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Robust Two-Layer Control of DC Microgrids with Fluctuating Constant Power Load Demand

Grigoris Michos, Pablo R. Baldovieso-Monasterios, George C. Konstantopoulos, Paul A. Trodden

Abstract—This paper proposes a cascaded control structure for the regulation of DC Microgrids (MG). It is well-known that the negative impedance characteristics of a constant power load (CPL) adversely affect the stability of the network, and can cause problems such as voltage collapse or damage the electronic components. To mitigate this, we propose a two-layer control structure, where at the inner layer, the proposed controller achieves fast tracking of the supplied reference points and ultimate boundedness of the trajectories in a desired set. The outer layer generates the inner layer reference points, accounts for system constraints, and introduces robustness of the voltage dynamics to unknown perturbations of the CPL demand. For the first time, an investigation of the nonlinear geometric behaviour of the CPL is carried out to derive necessary conditions that ensure boundedness of the network dynamics and feasible regulation to a desired equilibrium set. Finally, Control Lyapunov functions are formulated to prove the stability and estimate the region of attraction of the closed loop dynamics. A simulated scenario of a meshed MG network is presented to confirm the validity of the results.

I. Introduction

The concept of a MG brought a paradigm change to the architecture of conventional power networks [1]. The traditionally centralized structure has shifted to geographically decentralized clusters that are able to operate both in a grid-connected setting and isolated, known as islanded mode. The MG structure can be found in both AC and DC applications, however in many cases the use of a DC structure is often preferred because it provides higher efficiency and reliability, e.g. in High-Voltage-Direct-Current networks, aircrafts and transportation vehicles [2].

At the heart of the DC MG lies the bidirectional DC/DC converter, achieving integration of renewable energy sources in the network. These devices are utilized to achieve MG voltage/current regulation, power flow control and ensure that normal network operation is retained in the presence of external disturbances. One of the main challenges of DC/DC converter control is the instability caused by the nonlinearities of CPLs, [3], [4]. CPLs introduce negative impedance characteristics to the dynamics, which result in unstable equilibrium points. A plethora of studies have tried to address this problem,

with the majority adopting a linear control approach; see for example [5], [6] or [7]. The main drawback of linear approaches is the use of small-signal model analysis, which only guarantees local stability in an area around a desired operating point, where the linearization remains valid. In the presence of large load demand fluctuations, the nonlinearity of the CPL can drive the system outside the linearization region and destabilise the system. In light of this issue, more sophisticated control techniques have been developed that include the nonlinearity inside the control formulation. In [8], an adaptive backstepping technique was proposed for MGs feeding CPLs, where the estimation of the load demand is carried through a Kalman filter. A sliding-mode controller was presented in [9], where the sliding surface is designed as a linear combination of the voltage and current tracking errors. Stability of a boost converter feeding a CPL is demonstrated and the control scheme regulates the voltage to a desired reference point. Other methods using passivity theory [10], [11] or output regulation [12] have also been proposed. Enhancing the robustness of DC networks comprising of buck converters to unknown CPL demand is also a topic of rising interest. Passivity theory was used in [13] and [14] to establish robust stability of the network equilibria to perturbations of the load demand. This work was extended in [15] to achieve global robust stability in the case of constant-impedance-current loads. It has been shown that the adoption of a more advanced technique can enlarge the region of attraction and provide stronger stability guarantees with respect to changes in the load demand. However, the majority of the proposed techniques do not take system constraints into consideration, while the behaviour induced by the CPL is not extensively studied. A question that arises is whether the behaviour of the CPL can be included in the design of a unified constrained MG controller that guarantees stability and boundedness of the voltage and current states in the presence of unknown load demand perturbations.

The need to adopt a control scheme that is able to satisfy constraint sets arise naturally in MG control, most commonly in the form of actuation limits, current and voltage network capacities, or in order to prevent damage of electronic components. Therefore, the system is required to operate within a predefined operating range which can be translated to the form of constraints on both the current and the voltage of each DC/DC converter. One of the most popular and effective techniques for achieving control in the presence of constraints is Model

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Predictive Control (MPC). There is a rich literature behind robust MPC approaches, the majority of which follow a Tube MPC formulation. The term Tube MPC refers to a collection of control approaches that bound the trajectories of the uncertain system within a sequence of sets and regulate this sequence to desired terminal sets [16]. One of the most famous approaches dates back to the work of [17], which standardized the Tube MPC for linear systems. First, the uncertain system is decomposed into nominal and error dynamics, then an approximation of the minimum robust positive invariant (mRPI) set is calculated to bound the error, while the uncertain system is driven by regulating the nominal state trajectory in conjunction with feedback control on the error dynamics. Calculation of the mRPI set approximation requires an explicit form of the integral flow of the system, i.e. the solution of the ordinary differential equation describing the system dynamics, [18]. Calculating this would present a challenging task in the nonlinear setting, as an analytical solution of the dynamics may not even exist. A few approaches have focused on the nonlinear case, for example in [19] a linearization around each point in the horizon was proposed, while a feedback linearization was utilized in [20]. A tube nonlinear MPC (NMPC) was proposed in [21], however the calculation of the "restricted" nominal constraints is carried through simulations and lacks an analytic approach. The Contraction Theory is used to construct the tubes in [22] for a design of a distributed NMPC for dynamically decoupled subsystems. A method to optimize the tube size online was proposed in [23] by exploiting the structure of the adopted boundary layer sliding controller. An approach that constructs positive invariance sets for globally Lipschitz systems was proposed in [24], where the control action relies in computing a quadratic Lyapunov function for the system. However, imposing a globally Lipschitz condition restricts the scope of possible applications of this control method and may result in a conservative controller. It is evident from the literature that a robust NMPC scheme is a subject open for investigation, and is often reliant on the specific form of the dynamics. According to the authors' best knowledge, despite its distinct advantages, the extension of such a scheme to MGs is still on its early stages. In a previous work of the authors, a robust NMPC scheme for islanded DC MGs was developed in [25], where the inherent robustness properties of the nominal MPC are exploited to ensure recursive feasibility of the optimal control problem. It is shown that the system demonstrates a degree of robustness, by bounding the fluctuations of the power demand among the sampling intervals. However, only linear loads were considered in this study. This work was extended to include CPLs in the network in [26], where tools from economic MPC were used to guarantee the recursive feasibility of the proposed control scheme.

In this study we propose a unified approach for the MG regulation, i.e. we include both a current and a voltage controller. We extend our previous work, by relaxing the bound on the load demand fluctuations and investigate

its effect on the rest of the network. More specifically, we propose a robust NMPC scheme for regulating the voltage of the network in the presence of persisting, unknown, power demand fluctuations. We exploit the dynamic structure of the converter system, in order to parametrize a control law that bounds the system dynamics in a desired region. As it will be shown, this results in a Tube MPC performance but in a nonlinear setting, where the bound on the error is chosen based on the interplay of the control and drift vector fields. A preliminary version of this was recently proposed by the authors in [27], where local approximations techniques were used to show local input to state stability of the dynamics and establish a tube-like behaviour of the system trajectories. This paper significantly extends this work by employing candidate Lyapunov functions of the original nonlinear dynamics as opposed to using linearization techniques. In addition, we study the geometric effect of the CPL on the network dynamics, to characterize a positive invariant set under the solution of the voltage dynamics. Contrary to our previous work, this is done in a global sense, by studying the behaviour of the network equilibria. Thus, we avoid possible approximation errors due to linearization techniques and reduce the conservativeness of the controller. More specifically, the contributions of this work are:

- C1: We propose a modified version of the state-limiting PI, introduced in [28], in order to bound the converter current and introduce overcurrent converter protection during transient performance. Contrary to the literature, instead of a saturation unit, we employ a nonlinear parametrization of the input and show that by the proposed control law, the system flow is smooth and ultimately bounded in a predefined set, thus facilitating the stability analysis of the overall system.
- C2: For the first time, we investigate the geometric behaviour of the CPL and exploit the effect it has on the network to construct a robust positive invariant set of the system dynamics. Contrary to [27], this set is constructed using the displacement of the voltage equilibria due to the power demand instead of employing local linearizations. Then, we employ energy-like functions to establish ultimate boundedness and specify an explicit bound of the error trajectories, as opposed to relying on local approximations of the system's region of attraction. This way, we can characterize a positive invariant set in a global sense, provide stronger theoretical guarantees and reduce conservativeness by allowing larger unknown load demand perturbations.
- C3: We propose the adoption of an MPC control scheme to drive the nonlinear nominal dynamics to desired reference voltages. We expand the results from [27], by deducing a necessary condition of the augmented nominal control law that guarantees stability of the cascaded dynamics and recursive feasibility of the optimal control problem. Furthermore, we include

a result on the estimation of the region of attraction for the nominal dynamics.

