \tag{5.1}$$

More generally, given any $\underline{w} = s_{\alpha^{(1)}} s_{\alpha^{(2)}} \dots s_{\alpha^{(k)}}$ any expression of breadth $b(\underline{w}) = n$, we have an associated path $\mathsf{P}_{\underline{w}}$, and an element of the quiver Hecke algebra

$$e_{\mathsf{P}_{\underline{w}}} := e_{\mathrm{res}(\mathsf{P}_{\underline{w}})} = e_{\mathsf{P}_{\alpha^{(1)}}} \otimes e_{\mathsf{P}_{\alpha^{(2)}}} \otimes \dots \otimes e_{\mathsf{P}_{\alpha^{(k)}}} \in \mathcal{H}^{\sigma}_{nh}$$

and a $(\underline{w}, \underline{w})$-Soergel diagram

$$1_{\underline{w}} = 1_{\alpha^{(1)}} \otimes 1_{\alpha^{(2)}} \otimes \dots \otimes 1_{\alpha^{(k)}}$$

given by k vertical strands, coloured with $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}$ from left to right. We define

$$\Psi(1_{\underline{w}}) = e_{\mathsf{P}_{\underline{w}}}. \tag{5.2}$$

Example 5.1. Continuing with Fig. 1 and Examples 2.14 and 2.25, we let

$$\underline{w} = s_{\emptyset} s_{\emptyset} s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2} s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2}$$

$$\underline{w}' = s_{\varepsilon_3-\varepsilon_1} s_{\emptyset} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2} s_{\emptyset} s_{\varepsilon_3-\varepsilon_1} s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2}.$$

Recall these paths came from "inserting determinants" into the path in Fig. 1. We have that

$$\Psi(1_{\underline{w}}) = e_{0,1,2,4,0,1,3,4,2,3,1,2,0,4,3,0,2,1,0,1,4,3,4,2,3,1,2,0,4,3,0,2,1,0,1,4}$$

$$\Psi(1_{\underline{w}'}) = e_{0,1,4,0,3,4,2,1,0,2,3,4,4,1,0,4,0,3,2,3,4,3,4,2,3,1,2,0,4,3,0,2,1,0,1,4}.$$

Remark 5.2. For two paths S and T, we have that $\mathsf{S} \sim \mathsf{T}$ if and only if $\mathrm{res}(\mathsf{S}) = \mathrm{res}(\mathsf{T})$. Therefore if $\mathsf{S} \sim \mathsf{T}$ then $e_{\mathsf{T}} = e_{\mathsf{S}} e_{\mathsf{T}} = e_{\mathsf{S}}$. In particular $e_{\mathsf{P}_{\alpha}} = e_{\mathsf{P}_{\alpha}} e_{\mathsf{P}^{\flat}_{\alpha}} = e_{\mathsf{P}^{\flat}_{\alpha}}$.

Remark 5.3. We have defined two distinct paths P_{α} and $\mathsf{P}^{\flat}_{\alpha}$ which label the same idempotent, thus $e_{\mathsf{P}_{\emptyset}} \mathcal{H}^{\sigma}_{b_{\alpha}} e_{\mathsf{P}_{\alpha}} = e_{\mathsf{P}_{\emptyset}} \mathcal{H}^{\sigma}_{b_{\alpha}} e_{\mathsf{P}^{\flat}_{\alpha}}$. This apparent redundancy is required because we cannot directly compare $\mathsf{P}_{\emptyset}$ and P_{α} as they *do not have the same shape* — however, we can compare $\mathsf{P}_{\emptyset}$ and $\mathsf{P}^{\flat}_{\alpha}$ as they *do have the same shape*. Thus $\mathsf{P}^{\flat}_{\alpha}$ is required in order to define the spot-morphism. For the remainder of this section, we will restrict our attention to a subset of morphisms between paths of the same shape which form a set of monoidal generators of our truncated KLR algebra.

5.2. Local adjustments and isotopy

We will refer to the passage between alcove paths which differ only by occurrences of $s_{\emptyset} = 1$ (and their associated idempotents) as "adjustment". We wish to understand the morphism relating the paths $\mathsf{P}_{\alpha} \otimes \mathsf{P}_{\emptyset}$ and $\mathsf{P}_{\emptyset} \otimes \mathsf{P}_{\alpha}$.

Proposition 5.4. *The element $\psi^{\mathsf{P}_{\alpha\emptyset}}_{\mathsf{P}_{\emptyset\alpha}}$ is independent of the choice of reduced expression.*

Proof. There are precisely three crossings in $\psi^{\mathsf{P}_{\alpha\emptyset}}_{\mathsf{P}_{\emptyset\alpha}}$ of non-zero degree. Namely, the r-strand (for some $r \in \mathbb{Z}/e\mathbb{Z}$) connecting the $\mathsf{P}^{-1}_{\emptyset\alpha}(1, \varepsilon_i)$th top vertex to the $\mathsf{P}^{-1}_{\alpha\emptyset}(1, \varepsilon_i)$th bottom vertex crosses each of the strands connecting $\mathsf{P}^{-1}_{\emptyset\alpha}(q, \varepsilon_{i+1})$th top vertices to the $\mathsf{P}^{-1}_{\alpha\emptyset}(q, \varepsilon_{i+1})$th bottom vertices for $q = b_{\alpha} - 1, b_{\alpha}, b_{\alpha} + 1$ (of residues $r + 1$, r, and $r - 1$ respectively) precisely once with degrees $+1$, -2, and $+1$ respectively. Thus $\psi^{\mathsf{P}_{\alpha\emptyset}}_{\mathsf{P}_{\emptyset\alpha}}$ is residue-commutative and the result follows from Lemma 4.7. □

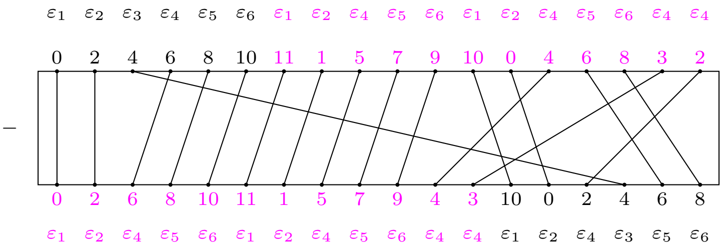


Fig. 18. We let $h = 1$, $\ell = 6$, $e = 12$, $\sigma = (0, 2, 4, 6, 8, 10)$ and $\boldsymbol{\alpha} = \varepsilon_3 - \varepsilon_4$. The adjustment term $\mathsf{adj}^{\emptyset\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\emptyset}$ is illustrated. The steps of the path $\mathsf{P}_{\boldsymbol{\alpha}}$ and $\mathsf{P}_{\emptyset}$ are coloured pink and black respectively within both $\mathsf{P}_{\boldsymbol{\alpha}\emptyset}$ (along the top of the diagram) and $\mathsf{P}_{\emptyset\boldsymbol{\alpha}}$ (along the bottom of the diagram).

Thus we are free to define the KLR-adjustment to be

$$\mathsf{adj}^{\boldsymbol{\alpha}\emptyset}_{\emptyset\boldsymbol{\alpha}} := \Upsilon^{\mathsf{P}_{\boldsymbol{\alpha}\emptyset}}_{\mathsf{P}_{\emptyset\boldsymbol{\alpha}}}$$

which is independent of the choice of reduced expression of the permutation. (See Fig. 18.)

Proposition 5.5. *We have that*

$$\mathsf{adj}^{\emptyset\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\emptyset} \circ e_{\mathsf{P}_{\boldsymbol{\alpha}\emptyset}} \circ \mathsf{adj}^{\boldsymbol{\alpha}\emptyset}_{\emptyset\boldsymbol{\alpha}} = e_{\mathsf{P}_{\emptyset\boldsymbol{\alpha}}} \qquad \textit{and} \qquad \mathsf{adj}^{\boldsymbol{\alpha}\emptyset}_{\emptyset\boldsymbol{\alpha}} \circ e_{\mathsf{P}_{\emptyset\boldsymbol{\alpha}}} \circ \mathsf{adj}^{\emptyset\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\emptyset} = e_{\mathsf{P}_{\boldsymbol{\alpha}\emptyset}}$$

and so adjustment is an invertible process.

Proof. The paths $\mathsf{P}_{\boldsymbol{\alpha}\emptyset}$ and $\mathsf{P}_{\emptyset\boldsymbol{\alpha}}$ satisfy the conditions of Proposition 4.4 and so the result follows. $\square$

Finally, we remark that the above adjustment can be generalised from the $b_\emptyset = 1$ case to the $b_{\varnothing} \geqslant 1$ case as follows. For $\underline{w} = s_{\boldsymbol{\alpha}} s_{\varnothing}$ with $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \Pi$ two (equal, adjacent, or non-adjacent) simple roots, we set

$$\mathsf{A}^{\varnothing\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\varnothing}(q) = \mathsf{P}_{q\emptyset} \otimes \mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{(b_{\boldsymbol{\gamma}}-q)\emptyset}$$

for $0 \leqslant q \leqslant b_{\boldsymbol{\gamma}}$ and we set

$$\mathsf{adj}^{\varnothing\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\varnothing}(q) = e_{\mathsf{A}^{\varnothing\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\varnothing}(q+1)} \left(e_{\mathsf{P}_{q\emptyset}} \otimes \mathsf{adj}^{\emptyset\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\emptyset} \otimes e_{\mathsf{P}_{(b_{\boldsymbol{\gamma}}-q-1)\emptyset}} \right) e_{\mathsf{A}^{\varnothing\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\varnothing}(q)}$$

and we define

$$\mathsf{adj}^{\varnothing\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\varnothing} = \mathsf{adj}^{\varnothing\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\varnothing}(b_{\boldsymbol{\gamma}} - 1) \ldots \mathsf{adj}^{\varnothing\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\varnothing}(1) \mathsf{adj}^{\varnothing\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\varnothing}(0).$$

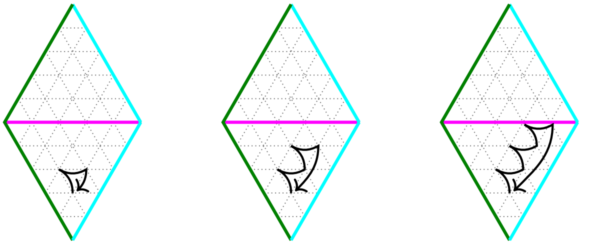


Fig. 19. An example timeline for the KLR spot. Fix $\ell = 1$ and $h = 3$ and $e = 5$ and $\alpha = \varepsilon_3 - \varepsilon_1$ (so that $b_\alpha = 3$). From left to right we picture $\mathsf{S}_{2,\alpha} = \mathsf{S}_\alpha(3) = \mathsf{P}_\emptyset$, $\mathsf{S}_{1,\alpha}$, $\mathsf{S}_{0,\alpha} = \mathsf{P}^\flat_\alpha$. We do not picture the $k = 2, 1, 0$ copies of the path $(+\varepsilon_1, +\varepsilon_2, +\varepsilon_3)$ at the start of each path, for ease of readability.

5.3. The KLR-spot diagram

We now define the spot path morphism. Recall that

$$\mathsf{P}_\emptyset = (\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_h)^{b_\alpha} \qquad \mathsf{P}^\flat_\alpha = (\varepsilon_1, \dots, \varepsilon_{i-1}, \widehat{\varepsilon_i}, \varepsilon_{i+1}, \dots, \varepsilon_h)^{b_\alpha} \boxtimes (\varepsilon_i)^{b_\alpha}$$

are both paths of the same shape. The permutation $w^{\mathsf{P}_\emptyset}_{\mathsf{P}^\flat_\alpha}$ is fully-commutative and so we are free to define the $\mathsf{KLR\text{-}spot}$ to be the elements

$$\mathsf{spot}^\emptyset_\alpha := \Upsilon^{\mathsf{P}_\emptyset}_{\mathsf{P}^\flat_\alpha} \qquad \mathsf{spot}^\alpha_\emptyset := \Upsilon^{\mathsf{P}^\flat_\alpha}_{\mathsf{P}_\emptyset}$$

which are both independent of choice of reduced expressions and both belong to $e_{\mathsf{P}_\alpha}\mathcal{H}^\sigma_{b_\alpha} e_{\mathsf{P}_\alpha} = e_{\mathsf{P}^\flat_\alpha}\mathcal{H}^\sigma_{b_\alpha} e_{\mathsf{P}^\flat_\alpha}$.

We wish to inductively pass between the paths $\mathsf{P}^\flat_\alpha$ and $\mathsf{P}_\emptyset$ by means of a visual timeline (pictured in Fig. 19). This allows us to factorise the KLR-spots and to simplify our proofs later on. To this end we define

$$\mathsf{S}_{q,\alpha} = \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha - q} \boxtimes \mathsf{P}_i^{b_\alpha - q} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_h)^q \boxtimes (\varepsilon_1, \dots, \varepsilon_{i-1}, \widehat{\varepsilon_i}, \varepsilon_{i+1}, \dots, \varepsilon_h)^{b_\alpha - q} \boxtimes (\varepsilon_i)^{b_\alpha - q}$$

for $0 \leqslant q \leqslant b_\alpha$ and we notice that $\mathsf{S}_{0,\alpha} = \mathsf{P}^\flat_\alpha$ and $\mathsf{S}_{b_\alpha,\alpha} = \mathsf{P}_\emptyset$. We define $\mathsf{spot}^\emptyset_\alpha(q)$ to be the element $\mathsf{spot}^\emptyset_\alpha(q) = \psi^{\mathsf{S}_{q+1,\alpha}}_{\mathsf{S}_{q,\alpha}}$ for $0 \leqslant q < b_\alpha$ and we factorise $\mathsf{spot}^\emptyset_\alpha$ as follows

$$\mathsf{spot}^\emptyset_\alpha := e_{\mathsf{P}_\emptyset} \circ \mathsf{spot}^\emptyset_\alpha(b_\alpha - 1) \circ \cdots \circ \mathsf{spot}^\emptyset_\alpha(1) \circ \mathsf{spot}^\emptyset_\alpha(0) \circ e_{\mathsf{P}^\flat_\alpha}.$$

Example 5.6. Let $h = 3$ and $\ell = 1$ and $e = 5$ and $\alpha = \varepsilon_3 - \varepsilon_1$. We have that $b_\alpha = 3$ and

$$\begin{aligned}
\mathsf{P}^\flat_\alpha = \mathsf{S}_{0,\alpha} &= (\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_3, \varepsilon_3)\\
\mathsf{S}_{1,\alpha} &= (\varepsilon_1, \varepsilon_2, \varepsilon_3) \boxtimes (\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_3)\\
\mathsf{S}_{2,\alpha} &= (\varepsilon_1, \varepsilon_2, \varepsilon_3) \boxtimes (\varepsilon_1, \varepsilon_2, \varepsilon_3) \boxtimes (\varepsilon_1, \varepsilon_2, \varepsilon_3)\\
\mathsf{P}_\emptyset = \mathsf{S}_{3,\alpha} &= (\varepsilon_1, \varepsilon_2, \varepsilon_3) \boxtimes (\varepsilon_1, \varepsilon_2, \varepsilon_3) \boxtimes (\varepsilon_1, \varepsilon_2, \varepsilon_3)
\end{aligned}$$

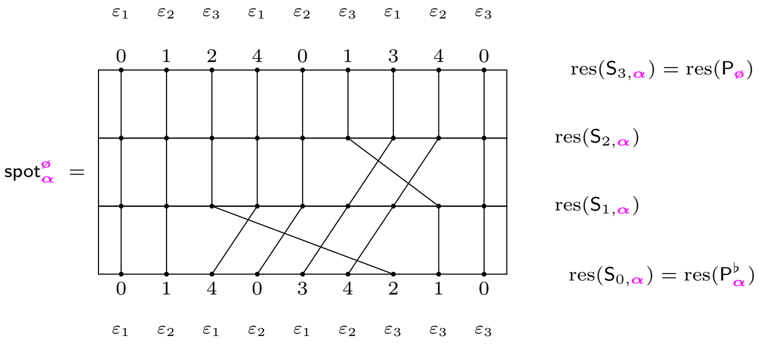


Fig. 20. The element $\mathsf{spot}^{\varnothing}_{\alpha}$ of Example 5.6. We have added the step labels on top and bottom so that one can appreciate that this element is a morphism between paths. However, we remark that while a necessary condition for a product of two KLR diagrams to be non-zero is that their residue sequences must coincide, the same is not true for their step labels (see Remark 5.2).

which are depicted in Fig. 19. Of course, $\mathsf{S}_{3,\alpha} = \mathsf{S}_{2,\alpha}$ in this case, but this is only because α is the affine root $\varepsilon_3 - \varepsilon_1$ with $3 = h$.

Remark 5.7. We have that $w^{\mathsf{S}_{\alpha,q+1}}_{\mathsf{S}_{\alpha,q}} = w^{qh+i}_{b_\alpha h - b_\alpha + q+1}$ for $0 \leqslant q < b_\alpha$, where the sub and superscripts correspond to

$$\mathsf{S}^{-1}_{q,\alpha}(q+1, \varepsilon_i) = qh + i \qquad \mathsf{S}^{-1}_{q+1,\alpha}(q+1, \varepsilon_i) = b_\alpha h - b_\alpha + q + 1$$

and so one can think of the spot morphism as successively removing each $+\varepsilon_i$ step from the latter path and adding it to the former. (See Fig. 20.)

Remark 5.8. The element $e_{\mathsf{S}_{q+1,\alpha}} \mathsf{spot}^{\varnothing}_{\alpha}(q) e_{\mathsf{S}_{q,\alpha}}$ is of degree 1 for $q = 0$ and degree 0 for $0 < q < b_\alpha$. The terms with $0 < q < b_\alpha$ are invertible by Proposition 4.4. Thus one can think of the $q = 0$ term as the real substance of $\mathsf{spot}^{\varnothing}_{\alpha}$. One should intuitively think of this degree contribution as coming from the fact that the path $\mathsf{S}_{0,\alpha}$ steps onto and off of a hyperplane but $\mathsf{S}_{1,\alpha}$ does not touch the hyperplane at any point. The diagram $\mathsf{spot}_{\alpha}(0)$ has a crossing involving the strand from the $\mathsf{S}^{-1}_{0,\alpha}(1, \varepsilon_i)$th node on the bottom edge to the $\mathsf{S}^{-1}_{1,\alpha}(1, \varepsilon_i)$th node on the top edge and the strand from the $\mathsf{S}^{-1}_{1,\alpha}(b_\alpha, \varepsilon_{i+1})$th node on the bottom edge to the $\mathsf{S}^{-1}_{0,\alpha}(b_\alpha, \varepsilon_{i+1})$th node on the top edge. See Fig. 19 for a visualisation.

5.4. The KLR-fork diagram

We wish to understand the morphism from $\mathsf{P}_\varnothing \otimes \mathsf{P}_\alpha$ to $\mathsf{P}_\alpha \otimes \mathsf{P}^\flat_\alpha$ (which are both paths of the same shape, so this makes sense). The permutation $w^{\mathsf{P}_\varnothing \otimes \mathsf{P}_\alpha}_{\mathsf{P}_\alpha \otimes \mathsf{P}^\flat_\alpha}$ is *not* fully commutative and so we must do a little work prior to our definition.

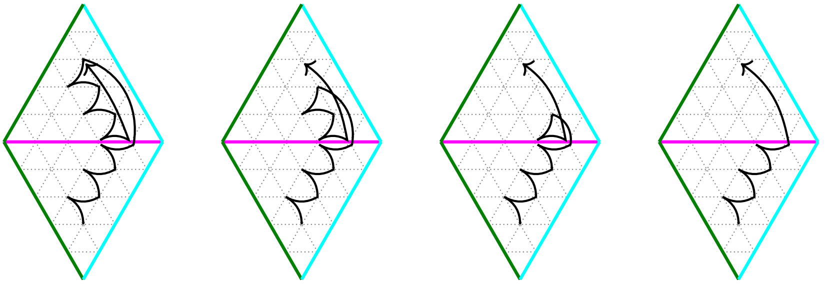


Fig. 21. An example of a timeline for the KLR fork. Fix $\ell = 1$ and $h = 3$ and $e = 5$ and $\boldsymbol{\alpha} = \varepsilon_3 - \varepsilon_1$ (so that $b_{\boldsymbol{\alpha}} = 3$). From left to right we picture the paths $\mathsf{F}_{0,\varnothing\boldsymbol{\alpha}} = \mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}}$, $\mathsf{F}_{1,\varnothing\boldsymbol{\alpha}}$, $\mathsf{F}_{2,\varnothing\boldsymbol{\alpha}}$, $\mathsf{F}_{3,\varnothing\boldsymbol{\alpha}} = \mathsf{P}_{\varnothing\boldsymbol{\alpha}}$. Notice that we do not picture the $q = 0, 1, 2, 3$ copies of the path $(+\varepsilon_1, +\varepsilon_2, +\varepsilon_3)$ at the start of each path, for ease of readability.

Proposition 5.9. *The elements* $\psi^{\mathsf{P}_\varnothing \otimes \mathsf{P}_{\boldsymbol{\alpha}}}_{\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}}}$ *and* $\psi^{\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}_\varnothing}_{\mathsf{P}^\flat_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}}$ *are independent of the reduced expressions.*

Proof. We focus on the former case, as the latter is similar. The element $w^{\mathsf{P}_\varnothing \otimes \mathsf{P}_{\boldsymbol{\alpha}}}_{\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}}}$ contains precisely $b_{\boldsymbol{\alpha}}$ crossings of strands with the same residue label: Namely for each $1 \leqslant q \leqslant b_{\boldsymbol{\alpha}}$ the strand connecting the top and bottom vertices labelled by the integers

$$\mathsf{P}^{-1}_{\varnothing\boldsymbol{\alpha}}(q, \varepsilon_i) = qh + i \qquad (\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}})^{-1}(q, \varepsilon_i) = b_{\boldsymbol{\alpha}} h + (q-1)(h-1) + \boldsymbol{\alpha}(i+1)$$

crosses the strand connecting the top and bottom vertices labelled by the integers

$$\mathsf{P}^{-1}_{\varnothing\boldsymbol{\alpha}}(b_{\boldsymbol{\alpha}} + q, \varepsilon_{i+1}) = b_{\boldsymbol{\alpha}} h + (q-1)(h-1) + \boldsymbol{\alpha}(i+1)$$
$$(\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}})^{-1}(b_{\boldsymbol{\alpha}} + q, \varepsilon_{i+1}) = b_{\boldsymbol{\alpha}} h - b_{\boldsymbol{\alpha}} + q.$$

The qth of these like-labelled crossings forms a braid with a third strand if and only if this third strand connects a top and bottom node labelled by the integers

$$\mathsf{P}^{-1}_{\varnothing\boldsymbol{\alpha}}(b_{\boldsymbol{\alpha}} + p, \varepsilon_j) = b_{\boldsymbol{\alpha}} h + (p-1)(h-1) + \boldsymbol{\alpha}(j)$$
$$(\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}})^{-1}(b_{\boldsymbol{\alpha}} + p, \varepsilon_j) = b_{\boldsymbol{\alpha}} h + (p-1)(h-1) + \boldsymbol{\alpha}(j)$$

for $\boldsymbol{\alpha}(j) \neq \boldsymbol{\alpha}(i+1)$ and $1 \leqslant p < q$ or $p = q$ and $\boldsymbol{\alpha}(j) < \boldsymbol{\alpha}(i+1)$. None of the resulting braids is bad; thus $\psi^{\mathsf{P}_\varnothing \otimes \mathsf{P}_{\boldsymbol{\alpha}}}_{\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}}}$ is residue-commutative and the result follows. □

Thus we are free to define the KLR-forks to be the elements

$$\mathsf{fork}^{\varnothing\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\boldsymbol{\alpha}} := \Upsilon^{\mathsf{P}_\varnothing \otimes \mathsf{P}_{\boldsymbol{\alpha}}}_{\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}}} \qquad \mathsf{fork}^{\boldsymbol{\alpha}\varnothing}_{\boldsymbol{\alpha}\boldsymbol{\alpha}} := \Upsilon^{\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}_\varnothing}_{\mathsf{P}^\flat_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}}$$

which are independent of the choice of reduced expressions. We reemphasise that $\mathrm{res}(\mathsf{P}_{\boldsymbol{\alpha}}) = \mathrm{res}(\mathsf{P}^\flat_{\boldsymbol{\alpha}})$, thus former element belongs to $(e_{\mathsf{P}_\varnothing} \otimes e_{\mathsf{P}_{\boldsymbol{\alpha}}})\mathcal{H}^\sigma_{b_{\boldsymbol{\alpha}}}(e_{\mathsf{P}_{\boldsymbol{\alpha}}} \otimes e_{\mathsf{P}_{\boldsymbol{\alpha}}}) = (e_{\mathsf{P}_\varnothing} \otimes e_{\mathsf{P}_{\boldsymbol{\alpha}}})\mathcal{H}^\sigma_{b_{\boldsymbol{\alpha}}}(e_{\mathsf{P}_{\boldsymbol{\alpha}}} \otimes e_{\mathsf{P}^\flat_{\boldsymbol{\alpha}}})$ (a similar statement holds for the latter element).

We wish to inductively pass between the paths $\mathsf{P}_{\alpha} \otimes \mathsf{P}^{\flat}_{\alpha}$ and $\mathsf{P}_{\varnothing\alpha}$ (respectively $\mathsf{P}^{\flat}_{\alpha} \otimes \mathsf{P}_{\alpha}$ and $\mathsf{P}_{\alpha\varnothing}$) by means of a visual timeline (as in Fig. 21). This allows us to factorise KLR-forks and to simplify our proofs later on. To this end we define

$$\mathsf{F}_{q,\varnothing\alpha} = \mathsf{P}^{q\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_{i+1}^{b_\alpha - q} \otimes_\alpha \mathsf{M}_i^{b_\alpha - q} \boxtimes \mathsf{P}_i^{b_\alpha}$$
$$\mathsf{F}_{q,\alpha\varnothing} = \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_i^{b_\alpha - q} \boxtimes \mathsf{M}_i^{b_\alpha - q} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} \otimes_\alpha \mathsf{P}^{q\emptyset}$$

and we remark that

$$\mathsf{F}_{0,\varnothing\alpha} = \mathsf{P}_{\alpha} \otimes \mathsf{P}^{\flat}_{\alpha} \quad \mathsf{F}_{b_\alpha,\varnothing\alpha} = \mathsf{P}_{\varnothing} \otimes \mathsf{P}_{\alpha} \quad \mathsf{F}_{0,\alpha\varnothing} = \mathsf{P}^{\flat}_{\alpha} \otimes \mathsf{P}_{\alpha} \quad \mathsf{F}_{b_\alpha,\alpha\varnothing} = \mathsf{P}_{\alpha} \otimes \mathsf{P}_{\varnothing}.$$

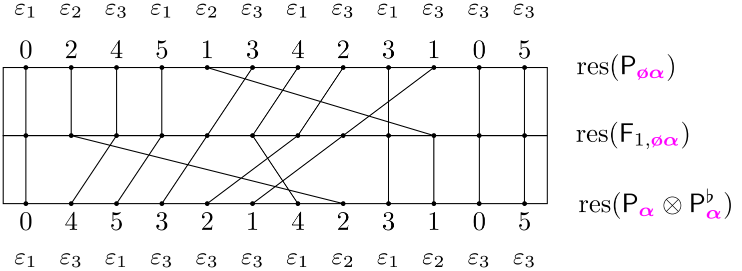
We define $\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}(q) = \Upsilon^{\mathsf{F}_{q,\varnothing\alpha}}_{\mathsf{F}_{q+1,\varnothing\alpha}}$ and $\mathsf{fork}^{\alpha\varnothing}_{\alpha\alpha}(q) = \Upsilon^{\mathsf{F}_{q,\alpha\varnothing}}_{\mathsf{F}_{q+1,\alpha\varnothing}}$ for $0 \leqslant k < b_\alpha$ and we factorise the KLR-forks as follows

$$\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} = e_{\mathsf{P}_{\varnothing\alpha}} \circ \mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}(b_\alpha - 1) \circ \cdots \circ \mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}(1) \circ \mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}(0) \circ e_{\mathsf{P}_\alpha \otimes \mathsf{P}^{\flat}_\alpha}$$
$$\mathsf{fork}^{\alpha\varnothing}_{\alpha\alpha} = e_{\mathsf{P}_{\alpha\varnothing}} \circ \mathsf{fork}^{\alpha\varnothing}_{\alpha\alpha}(b_\alpha - 1) \circ \cdots \circ \mathsf{fork}^{\alpha\varnothing}_{\alpha\alpha}(1) \circ \mathsf{fork}^{\alpha\varnothing}_{\alpha\alpha}(0) \circ e_{\mathsf{P}^{\flat}_\alpha \otimes \mathsf{P}_\alpha}.$$

Example 5.10. Let $h = 1$, $\ell = 3$, $e = 6$, $\sigma = (0, 2, 4) \in \mathbb{Z}^3$ and $\alpha = \varepsilon_2 - \varepsilon_3$ (thus $b_\alpha = 2$). We have

$$\begin{aligned}\mathsf{P}_{\alpha} \otimes \mathsf{P}^{\flat}_{\alpha} &= (\varepsilon_1, \varepsilon_3, \varepsilon_1, \varepsilon_3, \varepsilon_3, \varepsilon_3) \otimes (\varepsilon_1, \varepsilon_3, \varepsilon_1, \varepsilon_3, \varepsilon_2, \varepsilon_2)\\ &= (\varepsilon_1, \varepsilon_3, \varepsilon_1, \varepsilon_3, \varepsilon_3, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_3)\\ \mathsf{P}_{\varnothing\alpha} &= (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_3, \varepsilon_1, \varepsilon_3, \varepsilon_3, \varepsilon_3)\end{aligned}$$

are both dominant paths terminating at $(1^4 \mid 1^2 \mid 1^6) \in \mathscr{P}_{1,3}(12)$. The KLR-fork diagram is as follows

$\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} =$ 

The following proposition allows us to see that these two elements are essentially the same. We will see in the proof that the "timelines" for the fork generators allow us to proceed step-by-step (the steps are indexed by $b_\alpha \geqslant q \geqslant 1$).

Proposition 5.11. *Let* $\alpha \in \Pi$. *We have that* $\mathsf{fork}^{\alpha\varnothing}_{\alpha\alpha} = \mathsf{adj}^{\alpha\varnothing}_{\varnothing\alpha}\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}$.

Proof. We note that $\mathsf{A}^{\varnothing\alpha}_{\alpha\varnothing}(b_\alpha) = \mathsf{P}_\varnothing \otimes \mathsf{P}_\alpha = \mathsf{F}_{b_\alpha,\varnothing\alpha}$ and $\mathsf{A}^{\varnothing\alpha}_{\alpha\varnothing}(0) = \mathsf{P}_\alpha \otimes \mathsf{P}_\varnothing = \mathsf{F}_{0,\alpha\varnothing}$. We claim that

$$\mathsf{adj}^{\alpha\varnothing}_{\varnothing\alpha}(q-1) \circ \Upsilon^{\mathsf{A}^{\varnothing\alpha}_{\alpha\varnothing}(q)}_{\mathsf{F}_{q,\alpha\varnothing}} \circ \mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}(q-1) = \Upsilon^{\mathsf{A}^{\varnothing\alpha}_{\alpha\varnothing}(q-1)}_{\mathsf{F}_{q-1,\alpha\varnothing}} \tag{5.3}$$

for $b_\alpha \geqslant q \geqslant 1$. The result follows immediately from Proposition 5.9 once we have proven the claim. We label the top and bottom vertices of the lefthand-side of equation (5.3) by the paths $\mathsf{T}_q = \mathsf{A}^{\varnothing\alpha}_{\alpha\varnothing}(q)$ and $\mathsf{B}_q = \mathsf{F}_{q,\varnothing\alpha}$ respectively. We remark $\mathrm{res}(\mathsf{F}_{q,\varnothing\alpha}) = \mathrm{res}(\mathsf{F}_{q,\alpha\varnothing})$ (as these paths are obtained from each other by reflection) and so this labelling makes sense.

We now prove the claim. There are two strands in the concatenated diagram which do not respect step-labels. Namely, the r_q-strands (for some $r_q \in \mathbb{Z}/e\mathbb{Z}$) connecting the $\mathsf{T}_q^{-1}(q,\varepsilon_i)$ and $\mathsf{B}_q^{-1}(b_\alpha + q, \varepsilon_{i+1})$ top and bottom vertices and the strand connecting the $\mathsf{T}_q^{-1}(b_\alpha + q, \varepsilon_{i+1})$ and $\mathsf{B}_q^{-1}(q,\varepsilon_i)$ top and bottom vertices. There are four crossings of non-zero degree in the product, all of which involve the former, "distinguished", r_q-strand. Namely, the distinguished r_q-strand passes from $\mathsf{T}_q^{-1}(q,\varepsilon_i)$ to the left through the latter r_q-strand and then through the vertical (r_q+1)-strand connecting the $\mathsf{T}^{-1}(b_\alpha + q, \varepsilon_{i+1})$ and $\mathsf{B}^{-1}(b_\alpha + q, \varepsilon_{i+1})$ vertices before then passing back again through both these strands and terminating at $\mathsf{B}_q^{-1}(b_\alpha + q, \varepsilon_{i+1})$. (The distinguished strand crosses several other strands in the process, but the crossings are of degree zero and so can be undone trivially, by case 2 of relation (R4).) Using case 4 of relation (R4), we pull the distinguished r_q-strand rightwards through the $(r_q - 1)$-strand and hence change the sign and obtain a dot on the r_q-strand (the term with a dot on the $(r_q + 1)$-strand is zero by case 1 of relation (R4) and the commutativity relations). Using relation (R3), we pull the dot on the distinguished strand rightwards through the crossing of r_q-strands and hence undo this crossing, kill the dot, and change the sign again (the other term is again zero by case 1 of relation (R4) and the commutativity relations). The resulting diagram has no double-crossings and respects step labels and thus is equal to the righthand-side of Proposition 5.9, as required. □

5.5. The KLR hexagon diagram

We now define the hexagon in the KLR algebra. We let $\alpha, \beta \in \Pi$ label non-commuting reflections. We assume, without loss of generality, that $j = i+1$. We have two cases to consider: if $b_\alpha \geqslant b_\beta$ then we must deform the path $\mathsf{P}_{\alpha\beta\alpha}$ into the path $\mathsf{P}_{\varnothing-\varnothing} \otimes \mathsf{P}_{\beta\alpha\beta}$ and if $b_\alpha \leqslant b_\beta$ then we must deform the path $\mathsf{P}_{\varnothing-\varnothing} \otimes \mathsf{P}_{\alpha\beta\alpha}$ into the path $\mathsf{P}_{\beta\alpha\beta}$, where here $\varnothing - \varnothing := \emptyset^{b_\alpha - b_\beta}$.

