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Stability and the Separation Principle in Output-Feedback Stochastic MPC with Random Packet Losses

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Abstract: This paper considers a linear quadratic Gaussian (LQG) control problem with constraints on system inputs and random packet losses occurring on the communication channel between plant and controller. It is well known that, in the absence of constraints, the Separation Principle between estimator and controller holds when the channel employs a TCP-like protocol but not so under a UDP-like protocol. This paper gives a counterexample that shows that, under a model predictive control (MPC) scheme that handles the constraints, the Separation Principle does not hold even in the TCP-like case. Theoretical analysis characterizes and reveals a trade-off between estimation errors in the estimator and prediction errors in the controller. Counterintuitively, the poorer on-average performance of the estimator in the UDP case may be compensated by smaller prediction errors in the controller.

Keywords: Control and estimation with data loss; Predictive control; Kalman filtering; Stochastic optimal control problems.

1. INTRODUCTION

The Separation Principle is a cornerstone result in modern control theory (Georgiou and Lindquist, 2013). In its simplest form, it says that the design of a state-feedback controller and an output-measurement state estimator may be executed independently while guaranteeing stability of the overall loop. The focus of control research over the last few decades has arguably been allowed to focus more on the simpler case of state feedback because the Separation Principle is either known, or assumed, to hold in the considered setting; once a state feedback control law has been designed, a state estimator can always be designed and deployed on the real, output-measured system.

Another important consideration for modern control system design is the communication channel between controller and plant; in the advent of new and emerging control technologies such as smart grids, robotics, and advanced autonomous systems, it is a realistic proposition that controller and plant are not co-located and/or physically connected. In such cases, sensor measurements and control inputs are sent and received over communication channels and may be subject to noise, delays, and packet losses. It may be necessary to consider such effects when designing the controller and estimator.

An important line of research (Sinopoli et al., 2004, 2005), collected in (Schenato et al., 2007), discovered that whether the Separation Principle holds depends on the communication channel protocol employed. In particular, the authors considered a discrete-time linear time-invariant system with Gaussian process noise and sensing noise on the output measurement—a classical LQG-type problem—and modelled the channel between controller and plant as being subject to random packet losses. A key result was to show not just that the Separation Principle holds when the channel employs a TCP-like communication protocol—*i.e.* where an acknowledgement of a received packet is transmitted—but also that it *does not* hold when the channel is UDP-like, *i.e.* absent of any acknowledgement. Moreover, the *stability region* of the TCP-based controller strictly contains that of the UDP-based controller, in the sense that the former stabilizes an LTI system when the latter is unable to.

It is interesting to enquire whether the same result holds in the presence of constraints. In this paper, therefore, we consider the same LQG-type setting albeit with the addition of (general) constraints on the system inputs. To handle these constraints, we replace the classical LQG controller based on dynamic programming with a (conventional) certainty-equivalent stochastic model predictive controller, wherein the constrained optimal control problem is solved at each new state estimate computed by the Kalman filter. In other words, we consider a standard output-feedback MPC design that may be found in many industrial applications: MPC in the state estimates, with a Kalman filter in the loop.

Our contribution is to show the existence of a counterexample where the stability of the predictive controller under the TCP-like protocol depends on the gain of the estimator. Thus, the Separation Principle *does not* hold for the TCP-like case when constraints are present and the controller employs a receding horizon, in direct contrast to the fact that the Separation Principle does hold for a (static) finite-horizon implementation of LQG (Lim et al., 1996). While

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this is not surprising in itself—for it is well known that the Separation Principle does not hold in general in constrained MPC—an interesting observation is that the UDP-based controller stabilizes the same example. This establishes that UDP-based estimation and constrained control may outperform a TCP-based scheme and, moreover, confirms that the TCP-like stability region no longer contains the UDP-like stabilty region when constraints are present. Finally, we provide a theoretical analysis that shows an interesting relation and trade-off between the estimation errors and prediction errors in both schemes; owing to an information asymmetry between estimator and controller in the TCP-like case, the on-average poorer performance of the UDP-like estimator may be compensated for by smaller prediction errors in the controller.

Notation: $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers and $\mathbb{N}_{\geq 0}$ is the set of non-negative integers. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ are, respectively, the sets of real-valued n vectors and $n\times m$ matrices. The notation $x_{k+j|k}$ is the j-step ahead prediction of x made at time $k\in\mathbb{N}_{\geq 0}$. The notation x^+ is shorthand for x_{k+1} when $x=x_k$. The conditional expectation of x given y is $\mathbb{E}\{x\,|\,y\}$, and the unconditional expectation of x is $\mathbb{E}\{x\}$. The (multivariate) normal distribution with mean μ and covariance matrix Σ is denoted $\mathcal{N}(\mu,\Sigma)$. The (univariate) Bernoulli distribution with mean μ is denoted $\mathcal{B}(\mu)$. For a matrix Q, $Q\succeq 0$ denotes positive semidefiniteness and $Q\succ 0$ denotes positive definiteness. The L_p-norm is written as $\|x\|_p$, while $\|x\|$ denotes a generic vector norm. The quadratic form $x^\top Qx$ is written as $\|x\|_Q^2$. We recall that the expectation of the quadratic form $x^\top Qx$ is $\mathbb{E}\{x^\top Qx\}=\hat{x}^\top Q\hat{x}+\mathrm{tr}(Q\Sigma_{xx})$, where $\Sigma_{xy}:=\mathbb{E}\{(x-\hat{x})(y-\hat{y})^\top\}$ is the covariance matrix between x and y, and $\mathrm{tr}(A)$ is the trace of matrix A. A function $\alpha(\cdot):\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0)=0$. I_n refers for the $n\times n$ identity matrix.