- C4: We introduce a Control Lyapunov Function to prove the stability of the inner current dynamics and estimate the region of attraction of the respective equilibrium points. Then, we prove the stability of the cascaded dynamics and establish stronger stability guarantees compared to the linearization approaches, as these do not require explicit knowledge of the load demand and are not affected by linearization errors.

In what follows, in Section II we formulate the problem of controlling the DC MG network, in Section III we introduce the current controller and prove the ultimate boundedness of the current dynamics. In Section IV we formulate the voltage control that guarantees constraint satisfaction and robustness to constant power demand perturbations. Then, Section V provides the the stability results of the cascaded structure and in Section VI we demonstrate the application of the proposed control scheme with a simulation scenario.

A. Notation

A polytope $\mathcal{A} \subset \mathbb{R}^n$ with complexity n_c is characterized by $\mathcal{A} = \{x \in \mathbb{R}^n : Px \leq q\}$ with $P \in \mathbb{R}^{n_c \times n}$ and $q \in \mathbb{R}^{n_c}$. The notation $|\mathcal{A}|$ denotes the cardinality of \mathcal{A} . The closure, boundary and interior of a set \mathcal{A} are denoted as $\text{cl}(\mathcal{A})$, $\partial\mathcal{A}$ and $\text{int}(\mathcal{A})$ respectively. The Pontryagin Difference of polytopes \mathcal{A} and \mathcal{B} is defined as $\mathcal{A} \ominus \mathcal{B} = \{a \in \mathbb{R}^n : a + b \in \mathcal{A}, \forall b \in \mathcal{B}\}$. For a vector $x \in \mathbb{R}^n$ the notation $[x]$ denotes a diagonal matrix on $\mathbb{R}^{n \times n}$, where $x_i = [x]_{ii}$, for $i = 1, 2, \dots, n$. A sequence of length k is denoted as $\{x\}_k = \{x_1, x_2, \dots, x_k\}$. The Hausdorff distance between two sets $A, B \subset \mathbb{R}^n$ is $h(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} |a - b|^2, \sup_{b \in B} \inf_{a \in A} |a - b|^2\}$.

A topological space M is called a smooth manifold if for any $p \in M$ there exist an open set in a neighbourhood of p , $\mathcal{O} \subset M$ and diffeomorphism $\psi: \mathcal{O} \rightarrow \mathbb{R}^n$. The combination (ψ, \mathcal{O}) is called a coordinate chart of M . The tangent space on a point $p \in M$ is denoted as $T_p M$, and a vector field X is a map assigning a vector $X(p) \in T_p M$ at each point $p \in M$. A Riemannian metric G represents the inner product of the tangent space of M at x , where $G: T_x M \times T_x M \rightarrow \mathbb{R}$. The pair of a smooth manifold together with G is called a Riemannian manifold. Let $\gamma: [0, 1] \rightarrow M$ be a smooth function with $\gamma(0) = x_1$ and $\gamma(1) = x_2$ for $x_1, x_2 \in M$. The Riemannian distance between x_1 and x_2 is given by $d_R = \inf \ell_R(\gamma)$, where $\ell_R(\gamma)$ is the arc length of γ on M . The respective Riemannian norm is denoted as $\|\cdot\|_R$. The Lie derivative of a function $f \in C^\infty(M)$ along a vector field $X \in TM$ is given as $\mathcal{L}_X f = \sum_{i=1}^n X_i \frac{\partial f}{\partial x_i}$. The inner product between two vectors $a, b \in \mathbb{R}^n$ is denoted as $\langle a, b \rangle = \sum_{i=1}^n a_i b_i$.

A MG can be seen as an undirected and connected graph $\mathcal{G} = (\mathcal{M}, \mathcal{E})$ where the set of nodes \mathcal{M} represent a collection of power converters and local loads; the set of edges $\mathcal{E} \subseteq \mathcal{M} \times \mathcal{M}$ defining the MG topology is characterized by the node-edge matrix $\mathcal{B} \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{M}|}$ which

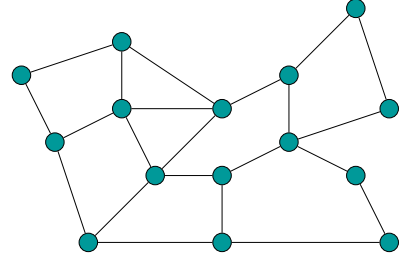


Fig. 1. Network topology of a meshed islanded DC MicroGrid.

for edge $\varepsilon = (i, j) \in \mathcal{E}$ involving nodes i and j can be defined as $[\mathcal{B}]_{ei} = 1$ if node i is the source of $e \in \mathcal{E}$, and $[\mathcal{B}]_{ej} = -1$ if node j is its sink, and zero otherwise. The weights of the edge ε , representing the admittances of the lines, are collected in the line admittances matrix R^{-1} .

We also use the following well-known result that we include here for reference.

Theorem 1 (Nagumo's Theorem). Let $\dot{x} = f(x)$, where the map f is at least once continuously differentiable and the solutions exists inside an open set $\mathcal{O} \subseteq \mathbb{R}^n$. Then, the closed subset $S \subset \mathcal{O}$ is positive invariant under the flow of the system if and only if

$$\liminf_{\lambda \rightarrow 0^+} \frac{d(S, x + \lambda f(x))}{\lambda} = 0$$

for all $x \in \partial S$, where $d(\cdot, \cdot)$ denotes the euclidean distance.

II. Problem Formulation

In this section we investigate the system of an islanded meshed DC Microgrid composed of n number of power converters, where each i^{th} converter is connected to local CPL, with $i \in \mathcal{M} = [1, 2, \dots, n]$. Our aim is to introduce robustness of the system operation to load demand fluctuations and ensure that the system dynamics are restricted in a predefined operational range. It is common to represent an islanded DC Microgrid as a meshed, connected, undirected graph, see Fig. 1. Each node models an interlinking DC/DC buck converter, integrating an energy source to the power network. The power consumption of the network is modelled by CPLs, connected to the output capacitor of each converter. This is represented in the converter circuit diagram as a controlled current source, see Fig. 2. In order to simplify the analysis, we only consider the pure resistive component of the lines. This is a common approach in DC MGs as it has been shown that the same stability results can be obtained by omitting the line inductance, see for example [29]. Therefore, we will consider the Kirchhoff model of the network dynamics given by

$$L \frac{di}{dt} = \nu E - ri - v, \quad (1a)$$

$$C \frac{dv}{dt} = i - i_o. \quad (1b)$$

The capacitance and inductance of each node are collected into diagonal matrices L and C respectively, r denotes a

diagonal matrix collecting the node parasitic resistances, and E denotes the input voltage of the converter. The states of the system are the capacitor voltage $v \in \mathbb{R}^n$ and the inductor current $i \in \mathbb{R}^n$, while the duty ratio of the switching mechanism $\nu \in [0, 1]$, is regarded as the input to the system.