Proposition 5.12. *The elements* $\psi^{\mathsf{P}_{\alpha\beta\alpha}}_{\mathsf{P}_{\varnothing-\varnothing}\otimes\mathsf{P}_{\beta\alpha\beta}}$ *and* $\psi^{\mathsf{P}_{\varnothing-\varnothing}\otimes\mathsf{P}_{\alpha\beta\alpha}}_{\mathsf{P}_{\beta\alpha\beta}}$ *are independent of the choice of reduced expressions for* $b_\alpha \geqslant b_\beta$ *and* $b_\beta \geqslant b_\alpha$*, respectively.*

Proof. We consider the first case as the second is similar. The bad triples of $\psi^{\mathsf{P}_{\alpha\beta\alpha}}_{\mathsf{P}_{\varnothing-\varnothing}\otimes\mathsf{P}_{\beta\alpha\beta}}$ are precisely the triples labelled by the integers

$$\mathsf{P}^{-1}_{\alpha\beta\alpha}(q,\varepsilon_i) < \mathsf{P}^{-1}_{\alpha\beta\alpha}(b_{\alpha\beta}+q\pm 1,\varepsilon_{i+2}) < \mathsf{P}^{-1}_{\alpha\beta\alpha}(b_\alpha+q,\varepsilon_{i+1})$$

for $1 \leqslant q \leqslant b_\alpha$, where the first and third steps have residue $r_q \in \mathbb{Z}/e\mathbb{Z}$ and the second has residue $r_{q\pm1} = r_q \mp 1 \in \mathbb{Z}/e\mathbb{Z}$. Thus it is enough to consider the subexpression, $\psi_{\underline{w}}$, formed from the union of the (r_q, r_{q+1})-strands for $0 \leqslant q \leqslant b_\alpha$ enumerated above. We set $\mathsf{T} = \mathsf{P}_{\alpha\beta\alpha}$ and $\mathsf{B} = \mathsf{P}_{\varnothing-\varnothing} \otimes \mathsf{P}_{\alpha\beta\alpha}$ and we let

$$\mathsf{t}_i(q) = \mathsf{T}^{-1}(q,\varepsilon_i) \quad \mathsf{t}_{i+1}(q) = \mathsf{T}^{-1}(b_\alpha+q,\varepsilon_{i+1}) \quad \mathsf{t}_{i+2}(q) = \mathsf{T}^{-1}(b_{\alpha\beta}+q,\varepsilon_{i+2})$$
$$\mathsf{b}_i(q) = \mathsf{B}^{-1}(q,\varepsilon_i) \quad \mathsf{b}_{i+1}(q) = \mathsf{B}^{-1}(b_\alpha+q,\varepsilon_{i+1}) \quad \mathsf{b}_{i+2}(q) = \mathsf{B}^{-1}(b_{\alpha\beta}+q,\varepsilon_{i+2})$$

for $0 \leqslant q \leqslant b_\alpha + 1$. We have that

$$\mathsf{t}_i(q) < \mathsf{t}_i(q+1) < \mathsf{t}_{i+2}(q) < \mathsf{t}_{i+2}(q+1) < \mathsf{t}_{i+1}(q) < \mathsf{t}_{i+1}(q+1)$$
$$\mathsf{b}_i(q) > \mathsf{b}_i(q+1) > \mathsf{b}_{i+2}(q) > \mathsf{b}_{i+2}(q+1) > \mathsf{b}_{i+1}(q) > \mathsf{b}_{i+1}(q+1)$$

for $1 \leqslant q \leqslant b_\alpha$ and

$$\mathsf{t}_i(1) < \mathsf{t}_{i+2}(0) < \mathsf{t}_{i+1}(1) \qquad \mathsf{t}_i(b_\alpha) < \mathsf{t}_{i+2}(b_\alpha+1) < \mathsf{t}_{i+1}(b_\alpha)$$
$$\mathsf{b}_i(1) > \mathsf{b}_{i+2}(0) > \mathsf{b}_{i+1}(1) \qquad \mathsf{b}_i(b_\alpha) > \mathsf{b}_{i+2}(b_\alpha+1) > \mathsf{b}_{i+1}(b_\alpha).$$

Thus the subexpression $\psi_{\underline{w}}$ is the nib truncation of a quasi-$(b_\alpha+2)$-expression for $w = (13) \in \mathfrak{S}_3$, which is independent of the choice of expression by Corollary 4.10. Thus the result follows. $\square$

We are now free to define the KLR-hexagon to be the element

$$\mathsf{hex}^{\alpha\beta\alpha}_{\beta\alpha\beta} := \Upsilon^{\mathsf{P}_{\alpha\beta\alpha}}_{\mathsf{P}_{\varnothing-\varnothing}\otimes\mathsf{P}_{\beta\alpha\beta}} \qquad \text{or} \qquad \mathsf{hex}^{\alpha\beta\alpha}_{\beta\alpha\beta} := \Upsilon^{\mathsf{P}_{\varnothing-\varnothing}\otimes\mathsf{P}_{\alpha\beta\alpha}}_{\mathsf{P}_{\beta\alpha\beta}}$$

for $b_\alpha \geqslant b_\beta$ or $b_\alpha \leqslant b_\beta$ respectively, which are independent of the choice of reduced expressions. See Fig. 23 for an example. We wish to inductively pass between the paths $\mathsf{P}_{\alpha\beta\alpha}$ and $\mathsf{P}_{\varnothing-\varnothing} \otimes \mathsf{P}_{\beta\alpha\beta}$ by means of a visual timeline (as in Fig. 22). This allows us to factorise the KLR-hexagon and to simplify our proofs later on. First assume that $b_\alpha \geqslant b_\beta$. We define $\mathsf{H}_{q,\alpha\beta\alpha}$ to be the path

$$\begin{cases} \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} \otimes_\alpha \mathsf{M}_{i+1}^{b_\beta-q} \boxtimes \mathsf{P}_{i+2}^{b_\beta} \otimes_\beta \mathsf{M}_{i,i+2}^{q} \boxtimes \mathsf{M}_i^{b_\alpha-q} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} & 0 \leqslant q \leqslant b_\beta \\ \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_{i+1}^{b_{\alpha\beta}-q} \otimes_\alpha \mathsf{P}_{i+2}^{b_\beta} \otimes_\beta \mathsf{M}_{i,i+2}^{b_\beta} \boxtimes \mathsf{M}_i^{b_\alpha-q} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} & b_\beta \leqslant q \leqslant b_\alpha \\ \mathsf{P}_{\varnothing} \boxtimes \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_{i+1}^{b_{\alpha\beta}-q} \otimes_\alpha \mathsf{P}_{i+2}^{b_\beta} \otimes_\beta \mathsf{M}_{i,i+2}^{b_\beta} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} \boxtimes \mathsf{P}_i^{q-b_\alpha} & b_\alpha \leqslant q \leqslant b_{\alpha\beta} \end{cases}$$

This is demonstrated in the first 5 paths in Fig. 22. We now come from the opposite side to meet in the middle. We define $\mathsf{H}_{q,\beta\alpha\beta}$ to be the path

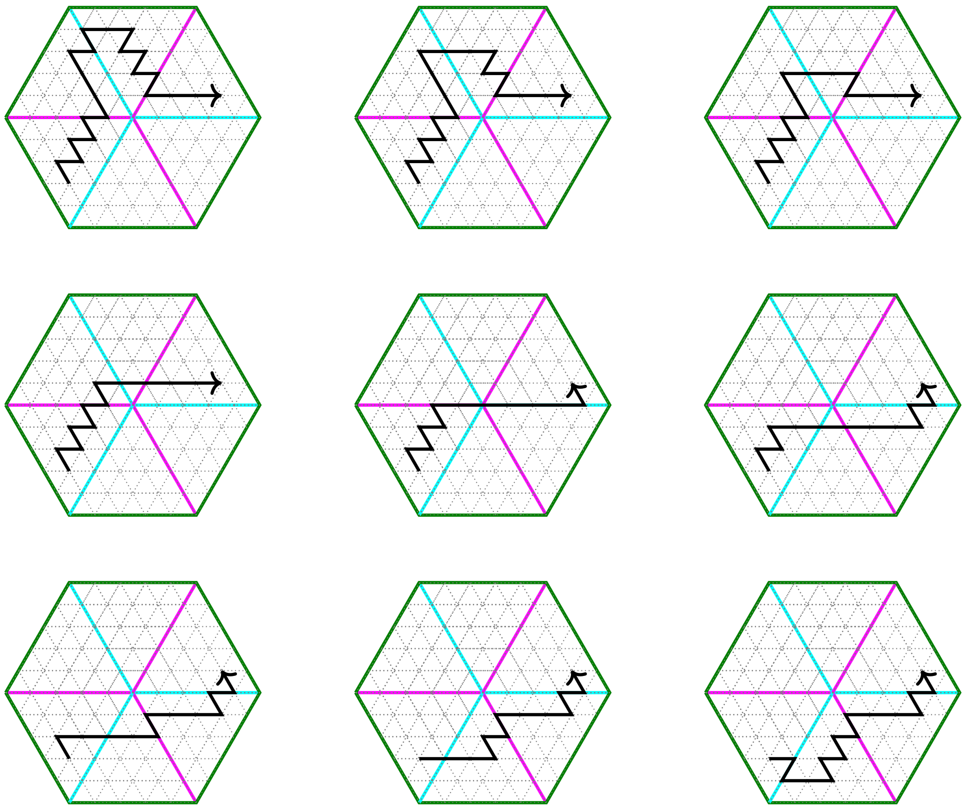


Fig. 22. An example of a timeline for the KLR hexagon. Mutating from $\mathsf{P}_{\alpha\beta\alpha}$ to $\mathsf{P}_{\varnothing-\varnothing}\otimes\mathsf{P}_{\beta\alpha\beta}$ for $b_\alpha \geqslant b_\beta$ (again we do not picture the determinant paths). Steps in the procedure should be read from left-to-right along successive rows (the paths are $\mathsf{H}_{0,\alpha\beta\alpha}$, $\mathsf{H}_{1,\alpha\beta\alpha}$, $\mathsf{H}_{2,\alpha\beta\alpha}$, $\mathsf{H}_{3,\alpha\beta\alpha}$, $\mathsf{H}_{4,\alpha\beta\alpha}$, $\mathsf{H}_{4,\alpha\beta\alpha} = \mathsf{P}_{\emptyset}\boxtimes\mathsf{H}_{4,\beta\alpha\beta}$, $\mathsf{H}_{3,\beta\alpha\beta}$, $\mathsf{H}_{2,\beta\alpha\beta}$, $\mathsf{H}_{1,\beta\alpha\beta}$, $\mathsf{H}_{0,\beta\alpha\beta}$).

$$\begin{cases} \mathsf{P}_{q\emptyset}\boxtimes\mathsf{M}_{i+1}^{b_\beta-q}\boxtimes\mathsf{M}_{i,i+1}^{q}\boxtimes\mathsf{P}_{i+2}^{b_\beta}\otimes_\beta\mathsf{M}_{i}^{b_\alpha-q}\boxtimes\mathsf{P}_{i+1}^{b_\alpha}\otimes_\alpha\mathsf{M}_{i+1}^{b_\beta}\boxtimes\mathsf{P}_{i+2}^{b_\beta} & 0\leqslant q\leqslant b_\beta \\ \mathsf{P}_{\varnothing}\boxtimes\mathsf{M}_{i}^{q-b_\beta}\boxtimes\mathsf{M}_{i,i+1}^{b_\beta}\boxtimes\mathsf{P}_{i+2}^{b_\beta}\otimes_\beta\mathsf{M}_{i}^{b_\alpha-q}\boxtimes\mathsf{P}_{i+1}^{b_\alpha}\otimes_\alpha\mathsf{M}_{i+1}^{b_\beta}\boxtimes\mathsf{P}_{i+2}^{b_\beta} & b_\beta\leqslant q\leqslant b_\alpha \\ \mathsf{P}_{\varnothing}\boxtimes\mathsf{M}_{i}^{q-b_\beta}\boxtimes\mathsf{P}_{i+2}^{b_\beta}\otimes_\beta\mathsf{M}_{i,i+2}^{q-b_\alpha}\boxtimes\mathsf{P}_{i+1}^{b_\alpha}\otimes_\alpha\mathsf{M}_{i+1}^{b_{\alpha\beta}-q}\boxtimes\mathsf{P}_{i+2}^{b_\beta} & b_\alpha\leqslant q\leqslant b_{\alpha\beta} \end{cases}$$

This is demonstrated in the final 5 paths in Fig. 22. While the definitions seem technical, one can intuitively think of this process as "flattening" the path layer-by-layer by means of the timeline depicted in Fig. 22. We see that $\mathsf{H}_{b_{\alpha\beta},\alpha\beta\alpha} = \mathsf{P}_{\varnothing-\varnothing}\boxtimes\mathsf{H}_{b_{\alpha\beta},\beta\alpha\beta}$.

We now assume that $b_\alpha \leqslant b_\beta$. We define $\mathsf{H}_{q,\alpha\beta\alpha}$ to be the path

$$\begin{cases} \mathsf{P}_{q\emptyset}\boxtimes\mathsf{M}_{i}^{b_\alpha}\boxtimes\mathsf{P}_{i+1}^{b_\alpha}\otimes_\alpha\mathsf{M}_{i+1}^{b_\beta-q}\boxtimes\mathsf{P}_{i+2}^{b_\beta}\otimes_\beta\mathsf{M}_{i,i+2}^{q}\boxtimes\mathsf{M}_{i}^{b_\alpha-q}\boxtimes\mathsf{P}_{i+1}^{b_\alpha} & 0\leqslant q\leqslant b_\alpha \\ \mathsf{P}_{\varnothing}\boxtimes\mathsf{M}_{i}^{b_\alpha}\boxtimes\mathsf{P}_{i+1}^{b_\alpha}\otimes_\alpha\mathsf{M}_{i+1}^{b_\beta-q}\boxtimes\mathsf{M}_{i,i+1}^{q-b_\alpha}\boxtimes\mathsf{P}_{i+2}^{b_\beta}\otimes_\beta\mathsf{M}_{i,i+2}^{b_\alpha}\boxtimes\mathsf{P}_{i+1}^{q}\boxtimes\mathsf{P}_{i}^{q-b_\alpha} & b_\alpha\leqslant q\leqslant b_\beta \\ \mathsf{P}_{\varnothing}\boxtimes\mathsf{M}_{i}^{b_\alpha}\boxtimes\mathsf{P}_{i+1}^{b_{\alpha\beta}-q}\otimes_\alpha\mathsf{M}_{i,i+1}^{b_\beta-b_\alpha}\boxtimes\mathsf{P}_{i+2}^{b_\beta}\otimes_\beta\mathsf{M}_{i,i+2}^{b_\alpha}\boxtimes\mathsf{P}_{i+1}^{b_\alpha}\boxtimes\mathsf{P}_{i}^{q-b_\alpha} & b_\beta\leqslant q\leqslant b_{\alpha\beta} \end{cases}$$

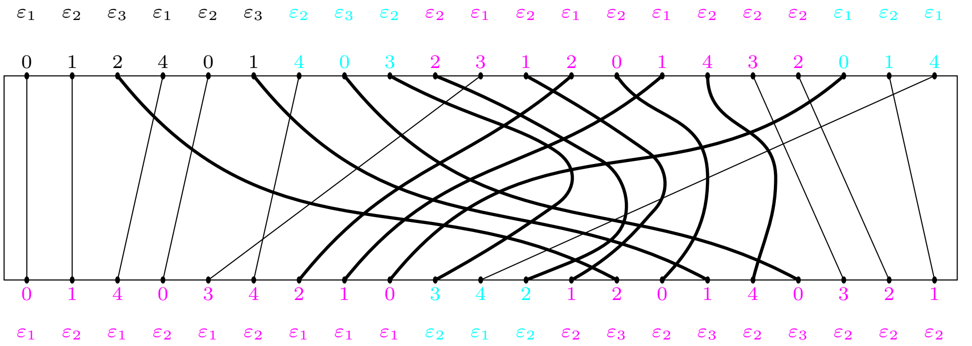


Fig. 23. Let $h = 3$, $\ell = 1$, $e = 5$ and $\alpha = \varepsilon_3 - \varepsilon_1$, $\beta = \varepsilon_1 - \varepsilon_2$. We depict the element $\mathsf{hex}^{\beta\alpha\beta}_{\alpha\beta\alpha}$ and highlight the dilated word $\mathrm{nib}(1,3)_5$ in bold. The reader should compare the 11 highlighted strands with the diagram from $\mathfrak{S}_1 1$ depicted in Fig. 15. (We have drawn all bad-crossing so that they bi-pass on the right.)

We now come from the opposite side to meet in the middle. We define $\mathsf{H}_{q,\beta\alpha\beta}$ to be the path

$$\begin{cases} \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}^{b_\beta - q}_{i+1} \boxtimes \mathsf{M}^{q}_{i,i+1} \boxtimes \mathsf{P}^{b_\beta}_{i+2} \otimes_\beta \mathsf{M}^{b_\alpha - q}_{i} \boxtimes \mathsf{P}^{b_\alpha}_{i+1} \otimes_\alpha \mathsf{M}^{b_\beta}_{i+1} \boxtimes \mathsf{P}^{b_\beta}_{i+2} & 0 \leqslant q \leqslant b_\alpha \\ \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}^{b_\beta - q}_{i+1} \boxtimes \mathsf{M}^{q}_{i,i+1} \boxtimes \mathsf{P}^{b_\beta}_{i+2} \otimes_\beta \mathsf{P}^{b_\alpha}_{i+1} \otimes_\alpha \mathsf{M}^{b_{\alpha\beta} - q}_{i+1} \boxtimes \mathsf{P}^{b_\beta}_{i+2} & b_\alpha \leqslant q \leqslant b_\beta \\ \mathsf{P}_{\varnothing} \boxtimes \mathsf{M}^{q - b_\beta}_{i} \boxtimes \mathsf{M}^{b_{\beta\beta} - q}_{i,i+1} \boxtimes \mathsf{P}^{b_\beta}_{i+2} \otimes_\beta \mathsf{P}^{b_\alpha}_{i+1} \otimes_\alpha \mathsf{M}^{b_{\alpha\beta} - q}_{i+1} \boxtimes \mathsf{P}^{b_\beta}_{i+2} & b_\beta \leqslant q \leqslant b_{\alpha\beta} \end{cases}$$

With our paths in place, this allows us to define

$$\mathsf{hex}^{\alpha\beta\alpha}(q) = \Upsilon^{\mathsf{H}_{q,\alpha\beta\alpha}}_{\mathsf{H}_{q+1,\alpha\beta\alpha}} \qquad \mathsf{hex}_{\beta\alpha\beta}(q) = \Upsilon^{\mathsf{H}_{q+1,\beta\alpha\beta}}_{\mathsf{H}_{q,\beta\alpha\beta}}$$

and we set

$$\mathsf{hex}^{\alpha\beta\alpha} = \prod_{b_{\alpha\beta} > q \geqslant 0} \mathsf{hex}^{\alpha\beta\alpha}(q) \qquad \mathsf{hex}_{\beta\alpha\beta} = \prod_{0 \leqslant q \leqslant b_{\alpha\beta}} \mathsf{hex}_{\beta\alpha\beta}(q)$$

which allows us to factorise the hexagon generators as follows

$$\mathsf{hex}^{\alpha\beta\alpha}_{\beta\alpha\beta} = \begin{cases} \mathsf{hex}^{\alpha\beta\alpha}(e_{\mathsf{P}_{\varnothing - \varnothing}} \otimes \mathsf{hex}_{\beta\alpha\beta}) & \text{for } b_\alpha \geqslant b_\beta \\ (e_{\mathsf{P}_{\varnothing - \varnothing}} \otimes \mathsf{hex}^{\alpha\beta\alpha})\mathsf{hex}_{\beta\alpha\beta} & \text{for } b_\alpha \leqslant b_\beta \end{cases}$$

and, finally, we define

$$\mathsf{hex}^{\varnothing\alpha\beta\alpha}_{\varnothing\beta\alpha\beta} = \begin{cases} e_{\mathsf{P}_\varnothing} \otimes \mathsf{hex}^{\alpha\beta\alpha}_{\beta\alpha\beta} & \text{if } b_\alpha \leq b_\beta \\ e_{\mathsf{P}_\varnothing} \otimes \mathsf{hex}^{\alpha\beta\alpha}_{\beta\alpha\beta} & \text{if } b_\alpha \geqslant b_\beta \end{cases}$$

the latter notation will be useful when we wish to consider products of such hexagons without assuming $b_\alpha \geqslant b_\beta$ or vice versa. Finally, the following shorthand will come in useful when addressing some of the relations in Section 6. Recall that adjustment is invertible. With this in mind, we set

$$\mathsf{hex}^{\underline{v}\beta\alpha\beta\underline{w}\varnothing}_{\underline{v}\alpha\beta\alpha\underline{w}\varnothing} = \mathsf{adj}^{\underline{v}\beta\alpha\beta\underline{w}\varnothing}_{\underline{v}\varnothing\beta\alpha\beta\underline{w}}(e_{\mathsf{P}_{\underline{v}}} \otimes \mathsf{hex}^{\varnothing\beta\alpha\beta}_{\varnothing\alpha\beta\alpha} \otimes e_{\mathsf{P}_{\underline{w}}})\mathsf{adj}^{\underline{v}\varnothing\alpha\beta\alpha\underline{w}}_{\underline{v}\alpha\beta\alpha\underline{w}\varnothing} = \Upsilon^{\underline{v}\beta\alpha\beta\underline{w}\varnothing}_{\underline{v}\alpha\beta\alpha\underline{w}\varnothing}$$

where the second equality follows by removing the resulting double-crossings using Proposition 4.4 in each case. Independence of the reduced expression follows from residue-commutativity of adjustment. Alternatively, the reader is invited to make minor modifications to the proof of Proposition 5.12.

5.6. The commuting strands diagram

Let $\gamma, \beta \in \Pi$ be roots labelling commuting reflections (in terms of convention 2.32, this is equivalent to $|k-j| > 1$). We wish to understand the morphism relating the paths $\mathsf{P}_\gamma \otimes \mathsf{P}_\beta$ to $\mathsf{P}_\beta \otimes \mathsf{P}_\gamma$. We suppose without loss of generality that $b_\gamma \geqslant b_\beta$.

Proposition 5.13. *The element $\psi^{\mathsf{P}_\gamma\otimes\mathsf{P}_\beta}_{\mathsf{P}_\beta\otimes\mathsf{P}_\gamma}$ is independent of the choice of reduced expression*

Proof. There are precisely $b_{\gamma\beta}$ like-labelled crossings. The first b_γ of these connect the $\mathsf{P}^{-1}_{\gamma\beta}(q, \varepsilon_j)$th and $\mathsf{P}^{-1}_{\gamma\beta}(b_\gamma + q, \varepsilon_{j+1})$th northern vertices to the $\mathsf{P}^{-1}_{\beta\gamma}(q, \varepsilon_j)$th and $\mathsf{P}^{-1}_{\beta\gamma}(b_\beta + q, \varepsilon_{j+1})$th southern vertices for $1 \leqslant q \leqslant b_\gamma$. The latter b_β of these connect the $\mathsf{P}^{-1}_{\gamma\beta}(b_\beta + q, \varepsilon_{k+1})$th and $\mathsf{P}^{-1}_{\gamma\beta}(q, \varepsilon_k)$th northern vertices to the $\mathsf{P}^{-1}_{\beta\gamma}(b_\beta + q, \varepsilon_{k+1})$th and $\mathsf{P}^{-1}_{\beta\gamma}(q, \varepsilon_k)$th southern vertices for $1 \leqslant q \leqslant b_\gamma$.

For $k \neq h$ (respectively $k = h$) each of the first $1 \leqslant q \leqslant b_\gamma$ (respectively $1 < q \leqslant b_\gamma$) like-labelled crossings forms a braid with precisely one other strand, namely that connecting the $\mathsf{P}^{-1}_{\gamma\beta}(b_\beta + q, \varepsilon_{k+1})$th top vertex to the $\mathsf{P}^{-1}_{\beta\gamma}(b_\beta + q, \varepsilon_{k+1})$th bottom vertex for $1 \leqslant q \leqslant b_\gamma$ (respectively $1 \leqslant q < b_\gamma$). This strand is of non-adjacent residue (by our assumption that γ and β label commuting reflections). The latter b_β cases can be treated similarly.

Thus each of the braids involving a like-labelled crossing (either totalling $b_{\beta\gamma}$ if $k, j \neq h$ or $b_{\beta\gamma} - 1$ otherwise) is residue-commutative. Thus $\psi^{\mathsf{P}_{\gamma\beta}}_{\mathsf{P}_{\beta\gamma}}$ is residue commutative and the result follows. $\square$

Thus we are free to define the KLR-commutator to be the element

$$\mathsf{com}^{\gamma\beta}_{\beta\gamma} := \Upsilon^{\mathsf{P}_\gamma\otimes\mathsf{P}_\beta}_{\mathsf{P}_\beta\otimes\mathsf{P}_\gamma}$$

which is independent of the choice of reduced expression. We wish to inductively pass between the paths $\mathsf{P}_\gamma \otimes \mathsf{P}_\beta$ and $\mathsf{P}_\beta \otimes \mathsf{P}_\gamma$ by means of a visual timeline (as in Fig. 24).

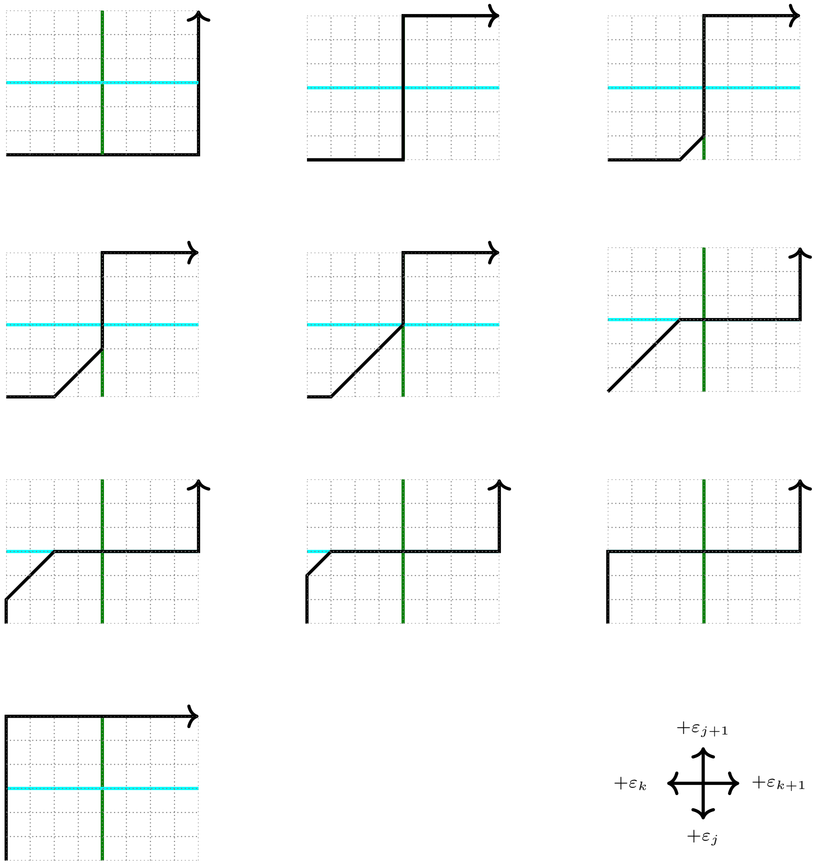


Fig. 24. An example timeline for the KLR commutator. We mutate from $\mathsf{P}^{\gamma\beta}$ to $\mathsf{P}_{\beta\gamma}$ for $b_\gamma = 4, b_\beta = 3$. Reading from left-to-right along successive rows the paths are $\mathsf{P}^{-1,\gamma\beta}$, $\mathsf{P}^{0,\gamma\beta}$, $\mathsf{P}^{1,\gamma\beta}$, $\mathsf{P}^{2,\gamma\beta}$, $\mathsf{P}^{3,\gamma\beta} = \mathsf{P}_{2,\beta\gamma}$, $\mathsf{P}_{1,\beta\gamma}$, $\mathsf{P}_{0,\beta\gamma}$, $\mathsf{P}_{-1,\beta\gamma}$. We draw paths in the projection onto $\mathbb{R}\{\varepsilon_j + \varepsilon_{j+1}, \varepsilon_k + \varepsilon_{k+1}\}$.

We define

$$\mathsf{C}^{q,\gamma\beta} = \begin{cases} \mathsf{M}_k^{b_\gamma} \boxtimes \mathsf{P}_{k+1}^{b_\gamma} \otimes_\gamma \mathsf{M}_j^{b_\beta} \boxtimes \mathsf{P}_{j+1}^{b_\beta} & \text{for } q = -1 \\ \mathsf{M}_k^{b_\gamma} \otimes_\gamma \mathsf{M}_j^{b_\beta} \boxtimes \mathsf{P}_{j+1}^{b_\beta} \boxtimes \mathsf{P}_k^{b_\gamma} & \text{for } q = 0 \\ \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_k^{b_\gamma - q} \otimes_\gamma \mathsf{M}_{k+1,j}^{q} \boxtimes \mathsf{M}_j^{b_\beta - q} \boxtimes \mathsf{P}_{j+1}^{b_\beta} \boxtimes \mathsf{P}_k^{b_\gamma} & \text{for } 0 < q \leqslant b_\beta \end{cases}$$

$$\mathsf{C}_{q,\beta\gamma} = \begin{cases} \mathsf{P}_{\emptyset} \otimes_\beta \mathsf{M}_k^{q - b_\beta} \boxtimes \mathsf{M}_{k,j+1}^{b_\beta} \boxtimes \mathsf{M}_k^{b_\gamma - q} \boxtimes \mathsf{P}_j^{b_\beta} \boxtimes \mathsf{P}_{k+1}^{b_\gamma} & \text{for } b_\gamma \geqslant q > b_\beta \\ \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_j^{b_\beta - q} \otimes_\beta \mathsf{M}_{k,j+1}^{q} \boxtimes \mathsf{M}_k^{b_\gamma - q} \boxtimes \mathsf{P}_j^{b_\beta} \boxtimes \mathsf{P}_{k+1}^{b_\gamma} & \text{for } b_\beta \geqslant q > 0 \\ \mathsf{M}_j^{b_\beta} \otimes_\beta \mathsf{M}_k^{b_\gamma} \boxtimes \mathsf{P}_j^{b_\beta} \boxtimes \mathsf{P}_{k+1}^{b_\gamma} & \text{for } q = 0 \\ \mathsf{M}_j^{b_\beta} \boxtimes \mathsf{P}_{k+1}^{b_\beta} \otimes_\beta \mathsf{M}_k^{b_\gamma} \boxtimes \mathsf{P}_{k+1}^{b_\gamma} & \text{for } q = -1 \end{cases}$$

and we note that $\mathsf{C}_{b_\gamma,\beta\gamma} = \mathsf{C}^{b_\beta,\gamma\beta}$ (to see this, note that the definition of the former contains a tensor product $\otimes_\gamma$ and the latter contains a tensor product $\otimes_\beta$ and this explains the differences in the subscripts). We now define

$$\mathsf{com}^{q,\gamma\beta} = \Upsilon^{\mathsf{C}^{q,\gamma\beta}}_{\mathsf{C}^{q+1,\gamma\beta}} \qquad \mathsf{com}_{q,\beta\gamma} = \Upsilon^{\mathsf{C}_{q+1,\beta\gamma}}_{\mathsf{C}_{q,\beta\gamma}}.$$

This allows us to factorise

$$\mathsf{com}^{\gamma\beta}_{\beta\gamma} = \mathsf{com}^{\gamma\beta}\mathsf{com}_{\beta\gamma} \qquad \mathsf{com}^{\gamma\beta} = \prod_{-1\leqslant q<b_\beta} \mathsf{com}^{q,\gamma\beta} \qquad \mathsf{com}_{\beta\gamma} = \prod_{b_\gamma>q\geqslant -1} \mathsf{com}_{q,\beta\gamma}.$$

The following notation will come in useful in Section 6

$$\mathsf{com}^{\underline{v}\gamma\beta\underline{w}}_{\underline{v}\beta\gamma\underline{w}} = e_{\mathsf{P}_{\underline{v}}} \otimes \mathsf{com}^{\underline{v}\gamma\beta\underline{w}}_{\underline{v}\beta\gamma\underline{w}} \otimes e_{\mathsf{P}_{\underline{w}}}.$$

5.7. The isomorphism

Finally, we now explicitly state the isomorphism. Our notation has been chosen so as to make this almost tautological at this point. We suppose that α and β (respectively β and γ) label non-commuting (respectively commuting) reflections. We define

$$\Psi : \mathscr{S}^{\rm br}_{\underline{h}}(n,\sigma) \longrightarrow \mathsf{f}_{n,\sigma}\left(\mathcal{H}^\sigma_n/\mathcal{H}^\sigma_n \mathsf{y}_{\underline{h}}\mathcal{H}^\sigma_n\right)\mathsf{f}_{n,\sigma} \tag{5.4}$$

to be the map defined on generators (and extended using vertical concatenation and contextualised horizontal concatenation) as follows

$$\Psi(1_\alpha) = e_{\mathsf{P}_\alpha} \quad \Psi(1_\emptyset) = e_{\mathsf{P}_\emptyset} \quad \Psi(1^{\emptyset\alpha}_{\alpha\emptyset}) = \mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset} \quad \Psi(\mathsf{SPOT}^{\emptyset}_{\alpha}) = \mathsf{spot}^{\emptyset}_{\alpha}$$
$$\Psi(\mathsf{FORK}^{\emptyset\alpha}_{\alpha\alpha}) = \mathsf{fork}^{\emptyset\alpha}_{\alpha\alpha} \quad \Psi(\mathsf{HEX}^{\beta\alpha\beta}_{\alpha\beta\alpha}) = \mathsf{hex}^{\beta\alpha\beta}_{\alpha\beta\alpha} \quad \Psi(\mathsf{COM}^{\gamma\beta}_{\beta\gamma}) = \mathsf{com}^{\gamma\beta}_{\beta\gamma}$$

and we extend this to the flips of these diagrams through their horizontal axes.

Remark 5.14. We note that our use of *contextualised* horizontal concatenation implies that equation (5.2) holds (see also Example 5.1).

6. Recasting the diagrammatic Bott–Samelson relations in the quiver Hecke algebra

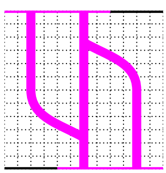
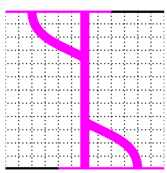
The purpose of this section is to recast Elias–Williamson's diagrammatic relations of Subsection 3.1 in the setting of the quiver Hecke algebra, thus verifying that the map Ψ_n is indeed a (graded) $\mathbb{Z}$-algebra homomorphism. We have already provided timelines which discretise each Soergel generator (which we think of as a continuous morphism between paths with a unique singularity, where the strands cross). We will verify most of the Soergel relations via a similar discretisation process which factorises the Soergel relation into simpler steps; we again record this is a visual timeline. We check each relation in turn, but leave it as an exercise for the reader to verify the flips of these relations through their vertical axes (the flips through horizontal axes follow immediately from the duality, $*$). We continue with the notations of Convention 2.32. Our relations fall into three categories:

- Products involving only hexagons, commutators, and adjustment generators. Simplifying such products is an inductive process. At each step, one simplifies a non-minimal expression (in the concatenated diagram) to a minimal one *without changing the underlying permutation*. This typically involves a single “distinguished” strand which double-crosses some other strands; these double-crossings can be undone using Proposition 4.4. (This preserves the parity of like-labelled crossings.)
- Products involving a fork or spot generator. Such generators reflect one of the indexing paths in an irreversible manner. Simplifying such products is an inductive process. At each step, one rewrites a *single pair of crossing strands* (in the concatenated permutation) which *do not respect step-labels of the reflected paths*. By undoing this crossing using relation (R3), we obtain the scalar -1 times a new diagram *which does* respect the new step-labelling for the reflected paths. (Thus changing the parity of like-labelled crossings and also changing the scalar ± 1.)
- Doubly spotted Soergel diagrams (such as the Demazure relations) for which we argue separately.

In each of the former two cases, we will decorate the top and bottom of the concatenated diagram with paths T and B (which we define case-by-case) and use the step-labelling from these paths to keep track of crossings of strands in the diagram.

6.1. *The double fork*

This leftmost relation in (S1) is incredibly simple to verify, and so there is no need to record this in a timeline. For $\alpha \in \Pi$, we must verify that

Ψ () $= \Psi$ () (6.1)

Thus we need to check that

$$\left(e_{\mathsf{P}_{\alpha}} \otimes \mathsf{fork}^{\alpha\varnothing}_{\alpha\alpha}\right) \circ \left(\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha} \otimes e_{\mathsf{P}_{\alpha}}\right) = \left(\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha} \otimes e_{\mathsf{P}_{\varnothing}}\right) \circ \left(e_{\mathsf{P}_{\varnothing}} \otimes \mathsf{fork}^{\alpha\varnothing}_{\alpha\alpha}\right). \tag{6.2}$$

The permutation underlying $e_{\mathsf{P}_{\alpha}} \otimes \mathsf{fork}^{\alpha\varnothing}_{\alpha\alpha}$ is the element $w^{\mathsf{T}}_{\mathsf{B}}$ indexed by the pair of paths

$$\mathsf{T} = \mathsf{P}_{\alpha} \otimes \mathsf{P}_{\alpha} \otimes \mathsf{P}_{\varnothing} \quad \text{and} \quad \mathsf{B} = \mathsf{P}_{\alpha} \otimes \mathsf{P}^{\flat}_{\alpha} \otimes \mathsf{P}_{\alpha}$$

which differ only by permuting the final $(b_{\alpha}h + b_{\alpha})$ steps. The permutation underlying $\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha} \otimes e_{\mathsf{P}_{\alpha}}$ is the element $w^{\mathsf{T}'}_{\mathsf{B}'}$ indexed by the pair of paths

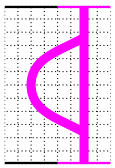
$$\mathsf{T}' = \mathsf{P}_{\alpha} \otimes \mathsf{P}^{\flat}_{\alpha} \otimes \mathsf{P}_{\alpha} \quad \text{and} \quad \mathsf{B}' = \mathsf{P}_{\varnothing} \otimes \mathsf{P}_{\alpha} \otimes \mathsf{P}_{\alpha},$$

which differ only by permuting the first $(b_{\alpha\alpha}h - b_{\alpha})$ steps. These elements of $\mathfrak{S}_{3b_{\alpha}h}$ commute as they permute disjoint subsets of $1, \dots, 3b_{\alpha}h$. Thus the elements $\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha} \otimes e_{\mathsf{P}_{\alpha}}$ and $e_{\mathsf{P}_{\alpha}} \otimes \mathsf{fork}^{\alpha\alpha}_{\alpha\varnothing}$ commute by relation (R2) (and the result follows immediately).

Remark 6.1. The reader might wonder why the element $w^{\mathsf{T}}_{\mathsf{B}}$ appears to permute a greater number of strands than $w^{\mathsf{T}'}_{\mathsf{B}'}$. This is because our distinguished choice of P_{α} has a total of $(b_{\alpha}h - b_{\alpha})$ steps below (or on) the α-hyperplane and b_{α} steps above the hyperplane.

6.2. The one-colour zero relation

We now consider the rightmost relation in (S1). For $\alpha \in \Pi$, we must verify that

$\Psi\Bigg($  $\Bigg) = \mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} \circ \mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha} = 0$ (6.3)

For $b_{\alpha} > q \geqslant 1$ the paths $\mathsf{F}_{q,\varnothing\alpha}$ and $\mathsf{F}_{q-1,\varnothing\alpha}$ are concatenates of a single α-crossing path and a single α-bouncing path. By Proposition 4.4 we have that

$$\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}(q)e_{\mathsf{F}_{q-1,\varnothing\alpha}}\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}(q) = e_{\mathsf{F}_{q,\varnothing\alpha}}$$

for $1 \leqslant q < b_{\alpha}$. We apply this from the centre of the product $\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} \circ \mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}$ which is equal to

$$e_{\mathsf{P}^{\varnothing\alpha}}\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}(b_{\alpha}-1)\cdots\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}(0)e_{\mathsf{P}^{\alpha\alpha}} \circ e_{\mathsf{P}^{\alpha\alpha}}\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}(0)\cdots\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}(b_{\alpha}-1)e_{\mathsf{P}^{\varnothing\alpha}}$$

until we obtain

$$\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} \circ \mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha} = e_{\mathsf{P}_{\varnothing\alpha}}\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}(b_{\alpha}-1)e_{\mathsf{F}_{b_{\alpha}-1,\varnothing\alpha}}\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}(b_{\alpha}-1)e_{\mathsf{P}_{\varnothing\alpha}}. \tag{6.4}$$

This is illustrated in Fig. 25.

We cannot apply Proposition 4.4 to the pair of paths $\mathsf{F}_{b_{\alpha}-1,\varnothing\alpha}$ and $\mathsf{F}_{b_{\alpha}-2,\varnothing\alpha}$ because the former path passes through the α-hyperplane once, whereas the latter passes through/bounces the α-hyperplane twice. There is a pair of double-crossing r-strand (for some $r \in \mathbb{Z}/e\mathbb{Z}$) between the $\mathsf{P}^{-1}_{\varnothing\alpha}(b_{\alpha}, \varepsilon_i)$th and $\mathsf{P}^{-1}_{\varnothing\alpha}(b_{\alpha\alpha}, \varepsilon_{i+1})$th top and bottom vertices in the diagram

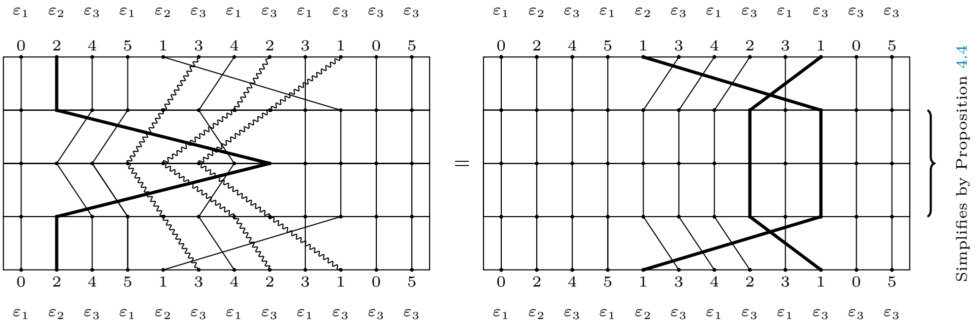


Fig. 25. Let $h = 1$, $\ell = 3$, $\sigma = (0, 2, 4)$ and $e = 6$. The lefthand-side is $\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}$; we apply Proposition 4.4 to undo the highlighted strands (compare the highlighted strands with the highlighted strands of the first diagram of Example 4.6). The thick double-crossing of strands in the rightmost diagram is zero by the first case of relation (R4) (after applying commutativity relations).

$$e_{\mathsf{P}_{\varnothing\alpha}}\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha}(b_\alpha - 1)e_{\mathsf{F}_{b_\alpha-1,\varnothing\alpha}}\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}(b_\alpha - 1)e_{\mathsf{P}_{\varnothing\alpha}}$$

This double-crossing of r-strands is not intersected by any strand of adjacent residue. Therefore the product is zero by the commutativity relations and the first case of relation (R4), as required.

6.3. Fork-spot contraction

We now consider the second relation depicted in (S1), namely

$$(\mathsf{spot}^{\varnothing}_{\alpha} \otimes e_{\mathsf{P}_\alpha}) \circ \mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha} = e_{\mathsf{P}_\varnothing} \otimes e_{\mathsf{P}_\alpha} \tag{6.5}$$

for $\alpha \in \Pi$. For $0 \leqslant q \leqslant b_\alpha$, we define the spot-fork path to be

$$\mathsf{FS}_{q,\alpha} = \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha} \otimes_\alpha \mathsf{P}_{i+1}^{b_\alpha - q} \otimes_\alpha \mathsf{M}_i^{b_\alpha - q} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} = \mathsf{P}^{q\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_i^{b_\alpha - q} \boxtimes \mathsf{M}_i^{b_\alpha - q} \boxtimes \mathsf{P}_{i+1}^{b_\alpha}$$

which is obtained from $\mathsf{F}_{q,\varnothing\alpha}$ by reflection by s_α (see Fig. 26). We note that $\mathsf{FS}_{b_\alpha,\alpha} = \mathsf{P}_\varnothing \otimes \mathsf{P}_\alpha$ and $\mathsf{FS}_{0,\alpha} = \mathsf{P}^\flat_\alpha \otimes \mathsf{P}_\alpha$. Thus these spot-fork paths allow us to iteratively prove equation (6.5), as we will see below.

The following example illustrates all of the important ideas in the proof of this relation (in particular, it illustrates our iterative approach using the fork-spot paths, examples of which are depicted in Fig. 26). These ideas will be used repeatedly when we consider (more complicated) relations in the remainder of this section.

