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ScienceDirect
IFAC PapersOnLine 56-2 (2023) 9540–9545

A non-iterative approach to Linear Quadratic Static Output Feedback

Hadriano Morales Escamilla * Paul Trodden *
Visakan Kadirkamanathan *

* Department of Automatic Control and Systems Engineering, ACSE, The University of Sheffield, Sheffield, UK (e-mail: h.m.escamilla@sheffield.ac.uk, p.trodden@sheffield.ac.uk, visakan@sheffield.ac.uk).

Abstract:
This paper considers the problem of static output feedback (SOF) synthesis for linear time-invariant (LTI) systems. Static output feedback, and more generally structured controller synthesis, is of special interest to any industrial application where a reduced-order controller is desired, e.g., high-order systems, or a specific structure is to be imposed, i.e., distributed/decentralised control. A simple two-step process is proposed, solving one Riccati equation and one optimisation problem with linear matrix inequality (LMI) constraints, enabled via a dilation using the distance to the full-state feedback optimum gain. Numerical analysis shows considerable computational savings, with negligible differences in the performance, when compared to current iterative methods.

Keywords: Output feedback control, Controller constraints and structure, Linear multivariable systems, Linear matrix inequalities, Optimal Control, Static controller synthesis, S-procedure.

1. INTRODUCTION

Static output feedback (SOF) synthesis for multi-variable linear systems is a key challenge in control design, and still remains to be one of the most important problems in this area: see Silva and Frezzatto (2021) for a recent example of robust SOF applied to linear-parameter-varying (LPV) systems, and Duan et al. (2022) for a recent investigation on SOF synthesis using gradient-based optimisation. The relevance of SOF stems not only from its importance for output feedback control, but also from its applicability to fixed-order controller synthesis (Dabboussi and Zrida, 2012), as well as to distributed/decentralised control design (Wang and Davidson, 1973). Extensive efforts have been made in the literature to solve this problem, and comprehensive reviews can be found in Syrmos et al. (1997) and Sadabadi and Peaucelle (2016).

Similar to the full-state feedback case, other goals can be sought in addition to stability (Skogestad and Postlethwaite, 2005): robustness, via an H_2/H_∞ formulation, and/or some other performance objective, e.g. via optimisation of a quadratic cost function. Most of the literature focuses on the robustness problem, while little to no attention has been paid to the 'simpler' optimal control case: the output-feedback version of the linear quadratic regulator (LQR).

The increased interest on robust approaches, moving on from LQR to H_2 formulation, was a logical step, as the latter can be interpreted as a generalisation of the former (Skogestad and Postlethwaite, 2005). However, LQR remains relevant in practice, given that more often than not the uncertainty matrix of the state equation is unknown, the tuning and synthesis of the gain is well-understood by the control community, and its simplicity and fast computation make it an appealing method to use during prototyping, or as benchmark for other methods.

Our proposed approach poses the linear quadratic static output feedback (LQ-SOF) problem in reference to the full-state-feedback LQR gain, avoiding iterations, and completing computation faster than current methods.

1.1 Linear quadratic static output feedback

One of the few and earliest attempts towards LQ-SOF control can be found in Levine and Athans (1970). Their proposed method formulates the gain as a function of two Lyapunov variables, related respectively with controllability and observability of the system. An iterative algorithm provides the solution to these variables. The method guarantees decay through the iterations, and it relies on a first guess of the output-feedback controller.

Trofino-Neto and Kucera (1993) present an alternative approach, defining the gain as a function of one Lyapunov variable, like in the state-feedback case, using the pseudo-inverse of the measurement matrix. The method then proposes to solve the resulting algebraic Riccati equation (ARE) using homotopy methods, for low-rank disturbances, as described in Richter et al. (1990). By then, the interest in the robust formulation (Doyle et al., 1989) overtook that of solving the LQ-SOF problem directly.

Regarding the robustness problem, the existing approaches can be classified in two main categories (Sadabadi and Peaucelle, 2016), depending on whether Lyapunov variables are used or not. Non-Lyapunov methods are not

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2023.10.254

discussed here for brevity, as these are not relevant to our method. The interested reader is referred to the work by Apkarian and Noll (2006) for one of the earliest non-Lyapunov-based alternatives, where a non-smooth optimisation algorithm is used to circumvent the non-convexity condition of the bilinear matrix inequality (BMI).

1.2 H_2/H_∞ Lyapunov-based methods

It is well-known that SOF synthesis leads to a BMI that cannot be linearised using standard strategies, such as change of variables, *Elimination Lemma*, or Schur complement (VanAntwerp and Braatz, 2000). Furthermore, solving the problem, which is non-convex due to the cross-products between the gain matrix and the Lyapunov variable (Sadabadi and Peaucelle, 2016), has been reported NP-hard (Blondel and Tsitsiklisz, 1997).