The connection between the nodes of the graphs can be represented by the weighted adjacency matrix $A(R) \in \mathbb{R}^{n \times n}$, where $a_{ij} = R_{ij}^{-1}$, with R_{ij}^{-1} the admittance of the line between nodes i and j , and $a_{ij} = 0$ if the edge (i, j) is not incident. The full topology of the network is represented by the Laplacian matrix $\mathcal{L} = [A(R)\mathbf{1}_n] - A(R)$. Therefore, the output network current can be modelled as

$$i_o = \mathcal{L}v + [v]^{-1}P, \quad (2)$$

where $P \in \mathbb{R}^n$ is the power demand vector. To allow the development of a robust control strategy, we assume that the load demands are bounded in a set $\mathbb{P} \subset \mathbb{R}^n$. In addition, the system is subjected to operational constraints, i.e. desired regions of voltage and current operation, denoted as \mathbb{X} and \mathbb{I} respectively. In order to ensure a smooth region of operation we invoke the following assumption on the structure of the constraint sets.

Assumption 1. The constraint sets \mathbb{X} , \mathbb{I} and load demand \mathbb{P} are polytopic and compact sets, where the origin is within the interior of \mathbb{P} and \mathbb{I} .

It is common to require the inner current and output voltage of a converter to operate in a predefined range, usually as a bounded deviation from a rated value. The above assumption translates this requirement into a mathematical notion, as well as ensures that the load demand acquires a finite maximum value. This property is used to ensure convexity of the optimization problem that is introduced in Section IV. In addition, a consequence of this assumption is that the origin is not necessarily included in the voltage constraint set. Thus, the voltage dynamics are smooth over the constraint set which allows us to consider the respective state subspace as a Riemannian manifold M embedded in \mathbb{R}^n . Compactness of the constraint set also implies that any closed subset of \mathbb{X} is also compact. As a result, any subspace of the metric space (\mathbb{X}, d_R) has the Heine-Borel property, i.e. the metric space is compact and complete. This is a particularly useful property that will assist us in the following sections to establish convergence of the system flow.

Linearity of (2) with respect to the load allows us to separate the load current into a nominal current depending on an a-priori known nominal load \bar{P} and an uncertain one parametrized by a deviation $\delta P = P - \bar{P}$. Then, we define the set of deviations from the nominal load demand as a disturbance set $\mathbb{W} := \{\delta P \in \mathbb{R}^n : \bar{P} + \delta P \in \mathbb{P}\}$. Following Assumption 1, the disturbance set \mathbb{W} inherits the same properties of \mathbb{P} . The control objective is to introduce robustness to the system with respect to fluctuations of the load demand δP from the known, constant value \bar{P} . To this aim, in the following sections we first introduce boundedness of each converter inner current state i_i . Then,

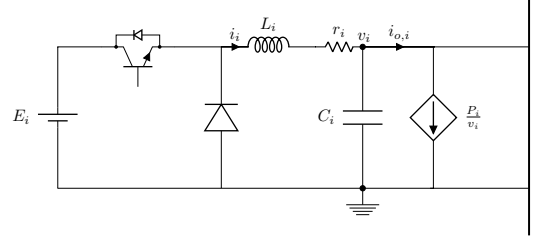


Fig. 2. Node circuit modelled as a DC/DC Buck converter connected in parallel to a local constant power load.

we decompose the true voltage dynamics into a nominal state, where the load demand is \bar{P} at all times, and an error between the true and the nominal voltage emanating from the fluctuations of the load demand δP from the nominal value \bar{P} . Finally, we introduce conditions on the choice of the tuning parameters such that the distance between the trajectory of the nominal and the true voltages is bounded, the network satisfies the desired operational constraints at all times and the stability of the overall system is guaranteed.

III. Inner Current Control

The current controller regulates the input current to a desired setpoint by adjusting the duty ratio ν . It is common in the literature to use some form of a saturated controller in order to achieve overcurrent protection of each power converter. To this aim, we employ a modified version of the state-limiting PI, firstly introduced in [28]. This ensures smoothness of the dynamics, thus avoiding performance degradation or instabilities caused by traditional saturated controllers, as highlighted in [30]. Moreover, it allows us to formulate an analytic procedure to choose the control parameters that achieve the desired transient performance and boundedness properties. The parametrization of the duty ratio ν is given by

$$\begin{aligned} \nu &= (v - k_p i + M\sigma)E^{-1} \\ M\dot{\sigma} &= k_I(\mathbf{1}_n - \sigma^2)(\hat{i} - i). \end{aligned} \quad (3)$$

resulting in closed loop current dynamics,

$$\begin{aligned} L\frac{di}{dt} &= -(k_p + r)i + M\sigma \\ M\dot{\sigma} &= k_I(\mathbf{1}_n - \sigma^2)(\hat{i} - i). \end{aligned} \quad (4)$$

where \hat{i} is the reference current, M , k_p , $k_I \in \mathbb{R}^{n \times n}$ are the tuning parameters and σ is the controller introduced integrator state. As shown in the following proposition, this control parametrization restricts the flow of the closed loop system in a control invariant subset of the constraint set $\mathbb{C} \subseteq \mathbb{I}$.

Proposition 1 (Current limitation). For each node $i \in \mathcal{M}$, the set $\mathbb{C} = [-I_{\max}, I_{\max}] \times [-1, 1]$ with $I_{\max, i} = \frac{M_i}{k_{p, i}}$ is control invariant for the current dynamics (4).

Proof. We begin by showing the boundedness of the integrator dynamics. For each individual component $\sigma_i(0) \in$

$[-1, 1]$, let there be time instants $\tau_1 > 0$ and $\tau_2 > \tau_1$ such that $|\sigma(\tau_1)| = 1$ and $|\sigma_i(\tau_2)| > 1$. By defining an energy function $E_{\sigma,i} = \frac{1}{2}\sigma_i^2$ with time derivative $\dot{E}_{\sigma,i} = \sigma_i \frac{k_{L,i}}{M_i}(1 - \sigma_i^2)(\dot{\hat{i}} - i)$, the derivative vanishes at point $|\sigma_i| = 1$, i.e. this point represents an equilibrium point of the integrator dynamics. Leveraging on the continuity property of (4) and the form of $E_{\sigma,i}$, then the trajectory cannot escape the ball defined by $|\sigma_i| = 1$, which leads into a contradiction on the existence of a point $\sigma_i(\tau_2)$ outside this ball.

Now we proceed with the boundedness of the current dynamics. We consider the \mathcal{C}^1 energy function of the inductor $E_{c,i} = \frac{1}{2}Li_i^2$, with time derivative

$$\dot{E}_{c,i} = -(k_{p,i} + r_i)i_i^2 + i_i M_i \sigma_i < -k_{p,i}|i_i|^2 + |i_i|M_i.$$

Following [31, Theorem 4.18], the flow of the system $\phi_c(i_i^0, t)$ for $i_i^0 \leq \frac{M_i}{k_{p,i}}$ is ultimately bounded with bound $I_{\max,i} = \frac{M_i}{k_{p,i}}$. \square

Therefore, the desired behaviour of the inner current loop can be achieved by adjusting the tuning parameters according to the desired bound $I_{\max,i} = \frac{M_i}{k_{p,i}}$. The stability of the system will be investigated in Section V, where we will discuss the stability properties of the overall cascaded dynamics. In addition, we note that the choice of the inner current bound needs to be made according to the required voltage constraint set \mathbb{X} , in order to satisfy the requirement $v_i \in (0, 1)$ for all $(i_i, \sigma_i) \in \mathbb{C}_i$ and $v_i \in \mathbb{X}_i$.