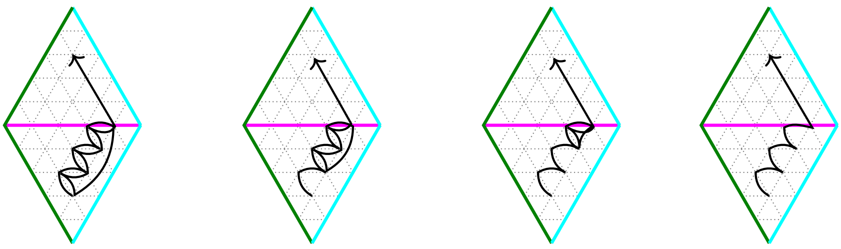
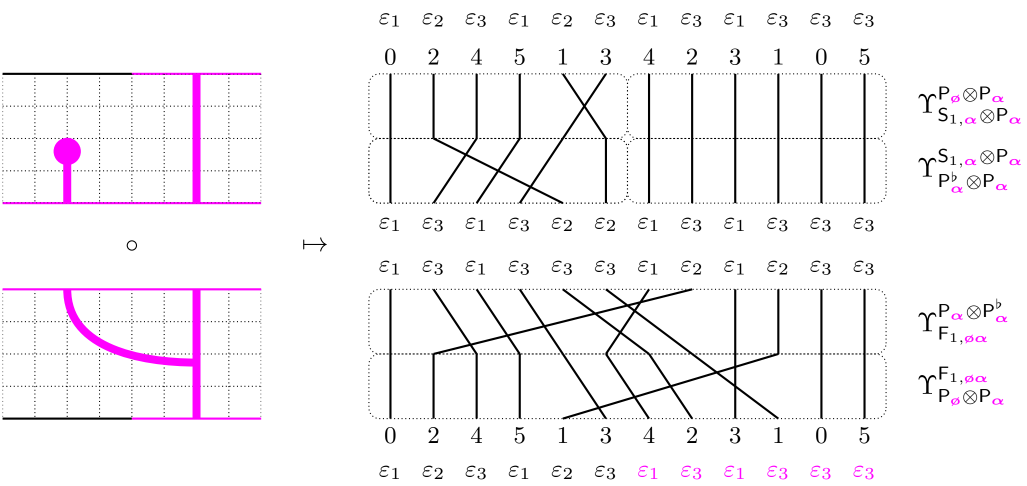


Fig. 26. An example of a timeline for the KLR spot-fork relation, with $\ell = 1$, $h = 3$, $e = 5$ and $\boldsymbol{\alpha} = \varepsilon_3 - \varepsilon_1$. From left to right we picture the paths $\mathsf{FS}_{0,\boldsymbol{\alpha}} = \mathsf{P}^{\flat}_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}$, $\mathsf{FS}_{1,\boldsymbol{\alpha}}$, $\mathsf{FS}_{2,\boldsymbol{\alpha}}$, $\mathsf{FS}_{3,\boldsymbol{\alpha}} = \mathsf{P}_{\varnothing} \otimes \mathsf{P}_{\boldsymbol{\alpha}}$.

Example 6.2. We set $\sigma = (0, 2, 4)$ and $e = 6$. We will consider the following product



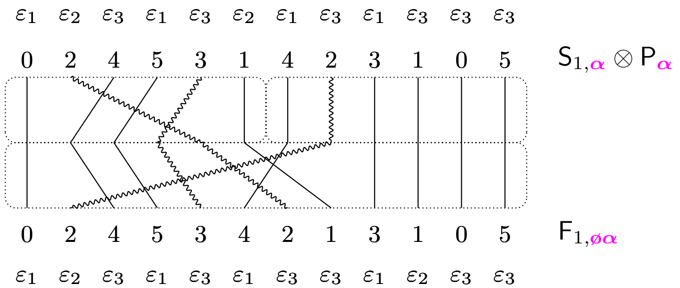
where we have emphasised the factorisation of spot and fork by recording the steps within these paths at top and bottom and the corresponding labelled $\Upsilon^{\mathsf{P}}_{\mathsf{Q}}$ elements for each layer of the righthand-side. We have also recorded the residues of paths (at the very top and bottom: $0, 2, 4, \dots$).

Notice that the path at the bottom of the spot-strand KLR-diagram is not the same as the path at the top of the fork KLR-diagram – however, the residue sequences are identical (simply trace through the residues on strands). We start at the middle of the product — that is we first compute

$$\Upsilon^{\mathsf{S}_{1,\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}}_{\mathsf{P}^{\flat}_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}} \circ \Upsilon^{\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}}_{\mathsf{F}_{1,\varnothing\boldsymbol{\alpha}}}$$

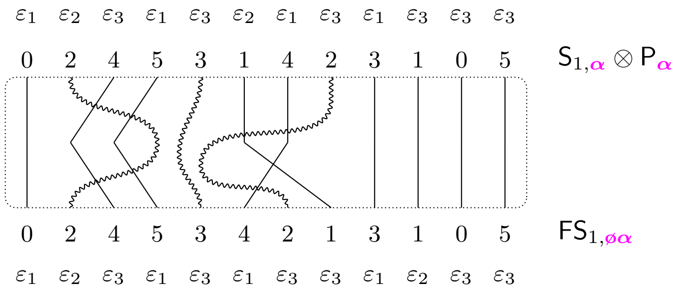
as follows: we first place the diagrams on top of each other recording the paths $\mathsf{S}_{1,\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}$ and $\mathsf{F}_{1,\varnothing\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}$ at the top and bottom of the diagram (notice that the permutation is not step-preserving) and we highlight the strands in the product which have crossings of non-zero degree

$$\Upsilon^{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\alpha}_{\mathsf{P}^\flat_\alpha\otimes\mathsf{P}_\alpha}\circ\Upsilon^{\mathsf{P}_\alpha\otimes\mathsf{P}_\alpha}_{\mathsf{F}_{1,\varnothing\alpha}} =$$



We apply relation (R5) to obtain two terms: the term in which we undo this highlighted braid and the other term which is equal to zero by Lemma 4.1. We relabel the bottom of the (non-zero) diagram by the folded fork path, $\mathsf{FS}_{1,\varnothing\alpha}$, and hence obtain

$$\Upsilon^{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\alpha}_{\mathsf{P}^\flat_\alpha\otimes\mathsf{P}_\alpha}\circ\Upsilon^{\mathsf{P}_\alpha\otimes\mathsf{P}_\alpha}_{\mathsf{F}_{1,\varnothing\alpha}} =$$



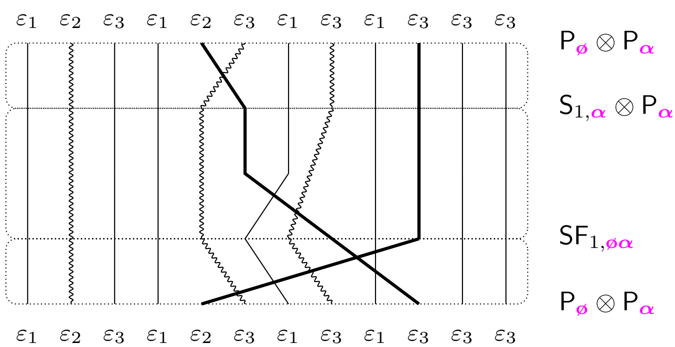
which we now observe is a step-preserving KLR diagram. We trivially undo the double-crossings in the above diagram (using Proposition 4.4) and hence obtain

$$\Upsilon^{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\alpha}_{\mathsf{P}^\flat_\alpha\otimes\mathsf{P}_\alpha}\circ\Upsilon^{\mathsf{P}_\alpha\otimes\mathsf{P}_\alpha}_{\mathsf{F}_{1,\varnothing\alpha}} = \Upsilon^{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\alpha}_{\mathsf{SF}_{1,\varnothing\alpha}}.$$

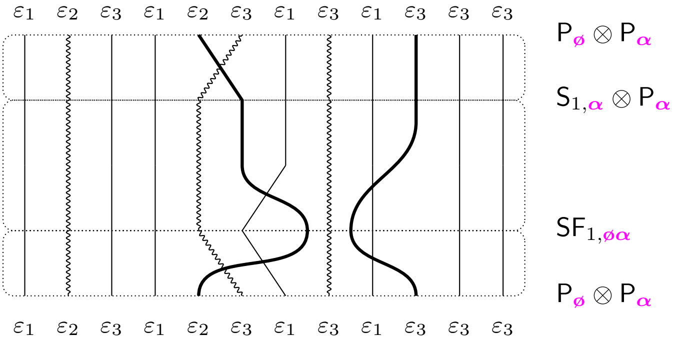
We now insert this back into the larger product (see also equation (6.6)) and hence obtain the following (not-step-preserving) KLR diagram of

$$(\mathsf{spot}^{\varnothing}_{\alpha}(1)\otimes e_{\mathsf{P}_\alpha})\circ\Upsilon^{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\alpha}_{\mathsf{SF}_{1,\varnothing\alpha}}\circ\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}(1)$$

which is equal to



where we have highlighted the wiggly strands from the previous step (to facilitate comparison) and we have emboldened the unique pair of crossing strands of the same residue. The rightmost wiggly strand and the pair of bold strands are the only strands have crossings of non-zero degree. We apply the same argument as above to undo this braid (we do not need to relabel the bottom of the diagram in this case, as the final fork-spot path is equal to $\mathsf{P}_{\varnothing} \otimes \mathsf{P}_{\alpha}$) and we hence obtain



which we now observe is a step-preserving KLR diagram. We trivially undo the double-crossings (using Proposition 4.4) and hence obtain

$$(\mathsf{spot}^{\varnothing}_{\alpha}(1) \otimes \mathsf{e}_{\mathsf{P}_{\alpha}}) \circ \Upsilon^{\mathsf{S}_{1,\alpha} \otimes \mathsf{P}_{\alpha}}_{\mathsf{SF}_{1,\varnothing\alpha}} \circ \mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}(1) = \mathsf{e}_{\mathsf{P}_{\varnothing} \otimes \mathsf{P}_{\alpha}}$$

as required.

What the above example illustrates is that we start at *the middle* of the product on the lefthand-side which is labelled by two *distinct paths* which have the same residue sequence, that is we start at the middle term in the product

$$(\mathsf{spot}^{\varnothing}_{\alpha} \otimes e_{\mathsf{P}_{\alpha}})\,(e_{\mathsf{P}^{\flat}_{\alpha} \otimes \mathsf{P}_{\alpha}} \circ e_{\mathsf{P}_{\alpha} \otimes \mathsf{P}^{\flat}_{\alpha}})\,\big(\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}\big)$$

where we note that $e_{\mathsf{P}^{\flat}_{\alpha} \otimes \mathsf{P}_{\alpha}} = e_{\mathsf{P}_{\alpha} \otimes \mathsf{P}^{\flat}_{\alpha}}$. Each iterative stage (of which there are two in Example 6.2) simply transforms a non-step-preserving KLR-permutation into a step-preserving one (by undoing all non-zero-degree crossings and relabelling). Thus the (seemingly technical) spot-fork paths become incredibly natural, as does their "time-line" construction (each stage corresponds to one KLR braid which we undo). Most beautifully of all: one should emphasise that the spot-fork path is simply the reflection of the fork path through the α-hyperplane (what else?!). This brings us to the general case:

Proposition 6.3. *For $\alpha \in \Pi$ and $0 \leqslant q < b_{\alpha}$ we have that*

$$(\mathsf{spot}^{\varnothing}_{\alpha}(q) \otimes e_{\mathsf{P}_{\alpha}}) \circ \Upsilon^{\mathsf{S}_{q,\alpha} \otimes \mathsf{P}_{\alpha}}_{\mathsf{FS}_{q,\alpha}} \circ \mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}(q) = \Upsilon^{\mathsf{S}_{q+1,\alpha} \otimes \mathsf{P}_{\alpha}}_{\mathsf{FS}_{q+1,\alpha}}. \tag{6.6}$$

Proof. We first note that the righthand-side is residue commutative (one can reindex the proof of Proposition 5.9). We decorate the top and bottom edges of the concatenated product on the lefthand-side of equation (6.6) with the tableaux $\mathsf{T}_q = \mathsf{S}_{q,\alpha} \otimes \mathsf{P}_\alpha$ and $\mathsf{B}_q = \mathsf{FS}_{q,\alpha}$ respectively for $0 \leqslant q < b_\alpha$. For each $0 \leqslant q < b_\alpha$, the product on the lefthand-side of equation (6.6) has a single pair of strands whose crossing if of degree -2: Namely, the strand Q_1 from connecting the $\mathsf{B}_q^{-1}(q+1, \varepsilon_i)$th bottom node to the $\mathsf{T}_q^{-1}(b_\alpha+q+1, \varepsilon_{i+1})$th top node and the strand Q_2 connecting the $\mathsf{B}_q^{-1}(b_\alpha+q+1, \varepsilon_{i+1})$th bottom node to the $\mathsf{T}_q^{-1}(q+1, \varepsilon_i)$th top node. The strands Q_1 and Q_2 are both of the same residue, $r_q \in \mathbb{Z}/e\mathbb{Z}$ say, and they cross each other exactly once. This crossing of r_q-strands is bi-passed on the left by the (r_q+1)-strand connecting the $\mathsf{B}_q^{-1}(b_\alpha+q, \varepsilon_i)$th bottom node to the $\mathsf{T}_q^{-1}(b_\alpha+q, \varepsilon_i)$th top node. We pull the (r_q-1)-strand through this crossing, using relation (R5). We hence obtain two terms: the term in which we undo this braid is equal to the righthand-side of equation (6.6) and the other term is equal to zero by Lemma 4.1. □

Equation (6.5) holds by iteratively applying Proposition 6.3 a total of b_α times, as in Example 6.2.

6.4. The spot and commutator

Let $\beta, \gamma \in \Pi$ label two commuting reflections, we now verify the leftmost relation in (S6), namely that

$$\mathsf{com}^{\gamma\beta}_{\beta\gamma}(\mathsf{spot}^{\beta}_{\varnothing} \otimes e_{\mathsf{P}_\gamma}) = \Upsilon^{\mathsf{P}_\gamma \otimes \mathsf{P}^\flat_\beta}_{\mathsf{P}_\varnothing \otimes \mathsf{P}_\gamma} = (e_{\mathsf{P}_\gamma} \otimes \mathsf{spot}^{\beta}_{\varnothing})\mathsf{adj}^{\gamma\varnothing}_{\varnothing\gamma} \tag{6.7}$$

where the righthand equality is immediate. We now set about proving the lefthand-equality. We assume that $b_\beta \leqslant b_\gamma$ (the other case is similar, but has fewer steps). We define

$$\mathsf{SC}_{q,\beta\gamma} = \begin{cases} \mathsf{P}_\varnothing \boxtimes \mathsf{M}_k^{q-b_\beta} \boxtimes \mathsf{M}_{k,j+1}^{b_\beta} \boxtimes \mathsf{M}_k^{b_\gamma - q} \boxtimes \mathsf{P}_j^{b_\beta} \boxtimes \mathsf{P}_{k+1}^{b_\gamma} & \text{for } b_\gamma \geqslant q > b_\beta \\ \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_j^{b_\beta - q} \boxtimes \mathsf{M}_{k,j+1}^{q} \boxtimes \mathsf{M}_k^{b_\gamma - q} \boxtimes \mathsf{P}_j^{b_\beta} \boxtimes \mathsf{P}_{k+1}^{b_\gamma} & \text{for } b_\beta \geqslant q > 0 \\ \mathsf{M}_j^{b_\beta} \boxtimes \mathsf{M}_k^{b_\gamma} \boxtimes \mathsf{P}_j^{b_\beta} \boxtimes \mathsf{P}_{k+1}^{b_\gamma} & \text{for } q = 0 \\ \mathsf{M}_j^{b_\beta} \boxtimes \mathsf{P}_{j+1}^{b_\beta} \boxtimes \mathsf{M}_k^{b_\gamma} \boxtimes \mathsf{P}_{k+1}^{b_\gamma} & \text{for } q = -1 \end{cases}$$

which is obtained from $\mathsf{C}_{q,\beta\gamma}$ by reflection through s_β. We invite the reader to draw an example of the timeline by reflecting the final four paths of Fig. 24 through s_β.

Proposition 6.4. *For $0 \leqslant q < b_\beta$, we have that*

$$\mathsf{com}_{\beta\gamma}(q) \circ \Upsilon^{\mathsf{SC}_{q,\beta\gamma}}_{\mathsf{S}_{q,\beta} \otimes \mathsf{P}_\gamma} \circ (\mathsf{spot}^{\beta}_{\varnothing}(q) \otimes e_{\mathsf{P}_\gamma}) = \Upsilon^{\mathsf{SC}_{q+1,\beta\gamma}}_{\mathsf{S}_{q+1,\beta} \otimes \mathsf{P}_\gamma} \tag{6.8}$$

(note that $\Upsilon^{\mathsf{SC}_{0,\beta\gamma}}_{\mathsf{S}_{0,\beta}\otimes\mathsf{P}_\gamma} = \mathsf{com}_{\beta\gamma}(-1)$*) and for* $b_\beta \leqslant q < b_\gamma$*, we have that*

$$\mathsf{com}_{\beta\gamma}(q) \circ \Upsilon^{\mathsf{SC}_{q,\beta\gamma}}_{\mathsf{P}_\varnothing\otimes\mathsf{P}_\gamma} = \Upsilon^{\mathsf{SC}_{q+1,\beta\gamma}}_{\mathsf{P}_\varnothing\otimes\mathsf{P}_\gamma}. \tag{6.9}$$

Proof. All these elements are residue commutative (by reindexing the proof of Proposition 5.13). We prove equation (6.8) and (6.9) by induction on $0 \leqslant q < b_\gamma$ (the $q = -1$ case is trivial). Label the top and bottom frames of the concatenated diagrams on the lefthand-side of equation (6.8) and (6.9) by the paths $\mathsf{T}_{q+1} = \mathsf{SC}_{q+1,\beta\gamma}$ and $\mathsf{B}_{q+1} = \mathsf{S}_{q+1,\varnothing} \otimes \mathsf{P}_\gamma$. The concatenated diagram on the lefthand-side of both equation (6.8) and equation (6.9) has a single crossing which does not preserve step labels. Namely the strands connecting the $\mathsf{T}_q^{-1}(q+1, \varepsilon_j)$th and $\mathsf{T}_q^{-1}(b_\beta + q + 1, \varepsilon_{j+1})$th top vertices to the $\mathsf{B}_q^{-1}(q+1, \varepsilon_j)$th and $\mathsf{B}_q^{-1}(b_\beta + q + 1, \varepsilon_{j+1})$th bottom vertices form an r_q-crossing, for some $r_q \in \mathbb{Z}/e\mathbb{Z}$ say, and these strands permute the labels $+\varepsilon_j$ and $+\varepsilon_{j+1}$. This crossing is bi-passed on the left by a strand connecting the $\mathsf{T}_q^{-1}(b_\beta + q, \varepsilon_{j+1})$th top and $\mathsf{B}_q^{-1}(b_\beta + q, \varepsilon_{j+1})$th bottom vertices. We undo this triple using case 2 of relation (R5) and hence obtain the righthand-side of equation (6.8) and (6.9). □

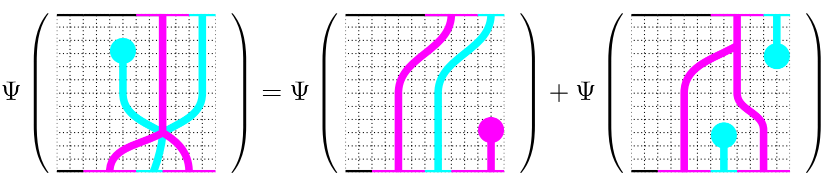
In order to deduce that equation (6.7) holds, we observe that

$$\mathsf{com}^{\gamma\beta} \circ (\mathsf{com}_{\beta\gamma}(\mathsf{spot}^{\beta}_{\varnothing} \otimes e_{\mathsf{P}_\gamma})) = \mathsf{com}^{\gamma\beta} \circ \Upsilon^{\mathsf{SC}_{b_\gamma,\beta\gamma}}_{\mathsf{P}_\varnothing\otimes\mathsf{P}_\gamma} = \Upsilon^{\mathsf{P}_\gamma\otimes\mathsf{P}^\flat_\beta}_{\mathsf{P}_\varnothing\otimes\mathsf{P}_\gamma}$$

as the lefthand-side of the final equality is minimal and respects step-labels.

6.5. The spot-hexagon

For $\alpha, \beta \in \Pi$ labelling two non-commuting reflections, we now check the rightmost relation in (S3), namely that



(6.10)

(and we leave it the reader to check the reflection of this relation through its vertical axis). In other words, we need to check that

$$(e_{\mathsf{P}_\varnothing} \otimes \mathsf{spot}^{\varnothing}_{\beta} \otimes e_{\mathsf{P}_{\alpha\beta}})\mathsf{hex}^{\varnothing\beta\alpha\beta}_{\varnothing\alpha\beta\alpha}$$

is equal to

$$\mathsf{adj}^{\varnothing\varnothing\alpha\beta}_{\varnothing\alpha\beta\varnothing}(e_{\mathsf{P}_{\varnothing\alpha\beta}} \otimes \mathsf{spot}^{\varnothing}_{\alpha}) + e_{\mathsf{P}_\varnothing} \otimes (\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} \otimes \mathsf{spot}^{\beta}_{\varnothing})\mathsf{adj}^{\alpha\alpha\varnothing}_{\alpha\varnothing\alpha}(e_{\mathsf{P}_\alpha} \otimes \mathsf{spot}^{\varnothing}_{\beta} \otimes e_{\mathsf{P}_\alpha})).$$

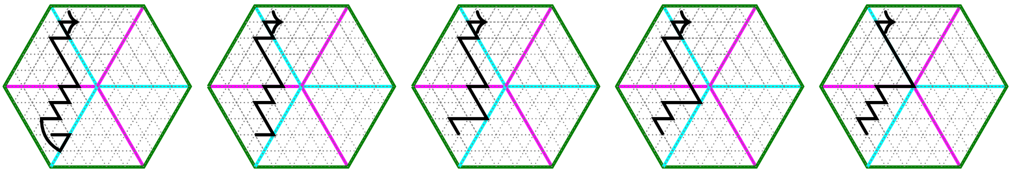


Fig. 27. An example of the tableaux $\mathsf{SH}_{q,\beta\alpha\beta}$ for $0 \leqslant q \leqslant b_{\alpha\beta}$. The reader should compare these reflected paths with the final five paths of Fig. 22.

We set $j = i+1$ so that $\alpha = \varepsilon_i - \varepsilon_{i+1}$, $\beta = \varepsilon_{i+1} - \varepsilon_{i+2}$. We will begin by considering the lefthand-side of the equation. In order to do this, we need to use the reflections of the braid $\mathsf{H}_{q,\beta\alpha\beta}$-paths for $0 \leqslant q \leqslant b_{\alpha\beta}$ through the first β-hyperplane which they come across (namely the hyperplane whose strand we are putting a spot on top of) and we remark that this path will have the *same residue sequence* as the original $\mathsf{H}_{q,\beta\alpha\beta}$-paths, but different step labelling. We define $\mathsf{SH}_{q,\beta\alpha\beta}$ to be the path

$$\begin{cases} \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_{i+1}^{b_\beta - q} \boxtimes \mathsf{M}_{i,i+1}^{q} \boxtimes \mathsf{P}_{i+1}^{b_\beta} \boxtimes \mathsf{M}_{i}^{b_\alpha - q} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} \otimes_\alpha \mathsf{M}_{i+1}^{b_\beta} \boxtimes \mathsf{P}_{i+2}^{b_\beta} & 0 \leqslant q \leqslant b_\beta \\ \mathsf{P}_{\emptyset} \boxtimes \mathsf{M}_{i}^{q-b_\beta} \boxtimes \mathsf{M}_{i,i+1}^{b_\beta} \boxtimes \mathsf{P}_{i+1}^{b_\beta} \boxtimes \mathsf{M}_{i}^{b_\alpha - q} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} \otimes_\alpha \mathsf{M}_{i+1}^{b_\beta} \boxtimes \mathsf{P}_{i+2}^{b_\beta} & b_\beta \leqslant q \leqslant b_\alpha \\ \mathsf{P}_{\emptyset} \boxtimes \mathsf{M}_{i}^{q-b_\beta} \boxtimes \mathsf{P}_{i+2}^{b_\beta} \boxtimes \mathsf{M}_{i,i+2}^{q-b_\alpha} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} \otimes_\alpha \mathsf{M}_{i+1}^{b_{\alpha\beta} - q} \boxtimes \mathsf{P}_{i+2}^{b_\beta} & b_\alpha \leqslant q \leqslant b_{\alpha\beta} \end{cases}$$

for $b_\alpha \geqslant b_\beta$ (the $b_\alpha < b_\beta$ case is similar). See Fig. 27 for an example.

Proposition 6.5. *We have that*

$$\left(e_{\mathsf{P}_\emptyset} \otimes \mathsf{spot}_\beta^\emptyset \otimes e_{\mathsf{P}_{\alpha\beta}}\right) \mathsf{hex}^{\emptyset\beta\alpha\beta} = \Upsilon^{\mathsf{P}_{\emptyset\emptyset\alpha\beta}}_{\mathsf{P}_\emptyset \otimes \mathsf{SH}_{b_{\alpha\beta},\beta\alpha\beta}} \tag{6.11}$$

Proof. First, we remark that the righthand-side of equation (6.11) is residue-commuting and so makes sense. For $0 \leqslant q < b_{\alpha\beta}$, we claim that

$$\left(e_{\mathsf{P}_\emptyset} \otimes \mathsf{spot}_\beta^\emptyset(q) \otimes e_{\mathsf{P}_{\alpha\beta}}\right) \Upsilon^{\mathsf{P}_\emptyset \otimes \mathsf{S}_{q,\beta} \otimes \mathsf{P}_{\alpha\beta}}_{\mathsf{P}_\emptyset \otimes \mathsf{SH}_{q,\beta\alpha\beta}} \mathsf{hex}^{\emptyset\beta\alpha\beta}(q) = \Upsilon^{\mathsf{P}_\emptyset \otimes \mathsf{S}_{q+1,\beta} \otimes \mathsf{P}_{\alpha\beta}}_{\mathsf{P}_\emptyset \otimes \mathsf{SH}_{q+1,\beta\alpha\beta}} \tag{6.12}$$

and we will we label the top and bottom of these diagrams according to the paths $\mathsf{T}_q = \mathsf{P}_\emptyset \otimes \mathsf{S}_{q+1,\beta} \otimes \mathsf{P}_{\alpha\beta}$ and $\mathsf{B}_q = \mathsf{P}_\emptyset \otimes \mathsf{SH}_{q+1,\beta\alpha\beta}$ respectively (with the convention that $\mathsf{S}_{q,\beta} = \mathsf{P}_\emptyset$ for $q \geqslant b_\beta$). Again, this element is residue-commuting and so there is no ambiguity here. In the concatenated diagram on the lefthand-side of equation (6.12), there is a single pair of strands, Q and Q' whose crossing if of degree -2 (of residue $r_q \in \mathbb{Z}/e\mathbb{Z}$, say); these strands connect the

$$\mathsf{T}_q^{-1}(b_\alpha + q + 1, \varepsilon_{i+1}) \quad \mathsf{T}_q^{-1}(b_{\alpha\beta} + q + 1, \varepsilon_{i+2})$$

top vertices and the

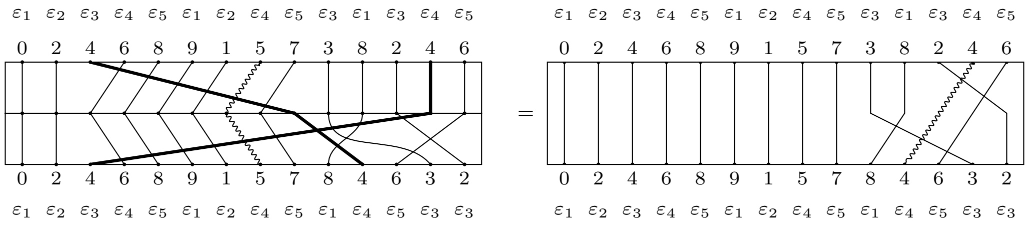


Fig. 28. The product $(\mathsf{spot}^{\varnothing}_{\beta}(0) \otimes e_{\mathsf{P}_{\alpha\beta}})\mathsf{hex}^{\beta\alpha\beta}(0)$ in the proof of Proposition 6.5 for $h = 1$, $\ell = 5$, $\kappa = (0, 2, 4, 6, 8)$, $e = 10$ and $\alpha = \varepsilon_2 - \varepsilon_3$, $\beta = \varepsilon_3 - \varepsilon_4$. The top path is $\mathsf{S}_{1,\alpha} \otimes \mathsf{M}_2$ and the bottom path is $\mathsf{SH}_{1,\beta\alpha\beta} \otimes \mathsf{M}_2$ (the prefix $\mathsf{P}_{\varnothing}$ and the remainder of the postfix $\mathsf{P}_{\alpha} = \mathsf{M}_2^{b_\alpha} \boxtimes \mathsf{P}_3^{b_\alpha}$ would not fit).

$$\mathsf{B}_q^{-1}(b_{\alpha} + q + 1, \varepsilon_{i+1}) \quad \mathsf{B}_q^{-1}(b_{\alpha\beta} + q + 1, \varepsilon_{i+2})$$

bottom vertices (thus crossing one another). This crossing of r_q-strands, Q and Q', is bi-passed on the left by the (r_q+1)-strand from $\mathsf{T}_q^{-1}(b_{\alpha\beta}+q, \varepsilon_{i+2})$ to $\mathsf{B}_q^{-1}(b_{\alpha\beta}+q, \varepsilon_{i+2})$.

Applying case 2 of relation (R5) to the concatenated diagram we obtain two terms: the term with the crossing is bi-passed on the right is zero by Lemma 4.1; the term in which we undo the crossing is equal to the righthand-side of equation (6.12) (since the resulting diagram is minimal). An example is given in Fig. 28. $\square$

We now wish to show that

$$\Upsilon^{\mathsf{P}_{\varnothing\beta\alpha\beta}}_{\mathsf{P}_{\varnothing}\otimes \mathsf{SH}_{b_{\alpha\beta},\beta\alpha\beta}} \mathsf{hex}_{\varnothing\alpha\beta\alpha}$$

is equal to

$$\mathsf{adj}^{\varnothing\varnothing\alpha\beta}_{\varnothing\alpha\beta\varnothing}(e_{\mathsf{P}_{\varnothing\alpha\beta}} \otimes \mathsf{spot}^{\varnothing}_{\alpha}) + e_{\mathsf{P}_{\varnothing}} \otimes (\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} \otimes \mathsf{spot}^{\beta}_{\varnothing})\mathsf{adj}^{\alpha\alpha\varnothing}_{\alpha\varnothing\alpha}(e_{\mathsf{P}_{\alpha}} \otimes \mathsf{spot}^{\varnothing}_{\beta} \otimes e_{\mathsf{P}_{\alpha}})).$$

In what follows, we assume that $b_{\alpha} \geqslant b_{\beta}$. In order to consider the first term, we use the reflections of the $\mathsf{H}_{q,\alpha\beta\alpha}$-paths for $0 \leqslant q \leqslant b_{\alpha\beta}$ through the final α-hyperplane which they come across (namely the hyperplane whose strand we are putting a spot on top of) and we remark that this path will have the *same residue sequence* as the original $\mathsf{H}_{q,\alpha\beta\alpha}$-paths but with a different step labelling. We define $\mathsf{S}_{\alpha}\mathsf{H}_{q,\alpha\beta\alpha}$ to be the path

$$\begin{cases} \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} \otimes_{\alpha} \mathsf{M}_{i+1}^{b_\beta - q} \boxtimes \mathsf{P}_{i+2}^{b_\beta} \otimes_{\beta} \mathsf{M}_{i,i+2}^{q} \boxtimes \mathsf{M}_i^{b_\alpha - q} \otimes_{\alpha} \mathsf{P}_{i+1}^{b_\alpha} & 0 \leqslant q \leqslant b_{\beta} \\ \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_{i+1}^{b_{\alpha\beta} - q} \otimes_{\alpha} \mathsf{P}_{i+2}^{b_\beta} \otimes_{\beta} \mathsf{M}_{i,i+2}^{b_\beta} \boxtimes \mathsf{M}_i^{b_\alpha - q} \otimes_{\alpha} \mathsf{P}_{i+1}^{b_\alpha} & b_{\beta} \leqslant q \leqslant b_{\alpha} \\ \mathsf{P}_{\varnothing} \boxtimes \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_{i+1}^{b_{\alpha\beta} - q} \otimes_{\alpha} \mathsf{P}_{i+2}^{b_\beta} \otimes_{\beta} \mathsf{M}_{i,i+2}^{b_\beta} \otimes_{\alpha} \mathsf{P}_{i+1}^{b_\alpha} \boxtimes \mathsf{P}_i^{q - b_\alpha} & b_{\alpha} \leqslant q \leqslant b_{\alpha\beta} \end{cases}$$

In order to consider the second term, we need the reflections of the $\mathsf{H}_{q,\alpha\beta\alpha}$-paths for $0 \leqslant q \leqslant b_{\alpha\beta}$ through the first β-hyperplane which they come across. We define $\mathsf{S}_{\beta}\mathsf{H}_{q,\alpha\beta\alpha}$ to be the path

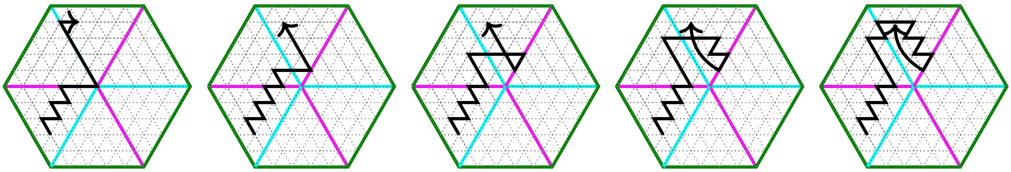


Fig. 29. An example of the paths $\mathsf{S}_{\alpha}\mathsf{H}_{q,\alpha\beta\alpha}$ for $b_{\alpha\beta} \geqslant q \geqslant 0$.

$$\begin{cases} \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} \otimes_\alpha \mathsf{M}_{i+1}^{b_\beta - q} \boxtimes \mathsf{P}_{i+2}^{b_\beta} \boxtimes \mathsf{M}_{i,i+2}^{q} \boxtimes \mathsf{M}_i^{b_\alpha - q} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} & 0 \leqslant q \leqslant b_\beta \\ \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_{i+1}^{b_{\alpha\beta} - q} \otimes_\alpha \mathsf{P}_{i+2}^{b_\beta} \boxtimes \mathsf{M}_{i,i+2}^{b_\beta} \boxtimes \mathsf{M}_i^{b_\alpha - q} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} & b_\beta \leqslant q \leqslant b_\alpha \\ \mathsf{P}_{\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha} \boxtimes \mathsf{P}_{i+1}^{b_{\alpha\beta} - q} \otimes_\alpha \mathsf{P}_{i+2}^{b_\beta} \boxtimes \mathsf{M}_{i,i+2}^{b_\beta} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} \boxtimes \mathsf{P}_i^{q - b_\alpha} & b_\alpha \leqslant q \leqslant b_{\alpha\beta} \end{cases}$$

See Fig. 29 for an example of the $\mathsf{S}_{\alpha}\mathsf{H}_{q,\alpha\beta\alpha}$ paths. We leave it as an exercise for the reader to draw the $\mathsf{S}_{\beta}\mathsf{H}_{q,\alpha\beta\alpha}$ paths. Finally, for the purposes of the proof we will also need the following "error path"

$$\mathsf{eS}_{\beta}\mathsf{H}_{\alpha\beta\alpha} = \mathsf{P}_{\emptyset} \boxtimes \mathsf{M}_i^{b_\alpha} \otimes_\alpha \mathsf{P}_{i+2}^{b_\beta - 1} \boxtimes \mathsf{M}_{i,i+2} \boxtimes \mathsf{P}_{i+2} \boxtimes \mathsf{M}_{i,i+2}^{b_\beta - 1} \boxtimes \mathsf{P}_{i+1}^{b_\alpha} \boxtimes \mathsf{P}_i^{b_\beta}$$

which one should compare with the final path (the $b_{\alpha\beta}$th case) above. One should repeat the above definitions for the $b_\alpha < b_\beta$ case.

Proposition 6.6. *We have that*

$$\Upsilon^{\mathsf{P}_{\emptyset\emptyset\alpha\beta}}_{\mathsf{P}_{\emptyset} \otimes \mathsf{SH}_{b_{\alpha\beta},\beta\alpha\beta}} \mathsf{hex}_{\emptyset\alpha\beta\alpha} = \Upsilon^{\mathsf{P}_{\emptyset} \otimes \mathsf{P}_{\emptyset} \otimes \mathsf{P}_{\alpha} \otimes \mathsf{P}_{\beta}}_{\mathsf{P}_{\emptyset} \otimes \mathsf{P}_{\alpha} \otimes \mathsf{P}_{\beta} \otimes \mathsf{P}^{\flat}_{\alpha}} + \Upsilon^{\mathsf{P}_{\emptyset} \otimes \mathsf{P}_{\emptyset} \otimes \mathsf{P}^{\flat}_{\alpha} \otimes \mathsf{P}^{\flat}_{\beta}}_{\mathsf{P}_{\emptyset} \otimes \mathsf{P}_{\alpha} \otimes \mathsf{P}^{\flat}_{\beta} \otimes \mathsf{P}_{\alpha}}. \tag{6.13}$$

Proof. First, we remark that both terms on the righthand-side of equation (6.13) are residue-commuting. We suppose $b_\alpha \geqslant b_\beta$ as the other case is similar. We observe that

$$\Upsilon^{\mathsf{P}_{\emptyset\emptyset\alpha\beta}}_{\mathsf{P}_{\emptyset} \otimes \mathsf{SH}_{b_{\alpha\beta},\beta\alpha\beta}} = \Upsilon^{\mathsf{P}_{\emptyset} \otimes \mathsf{P}_{\emptyset} \otimes \mathsf{P}_{\alpha} \otimes \mathsf{P}_{\beta}}_{\mathsf{P}_{\emptyset} \otimes \mathsf{S}_{\alpha}\mathsf{H}_{b_{\alpha\beta},\beta\alpha\beta}} = \Upsilon^{\mathsf{P}_{\emptyset} \otimes \mathsf{P}_{\emptyset} \otimes \mathsf{P}^{\flat}_{\alpha} \otimes \mathsf{P}^{\flat}_{\beta}}_{\mathsf{P}_{\emptyset} \otimes \mathsf{S}_{\beta}\mathsf{H}_{b_{\alpha\beta},\beta\alpha\beta}}$$

as the underlying permutations (and residue sequences) are all identical. We set

$$\mathsf{T}_{\alpha} = \mathsf{P}_{\emptyset\emptyset} \otimes \mathsf{P}_{\alpha} \otimes \mathsf{P}_{\beta} \qquad \mathsf{T}_{\beta} = \mathsf{P}_{\emptyset\emptyset} \otimes \mathsf{P}^{\flat}_{\alpha} \otimes \mathsf{P}^{\flat}_{\beta}$$
$$\mathsf{B}_{q,\alpha} = \mathsf{P}_{\emptyset} \otimes \mathsf{S}_{\alpha}\mathsf{H}_{q+1,\alpha\beta\alpha} \qquad \mathsf{B}_{q,\beta} = \mathsf{P}_{\emptyset} \otimes \mathsf{S}_{\beta}\mathsf{H}_{q,\alpha\beta\alpha}$$

for $b_{\alpha\beta} > q \geqslant 0$. We first consider the $q = b_{\alpha\beta} - 1$ case. The concatenated diagram

$$\Upsilon^{\mathsf{P}_{\emptyset\emptyset\alpha\beta}}_{\mathsf{P}_{\emptyset} \otimes \mathsf{SH}_{b_{\alpha\beta},\beta\alpha\beta}} (e_{\mathsf{P}_{\emptyset}} \otimes \mathsf{hex}_{\alpha\beta\alpha}(b_{\alpha\beta} - 1))$$

contains a single like-labelled crossing of $r_{b_{\alpha\beta}-1}$-strands connecting the pair

$$\mathsf{T}^{-1}_{\alpha}(b_{\alpha\beta\alpha}+1,\varepsilon_{i+1})=\mathsf{T}^{-1}_{\beta}(b_{\alpha\beta}+1,\varepsilon_{i}) \qquad \mathsf{T}^{-1}_{\alpha}(2b_{\alpha\beta}+1,\varepsilon_{i+2})=\mathsf{T}^{-1}_{\beta}(b_{\alpha\beta\alpha}+1,\varepsilon_{i+1})$$

of top vertices to the pair of

$$\mathsf{B}^{-1}_{\alpha}(2b_{\alpha\beta}+1,\varepsilon_{i+2})=\mathsf{B}^{-1}_{\beta}(b_{\alpha\beta\alpha}+1,\varepsilon_{i+1}) \qquad \mathsf{B}^{-1}_{\alpha}(b_{\alpha\beta\alpha}+1,\varepsilon_{i+1})=\mathsf{B}^{-1}_{\beta}(b_{\alpha\beta}+1,\varepsilon_{i})$$

These $r_{b_{\alpha\beta}-1}$-crossing strands are bi-passed on the left by the $r_{b_{\alpha\beta}}$-strand connecting the

$$\mathsf{T}^{-1}_{\alpha}(2b_{\alpha\beta},\varepsilon_{i+2})=\mathsf{T}^{-1}_{\beta}(2b_{\alpha\beta},\varepsilon_{i+2}) \qquad \mathsf{B}^{-1}_{\alpha}(2b_{\alpha\beta},\varepsilon_{i+2})=\mathsf{B}^{-1}_{\beta}(2b_{\alpha\beta},\varepsilon_{i+2})$$

top and bottom vertices. We apply case 2 of relation (R5) to this triple of strands and hence obtain

$$\Upsilon^{\mathsf{P}_{\varnothing\varnothing\alpha\beta}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{SH}_{b_{\alpha\beta},\beta\alpha\beta}}\mathsf{hex}_{\varnothing\alpha\beta\alpha}(b_{\alpha\beta}-1)=\Upsilon^{\mathsf{P}_{\varnothing}\otimes\mathsf{P}_{\varnothing}\otimes\mathsf{P}_{\alpha}\otimes\mathsf{P}_{\beta}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{S}_{\alpha}\mathsf{H}_{b_{\alpha\beta}-1,\beta\alpha\beta}}+\Upsilon^{\mathsf{P}_{\varnothing}\otimes\mathsf{P}_{\varnothing}\otimes\mathsf{P}^{\flat}_{\alpha}\otimes\mathsf{P}^{\flat}_{\beta}}_{\mathsf{P}_{\varnothing}\otimes e\mathsf{S}_{\beta}\mathsf{H}_{\alpha\beta\alpha}}\Upsilon^{\mathsf{P}_{\varnothing}\otimes e\mathsf{S}_{\beta}\mathsf{H}_{\alpha\beta\alpha}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{S}_{\beta}\mathsf{H}_{b_{\alpha\beta}-1,\beta\alpha\beta}} \tag{6.14}$$

where in the first term we have undone the triple-crossing and in the second "error" term the $r_{b_{\alpha\beta}}$-strand bi-passes the crossing to the right (and is labelled by the "error path"). We are now ready to consider the $b_{\alpha\beta}-1>q\geqslant 0$ cases — which we do separately for α and β, in turn.