Initial efforts to address the non-convexity turned to iterative algorithms, which render the problem convex by freezing one of the variables at each step, while optimising over the other one (El Ghaoui and Balakrishnan, 1994). Alternative iterative processes are presented in Iwasaki et al. (1994), for the H_2 problem, and in Cao et al. (1998), for the H_∞ problem. These are solved by iterating between two Lyapunov variables (the total cost matrix and its inverse). Iwasaki et al. (1994) mention LQR as a particularisation of the H_2 problem for the full-state-feedback case.

A different convexification of the problem is achieved using the *S-procedure*, which in its initial formulation (de Oliveira et al., 1999) decouples the controller gain from the Lyapunov variable exploiting the necessity conditions in the *Projection Lemma* (Gahinet and Apkarian, 1994). This method is well-documented in Ebihara et al. (2015), where it is observed that the dilated LMI obtained is computationally more tractable thanks to the additional degrees of freedom introduced by the slack variables.

Representative examples using the *S-procedure* are found in Peaucelle and Arzelier (2001), where the H_2 problem is addressed using an iterative approach with dilated LMIs, or in Sadabadi and Karimi (2015), where the H_∞ problem is solved using an also iterative approach with a different dilation, inspired by the *Positive Real Lemma*.

Iterative approaches are highly dependent on the stopping criterion, and thus special care is to be taken when defining this, in order to avoid running into large computation times. It is also recognised that the solution reached with these methods is highly dependent on the initial guess (Sadabadi and Peaucelle, 2016). Most methods found in the literature also acknowledge that the problem is still open, and that the solution offered might be sub-optimal.

1.3 Aim and contribution of this paper

In this paper we consider the linear quadratic static output feedback (LQ-SOF) synthesis, which is the equivalent of LQR for the SOF case. This allows us to focus on the main practical challenge: the linearisation of the bilinear matrix inequality (BMI) that arises in SOF problems, and to draw particular insights into the conservativeness of the approach selected.

Our method dilates the original BMI to an LMI by means of first formulating the output-feedback gain in reference to the full-state-feedback gain. This makes the resulting state matrix, which multiplies the Lyapunov variable, both stable and optimal in the LQR sense.

This differs from Sadabadi and Karimi (2015), which expresses the LMI in terms of a generic state-feedback gain. Though this detail seems minor, it allows us to (i) gain explicit insight into how the LQ-SOF gain compares to the LQR one and, more significantly, (ii) solve the SOF problem by solving one ARE followed by one LMI instead of a sequence of LMIs solved until convergence.

2. PRELIMINARIES

This section defines the notation, the control problem addressed, useful lemmas used in the paper, and the adaptation to LQR of two results from the robust literature that serve as benchmark for the method presented here, given the similarities with the dilations used in this paper.

2.1 Notation

Throughout the paper, the standard mathematical notation is used, with lower case variables, e.g. x, representing arrays, uppercase matrix variables, e.g. A, and the symbols $\mathbb{R}$ and $\mathbb{S}$, are sets of real and symmetric numbers respectively. The dimensions of the sets are specified as exponents, and any restrictions as subscripts, e.g. $\mathbb{S}^{n_x}_{++}$ ($\mathbb{S}^{n_x}_{+}$) is the set of positive definite (positive semi-definite) symmetric matrices with n_x rows.

The sum of a square matrix with its transpose is represented by $\{A\}^{\mathrm{S}} = A + A^T$. The definiteness and semi-definiteness of a matrix are represented by the mathematical symbols $\prec$ and $\preceq$. Symbol I refers to the identity matrix of appropriate dimensions. Finally, the symbol $\star$ in the upper-diagonal blocks is used to indicate their equivalence to the transpose of the corresponding lower-diagonal blocks in a symmetric matrix.

$$\begin{bmatrix} A & \star \\ B & C \end{bmatrix} \equiv \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} . \tag{1}$$

2.2 The linear quadratic static output feedback problem

Let us consider the following LTI continuous-time system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \ , \\ y(t) = Cx(t) \ , \end{cases} \tag{2}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, and $y \in \mathbb{R}^{n_y}$ represent respectively the state, input, and measurement of the system, while matrices A, B, and C are of the appropriate dimensions.

The LQ-SOF control problem consists in designing a linear controller gain $K \in \mathbb{R}^{n_u \times n_y}$, such that the control law $u = Ky$ stabilises the system in (2), while minimising the cost function J_∞ defined in (3).

$$J_\infty = \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \ , \tag{3}$$

where $Q \in \mathbb{S}_{+}^{n_x}$ is the penalty in the states, $R \in \mathbb{S}_{++}^{n_x}$ is the penalty in the inputs, and $S \in \mathbb{R}^{n_x \times n_u}$ is the cross-penalty. Once K is designed, the closed-loop dynamics of the system are defined by (4), and the optimisation cost as per (5).

$$\dot{x}(t) = (A + BKC)x(t) \ , \quad (4)$$

$$J_\infty = \int_0^\infty x(t)^T \mathcal{Q}_{KC} \ x(t)dt \ , \quad (5)$$

where $\mathcal{Q}_{KC}$ is obtained by setting $\mathcal{K} = KC$ in the definition of $\mathcal{Q}_\mathcal{K}$ in (6).

$$\mathcal{Q}_\mathcal{K} = Q + S\mathcal{K} + \mathcal{K}^T S^T + \mathcal{K}^T R \mathcal{K} \ . \quad (6)$$

For clarity, the time-dependency of all signals will be omitted from the notation hereafter.