IV. Voltage Control

In this section, we formulate the robust control scheme for the voltage node dynamics. To this aim, we separate the dynamics into a nominal and an uncertain part. In the following, we will show that the uncertain part of the dynamics can be bounded within a positive invariant set, leveraging on the geometric properties of the closed loop vectorfield. Then, the nominal dynamics are shown to act as a driving subsystem, regulating the true system to some desired reference behaviour. We will use the notion of control sets to guarantee constraint satisfaction of the nominal subsystem, which, by an appropriate parametrization of the constraint sets, implies constraint satisfaction of $\mathbb{Y} = \mathbb{X} \times \mathbb{C}$ by the uncertain dynamics. In this study, we assume that the current operates in a faster time-scale than the voltage. This is a common assumption in the literature, see for example [32] or [33], as it allows to study each of the dynamic components separately and thus significantly simplifies the analysis. This leads to a cascaded structure where the input current in (1b) represents a piecewise constant reference and is considered as the input to the voltage dynamics. Following [34, Theorem 4.4] and [35, Appendix A] the reachability, and consequently convergence, properties are preserved under the replacement of integrable smooth controls with piecewise constant controls. Having established boundedness of the solution of (4), we formalize the time-scale separation by invoking the following assumption.

Assumption 2 (Time-scale separation). The network parameters satisfy

$$\max_{i \in \mathcal{M}} \left\{ \frac{L_i}{k_{p,i} + r_i}, \frac{k_{p,i}}{k_{L,i}} \right\} \ll \min_{i \in \mathcal{M}} \left\{ \frac{C_i P_i}{(I_{\max,i})^2} \right\}$$

Remark 1. As a consequence of Assumption 1, the load demand P_i is bounded and thus one can enforce the time-scale separation of the node dynamics by an appropriate selection of the tuning parameters, as shown in Assumption 2 where we provide explicit time constants derived using singular perturbation analysis. We refer the interested reader to [31, Chapter 11] for a detailed analysis.

A. Formulation and boundedness of the error dynamics

Following Assumption 2, we begin by writing the node voltage uncertain system

$$C\dot{v} = -\mathcal{L}v - [v]^{-1}\bar{P} - [v]^{-1}\delta P + \hat{i} \quad (5)$$

and define the error term $e = v - z$, that separates the nominal part of the dynamics from the uncertain. Our aim is to use the nominal state z as a driving state of the system's flow $\phi_v(\cdot)$, while restricting the effect of the disturbance such that the distance between the flow of the uncertain system and the nominal satisfy $d_R(\phi_v, \phi_z) \leq \varepsilon$ for some $\varepsilon > 0$. We define the control policy

$$\hat{i}(e) = -Ke + u \quad (6)$$

where u is the nominal control policy that will be defined later. Then, isolating the nominal part of (5), results in

$$C\dot{z} = -\mathcal{L}z - [z]^{-1}\bar{P} + u. \quad (7)$$

Therefore, the evolution of the error dynamics can be described by the ordinary differential equation

$$C\dot{e} = -\mathcal{L}e - Ke - [e + z]^{-1}\bar{P} + [z]^{-1}\bar{P} - [e + z]^{-1}\delta P. \quad (8)$$

We define the vector field map $\gamma: M \rightarrow TM$,

$$C\dot{e} = \gamma(e) \quad (9)$$

where for a constant power demand deviation δP_s and converged nominal dynamics to a point \hat{z} , we have

$$\gamma(\hat{e}) = 0. \quad (10)$$

Many studies have been focused on the existence of real solutions for the voltage equilibrium map usually deriving a necessary condition to be satisfied by the system parameters, see for example [36]. It is shown that, if the necessary conditions are met, the equilibrium map is a diffeomorphism where a high and a low voltage solution exist, see [37] or [38]. Note that $\gamma(\hat{e}) = 0$ is the result of the state transformation $v = e + z$, similar results with the voltage v equilibrium map can be obtained for $\hat{e} = \gamma^{-1}(0)$. Therefore, the discontinuity of the true voltage dynamics is translated to the critical point $e = -z$ of the error state space. The second solution of $\gamma(\hat{e}) = 0$ is then the steady state displacement caused by the load demand perturbations δP between the true voltage trajectory and

the nominal. In the following, we intend to use the imposed properties of the vector field $\gamma(\cdot)$ in an area around the origin to restrict the error flow ϕ_e in a positive invariant set. Considering the solution of $\hat{e} = \gamma^{-1}(0)$ closest to the origin, the maximum displacement due to the load demand can be found by

$$e^m = \max_{\delta P \in \mathbb{W}} \left(\left\| \gamma^{-1}(0) \right\| \right). \quad (11)$$

Next we define the polytope,

$$N := \{e \in M : P_e \leq \xi(e^m)\} \quad (12)$$

where $P = [I_n \ -I_n]^\top$ and $\xi(e^m) = [\mathbf{1}_n e^m \ \mathbf{1}_n e^m]^\top$, that is the set N is a n -dimensional ‘‘box’’ around the origin. The question that arises is under which conditions does the system guarantee the desired theoretic properties, i.e. existence of a unique solution of the error dynamics in N , as well as positive invariance of N . The former is addressed in the following result.

Lemma 1 (Lipschitz continuity). Consider the error dynamics given in (8) and bounded positive nominal voltage $\underline{z} \leq z \leq \bar{z}$. Given a constant power load demand $P_s \in \{\bar{P}\} \oplus \mathbb{W}$ and a scalar positive definite matrix K , the map $\gamma(\cdot)$ is Lipschitz continuous on N , with Lipschitz constant

$$\mathcal{K} = \left| \lambda_{\max} \left(-K + [-\mathbf{1}_n e^m + \underline{z}]^{-2} [P_s] \right) \right|.$$

Proof. We begin by finding the Lie derivative for a constant power demand $P_s \in \{\bar{P}\} \oplus \mathbb{W}$

$$\mathcal{L}ie(\gamma) = -(\mathcal{L} + K) + [e + \hat{z}]^{-2} [P_s].$$

Due to the fact that \underline{z} and e^m are positive values, the matrix ℓ^2 -norm is given by

$$\begin{aligned} \left\| \mathcal{L}ie(\gamma) \right\|_{\ell^2} &= \sup_{e \in N} \left\| \mathcal{L}ie(\gamma) \right\|_{\ell^2} \\ &= \left\| -(\mathcal{L} + K) + [-\mathbf{1}_n e^m + \underline{z}]^{-2} [P_s] \right\|_{\ell^2} \\ &\leq \left\| -(\mathcal{L} + K) + [-\mathbf{1}_n e^m + \underline{z}]^{-2} [P_s] \right\|_{\ell^2}. \end{aligned}$$

We define $\mathcal{K} = \left\| -(\mathcal{L} + K) + [-\mathbf{1}_n e^m + \underline{z}]^{-2} [P_s] \right\|_{\ell^2}$ and note that the resulting matrix within the norm is symmetric. Then, for a symmetric matrix the largest singular value is given by its spectral radius, i.e.

$$\mathcal{K} = \left| \lambda_{\max} \left(-(\mathcal{L} + K) + [-\mathbf{1}_n e^m + \underline{z}]^{-2} [P_s] \right) \right|.$$

By the properties of the Laplacian matrix we have $\lambda_n \geq \dots \geq \lambda_2 \geq \lambda_1 = 0$ and thus $\lambda_{\max}(-\mathcal{L}) = 0$. The above takes the form,

$$\mathcal{K} = \left| \lambda_{\max} \left(-K + [-\mathbf{1}_n e^m + \underline{z}]^{-2} [P_s] \right) \right|.$$

This establishes the desired upper bound on the norm of the Lie derivative, i.e. $\left\| \mathcal{L}ie(\gamma) \right\|_{\ell^2} \leq L$. Therefore, for two

infinitesimally different $e_1, e_2 \in N$ and a $\xi \in (e_1, e_2)$, an application of the mean value theorem states

$$\begin{aligned} \left\| \gamma(e_1) - \gamma(e_2) \right\|_{\ell^2} &= \left\| \mathcal{L}ie|_{\xi}(\gamma) \right\|_{\ell^2} \|e_1 - e_2\|_{\ell^2} \\ &\leq \mathcal{K} \|e_1 - e_2\|_{\ell^2}. \end{aligned}$$

Therefore, the vector field is Lipschitz continuous on N with Lipschitz constant \mathcal{K} . \square

Having established the Lipschitz continuity of the system flow inside N , we now are able to investigate the stability properties of the equilibrium $\hat{e} \in N$. This is demonstrated in the following result.