Case α. We first consider the first term on the righthand-side of equation (6.14). We claim that

$$\Upsilon^{\mathsf{P}_{\varnothing}\otimes\mathsf{P}_{\varnothing}\otimes\mathsf{P}_{\alpha}\otimes\mathsf{P}_{\beta}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{S}_{\alpha}\mathsf{H}_{q+1,\alpha\beta\alpha}}\mathsf{hex}_{\varnothing\alpha\beta\alpha}(q)=\Upsilon^{\mathsf{P}_{\varnothing\varnothing}\otimes\mathsf{P}_{\alpha}\otimes\mathsf{P}_{\beta}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{S}_{\alpha}\mathsf{H}_{q,\alpha\beta\alpha}} \tag{6.15}$$

for $b_{\alpha\beta}-1>q\geqslant 0$. For each $b_{\alpha\beta}>q\geqslant b_{\alpha}$ the concatenated diagram in equation (6.15) contains a single like-labelled crossing of r_q-strands (for some $r_q\in\mathbb{Z}/e\mathbb{Z}$ say) connecting the pair

$$\mathsf{T}^{-1}_{\alpha}(2b_{\beta}+3b_{\alpha}-q,\varepsilon_{i+1}) \qquad \mathsf{T}^{-1}_{\alpha}(3b_{\beta}+3b_{\alpha}-q,\varepsilon_{i+2})$$

of top vertices to the pair of

$$\mathsf{B}^{-1}_{\alpha}(3b_{\beta}+3b_{\alpha}-q,\varepsilon_{i+2}) \qquad \mathsf{B}^{-1}_{\alpha}(3b_{\beta}+3b_{\alpha}-q,\varepsilon_{i+1})$$

bottom vertices, respectively. For $b_{\alpha\beta}-1>q\geqslant b_{\alpha}$ the aforementioned (unique) pair of crossing r_q-strands in

$$\Upsilon^{\mathsf{P}_{\varnothing\varnothing}\otimes\mathsf{P}_{\alpha}\otimes\mathsf{P}_{\beta}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{S}_{\alpha}\mathsf{H}_{q+1,\alpha\beta\alpha}}\mathsf{hex}_{\varnothing\alpha\beta\alpha}(q)=\Upsilon^{\mathsf{P}_{\varnothing\varnothing}\otimes\mathsf{P}_{\alpha}\otimes\mathsf{P}_{\beta}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{S}_{\alpha}\mathsf{H}_{q,\alpha\beta\alpha}}$$

is bi-passed on the left by the r_{q+1}-strand connecting $\mathsf{T}^{-1}_{\alpha}(3b_{\beta}+3b_{\alpha}-q-1,\varepsilon_{i+2})$ and $\mathsf{B}^{-1}_{\alpha}(3b_{\beta}+3b_{\alpha}-q-1,\varepsilon_{i+2})$ top and bottom vertices. Applying case 2 of relation (R5) we undo this triple crossing (the other term is zero by Lemma 4.1) as required. Now for $b_{\alpha} > q \geqslant 0$ the concatenated product on the lefthand-side of equation (6.15) is both minimal and step-preserving and so the claim follows.

Case β. We now consider the second term on the right of equation (6.14). We have that

$$\Upsilon^{\mathsf{P}_{\varnothing}\otimes \mathsf{eS}_{\beta}\mathsf{H}_{\alpha\beta\alpha}}_{\mathsf{P}_{\varnothing}\otimes \mathsf{S}_{\beta}\mathsf{H}_{q+1,\alpha\beta\alpha}} \mathsf{hex}_{\varnothing\alpha\beta\alpha}(q) = \Upsilon^{\mathsf{P}_{\varnothing}\otimes \mathsf{eS}_{\beta}\mathsf{H}_{\alpha\beta\alpha}}_{\mathsf{P}_{\varnothing}\otimes \mathsf{S}_{\beta}\mathsf{H}_{q,\alpha\beta\alpha}}$$

for $b_{\alpha\beta} - 1 > q \geqslant b_{\alpha}$ as the lefthand-side is minimal and step-preserving. Now, we claim that

$$\Upsilon^{\mathsf{P}_{\varnothing}\otimes \mathsf{P}_{\varnothing}\otimes \mathsf{P}^{\flat}_{\alpha}\otimes \mathsf{P}^{\flat}_{\beta}}_{\mathsf{P}_{\varnothing}\otimes \mathsf{eS}_{\beta}\mathsf{H}_{\alpha\beta\alpha}} \Upsilon^{\mathsf{P}_{\varnothing}\otimes \mathsf{eS}_{\beta}\mathsf{H}_{\alpha\beta\alpha}}_{\mathsf{P}_{\varnothing}\otimes \mathsf{S}_{\beta}\mathsf{H}_{b_{\alpha},\alpha\beta\alpha}} \mathsf{hex}_{\varnothing\alpha\beta\alpha}(b_{\alpha} - 1) = \Upsilon^{\mathsf{P}_{\varnothing}\otimes \mathsf{P}_{\varnothing}\otimes \mathsf{P}^{\flat}_{\alpha}\otimes \mathsf{P}^{\flat}_{\beta}}_{\mathsf{P}_{\varnothing}\otimes \mathsf{S}_{\beta}\mathsf{H}_{b_{\alpha}-1,\alpha\beta\alpha}} \tag{6.16}$$

and that

$$\Upsilon^{\mathsf{P}_{\varnothing}\otimes \mathsf{P}_{\varnothing}\otimes \mathsf{P}^{\flat}_{\alpha}\otimes \mathsf{P}^{\flat}_{\beta}}_{\mathsf{P}_{\varnothing}\otimes \mathsf{S}_{\beta}\mathsf{H}_{q+1,\beta\alpha\beta}} \mathsf{hex}_{\varnothing\alpha\beta\alpha}(q) = \Upsilon^{\mathsf{P}_{\varnothing}\otimes \mathsf{P}_{\varnothing}\otimes \mathsf{P}^{\flat}_{\alpha}\otimes \mathsf{P}^{\flat}_{\beta}}_{\mathsf{P}_{\varnothing}\otimes \mathsf{S}_{\beta}\mathsf{H}_{q,\alpha\beta\alpha}} \tag{6.17}$$

for $b_{\alpha} - 1 > q \geqslant 0$. For each $b_{\alpha} \geqslant q \geqslant 0$ the concatenated diagram on the lefthand-side of equation (6.16) and (6.17) contains a crossing pair of r_q-strands connecting the

$$\mathsf{T}^{-1}_{\beta}(b_{\beta} + q + 1, \varepsilon_i) \quad \mathsf{T}^{-1}_{\beta}(2b_{\beta} + b_{\alpha} + q + 1, \varepsilon_{i+2})$$

and

$$\mathsf{B}^{-1}_{\beta}(2b_{\beta} + b_{\alpha} + q + 1, \varepsilon_{i+2}) \quad \mathsf{B}^{-1}_{\beta}(b_{\beta} + q + 1, \varepsilon_i)$$

top and bottom vertices, respectively (note that this crossing does not respect step labels). This r_q-crossing is bi-passed on the right by the $(r_q - 1)$-strand connecting the

$$\mathsf{T}^{-1}_{\beta}(b_{\beta} + q + 2, \varepsilon_i) \qquad \mathsf{B}^{-1}_{\beta}(b_{\beta} + q + 2, \varepsilon_i)$$

top and bottom vertices. We undo this triple-crossing using case 1 of relation (R5) (the other term is zero by Lemma 4.1). The concatenated product is minimal and step-preserving, as required. □

Finally, in order to deduce equation (6.10), we observe that

$$\mathsf{adj}^{\varnothing\varnothing\alpha\beta}_{\varnothing\alpha\beta\varnothing}(e_{\mathsf{P}_{\varnothing\alpha\beta}} \otimes \mathsf{spot}^{\varnothing}_{\alpha}) = \Upsilon^{\mathsf{P}_{\varnothing\varnothing\alpha}\otimes \mathsf{P}_{\beta}}_{\mathsf{P}_{\varnothing\alpha\beta}\otimes \mathsf{P}^{\flat}_{\alpha}}$$

$$e_{\mathsf{P}_{\varnothing}} \otimes ((\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} \otimes \mathsf{spot}^{\beta}_{\varnothing})\mathsf{adj}^{\alpha\alpha\varnothing}_{\alpha\varnothing\alpha}(e_{\mathsf{P}_{\alpha}} \otimes \mathsf{spot}^{\varnothing}_{\beta} \otimes e_{\mathsf{P}_{\alpha}})) = \Upsilon^{\mathsf{P}_{\varnothing\varnothing}\otimes \mathsf{P}^{\flat}_{\alpha}\otimes \mathsf{P}^{\flat}_{\beta}}_{\mathsf{P}_{\varnothing\alpha}\otimes \mathsf{P}^{\flat}_{\beta}\otimes \mathsf{P}_{\alpha}}$$

as the concatenated diagrams are minimal, step-preserving, and residue-commutative.

6.6. The fork-hexagon

For $\alpha, \beta \in \Pi$ labelling two non-commuting reflections, we now check the leftmost relation in (S3), namely that

$$(e_{\mathsf{P}_{\emptyset\emptyset}} \otimes \mathsf{hex}^{\emptyset\beta\alpha\beta}_{\emptyset\alpha\beta\alpha})(e_{\mathsf{P}_{\emptyset\emptyset}} \otimes \mathsf{fork}^{\emptyset\alpha}_{\alpha\alpha} \otimes e_{\mathsf{P}_{\beta\alpha}})\mathsf{adj}^{\emptyset\emptyset\alpha\alpha\beta\alpha}_{\emptyset\alpha\emptyset\alpha\beta\alpha}(e_{\mathsf{P}_{\emptyset\alpha}} \otimes \mathsf{hex}^{\emptyset\alpha\beta\alpha}_{\emptyset\beta\alpha\beta}) \tag{6.18}$$

is equal to

$$\mathsf{adj}^{\emptyset\emptyset\emptyset\beta\alpha\beta}_{\emptyset\emptyset\beta\alpha\emptyset\beta}(e_{\mathsf{P}_{\emptyset\emptyset\beta\alpha}} \otimes \mathsf{fork}^{\emptyset\beta}_{\beta\beta})(e_{\mathsf{P}_{\emptyset}} \otimes \mathsf{hex}^{\emptyset\beta\alpha\beta}_{\emptyset\alpha\beta\alpha} \otimes e_{\mathsf{P}_{\beta}})\mathsf{adj}^{\emptyset\emptyset\alpha\beta\alpha\beta}_{\emptyset\alpha\emptyset\beta\alpha\beta} \tag{6.19}$$

Unlike earlier sections, we find that neither of (6.18) or (6.19) is of minimal length. We again set $j = i + 1$. First assume that $b_\alpha \geqslant b_\beta$. For (6.18), we must simplify the middle of the diagram. We define $\mathsf{FH}_{q,\alpha\beta\alpha}$ to be the path

$$\begin{cases} \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}^{b_\alpha}_i \boxtimes \mathsf{P}^{b_\alpha}_i \boxtimes \mathsf{M}^{b_\beta - q}_{i+1} \boxtimes \mathsf{P}^{b_\beta}_{i+2} \otimes_\beta \mathsf{M}^{q}_{i,i+2} \boxtimes \mathsf{M}^{b_\alpha - q}_i \boxtimes \mathsf{P}^{b_\alpha}_{i+1} & 0 \leqslant q \leqslant b_\beta \\ \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}^{b_\alpha}_i \boxtimes \mathsf{P}^{b_{\alpha\beta} - q}_i \boxtimes \mathsf{P}^{b_\beta}_{i+2} \otimes_\beta \mathsf{M}^{b_\beta}_{i,i+2} \boxtimes \mathsf{M}^{b_\alpha - q}_i \boxtimes \mathsf{P}^{b_\alpha}_{i+1} & b_\beta \leqslant q \leqslant b_\alpha \\ \mathsf{P}_{\emptyset} \boxtimes \mathsf{M}^{b_\alpha}_i \boxtimes \mathsf{P}^{b_{\alpha\beta} - q}_i \boxtimes \mathsf{P}^{b_\beta}_{i+2} \otimes_\beta \mathsf{M}^{b_\beta}_{i,i+2} \boxtimes \mathsf{P}^{b_\alpha}_{i+1} \boxtimes \mathsf{P}^{q - b_\alpha}_i & b_\alpha \leqslant q \leqslant b_{\alpha\beta} \end{cases}$$

We have that $\mathsf{FH}_{q,\alpha\beta\alpha} \sim \mathsf{H}_{q,\alpha\beta\alpha}$ because the former is obtained from the latter by reflection through the first α-hyperplane it crosses, this is depicted in Fig. 30. Similarly, we define $\mathsf{FH}_{q,\beta\alpha\beta}$ to be the path

$$\begin{cases} \mathsf{P}_{q\emptyset} \boxtimes \mathsf{M}^{b_\beta - q}_{i+1} \boxtimes \mathsf{M}^{q}_{i,i+1} \boxtimes \mathsf{P}^{b_\beta}_{i+2} \otimes_\beta \mathsf{M}^{b_\alpha - q}_i \boxtimes \mathsf{P}^{b_\alpha}_{i+1} \otimes_\alpha \mathsf{M}^{b_\beta}_{i+1} \otimes_\beta \mathsf{P}^{b_\beta}_{i+2} & 0 \leqslant q \leqslant b_\beta \\ \mathsf{P}_{\emptyset} \boxtimes \mathsf{M}^{q - b_\beta}_i \boxtimes \mathsf{M}^{b_\beta}_{i,i+1} \boxtimes \mathsf{P}^{b_\beta}_{i+2} \otimes_\beta \mathsf{M}^{b_\alpha - q}_i \boxtimes \mathsf{P}^{b_\alpha}_{i+1} \otimes_\alpha \mathsf{M}^{b_\beta}_{i+1} \otimes_\beta \mathsf{P}^{b_\beta}_{i+2} & b_\beta \leqslant q \leqslant b_\alpha \\ \mathsf{P}_{\emptyset} \boxtimes \mathsf{M}^{q - b_\beta}_i \boxtimes \mathsf{P}^{b_\beta}_{i+2} \otimes_\beta \mathsf{M}^{q - b_\alpha}_{i,i+2} \boxtimes \mathsf{P}^{b_\alpha}_{i+1} \otimes_\alpha \mathsf{M}^{b_{\alpha\beta} - q}_{i+1} \otimes_\beta \mathsf{P}^{b_\beta}_{i+2} & b_\alpha \leqslant q \leqslant b_{\alpha\beta} \end{cases}$$

We have that $\mathsf{FH}_{q,\beta\alpha\beta} \sim \mathsf{H}_{q,\beta\alpha\beta}$ because the former is obtained from the latter by reflection through the final β-hyperplane it crosses. We note that $\mathsf{FH}_{b_{\alpha\beta},\alpha\beta\alpha} = \mathsf{P}_{\emptyset - \emptyset} \boxtimes \mathsf{FH}_{b_{\alpha\beta},\beta\alpha\beta}$. One can define the paths $\mathsf{FH}_{q,\alpha\beta\alpha}$ and $\mathsf{FH}_{q,\beta\alpha\beta}$ for $b_\alpha < b_\beta$ in an entirely analogous fashion.

Proposition 6.7. *The element $\Upsilon^{\mathsf{P}_{\emptyset\emptyset\emptyset\beta\alpha\beta}}_{\mathsf{P}_{\emptyset\alpha\emptyset\beta\alpha} \otimes \mathsf{P}^{\flat}_{\beta}}$ is independent of the choice of reduced expression.*

Proof. We proceed as in the proof of Proposition 5.12. We set $\mathsf{T} = \mathsf{P}_{\emptyset\emptyset\emptyset\beta\alpha\beta}$ and $\mathsf{B} = \mathsf{P}_{\emptyset\alpha\emptyset\beta\alpha} \otimes \mathsf{P}^{\flat}_{\beta}$. For $0 \leqslant q \leqslant b_\alpha + 1$, we set

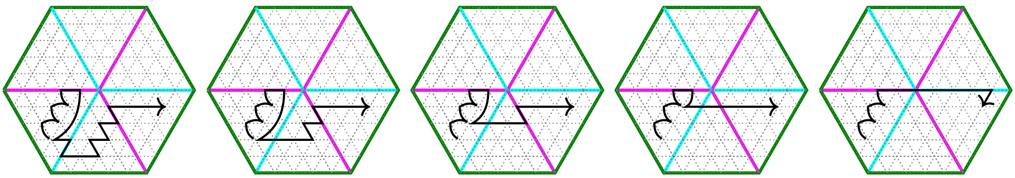


Fig. 30. An example of the tableaux $\mathsf{FH}_{q,\alpha\beta\alpha}$ for $b_{\alpha\beta \geqslant q \geqslant 0}$. We note that $\mathsf{FH}_{b_{\alpha\beta},\alpha\beta\alpha} = \mathsf{FH}_{b_{\alpha\beta},\beta\alpha\beta}$. The reader should compare these reflected paths with the first five paths of Fig. 22.

$$\begin{array}{ll} \mathsf{t}_i(q) = \mathsf{T}^{-1}(b_{\alpha\beta} + q, \varepsilon_i) & \mathsf{t}_{i+1}(q) = \mathsf{T}^{-1}(b_{\alpha\alpha} + q, \varepsilon_{i+1}) \\ \mathsf{t}_{i+2}(q) = \mathsf{T}^{-1}(b_{\alpha\beta\alpha} + q, \varepsilon_{i+2}) & \\ \mathsf{b}_i(q) = \mathsf{B}^{-1}(b_{\alpha\beta} + q, \varepsilon_{i+1}) & \mathsf{b}_{i+1}(q) = \mathsf{B}^{-1}(b_{\alpha\alpha\alpha} + q, \varepsilon_{i+1}) \\ \mathsf{b}_{i+2}(q) = \mathsf{B}^{-1}(b_{\alpha\beta\alpha} + q, \varepsilon_{i+2}). & \end{array}$$

We have that

$$\mathsf{t}_i(q) < \mathsf{t}_i(q+1) < \mathsf{t}_{i+2}(q) < \mathsf{t}_{i+2}(q+1) < \mathsf{t}_{i+1}(q) < \mathsf{t}_{i+1}(q+1)$$

$$\mathsf{b}_i(q) > \mathsf{b}_i(q+1) > \mathsf{b}_{i+2}(q) > \mathsf{b}_{i+2}(q+1) > \mathsf{b}_{i+1}(q) > \mathsf{b}_{i+1}(q+1)$$

for $1 \leqslant q \leqslant b_\alpha$ and

$$\mathsf{t}_i(1) < \mathsf{t}_{i+2}(0) < \mathsf{t}_{i+1}(1) \qquad \mathsf{t}_i(b_\alpha) < \mathsf{t}_{i+2}(b_\alpha + 1) < \mathsf{t}_{i+1}(b_\alpha)$$

$$\mathsf{b}_i(1) > \mathsf{b}_{i+2}(0) > \mathsf{b}_{i+1}(1) \qquad \mathsf{b}_i(b_\alpha) > \mathsf{b}_{i+2}(b_\alpha + 1) > \mathsf{b}_{i+1}(b_\alpha).$$

Thus the subexpression $\psi_{\underline{w}}$ is the nib truncation of a quasi-$(b_\alpha + 2)$-expression for $w = (13)$, which is independent of the choice of expression by Corollary 4.10. Thus the result follows. □

Proposition 6.8. *We have that*

$$(e_{\mathsf{P}_{\varnothing\varnothing}} \otimes \mathsf{hex}^{\varnothing\beta\alpha\beta}_{\varnothing\alpha\beta\alpha})(e_{\mathsf{P}_{\varnothing\varnothing}} \otimes \mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} \otimes e_{\mathsf{P}_{\beta\alpha}})\mathsf{adj}^{\varnothing\varnothing\alpha\alpha\beta\alpha}_{\varnothing\alpha\varnothing\alpha\beta\alpha}(e_{\mathsf{P}_{\varnothing\alpha}} \otimes \mathsf{hex}^{\varnothing\alpha\beta\alpha}_{\varnothing\beta\alpha\beta}) = \Upsilon^{\mathsf{P}_{\varnothing\varnothing\varnothing\beta\alpha\beta}}_{\mathsf{P}_{\varnothing\alpha\varnothing\beta\alpha} \otimes \mathsf{P}^\flat_\beta}$$

Proof. For $0 \leqslant q < b_{\beta\alpha}$, we claim that

$$(e_{\mathsf{P}_\varnothing} \otimes \mathsf{hex}_{\varnothing\alpha\beta\alpha}(q))(e_{\mathsf{P}_\varnothing} \otimes \mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} \otimes e_{\mathsf{P}_{\beta\alpha}})\mathsf{adj}^{\varnothing\alpha\alpha\beta\alpha}_{\alpha\varnothing\alpha\beta\alpha}(e_{\mathsf{P}_{\varnothing\alpha}} \otimes \mathsf{hex}^{\varnothing\alpha\beta\alpha}(q)) = \Upsilon^{\mathsf{P}_{\varnothing\varnothing} \otimes \mathsf{H}_{q,\alpha\beta\alpha}}_{\mathsf{P}_{\alpha\varnothing} \otimes \mathsf{FH}_{q,\beta\alpha\beta}}$$

and the statement of the proposition will immediately follow. We now prove our claim. We set $\mathsf{T}_q = \mathsf{P}_{\varnothing\varnothing} \otimes \mathsf{hex}_{\alpha\beta\alpha}(q)$ and $\mathsf{B}_q = \mathsf{P}_{\alpha\varnothing} \otimes \mathsf{FH}_{q,\alpha\beta\alpha}$. We consider the strand, Q, from $\mathsf{T}_q^{-1}(b_{\alpha\beta} + q, \varepsilon_i)$ on the top edge to $\mathsf{B}_q^{-1}(b_{\alpha\beta\alpha} + q, \varepsilon_{i+1})$ on the bottom edge of the diagram

$$(e_{\mathsf{P}_\varnothing} \otimes \mathsf{hex}_{\alpha\beta\alpha}(q)) \circ (\mathsf{fork}^{\varnothing\alpha}_{\alpha\alpha} \otimes e_{\mathsf{P}_{\alpha\beta}}) \circ (e_{\mathsf{P}_\alpha} \otimes \mathsf{hex}^{\alpha\beta\alpha}(q))$$

for $0 \leqslant q < b_{\alpha\beta}$. We wish to consider the non-zero degree crossings of the r_q-strand Q within the diagram. These are with the strands $\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3, \mathscr{Q}_4, \mathscr{Q}_5, \mathscr{Q}_6, \mathscr{Q}_7$ connecting the

$$\mathsf{T}^{-1}_{q+1}(b_{\alpha\beta} + q - 1, \varepsilon_i),\ \mathsf{T}^{-1}_{q+1}(b_{\alpha\beta\alpha} + q, \varepsilon_{i+1}),\ \mathsf{T}^{-1}_{q+1}(b_{\alpha\beta\alpha} + q + 1, \varepsilon_{i+1}),\\ \mathsf{T}^{-1}_{q+1}(b_{\alpha\beta\alpha} + q + 2, \varepsilon_{i+1})\\ \mathsf{T}^{-1}_{q+1}(b_{\alpha\beta\alpha\beta} + q + 1, \varepsilon_{i+2}),\ \mathsf{T}^{-1}_{q+1}(b_{\alpha\beta\alpha\beta} + q + 2, \varepsilon_{i+2}),\ \mathsf{T}^{-1}_{q+1}(b_{\alpha\beta\alpha\beta} + q + 3, \varepsilon_{i+2})$$

top vertices (which are ordered in increasingly from left to right) to the

$$\mathsf{B}^{-1}_{q+1}(b_{\alpha\beta} + q, \varepsilon_i),\ \mathsf{B}^{-1}_{q+1}(b_{\alpha\beta\alpha} + q, \varepsilon_{i+1}),\ \mathsf{B}^{-1}_{q+1}(b_{\alpha\beta} + q + 1, \varepsilon_i),\ \mathsf{B}^{-1}_{q+1}(b_{\alpha\beta\alpha} + q + 2, \varepsilon_{i+1})\\ \mathsf{B}^{-1}_{q+1}(b_{\alpha\beta\alpha\beta} + q + 1, \varepsilon_{i+2}),\ \mathsf{B}^{-1}_{q+1}(b_{\alpha\beta\alpha\beta} + q + 2, \varepsilon_{i+2}),\ \mathsf{B}^{-1}_{q+1}(b_{\alpha\beta\alpha\beta} + q + 3, \varepsilon_{i+2})$$

bottom vertices, respectively. The residues of these strands are $r_q + 1, r_q + 1, r_q, r_q - 1$ for the first row and $r_q + 1, r_q, r_q - 1$ or the second row. We have that

$$\mathsf{T}^{-1}_{q+1}(b_{\alpha\beta} + q - 1, \varepsilon_i) < \mathsf{T}^{-1}_{q+1}(b_{\alpha\beta\alpha} + q, \varepsilon_{i+1})\\ \mathsf{B}^{-1}_{q+1}(b_{\alpha\beta} + q, \varepsilon_i) > \mathsf{B}^{-1}_{q+1}(b_{\alpha\beta\alpha} + q, \varepsilon_{i+1})$$

and so the pair of strands $\mathscr{Q}_1$ and $\mathscr{Q}_2$ form a crossing of $(r_q + 1)$-strands. The strand Q crosses $\mathscr{Q}_1$ and $\mathscr{Q}_2$ exactly once each. The remaining 5 strands are all vertical lines (in other words their top and bottom vertices coincide). The strand Q crosses each of these vertical strands twice. (Thus the total degree contribution of these crossings is zero.)

We undo the crossing of Q with the triple of strands $\mathscr{Q}_5, \mathscr{Q}_6, \mathscr{Q}_7$ as in the proof of Proposition 4.4. Pull the Q strand through $\mathscr{Q}_4$ using case 4 of relation (R4) at the expense of acquiring a dot on Q (the other term is zero by case 1 of relation (R4)) we then pull the dot on Q upwards through the crossing of Q and $\mathscr{Q}_3$ using relation (R3) and obtain two terms: the first term, in which the dot has passed through the crossing, is zero by case 1 of relation (R4); in the second term, in which we undo one (of the two) crossings between Q and $\mathscr{Q}_3$, is equal to $\psi^{\mathsf{P}_{\varnothing} \otimes \mathsf{H}_{q,\alpha\beta\alpha}}_{\mathsf{P}_{\alpha} \otimes \mathsf{FH}_{q,\beta\alpha\beta}}$ as required.

Now suppose $b_{\alpha} \leqslant q < b_{\alpha\beta}$. The r_q-strand connecting the $\mathsf{B}^{-1}(4b_{\alpha} + 2b_{\beta} - q, \varepsilon_{i+1})$ and $\mathsf{T}^{-1}(4b_{\alpha} + 2b_{\beta} - q, \varepsilon_{i+1})$ top and bottom nodes double-crosses the $(r_q + 1)$- r_q- and $(r_q - 1)$- strands connecting the $\mathsf{T}^{-1}(4b_{\alpha} + 3b_{\beta} - q - 1, \varepsilon_{i+2})$, $\mathsf{T}^{-1}(4b_{\alpha} + 3b_{\beta} - q, \varepsilon_{i+2})$, $\mathsf{T}^{-1}(4b_{\alpha} + 3b_{\beta} - q + 1, \varepsilon_{i+2})$ top vertices to the $\mathsf{B}^{-1}(4b_{\alpha} + 3b_{\beta} - q - 1, \varepsilon_{i+2})$, $\mathsf{B}^{-1}(4b_{\alpha} + 3b_{\beta} - q, \varepsilon_{i+2})$, $\mathsf{B}^{-1}(4b_{\alpha} + 3b_{\beta} - q + 1, \varepsilon_{i+2})$ bottom vertices. We undo these double-crossings as in the proof of Proposition 4.4. □

Proposition 6.9. *We have that*

$$\mathsf{adj}^{\varnothing\varnothing\varnothing\beta\alpha\beta}_{\varnothing\varnothing\beta\alpha\varnothing\beta}(e_{\mathsf{P}_{\varnothing\varnothing\beta\alpha}} \otimes \mathsf{fork}^{\varnothing\beta}_{\beta\beta})\mathsf{hex}^{\varnothing\varnothing\beta\alpha\beta\beta}_{\varnothing\varnothing\alpha\beta\alpha\beta}\mathsf{adj}^{\varnothing\varnothing\alpha\beta\alpha\beta}_{\varnothing\alpha\varnothing\beta\alpha\beta} = \Upsilon^{\mathsf{P}_{\varnothing\varnothing\varnothing\beta\alpha\beta}}_{\mathsf{P}_{\varnothing\alpha\varnothing\beta\alpha} \otimes \mathsf{P}^{\flat}_{\beta}}. \tag{6.20}$$

Proof. For $0 \leqslant q \leqslant b_{\alpha\beta}$, we claim that

$$\Upsilon^{\mathsf{P}_{\varnothing\varnothing\varnothing\beta\alpha\beta}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{H}_{q,\varnothing\alpha\beta\alpha}\otimes\mathsf{P}_{\beta}}(e_{\mathsf{P}_{\varnothing}} \otimes \mathsf{hex}_{\varnothing\alpha\beta\alpha}(q+1) \otimes e_{\mathsf{P}_{\beta}}) = \Upsilon^{\mathsf{P}_{\varnothing\varnothing\varnothing\beta\alpha\beta}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{H}_{q,\varnothing\alpha\beta\alpha}\otimes\mathsf{P}_{\beta}}. \tag{6.21}$$

We decorate the top and bottom edges of the concatenated diagram in equation (6.21) by the paths $\mathsf{T} = \mathsf{P}_{\varnothing\varnothing\varnothing\beta\alpha\beta}$ and $\mathsf{B}_{q+1} = \mathsf{P}_{\varnothing} \otimes \mathsf{H}_{q+1,\varnothing\alpha\beta\alpha} \otimes \mathsf{P}_{\beta}$. For each $0 \leqslant q < b_{\beta}$ the strand (of residue $r_q \in \mathbb{Z}/e\mathbb{Z}$, say) connecting the top $\mathsf{T}^{-1}(b_{\alpha\beta} + q, \varepsilon_{i+1}))$th and $\mathsf{B}_q^{-1}(b_{\alpha\beta} + q, \varepsilon_{i+1})$th bottom vertices (both of which are equal to $(b_{\alpha\beta} + q)h + \emptyset(i+1))$ of the concatenated diagram has double-crossings of non-zero degree with three strands of residues $r_q + 1$, r_q and $r_q - 1$ connecting the $\mathsf{T}^{-1}(b_{\beta\alpha\beta} - 1 + q, \varepsilon_{i+2})$th, $\mathsf{T}^{-1}(b_{\beta\alpha\beta} + q, \varepsilon_{i+2})$th, and $\mathsf{T}^{-1}(b_{\beta\alpha\beta} + q + 1, \varepsilon_{i+2})$th top vertices to the $\mathsf{B}_q^{-1}(b_{\beta\alpha\beta} - 1 + q, \varepsilon_{i+2})$th, $\mathsf{B}_q^{-1}(b_{\beta\alpha\beta} + q, \varepsilon_{i+2})$th, and $\mathsf{B}_q^{-1}(b_{\beta\alpha\beta} + q + 1, \varepsilon_{i+2})$th bottom vertices respectively; we undo these crossings using Proposition 4.4. Now, for $b_{\beta} \leqslant q < b_{\alpha\beta}$ the claim is immediate as the concatenated diagram is step-preserving and has minimal length. Finally, we substitute equation (6.21) into equation (6.20) and the resulting diagram is again step-preserving and has minimal length and the result follows. □

6.7. The tetrahedron relation

We now check that the image of relation (S8) holds in the quiver Hecke algebra. Our aim is to show that

$$\mathsf{hex}^{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}_{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}\mathsf{hex}^{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}_{\alpha\gamma\beta\alpha\beta\gamma\varnothing\varnothing\varnothing}\mathsf{com}^{\alpha\gamma\beta\alpha\beta\gamma\varnothing\varnothing\varnothing}_{\alpha\beta\gamma\alpha\gamma\beta\varnothing\varnothing\varnothing}\mathsf{hex}^{\alpha\beta\gamma\alpha\gamma\beta\varnothing\varnothing\varnothing}_{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}\mathsf{hex}^{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}_{\beta\alpha\beta\gamma\alpha\beta\varnothing\varnothing\varnothing}\mathsf{com}^{\beta\alpha\beta\gamma\alpha\beta\varnothing\varnothing\varnothing}_{\beta\alpha\gamma\beta\alpha\beta\varnothing\varnothing\varnothing}$$

is equal to

$$\mathsf{com}^{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}_{\gamma\alpha\beta\gamma\alpha\gamma\varnothing\varnothing\varnothing}\mathsf{hex}^{\alpha\beta\gamma\alpha\gamma\varnothing\varnothing\varnothing\gamma}_{\gamma\alpha\beta\alpha\gamma\alpha\varnothing\varnothing\varnothing}\mathsf{hex}^{\gamma\alpha\beta\alpha\gamma\alpha\varnothing\varnothing\varnothing}_{\beta\alpha\beta\gamma\alpha\varnothing\varnothing\varnothing\gamma}\mathsf{com}^{\gamma\beta\alpha\beta\gamma\alpha\varnothing\varnothing\varnothing}_{\beta\gamma\alpha\gamma\beta\alpha\varnothing\varnothing\varnothing}\mathsf{hex}^{\beta\gamma\alpha\gamma\beta\alpha\varnothing\varnothing\varnothing}_{\varnothing\beta\alpha\gamma\alpha\beta\alpha\varnothing\varnothing}\mathsf{hex}^{\beta\alpha\gamma\alpha\beta\alpha\varnothing\varnothing\varnothing}_{\beta\alpha\gamma\beta\alpha\beta\varnothing\varnothing\varnothing}.$$

Proposition 6.10. *The element* $\psi^{\mathsf{P}_{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\beta\alpha\gamma\beta\alpha\beta\varnothing\varnothing\varnothing}}$ *is independent of the choice of reduced expression.*

Proof. For notational ease, we let $j = i+1$ and $k = i-1$ and we decorate the top and bottom edges with $\mathsf{T} = \mathsf{P}_{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}$ and $\mathsf{B} = \mathsf{P}_{\beta\alpha\gamma\beta\alpha\beta\varnothing\varnothing\varnothing}$ respectively. For each $b_{\beta} \leqslant q \leqslant b_{\alpha\beta} + 1$, we consider the collection of permutations w_q formed from the r_q-strands connecting each of the

$$\mathsf{B}_{i-1}(q) = \mathsf{B}^{-1}(q, \varepsilon_{i-1}) \qquad \mathsf{B}_{i}(q) = \mathsf{B}^{-1}(b_{\gamma} + q, \varepsilon_{i})$$
$$\mathsf{B}_{i+1}(q) = \mathsf{B}^{-1}(b_{\alpha\gamma} + q, \varepsilon_{i+1}) \qquad \mathsf{B}_{i+1}(q) = \mathsf{B}^{-1}(b_{\alpha\beta\gamma} + q, \varepsilon_{i+2})$$

bottom vertices to

$$\mathsf{T}_{i-1}(q) = \mathsf{T}^{-1}(q, \varepsilon_{i-1}) \qquad \mathsf{T}_{i}(q) = \mathsf{T}^{-1}(b_{\gamma} + q, \varepsilon_{i})$$
$$\mathsf{T}_{i+1}(q) = \mathsf{T}^{-1}(b_{\alpha\gamma} + q, \varepsilon_{i+1}) \qquad \mathsf{T}_{i+1}(q) = \mathsf{T}^{-1}(b_{\alpha\beta\gamma} + q, \varepsilon_{i+2})$$

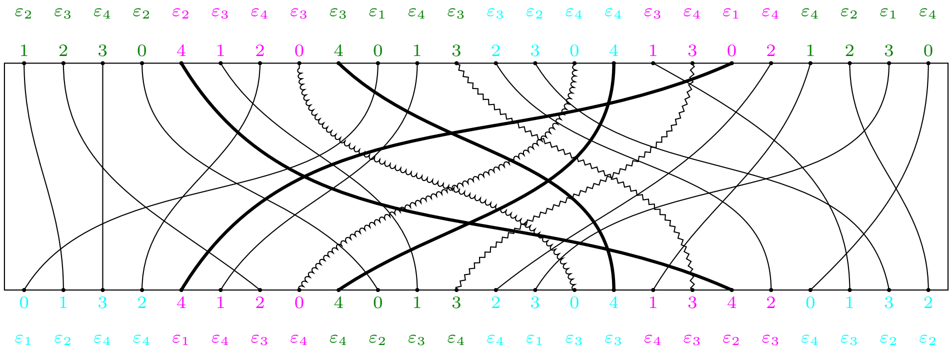


Fig. 31. The element $\psi^{\mathsf{P}_{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\beta\alpha\gamma\beta\alpha\beta\varnothing\varnothing\varnothing}}$ for $p = 5$, $h = 3$, $\ell = 1$ and $\boldsymbol{\alpha} = \varepsilon_2 - \varepsilon_3$, $\boldsymbol{\beta} = \varepsilon_3 - \varepsilon_4$, $\boldsymbol{\gamma} = \varepsilon_1 - \varepsilon_2$. The thick black 4-strands form a $\underline{w} = s_3s_2s_1s_3s_2s_3$ braid. Together with the wiggly strands, these form a subexpression nib$\psi_{\underline{w}_3}$ containing all bad crossings.

top vertices respectively. By definition $r_q = r_{q+1}+1$ for $b_{\beta} \leqslant q < b_{\alpha\beta}+1$. We let $\underline{w}$ denote the subexpression consisting of all strands from (the union of) the w_q-subexpressions for $b_{\beta} \leqslant q \leqslant b_{\alpha\beta} + 1$. One can verify, simply by looking at the paths T and B (and their residue sequences) that any bad-crossing in w belongs to $\psi_{\mathrm{nib}(\underline{w})}$. We have that

$$\begin{aligned}&\mathsf{B}_{i-1}(q) < \mathsf{B}_{i-1}(q+1) < \mathsf{B}_{i+2}(q) < \mathsf{B}_{i+2}(q+1) < \mathsf{B}_{i+1}(q) < \mathsf{B}_{i+1}(q+1) < \mathsf{B}_{i}(q)\\ &\quad < \mathsf{B}_{i}(q+1)\\ &\mathsf{T}_{i-1}(q) > \mathsf{T}_{i-1}(q+1) > \mathsf{T}_{i+2}(q) > \mathsf{T}_{i+2}(q+1) > \mathsf{T}_{i+1}(q) > \mathsf{T}_{i+1}(q+1) > \mathsf{T}_{i}(q)\\ &\quad > \mathsf{T}_{i}(q+1),\end{aligned}$$

for $b_{\beta} < q < b_{\alpha\beta}$. In other words, the r_q-strands for $b_{\beta} < q \leqslant b_{\alpha\beta}$ form a $\psi_{(1,4)(2,3)_{b_{\alpha}}}$ braid (and thus this subexpression is quasi-dilated and of breadth b_{α}). We now restrict to the case $q = b_{\beta}$, as the $q = b_{\alpha\beta} + 1$ is similar. We have that

$$\begin{aligned}&\mathsf{B}_{i-1}(b_{\beta}+1) < \mathsf{B}_{i+2}(b_{\beta}) < \mathsf{B}_{i+2}(b_{\beta}+1) < \mathsf{B}_{i+1}(b_{\beta}) < \mathsf{B}_{i+1}(b_{\beta}+1) < \mathsf{B}_{i}(b_{\beta}+1)\\ &\mathsf{T}_{i-1}(b_{\beta}+1) > \mathsf{T}_{i+2}(b_{\beta}) > \mathsf{T}_{i+2}(b_{\beta}+1) > \mathsf{T}_{i+1}(b_{\beta}) > \mathsf{T}_{i+1}(b_{\beta}+1) > \mathsf{T}_{i}(b_{\beta}+1).\end{aligned}$$

(We have not considered the strands connecting $\mathsf{B}_{i-1}(b_{\beta})$ and $\mathsf{T}_{i-1}(b_{\beta})$ or $\mathsf{B}_{i}(b_{\beta})$ and $\mathsf{T}_{i}(b_{\beta})$ as these were removed under the nib truncation map.) Thus $\psi_{\mathrm{nib}(\underline{w})}$ is independent of the choice of expression by Corollary 4.10 and the result follows. See Fig. 31 for an example. □

Proposition 6.11. *We have that* $\Upsilon^{\mathsf{P}_{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\beta\alpha\gamma\beta\alpha\beta\varnothing\varnothing\varnothing\varnothing}}$ *is equal to both*

$$\mathsf{hex}^{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}_{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}\mathsf{hex}^{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}_{\alpha\gamma\beta\alpha\beta\gamma\varnothing\varnothing\varnothing}\mathsf{com}^{\alpha\gamma\beta\alpha\beta\gamma\varnothing\varnothing\varnothing}_{\alpha\beta\gamma\alpha\gamma\beta\varnothing\varnothing\varnothing}\mathsf{hex}^{\alpha\beta\gamma\alpha\gamma\beta\varnothing\varnothing\varnothing}_{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}\mathsf{hex}^{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}_{\beta\alpha\beta\gamma\alpha\beta\varnothing\varnothing\varnothing}\mathsf{com}^{\beta\alpha\beta\gamma\alpha\beta\varnothing\varnothing\varnothing}_{\beta\alpha\gamma\beta\alpha\beta\varnothing\varnothing\varnothing}$$

and

$$\mathsf{com}^{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}_{\gamma\alpha\beta\gamma\alpha\gamma\varnothing\varnothing\varnothing}\mathsf{hex}^{\alpha\beta\gamma\alpha\gamma\varnothing\varnothing\varnothing\gamma}_{\gamma\alpha\beta\alpha\gamma\alpha\varnothing\varnothing\varnothing}\mathsf{hex}^{\gamma\alpha\beta\alpha\gamma\alpha\varnothing\varnothing\varnothing}_{\beta\alpha\beta\gamma\alpha\varnothing\varnothing\varnothing\gamma}\mathsf{com}^{\gamma\beta\alpha\beta\gamma\alpha\varnothing\varnothing\varnothing}_{\beta\gamma\alpha\gamma\beta\alpha\varnothing\varnothing\varnothing}\mathsf{hex}^{\beta\gamma\alpha\gamma\beta\alpha\varnothing\varnothing\varnothing}_{\varnothing\beta\alpha\gamma\alpha\beta\alpha\varnothing\varnothing}\mathsf{hex}^{\beta\alpha\gamma\alpha\beta\alpha\varnothing\varnothing\varnothing}_{\beta\alpha\gamma\beta\alpha\beta\varnothing\varnothing\varnothing}.$$

Proof. We set $k = i - 1$, $j = i + 1$. We will prove the first equality as the second is very similar (for more details, see Remark 6.12). We proceed from the centre of the diagram, considering the first pair of hexagons (on top and bottom of a pair of commutators), the second pairs of hexagons (on top and bottom of the previous product) and then finally the last commutator (below the previous product).