2. PAPER SETTING AND AIMS

We consider the following discrete-time linear system

$$x_{k+1}=Ax_k+\nu_k Bu_k+w_k \tag{1a}$$

$$y_k=\gamma_k Cx_k+s_k \tag{1b}$$

where $x_k\in\mathbb{R}^n$, $u_k\in\mathbb{R}^m$, and $y_k\in\mathbb{R}^p$ are, respectively, the state, input, and output of the system at sample time $k\in\mathbb{N}_{\geq 0}$. The system is subject to uncertainty in the form of (i) process noise w_k and measurement noise s_k, and (ii) random packet losses affecting the input and output channels, via the variables ν_k and γ_k. The system input, u_k, is constrained to take values in a set $\mathcal{U}\subset\mathbb{R}^m$ but the states and outputs are unconstrained.

We make the following standing assumptions.

Assumption 1. The matrices A, B and C are known, the pair (A,B) is stabilizable, and the pair (C,A) is observable.

Assumption 2. The set $\mathcal{U}$ is known and compact, containing the origin in its interior.

Assumption 3. The process noise $w_k\in\mathbb{R}^n$ and measurement noise $s_k\in\mathbb{R}^p$ are independent and identically distributed (i.i.d.) random variables, with $w_k\sim\mathcal{N}(0,Q_w)$ and $s_k\sim\mathcal{N}(0,R_s)$.

Assumption 4. The input packet loss variable $\nu_k\in\{0,1\}$ and output packet loss variable $\gamma_k\in\{0,1\}$ are i.i.d random variables with $\nu_k\sim\mathcal{B}(\bar{\nu})$ and $\gamma_k\sim\mathcal{B}(\bar{\gamma})$, where $\bar{\nu}$ and $\bar{\gamma}$ are the respective probabilities of successful packet delivery.

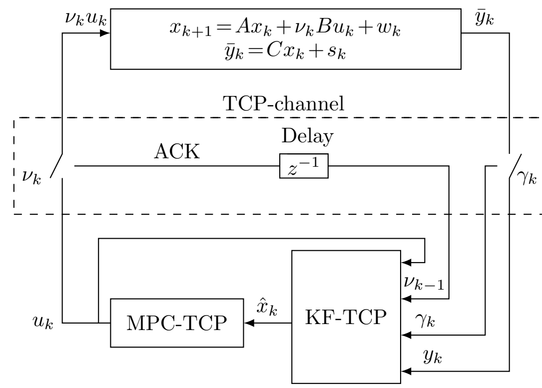


Fig. 1. Paper setting, including the uncertain system, TCP-like channel, and control and estimation modules.

Assumption 5. The information set available to the controller at time $k\in\mathbb{N}_{\geq 0}$ is

$$\mathcal{I}_k=\begin{cases}\mathcal{F}_k:=\{\mathbf{y}_k,\boldsymbol{\gamma}_k,\boldsymbol{\nu}_{k-1}\} & \text{TCP-like protocol}\\ \mathcal{G}_k:=\{\mathbf{y}_k,\boldsymbol{\gamma}_k\} & \text{UDP-like protocol}\end{cases} \tag{2}$$

where $\mathbf{y}_k=\{y_k,y_{k-1},\cdots,y_1\}$, $\boldsymbol{\gamma}_k=\{\gamma_k,\gamma_{k-1},\cdots,\gamma_1\}$, and $\boldsymbol{\nu}_k=\{\nu_k,\nu_{k-1},\cdots,\nu_1\}$.

Assumptions 1 and 2 are mild and standard. Assumptions 3–5 imply the same setting studied in the literature (e.g. Sinopoli et al. (2004); Schenato et al. (2007)), wherein the actuation and sensing channels are either TCP-like—in which an acknowledgement (ACK) of successful or unsuccessful packet delivery is sent—or UDP-like where no such acknowledgement is sent. Fig. 1 illustrates the setup in the TCP-like case, showing also the controller (MPC-TCP) and estimator (KF-TCP); the lack of state measurements (Assumption 5) motivates the need for the latter. The difference between this setup and that of Schenato et al. (2007) is the presence of input constraints. The control objective is to regulate the state x to (a neighbourhood of) the origin, despite uncertainty and packet losses, while meeting these constraints.

We remark that even though only input constraints are considered and the system is linear, this is not a trivial problem; a common approach to establishing stability even with *state* measurements available is to assume the existence of a *global* Control Lyapunov Function (CLF) (Rawlings and Mayne, 2009), which is restrictive in the presence of constraints. The aim of this paper is to analyse the closed-loop stability of the system in the described setting, and establish if, and under which conditions, the Separation Principle—which allows independent design of estimator and controller—holds.