Proposition 2 (Stability and Positive Invariance of the error dynamics). For bounded local nominal dynamics $\underline{z}_i \leq z_i \leq \bar{z}_i$, the error dynamics of every node $i \in \mathcal{N}$, admit an asymptotically stable equilibrium point in N , if the local feedback gain satisfies

$$K_{ii} > \max \left\{ \frac{\bar{P}_i}{z_i e_i^m}, \frac{P_i^m}{(z_i - e_i^m)^2} \right\}$$

with $P_i^m = \max(|\bar{P}_i + \delta P_i|)$ and $\delta P_i \in \mathbb{W}_i$. In addition, the set N is positive invariant under the flow of the system $\phi_e(e^0, t)$, for all $t > 0$ and $e^0 \in N$.

Proof. For any equilibrium point $\hat{e} \in N$ and respective constant load demand \hat{P} , the resulting Jacobian matrix takes the form

$$J = C^{-1}(-(\mathcal{L} + K) + [(z + \hat{e})^2]^{-1} \hat{P}).$$

In order for the equilibrium point \hat{e} to be stable, the respective Jacobian matrix J needs to be Hurwitz, i.e. to have negative eigenvalues. Using Lyapunov theory, it suffices to investigate the largest eigenvalue of the matrix

$$-(\mathcal{L} + K) + [(z + \hat{e})^2]^{-1} \hat{P}.$$

Hence, it is sufficient to show that $\lambda_{\max}(-K + [(z + \hat{e})^2]^{-1} \hat{P}) < 0$, i.e.

$$-K_{ii} + \frac{\hat{P}_i}{(z_i + \hat{e}_i)^2} < 0$$

where we have used the fact that $\lambda_{\max}(-\mathcal{L}) = 0$. In order to ensure that a similar condition holds for any $\hat{e} \in N$ and any $\hat{P} \in \mathbb{P}$, we note that

$$\frac{P_i^m}{(z_i - e_i^m)^2} \geq \frac{\hat{P}_i}{(z_i + \hat{e}_i)^2},$$

holds for all $\hat{e} \in N$. Therefore, we require

$$K_{ii} > \frac{P_i^m}{(z_i - e_i^m)^2}$$

where we have used the fact that $\underline{z}_i \leq z_i \leq \bar{z}_i$, where the lower bound is positive. Following the definition of N , for some point at ∂N we have that $e_i = e^m$ for $i \in [1, \dots, n]$ and $e_j \leq e^m$ for $j \neq i$. We consider the quadratic function $V = \frac{1}{2} e^\top C e$ and our aim is to show that $\dot{V}|_{e \in \partial N}$ is negative outside the set N . Since $e_i \leq e^m$ for all $i \in [1, \dots, n]$, it

suffices to show $\dot{V}_i|_{e_i \in \partial N_i}$ is non-positive in both cases of $e_i = e_i^m$ and $e_i = -e_i^m$, i.e. we require

$$\dot{V}_i|_{e_i \in \partial N} = e_i \gamma_i(e_i)$$

or

$$e_i \left(-(\mathcal{L}_{ii} + K_{ii})e_i - \sum_{j \in \mathcal{N}_i} \mathcal{L}_{ij}e_j + \frac{\bar{P}_i}{z_i} - \frac{P_i}{e_i + z_i} \right)$$

to be non-positive, where $\mathcal{N}_i = \{j \in \mathcal{M} : \mathcal{L}_{ij} \neq 0, i \neq j\}$ denotes the set of neighbours of the i^{th} node, i.e. the nodes of the network where there exists a direct line connecting the two nodes. We bifurcate the analysis in two cases: (a) $e_i = e^m$ and (b) $e_i = -e^m$. In case of (a) we need the term inside the parenthesis to be negative. Since $e^m \geq e_j$, the diagonal Laplacian term dominates the off-diagonal elements. Comparing the lower bound of the feedback K_{ii} , it can be seen that when the gain obtains its lower value, it is not necessarily true that $-K_{ii}e_i + \frac{\bar{P}_i}{z_i} \leq 0$, unless the deviation between the maximum load demand and the nominal is ‘‘large enough’’. In order to ensure that the desired result holds in every case, we need to impose an additional condition on the feedback,

$$K_{ii} > \max \left\{ \frac{\bar{P}_i}{z_i e_i^m}, \frac{P_i^m}{(z_i - e_i^m)^2} \right\}.$$

Then, the desired inequality $\dot{V}_i|_{e_i = e_i^m} \leq 0$ holds as the expression inside the parenthesis is a summation of strictly negative elements. In case of (b), we note that, by construction, at the point $e = -\mathbf{1}_n e^m$ the velocity is given by

$$\underbrace{\mathcal{L}_{ii}e^m + \sum_{j \in \mathcal{N}_i} \mathcal{L}_{ij}e^m}_{=0} + K_{ii}e^m + \frac{\bar{P}_i}{z_i} - \frac{P_i^m}{-e^m + z_i} = 0,$$

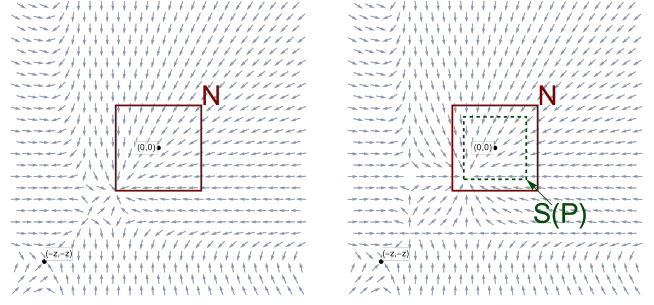
when K_{ii} , z_i obtain their respective lower bound and the load demand P_i is maximized. Then, for any $e_j \geq -e^m$, $P_i \leq P_i^m$ and $z_i \geq \underline{z}_i$ we have that

$$\begin{aligned} & \overbrace{\mathcal{L}_{ii}e^m - \sum_{j \in \mathcal{N}_i} \mathcal{L}_{ij}e_j}^{\geq 0} + K_{ii}e^m + \frac{\bar{P}_i}{z_i} - \frac{P_i}{-e^m + z_i} \\ & \geq K_{ii}e^m + \frac{\bar{P}_i}{\underline{z}_i} - \frac{P_i^m}{-e^m + \underline{z}_i} \end{aligned}$$

and therefore we obtain the desired inequality

$$\mathcal{L}_{ii}e^m - \sum_{j \in \mathcal{N}_i} \mathcal{L}_{ij}e_j + K_{ii}e^m + \frac{\bar{P}_i}{z_i} - \frac{P_i}{-e^m + z_i} \geq 0,$$

which ensures $\dot{V} \leq 0$. Therefore, the convex nature of set N allows us to conclude that the velocity vector assigned to every point $e \in \partial N$ by the map $\gamma: N \rightarrow TN$ is sub-tangential to the set N and by direct application of the Nagumo’s theorem, the set N is positive invariant under the flow of the error dynamics with $\phi_e: N \rightarrow N$. \square



(a) Given a constant power demand P^m , any solution starting in N will remain in N and converge to ∂N . (b) For a power demand $P \in \mathbb{P}$ the region of the dynamic map shrinks within the set $S(P) \subset N$.

Fig. 3. Vector field and positive invariant set N for a two node system.