Step 1. We add the first pair of hexagonal generators symmetrically as follows

$$\mathsf{hex}^{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}_{\alpha\gamma\beta\alpha\beta\gamma\varnothing\varnothing\varnothing}(e_{\mathsf{P}_{\alpha}} \otimes \mathsf{com}^{\gamma\beta\alpha\beta\gamma}_{\beta\gamma\alpha\gamma\beta} \otimes e_{\mathsf{P}_{\varnothing\varnothing\varnothing}})\mathsf{hex}^{\alpha\beta\gamma\alpha\gamma\beta\varnothing\varnothing\varnothing}_{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing} = \Upsilon^{\mathsf{P}_{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}}. \tag{6.22}$$

The only points worth bearing in mind are (i) double-crossings strands of non-adjacent residue can be undone trivially and (ii) that the implicit adjustments in the definitions of $\mathsf{hex}^{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}_{\alpha\gamma\beta\alpha\beta\gamma\varnothing\varnothing\varnothing}$ and $\mathsf{hex}^{\alpha\beta\gamma\alpha\gamma\beta\varnothing\varnothing\varnothing}_{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}$ will give rise to (a total of $|b_{\alpha}-b_{\beta}|+|b_{\alpha}-b_{\gamma}|+|b_{\beta}-b_{\gamma}|$) double-crossings which can be undone as in the proof of Proposition 4.4.

Step 2. We now add the next pair of hexagonal generators symmetrically to the diagram, $\Upsilon^{\mathsf{P}_{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}}$, output by the previous step in the procedure. We first note that

$$\mathsf{adj}^{\mathsf{P}_{\varnothing}\otimes\mathsf{H}_{0,\alpha\gamma\alpha}\otimes\mathsf{P}_{\beta\alpha\gamma\varnothing\varnothing}}_{\mathsf{P}_{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}} \circ \Upsilon^{\mathsf{P}_{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}} \circ \mathsf{adj}^{\mathsf{P}_{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{H}_{0,\alpha\beta\alpha}\otimes\mathsf{P}_{\gamma\alpha\beta\varnothing\varnothing}} = \Upsilon^{\mathsf{P}_{\varnothing}\otimes\mathsf{H}_{0,\alpha\gamma\alpha}\otimes\mathsf{P}_{\beta\alpha\gamma\varnothing\varnothing}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{H}_{0,\alpha\beta\alpha}\otimes\mathsf{P}_{\gamma\alpha\beta\varnothing\varnothing}}$$

again by (a total of $|b_{\beta} - b_{\gamma}|$ applications of) Proposition 4.4. We claim that

$$\big(\mathsf{hex}^{\varnothing\alpha\gamma\alpha}(q)\otimes e_{\mathsf{P}_{\beta\alpha\gamma\varnothing\varnothing}}\big)\Upsilon^{\mathsf{P}_{\varnothing}\otimes\mathsf{H}_{q,\alpha\gamma\alpha}\otimes\mathsf{P}_{\beta\alpha\gamma\varnothing\varnothing}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{H}_{q,\alpha\beta\alpha}\otimes\mathsf{P}_{\gamma\alpha\beta\varnothing\varnothing}}\big(\mathsf{hex}_{\varnothing\alpha\beta\alpha}(q)\otimes e_{\mathsf{P}_{\gamma\alpha\beta\varnothing\varnothing}}\big) = \Upsilon^{\mathsf{P}_{\varnothing}\otimes\mathsf{H}_{q+1,\alpha\gamma\alpha}\otimes\mathsf{P}_{\beta\alpha\gamma\varnothing\varnothing}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{H}_{q+1,\alpha\beta\alpha}\otimes\mathsf{P}_{\gamma\alpha\beta\varnothing\varnothing}} \tag{6.23}$$

for $0 \leqslant q < \max\{b_{\beta}, b_{\gamma}\} + b_{\alpha}$. For $0 \leqslant q \leqslant b_{\alpha} + |b_{\beta} - b_{\gamma}|$ the concatenated diagram on the lefthand-side of equation (6.23) contains a distinguished strand connecting the $\mathsf{T}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + q + 1, \varepsilon_i)$ top and $\mathsf{B}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + q + 1, \varepsilon_i)$ bottom vertices. For $0 \leqslant q \leqslant b_{\alpha} + |b_{\beta} - b_{\gamma}|$ the distinguished strand passes from left to right and back again, thus admitting a double-crossing with each of the $(r_q - 1)$-, r_q-, $(r_q + 1)$-strands connecting the

$$\mathsf{T}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha} + q, \varepsilon_{i+1}) \qquad \mathsf{T}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha} + q + 1, \varepsilon_{i+1})$$
$$\mathsf{T}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha} + q + 2, \varepsilon_{i+1})$$

top vertices to the

$$\mathsf{B}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha} + q, \varepsilon_{i+1}) \qquad \mathsf{B}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha} + q + 1, \varepsilon_{i+1})$$
$$\mathsf{B}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha} + q + 2, \varepsilon_{i+1})$$

bottom vertices. For $|b_{\beta} - b_{\gamma}| \leqslant q \leqslant b_{\alpha} + |b_{\beta} - b_{\gamma}|$ the distinguished strand *also* admits a double-crossing with each of the $(r_q - 1)$-, r_q-, $(r_q + 1)$-strands connecting the

$$\mathsf{T}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha\beta} + q, \varepsilon_{i+2}) \qquad \mathsf{T}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha\beta} + q + 1, \varepsilon_{i+2})$$
$$\mathsf{T}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha\beta} + q + 2, \varepsilon_{i+2})$$

top vertices to the

$$\mathsf{B}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha\beta} + q, \varepsilon_{i+2}) \qquad \mathsf{B}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha\beta} + q + 1, \varepsilon_{i+2})$$
$$\mathsf{B}^{-1}(\min\{b_{\beta}, b_{\gamma}\} + b_{\alpha\beta} + q + 2, \varepsilon_{i+2})$$

bottom vertices. Note we have broken these strands into two triples. For $0 \leqslant q \leqslant b_{\alpha} + |b_{\beta} - b_{\gamma}|$ we undo the double-crossing of the distinguished strand with the former triple using a single application of Proposition 4.4. For $|b_{\beta} - b_{\gamma}| \leqslant q \leqslant b_{\alpha} + |b_{\beta} - b_{\gamma}|$ we undo the double-crossing of the distinguished strand with the latter triple and then the former triple as in the proof of Proposition 4.4. Thus equation (6.23) follows. If $b_{\beta} > b_{\gamma}$ (respectively $b_{\gamma} > b_{\beta}$) we must now multiply on the bottom (respectively top) by the remaining terms to obtain a minimal, step-preserving diagram. We hence deduce that

$$(\mathsf{hex}_{\varnothing\alpha\gamma\alpha} \otimes e_{\mathsf{P}_{\beta\alpha\gamma\varnothing\varnothing}})\Upsilon^{\mathsf{P}_{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}}(\mathsf{hex}_{\varnothing\alpha\beta\alpha} \otimes e_{\mathsf{P}_{\gamma\alpha\beta\varnothing\varnothing}}) = \Upsilon^{\mathsf{H}_{b_{\alpha\gamma},\varnothing\alpha\gamma\alpha}\otimes\mathsf{P}_{\beta\alpha\gamma\varnothing\varnothing}}_{\mathsf{H}_{b_{\alpha\beta},\varnothing\alpha\beta\alpha}\otimes\mathsf{P}_{\gamma\alpha\beta\varnothing\varnothing}}.$$

We now multiply on the top and bottom by the other "halves" of the hexagonal generators to get

$$\mathsf{hex}^{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}_{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}\Upsilon^{\mathsf{P}_{\alpha\gamma\alpha\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}}\mathsf{hex}^{\alpha\beta\alpha\gamma\alpha\beta\varnothing\varnothing\varnothing}_{\beta\alpha\beta\gamma\alpha\beta\varnothing\varnothing\varnothing} = \Upsilon^{\mathsf{P}_{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\beta\alpha\beta\gamma\alpha\beta\varnothing\varnothing\varnothing}} \tag{6.24}$$

where here the hexagonal terms are minimal and step-preserving, but we must again undo any double-crossings arising from adjustments as in the proof of Proposition 4.4. We emphasise that the righthand-side of equation (6.24) is independent of the choice of reduced expression, which can be shown in a similar fashion to Proposition 6.10.

Step 3. For $0 \leqslant q < b_{\beta\gamma}$, we claim that

$$\Upsilon^{\mathsf{P}_{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\beta\alpha}\otimes\mathsf{C}^{q,\beta\gamma}\otimes e_{\mathsf{P}_{\alpha\beta\varnothing\varnothing\varnothing}}}(e_{\mathsf{P}_{\beta\alpha}} \otimes \mathsf{com}^{\beta\gamma}(q) \otimes e_{\mathsf{P}_{\alpha\beta\varnothing\varnothing\varnothing}}) = \Upsilon^{\mathsf{P}_{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\beta\alpha}\otimes\mathsf{C}^{q+1,\beta\gamma}\otimes\mathsf{P}_{\alpha\beta\varnothing\varnothing\varnothing}}$$

and for $b_{\beta\gamma} \geqslant q > 0$, we claim that

$$\Upsilon^{\mathsf{P}_{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\beta\alpha}\otimes\mathsf{C}_{q,\gamma\beta}\otimes e_{\mathsf{P}_{\alpha\beta\varnothing\varnothing\varnothing}}}(e_{\mathsf{P}_{\beta\alpha}} \otimes \mathsf{com}_{\gamma\beta}(q) \otimes e_{\mathsf{P}_{\alpha\beta\varnothing\varnothing\varnothing}}) = \Upsilon^{\mathsf{P}_{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\beta\alpha}\otimes\mathsf{C}_{\gamma\beta}(q-1)\otimes\mathsf{P}_{\alpha\beta\varnothing\varnothing\varnothing}}.$$

We consider the former product, as the latter is similar. If $b_{\gamma} > b_{\beta}$, then the concatenated diagram is minimal and step-preserving. If $b_{\gamma} \leqslant b_{\beta}$ then the r_q-braid connecting the strands

$$\mathsf{T}^{-1}(q+1, \varepsilon_{i-1}) \quad \mathsf{T}^{-1}(b_{\gamma} + q + 1, \varepsilon_{i}) \quad \mathsf{T}^{-1}(b_{\alpha\gamma} + q + 1, \varepsilon_{i+1}) \quad \mathsf{T}^{-1}(b_{\alpha\beta\gamma} + q + 1, \varepsilon_{i+2})$$
$$\mathsf{B}^{-1}(q+1, \varepsilon_{i-1}) \quad \mathsf{B}^{-1}(b_{\gamma} + q + 1, \varepsilon_{i}) \quad \mathsf{B}^{-1}(b_{\alpha\gamma} + q + 1, \varepsilon_{i+1}) \quad \mathsf{B}^{-1}(b_{\alpha\beta\gamma} + q + 1, \varepsilon_{i+2})$$

top and bottom vertices form the non-minimal expression $(s_2s_1s_3s_2s_3)s_3$ (the bracketed term belongs to the multiplicand $\Upsilon^{\mathsf{P}_{\gamma\alpha\gamma\beta\alpha\gamma\varnothing\varnothing\varnothing}}_{\mathsf{P}_{\beta\alpha\beta\gamma\alpha\beta\varnothing\varnothing\varnothing}}$ and so can be chosen arbitrarily, we have chosen the simplest form for what follows). The r_q-strand with label ε_i double-crosses the (r_q-1)-strand connecting the $\mathsf{T}^{-1}(b_{\alpha\gamma}+q+2,\varepsilon_{i+1})$ and $\mathsf{B}^{-1}(b_{\alpha\gamma}+q+2,\varepsilon_{i+1})$ top and bottom vertices. We undo this double-crossing at the expense of placing a KLR dot on the r_q-strand (the other term is zero, by case 1 of equation (R4)). We then pull this dot through the r_q-crossing labelled by the ε_i and ε_{i+2} strands and hence undoing the bottommost crossing (the other, dotted, term is zero, again by case 1 of equation (R4)). Thus our r_q-braid now forms the non-minimal expression $s_2s_1s_3s_2s_3$. The r_q-crossing of strands connecting the

$$\mathsf{T}^{-1}(b_{\alpha\gamma}+q+1,\varepsilon_{i+1}),\ \mathsf{T}^{-1}(b_{\beta}+q+1,\varepsilon_{i}),\ \mathsf{B}^{-1}(b_{\beta}+q+1,\varepsilon_{i}),\ \mathsf{B}^{-1}(b_{\alpha\gamma}+q+1,\varepsilon_{i+1})$$

top and bottom vertices is bi-passed on the left by the (r_q+1)-strand connecting the $\mathsf{T}^{-1}(b_{\alpha\gamma}+q,\varepsilon_{i+1})$ and $\mathsf{B}^{-1}(b_{\alpha\gamma}+q,\varepsilon_{i+1})$ vertices. We pull this (r_q+1)-strand through this crossing using relation (R5) and hence obtain the diagram in which the crossing is undone (at the expense of another term, which is zero by Lemma 4.1). Thus our r_q-braid now forms the minimal expression $s_2s_1s_3s_2$, and the diagram is minimal and step-preserving, as required. □

Remark 6.12. The reader should note that in equation (S8), the righthand-side is obtained by first flipping the lefthand-side through the horizontal and vertical axes and then swapping the β and γ labels. The "very similar" proof of the second equality in Proposition 6.11 amounts to rewriting the above argument but with indices of the crossing-strands determined by the horizontal and vertical flips and recolouring (swap mentions of b_β and b_γ) of the indices in the proof above.

6.8. The tricoloured commutativity relations

We now verify the two relations depicted in (S7). Namely, we will show that

$$\begin{aligned}\Upsilon^{\varnothing\alpha\beta\alpha\delta}_{\delta\varnothing\beta\alpha\beta} &= \mathsf{hex}^{\varnothing\alpha\beta\alpha\delta}_{\varnothing\beta\alpha\beta\delta}\mathsf{com}^{\varnothing\beta\alpha\beta\delta}_{\varnothing\beta\alpha\delta\beta}\mathsf{com}^{\varnothing\beta\alpha\delta\beta}_{\varnothing\beta\delta\alpha\beta}\mathsf{com}^{\varnothing\beta\delta\alpha\beta}_{\varnothing\delta\beta\alpha\beta}\mathsf{adj}^{\varnothing\delta\beta\alpha\beta}_{\delta\varnothing\beta\alpha\beta} \\ &= \mathsf{com}^{\varnothing\alpha\beta\alpha\delta}_{\varnothing\alpha\beta\delta\alpha}\mathsf{com}^{\varnothing\alpha\beta\delta\alpha}_{\varnothing\alpha\delta\beta\alpha}\mathsf{com}^{\varnothing\alpha\delta\beta\alpha}_{\varnothing\delta\alpha\beta\alpha}\mathsf{adj}^{\varnothing\delta\alpha\beta\alpha}_{\delta\varnothing\alpha\beta\alpha}\mathsf{hex}^{\delta\varnothing\alpha\beta\alpha}_{\delta\varnothing\beta\alpha\beta}\end{aligned} \tag{6.25}$$

and we have that

$$\Upsilon^{\beta\gamma\delta}_{\delta\gamma\beta} = \mathsf{com}^{\beta\gamma\delta}_{\beta\delta\gamma}\mathsf{com}^{\beta\delta\gamma}_{\delta\beta\gamma}\mathsf{com}^{\delta\beta\gamma}_{\delta\gamma\beta} = \mathsf{com}^{\beta\gamma\delta}_{\gamma\beta\delta}\mathsf{com}^{\gamma\beta\delta}_{\beta\delta\gamma}\mathsf{com}^{\beta\delta\gamma}_{\delta\gamma\beta}. \tag{6.26}$$

We suppress mention of crossing which can be undone using the commutativity KLR relations in what follows.

Consider the former product in equation (6.25). For $1 \leqslant q \leqslant b_\delta$ the strand connecting the $\mathsf{P}^{-1}_{\varnothing\alpha\beta\alpha\delta}(q,\varepsilon_j)$ and $\mathsf{P}^{-1}_{\delta\varnothing\beta\alpha\beta}(q,\varepsilon_j)$ northern and southern vertices double-crosses the strands connecting each of the $\mathsf{P}^{-1}_{\varnothing\alpha\beta\alpha\delta}(b_\beta + p, \varepsilon_{j+1})$ and $\mathsf{P}^{-1}_{\delta\varnothing\beta\alpha\beta}(b_\beta + p, \varepsilon_{j+1})$ northern and southern vertices for $p = q-1, q, q+1$. Now consider the latter product of equation (6.25). For $1 \leqslant q \leqslant b_\delta$ the strand connecting the $\mathsf{P}^{-1}_{\varnothing\alpha\beta\alpha\delta}(b_{\alpha\beta\alpha} + q, \varepsilon_j)$ and $\mathsf{P}^{-1}_{\delta\varnothing\beta\alpha\beta}(b_{\alpha\beta\alpha} + q, \varepsilon_j)$ northern and southern vertices double-crosses the strands connecting each of the $\mathsf{P}^{-1}_{\varnothing\alpha\beta\alpha\delta}(b_{\alpha\beta\alpha\beta} + p, \varepsilon_{j+1})$ and $\mathsf{P}^{-1}_{\delta\varnothing\beta\alpha\beta}(b_{\alpha\beta\alpha\beta} + p, \varepsilon_{j+1})$ northern and southern vertices for $p = q-1, q, q+1$. For each $1 \leqslant q \leqslant b_\delta$ we can undo these crossings using Proposition 4.4.

Consider the former product in equation (6.26). For $1 \leqslant q \leqslant \min\{b_\beta, b_\delta\}$ the strand connecting the $\mathsf{P}^{-1}_{\beta\gamma\delta}(q, \varepsilon_k)$ and $\mathsf{P}^{-1}_{\delta\gamma\beta}(q, \varepsilon_k)$ northern and southern vertices double-crosses the strands connecting each of the $\mathsf{P}^{-1}_{\beta\gamma\delta}(b_\gamma + p, \varepsilon_{k+1})$ and $\mathsf{P}^{-1}_{\delta\gamma\beta}(b_\gamma + p, \varepsilon_{k+1})$ northern and southern vertices for $p = q-1, q, q+1$. Now consider the latter product in equation (6.26). For $0 \leqslant q < \min\{b_\beta, b_\delta\}$ the strand connecting the $\mathsf{P}^{-1}_{\beta\gamma\delta}(b_{\beta\gamma\delta} - q, \varepsilon_k)$ and $\mathsf{P}^{-1}_{\delta\gamma\beta}(b_{\beta\gamma\delta} - q, \varepsilon_k)$ northern and southern vertices double-crosses the strands connecting each of the $\mathsf{P}^{-1}_{\beta\gamma\delta}(b_{\beta\gamma\gamma\delta} - p, \varepsilon_{k+1})$ and $\mathsf{P}^{-1}_{\delta\gamma\beta}(b_{\beta\gamma\gamma\delta} - p, \varepsilon_{k+1})$ northern and southern vertices for $p = q+1, q, q-1$. For each $0 \leqslant q < \min\{b_\beta, b_\delta\}$ we can undo these crossings using Proposition 4.4.

Thus we obtain the desired equalities and the image of relation (S7) holds.

6.9. The fork and commutator

Let $\gamma, \beta \in \Pi$ label two commuting reflections, we now verify the middle relation depicted in (S6), namely that

$$\Upsilon^{\mathsf{P}_{\beta\varnothing\gamma}}_{\mathsf{P}_\gamma \otimes \mathsf{P}^\flat_\gamma \otimes \mathsf{P}_\beta} = (e_{\mathsf{P}_\beta} \otimes \mathsf{fork}^{\varnothing\gamma}_{\gamma\gamma})(\mathsf{com}^{\beta\gamma}_{\gamma\beta} \otimes e_{\mathsf{P}_\gamma})(e_{\mathsf{P}_\gamma} \otimes \mathsf{com}^{\beta\gamma}_{\gamma\beta})$$
$$= (\mathsf{adj}^{\beta\varnothing}_{\varnothing\beta} \otimes e_{\mathsf{P}_\gamma})(e_{\mathsf{P}_\varnothing} \otimes \mathsf{com}^{\beta\gamma}_{\gamma\beta})(\mathsf{fork}^{\varnothing\gamma}_{\gamma\gamma} \otimes e_{\mathsf{P}_\beta})$$

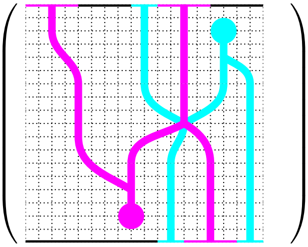
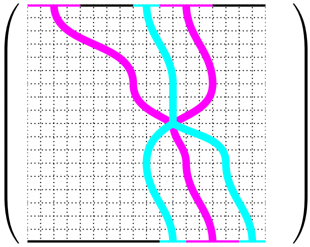
as both products produce minimal, step-preserving, and residue commutative elements (after undoing any double-crossings of non-adjacent residue using the commutativity relations).

6.10. Naturality of adjustment

For each generator, we must check the corresponding adjustment naturality relation pictured in Figs. 6 and 7. For the unique one-sided naturality relation, $(\mathsf{spot}^{\varnothing}_{\alpha} \otimes e_{\mathsf{P}_\varnothing})\mathsf{adj}^{\alpha\varnothing}_{\varnothing\alpha} = e_{\mathsf{P}_\varnothing} \otimes \mathsf{spot}^{\varnothing}_{\alpha}$, this follows by a generalisation of the proof of Proposition 5.11. The remaining relations all follow from Proposition 4.4.

6.11. Cyclicity

Given $\alpha, \beta \in \Pi$ labelling a pair of non-commuting reflections, we now verify relation (S4), namely that

Ψ  $=\Psi$  . (6.27)

The lefthand-side of equation (6.27) is equal to

$$(e_{\mathsf{P}_{\alpha\varnothing\beta\alpha}} \otimes (\mathsf{spot}^{\varnothing}_{\beta}\otimes e_{\mathsf{P}_{\varnothing}})\mathsf{fork}^{\beta\varnothing}_{\beta\beta})\mathsf{hex}^{\alpha\varnothing\beta\alpha\beta\beta}_{\alpha\varnothing\alpha\beta\alpha\beta}((\mathsf{adj}^{\alpha\varnothing\alpha}_{\varnothing\alpha\alpha}(e_{\mathsf{P}_{\varnothing}}\otimes(\mathsf{fork}^{\alpha\alpha}_{\varnothing\alpha}(e_{\mathsf{P}_{\varnothing}}\otimes \mathsf{spot}^{\alpha}_{\varnothing}))))\otimes e_{\mathsf{P}_{\beta\alpha\beta}})$$

which is minimal and step-preserving and so is equal to $\Upsilon^{\alpha\varnothing\beta\alpha\beta\varnothing}_{\varnothing\varnothing\varnothing\beta\alpha\beta}$ (which is independent of the choice of reduced expression by simply re-indexing the proof of Proposition 5.12). The righthand-side of equation (6.27) is equal to

$$\mathsf{adj}^{\alpha\varnothing\beta\alpha\varnothing\varnothing}_{\varnothing\varnothing\alpha\beta\alpha\varnothing}(e_{\mathsf{P}_{\varnothing}}\otimes \mathsf{hex}^{\varnothing\alpha\beta\alpha}_{\varnothing\beta\alpha\beta}\otimes e_{\mathsf{P}_{\varnothing}})(e_{\mathsf{P}_{\varnothing\varnothing}}\otimes \mathsf{adj}^{\beta\alpha\beta\varnothing}_{\varnothing\beta\alpha\beta}). \tag{6.28}$$

It will suffice to show that

$$(\mathsf{hex}_{\beta\alpha\beta}\otimes e_{\mathsf{P}_{\emptyset}})\mathsf{adj}^{\beta\alpha\beta\emptyset}_{\emptyset\beta\alpha\beta}=\Upsilon^{\mathsf{H}_{b_{\alpha\beta},\beta\alpha\beta}\otimes\mathsf{P}_{\emptyset}}_{\emptyset\varnothing\alpha\beta\alpha} \tag{6.29}$$

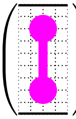
as b_{β} applications of this will simplify equation (6.28) so that it is minimal and step-preserving. The lefthand-side of equation (6.29) contains an r-strand from $\mathsf{H}^{-1}_{q,\alpha\beta\alpha}(q+1,\varepsilon_{i+1})$ to $\mathsf{P}^{-1}_{\emptyset\varnothing\alpha\beta\alpha}(q+1,\varepsilon_{i+1})$ which double-crosses the strands connecting the top and bottom vertices

$$\begin{array}{ccc}\mathsf{H}^{-1}_{q,\alpha\beta\alpha}(b_{\alpha}+q,\varepsilon_i) & \mathsf{H}^{-1}_{q,\alpha\beta\alpha}(b_{\alpha}+q+1,\varepsilon_i) & \mathsf{H}^{-1}_{q,\alpha\beta\alpha}(b_{\alpha}+q+2,\varepsilon_i)\\ \mathsf{P}^{-1}_{\emptyset\varnothing\alpha\beta\alpha}(b_{\alpha}+q,\varepsilon_i) & \mathsf{P}^{-1}_{\emptyset\varnothing\alpha\beta\alpha}(b_{\alpha}+q+1,\varepsilon_i) & \mathsf{P}^{-1}_{\emptyset\varnothing\alpha\beta\alpha}(b_{\alpha}+q+2,\varepsilon_i),\end{array}$$

respectively. We undo these double-crossings as in the proof of Proposition 4.4 to obtain $\Upsilon^{\alpha\varnothing\beta\alpha\beta\varnothing}_{\varnothing\varnothing\varnothing\beta\alpha\beta}$.

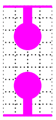
6.12. *Some results concerning doubly-spotted Soergel diagrams*

The remainder of this section is dedicated to proving results involving the "doubly-spotted" Soergel diagrams. These proofs are of a different flavour to the "timeline" proofs considered above. We shall see that each Soergel spot diagram roughly corresponds to "half" of a KLR dotted diagram. This idea is easiest to see through its manifestation in the grading (Soergel spots have degree 1, whereas KLR dots have degree 2). We have that

$$\Psi\left(\right.$$  $$\left.\right) = e_{\mathsf{P}_{\varnothing}} \left(\prod_{b_{\alpha} > q \geqslant 0} \psi^{qh+i}_{b_{\alpha}h - b_{\alpha} + q + 1} \right) e_{\mathsf{P}_{\alpha}} \left(\prod_{0 \leqslant q < b_{\alpha}} \psi^{b_{\alpha}h - b_{\alpha} + q + 1}_{qh+i} \right) e_{\mathsf{P}_{\varnothing}}$$

$$= e_{\mathsf{P}_{\varnothing}} (y_{b_{\alpha}h - h + \emptyset(i+1)} - y_i) e_{\mathsf{P}_{\varnothing}} \tag{6.30}$$

by relation (R4); this is easily seen from the fact that the only crossings of non-zero degree are a double-crossing of strands which begin and end at the $\mathsf{P}^{-1}_{\varnothing}(b_{\alpha}, \varepsilon_{i+1}) = (b_{\alpha}h - h + \emptyset(i+1))$ and $\mathsf{P}^{-1}_{\varnothing}(1, \varepsilon_i) = i$ points on the top and bottom edges of the diagram (and application of case 3 of relation (R4)). Arguing similarly, one has that

$$\Psi\left(\right.$$  $$\left.\right) = e_{\mathsf{P}_{\alpha}} \left(\prod_{0 \leqslant q < b_{\alpha}} \psi^{b_{\alpha}h - b_{\alpha} + q + 1}_{qh+i} \right) e_{\mathsf{P}_{\varnothing}} \left(\prod_{b_{\alpha} > q \geqslant 0} \psi^{qh+i}_{b_{\alpha}h - b_{\alpha} + q + 1} \right) e_{\mathsf{P}_{\alpha}}$$

$$= e_{\mathsf{P}_{\alpha}} (y_{b_{\alpha}h - b_{\alpha} - h + 1 + \alpha(i+1)} - y_{b_{\alpha}h - b_{\alpha} + 1}) e_{\mathsf{P}_{\alpha}}. \tag{6.31}$$

Proposition 6.13. *Let $\alpha = \varepsilon_i - \varepsilon_{i+1}, \gamma = \varepsilon_k - \varepsilon_{k+1} \in \Pi$ with $b_{\alpha} > 1$ and $0 \leqslant q < b_{\gamma}$. We have that*

$$y_{\emptyset(i+1)} e_{\mathsf{P}_{\emptyset\emptyset}} = y_{h+\emptyset(i+1)} e_{\mathsf{P}_{\emptyset\emptyset}} \qquad y_{\emptyset(i)} e_{\mathsf{P}_{\emptyset\emptyset}} = y_{h+\emptyset(i)} e_{\mathsf{P}_{\emptyset\emptyset}} \tag{6.32}$$

$$y_{h+\gamma(i+1)} e_{\mathsf{P}_{\emptyset\gamma}} = y_{\emptyset(i+1)} e_{\mathsf{P}_{\emptyset\gamma}} \qquad y_{h+\gamma(i)} e_{\mathsf{P}_{\emptyset\gamma}} = y_{\emptyset(i)} e_{\mathsf{P}_{\emptyset\gamma}} \tag{6.33}$$

$$y_{q(h-1)+\gamma(i+1)} e_{\mathsf{P}_{\gamma}} = y_{(q+1)(h-1)+\gamma(i+1)} e_{\mathsf{P}_{\gamma}} \qquad y_{q(h-1)+\gamma(i)} e_{\mathsf{P}_{\gamma}} = y_{(q+1)(h-1)+\gamma(i)} e_{\mathsf{P}_{\gamma}} \tag{6.34}$$

whenever the indices are defined (cross reference Definition 2.31).

Proof. We prove both cases of equation (6.32), the other pairs of cases are similar. Our assumption that $b_{\alpha} > 1$ implies that the residues of the ith and $(i+1)$th strands are *non-adjacent* and similarly that the $(h+\emptyset(i))$th and $(h+\emptyset(i+1))$th strands are *non-adjacent* (this is not true if $b_{\alpha} = 1$). Therefore we have that

$$0 = \psi^{i}_{h+i} e_{\mathsf{P}_{\emptyset\emptyset}} \psi^{h+i}_{i} = (y_i - y_{h+i}) e_{\mathsf{P}_{\emptyset\emptyset}}, \quad 0 = \psi^{h+\emptyset(i+1)}_{\emptyset(i+1)} e_{\mathsf{P}_{\emptyset\emptyset}} \psi^{\emptyset(i+1)}_{h+\emptyset(i+1)} = (y_i - y_{h+i}) e_{\mathsf{P}_{\emptyset\emptyset}}$$

where in both cases, the first and second equalities follow from Lemma 4.1 and the final case of relation (R4). □

Proposition 6.14. *Let $\alpha = \varepsilon_i - \varepsilon_{i+1}, \gamma = \varepsilon_k - \varepsilon_{k+1} \in \Pi$ with $b_{\alpha} = 1$ and $0 \leqslant q < b_{\gamma}$. We have that*

$$(y_i - y_{i+1}) e_{\mathsf{P}_{\emptyset\emptyset}} = (y_{h+i} - y_{h+i+1}) e_{\mathsf{P}_{\emptyset\emptyset}}$$

$$(y_{h+\gamma(i+1)} - y_{h+\gamma(i)}) e_{\mathsf{P}_{\emptyset\gamma}} = (y_{i+1} - y_i) e_{\mathsf{P}_{\emptyset\gamma}}$$

$$(y_{q(h-1)+\gamma(i)} - y_{q(h-1)+\gamma(i+1)}) e_{\mathsf{P}_{\gamma}} = (y_{(q+1)(h-1)+\gamma(i)} - y_{(q+1)(h-1)+\gamma(i+1)}) e_{\mathsf{P}_{\gamma}}$$

whenever the indices are defined (cross reference Definition 2.31).