3. OUTPUT-FEEDBACK STOCHASTIC MPC FORMULATION

The control scheme we study is composed of two steps: first, a Kalman filter performs state estimation conditioned on the information set $\mathcal{I}_k$ (Schenato et al., 2007). Subsequently, a stochastic model predictive controller computes and sends an optimal control input to the plant, based on minimizing the expectation of a cost function conditioned on the state estimate and covariance.

3.1 Estimator formulation

We briefly recall the Kalman filter conditioned on either TCP-like or UDP-like information sets, as given in (Schenato et al., 2007). Let

$$\hat{x}_k := \mathbb{E}\{x_k \,|\, \mathcal{I}_k\}, \tag{3a}$$
$$e_k := x_k - \hat{x}_k, \tag{3b}$$
$$P_k := \mathbb{E}\{e_k e_k^\top \,|\, \mathcal{I}_k\}. \tag{3c}$$

Considering the problem of estimating the state x_k at time k, the two cases differ on whether the value of ν_{k-1} is available to perform the innovation step.

TCP-like protocol:

$$\begin{aligned}\hat{x}_{k|k-1} &= \mathbb{E}\{x_k \,|\, \mathcal{F}_{k-1}, \nu_{k-1}\} \\ &= A\hat{x}_{k-1|k-1} + \nu_{k-1} B u_{k-1},\end{aligned} \tag{4a}$$
$$e_{k|k-1} = A e_{k-1} + w_{k-1}, \tag{4b}$$
$$P_{k|k-1} = A P_{k-1} A^\top + Q_w. \tag{4c}$$

UDP-like protocol:

$$\hat{x}_{k|k-1} = \mathbb{E}\{x_k \,|\, \mathcal{G}_{k-1}\} = A\hat{x}_{k-1|k-1} + \bar{\nu} B u_{k-1}, \tag{5a}$$
$$e_{k|k-1} = A e_{k-1} + (\nu_{k-1} - \bar{\nu}) B u_{k-1} + w_{k-1}, \tag{5b}$$
$$P_{k|k-1} = A P_{k-1} A^\top + \bar{\nu}(1-\bar{\nu}) B u_{k-1} u_{k-1}^\top B^\top + Q_w. \tag{5c}$$

In both cases, and because y_k, γ_k, w_k and $\mathcal{I}_k$ are independent, the correction step gives

$$\hat{x}_k = \hat{x}_{k|k-1} + \gamma_k K_k (y_k - C\hat{x}_{k|k-1}), \tag{6a}$$
$$e_k = (I - \gamma_k K_k C) e_{k|k-1} - \gamma_k K_k s_k, \tag{6b}$$
$$P_k = (I - \gamma_k K_k C) P_{k|k-1}, \tag{6c}$$
$$K_k = P_{k|k-1} C^\top (C P_{k|k-1} C^\top + R_s)^{-1}. \tag{6d}$$

In both cases the Kalman gain K_k is time-varying and stochastic, given its dependency on γ_k; it is well known that K_k, even for a stable process, does not converge to a steady value (Schenato et al., 2007).

It is also well known and easy to see from the innovation equations that the state error covariance $P_{k|k-1}$ is independent of the control input in the TCP-like case but not so in the UDP-like case. Indeed, a cornerstone result of Schenato et al. (2007) and its underlying work was to establish that the Separation Principle holds in the TCP-like case but does not in the UDP-like case. In particular, Schenato et al. (2007) considered a classical LQG setup and showed that in the TCP-like case the optimal controller is a linear function of the state estimate and the optimal estimator is independent of this; on the other hand, the optimal controller in the UDP-like case is a *nonlinear* function of the state estimate and the optimal estimator depends in a non-straightforward way on this control law.

We aim to study the same issue, albeit in the context of an *input-constrained* LQG setting. To deal with the input constraints in a systematic manner, we employ a conventional model predictive controller in the loop. The next subsection describes the formulation of the controller.

3.2 Controller formulation

The stochastic optimal control problem we consider, for the system at a state x_k and the information $\mathcal{I}_k$ available to the controller, is

$$V_N^0(\mathcal{I}_k) = \min_{\mathbf{u}_k \in \mathbb{U}} \mathbb{E}\{J_N(x_k, \mathbf{u}_k, \boldsymbol{\nu}_{\cdot|k}) \,|\, \mathcal{I}_k\} \tag{7}$$

where the decision variable

$$\mathbf{u}_k := \{u_{k|k}, u_{k+1|k}, \dots, u_{k+N-1|k}\}, \tag{8}$$

is the finite sequence of future control inputs, selected such that it lies in the constraint set

$$\mathbb{U} := \mathcal{U} \times \cdots \times \mathcal{U} \tag{9}$$

and minimizes the expectation of a cost function

$$\begin{aligned}J_N(x_k, \mathbf{u}_k, \boldsymbol{\nu}_{\cdot|k}) &:= \beta V_f(x_{k+N|k}) \\ &\quad + \sum_{j=0}^{N-1} \ell(x_{k+j|k}, \nu_{k+j|k} u_{k+j|k})\end{aligned} \tag{10a}$$

with

$$\ell(x,u) = \|x\|_Q^2 + \|u\|_R^2, \tag{10b}$$
$$V_f(x) = \|x\|_{Q_f}^2. \tag{10c}$$

Assumption 6. $Q \succ 0$, $R \succ 0$, $Q_f \succ 0$ and $\beta \geq 1$.