The invariance of set N combined with the asymptotic stability of every equilibrium point in N implies the following result.

Corollary 1. The set N is within the region of attraction of system (8).

Remark 2. The above results reveal the immediate effect of the feedback gain K on the degree of conservativeness of N . Higher values of K result in higher magnitudes of the velocity introduced by the linear part and therefore a decrease in the maximum displacement of the equilibrium point from the origin. Therefore, there exists a subset of N parametrized by K , $\tilde{N}(K) \subset N$, such that \tilde{N} is positive invariant and contains the limit of the flow for any $e_0 \in N$, i.e. $\lim_{t \rightarrow \infty} \phi_e(e_0, t) \in \tilde{N}(K)$. However, while a high feedback gain would diminish the effect of the load demand on the voltage dynamics, this has a negative effect on the converter current steady state, as can be seen by the reference current parametrization $\hat{i} = -Ke + u$. Higher values of the feedback introduce larger deviations of the reference current \hat{i} from the desired reference u , revealing an inherent trade-off on the choice of the system parameters.

The results are illustrated in Fig. 3, where the set $S(P)$ is a polytope similar to N but parametrized by an equilibrium point corresponding to some load demand with $P_i < P_i^m$. It is seen that the vector field drives the flow on the boundary of $S(P)$, while N is positive invariant. The above results also suggests that the set $\Omega = N \oplus \hat{z}$ is positive invariant for the uncertain system. A question that arises is that under which conditions, the uncertain state flow ϕ_v , parametrized by the control input \hat{i} , is driven in Ω , while $v \in \mathbb{X}$ is satisfied at all times. One way to guarantee this is to prove the existence of a robust control invariant set (RCI) for the uncertain voltage dynamics within the constraint set. To this end, we recall the following definition from [39], adapted to our setting.

Definition 1 (Robust Control Invariant Set). The set \mathcal{R} is said to be robust control invariant for the system (5), if

there exists a control $\hat{i} \in \mathbb{C}$ such that for all initial states $v^0 \in \mathcal{R}$ and any $w \in \mathbb{W}$ the flow of the system satisfies $v = \phi(v^0, \hat{i}, w, T) \in \mathcal{R}$, for all $T > 0$.

Using this notion of RCI sets, we require

$$\Omega \subseteq \mathcal{R} \subseteq \mathbb{X}.$$

Following the previous analysis, we recall that $\Omega = N \oplus \hat{z}$ with N being a positive invariant set, therefore, we can formulate the above question in the nominal setting, by defining the nominal constraint sets as

$$\mathbb{Z} := \mathbb{X} \ominus N, \quad \mathbb{U} := \mathbb{C} \ominus (-K)N.$$

The above indicates that the input constraint set is “shrinking” with larger choices of the control gain K , while on the other hand the nominal state constraint set \mathbb{Z} approaches the size of the original constraint set \mathbb{X} . As it was also discussed in Remark 2, this is the expected trade-off that needs to be considered when choosing the tuning parameter K . The aim of this interplay is to proportionally distribute the high-transients effect of the CPL between the current and the voltage trajectories, in order to satisfy the operational constraints.

Furthermore, we note that the computation of the positive invariant set can be done offline and only utilizes the information regarding the bound on the load demand; this significantly simplifies the development of a control scheme for the nominal voltage dynamics and substantially reduces conservativeness stemming from the need of instantaneous load measurements during operation.

B. Formulation of the nominal voltage controller

The problem now becomes of choosing references within the nominal constraint set, i.e. $\hat{z} \in \mathbb{Z}$, and proving the existence of a control invariant set for the nominal dynamics. While the former can be trivially satisfied, the latter requires the association of a control policy with the candidate control invariant set. A common way to resolve this, is by employing an MPC control scheme. In MPC, the control policy is generated by solving a finite horizon optimal control problem subject to the system constraints and dynamics. The cost function is often adopted as a quadratic function penalizing the deviation of the system current state from a reference point, thus achieving regulation of the flow to a desired equilibrium. At each time instant, a sequence of control actions is generated and the first element is used as an input to the system, while the rest of the sequence is discarded. This process is repeated in the next time instant, thus achieving the receding horizon implementation. Efficient techniques have been developed to solve the continuous-time counterpart of the optimal control problem that usually involve an approximation of the solutions using a numerical solver. Some of the most commonly adopted techniques are the interior point method (IPOPT) or sequential quadratic programming (SQP), see [40, Chapter 10] for a detailed analysis.

One way to achieve a stabilizing and recursively feasible controller is by adopting a positive definite cost, compact constraint sets and invariant terminal ingredients [17], [41]. Noting Assumption 1, it is straightforward to satisfy the first two requirements. However, the CPL destabilizes every fixed point $\hat{z} \in \mathbb{Z}$. To overcome this, we define

$$u = -K_z z + \eta^o,$$

where η^o is the optimal control policy generated by solving the MPC problem. Thus, the nominal dynamics take the form

$$C\dot{z} = -(\mathcal{L} + K_z)z - \frac{\bar{P}}{z} + \eta^o. \quad (13)$$

We provide the following result to establish asymptotic stability of the terminal dynamics.

Proposition 3 (Stability of the terminal dynamics). Considering the nominal system (13), there exists a $\delta > 0$ such that the fixed point \hat{z} is an asymptotically stable equilibrium with region of attraction $\mathcal{A} = \{z \in M : \|z - \hat{z}\| \leq \delta\}$, if and only if,

$$K_{z,i} > \frac{\bar{P}_i}{\hat{z}_i^2},$$

for all $\hat{z}_i \in \mathbb{Z}_i$.

Proof. To investigate the stability of the fixed point \hat{z} we linearize (13) and find the Jacobian matrix

$$J = \nabla_{z=\hat{z}} \dot{z} = C^{-1}(-\mathcal{L} + [\hat{z}^2]^{-1} \bar{P} - K_z).$$

The point \hat{z} is asymptotically stable if and only if the Jacobian is a Hurwitz matrix. Exploiting the properties of the Laplacian matrix we can define a worst case scenario at $\lambda_{\max}(-\mathcal{L}) = 0$. Then, in combination with the fact that the rest of the terms are symmetric matrices, we can deduce the scalar condition

$$-K_{z,i} + \frac{\bar{P}_i}{\hat{z}_i^2} < 0$$

Which results in the required inequality

$$K_{z,i} > \frac{\bar{P}_i}{\hat{z}_i^2},$$

for all $\hat{z}_i \in \mathbb{Z}_i$. Shifting the axes to the desired point \hat{z} the new dynamics can be described by

$$\dot{\tilde{z}} = J\tilde{z} + g(\tilde{z})$$

where $\tilde{z} = z - \hat{z}$ and for some $\varepsilon > 0$ and $\delta > 0$, $g(\tilde{z})$ satisfies $\|g(\tilde{z})\| \leq \varepsilon \|\tilde{z}\|$ in some neighbourhood of the origin, i.e. $\|\tilde{z}\| \leq \delta$. Then, let $V(\tilde{z}) = \tilde{z}^\top \mathcal{P} \tilde{z}$ be a Lyapunov candidate function, where \mathcal{P} is a solution of $\mathcal{P}J + J^\top \mathcal{P} = -Q$ with $Q \succ 0$. Note that existence and uniqueness of \mathcal{P} is guaranteed due to J being a Hurwitz matrix [31, Theorem 4.6]. The time derivative of $V(\cdot)$ results in,

$$\dot{V} = \tilde{z}^\top \mathcal{P} (J\tilde{z} + g(\tilde{z})) + (\tilde{z}^\top J^\top + g^\top(\tilde{z})) \mathcal{P} \tilde{z} \quad (14)$$

$$= -\tilde{z}^\top Q \tilde{z} + 2\tilde{z}^\top \mathcal{P} g(\tilde{z}) \quad (15)$$

$$\leq (\lambda_{\min}(Q) + 2\|\mathcal{P}\|\varepsilon) \|\tilde{z}\|^2 \quad (16)$$

where we have used the fact that $\|g(\tilde{z})\| \leq \varepsilon\|\tilde{z}\|$ and $\tilde{z}^\top Q \tilde{z} > \lambda_{\min}(Q)\|\tilde{z}\|^2$. Now, for $\varepsilon < \frac{\lambda_{\min}(Q)}{2\|\mathcal{P}\|}$, the function $V(\tilde{z}) = \tilde{z}^\top \mathcal{P} \tilde{z}$ is a Lyapunov function for the shifted system as $\dot{V} \leq 0$ and thus there exists a $\delta > 0$ such that the region of attraction of \hat{z} is $\mathcal{A} = \{z \in M: \|z - \hat{z}\| \leq \delta\}$. \square

Corollary 2. Any level set \mathbb{Z}_0 satisfying $\mathbb{Z}_0 = \beta\mathcal{A}$ for $0 \leq \beta \leq 1$ is positive invariant for the dynamics (13).