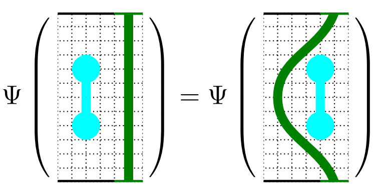
Proof. We prove the first equality as the other cases are similar. Since $b_{\alpha} = 1$, we have that $\emptyset(i) = i$ and $\emptyset(i+1) = i+1$ (in other words, $i \neq h$) and are of adjacent residue. We have that

$$\begin{aligned}
(y_{h+i+1} - y_{h+i})e_{\mathsf{P}_{\emptyset\emptyset}} &= e_{\mathsf{P}_{\emptyset\emptyset}} \psi^{h+i+1}_{h+i} \psi^{h+i}_{h+i+1} e_{\mathsf{P}_{\emptyset\emptyset}} \\
&= (e_{\mathsf{P}_{\emptyset\emptyset}} \psi^{h+i+1}_{h+i}) \psi^{h+i}_{i+2} \psi^{i+2}_{h+i} (\psi^{h+i}_{h+i+1} e_{\mathsf{P}_{\emptyset\emptyset}}) \\
&= (e_{\mathsf{P}_{\emptyset\emptyset}} \psi^{h+i+1}_{h+i} \psi^{h+i}_{i+2})(\psi_{i+1}\psi_i\psi_{i+1} + \psi_i\psi_{i+1}\psi_i)(\psi^{i+2}_{h+i}\psi^{h+i}_{h+i+1} e_{\mathsf{P}_{\emptyset\emptyset}}) \\
&= (e_{\mathsf{P}_{\emptyset\emptyset}} \psi^{h+i+1}_{h+i} \psi^{h+i}_{i+2})\psi_i\psi_{i+1}\psi_i(\psi^{i+2}_{h+i}\psi^{h+i}_{h+i+1} e_{\mathsf{P}_{\emptyset\emptyset}}) \\
&= (e_{\mathsf{P}_{\emptyset\emptyset}} \psi_i \psi^{h+i+1}_{h+i}) \psi^{h+i}_{i+2} \psi_{i+1} \psi^{i+2}_{h+i} (\psi^{h+i}_{h+i+1} \psi_i e_{\mathsf{P}_{\emptyset\emptyset}}) \\
&= (e_{\mathsf{P}_{\emptyset\emptyset}} \psi_i \psi^{h+i+1}_{h+i}) \psi^{i+1}_{h+i-1} \psi_{h+i-1} \psi^{h+i-1}_{i+1}) \psi^{h+i}_{h+i+1} \psi_i e_{\mathsf{P}_{\emptyset\emptyset}}) \\
&= (e_{\mathsf{P}_{\emptyset\emptyset}} \psi_i \psi^{i+1}_{h+i-1}) \psi^{h+i+1}_{h+i} \psi_{h+i-1} \psi^{h+i}_{h+i+1} (\psi^{h+i-1}_{i+1} \psi_i e_{\mathsf{P}_{\emptyset\emptyset}}) \\
&= (e_{\mathsf{P}_{\emptyset\emptyset}} \psi_i \psi^{i+1}_{h+i-1})(1 + \psi_{h+i-1}\psi_{h+i}\psi_{h+i-1})(\psi^{h+i-1}_{i+1} \psi_i e_{\mathsf{P}_{\emptyset\emptyset}}) \\
&= (e_{\mathsf{P}_{\emptyset\emptyset}} \psi_i \psi^{i+1}_{h+i-1})(\psi^{h+i-1}_{i+1} \psi_i e_{\mathsf{P}_{\emptyset\emptyset}}) \\
&= e_{\mathsf{P}_{\emptyset\emptyset}} \psi_i \psi_i e_{\mathsf{P}_{\emptyset\emptyset}} \\
&= e_{\mathsf{P}_{\emptyset\emptyset}} (y_{i+1} - y_i) e_{\mathsf{P}_{\emptyset\emptyset}}
\end{aligned}$$

where the first equality holds by the third case of relation (R4), the second holds by the second case of relation (R4) (the commuting version), the third holds by case 2 of relation (R5), the fourth holds by Lemma 4.1, and the fifth to the seventh by the second case of relation (R4) (the commuting version), and the eighth by the first case of (R5), and the ninth by Lemma 4.1, the tenth by the second case of relation (R4) (the commuting version), and the eleventh by the third case of relation (R4). □

6.13. The barbell and commutator

For $\beta, \gamma \in \Pi$ labelling two commuting reflections, we check that



(6.35)

In other words,

$$(\mathsf{spot}^{\emptyset}_{\beta}\mathsf{spot}^{\beta}_{\emptyset}) \otimes e_{\mathsf{P}_{\gamma}} = \mathsf{adj}^{\emptyset\gamma}_{\gamma\emptyset}(e_{\mathsf{P}_{\gamma}} \otimes (\mathsf{spot}^{\emptyset}_{\beta}\mathsf{spot}^{\beta}_{\emptyset}))\mathsf{adj}^{\gamma\emptyset}_{\emptyset\gamma}.$$

This relation is very simple to check. We have that

$$\begin{aligned}\mathsf{adj}^{\emptyset\gamma}_{\gamma\emptyset}(e_{\mathsf{P}_\gamma} \otimes (\mathsf{spot}^{\emptyset}_{\beta}\mathsf{spot}^{\beta}_{\emptyset}))\mathsf{adj}^{\gamma\emptyset}_{\emptyset\gamma} &= \mathsf{adj}^{\emptyset\gamma}_{\gamma\emptyset}(y_{b_{\gamma\beta}h-h+\emptyset(j+1)} - y_{b_\gamma h+j})\mathsf{adj}^{\gamma\emptyset}_{\emptyset\gamma}e_{\mathsf{P}_{\emptyset\gamma}}\\ &= (y_{b_{\gamma\beta}h-h+1-b_\gamma+\gamma(j+1)} - y_{b_\gamma h+\gamma(j)})\mathsf{adj}^{\emptyset\gamma}_{\gamma\emptyset}\mathsf{adj}^{\gamma\emptyset}_{\emptyset\gamma}e_{\mathsf{P}_{\emptyset\gamma}}\\ &= (y_{b_{\gamma\beta}h-h+1-b_\gamma+\gamma(j+1)} - y_{b_\gamma h+\gamma(j)})e_{\mathsf{P}_{\emptyset\gamma}}\\ &= (y_{b_\beta h-h+\emptyset(j+1)} - y_j)e_{\mathsf{P}_{\emptyset\gamma}}\end{aligned}$$

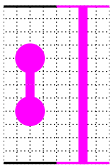
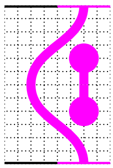
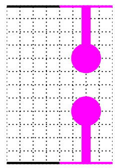
where the first equality follows from equation (6.31), the second equality follows from the commuting cases of relations (R3) and (R2), the third equality follows from Proposition 4.4, the fourth equality follows from applying Propositions 6.13 and 6.14. Again by equation (6.31), we have that

$$(\mathsf{spot}^{\emptyset}_{\beta}\mathsf{spot}^{\beta}_{\emptyset}) \otimes e_{\mathsf{P}_\gamma} = (y_{b_\beta h-h+\emptyset(j+1)} - y_j)e_{\mathsf{P}_{\emptyset\gamma}}$$

as required.

6.14. The one colour Demazure relation

We now verify (S2), namely that

$\Psi\Bigg($  $\Bigg) + \Psi\Bigg($  $\Bigg) = 2\Psi\Bigg($  $\Bigg).$ (6.36)

for $\alpha \in \Pi$. In other words, we must check that

$$(\mathsf{spot}^{\emptyset}_{\alpha}\mathsf{spot}^{\alpha}_{\emptyset}) \otimes e_{\mathsf{P}_\alpha} + \mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}(e_{\mathsf{P}_\alpha} \otimes \mathsf{spot}^{\emptyset}_{\alpha}\mathsf{spot}^{\alpha}_{\emptyset})\mathsf{adj}^{\alpha\emptyset}_{\emptyset\alpha} = 2(e_{\mathsf{P}_\emptyset} \otimes \mathsf{spot}^{\alpha}_{\emptyset}\mathsf{spot}^{\emptyset}_{\alpha})$$

Substituting equation (6.30) and (6.31) into the above, we must show that

$$\begin{aligned}&e_{\mathsf{P}_{\emptyset\alpha}}\left(y_{b_\alpha h-h+\emptyset(i+1)} - y_i + \mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}(y_{b_{\alpha\alpha}h-h+\emptyset(i+1)} - y_{b_\alpha h+i})\mathsf{adj}^{\alpha\emptyset}_{\emptyset\alpha}\right)e_{\mathsf{P}_{\emptyset\alpha}}\\ &= 2e_{\mathsf{P}_{\emptyset\alpha}}(y_{b_{\alpha\alpha}h-b_\alpha-h+1+\alpha(i+1)} - y_{b_{\alpha\alpha}h-b_\alpha+1})e_{\mathsf{P}_{\emptyset\alpha}}.\end{aligned} \tag{6.37}$$

This leads us to consider the effect of passing dots through the adjustment terms.

Proposition 6.15. *Let $\alpha \in \Pi$. We have that*

$$\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}y_{b_\alpha h+i}\mathsf{adj}^{\alpha\emptyset}_{\emptyset\alpha} = y_{b_{\alpha\alpha}h-b_\alpha+1}e_{\mathsf{P}_{\emptyset\alpha}} \tag{6.38}$$

$$\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}(b_\alpha-2)\dots\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}(0)y_{b_{\alpha\alpha}h-h+\emptyset(i+1)}\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}(0)\dots\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}(b_\alpha-2) = y_{b_{\alpha\alpha}h-h+\emptyset(i+1)}e_{\mathsf{P}_{\emptyset\alpha}}. \tag{6.39}$$

Proof. By the commuting case of relation (R2), we have that the lefthand-sides of equation (6.38) and (6.39) are equal to $y_{b_{\alpha\alpha}h-b_\alpha+1}\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}\mathsf{adj}^{\alpha\emptyset}_{\emptyset\alpha}$ and $y_{b_{\alpha\alpha}h-h+\emptyset(i+1)}\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}(b_\alpha - 2)\dots\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}(b_\alpha-2)$ respectively. The result then follows by Proposition 4.4. □

In equation (6.39) we pulled the dot through most of the adjustment term; in equation (6.40) below, we pull the dot through the final adjustment term. Equation (6.41) has an almost identical proof and so we record it here, for convenience.

Proposition 6.16. *Let $\alpha \in \Pi$. We have that*

$$\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset} y_{b_\alpha h+\emptyset(i+1)} \mathsf{adj}^{\alpha\emptyset}_{\emptyset\alpha} = \left(y_i + y_{b_\alpha h-b_\alpha+1+\alpha(i+1)} - y_{b_\alpha h+h-b_\alpha+1}\right) e_{\mathsf{P}_{\emptyset\alpha}} \tag{6.40}$$

$$\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset} y_{b_\alpha h-b_\alpha+1} \mathsf{adj}^{\alpha\emptyset}_{\emptyset\alpha} = y_{b_\alpha h+h-b_\alpha+1} e_{\mathsf{P}_{\emptyset\alpha}}. \tag{6.41}$$

Proof. We first prove equation (6.40). The dotted strand in the concatenated diagram on the left of equation (6.40) connects the $i = \mathsf{P}^{-1}_{\emptyset\alpha}(1, \varepsilon_i)$ top and bottom vertices, by way of the $b_\alpha h + \emptyset(i+1) = \mathsf{P}^{-1}_{\alpha\emptyset}(1, \varepsilon_{i+1})$ vertex in the centre of the diagram. We suppose this dotted strand is of residue $r \in \mathbb{Z}/e\mathbb{Z}$, say. This dotted strand crosses a single strand of the same residue: namely, the strand connecting the $\mathsf{P}^{-1}_{\emptyset\alpha}(b_\alpha + 1, \varepsilon_{i+1})$th vertices on the top and bottom edges. By relation (R3), we can pull the dot upwards along its strand and through this crossing at the expense of an error term. We thus obtain

$$\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset} y_{b_\alpha h+\emptyset(i+1)} \mathsf{adj}^{\alpha\emptyset}_{\emptyset\alpha} = e_{\mathsf{P}_{\alpha\emptyset}}\left(y_i \psi^{\mathsf{P}_{\emptyset\alpha}}_{\mathsf{P}_{\alpha\emptyset}} \psi^{\mathsf{P}_{\alpha\emptyset}}_{\mathsf{P}_{\emptyset\alpha}}\right) e_{\mathsf{P}_{\alpha\emptyset}} + e_{\mathsf{P}_{\alpha\emptyset}}\left(\psi^{\mathsf{P}_\emptyset\otimes \mathsf{S}_{0,\alpha}}_{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\emptyset} \psi^{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\emptyset}_{\mathsf{S}_{0,\alpha}\otimes\mathsf{P}_\emptyset} \psi^{\mathsf{P}_{\alpha\emptyset}}_{\mathsf{P}_{\emptyset\alpha}}\right) e_{\mathsf{P}_{\alpha\emptyset}} \tag{6.42}$$

(we note that $\mathsf{S}_{0,\alpha} = \mathsf{P}^\flat_\alpha$). The first term in equation (6.42) is equal to $y_i e_{\mathsf{P}_{\emptyset\alpha}}$ by Proposition 4.4 (and this is equal to the leftmost term on the righthand-side of equation (6.40)). We now consider the latter term. We label the top and bottom edges by $\mathsf{T} = \mathsf{P}_\emptyset \otimes \mathsf{P}^\flat_\alpha$ and $\mathsf{B} = \mathsf{P}_\emptyset \otimes \mathsf{P}_\alpha$. There is a unique crossing of strands of the same residue in the diagram

$$e_{\mathsf{P}_{\emptyset\alpha}}\left(\psi^{\mathsf{P}_\emptyset\otimes \mathsf{S}_{0,\alpha}}_{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\emptyset} \circ \psi^{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\emptyset}_{\mathsf{S}_{0,\alpha}\otimes\mathsf{P}_\emptyset} \circ \psi^{\mathsf{P}_{\alpha\emptyset}}_{\mathsf{P}_{\emptyset\alpha}}\right) e_{\mathsf{P}_{\emptyset\alpha}}$$

namely the r-strands connecting the $i = \mathsf{T}^{-1}(1, \varepsilon_i)$ and $\mathsf{B}^{-1}(b_\alpha + 1, \varepsilon_{i+1})$ vertices on the top and bottom edges of the diagram. This crossing of strands is bi-passed on the left by the $(r+1)$-strand connecting the $\mathsf{T}^{-1}(b_\alpha, \varepsilon_{i+1}) = \mathsf{B}^{-1}(b_\alpha, \varepsilon_{i+1})$ top and bottom vertices. We pull this $(r+1)$-strand to the right through the crossing r-strands using case 2 of relation (R5) (and the commuting relations). We hence undo this crossing and obtain

$$e_{\mathsf{P}_{\emptyset\alpha}}\left(\psi^{\mathsf{P}_\emptyset\otimes \mathsf{S}_{0,\alpha}}_{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\emptyset} \psi^{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\emptyset}_{\mathsf{P}_\emptyset\otimes\mathsf{S}_{0,\alpha}}\right) e_{\mathsf{P}_{\emptyset\alpha}}$$

(the other term depicted in equation (R5) is zero by Lemma 4.1). Now, this diagram contains a double-crossing of the r-strand connecting the $(\mathsf{P}_\emptyset \otimes \mathsf{P}^\flat_\alpha)^{-1}(b_\alpha + 1, \varepsilon_{i+1})$ top and bottom vertices and the $(r-1)$-strand connecting the $(\mathsf{P}_\emptyset \otimes \mathsf{P}^\flat_\alpha)^{-1}(2, \varepsilon_i)$ top and bottom vertices. We undo this double-crossing using case 4 of relation (R4) (and the commutativity relations) to obtain

$$e_{\mathsf{P}_{\emptyset\alpha}}(y_{b_\alpha h-b_\alpha+1+\alpha(i+1)} - y_{b_\alpha h+h-b_\alpha+1})e_{\mathsf{P}_{\emptyset\alpha}} \tag{6.43}$$

and so equation (6.40) follows. Regarding the enumeration above, we note that

$$(\mathsf{P}_\emptyset \otimes \mathsf{P}^\flat_\alpha)^{-1}(b_\alpha+1, \varepsilon_{i+1}) = b_\alpha h - b_\alpha + 1 + \alpha(i+1)$$
$$(\mathsf{P}_\emptyset \otimes \mathsf{P}^\flat_\alpha)^{-1}(2, \varepsilon_i) = b_\alpha h + h - b_\alpha + 1.$$

Now we turn to equation (6.41). We push the KLR-dot upwards along its strand using (R3) to obtain

$$e_{\mathsf{P}_{\emptyset\alpha}}\left(y_{b_\alpha h-b_\alpha+1+\alpha(i+1)}\psi^{\mathsf{P}_{\emptyset\alpha}}_{\mathsf{P}_{\alpha\emptyset}}\psi^{\mathsf{P}_{\alpha\emptyset}}_{\mathsf{P}_{\emptyset\alpha}}\right)e_{\mathsf{P}_{\emptyset\alpha}} - e_{\mathsf{P}_{\emptyset\alpha}}\left(\psi^{\mathsf{P}_\emptyset\otimes\mathsf{S}_{0,\alpha}}_{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\emptyset}\psi^{\mathsf{S}_{1,\alpha}\otimes\mathsf{P}_\emptyset}_{\mathsf{S}_{0,\alpha}\otimes\mathsf{P}_\emptyset}\circ\psi^{\mathsf{P}_{\alpha\emptyset}}_{\mathsf{P}_{\emptyset\alpha}}\right)e_{\mathsf{P}_{\emptyset\alpha}}. \tag{6.44}$$

The first term is equal to $y_{b_\alpha h-b_\alpha+1+\alpha(i+1)}e_{\mathsf{P}_{\emptyset\alpha}}$ (again this follows by Proposition 4.4). The second term is identical to the second term in equation (6.42) and so is equal to equation (6.43) but with negative coefficient. Thus we can rewrite equation (6.44) in the form

$$e_{\mathsf{P}_{\emptyset\alpha}}\left(y_{b_\alpha h-b_\alpha+1+\alpha(i+1)} - (y_{b_\alpha h-b_\alpha+1+\alpha(i+1)} - y_{b_\alpha h+h-b_\alpha+1})\right)e_{\mathsf{P}_{\emptyset\alpha}}$$

and equation (6.41) follows. □

We now gather together our conclusions from Propositions 6.15 and 6.16 (shifting the indexing where necessary) in order to prove equation (6.36). We have that $(\mathsf{spot}^{\emptyset}_{\alpha}\mathsf{spot}^{\alpha}_{\emptyset})\otimes e_{\mathsf{P}_\alpha}$ is equal to

$$e_{\mathsf{P}_{\emptyset\alpha}}(y_{b_\alpha h-h+\emptyset(i+1)} - y_i)e_{\mathsf{P}_{\emptyset\alpha}}$$

and $\mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}(e_{\mathsf{P}_\alpha}\otimes\mathsf{spot}^{\emptyset}_{\alpha}\mathsf{spot}^{\alpha}_{\emptyset})\mathsf{adj}^{\alpha\emptyset}_{\emptyset\alpha}$ is equal to

$$-e_{\mathsf{P}_{\emptyset\alpha}}y_{b_{\alpha\alpha}h-b_\alpha+1}e_{\mathsf{P}_{\emptyset\alpha}} + e_{\mathsf{P}_{\emptyset\alpha}}\left(y_{b_\alpha h-h+i} + y_{b_{\alpha\alpha}h-b_\alpha-h+1+\alpha(i+1)} - y_{b_{\alpha\alpha}h-b_\alpha+1}\right)e_{\mathsf{P}_{\emptyset\alpha}}$$

By Propositions 6.13 and 6.14, we have that

$$y_{b_\alpha h-h+\emptyset(i+1)}e_{\mathsf{P}_{\emptyset\alpha}} = y_{b_{\alpha\alpha}h-b_\alpha-h+1+\alpha(i+1)}e_{\mathsf{P}_{\emptyset\alpha}}$$

for $b_\alpha \geqslant 1$ and by Proposition 6.13 we have that

$$y_ie_{\mathsf{P}_{\emptyset\alpha}} = y_{b_\alpha h-h+i}e_{\mathsf{P}_{\emptyset\alpha}}$$

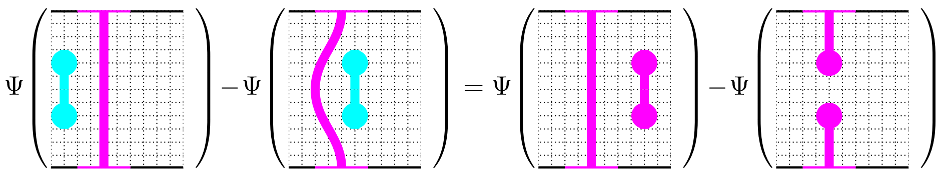
for $b_\alpha > 1$ (we note that this latter statement is vacuous if $b_\alpha = 1$ as the subscripts are equal). The former pair of terms sum up and the latter cancel, so we obtain

$$(\mathsf{spot}^{\emptyset}_{\alpha}\mathsf{spot}^{\alpha}_{\emptyset}) \otimes e_{\mathsf{P}_{\alpha}} + \mathsf{adj}^{\emptyset\alpha}_{\alpha\emptyset}(e_{\mathsf{P}_{\alpha}} \otimes \mathsf{spot}^{\emptyset}_{\alpha}\mathsf{spot}^{\alpha}_{\emptyset})\mathsf{adj}^{\alpha\emptyset}_{\emptyset\alpha}$$
$$= 2e_{\mathsf{P}_{\emptyset\alpha}}(y_{b_{\alpha}h-h+\emptyset(i+1)} - y_{b_{\alpha\alpha}h-b_{\alpha}+1})e_{\mathsf{P}_{\emptyset\alpha}}$$

Hence equation (6.37) holds by a further application of Propositions 6.13 and 6.14.

6.15. Two colour Demazure

For $\alpha, \beta \in \Pi$ labelling two non-commuting reflections, we now verify relation (S5), namely that



(6.45)

We assume that the rank of Φ is at least 2. The reader is invited to check the rank 1 case separately (here the scalar 2 appears due to certain coincidences in the arithmetic).

Proposition 6.17. *Let $\alpha \in \Pi$. If $b_{\alpha} > 1$, we have that*

$$y_{b_{\alpha}h+h+\emptyset(i+1)}e_{\mathsf{P}_{\emptyset\alpha\emptyset}} = (y_i + y_{\emptyset(i+1)} - y_{b_{\alpha}h+h-b_{\alpha}+1})e_{\mathsf{P}_{\emptyset\alpha\emptyset}}$$
$$y_{b_{\alpha}h+h+i}e_{\mathsf{P}_{\emptyset\alpha\emptyset}} = y_{b_{\alpha}h+h}e_{\mathsf{P}_{\emptyset\alpha\emptyset}} = y_{b_{\alpha}h+h-b_{\alpha}+1}e_{\mathsf{P}_{\emptyset\alpha\emptyset}}$$

and if $b_{\alpha} = 1$ we have that

$$(y_{b_{\alpha}h+h+i} - y_{b_{\alpha}h+h+\emptyset(i+1)})e_{\mathsf{P}_{\emptyset\alpha\emptyset}} = (2y_{b_{\alpha}h+h-b_{\alpha}+1} - y_i - y_{\emptyset(i+1)})e_{\mathsf{P}_{\emptyset\alpha\emptyset}}.$$

Proof. We check the $b_{\alpha} > 1$ case as the other is similar. The second equality follows as in the proof of Proposition 6.13. We now consider the first equality. We momentarily drop the prefix $\mathsf{P}_{\emptyset}$ to the path $\mathsf{P}_{\emptyset\alpha\emptyset}$ for the sake of more manageable indices. Since $b_{\alpha} > 1$ we can pull the vertical strand connecting the $b_{\alpha}h + \emptyset(i+1)$ top and bottom vertices leftwards until we reach a strand of adjacent residue (namely the $(b_{\alpha}h - b_{\alpha} + 2)$th strand) as follows

$$e_{\mathsf{P}_{\alpha\emptyset}} = e_{\mathsf{P}_{\alpha\emptyset}}\psi^{b_{\alpha}h+\emptyset(i+1)}_{b_{\alpha}h-b_{\alpha}+3}\psi^{b_{\alpha}h-b_{\alpha}+3}_{b_{\alpha}h+\emptyset(i+1)}e_{\mathsf{P}_{\alpha\emptyset}}$$

we can rewrite the centre of the diagram which using the braid relation as follows,

$$e_{\mathsf{P}_{\alpha\emptyset}}\psi^{b_{\alpha}h+\emptyset(i+1)}_{b_{\alpha}h-b_{\alpha}+3}(\psi_{b_{\alpha}h-b_{\alpha}+2}\psi_{b_{\alpha}h-b_{\alpha}+1}\psi_{b_{\alpha}h-b_{\alpha}+2} -$$
$$\psi_{b_{\alpha}h-b_{\alpha}+1}\psi_{b_{\alpha}h-b_{\alpha}+2}\psi_{b_{\alpha}h-b_{\alpha}+1})\psi^{b_{\alpha}h-b_{\alpha}+3}_{b_{\alpha}h+\emptyset(i+1)}e_{\mathsf{P}_{\alpha\emptyset}}$$

where the latter term is zero by Lemma 4.1 and so this simplifies to

$$e_{\mathsf{P}_{\alpha\emptyset}}\psi^{b_\alpha h+\emptyset(i+1)}_{b_\alpha h-b_\alpha+2}(\psi_{b_\alpha h-b_\alpha+1})\psi^{b_\alpha h-b_\alpha+2}_{b_\alpha h+\emptyset(i+1)}e_{\mathsf{P}_{\alpha\emptyset}}$$

now we use the non-commuting version of relation (R2) together with case 1 of relation (R4) to rewrite the middlemost crossing as a double-crossing with a KLR-dot,

$$-e_{\mathsf{P}_{\alpha\emptyset}}\psi^{b_\alpha h+\emptyset(i+1)}_{b_\alpha h-b_\alpha+2}(\psi_{b_\alpha h-b_\alpha+1}y_{b_\alpha h-b_\alpha+1}\psi_{b_\alpha h-b_\alpha+1})\psi^{b_\alpha h-b_\alpha+2}_{b_\alpha h+\emptyset(i+1)}e_{\mathsf{P}_{\alpha\emptyset}},$$

we pull the dotted strand leftwards through the next strand of adjacent residue (namely the $((b_\alpha-1)(h-1)+\alpha(i+1))$th strand) using the commutativity relations and case 4 of relation (R4) to obtain

$$e_{\mathsf{P}_{\alpha\emptyset}}\psi^{b_\alpha h+\emptyset(i+1)}_{b_\alpha h-b_\alpha+2}(y_{(b_\alpha-1)(h-1)+\alpha(i+1)}+ \psi^{b_\alpha h-b_\alpha+2}_{(b_\alpha-1)(h-1)+\alpha(i+1)}\psi^{(b_\alpha-1)(h-1)+\alpha(i+1)}_{b_\alpha h-b_\alpha+2})\psi^{b_\alpha h-b_\alpha+2}_{b_\alpha h+\emptyset(i+1)}e_{\mathsf{P}_{\alpha\emptyset}}$$

where the first summand is zero by case 1 of relation (R4) and the latter term is equal to

$$e_{\mathsf{P}_{\alpha\emptyset}}\psi^{b_\alpha h+\emptyset(i+1)}_{(b_\alpha-1)(h-1)+\alpha(i+1)}\psi^{(b_\alpha-1)(h-1)+\alpha(i+1)}_{b_\alpha h+\emptyset(i+1)}e_{\mathsf{P}_{\alpha\emptyset}}.$$

Now we concatenate on the left by $\mathsf{P}_\emptyset$ and then multiply by $y_{b_\alpha h+h+\emptyset(i+1)}$ to obtain

$$y_{b_\alpha h+h+\emptyset(i+1)}e_{\mathsf{P}_{\emptyset\alpha\emptyset}} = y_{b_\alpha h+h+\emptyset(i+1)}e_{\mathsf{P}_{\emptyset\alpha\emptyset}}\psi^{b_\alpha h+h+\emptyset(i+1)}_{b_\alpha h-b_\alpha+1+\alpha(i+1)}\psi^{b_\alpha h-b_\alpha+1+\alpha(i+1)}_{b_\alpha h+h+\emptyset(i+1)}e_{\mathsf{P}_{\emptyset\alpha\emptyset}} \tag{6.46}$$

which by relation (R4) is equal to

$$e_{\mathsf{P}_{\emptyset\alpha\emptyset}}(\psi^{b_\alpha h+h+\emptyset(i+1)}_{b_\alpha h-b_\alpha+1+\alpha(i+1)}y_{b_\alpha h-b_\alpha+1+\alpha(i+1)}+ \psi^{b_\alpha h+h+\emptyset(i+1)}_{b_\alpha h+h-b_\alpha+2}\psi^{b_\alpha h+h-b_\alpha+1}_{b_\alpha h-b_\alpha+1+\alpha(i+1)})\psi^{b_\alpha h-b_\alpha+1+\alpha(i+1)}_{b_\alpha h+h+\emptyset(i+1)}e_{\mathsf{P}_{\emptyset\alpha\emptyset}}.$$

We consider the first term in the sum first. By the commuting relations, this term is equal to

$$e_{\mathsf{P}_{\emptyset\alpha\emptyset}}(\psi^{b_\alpha h+h+\emptyset(i+1)}_{h+i}y_{h+i}\psi^{h+i}_{b_\alpha h+h+\emptyset(i+1)})e_{\mathsf{P}_{\emptyset\alpha\emptyset}}$$

and by Proposition 6.13 this is equal to

$$e_{\mathsf{P}_{\emptyset\alpha\emptyset}}(\psi^{b_\alpha h+h+\emptyset(i+1)}_{h+i}y_i\psi^{h+\emptyset(i+1)}_{b_\alpha h+h+\emptyset(i+1)})e_{\mathsf{P}_{\emptyset\alpha\emptyset}}$$

and now, having moved this KLR-dot a total of h strands leftward, we can apply the commutativity relations again to obtain

$$y_ie_{\mathsf{P}_{\emptyset\alpha\emptyset}}(\psi^{b_\alpha h+h+\emptyset(i+1)}_{b_\alpha h-b_\alpha+1+\alpha(i+1)}\psi^{b_\alpha h-b_\alpha+1+\alpha(i+1)}_{b_\alpha h+h+\emptyset(i+1)}e_{\mathsf{P}_{\emptyset\alpha\emptyset}}) = y_ie_{\mathsf{P}_{\emptyset\alpha\emptyset}} \tag{6.47}$$

where the final equality follows by equation (6.46). We now turn to the second term in the above sum, namely

$$e_{\mathsf{P}_{\emptyset\alpha\emptyset}}\psi^{b_\alpha h+h+\emptyset(i+1)}_{b_\alpha h+h-b_\alpha+2}\psi^{b_\alpha h+h-b_\alpha+1}_{b_\alpha h-b_\alpha+1+\alpha(i+1)}\psi^{b_\alpha h-b_\alpha+1+\alpha(i+1)}_{b_\alpha h+h+\emptyset(i+1)}e_{\mathsf{P}_{\emptyset\alpha\emptyset}}.$$

This has a crossing of like-labelled strands (of residue $r\in\mathbb{Z}/e\mathbb{Z}$) connecting the $(b_\alpha h+\emptyset(i+1))$th and $(b_\alpha h-b_\alpha+1)$th top and bottom vertices. This crossing is bi-passed on the right by the $(r-1)$-strand connecting the $(b_\alpha h-b_\alpha+2)$th top and bottom vertices. We undo this braid using case 1 of relation (R5) to obtain

$$e_{\mathsf{P}_{\emptyset\alpha\emptyset}}(\psi^{b_\alpha h+h+\emptyset(i+1)}_{b_\alpha h+h-b_\alpha+2}\psi^{b_\alpha h+h-b_\alpha+1}_{b_\alpha h-b_\alpha+1+\alpha(i+1)})(\psi^{b_\alpha h-b_\alpha+1+\alpha(i+1)}_{b_\alpha h+h-b_\alpha+1}\psi^{b_\alpha h+h-b_\alpha+2}_{b_\alpha h+h+\emptyset(i+1)})e_{\mathsf{P}_{\emptyset\alpha\emptyset}}$$

where the other term in relation (R5) is zero by Lemma 4.1. This diagram contains a single double-crossing of adjacent residues, which we undo using case 4 of relation (R4) (and we undo all the other crossings using the commutativity relation) to obtain

$$e_{\mathsf{P}_{\emptyset\alpha\emptyset}}(y_{b_\alpha h-b_\alpha+1+\alpha(i+1)}-y_{b_\alpha h-b_\alpha+1})e_{\mathsf{P}_{\alpha\emptyset}}=e_{\mathsf{P}_{\alpha\emptyset}}(y_{\emptyset(i+1)}-y_{b_\alpha h-b_\alpha+1})e_{\mathsf{P}_{\emptyset\alpha\emptyset}}\tag{6.48}$$

where the final equality follows by Proposition 6.13. The result follows by summing over equation (6.47) and (6.48). □

Proposition 6.18. *Let* $\alpha=\varepsilon_i-\varepsilon_{i+1}, \beta=\varepsilon_{i+1}-\varepsilon_{i+2}\in\Pi$. *We have that*

$$\begin{aligned}&(\mathsf{spot}^{\varnothing}_{\beta}\mathsf{spot}^{\beta}_{\varnothing})\otimes e_{\mathsf{P}_{\alpha\varnothing}}-(\mathsf{adj}^{\varnothing\alpha}_{\alpha\varnothing}\otimes e_{\mathsf{P}_\varnothing})(e_{\mathsf{P}_\alpha}\otimes\mathsf{spot}^{\varnothing}_{\beta}\mathsf{spot}^{\beta}_{\varnothing}\otimes e_{\mathsf{P}_\varnothing})(\mathsf{adj}^{\alpha\varnothing}_{\varnothing\alpha}\otimes e_{\mathsf{P}_\varnothing})\\&=e_{\mathsf{P}_\alpha}(y_i-y_{b_{\alpha\beta}h})e_{\mathsf{P}_\alpha}\\&=e_{\mathsf{P}_{\varnothing\alpha}}\otimes(\mathsf{spot}^{\varnothing}_{\alpha}\mathsf{spot}^{\alpha}_{\varnothing})-e_{\mathsf{P}_\varnothing}\otimes(\mathsf{spot}^{\alpha}_{\varnothing}\mathsf{spot}^{\varnothing}_{\alpha})\otimes e_{\mathsf{P}_\varnothing}.\end{aligned}$$

Proof. Substituting equation (6.30) and (6.31) into the third line, we obtain

$$e_{\mathsf{P}_{\varnothing\alpha\varnothing}}(y_{b_{\alpha\beta\alpha}h-h+\emptyset(i+1)}-y_{b_{\alpha\beta}h+i}-y_{b_{\alpha\beta}h-b_\alpha-h+1+\alpha(i+1)}+y_{b_{\alpha\beta}h-b_\alpha+1})e_{\mathsf{P}_{\varnothing\alpha\varnothing}}.$$

We apply Proposition 6.17 to the first term in the sum and then cancelling terms using Propositions 6.13 and 6.14. Substituting equation (6.30) and (6.31) into the first line, we obtain

$$e_{\mathsf{P}_{\varnothing\alpha\varnothing}}\left(y_{b_\beta h-h+\emptyset(i+2)}-y_{\emptyset(i+1)}-\mathsf{adj}^{\varnothing\alpha\varnothing}_{\alpha\varnothing\varnothing}(y_{b_{\alpha\beta}h-h+\emptyset(i+2)}-y_{b_\alpha h+\emptyset(i+1)})\mathsf{adj}^{\alpha\varnothing\varnothing}_{\varnothing\alpha\varnothing}\right)e_{\mathsf{P}_{\varnothing\alpha\varnothing}}.\tag{6.49}$$

We have that

$$e_{\mathsf{P}_{\varnothing\alpha\varnothing}}\mathsf{adj}^{\varnothing\alpha\varnothing}_{\alpha\varnothing\varnothing}y_{b_{\alpha\beta}h-h+\emptyset(i+2)}\mathsf{adj}^{\alpha\varnothing\varnothing}_{\varnothing\alpha\varnothing}e_{\mathsf{P}_{\varnothing\alpha\varnothing}}=y_{b_\alpha h-b_\alpha-h+1+\alpha(i+2)}=y_{b_\beta h-h+\emptyset(i+2)}\tag{6.50}$$

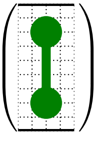
where the first equality follows from the commuting KLR-dot relation (R3) and the latter follows from Propositions 6.13 and 6.14. We also have that

$$
\begin{aligned}
\mathsf{adj}^{\varnothing\alpha\varnothing}_{\alpha\varnothing\varnothing} y_{b_\alpha h+\emptyset(i+1)} \mathsf{adj}^{\alpha\varnothing\varnothing}_{\varnothing\alpha\varnothing} &= e_{\mathsf{P}_{\varnothing\alpha\varnothing}} \left(y_{b_\beta h-h+i} + y_{b_{\alpha\beta} h-h-b_\alpha+1+\alpha(i+1)} - y_{b_{\alpha\beta} h-b_\alpha+1} \right) e_{\mathsf{P}_{\varnothing\alpha\varnothing}} \\
&= e_{\mathsf{P}_{\varnothing\alpha\varnothing}} \left(y_i + y_{\emptyset(i+1)} - y_{b_{\alpha\beta} h} \right) e_{\mathsf{P}_{\varnothing\alpha\varnothing}}
\end{aligned} \tag{6.51}
$$

where the first equality follows from Proposition 6.16 and the second by Propositions 6.13 and 6.14. Thus substituting equation (6.50) and (6.51) in to equation (6.49), the first equality follows. □

6.16. The cyclotomic relation

We now verify relation (S9). We have that $\Psi(1_\alpha) = e_{\mathsf{P}_\alpha}$ for any $\alpha \in \Pi$. If the α-hyperplane is a wall of the dominant region, then the tableau P_α is *non-standard* and therefore $e_{\mathsf{P}_\alpha} = 0$ by Lemma 4.1. Now, let $\gamma \in \Pi$ be arbitrary. By equation (6.30), we have that

$\Psi\Bigg($  $\Bigg) = e_{\mathsf{P}_\varnothing} \big(y_{b_\gamma h-h+\emptyset(k+1)} - y_k \big) e_{\mathsf{P}_\varnothing} = e_{\mathsf{P}_\varnothing} \big(y_{\emptyset(k+1)} - y_k \big) e_{\mathsf{P}_\varnothing}$

where the latter equality follows from Propositions 6.13 and 6.14. If $x \equiv 1$ modulo h, then

$$
y_x e_{\mathsf{P}_\varnothing} = e_{\mathsf{P}_\varnothing} (\psi_1^x y_1 \psi_x^1) e_{\mathsf{P}_\varnothing} = 0 \tag{6.52}
$$

by relation (3.4). If not, then by relation (R4) we have that

$$
y_x e_{\mathsf{P}_\varnothing} = y_{x-1} e_{\mathsf{P}_\varnothing} - e_{\mathsf{P}_\varnothing} \psi_x \psi_x e_{\mathsf{P}_\varnothing} \tag{6.53}
$$

where the latter term is zero by Remark 3.25 (as $(\varepsilon_1, \dots, \varepsilon_{x-1}, \varepsilon_{x+1}, \varepsilon_x, \varepsilon_{x+2}, \dots, \varepsilon_h)$ is non-standard for $b_\gamma = 1$). Thus the cyclotomic relation holds (as we can apply equation (6.53) as many times as necessary and then apply equation (6.52)).

7. Decomposition numbers of cyclotomic Hecke algebras

In this section we recall the construction of the graded cellular and “light leaves” bases for the algebras $\mathscr{S}^{\mathrm{br}}_{\underline{h}}(n,\sigma)$, our quotient algebras $\mathcal{H}_n^\sigma / \mathcal{H}_n^\sigma \mathsf{y}_{\underline{h}} \mathcal{H}_n^\sigma$, and their truncations. We show that the homomorphism Ψ preserves these $\mathbb{Z}$-bases (trivially, by definition) and hence deduce that Ψ is indeed an isomorphism and hence prove Theorems A and B of the introduction.

7.1. Why is it enough to consider the truncated algebras?

Thus far in the paper, we have truncated to consider paths which terminate at a point $\lambda \in \mathscr{P}_{\underline{h}}(n,\sigma) \subseteq \mathscr{P}_{\underline{h}}(n)$. This is, in general, a proper co-saturated subset of the principal linkage class of multipartitions for a given $n \in \mathbb{Z}_{\geqslant 0}$.

Theorem 7.1 *([4, Corollary 2.14]). For each λ, we fix $\mathsf{P}_\lambda \in \mathrm{Std}(\lambda)$ a choice of reduced path. The algebra $\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{y}_{\underline{h}} \mathcal{H}_n^\sigma$ is quasi-hereditary with graded cellular basis*

$$\{\psi^{\mathsf{T}}_{\mathsf{P}_\lambda}\psi^{\mathsf{P}_\lambda}_{\mathsf{B}} \mid \mathsf{T},\mathsf{B} \in \mathrm{Std}(\lambda), \lambda \in \mathscr{P}_{\underline{h}}(n)\}$$

with respect to the reverse cylindric order on $\mathscr{P}_{\underline{h}}(n)$ (see [4, Definition 1.3], but for the subset $\mathscr{P}_{\underline{h}}(n,\sigma) \subseteq \mathscr{P}_{\underline{h}}(n)$ is a refinement of the opposite of the Bruhat ordering on their alcoves) and the anti-involution, $*$, given by flipping a diagram through the horizontal axis.

Remark 7.2. In [4, Corollary 2.14] it is not explicitly stated that the algebra is quasi-hereditary. However, this is immediate from the fact that each layer in the cell-filtration has an idempotent e_{P_λ} for $\lambda \in \mathscr{P}_{\underline{h}}(n)$ (and standard facts about cellular algebras).

Remark 7.3. In the case of the Hecke algebra of the symmetric group, the basis of [4, Corollary 2.14] is equivalent (via uni-triangular change of basis with respect to the dominance ordering) to the cellular basis of Hu–Mathas [14].