The expectation in (7) is to be taken over predicted states and the actuation channel packet loss variable ν:

$$\boldsymbol{\nu}_{\cdot|k} := \{\nu_{k|k}, \nu_{k+1|k}, \dots\}. \tag{11}$$

This motivates the consideration of the two different information sets, TCP-like and UDP-like, and how they affect the formulation of the optimal control problem.

- In the UDP-like case, $\mathcal{I}_k = \mathcal{G}_k = \{\mathbf{y}_k, \boldsymbol{\gamma_k}\}$ contains no additional information on which to condition the expectation in (7) beyond the state estimate and covariance—provided by the estimator—and, as in the UDP-like estimator, the expected value $\mathbb{E}\{\nu\} = \bar{\nu}$.
- In the TCP-like case, $\mathcal{I}_k = \mathcal{F}_k = \{\mathbf{y}_k, \boldsymbol{\gamma_k}, \boldsymbol{\nu}_{k-1}\}$ contains information of the past realizations of ν. The predictive control formulation would require, however, information on *future* realizations of ν_k if the use of the expected value $\mathbb{E}\{\nu\} = \bar{\nu}$ is to be avoided.

In both cases, therefore, and since ν is i.i.d. with $\mathbb{E}\{\nu u\} = \bar{\nu} u$, the expectations over $\nu_{k+j|k}$ are replaced by $\bar{\nu}$:

$$\mathbb{E}\{x_{k+j+1} \,|\, \mathcal{I}_k\} = z_{k+j+1|k} = A z_{k+j|k} + \bar{\nu} B u_{k+j|k} \tag{12a}$$
$$\mathcal{P}_{k+j+1|k} = A \mathcal{P}_{k+j|k} A^\top + Q_w \tag{12b}$$

with $z_{k|k} = \hat{x}_k$ and $\mathcal{P}_{k|k} = P_k$. Note that we use $z_{\cdot|k}$ and $\mathcal{P}_{\cdot|k}$ to denote open-loop predictions by the controller, and reserve $\hat{x}_{\cdot|k}$ and $P_{\cdot|k}$ for the estimator; as we will show, the prediction $z_{k+1|k}$ is not necessarily equal to innovation $\hat{x}_{k+1|k}$.

The optimal control problem (7) may be rewritten in the deterministic form

$$V_N^0(\mathcal{I}_k) = V_N^0(\hat{x}_k, P_k) = \min_{\mathbf{u}_k \in \mathcal{U}} J_N(\hat{x}_k, \mathbf{u}_k, \bar{\nu}) + c(P_k) \tag{13}$$

subject to, for $j \in \mathbb{N}_{[0,N-1]}$,

$$z_{k|k} = \hat{x}_k, \tag{14a}$$
$$z_{k+j+1|k} = A z_{k+j|k} + \bar{\nu} B u_{k+j|k}, \tag{14b}$$
$$u_{k+j|k} \in \mathcal{U}, \tag{14c}$$

where

$$c(P_k) = \sum_{j=0}^{N} \operatorname{tr}(Q_j \mathcal{P}_{k+j|k}), \quad Q_j = \begin{cases} Q & j \in \mathbb{N}_{[0,N-1]} \\ \beta Q_f & j = N \end{cases}$$

is a constant term that may be omitted from the optimization, but is required to determine the value function.

Remark 1. The additional information contained in the TCP-like case benefits the estimator but provides no additional information for use by the predictive controller. Problem (13) subject to (14) is therefore of a form close to a conventional input-constrained MPC problem; the only difference is the inclusion of the mean of the input packet loss variable in the dynamic model.

Remark 2. It is well known (Heirung et al., 2018) that the value of $c(P_k)$ may be reduced by parametrizing the control input as $u_{k+j|k} = Kz_{k+j|k} + v_{k+j|k}$, K stabilizing for (A, B). This, however, replaces pure input constraints with state constraints, resulting in the recursive feasibility of the controller being non-trivial to establish.

Solving $\mathbb{P}_N(\hat{x}_k, P_k)$ yields the (unique) optimal solution

$$\mathbf{u}_k^0(\hat{x}_k) = \left\{ u_{k|k}^0(\hat{x}_k), \ldots, u_{k+N-1|k}^0(\hat{x}_k) \right\}, \tag{15}$$

with associated optimal cost value $V_N^0(\hat{x}_k, P_k)$; the former does not depend on P_k but the latter does. The application of the first control in the optimal sequence to the plant, followed by a repetition of the process at the next sampling instant, defines the implicit control law

$$u_k = \kappa_N(\hat{x}_k) := u_{k|k}^0(\hat{x}_k). \tag{16}$$

In view of the lack of state constraints, the domain of the value function $V_N^0(\cdot, P)$ and control law $\kappa_N(\cdot)$ is the whole state space, meaning that recursive feasibility of the optimal control problem is trivially established. Stability of the closed-loop system, including the KF in the loop, is much harder to establish, exacerbated by the lack of terminal state constraints (Rawlings and Mayne, 2009).

4. STABILITY AND THE SEPARATION PRINCIPLE

We open this section with an interesting example. Under a particular choice of parameters, we find an instance where the TCP-like scheme loses stability while the UDP-like one retains it. We find the stability of the TCP-like controller depends on the gain of the estimator, showing that the controller and estimator cannot necessarily be designed separately. This serves as a counterexample to show that the Separation Principle does not necessarily hold in the presence of constraints.