Following the above result above, we formalize the definition of the optimization problem as

$$J^o(z(0), \hat{z}, \hat{\eta}) = \min_{\eta} \int_{t_0}^{t_f} \ell(z - \hat{z}, \eta - \hat{\eta}) dt + J_f(z_f - \hat{z}) \quad (17)$$

subject to the constraints,

$$\begin{aligned} z(0) - z &\in N, \\ C\dot{z} &= -(\mathcal{L} + K_z)z - [z]^{-1}\bar{P} + \eta, \\ (z, \eta) &\in \mathbb{Z} \times \mathbb{U} \oplus K_z\mathbb{Z}, \quad z_f \in \mathbb{Z}_f, \end{aligned} \quad (18)$$

where $\ell(\cdot)$ is a positive definite function. The resulting control is applied in a receding horizon fashion, where at some time t_1 the first element of the resulting optimal control sequence is used as input to the system and the process is repeated at the next sampling instant $t_2 = t_1 + T$. Note that a direct consequence of Proposition 3 and Corollary 2, is that choosing $\mathbb{Z}_f = \mathbb{Z}_0$ and $J_f = \frac{1}{2}(z_f - \hat{z})^\top \mathcal{P}(z_f - \hat{z})$ leads to recursive feasibility of (17) provided that the problem is feasible at some initial time t_0 . In addition, any sublevel set of the stabilizing terminal cost function is control invariant for the nominal dynamics. For completeness purposes, we will provide here a sketch of the recursive feasibility result and we refer the interested reader to [40, Chapter 5] for a detailed analysis.

Proposition 4 (Recursive Feasibility of OCP). Let the problem (17) be feasible at some initial state z_0 and time $t_0 > 0$. Then, the problem remains feasible for all $t > t_0$.

Proof. Let $\mathbf{z}(z_0, t_0)$ and $\boldsymbol{\eta}(z_0, t_0)$ denote the feasible solutions of the optimal control problem respectively, with initial state z_0 and time t_0 . In addition, let $\mathbf{z}(\phi_z(z_0, t_0 + T), t_0 + T)$ and $\boldsymbol{\eta}(\phi_z(z_0, t_0 + T), t_0 + T)$ be the candidate state and control sequences at the next immediate sampling instant, i.e. at time $t_0 + T$. Assuming no additive uncertainties to the system, following Proposition 3 and Corollary 2, the candidate solutions at time $t_0 + T$ are formulated as the tails of the ones at time t_0 where the predicted state at the end of the horizon satisfies $z_f(t_0 + T + t_f) \in \mathbb{Z}_f$. Positive invariance of \mathbb{Z}_f implies that $\mathbf{z}(\phi_z(z_0, t_0 + T), t_0 + T)$ and $\boldsymbol{\eta}(\phi_z(z_0, t_0 + T), t_0 + T)$ are feasible and the problem is recursively feasible for all $t > t_0$. \square

V. Stability analysis of the cascaded dynamics

Following the analysis of the previous sections, the problem has been shifted to regulating the following

cascaded dynamics,

$$L \frac{di}{dt} = -(k_p + r)i + M\sigma \quad (19a)$$

$$M \frac{d\sigma}{dt} = k_I(\mathbf{1}_n - \sigma^2)(u - Ke - i) \quad (19b)$$

$$C \frac{dz}{dt} = -\mathcal{L}z - [z]^{-1}\bar{P} - u \quad (19c)$$

$$C \frac{de}{dt} = -\mathcal{L}e - Ke - [e + z]^{-1}\bar{P} + [z]^{-1}\bar{P} - [e + z]^{-1}\delta P. \quad (19d)$$

The stability of a cascaded structure has been thoroughly investigated in the literature, see [42]. The conventional procedure is to separate the dynamics into a driving subsystem and a driven one, where the state of the former is considered an input to the latter. Then, asymptotic stability of the overall dynamics follows from asymptotic stability of the driving subsystem and asymptotic stability or some form of boundedness of the driven dynamics. Given a desired reference nominal voltage z^{ss} , we define the equilibrium set for (19) with state vector $x = (i, \sigma, z, e)$ as

$$\mathcal{E} := \{\hat{i}, \hat{\sigma}, \hat{z}, \hat{e} \in \mathbb{R}^n: \hat{i} = u - K\hat{e}, \hat{\sigma} = M^{-1}(k_p + r)\hat{i}, \hat{z} = z^{ss}, \hat{e} \in N\}. \quad (20)$$

We begin the stability analysis by showing asymptotic stability of the node current.

Theorem 2 (Lyapunov stability of the driving dynamics). For every node $i \in \mathcal{M}$, the \mathcal{C}^1 function $\mathcal{W}_i: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$,

$$\begin{aligned} \mathcal{W}(i_i, \sigma_i) &= \frac{1}{2}L_i(i_i - \hat{i}_i)^2 + \\ &\quad \frac{M_i^2 - M_i(k_{p,i} + r_i)\hat{i}_i}{k_{I,i}} \ln \frac{1}{1 - \sigma_i^2} + \\ &\quad \frac{M_i(k_{p,i} + r_i)\hat{i}_i}{k_{I,i}} \ln \frac{1}{1 + \sigma_i}, \end{aligned} \quad (21)$$

is a Control Lyapunov Function for the subsystem (19a), (19b), and the subsystem is asymptotically stable with equilibrium point described by (20) and region of attraction $\mathbb{C} = [-I_{\max}, I_{\max}] \times [-1, 1]$.

Proof. Following the definition of \mathcal{W} , it can be seen that it is positive definite. Therefore, all we need to show is negative definiteness of the first derivative. Following the assumption on the time scale separation of the dynamics, the input current is constant, i.e. $\hat{i}_i = \bar{i}$. Then, the time derivative of \mathcal{W}_i results in

$$\begin{aligned} \dot{\mathcal{W}}_i &= L_i \frac{di_i}{dt} (i_i - \bar{i}_i) + \frac{M_i^2 - M_i(k_{p,i} + r_i)\bar{i}_i}{2k_{I,i}} \frac{2\sigma_i}{1 - \sigma_i^2} \frac{d\sigma_i}{dt} \\ &\quad - \frac{M_i(k_{p,i} + r_i)\bar{i}_i}{k_{I,i}(1 + \sigma_i)} \frac{d\sigma_i}{dt} \\ &= -(k_{p,i} + r_i)(i_i - \bar{i}_i)^2, \end{aligned}$$

where

$$\dot{\mathcal{W}}_i = -(k_{p,i} + r_i)(i_i - \bar{i}_i)^2 \leq 0$$

for $k_{p,i}, r_i > 0$. Following La Salle's invariance theorem and noting that the derivative vanishes only at $i \in \mathcal{E}$, the

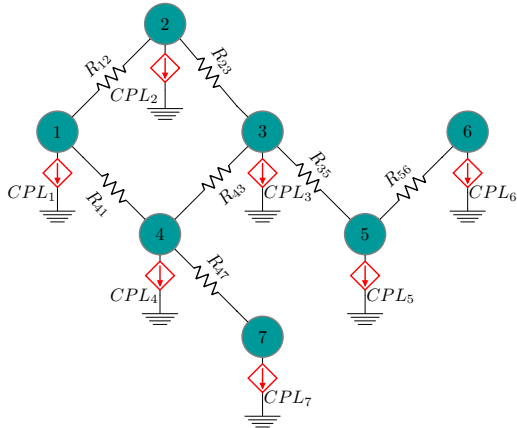


Fig. 4. Network topology of a meshed islanded MicroGrid.

driving subsystem is asymptotically stable with respect to the equilibrium set \mathcal{E} . In addition, following Prop.1 the region of attraction for the current equilibria is the set \mathcal{C} . \square

The next Theorem combines the aforementioned results to establish the stability of the cascaded dynamics under the proposed control scheme.