Example 7.4. Let $\lambda = (3^n, 1^{15})$ with $n \geqslant 0$. The first $n = 0, 1, 2, 3, 4, 5$ partitions in this sequence are (1^{15}), $(3, 1^{15})$, $(3^2, 1^{15})$, $(3^3, 1^{15})$, $(3^4, 1^{15})$ and $(3^5, 1^{15})$, all of which label simple modules which belong to the principal blocks of their corresponding group algebras. In fact, they all label the same point, in the alcove $s_{\varepsilon_3-\varepsilon_1}s_{\varepsilon_1-\varepsilon_2}s_{\varepsilon_2-\varepsilon_3}s_{\varepsilon_3-\varepsilon_1}s_{\varepsilon_1-\varepsilon_2}s_{\varepsilon_2-\varepsilon_3}A_0$, in the projection onto 2-dimensional space in Fig. 1. However, $\mathrm{Std}_{n,\sigma}(\lambda) = \emptyset$ for the first five of these partitions. For $\lambda = (3^n, 1^{15})$ with $n \geqslant 5$ we have that $\mathrm{Std}_{n,\sigma}(\lambda) \neq \emptyset$. Thus, one might be forgiven in thinking that our Theorem A only allows us to see λ for $n \geqslant 5$. This is, in fact, not the case as we shall soon see.

Proposition 7.5. *Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we set $\det_h(\lambda) = (h, \lambda_1, \lambda_2, \dots)$. We have an injective map of partially ordered sets $\det_h : \mathscr{P}_{\underline{h}}(n) \hookrightarrow \mathscr{P}_{\underline{h}}(n+h)$ given by*

$$\det_h(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell-1)}) = (\det_{h_0}(\lambda^{(0)}), \det_{h_1}(\lambda^{(1)}), \dots, \det_{h_{\ell-1}}(\lambda^{(\ell-1)}))$$

and $\det_h(\mathscr{P}_{\underline{h}}(n)) \subseteq \mathscr{P}_{\underline{h}}(n+h)$ is a co-saturated subset. We have an isomorphism of graded $\mathbb{Z}$-algebras

$$\sum_{\mathsf{B},\mathsf{T} \in \mathrm{Std}_n} e_{\mathsf{T}}(\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{y}_{\underline{h}}\mathcal{H}_n^\sigma)e_{\mathsf{B}} \cong \sum_{\mathsf{B},\mathsf{T} \in \mathrm{Std}_n} e_{\mathsf{P}_\emptyset \otimes \mathsf{T}}(\mathcal{H}_{n+h}^\sigma/\mathcal{H}_{n+h}^\sigma \mathsf{y}_{\underline{h}}\mathcal{H}_{n+h}^\sigma)e_{\mathsf{P}_\emptyset \otimes \mathsf{B}} \tag{7.1}$$

where $\mathrm{Std}_n = \cup_{\lambda \in \mathscr{P}_{\underline{h}}(n)} \mathrm{Std}(\lambda)$.

Proof. On the level of graded $\mathbb{Z}$-modules the isomorphism, ϕ say, is clear. The local KLR relations also go through easily. We have that

$$\phi(y_1 e_{\mathsf{P}}) = y_{h+1} e_{\mathsf{P}_\emptyset \otimes \mathsf{P}} = y_1 e_{\mathsf{P}_\emptyset \otimes \mathsf{P}} = 0 = y_1 e_{\mathsf{P}} \tag{7.2}$$

where the second equality follows using the same argument as Propositions 6.13 and 6.14 and the other equalities all hold by definition. We further note that P is dominant path if and only if $\mathsf{P}_\emptyset \otimes \mathsf{P}$ is a dominant path. Thus the cyclotomic relation follows from equation (7.2) and Remark 3.25. □

We wish to only explicitly consider the principal linkage class, but to make deductions for all regular linkage classes. This is a standard Lie theoretic trick known as the translation principle. Given $\Gamma \subseteq \mathscr{P}_{\underline{h}}(n)$ any co-saturated subset and $r \in \mathbb{Z}/e\mathbb{Z}$ we let

$$e_\Gamma = \sum_{\substack{\mathsf{P} \in \mathrm{Std}(\mu) \\ \mu \in \Gamma}} e_{\mathsf{P}} \qquad \mathsf{E}_r = \sum_{i_1,\dots,i_n \in \mathbb{Z}/e\mathbb{Z}} e(i_1,\dots,i_n,r)$$

denote the corresponding idempotents. Given $\lambda \in \mathscr{P}_{\underline{h}}(n)$ we set $\Lambda = (\widehat{\mathfrak{S}}_h \cdot \lambda) \cap \mathscr{P}_{\underline{h}}(n)$. Since every λ belongs to some linkage class, we have that $\mathscr{P}_{\underline{h}}(n) = \Lambda' \cup \Lambda'' \cup \dots$ and we have a corresponding decomposition

$$\mathcal{H}_n^\sigma / \mathcal{H}_n^\sigma \mathsf{y}_{\underline{h}} \mathcal{H}_n^\sigma = \mathcal{H}_n^{\Lambda',\sigma} \oplus \mathcal{H}_n^{\Lambda'',\sigma} \oplus \dots \quad \text{where} \quad \mathcal{H}_n^{\Lambda,\sigma} = e_\Lambda (\mathcal{H}_n^\sigma / \mathcal{H}_n^\sigma \mathsf{y}_{\underline{h}} \mathcal{H}_n^\sigma) e_\Lambda$$

and similarly for the primed cases. Now, we let $\square$ denote an addable node of the Young diagram multipartition $\lambda \in \mathscr{P}_{\underline{h}}(n)$, that is we suppose that $\lambda \cup \square = \lambda'$ for some $\lambda' \in \mathscr{P}_{\underline{h}}(n+1)$.

Proposition 7.6. *Suppose that* $\lambda \in \mathscr{P}_{\underline{h}}(n)$ *and* $\lambda + \square = \lambda' \in \mathscr{P}_{\underline{h}}(n+1)$ *are* σ*-regular and* $\square$ *is of residue* $r \in \mathbb{Z}/e\mathbb{Z}$ *say. We have an injective map*

$$\varphi : \Lambda \hookrightarrow \Lambda' \qquad \varphi(\mu) = \mu + \square$$

for $\square$ *the unique addable node of residue* $r \in \mathbb{Z}/e\mathbb{Z}$*. The image,* $\varphi(\Lambda)$*, is a co-saturated subset of* Λ'*. We have an isomorphism of graded* $\mathbb{Z}$*-algebras:*

$$\mathcal{H}_n^{\Lambda,\sigma} \cong \mathsf{E}_r (e_{\varphi(\Lambda)} \mathcal{H}_{n+1}^{\Lambda'} e_{\varphi(\Lambda)}) \mathsf{E}_r \tag{7.3}$$

and this preserves the cellular structure.

Proof. Since both λ and $\lambda + \square$ are both e-regular, there is a bijection between the path bases of the algebras in equation (7.3). (Note that if λ were on a hyperplane and $\lambda + \square$

in an alcove, then the number of paths would double.) Thus we need only check that this $\mathbb{Z}$-module homomorphism lifts to an algebra homomorphism. However this is obvious, as all we have done is add a single strand (of residue $r \in \mathbb{Z}/e\mathbb{Z}$) to the righthand-side of the diagram and this preserves the multiplication. $\square$

Thus any regular block of $\mathcal{H}_N^\sigma/\mathcal{H}_N^\sigma \mathsf{y}_{\underline{h}} \mathcal{H}_N^\sigma$ is isomorphic to a co-saturated idempotent subalgebra of $\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{y}_{\underline{h}} \mathcal{H}_n^\sigma$ for some $n \geqslant N$. Such truncations preserve decomposition numbers [10, Appendix] and much cohomological structure and so it suffices to consider only these truncated algebras (which is precisely what we have done thus far in the paper!).

7.2. Bases of diagrammatic algebras

For $\lambda, \mu \in \mathscr{P}_{\underline{h}}(n,\sigma)$, we choose reduced paths $\mathsf{P}_{\underline{w}} \in \mathrm{Std}_{n,\sigma}(\lambda)$ and $\mathsf{P}_{\underline{v}} \in \mathrm{Std}_{n,\sigma}(\mu)$ which will remain fixed for the remainder of this section. We remind the reader that this implicitly says that $\lambda \in wA_0$ and $\mu \in vA_0$. We have shown that the map

$$\Psi : \mathscr{S}_{\underline{h}}^{\mathrm{br}}(n,\sigma) \to \mathsf{f}_{n,\sigma}(\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{y}_{\underline{h}} \mathcal{H}_n^\sigma)\mathsf{f}_{n,\sigma}$$

is a graded $\mathbb{Z}$-algebra homomorphism. It remains to show that this map is an isomorphism. Let $\lambda \in \mathscr{P}_{\underline{h}}(n,\sigma)$. Given any reduced path $\mathsf{P}_{\underline{w}} \in \mathrm{Std}_{n,\sigma}(\lambda)$ and any (not necessarily reduced) $\mathsf{Q} \in \mathrm{Std}_{n,\sigma}(\lambda)$ we will inductively construct elements

$$C_{\mathsf{Q}}^{\mathsf{P}} \in 1_{\mathsf{P}} \mathscr{S}_{\underline{h}}^{\mathrm{br}}(n,\sigma) 1_{\mathsf{Q}} \qquad c_{\mathsf{Q}}^{\mathsf{P}} \in e_{\mathsf{P}}(\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{y}_{\underline{h}} \mathcal{H}_n^\sigma) e_{\mathsf{Q}}$$

which provide (cellular) $\mathbb{Z}$-bases of both algebras which match up under the homomorphism, thus proving that Φ is indeed an isomorphism.

We can extend a path $\mathsf{Q}' \in \mathrm{Std}_{n,\sigma}(\lambda)$ to obtain a new path Q in one of three possible ways

$$\mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\boldsymbol{\alpha}} \qquad \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\boldsymbol{\alpha}}^{\flat} \qquad \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\emptyset}$$

for some $\boldsymbol{\alpha} \in \Pi$. The first two cases each subdivide into a further two cases based on whether $\boldsymbol{\alpha}$ is an upper or lower wall of the alcove containing λ. These four cases are pictured in Fig. 32 (for $\mathsf{P}_{\emptyset}$ we refer the reader to Fig. 4). Any two reduced paths $\mathsf{P}_{\underline{w}}, \mathsf{P}_{\underline{v}} \in \mathrm{Std}_{n,\sigma}(\lambda)$ can be obtained from one another by some iterated application of hexagon and commutativity permutations. We let

$$\mathsf{rex}_{\mathsf{P}_{\underline{w}}}^{\mathsf{P}_{\underline{v}}} \qquad \mathsf{REX}_{\mathsf{P}_{\underline{w}}}^{\mathsf{P}_{\underline{v}}}$$

denote the corresponding path-morphism in the algebras $\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{y}_{\underline{h}} \mathcal{H}_n^\sigma$ and $\mathscr{S}_{\underline{h}}^{\mathrm{br}}(n,\sigma)$, respectively (so-named as they permute reduced expressions). In the following construction, we will assume that the elements $c_{\mathsf{Q}'}^{\mathsf{P}'}$ and $C_{\mathsf{Q}'}^{\mathsf{P}'}$ exist for any choice of reduced path

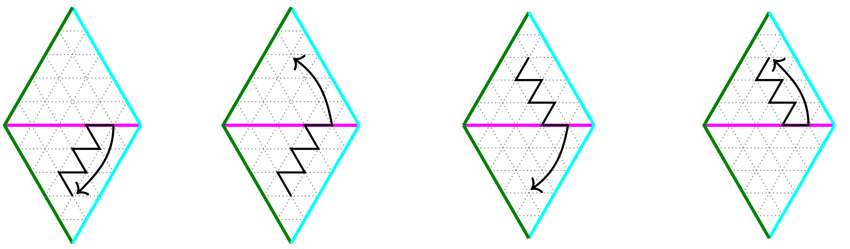


Fig. 32. The first (respectively last) two paths are P_{α} and $\mathsf{P}^{\flat}_{\alpha}$ originating in an alcove with α labelling an upper (respectively lower) wall. The origin lies below the α-hyperplane. We call these paths U_0, U_1, D_0, and D_1 respectively.

P'. We then extend P' using one of the U_0, U_1, D_0, and D_1 paths (which puts a restriction on the form of the reduced expression) but then use a "rex move" to remove obtain elements $c^{\mathsf{P}}_{\mathsf{Q}}$ and $C^{\mathsf{P}}_{\mathsf{Q}}$ for P an arbitrary reduced expression.

Definition 7.7. Suppose that λ belongs to an alcove which has a hyperplane labelled by α as an upper alcove wall. Let $\mathsf{Q}' \in \mathrm{Std}_{n,\sigma}(\lambda)$. If $\mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\alpha}$ then we set $\deg(\mathsf{Q}) = \deg(\mathsf{Q}')$ and we define

$$C^{\mathsf{P}}_{\mathsf{Q}} = \mathsf{REX}^{\mathsf{P}}_{\mathsf{P}' \otimes \mathsf{P}_{\alpha}}(C^{\mathsf{P}'}_{\mathsf{Q}'} \otimes 1_{\alpha}) \qquad c^{\mathsf{P}}_{\mathsf{Q}} = \mathsf{rex}^{\mathsf{P}}_{\mathsf{P}' \otimes \mathsf{P}_{\alpha}}(c^{\mathsf{P}'}_{\mathsf{Q}'} \otimes e_{\mathsf{P}_{\alpha}}).$$

If $\mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}^{\flat}_{\alpha}$ then we set $\deg(\mathsf{Q}) = \deg(\mathsf{Q}') + 1$ and we define

$$C^{\mathsf{P}}_{\mathsf{Q}} = \mathsf{REX}^{\mathsf{P}}_{\mathsf{P}' \otimes \mathsf{P}_{\varnothing}}(C^{\mathsf{P}'}_{\mathsf{Q}'} \otimes \mathsf{SPOT}^{\varnothing}_{\alpha}) \qquad c^{\mathsf{P}}_{\mathsf{Q}} = \mathsf{rex}^{\mathsf{P}}_{\mathsf{P}' \otimes \mathsf{P}_{\varnothing}}(c^{\mathsf{P}'}_{\mathsf{Q}'} \otimes \mathsf{spot}^{\varnothing}_{\alpha}).$$

Now suppose that λ belongs to an alcove which has a hyperplane labelled by α as a lower alcove wall. Thus we can choose $\mathsf{P}_{\underline{v}} \otimes \mathsf{P}_{\alpha} = \mathsf{P}' \in \mathrm{Std}(\lambda)$. For $\mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\alpha}$, we set $\deg(\mathsf{Q}) = \deg(\mathsf{Q}')$ and define

$$\begin{aligned} C^{\mathsf{P}}_{\mathsf{Q}} &= \mathsf{REX}^{\mathsf{P}}_{\mathsf{P}_{\underline{v}\varnothing\varnothing}}\big(1_{\underline{v}} \otimes (\mathsf{SPOT}^{\varnothing}_{\alpha} \circ \mathsf{FORK}^{\alpha\varnothing}_{\alpha\alpha})\big)\big(C^{\mathsf{P}'}_{\mathsf{Q}'} \otimes 1_{\alpha}\big) \\ c^{\mathsf{P}}_{\mathsf{Q}} &= \mathsf{rex}^{\mathsf{P}}_{\mathsf{P}_{\underline{v}\varnothing\varnothing}}\big(e_{\mathsf{P}_{\underline{v}}} \otimes (\mathsf{spot}^{\varnothing}_{\alpha} \circ \mathsf{fork}^{\alpha\varnothing}_{\alpha\alpha})\big)\big(c^{\mathsf{P}'}_{\mathsf{Q}'} \otimes e_{\mathsf{P}_{\alpha}}\big) \end{aligned}$$

and if $\mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}^{\flat}_{\alpha}$ then we set $\deg(\mathsf{Q}) = \deg(\mathsf{Q}') - 1$ and we define

$$\begin{aligned} C^{\mathsf{P}}_{\mathsf{Q}} &= \mathsf{REX}^{\mathsf{P}}_{\mathsf{P}_{\underline{v}\alpha\varnothing}}\big(1_{\underline{v}} \otimes \mathsf{FORK}^{\alpha\varnothing}_{\alpha\alpha}\big)\big(C^{\mathsf{P}'}_{\mathsf{Q}'} \otimes 1_{\alpha}\big) \\ c^{\mathsf{P}}_{\mathsf{Q}} &= \mathsf{rex}^{\mathsf{P}}_{\mathsf{P}_{\underline{v}\alpha\varnothing}}\big(e_{\mathsf{P}_{\underline{v}}} \otimes \mathsf{fork}^{\alpha\varnothing}_{\alpha\alpha}\big)\big(c^{\mathsf{P}'}_{\mathsf{Q}'} \otimes e_{\mathsf{P}_{\alpha}}\big). \end{aligned}$$

In each of the four cases above, the path P is a reduced path by construction (and our assumption that P' is reduced). We remark that the degree of the path, Q, is equal to the degree of both the elements $c^{\mathsf{P}}_{\mathsf{Q}}$ and $C^{\mathsf{P}}_{\mathsf{Q}}$ (recall that P is a path associated to a reduced word and so is of degree zero).

Theorem 7.8 *(Light leaves basis, [13,18]). For each $\lambda \in \mathscr{P}_{\underline{h}}(n,\sigma)$, we fix an arbitrary reduced path $\mathsf{P}_{\underline{w}} \in \mathrm{Std}_{n,\sigma}(\lambda)$. The algebra $\mathscr{S}^{\mathrm{br}}_{\underline{h}}(n,\sigma)$ is quasi-hereditary with graded integral cellular basis*

$$\{C^{\mathsf{P}}_{\mathsf{P}_{\underline{w}}} C^{\mathsf{P}_{\underline{w}}}_{\mathsf{Q}} \mid \mathsf{P},\mathsf{Q} \in \mathrm{Std}_{n,\sigma}(\lambda), \lambda \in \mathscr{P}_{\underline{h}}(n,\sigma)\}$$

with respect to the Bruhat ordering $\triangleright$ on $\mathscr{P}_{\underline{h}}(n,\sigma)$, the anti-involution $*$ given by flipping a diagram through the horizontal axis and the map $\deg : \mathrm{Std}_{n,\sigma}(\lambda) \to \mathbb{Z}$.

We recalled a general construction of a cellular basis of $\mathcal{H}^\sigma_n/\mathcal{H}^\sigma_n \mathsf{y}_{\underline{h}} \mathcal{H}^\sigma_n$ in Theorem 7.1 subject to choosing the reduced expressions. This provides a cellular basis of $\mathsf{f}_{n,\sigma}\mathcal{H}^\sigma_n/\mathcal{H}^\sigma_n \mathsf{y}_{\underline{h}} \mathcal{H}^\sigma_n \mathsf{f}_{n,\sigma}$ by idempotent truncation. Choosing our expressions so as to be compatible with Theorem 7.8 through the map Ψ, we obtain the following.

Theorem 7.9 *(Light leaves basis, [4, Theorem 3.12]). For each $\lambda \in \mathscr{P}_{\underline{h}}(n,\sigma)$, choose an arbitrary reduced path $\mathsf{P}_{\underline{w}} \in \mathrm{Std}_{n,\sigma}(\lambda)$. The algebra $\mathsf{f}_{n,\sigma}(\mathcal{H}^\sigma_n/\mathcal{H}^\sigma_n \mathsf{y}_{\underline{h}} \mathcal{H}^\sigma_n)\mathsf{f}_{n,\sigma}$ is quasi-hereditary with graded integral cellular basis*

$$\{c^{\mathsf{P}}_{\mathsf{P}_{\underline{w}}} c^{\mathsf{P}_{\underline{w}}}_{\mathsf{Q}} \mid \mathsf{P},\mathsf{Q} \in \mathrm{Std}_{n,\sigma}(\lambda), \lambda \in \mathscr{P}_{\underline{h}}(n,\sigma)\}$$

with respect to the Bruhat ordering $\triangleright$ on $\mathscr{P}_{\underline{h}}(n,\sigma)$, the anti-involution $*$ given by flipping a diagram through the horizontal axis and the map $\deg : \mathrm{Std}_{n,\sigma}(\lambda) \to \mathbb{Z}$.

Theorem 7.10. *Let $\sigma \in \mathbb{Z}^\ell$ and $e \in \mathbb{Z}_{>1}$ and suppose that $\underline{h} \in \mathbb{Z}^\ell_{\geqslant 0}$ is (σ,e)-admissible. We have a canonical isomorphism of graded $\mathbb{Z}$-algebras,*

$$\mathsf{f}^+_{n,\sigma}\left(\mathcal{H}^\sigma_n/\mathcal{H}^\sigma_n \mathsf{y}_{\underline{h}} \mathcal{H}^\sigma_n\right)\mathsf{f}^+_{n,\sigma} \cong \mathrm{End}_{\mathcal{D}^{\mathrm{asph},\oplus}_{\mathrm{BS}}(A_{h_0}\times\ldots\times A_{h_{\ell-1}}\backslash \widehat{A}_{h_0+\cdots+h_{\ell-1}})}\left(\oplus_{\underline{w}\in\Lambda(n,\sigma)} B_{\underline{w}}\right).$$

That is, Theorem A of the introduction holds.

Proof. In Section 5 we defined a map from $\mathscr{S}^{\mathrm{br}}_{\underline{h}}(n)$ to $\mathcal{H}^\sigma_n/\mathcal{H}^\sigma_n \mathsf{y}_{\underline{h}} \mathcal{H}^\sigma_n$ via the generators of the former algebra. In Section 6 we showed that this map was a homomorphism by verifying that the relations for $\mathscr{S}^{\mathrm{br}}_{\underline{h}}(n)$ held in the image of the homomorphism. Now, the construction of the light leaves bases in $\mathscr{S}^{\mathrm{br}}_{\underline{h}}(n)$ (respectively $\mathcal{H}^\sigma_n$) is given in terms of the generator (respectively their images). Thus the map preserves the $\mathbb{Z}$-bases and hence is an isomorphism. Thus the result follows from Proposition 3.16. □

An earlier attempt to solve the Libedinsky–Plaza conjecture for the classical blob algebra (the case of $h = 1$ and $\ell = 2$) has already led to a deeper understanding of structure of the diagrammatic Soergel category [19]. We remark that their is no obvious intersection between their results and ours (they do not succeed in proving the $h = 1$ and $\ell = 2$ case, but nor do our results imply theirs).

7.3. Decomposition numbers of Hecke algebras

For $\lambda, \mu \in \mathscr{P}_{\underline{h}}(n,\sigma)$, we reiterate that we have chosen to fix reduced paths $\mathsf{P}_{\underline{w}} \in \mathrm{Std}_{n,\sigma}(\lambda)$ and $\mathsf{P}_{\underline{v}} \in \mathrm{Std}_{n,\sigma}(\mu)$. We define one-sided ideals

$$\mathscr{S}^{\trianglerighteq \underline{v}}_{n,\sigma} = \mathscr{S}^{\mathrm{br}}_{\underline{h}}(n,\sigma)1_{\mathsf{P}_{\underline{v}}} \qquad \mathscr{S}^{\triangleright \underline{w}}_{n,\sigma} = \mathscr{S}^{\trianglerighteq \underline{w}}_{n,\sigma} \cap \mathbb{Z}\{C^{\mathsf{T}}_{\mathsf{P}_{\underline{v}}} C^{\mathsf{P}_{\underline{v}}}_{\mathsf{B}} \mid \mathsf{T},\mathsf{B} \in \mathrm{Std}_{n,\sigma}(\mu), \mu \triangleright \lambda\}$$
$$\mathcal{H}^{\trianglerighteq \mu}_{+} = \mathscr{S}^{\mathrm{br}}_{\underline{h}}(n)e_{\mathsf{P}_{\underline{v}}} \qquad \mathcal{H}^{\triangleright \lambda}_{+} = \mathcal{H}^{\trianglerighteq \lambda}_{+} \cap \mathbb{Z}\{c^{\mathsf{T}}_{\mathsf{P}_{\underline{v}}} C^{\mathsf{P}_{\underline{v}}}_{\mathsf{B}} \mid \mathsf{T},\mathsf{B} \in \mathrm{Std}_{n,\sigma}(\mu), \mu \triangleright \lambda\}$$

and we define the $\mathsf{standard}$ modules of $\mathscr{S}^{\mathrm{br}}_{\underline{h}}(n,\sigma)$ and $\mathsf{f}_{n,\sigma}(\mathcal{H}^{\sigma}_{n}/\mathcal{H}^{\sigma}_{n}\mathsf{y}_{\underline{h}}\mathcal{H}^{\sigma}_{n})\mathsf{f}_{n,\sigma}$ by considering the resulting subquotients. The light leaves construction gives us explicit bases of these quotients as follows

$$\Delta_{\mathbb{Z}}(\underline{w}) = \{C^{\mathsf{S}}_{\mathsf{P}_{\underline{w}}} + \mathscr{S}^{\triangleright \lambda}_{n,\sigma} \mid \mathsf{S} \in \mathrm{Std}_{+}(\lambda)\} \qquad \mathsf{f}_{n,\sigma}\mathbf{S}_{\mathbb{Z}}(\lambda) = \{c^{\mathsf{S}}_{\mathsf{P}_{\underline{w}}} + \mathcal{H}^{\triangleright \lambda} \mid \mathsf{S} \in \mathrm{Std}_{+}(\lambda)\} \tag{7.4}$$

respectively for $\lambda \in \mathscr{P}_{\underline{h}}(n,\sigma)$. The modules $\mathsf{f}_{n,\sigma}\mathbf{S}_{\mathbb{Z}}(\lambda)$ are obtained by truncating the cell modules ($\mathbf{S}_{\mathbb{Z}}(\lambda)$, say) for the cellular structure in Theorem 7.1. For $\Bbbk$ a field, we define

$$\Delta_{\Bbbk}(\underline{w}) = \Delta_{\mathbb{Z}}(\underline{w}) \otimes_{\mathbb{Z}} \Bbbk \qquad \mathsf{f}_{n,\sigma}\mathbf{S}_{\Bbbk}(\lambda) = \mathsf{f}_{n,\sigma}\mathbf{S}_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} \Bbbk.$$

We recall that the cellular structure allows us to define bilinear forms, for each $\lambda \in \mathscr{P}_{\underline{h}}(n)$, there are bilinear forms $\langle\ ,\ \rangle^{\lambda}_{\mathscr{S}}$ and $\langle\ ,\ \rangle^{\lambda}_{\mathcal{H}}$ on $\Delta(\lambda)$ and $\mathsf{f}_{n,\sigma}\mathbf{S}_{\Bbbk}(\lambda)$ respectively, which are determined by

$$\begin{aligned} C^{\mathsf{P}_{\underline{w}}}_{\mathsf{P}} C^{\mathsf{Q}}_{\mathsf{P}_{\underline{w}}} &\equiv \langle C^{\mathsf{P}}_{\mathsf{P}_{\underline{w}}}, C^{\mathsf{Q}}_{\mathsf{P}_{\underline{w}}}\rangle^{\lambda}_{\mathscr{S}}\, 1_{\underline{w}} \pmod{\mathscr{S}^{\triangleright \lambda}_{n,\sigma}} \\ c^{\mathsf{P}_{\underline{w}}}_{\mathsf{P}} c^{\mathsf{Q}}_{\mathsf{P}_{\underline{w}}} &\equiv \langle c^{\mathsf{P}}_{\mathsf{P}_{\underline{w}}}, c^{\mathsf{Q}}_{\mathsf{P}_{\underline{w}}}\rangle^{\lambda}_{\mathcal{H}}\, e_{\mathsf{P}_{\underline{w}}} \pmod{\mathcal{H}^{\triangleright \lambda}_{n,\sigma}} \end{aligned} \tag{7.5}$$

for any $\mathsf{P},\mathsf{Q},\mathsf{P}_{\underline{w}},\mathsf{P}_{\underline{w}} \in \mathrm{Std}(\lambda)$. Factoring out by the radicals of these forms, we obtain a complete set of non-isomorphic simple modules for $\mathscr{S}^{\mathrm{br}}_{\underline{h}}(n,\sigma)$ and $\mathcal{H}^{\sigma}_{n}/\mathcal{H}^{\sigma}_{n}\mathsf{y}_{\underline{h}}\mathcal{H}^{\sigma}_{n}$ as follows

$$L_{\Bbbk}(\underline{w}) = \Delta_{\Bbbk}(\underline{w})/\mathrm{rad}(\Delta_{\Bbbk}(\underline{w})) \qquad \mathsf{f}_{n,\sigma}\mathbf{D}_{\Bbbk}(\lambda) = \mathsf{f}_{n,\sigma}\mathbf{S}_{\Bbbk}(\lambda)/\mathrm{rad}(\mathsf{f}_{n,\sigma}\mathbf{S}_{\Bbbk}(\lambda))$$

respectively for $\lambda \in \mathscr{P}^{+}_{\underline{h}}(n)$. Finally, the projective indecomposable modules are as follows,

$$\mathscr{S}^{\trianglerighteq \underline{v}}_{n,\sigma} = \bigoplus_{w \leqslant v} \dim_t(1_{\underline{v}} L_{\Bbbk}(\underline{w}))P_{\Bbbk}(\underline{w}) \quad \mathcal{H}^{\trianglerighteq \mu}_{n,\sigma} = \bigoplus_{\lambda \trianglerighteq \mu} \dim_t(e_{\mathsf{P}_{\mu}}\mathbf{D}_{\Bbbk}(\lambda))\mathbf{P}_{\Bbbk}(\lambda). \tag{7.6}$$

The isomorphism, Ψ, preserves standard, simple, and projective modules.

The categorical (rather than geometric) definition of p-Kazhdan–Lusztig polynomials is given via the *diagrammatic character* of [13, Definition 6.23]. This graded character is defined in terms of dimensions of certain weight spaces in the light leaves basis. Using the identifications of equation (7.4) and (7.6), the definition of the anti-spherical p-Kazhdan–Lusztig polynomial, ${}^p n_{v,w}(t)$, is as follows,

$$ {}^p n_{v,w}(t) := \dim_t \operatorname{Hom}_{\mathscr{S}^{\rm br}_{\underline{h}}(n,\sigma)}(P(\underline{v}), \Delta(\underline{w})) = \sum_{k\in\mathbb{Z}} \dim[\Delta_{\Bbbk}(\underline{w}) : L_{\Bbbk}(\underline{v})\langle k\rangle] t^k $$

for $v, w \in \Lambda(n,\sigma)$. We claim no originality in this observation and refer to [24, Theorem 4.8] for more details. Through our isomorphism this allows us to see that the graded decomposition numbers of symmetric groups and more general cyclotomic Hecke algebras are *tautologically equal to the associated p-Kazhdan–Lusztig polynomials* as follows,

$$ {}^p n_{v,w}(t) = \sum_{k\in\mathbb{Z}} \dim[\Delta_{\Bbbk}(\underline{w}) : L_{\Bbbk}(\underline{v})\langle k\rangle] t^k = \sum_{k\in\mathbb{Z}} \dim_t[\mathsf{f}_{n,\sigma}\mathbf{S}_{\Bbbk}(\lambda) : \mathsf{f}_{n,\sigma}\mathbf{D}_{\Bbbk}(\mu)\langle k\rangle] t^k $$

for $\lambda, \mu \in \mathscr{P}_{\underline{h}}(n,\sigma)$ where the equality follows immediately from our isomorphism. Finally, we remind the reader that truncation by $\mathsf{f}_{n,\sigma}$ is to a co-saturated subset of weights and so preserves the decomposition matrices of these algebras, see for example [10, Appendix]

7.4. Counterexamples to Lusztig's conjecture and intersection forms

In [28], the counterexamples to Soergel's conjecture are presented in the classical (rather than diagrammatic) language of intersection forms associated to the fibre of a Bott–Samelson resolution of a Schubert varieties. However, Williamson emphasises that all his calculations were done using the equivalent diagrammatic setting of the light leaves basis, which is *"explicit and amenable to computation"*. Moreover, Williamson's counterexamples are dependent on the diagrammatics because it is only *"from the diagrammatic approach [that] it is clear that [the intersection form] $I^{\Bbbk}_{x,\underline{w},d}$ is defined over $\mathbb{Z}$"* in the first place (see Section 3 of [28] for more details). In terms of the light leaves cellular basis, Williamson's calculation makes a clever choice of a pair of partitions λ, μ (equivalently, words $w, v \in \widehat{\mathfrak{S}}_h$ labelling the alcoves containing these partitions) for which there exists a unique element $\mathsf{Q} \in \mathrm{Std}_{n,\sigma}(\lambda)$ such that $\mathsf{Q} \sim \mathsf{P}_{\underline{v}} \in \mathrm{Std}_{n,\sigma}(\mu)$. By highest weight theory, we have that

$$ d_{\lambda\mu}(t) = \begin{cases} t^{\deg(\mathsf{Q})} & \text{if } \langle C^{\mathsf{Q}}_{\mathsf{P}_{\underline{w}}}, C^{\mathsf{Q}}_{\mathsf{P}_{\underline{w}}}\rangle^{\lambda}_{\mathscr{S}} = 0 \in \Bbbk \\ 0 & \text{otherwise} \end{cases} $$

and Williamson proved for $\lambda, \mu \in \mathscr{P}_{h,1}(n)$ (a pair from "around the Steinberg weight") that the form is zero for certain primes $p > h$ whereas it is equal to 1 for $\Bbbk = \mathbb{C}$ (and hence disproved Lusztig's and James' conjectures).

Now, clearly the Gram matrices of the bilinear forms in equation (7.5) are preserved under isomorphism. Thus applying our isomorphism (and Brundan–Kleshchev's [8]) one can view Williamson's counterexamples as being found entirely within the context of the symmetric group. More generally, we deduce the following:

Theorem 7.11. *Theorem B of the introduction holds.*

Acknowledgments

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Appendix A. Weakly graded monoidal categories

In this appendix we describe the framework for constructing the breadth-enhanced diagrammatic Bott–Samelson endomorphism algebras. Informally, "breadth-enhanced" means that we record and keep track of the "breadth" of Soergel diagrams, including the "blank spaces" between strands. This is contrary to the usual working assumption that Soergel diagrams are defined only up to isotopy. We will say a few words for why we have chosen to break this convention in this paper.

Soergel diagrams and KLR diagrams have an important fundamental difference. KLR diagrams, which are essentially decorated wiring diagrams, always have the same number of nodes on the top and bottom edges. By contrast, the top and bottom edges of a Soergel diagram may not have the same number of nodes. This basic observation is enough to ensure that a Soergel diagram cannot correspond to only one KLR diagram under the isomorphism in the main theorem. For example, suppose the isomorphism maps the α-coloured spot diagram to a KLR diagram spot_α, with bottom edge P and top edge Q. Then the empty Soergel diagram (with no strands at all) should map to the KLR idempotent e_{Q}. However it is also clear that the empty Soergel diagram should correspond to the empty KLR diagram.

The breadth-enhanced diagrammatic Bott–Samelson endomorphism algebra introduces new idempotents, indexed by expressions in the extended alphabet $S \cup \{\emptyset\}$. This ensures that the isomorphism is well defined, with each breadth-enhanced Soergel diagram corresponding to a single KLR diagram. The breadth of a breadth-enhanced Soergel diagram is simply the number of strands of the corresponding KLR diagram, divided by h. We draw breadth-enhanced Soergel diagrams so that the width is proportional to the breadth. In particular, we write $1_\emptyset$ to indicate the empty Soergel diagram of breadth 1 (i.e. a "blank space"), which corresponds to the KLR idempotent $e_{\mathsf{P}_\emptyset}$ with h strands. The breadth-enhanced algebras are Morita equivalent to the usual diagrammatic Bott–Samelson endomorphism algebras, by simply truncating with respect to the idempotents indexed by expressions which do not contain $\emptyset$. Thus once we prove the isomorphism for the breadth-enhanced algebras, we immediately obtain an isomorphism for the usual Bott–Samelson algebras.

The machinery for building breadth-enhanced algebras is the notion of a weakly graded monoidal category. Weakly graded monoidal categories can be thought of as generalizations of graded monoidal categories, with the grade shifts represented by tensoring with a fixed shifting object. The construction of breadth-enhanced algebras is then analogous to defining a graded category from a non-graded category by concentrating the objects in certain fixed degrees.

We have chosen to write this appendix using the categorical (rather than the algebraic) perspective. We hope that this will make the results more applicable and the proofs easier to read. All the categories below will be assumed to be small. We will also use "monoidal" to mean "strict monoidal" unless stated otherwise. It is probably possible to generalize everything to arbitrary monoidal categories, but this will not be necessary for our purposes.

A.1. Definition and examples

Definition A.1. A weakly graded monoidal category is a monoidal category $(\mathcal{A}, \otimes)$ together with an object in the Drinfeld centre with trivial self-braiding. This consists of the following data:

- an object I in $\mathcal{A}$ called the shifting object;
- for each object X in $\mathcal{A}$, an isomorphism $s_X : X \otimes I \xrightarrow{\sim} I \otimes X$ called a simple adjustment

such that

(WG1) the simple adjustments $\{s_X\}$ are the components of a natural isomorphism $s : (-) \otimes I \Rightarrow I \otimes (-)$;

(WG2) for any objects X, Y in $\mathcal{A}$ the following diagram commutes

$$\begin{array}{ccc} X \otimes Y \otimes I & \xrightarrow{\quad s_{X\otimes Y} \quad} & I \otimes X \otimes Y \\ & {\scriptstyle 1_X \otimes s_Y} \searrow \qquad \nearrow {\scriptstyle s_X \otimes 1_Y} & \\ & X \otimes I \otimes Y & \end{array}$$

(WG3) we have $s_I = 1_{I \otimes I}$.

Example A.2. Suppose $\mathcal{A}^\bullet$ is a graded monoidal category, i.e. a monoidal category whose Hom-spaces are graded modules. For the moment, let us drop the assumption of strictness and suppose that $\mathcal{A}^\bullet$ is strictly associative, but with non-trivial unitors. In the usual way we may construct a new category $\mathcal{A}$ by adding grade shifts and restricting to homogeneous morphisms. More precisely, the objects of $\mathcal{A}$ are the formal symbols $X(m)$ for each object X of $\mathcal{A}^\bullet$ and each $m \in \mathbb{Z}$, and the Hom-spaces are

$$\mathrm{Hom}_{\mathcal{A}}(X(m), Y(n)) = \mathrm{Hom}^{n-m}_{\mathcal{A}^\bullet}(X, Y).$$

It is clear that the grade shift (1) is an autoequivalence of $\mathcal{A}$. Moreover, the tensor product $X(m) \otimes Y(n) = (X \otimes Y)(m+n)$ gives $\mathcal{A}$ the structure of a monoidal category. Now let $\mathbb{1}$ be the identity object in $\mathcal{A}^\bullet$ and set $I = \mathbb{1}(1)$. We observe that

$$X(m) \otimes \mathbb{1} = (X \otimes \mathbb{1})(1) \xrightarrow{\rho_X(1)} X(m+1) \xleftarrow{\lambda_X(1)} (\mathbb{1} \otimes X)(1) = \mathbb{1} \otimes X(m),$$

and it is straightforward to check that the isomorphisms $s_{X(m)} = \lambda_{X(m)}(1)^{-1} \circ \rho_{X(m)}(1)$ satisfy axioms (WG1)–(WG3). Thus $\mathcal{A}$ has the structure of a weakly graded monoidal category.

The main result which we will need in the next subsection is a coherence theorem for weakly graded monoidal categories. Roughly, coherence for weakly graded monoidal categories means that every diagram built up from s and identity morphisms (using composition and tensor products) commutes. The precise formulation of coherence requires some combinatorial constructions, which we describe below. Let $\mathscr{W}$ be the set of non-empty words in the symbols e and x. We define the following semigroup homomorphisms length : $\mathscr{W} \to \mathbb{Z}_{\geqslant 0}$ and breadth : $\mathscr{W} \to \mathbb{Z}_{\geqslant 0}$ on the generators:

$$\begin{aligned} \mathrm{length}(e) &= 0 \qquad & \mathrm{breadth}(e) &= 1 \\ \mathrm{length}(x) &= 1 \qquad & \mathrm{breadth}(x) &= 0. \end{aligned}$$

For $w \in \mathscr{W}$ of length n, we can associate a functor $w_{\mathcal{A}} : \mathcal{A}^n \to \mathcal{A}$ by replacing each e with the object I, each x with the identity functor $1_{\mathcal{A}}$, and tensoring the resulting sequence. More formally, we fix

$$\begin{aligned} e_{\mathcal{A}} : * &\longrightarrow \mathcal{A} \qquad & x_{\mathcal{A}} : \mathcal{A} &\longrightarrow \mathcal{A} \\ * &\longmapsto I & A &\longmapsto A \end{aligned}$$

and inductively define

$$\begin{aligned} (ew)_{\mathcal{A}} : \mathcal{A}^n &\longrightarrow \mathcal{A} \qquad & (xw)_{\mathcal{A}} : \mathcal{A}^{n+1} &\longrightarrow \mathcal{A} \\ (A_1, \ldots, A_n) &\longmapsto I \otimes w_{\mathcal{A}}(A_1, \ldots, A_n) \qquad & (A_1, \ldots, A_{n+1}) &\longmapsto A_1 \otimes w_{\mathcal{A}}(A_2, \ldots, A_{n+1}) \end{aligned}$$

where $n = \mathrm{length}(w)$.