4.1 A counterexample

Consider a system with

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad C = [1 \ 0],$$

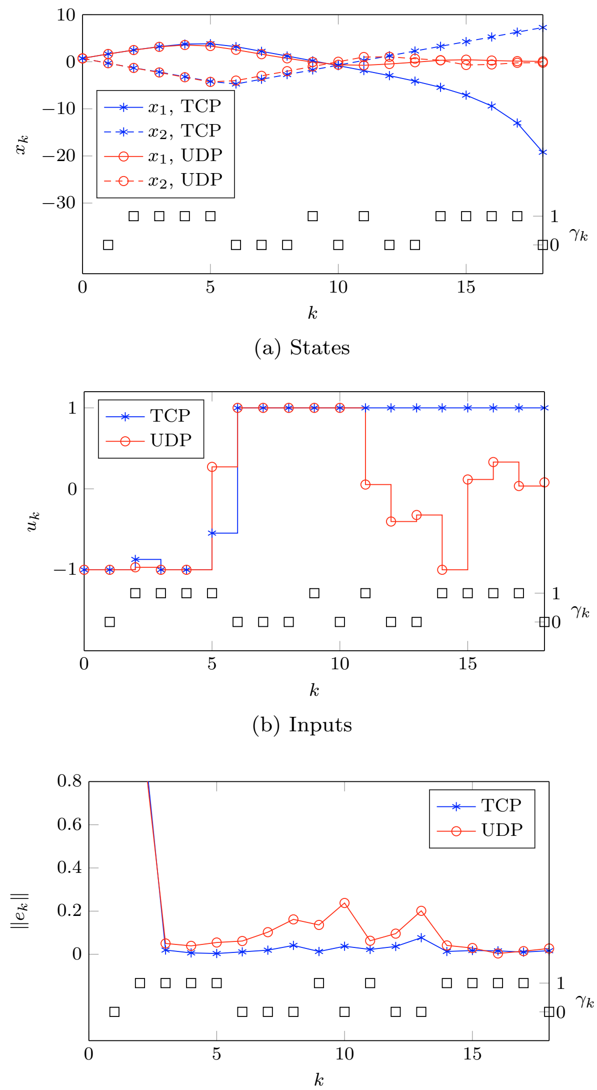
noise covariances $Q_w = 0.0001 I_2$, $R_s = 0.0001 I_1$, and input constraint set $\mathcal{U} = \{u : |u| \leq 1\}$. The expected values of ν and γ are $\bar{\nu} = 0.95$ and $\bar{\gamma} = 0.7$.

We design the controller with $Q = I$, $R = 1$, $N = 3$, $\beta = 1$,

$$Q_f = \begin{bmatrix} 12.7021 & 4.8583 \\ 4.8583 & 3.7103 \end{bmatrix}.$$

It is easily verified that this choice satisfies Assumption 7, given in the next section.

Let $x_0 = [0.731 \ 0.7]^\top$, the initial state estimate $\hat{x}_0 = [3.66 \ 0.7]^\top$, and the covariance $P_0 = \mathrm{diag}(2.6, 2)$. Fig. 2 shows the true state and estimation error trajectories under



(a) States

(b) Inputs

(c) Estimation errors

Fig. 2. True state trajectories, applied controls and estimation errors under $u = \kappa_N(\hat{x})$ with TCP-like and UDP-like estimation.

TCP-like and UDP-like state estimation schemes, with $\nu_k = 1, k = 0, 1, \ldots$ (*i.e.* no packet losses on the actuation channel) and

$$\{\gamma_k\} = \{0,1,1,1,1,0,0,0,1,0,1,0,0,1,1,1,1,0,0\}.$$

Note the TCP-based trajectory loses stability but the UDP-based solution maintains stability. Nevertheless, the estimation error in the TCP-like scheme is generally smaller. It should be emphasized that when there are no dropped packets at all ($\gamma_k = 1$), both controllers maintain stability.

Changing the initial covariance to $P_0 = \mathrm{diag}(8.579, 0)$, and repeating the same simulation under the same realizations of random variables, finds that both controllers maintain stability. Since K_k depends on P_0, this shows that the stability of the TCP-based controller can depend on the estimator gain.

4.2 Analysis

Adhering to the aim of analysing stability of a formulation that omits state constraints, we consider the use of just the terminal cost $\beta V_f(\cdot)$ and horizon length N as stabilizing ingredients. These and the cost function are supposed to satisfy certain assumptions (Limon et al., 2006), outlined in the next subsection.

Preliminaries: stability without a terminal set.

Assumption 7. The matrix $Q_f \succ 0$ is such that

$$(A+\bar{\nu}BK_f)^\top Q_f(A+\bar{\nu}BK_f)-Q_f=-(Q+\bar{\nu}^2K_f^\top RK_f)$$

for some K_f that stabilizes the pair (A,B).

The assumption says that $V_f(x)=x^\top Q_f x$ is a CLF for the expected terminal dynamics; it follows that $\beta V_f(\cdot),\beta\geq 1$, is also a CLF.