Theorem 3 (Asymptotic stability of the cascaded dynamics). The cascaded dynamics (19) are asymptotically stable with $\lim_{t \rightarrow \infty} \phi(x, u, w, t) \in \mathcal{E}$.

Proof. Asymptotic stability of the driving subsystem follows from Theorem 2. In addition, following Proposition 4, the dynamics described by (19c) are also asymptotically stable, where the proof follows the common approach of exploiting the recursive feasibility properties of the optimal control problem and using the cost function as a Lyapunov function for the system. Finally, boundedness of (19d) follows from Proposition 2. Thus, the cascaded dynamics are an interconnection of Lyapunov stable driving subsystems (19a), (19b), (19c) and a bounded driven subsystem (19d). Then, according to [42], the equilibrium points in (20) are asymptotically stable for the cascaded dynamics (19). \square

VI. Simulations

In this section we demonstrate the proposed control scheme in a simulated scenario of a seven node meshed network, see Fig. 4. We require the nominal voltage to reach given references, while satisfying the "tighter" constraint sets, hence the uncertain voltage to always remain within the original constraint set \mathbb{X} . The node voltage evolution is depicted in Fig. 5, where each node voltage is always contained within the respective set N_i . The current trajectories of Node 1, 3 and 7, along with the respective generated references, are shown in Fig. 6. It is seen that both the trajectories and the references are contained within the desired constraint set, validating the analysis developed in the previous sections. In addition, we note that the fluctuations of the current references are

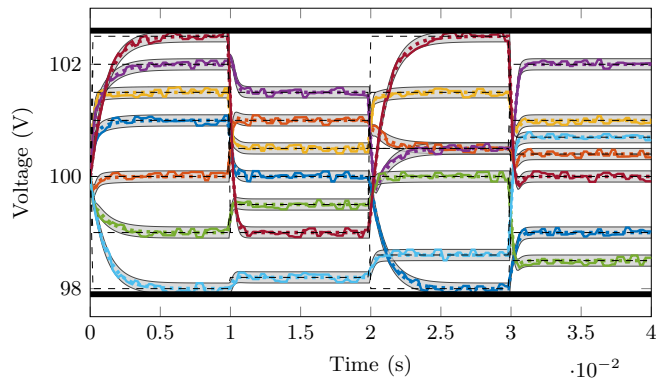


Fig. 5. Voltage and nominal voltage trajectories for Node 1 (—, ····), Node 2 (—, ····), Node 3 (—, ····), Node 4 (—, ····), Node 5 (—, ····), Node 6 (—, ····) and Node 7 (—, ····) respectively. The constrained region is represented with black solid lines (—) and the voltage references with black dashed lines (---). The voltage trajectories are within the respective S_i (□) at all times.

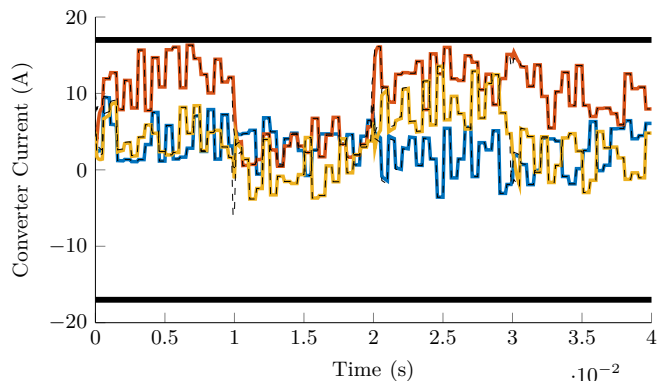


Fig. 6. Converter Current trajectories for Node 1 (—), Node 3 (—) and Node 7 (—) respectively. The constrained region \mathcal{C} is represented by black solid lines (—) and the current references \hat{i}_i by black dashed lines (---).

a result of the varying load demand which also affects the error dynamics. Here, the choice to allow larger current fluctuations is made, however, as it was demonstrated by the previous sections, one may choose to limit the fluctuations of the input currents by enlarging the set N , hence allowing larger deviations between the nominal and uncertain node voltages.

The rated voltage is set to $v^* = 100V$. The current loop control parameters are chosen as $k_{p,i} = 600$, $M_i = 9000$ resulting in a maximum current $I_{\max,i} = 17A$. The nominal power demand is $\bar{P}_i = 500W$ and we bound the maximum deviation at $|\delta P_i| \leq 500$, see Fig. 7. We furthermore parametrise the voltage constraint set as $\mathbb{X}_i = \{v_i \in \mathbb{R}^n : 97.9 \leq v_i \leq 102.6\}$. A quadratic cost functional is chosen of the form $\ell(z - \hat{z}, \eta - \hat{\eta}) = \|z - \hat{z}\|^2 + \|\eta - \hat{\eta}\|^2$ and the problem is solved using the SQP solver and "ICLOCS" MATLAB toolbox provided by [43]. Finally, we choose the voltage control parameters as $K_i = 50$ and $K_{z,i} = 4$.

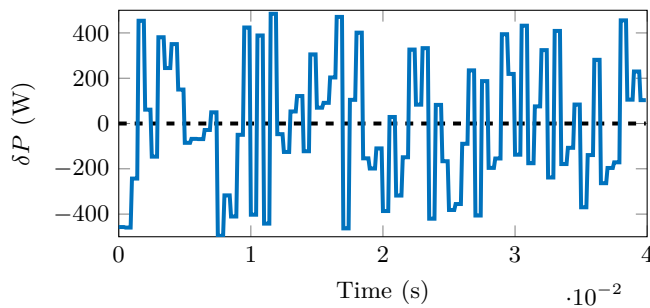


Fig. 7. Load fluctuations around the nominal value \bar{P} for Node 1. A similar load profile is shared among the rest of the Nodes.

VII. Conclusions

In this study we propose a robust control scheme that ensures stability of the closed loop system in the presence of fluctuating load demand in a meshed DC Microgrid. As it was shown, a “tube” behaviour naturally arises from the interplay of the control policy with the rest of the dynamics, allowing us to constraint the uncertain system trajectory within a predefined positive invariant set centred on a generated nominal trajectory. Furthermore, an inherent trade-off was revealed between the choice of the tube size and the availability of the current values that reside within the operational constraints and satisfy the load demand. We employed a MPC scheme to generate the nominal voltage trajectory and include the constraint sets in the control design process. To ensure recursive feasibility of the receding horizon optimization problem, the nominal control action was parametrized by a feedback gain that ensures positive invariance of the terminal ingredients. Finally, demonstration of the results was given by a simulation scenario on a seven node network with local constant power loads.

In future versions, the potential of extending the current approach to Plug & Play scenarios will be investigated. Additionally, we will focus on enlarging the feasibility region of the optimization problem by allowing the constraint sets to be time-varying.

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