Theorem A.3. *Let $u, v \in \mathscr{W}$ such that* length(u) = length(v) *and* breadth(u) = breadth(v). *There is a unique natural isomorphism $u_{\mathcal{A}} \cong v_{\mathcal{A}}$ built up from tensor products and compositions of components of s, s^{-1}, and the identity.*

We will defer the proof to the end of this appendix.

We call a component of any natural isomorphism arising from Theorem A.3 an adjustment. For two morphisms $f : X \to Y$ and $g : Z \to W$ we write $f \sim g$ and say that f and g are adjustment equivalent if there exist adjustments

$$q : X \xrightarrow{\sim} Z \qquad r : Y \xrightarrow{\sim} W$$

such that $g = r \circ f \circ q^{-1}$.

Example A.4. For any morphism $f : X \to Y$ in $\mathcal{A}$, we have $f \otimes 1_I \sim 1_I \otimes f$, because

$$f \otimes 1_I = s_Y^{-1} \circ (1_I \otimes f) \circ s_X$$

by the naturality of simple adjustments.

A.2. Breadth grading

Suppose $\mathcal{A}$ is a monoidal category. Assuming $\mathcal{A}$ is small, the set $\mathrm{Ob}(\mathcal{A})$ has the structure of a monoid. We call a monoidal homomorphism $b : \mathrm{Ob}(\mathcal{A}) \to \mathbb{Z}_{\geqslant 0}$ a breadth function.

Definition A.5. Let $\mathcal{A}$ be a monoidal category with a breadth function b. The weak grading of $\mathcal{A}$ concentrated in breadth b is the following weakly graded monoidal category $\mathcal{A}[b]$.

Objects The objects of $\mathcal{A}[b]$ are formal free tensor products of objects in $\mathcal{A}$ and a new object I. In other words, each object X in $\mathcal{A}[b]$ is a formal sequence

$$I^{\otimes r_0} \otimes X_1 \otimes I^{\otimes r_1} \otimes X_2 \otimes \cdots \otimes I^{\otimes r_{m-1}} X_m \otimes I^{\otimes r_m}$$

for some non-negative integers r_0, r_m, positive integers $r_1, r_2, \dots, r_{m-1}$, and non-identity objects $X_1, X_2, \dots, X_m$ in $\mathcal{A}$. The tensor product on objects in $\mathcal{A}$ extends in the obvious way to objects in $\mathcal{A}[b]$. We also extend the breadth function b to a monoidal homomorphism $b : \mathrm{Ob}(\mathcal{A}[b]) \to \mathbb{Z}_{\geqslant 0}$ by fixing $b(I) = 1$.

Morphisms For any object X of the above form write X' for the object

$$X_1 \otimes X_2 \otimes \cdots \otimes X_m$$

in $\mathcal{A}$. We define

$$\operatorname{Hom}_{\mathcal{A}[b]}(X,Y) = \begin{cases} \operatorname{Hom}_{\mathcal{A}}(X',Y') & \text{if } b(X)=b(Y), \\ 0 & \text{otherwise.} \end{cases}$$

Composition and tensor products follow from those in $\mathcal{A}$.

Weak grading For X an object in $\mathcal{A}[b]$, the natural isomorphism $s_X : X \otimes I \to I \otimes X$ in $\mathcal{A}[b]$ corresponding to the identity morphism $1_{X'}$ in $\mathcal{A}$ gives $\mathcal{A}[b]$ the structure of a weakly graded monoidal category.

If $f : X \to Y$ is a morphism in $\mathcal{A}[b]$, write $f' : X' \to Y'$ for the corresponding morphism in $\mathcal{A}$. It is easy to check that this mapping is functorial. We write $b(f)$ for the non-negative integer $b(X) = b(Y)$.

Remark A.6. The category $\mathcal{A}[b]$ is the weak graded analogue of the following graded construction. For a monoidal category $\mathcal{A}$ with a breadth function b, define a grading by setting $\deg f = b(X) - b(Y)$ for each morphism $f : X \to Y$. As in Example A.2, we add grade shifts and restrict to homogeneous morphisms to obtain the category $\mathcal{A}\langle b\rangle$. We may extend the breadth function b to all of $\mathcal{A}\langle b\rangle$ as above. For any morphism $g : U \to V$ in $\mathcal{A}\langle b\rangle$, we have $0 = \deg g = b(U) - b(V)$, which allows us to define the breadth of g to be $b(g) = b(U) = b(V)$ as in the weakly graded case.

Our naming convention for $\mathcal{A}[b]$ (“concentrated in breadth b”) comes from a special case of the above graded construction. If $\mathcal{A}$ is a category of modules over some ring R, then we may equivalently construct the grading by considering R to be a graded ring concentrated in degree 0 and each object X to be concentrated in degree $-b(X)$.

As a consequence of our coherence result, there is an alternative presentation of $\mathcal{A}[b]$ in terms of generators and relations. First we introduce a way of embedding morphisms from $\mathcal{A}$ into $\mathcal{A}[b]$.

Definition A.7. Let $f : U \to V$ be a morphism in $\mathcal{A}$. The (left) minimal breadth representative of f is the morphism $g : X \to Y$ in $\mathcal{A}[b]$ such that $g' = f$ and

$$X = I^{\otimes \max(0, b(V)-b(U))} \otimes U, \qquad Y = I^{\otimes \max(0, b(U)-b(V))} \otimes V.$$

Theorem A.8. *Let $\mathcal{M}$ be the set of all minimal breadth representatives of morphisms in $\mathcal{A}$. The category $\mathcal{A}[b]$ is generated as a monoidal category by the morphisms*

$$\{1_I\} \cup \{s_X : X \in \operatorname{Ob}(\mathcal{A})\} \cup \mathcal{M}$$

subject to the following relations:

- *the usual weak grading axioms (WG1)–(WG3);*
- *for morphisms $f : X \longrightarrow Y, g : Z \longrightarrow W, h : U \longrightarrow V$ in $\mathcal{M}$ such that $f' \circ g' = h'$, we have*

$$(1_I^{\otimes \max(0,b(g)-b(f))} \otimes f) \circ (1_I^{\otimes \max(0,b(f)-b(g))} \otimes g) \sim 1_I^{\otimes \max(b(f),b(g))-b(h)} \otimes h;$$

- *for morphisms* $f : X \longrightarrow Y, g : Z \longrightarrow W, h : U \longrightarrow V$ *in* $\mathcal{M}$ *such that* $f' \otimes g' = h'$, *we have*

$$f \otimes g \sim 1_I^{\otimes b(f)+b(g)-b(h)} \otimes h.$$

Proof. Let $\mathcal{B}$ be the monoidal category defined by the above generators and relations. It is clear that the same relations hold in $\mathcal{A}[b]$, so there is a functor $\mathcal{B} \to \mathcal{A}[b]$. It is enough to show that this functor is full and faithful. Let X, Y be objects in $\mathcal{B}$ such that $b(X) = b(Y)$. We will show that any morphism $X \to Y$ can be written in the form

$$q \circ (1_I^{b(X)-\max(b(X'),b(Y'))} \otimes f) \circ p^{-1},$$

where p, q are adjustments and f is a minimal breadth representative. In other words, we will show that every morphism in $\mathcal{B}$ is adjustment equivalent to the tensor product of a minimal breadth representative and some number of copies of 1_I. This automatically gives fullness and faithfulness of the functor above, which proves the result. Since the generating morphisms of $\mathcal{B}$ are all already of this form, it is enough to show that any composition or tensor product of two morphisms of this form is again of this form. Now, consider a composition

$$q \circ (1_I^{\otimes m} \otimes f) \circ p^{-1} \circ t \circ (1_I^{\otimes n} \otimes g) \circ r^{-1}$$

of two morphisms of the above form. Both f and g are minimal breadth representatives, so their domains and codomains are "left-adjusted", i.e. of the form $I^{\otimes l} \otimes U$ for some object U in $\mathcal{A}$ and some non-negative integer l. The adjustment $p^{-1} \circ t$ is an isomorphism between $I^{\otimes n} \otimes \mathsf{cod} g$ and $I^{\otimes m} \otimes \mathsf{dom} f$ which are both left-adjusted, so in fact they must be equal. By Theorem A.3 we must have $p = t$, so the composition above equals

$$\begin{aligned} q \circ (1_I^{\otimes m} \otimes f) \circ (1_I^{\otimes n} \otimes g) \circ r^{-1} &= q \circ (1_I^{\otimes (m-j)} \otimes 1_I^j \otimes f) \circ (1_I^{\otimes (n-k)} \otimes 1_I^{\otimes k} \otimes g) \circ r^{-1} \\ &\sim q \circ (1_I^{\otimes (m-j)} \otimes h) \circ r^{-1} \end{aligned}$$

where $j = \max(0, b(g) - b(f))$, $k = \max(0, b(f) - b(g))$, and h is the minimal breadth representative of $f' \circ g'$. Similarly, consider a tensor product of two morphisms of the above form. We have

$$\begin{aligned} &(q \circ (1_I^{\otimes m} \otimes f) \circ p^{-1}) \otimes (t \circ (1_I^{\otimes n} \otimes g) \circ r^{-1}) \\ &= (q \otimes t) \circ (1_I^{\otimes m} \otimes f \otimes 1_I^{\otimes n} \otimes g) \circ (p^{-1} \otimes r^{-1}) \\ &\sim (q \otimes t) \circ (1_I^{\otimes (m+n)} \otimes f \otimes g) \circ (p^{-1} \otimes r^{-1}) \\ &\sim (q \otimes t) \circ (1_I^{\otimes (m+n+b(f)+b(g)-b(h))} \otimes h) \circ (p^{-1} \otimes r^{-1}), \end{aligned}$$

where h is the minimal breadth representative of $f' \otimes g'$. $\square$

A.3. Proof of coherence

We conclude with the proof of the coherence theorem for weakly graded monoidal categories (Theorem A.3). The strategy is broadly similar to Mac Lane's proof of the coherence theorem for monoidal categories [20, VII.2]. This involves first proving the result for a single object X in the category $\mathcal{A}$, and then extending to all of $\mathcal{A}$.

Now let $\mathscr{S}$ be the set of words in the symbols $\{\sigma_w, \sigma_w^{-1} : w \in \mathscr{W}\} \cup \{\iota_e, \iota_x\}$ defined inductively as follows. For any $w \in \mathscr{W}$ we have $\sigma_w, \sigma_w^{-1} \in \mathscr{S}$. Moreover, for any $\alpha \in \mathscr{S}$ and $w \in \mathscr{W}$ we also have $\iota_e\alpha, \iota_x\alpha \in \mathscr{S}$ and $\alpha\iota_e, \alpha\iota_x \in \mathscr{S}$. For convenience we write ι_w for $\iota_{w_1}\iota_{w_2}\cdots\iota_{w_m}$, where $w = w_1w_2\cdots w_n$ is a word in $\mathscr{W}$. We inductively define $\mathsf{dom} : \mathscr{S} \to \mathscr{W}$ and $\mathsf{cod} : \mathscr{S} \to \mathscr{W}$ as follows:

$$\begin{aligned} \mathsf{dom}(\sigma_w) &= we & \mathsf{cod}(\sigma_w) &= ew \\ \mathsf{dom}(\sigma_w^{-1}) &= ew & \mathsf{cod}(\sigma_w^{-1}) &= we \\ \mathsf{dom}(\iota_w\alpha) &= w\mathsf{dom}(\alpha) & \mathsf{cod}(\iota_w\alpha) &= w\mathsf{cod}(\alpha) \\ \mathsf{dom}(\alpha\iota_w) &= \mathsf{dom}(\alpha)w & \mathsf{cod}(\alpha\iota_w) &= \mathsf{cod}(\alpha)w \end{aligned}$$

Let $\mathscr{G}$ be the quiver with vertices given by $\mathscr{W}$ and arrows given by $\mathscr{S}$. It is easy to verify that for any word in $\alpha \in \mathscr{S}$, $\text{length}(\mathsf{dom}(\alpha)) = \text{length}(\mathsf{cod}(\alpha))$ and $\text{breadth}(\mathsf{dom}(\alpha)) = \text{breadth}(\mathsf{cod}(\alpha))$. Thus the graph $\mathscr{G}$ has components $\mathscr{G}_{n,k}$ whose vertices $\mathscr{W}_{n,k}$ consist of words of length n and breadth k.

Now let $\mathcal{A}$ be a weakly graded monoidal category. We fix an object X in $\mathcal{A}$ and set

$$\begin{aligned} \mathscr{I}_X(e) &= I & \mathscr{I}_X(x) &= X \\ \mathscr{I}_X(ew) &= I \otimes \mathscr{I}_X(w) & \mathscr{I}_X(xw) &= X \otimes \mathscr{I}_X(w) \\ \mathscr{I}_X(\sigma_w) &= s_{w_X} & \mathscr{I}_X(\sigma_w^{-1}) &= s_{w_X}^{-1} \\ \mathscr{I}_X(\iota_w\alpha) &= 1_{w_X} \otimes \mathscr{I}_X(\alpha) & \mathscr{I}_X(\alpha\iota_w) &= \mathscr{I}_X(\alpha) \otimes 1_{w_X} \end{aligned}$$

Proposition A.9. *Let $u, v \in \mathscr{W}$ such that* $\text{length}(u) = \text{length}(v)$ *and* $\text{breadth}(u) = \text{breadth}(v)$*. Suppose $\alpha_1 \circ \cdots \circ \alpha_m$ and $\alpha'_1 \circ \cdots \circ \alpha'_{m'}$ are two paths in $\mathscr{G}$ from u to v. Then*

$$\mathscr{I}_X(\alpha_m) \circ \cdots \circ \mathscr{I}_X(\alpha_1) = \mathscr{I}_X(\alpha'_{m'}) \circ \cdots \circ \mathscr{I}_X(\alpha'_1).$$

Proof. Let $n = \text{length}(u) = \text{length}(v)$ and $k = \text{breadth}(u) = \text{breadth}(v)$. We will pivot on the sink vertex $w^{(n,k)} = e^kx^n$ in the component $\mathscr{G}_{n,k}$. Every nonempty word in S contains exactly one symbol of the form σ_w or σ_w^{-1} for $w \in W$. Call such words directed or anti-directed respectively. It is easy to check that for any two directed words α, α' with the same domain and codomain, we must have $\mathscr{I}_X(\alpha) = \mathscr{I}_X(\alpha')$.

We inductively define a function $\rho : W \to \mathbb{Z}_{\geqslant 0}$ by

$$\rho(e) = 0 \qquad \rho(x) = 0 \qquad \rho(ew) = \rho(w) \qquad \rho(xw) = \rho(w) + \text{breadth}(w).$$

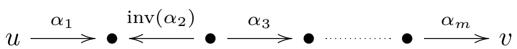
We also inductively define a function $\text{can}_{n,k}$ mapping words in $W_{n,k}$ to directed paths in $\mathscr{G}_{n,k}$ by

$$\text{can}_{0,1}(e) = \emptyset \quad \text{can}_{1,0}(x) = \emptyset \quad \text{can}_{n,k}(ew) = \iota_e \text{can}_{n,k-1}(w)$$
$$\text{can}_{n,k}(xw) = (\iota_e^{k-1}\sigma_x\iota_x^{n-1}) \circ \cdots \circ (\iota_e\sigma_x\iota_e^{k-2}\iota_x^{n-1}) \circ (\sigma_x\iota_e^{k-1}\iota_x^{n-1}) \circ (\iota_x \text{can}_{n-1,k}(w))$$

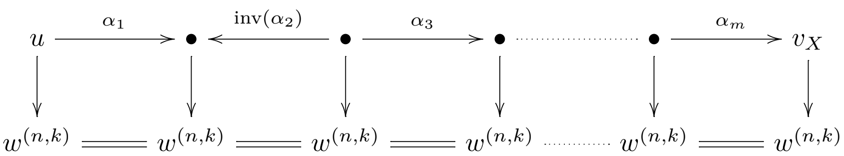
It can be shown that $\text{can}_{n,k}(w)$ is the longest directed path in $\mathscr{G}_{n,k}$ from w to $w^{(n,k)}$, and that $\rho(w) = \text{length}(\text{can}_{n,k}(w))$.

Lemma A.10. *For any $u \in \mathscr{W}_{n,k}$, $\mathscr{J}_X$ maps all directed paths from u to $w^{(n,k)}$ to the same morphism.*

Before we prove this lemma, we will show that the proposition follows from it almost immediately. For $\alpha \in \mathscr{S}$ let $\text{inv}(\alpha)$ be the word obtained by switching the symbols $\sigma_w \leftrightarrow \sigma_w^{-1}$. Clearly $\mathscr{J}_X(\text{inv}(\alpha)) = \mathscr{J}_X(\alpha)^{-1}$, and we may write any anti-directed word as the formal inverse of a directed word. Let us write the path $\alpha_m \circ \cdots \circ \alpha_1$ from u to v in this manner, using formal inverses of directed words for any anti-directed word that appears. For example, if α_2 is the only anti-directed word in this path, we write:



Now draw canonical paths downwards to $w^{(n,k)}$ underneath each of these objects:

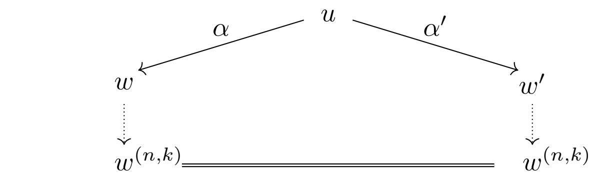


After applying $\mathscr{J}_X$, each square commutes by the above lemma, so

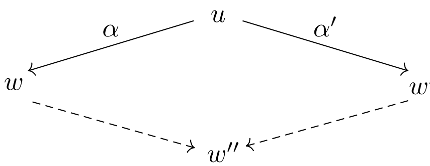
$$\mathscr{J}_X(\alpha_m) \circ \cdots \circ \mathscr{J}_X(\alpha_1) = \mathscr{J}_X(\text{can}_{n,k}(v))^{-1} \circ \mathscr{J}_X(\text{can}_{n,k}(u)).$$

Since the right-hand side only depends on u and v, we are done. □

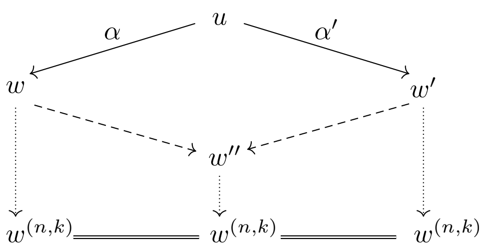
Proof of Lemma A.10. We induct on $\rho(u)$, n and k. Suppose we have two directed paths from u to $w_{n,k}$ which start with α and α' respectively.



As $\rho(w) < \rho(u)$, we are then done by induction. Otherwise, suppose $w \neq w'$ and $\alpha \neq \alpha'$. It is enough to find some $w'' \in W$ and some paths from w and w' to w'' such that the following diamond

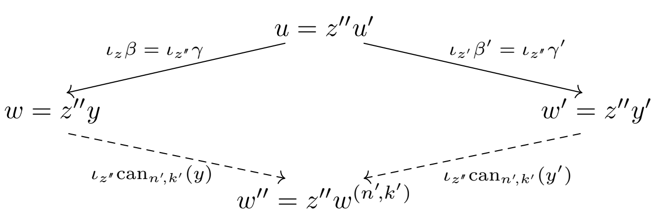


commutes after applying $\mathscr{J}_X$. For if so, then $\rho(w'') < \rho(u)$, and by induction the trapezoids in the following diagram



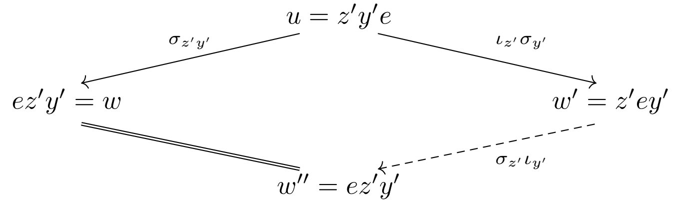
commute after applying $\mathscr{J}_X$, and therefore the whole diagram commutes.

Case 1. If $\alpha = \iota_z\beta$ and $\alpha' = \iota_{z'}\beta'$ for some $z, z' \in \mathscr{W}$ and $\beta, \beta' \in \mathscr{S}$, then both z and z' begin with some non-empty word z''. Thus u, w, and w' also begin with z'', and we can write α and α' as $\iota_{z''}\gamma$ and $\iota_{z''}\gamma'$ respectively. Let $u' = \mathsf{dom}(\gamma)$, $y = \mathsf{cod}(\gamma)$, and $y' = \mathsf{cod}(\gamma')$, and let n' and k' be the length and breadth of y (or y') respectively. Since y is a strict subword of w, we must have $n' < n$ or $k' < k$. Taking $w'' = z''w^{(n',k')}$ we obtain the following diamond

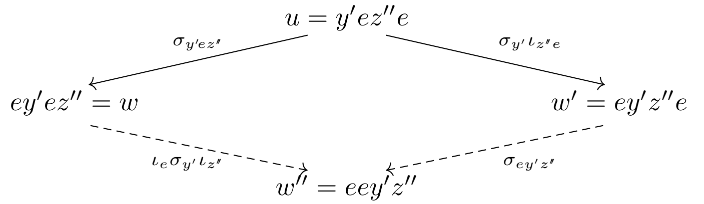


which commutes after applying $\mathscr{J}_X$ by induction on n and k. A similar proof works if $\alpha = \beta\iota_z$ and $\alpha' = \beta'\iota_{z'}$ for some $z, z'' \in \mathscr{W}$ and $\beta, \beta' \in \mathscr{S}$.

Cases 2 & 3. The next cases to consider occur when one of α or α' is σ_y for some $y \in \mathscr{W}$. Without loss of generality suppose $\alpha = \sigma_y$. If α' is of the form $\iota_{z'}\sigma_{y'}$ for some $y', z' \in \mathscr{W}$ then we must have $y = z'y'$ and thus $u = ye = z'y'e$. Taking $w'' = ez'y'$ we obtain the following diamond

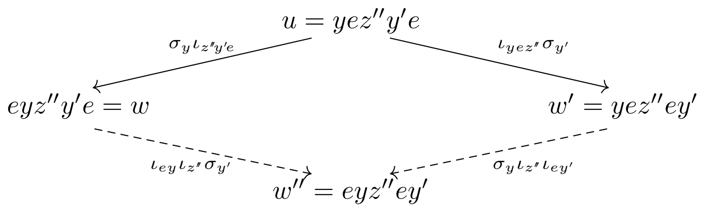


which commutes after applying $\mathscr{J}_X$ by (WG2). On the other hand, if α' is of the form $\sigma_{y'}\iota_{z'}$ for some $y', z' \in W$, then we must have $z' = z''e$ for some $z'' \in W$, and thus $y = y'ez''$. Taking $w'' = eey'z''$ we obtain the following diagram



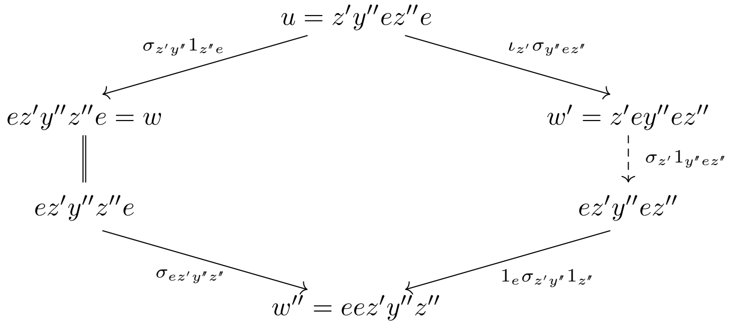
which commutes after applying $\mathscr{J}_X$, by the naturality of s.

Cases 4 & 5. The last cases are when $\alpha = \sigma_y\iota_z$ and $\alpha' = \iota_{z'}\sigma_{y'}$ for some $y, y', z, z' \in W$, so that $u = yez = z'y'e$. Suppose first that z' starts with ye. Then there is some $z'' \in W$ such that $z' = yez''$. Using $yez = z'y'e$ it is also clear that $z = z''y'e$ too. Taking $w'' = eyz''ey'$ we obtain the diamond



which commutes after applying $\mathscr{J}_X$ by bifunctoriality of the tensor product. On the other hand, if ye starts with z', then there exists some $y'' \in W$ such that $y = z'y''$. This also implies that $y'e$ ends with z, so there also exists some $z'' \in W$ such that $z = z''e$.

This means that $y' = y''ez''$. This time we complete the diamond in two steps. First, we compose $\iota_{z'}\sigma_{y''ez''}$ with $\sigma_{z'}\iota_{y''ez''}$. By (WG2) of a weak grading, this composition equals $\sigma_{z'y''ez''}$. Thus we have reduced to a previous case and so we are done.



□

To extend to the full coherence theorem, we consider objects in a higher category.

Proof of Theorem A.3. Let Iter($\mathcal{A}$) be the category of functors of the form $\mathcal{A}^n \to \mathcal{A}$, where n is a non-negative integer. It is clear that Iter($\mathcal{A}$) is also monoidal, with the tensor product of two functors $F : \mathcal{A}^m \to \mathcal{A}$ and $G : \mathcal{A}^n \to \mathcal{A}$ defined to be

$$(F \otimes G) : \mathcal{A}^{m+n} \longrightarrow \mathcal{A}, \;\; (A_1, \ldots, A_{m+n}) \longmapsto F(A_1, \ldots, A_m) \otimes G(A_{m+1}, \ldots, A_{m+n})$$

We observe that $w_{\mathcal{A}}$ is precisely $\mathscr{I}_{1_{\mathcal{A}}}(w)$ as defined above, where we consider the identity functor $1_{\mathcal{A}}$ as an object in Iter($\mathcal{A}$). Applying $\mathscr{I}_{1_{\mathcal{A}}}$ to any path between u and v gives an isomorphism in Iter($\mathcal{A}$) between $u_{\mathcal{A}}$ and $v_{\mathcal{A}}$, or in other words, a natural isomorphism between the two functors. Uniqueness of this natural isomorphism follows from Proposition A.9. □

Appendix B. List of symbols

For the convenience of the reader we list the symbols used in the main body of the paper in three categories: those corresponding to the general setup and basic combinatorics; those corresponding to the geometry and choice of paths; and those corresponding to the various algebras of interest (Tables 1–3). As Appendix A is relatively short and self-contained we omit those symbols here.

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Table 1
General symbols.

Symbol	§§	Symbol	§§	Symbol	§§
h	2	ℓ	2	e	2
σ	2	λ	2.1	$\lvert\lambda\rvert$	2.1
$\lambda^{(i)}$	2.1	$\mathrm{ct}(r,c,m)$	2.1	$\mathrm{res}(r,c,m)$	2.1
$\mathscr{P}_\ell(n)$	2.1	T_λ	2.1	$\mathscr{P}_{\underline{h}}(n)$	2.1
$\mathrm{Std}(\lambda)$	2.1	$\varnothing$	2.1	$\mathfrak{S}_h$	2.2
$\mathfrak{S}_f$	2.2	$\widehat{\mathfrak{S}}_h$	2.2	$\mathfrak{S}^f$	2.2
$\underline{w}$	2.4	$\overline{w}$	2.4	$r_h(t)$	2.4
$\boldsymbol{\alpha}(p)$	2.4	$\emptyset(q)$	2.4	$\underline{i}$	3.3
s_r	3.3	$s_r(\underline{i})$	3.3	w^p_q	3.3
$(i,i+1)_b$	4.3	$\underline{w}_b$	4.3	B	4.3
$\mathrm{nib}(\underline{w})$	4.3	$\mathrm{nib}(\underline{i})$	4.3	$\det_{\underline{h}}$	7.1
Std_n	7.1	Γ	7.1	$\square$	7.1

Table 2
Geometry and paths.

Symbol	§§	Symbol	§§	Symbol	§§
ε_i	2.2	$\mathbb{E}_{\underline{h}}$	2.2	$\overline{\mathbb{E}}_{\underline{h}}$	2.1
$\langle\ ,\ \rangle$	2.2	$(-)^T$	2.2	Φ	2.2
Φ_0	2.2	Δ	2.2	Δ_0	2.2
α	2.2	$s_{\alpha,re}$	2.2	α_0	2.2
Π	2.2	S	2.2	s_α	2.2
ρ	2.2	$w\cdot x$	2.2	$\mathbb{E}(\alpha,re)$	2.2
$\mathbb{E}^>(\alpha,re)$	2.2	$\mathbb{E}^<(\alpha,re)$	2.2	$\overline{\mathbb{E}}^+_{\underline{h}}$	2.2
Alc	2.2	A_0	2.2	$\overline{\mathbb{E}}(\alpha)$	2.2
As	2.2	$\ell_\alpha(\lambda)$	2.2	$\mathsf{P}(k)$	2.3
P	2.3	$\mathrm{Path}(\lambda)$	2.3	$\mathrm{Path}_{\underline{h}}(\lambda)$	2.3
$\mathsf{P}^{-1}(r,\varepsilon_i)$	2.3	$\mathrm{res}(\mathsf{P})$	2.3	$\mathrm{res}_{\mathsf{P}}(i)$	2.3
$\mathsf{P}\sim\mathsf{Q}$	2.3	δ	2.4	δ_n	2.4
$\mathsf{P}\boxtimes\mathsf{Q}$	2.4	$\mathsf{P}\otimes_w\mathsf{Q}$	2.4	$\mathsf{P}\otimes\mathsf{Q}$	2.4
$\mathsf{P}\otimes_{\boldsymbol{\alpha}}\mathsf{Q}$	2.4	$b_{\boldsymbol{\alpha}}$	2.4	$b_\emptyset$	2.4
$\delta_{\boldsymbol{\alpha}}$	2.4	$b_{\boldsymbol{\alpha}\boldsymbol{\beta}}$	2.4	M_i	2.4
P_i	2.4	$\mathsf{P}_{\boldsymbol{\alpha}}$	2.4	$\mathsf{P}_\emptyset$	2.4
$\mathsf{P}_{\boldsymbol{\varnothing}}$	2.4	$\mathsf{M}_{i,j}$	2.4	$\mathsf{P}_{\underline{w}}$	2.4
$\mathsf{P}^\flat_{\boldsymbol{\alpha}}$	2.4	$\mathsf{breadth}_\sigma(\underline{w})$	2.4	$\Lambda(n,\sigma)$	2.4
$\Lambda^+(n,\sigma)$	2.4	$\mathscr{P}_{\underline{h}}(n,\sigma)$	2.4	$\mathrm{Std}_{n,\sigma}(\lambda)$	2.4
$\mathrm{Std}^+_{n,\sigma}(\lambda)$	2.4	$w^{\mathsf{S}}_{\mathsf{T}}$	2.4	$\boldsymbol{\alpha}$	3
$\boldsymbol{\beta}$	3	$\boldsymbol{\gamma}$	3	δ	3
T	5	B	5	$\mathsf{A}^{\boldsymbol{\varnothing}\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\boldsymbol{\varnothing}}(q)$	5.2
$\mathsf{S}_{q,\boldsymbol{\alpha}}$	5.3	$\mathsf{F}_{q,\boldsymbol{\varnothing}\boldsymbol{\alpha}}$	5.4	$\mathsf{F}_{q,\boldsymbol{\alpha}\boldsymbol{\varnothing}}$	5.4
$\boldsymbol{\varnothing}-\boldsymbol{\varnothing}$	5.5	$\mathsf{H}_{q,\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\alpha}}$	5.5	$\mathsf{H}_{q,\boldsymbol{\beta}\boldsymbol{\alpha}\boldsymbol{\beta}}$	5.5
$\mathsf{C}^{q,\boldsymbol{\gamma}\boldsymbol{\beta}}$	5.6	$\mathsf{C}_{q,\boldsymbol{\beta}\boldsymbol{\gamma}}$	5.6	$\mathsf{FS}_{q,\boldsymbol{\alpha}}$	6.3
$\mathsf{SC}_{q,\boldsymbol{\beta}\boldsymbol{\gamma}}$	6.4	$\mathsf{SH}_{q,\boldsymbol{\beta}\boldsymbol{\alpha}\boldsymbol{\beta}}$	6.5	$\mathsf{S}_{\boldsymbol{\alpha}}\mathsf{H}_{q,\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\alpha}}$	6.5
$\mathsf{S}_{\boldsymbol{\beta}}\mathsf{H}_{q,\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\alpha}}$	6.5	$\mathsf{eS}_{\boldsymbol{\beta}}\mathsf{H}_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\alpha}}$	6.5	$\mathsf{FH}_{q,\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\alpha}}$	6.6
$\mathsf{FH}_{q,\boldsymbol{\beta}\boldsymbol{\alpha}\boldsymbol{\beta}}$	6.6	P_λ	7.1	Λ	7.1
U_0	7.2	U_1	7.2	D_0	7.2
D_1	7.2	${}^p n_{v,w}(t)$	7.3		

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Table 3
Algebras and elements.

Symbol	§§	Symbol	§§	Symbol	§§
$\mathsf{1}_{\alpha}$	3.1	$\mathsf{1}_{\underline{w}}$	3.1	$\mathscr{S}(n,\sigma)$	3.1
$\mathscr{S}_{\underline{h}}(n,\sigma)$	3.1	$*$	3.1	$\mathsf{1}_{\alpha}$	3.2
$\mathsf{1}_{\emptyset}$	3.2	$\mathsf{1}_{\underline{w}}$	3.2	$\mathsf{1}_{\underline{w}'}^{\underline{w}}$	3.2
$\mathsf{1}_{\emptyset\alpha}^{\alpha\emptyset}$	3.2	$\mathsf{SPOT}_{\alpha}^{\varnothing}$	3.2	$\mathsf{FORK}_{\alpha\alpha}^{\varnothing\alpha}$	3.2
$\mathsf{HEX}_{\alpha\beta\alpha}^{\beta\alpha\beta}$	3.2	$\mathsf{COMM}_{\beta\gamma}^{\gamma\beta}$	3.2	$\mathscr{S}^{\mathrm{br}}(n,\sigma)$	3.2
$\mathscr{S}_{\underline{h}}^{\mathrm{br}}(n,\sigma)$	3.2	$\Lambda^{+}(\leqslant n,\sigma)$	3.2	$\mathsf{1}_{n,\sigma}^{+}$	3.2
$\mathcal{H}_n$	3.3	$e_{\underline{i}}$	3.3	y_i	3.3
ψ_i	3.3	$\circ$	3.3	$*$	3.3
$\mathcal{H}_n^{\sigma}$	3.3	ψ_q^p	3.3	$\psi_{\underline{w}}$	3.3
deg	3.3	e_{S}	3.3	$\psi_{\mathsf{T}}^{\mathsf{S}}$	3.3
$\mathsf{y}_{\underline{h}}$	3.3	$\mathsf{f}_{n,\sigma}^{+}$	3.3	$\mathsf{f}_{n,\sigma}$	3.3
$\psi_{\mathsf{Q}}^{\mathsf{P}} \boxtimes \psi_{\mathsf{Q}'}^{\mathsf{P}'}$	3.3	$\psi_{\mathsf{Q}}^{\mathsf{P}} \otimes \psi_{\mathsf{Q}'}^{\mathsf{P}'}$	3.3	$\mathrm{nib}(\psi_{\underline{w}} e_{\underline{i}})$	4.3
$\psi_{[b,q]}$	4.4	Ω_q	4.4	$\Upsilon_{\underline{w}}$	5
$\Upsilon_{\mathsf{Q}}^{\mathsf{P}}$	5	$\mathsf{adj}_{\emptyset\alpha}^{\alpha\emptyset}$	5.2	$\mathsf{adj}_{\alpha\varnothing}^{\varnothing\alpha}(q)$	5.2
$\mathsf{adj}_{\alpha\varnothing}^{\varnothing\alpha}$	5.2	$\mathsf{spot}_{\alpha}^{\varnothing}$	5.3	$\mathsf{spot}_{\alpha}^{\varnothing}(q)$	5.3
$\mathsf{fork}_{\alpha\alpha}^{\varnothing\alpha}$	5.4	$\mathsf{fork}_{\alpha\alpha}^{\alpha\varnothing}$	5.4	$\mathsf{fork}_{\alpha\alpha}^{\varnothing\alpha}(q)$	5.4
$\mathsf{fork}_{\alpha\alpha}^{\alpha\varnothing}(q)$	5.4	$\mathsf{hex}_{\beta\alpha\beta}^{\alpha\beta\alpha}$	5.5	$\mathsf{hex}^{\alpha\beta\alpha}(q)$	5.5
$\mathsf{hex}_{\beta\alpha\beta}(q)$	5.5	$\mathsf{hex}^{\alpha\beta\alpha}$	5.5	$\mathsf{hex}_{\beta\alpha\beta}$	5.5
$\mathsf{hex}_{\varnothing\beta\alpha\beta}^{\varnothing\alpha\beta\alpha}$	5.5	$\mathsf{hex}_{\underline{v}\alpha\beta\alpha\underline{w}\varnothing}^{\underline{v}\beta\alpha\beta\underline{w}\varnothing}$	5.5	$\mathsf{com}_{\beta\gamma}^{\gamma\beta}$	5.6
$\mathsf{com}^{q,\gamma\beta}$	5.6	$\mathsf{com}_{q,\beta\gamma}$	5.6	$\mathsf{com}^{\gamma\beta}$	5.6
$\mathsf{com}_{\beta\gamma}$	5.6	$\mathsf{com}_{\underline{v}\beta\gamma\underline{w}}^{\underline{v}\gamma\beta\underline{w}}$	5.6	e_{Γ}	7.1
E_r	7.1	$\mathsf{rex}_{\mathsf{P}_{\underline{w}}}^{\mathsf{P}_{\underline{v}}}$	7.2	$\mathsf{REX}_{\mathsf{P}_{\underline{w}}}^{\mathsf{P}_{\underline{v}}}$	7.2
$C_{\mathsf{Q}}^{\mathsf{P}}$	7.2	$c_{\mathsf{Q}}^{\mathsf{P}}$	7.2	$\mathscr{S}_{n,\sigma}^{\trianglerighteq \underline{v}}$	7.3
$\mathscr{S}_{n,\sigma}^{\triangleright \underline{w}}$	7.3	$\mathcal{H}_{+}^{\trianglerighteq \mu}$	7.3	$\mathcal{H}_{+}^{\triangleright \lambda}$	7.3
$\Delta_{\mathbb{Z}}(\underline{w})$	7.3	$\mathbf{S}_{\mathbb{Z}}(\lambda)$	7.3	$\mathsf{f}_{n,\sigma}\mathbf{S}_{\mathbb{Z}}(\lambda)$	7.3
$\Delta_{\Bbbk}(\underline{w})$	7.3	$\mathsf{f}_{n,\sigma}\mathbf{S}_{\Bbbk}(\lambda)$	7.3	$\langle\ ,\ \rangle_{\mathscr{S}}^{\lambda}$	7.3
$\langle\ ,\ \rangle_{\mathcal{H}}^{\lambda}$	7.3	$L_{\Bbbk}(\underline{w})$	7.3	$\mathsf{f}_{n,\sigma}\mathbf{D}_{\Bbbk}(\lambda)$	7.3
$P_{\Bbbk}(\underline{w})$	7.3	$\mathbf{P}_{\Bbbk}(\lambda)$	7.3		

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