Assumption 8. Let $d_1>0$ be such that $K_f x\in\mathcal{U}$ for all

$$x\in\mathcal{X}_f(d_1):=\{x: V_f(x)\leq d_1\}. \tag{17}$$

Such a d_1 is guaranteed to exist in view of the positive definiteness of Q_f (Assumption 6) and the fact that $\mathcal{U}$ contains the origin in its interior (Assumption 2).

The following result is an immediate consequence of these two assumptions.

Lemma 1. For all $x\in\mathcal{X}_f(d_1)$,

$$V_f(Ax+\bar{\nu}BK_f x)-V_f(x)\leq -\ell(x,\bar{\nu}K_f x). \tag{18}$$

We require one more assumption (Limon et al., 2006):

Assumption 9. Let $d_2>0$ be such that

$$d_2\leq \ell(x,\bar{\nu}u) \tag{19}$$

for all $x\notin\mathcal{X}_f(d_1)$ and $u\in\mathcal{U}$, and the given $\bar{\nu}\in(0,1)$.

Such a d_2 is guaranteed to exist in view of positive definiteness of Q and R (Assumption 6) and compactness of $\mathcal{U}$ (Assumption 2).

Finally, we define the following set of states:

$$\Gamma_N^\beta := \left\{x\in\mathbb{R}^n : J_N^0(x,\mathbf{u}^0(z),\bar{\nu})\leq \beta d_1+Nd_2\right\}. \tag{20}$$

By construction,

$$\hat{x}_k=\mathbb{E}\{x_k\,|\,\mathcal{I}_k\}\in\Gamma_N^\beta \iff J_N(\hat{x}_k,\mathbf{u}_k^0,\bar{\nu})\leq\beta d_1+Nd_2.$$

The following result is adapted from Limon et al. (2006), and concerns the evolution of the system $x^+=Ax+Bu$ and optimal cost function $J_N(x,\mathbf{u}^0(x),\bar{\nu})$ when the loop is closed with $u=\kappa_N(x)$.

Lemma 2. If $x\in\Gamma_N^\beta$, then $x^+=Ax+\bar{\nu}B\kappa_N(x)\in\Gamma_N^\beta$ and

$$J_N(x^+,\mathbf{u}^0(x^+),\bar{\nu})-J_N(x,\mathbf{u}^0(x),\bar{\nu})\leq -\ell(x,\bar{\nu}\kappa_N(x)).$$

Closed-loop analysis: prediction and estimation errors. In our setting, Lemma 2 means

$$\begin{aligned}\hat{x}_k=\mathbb{E}\{x_k\,|\,\mathcal{I}_k\}\in\Gamma_N^\beta \implies\\ z_{k+1|k}=\mathbb{E}\{x_{k+1}\,|\,\mathcal{I}_k\}=A\hat{x}_k+\bar{\nu}B\kappa(\hat{x}_k)\in\Gamma_N^\beta.\end{aligned} \tag{21}$$

This successor state $z_{k+1|k}=\mathbb{E}\{x_{k+1}\,|\,\mathcal{I}_k\}$ is, however, the state estimate predicted by the controller at time k, using information $\mathcal{I}_k$, while the actual estimated state determined by the estimator at time $k+1$ is $\hat{x}_{k+1}=\mathbb{E}\{x_{k+1}\,|\,\mathcal{I}_{k+1}\}$:

$$\hat{x}_{k+1}=\hat{x}_{k+1|k}+\gamma_{k+1}K_{k+1}\big(y_{k+1}-C\hat{x}_{k+1|k}\big)$$

We note that $\hat{x}_{k+1|k}$ in this equation is not necessarily equal to the $z_{k+1|k}$ predicted by the controller:

$$\begin{aligned} z_{k+1|k}&=A\hat{x}_k+\bar{\nu}Bu^0_{k|k} && \text{MPC prediction}\\ \hat{x}_{k+1|k}&=A\hat{x}_k+\nu_kBu^0_{k|k}\neq z_{k+1|k} && \text{TCP-like innovation}\\ \hat{x}_{k+1|k}&=A\hat{x}_k+\bar{\nu}Bu^0_{k|k}= z_{k+1|k} && \text{UDP-like innovation}\end{aligned}$$

Therefore, the error between the one-step ahead state prediction and the new state estimate

$$\varepsilon_{k+1}:=\hat{x}_{k+1}-z_{k+1|k} \tag{22}$$

differs according to the protocol employed.

TCP-like protocol:

Since $\hat{x}_{k+1|k}$ is computed (at time $k+1$) using the available ν_k but the prediction $z_{k+1|k}$ used only $\bar{\nu}$, we have

$$\begin{aligned}\varepsilon^{\text{tcp}}_{k+1}=&(\nu_k-\bar{\nu})Bu^0_{k|k}\\ &+\gamma_{k+1}K_{k+1}C(Ae_k+w_k)+\gamma_{k+1}K_{k+1}s_{k+1}.\end{aligned} \tag{23}$$

while the estimation error is

$$e^{\text{tcp}}_{k+1}=(I-\gamma_{k+1}K_{k+1}C)(Ae_k+w_k)-\gamma_{k+1}K_{k+1}s_{k+1}.$$

Thus note that

$$e^{\text{tcp}}_{k+1}+\varepsilon^{\text{tcp}}_{k+1}=Ae_k+(\nu_k-\bar{\nu})Bu^0_{k|k}+w_k. \tag{24}$$

UDP-like protocol:

In this case both $\hat{x}_{k+1|k}$ and the prediction $z_{k+1|k}$ are computed using knowledge of only $\bar{\nu}$:

$$\begin{aligned}\varepsilon^{\text{udp}}_{k+1}=&\gamma_{k+1}K_{k+1}C(\nu_k-\bar{\nu})Bu^0_{k|k}\\ &+\gamma_{k+1}K_{k+1}C(Ae_k+w_k)+\gamma_{k+1}K_{k+1}s_{k+1}.\end{aligned} \tag{25}$$

The estimation error is

$$\begin{aligned}e^{\text{udp}}_{k+1}=&(I-\gamma_{k+1}K_{k+1}C)(\nu_k-\bar{\nu})Bu^0_{k|k}\\ &+(I-\gamma_{k+1}K_{k+1}C)(Ae_k+w_k)-\gamma_{k+1}K_{k+1}s_{k+1}.\end{aligned}$$

Thus note that, again,

$$e^{\text{udp}}_{k+1}+\varepsilon^{\text{udp}}_{k+1}=Ae_k+(\nu_k-\bar{\nu})Bu^0_{k|k}+w_k. \tag{26}$$

This, together with the fact that if x_k and e_k are given, then $u^0_{k|k}=\kappa_N(\hat{x}_k=x_k-e_k)$ is the same control in both UDP and TCP cases, proves the following.

Proposition 1. For a given x_k and e_k, the following statements are true:

$$e^{\text{udp}}_{k+1}+\varepsilon^{\text{udp}}_{k+1}=e^{\text{tcp}}_{k+1}+\varepsilon^{\text{tcp}}_{k+1} \tag{27}$$

and

$$e^{\text{udp}}_{k+1}=e^{\text{tcp}}_{k+1}+(I-\gamma_{k+1}K_{k+1}C)(\nu_k-\bar{\nu})Bu^0_{k|k} \tag{28}$$

$$\varepsilon^{\text{udp}}_{k+1}=\varepsilon^{\text{tcp}}_{k+1}-(I-\gamma_{k+1}K_{k+1}C)(\nu_k-\bar{\nu})Bu^0_{k|k} \tag{29}$$

The result characterizes a trade-off between the estimation error and prediction error depending on the channel protocol employed; if the effect of the input $u^0_{k|k}$ is to increase the estimation error in the UCP case compared with the TCP case, then a counter effect is to reduce the prediction error by the same margin. It is also worth pointing out that

$$\begin{aligned}e_{k+1}+\varepsilon_{k+1}&=(x_{k+1}-\hat{x}_{k+1})+(\hat{x}_{k+1}-z_{k+1|k})\\ &=x_{k+1}-z_{k+1|k}\end{aligned} \tag{30}$$

so this quantity represents the (unknown) total error between true state and MPC prediction.

It is now clear that monotonicity of the value function cannot be assured since, in general, $\hat{x}_{k+1}\neq z_{k+1|k}$. Indeed, we may write for the cost function (Rawlings and Mayne, 2009)

$$J_N(\tilde{x}_{k+1},\mathbf{u}^0_{k+1}(\tilde{x}_{k+1}),\bar{\nu}) \le J_N(z_{k+1|k},\mathbf{u}^0_{k+1}(z_{k+1|k}),\bar{\nu}) + \sigma(\|\varepsilon_{k+1}\|)$$

where $\sigma(\cdot)$ is a function of class $\mathcal{K}$, and so, for all $\hat{x}_k \in \Gamma^\beta_N$,

$$J_N(\tilde{x}_{k+1},\mathbf{u}^0_{k+1}(\tilde{x}_{k+1}),\bar{\nu}) - J_N(\hat{x}_k,\mathbf{u}^0_k(\hat{x}_k),\bar{\nu}) \le -\ell(\hat{x}_k,\bar{\nu}\kappa_N(\hat{x}_k)) + \sigma(\|\varepsilon_{k+1}\|). \quad (31)$$

It is then of interest to determine when $\|\varepsilon_{k+1}\|$ is zero (or small), in order that

$$\tilde{x}_k \in \Gamma^\beta_N \implies \tilde{x}_{k+1} \in \Gamma^\beta_N \quad (32)$$

as a key step towards ensuring stability of the controller.

The next result is an immediate result of the developed expressions (23) and (25).

Proposition 2. If $\gamma_{k+1} = 0$ then

(1) (UDP-like) $\varepsilon^{\mathrm{udp}}_{k+1} = 0$ necessarily, so for all $\hat{x}_k \in \Gamma^\beta_N$,

$$J_N(\tilde{x}_{k+1},\mathbf{u}^0_{k+1}(\tilde{x}_{k+1}),\bar{\nu}) - J_N(\hat{x}_k,\mathbf{u}^0_k(\hat{x}_k),\bar{\nu}) \le -\ell(\hat{x}_k,\bar{\nu}\kappa_N(\hat{x}_k)), \quad (33)$$

and (32) holds. However,

$$e^{\mathrm{udp}}_{k+1} = Ae_k + (\nu_k - \bar{\nu})Bu^0_{k|k} + w_k.$$

(2) (TCP-like) $\varepsilon^{\mathrm{tcp}}_{k+1} = (\nu_k - \bar{\nu})Bu^0_{k|k} \ne 0$ whenever $Bu^0_{k|k} \ne 0$. Therefore, for all $\hat{x}_k \in \Gamma^\beta_N$,

$$J_N(\tilde{x}_{k+1},\mathbf{u}^0_{k+1}(\tilde{x}_{k+1}),\bar{\nu}) - J_N(\hat{x}_k,\mathbf{u}^0_k(\hat{x}_k),\bar{\nu}) \le -\ell(\hat{x}_k,\bar{\nu}\kappa_N(\hat{x}_k)) + \sigma\Big(\|(\nu_k - \bar{\nu})Bu^0_{k|k}\|\Big), \quad (34)$$

and (32) does not necessarily hold. Moreover,

$$e^{\mathrm{tcp}}_{k+1} = Ae_k + w_k.$$

This result depicts a kind of reverse separation principle wherein, in the case of sensor dropouts, the UDP-MPC cost function enjoys a monotonic decrease, independent of the estimator, if $\hat{x}_k \in \Gamma^\beta_N$. The estimator performance is, however, dependent on the control input. In the TCP case, on the other hand, the estimator is independent of the controller (*c.f.* the separation observed by Schenato et al. (2007)), but the monotonicity of the controller cost function is now assured only for suitably small inputs.

4.3 Revisiting the counterexample

The designed controller in Section 4.1 satisfies Assumptions 6–9, the latter with $d_1 = 1.85$ and $d_2 = 1.2$; therefore, with $N = 3$ and $\beta = 1$,

$$\Gamma^1_3 = \{x : J_N(x,\mathbf{u}^0(x),\bar{\nu}) \le 5.45\}.$$

The initial state estimate $\hat{x}_0 \notin \Gamma^1_3$; however, in the UDP case the state estimate enters Γ^1_3 at $k = 11$ and remains therein—see Fig. 3. It can be seen that whenever $\gamma_{k+1} = 0$, $J_N(\hat{x}_{k+1},\mathbf{u}^0_{k+1}(\hat{x}_{k+1}),\bar{\nu}) < J_N(\hat{x}_k,\mathbf{u}^0_k(\hat{x}_k),\bar{\nu})$ (Proposition 2). The increase at $k = 14$ (when γ_k rises from 0 to 1) is explained by Proposition 1: the state estimate $\hat{x}_{14}$ is improved over the prediction $\bar{x}_{14|13} = z_{14|13}$ at the expense of higher prediction error.

In the TCP case, the state estimate never enters Γ^1_3; the cost reaches a minimum of 41.16 before diverging. However, increasing N to 4 results in the state estimate entering

$$\Gamma^1_4 = \{x : J_N(x,\mathbf{u}^0(x),\bar{\nu}) \le 6.65\}$$

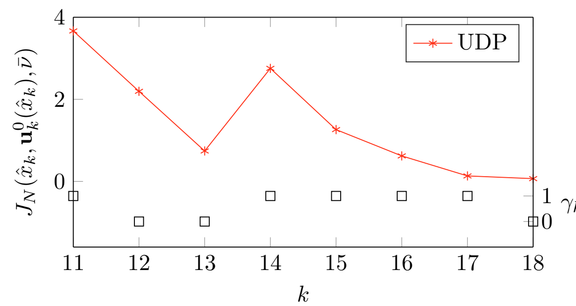


Fig. 3. Optimal cost value in the UDP case.

at $k = 9$, and subsequently maintaining stability.

5. CONCLUSIONS

This paper has considered an input-constrained LQG-type problem under random packet losses on the sensing and actuation channels. A counterexample established that, unlike in the unconstrained case, the Separation Principle does not hold when a TCP-like protocol is employed on the channels. Further analysis identified a relationship between the estimation errors and controller prediction errors, suggesting that controller performance may be worsened by improving estimation performance.

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REFERENCES

Georgiou, T.T. and Lindquist, A. (2013). The separation principle in stochastic control, redux. *IEEE Transactions on Automatic Control*, 58(10), 2481–2494.

Heirung, T.A.N., Paulson, J.A., O'Leary, J., and Mesbah, A. (2018). Stochastic model predictive control – how does it work? *Computers & Chemical Engineering*, 114, 158–170.

Lim, A.E., Moore, J.B., and Faybusovich, L. (1996). Separation theorem for linearly constrained LQG optimal control. *Systems & Control Letters*, 28(4), 227–235.

Limon, D., Alamo, T., Salas, F., and Camacho, E. (2006). On the stability of constrained MPC without terminal constraint. *IEEE Transactions on Automatic Control*, 51(5), 832–836.

Rawlings, J.B. and Mayne, D.Q. (2009). *Model Predictive Control: Theory and Design*. Nob Hill.

Schenato, L., Sinopoli, B., Franceschetti, M., Poolla, K., and Sastry, S.S. (2007). Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, 95(1), 163–187.

Sinopoli, B., Schenato, L., Franceschetti, M., Poolla, K., Jordan, M.I., and Sastry, S.S. (2004). Kalman filtering with intermittent observations. *IEEE Transactions on Automatic Control*, 49(9), 1453–1464.

Sinopoli, B., Schenato, L., Franceschetti, M., Poolla, K., and Sastry, S.S. (2005). Optimal control with unreliable communication: the TCP case. In *Proceedings of the 2005 American Control Conference*.