2.3 Useful lemmas

Lemma 1. LQR control. It is well-known (Boyd et al., 1994) that solving the Riccati inequality in (7) provides the optimal state-feedback control gain in the LQR sense.

$$\begin{bmatrix} I \\ \mathcal{K} \end{bmatrix}^T \begin{bmatrix} A^T P + PA + Q & \star \\ B^T P + S^T & R \end{bmatrix} \begin{bmatrix} I \\ \mathcal{K} \end{bmatrix} \preceq 0 \ , \quad (7)$$

where $P \in \mathbb{S}_{++}^{n_x}$ is the Lyapunov variable, and $\mathcal{K} \in \mathbb{R}^{n_u \times n_x}$ is the state-feedback gain. The optimum is achieved with P_o and $\mathcal{K}_o$ such that the equality holds.

Lemma 2. LQ-SOF. The pair (P, K) that minimises the trace of P subject to (8) is the optimum solution to the LQ-SOF problem.

$$\begin{bmatrix} I \\ KC \end{bmatrix}^T \begin{bmatrix} A^T P + PA + Q & \star \\ B^T P + S^T & R \end{bmatrix} \begin{bmatrix} I \\ KC \end{bmatrix} \preceq 0 \ , \quad (8)$$

where $K \in \mathbb{R}^{n_u \times n_y}$ is the SOF gain, and $P \in \mathbb{S}_{++}^{n_x}$. The proof follows from setting $\mathcal{K} = KC$ in (7).

Lemma 3. The Projection Lemma. Given some matrices $\Psi \in \mathbb{S}^q$, $\Omega \in \mathbb{R}^{m \times q}$, and $\Phi \in \mathbb{R}^{m \times q}$, LMI conditions in (9) and (10) are equivalent.

$$\Psi + \Phi^T \Omega + \Omega^T \Phi \preceq 0 \ . \quad (9)$$

$$\begin{cases} \mathcal{N}_\Omega^T \Psi \mathcal{N}_\Omega \preceq 0 \ , \\ \mathcal{N}_\Phi^T \Psi \mathcal{N}_\Phi \preceq 0 \ , \end{cases} \quad (10)$$

where $\mathcal{N}_\Omega$ and $\mathcal{N}_\Phi$ are the right null-spaces of Ω and Φ respectively. The proof can be found in Gahinet and Apkarian (1994).

2.4 Peaucelle and Arzelier's method

Peaucelle and Arzelier (2001) propose an iterative approach that is initialised with a $\mathcal{K}_i$ solution to (7). It follows by first freezing $\mathcal{K}_i$ and solving for minimum $\Upsilon_\mathcal{K} = trace(P)$, varying P, X, and Y, subject to (11).

$$\begin{bmatrix} \{PA - \mathcal{K}^T YC\}^\mathrm{S} + Q & \star \\ B^T P + S^T + YC + X^T \mathcal{K} & R - \{X\}^\mathrm{S} \end{bmatrix} \preceq 0 \ , \quad (11)$$

where the LMI has been adapted from their *Theorem 1* to the LQ-SOF case, using our naming convention. Next, X_i and Y_i are frozen, and the problem is solved again for minimum $\Upsilon_{X,Y} = trace(P)$ by varying P and $\mathcal{K}$.

Convergence is tested by closeness of $\Upsilon_\mathcal{K}$ and $\Upsilon_{X,Y}$. If the criterion is not met, the two problems are solved again, first freezing $\mathcal{K}_{i+1}$, and then X_{i+1} and Y_{i+1}. Once converged, the SOF gain is obtained as $K = X^{-1}Y$. The solution solves the LQ-SOF problem as proven in Peaucelle and Arzelier (2001) for the H_2-SOF case.

The Peaucelle and Arzelier (2001) method requires to solve a first LMI problem for the initial condition, and two additional LMI problems in each iteration. Convergence is guaranteed locally, after a number of iterations a-priori unknown.

2.5 Sadabadi and Karimi's method

In Sadabadi and Karimi (2015) an iterative approach is also proposed. The initialisation is performed by solving a different LMI problem, described in their *Theorem 4*, resulting in an initial state-feedback gain $\mathcal{K}_i$. This gain is then used to solve for minimum $\Upsilon_i = trace(P)$ by freezing $\mathcal{K} = \mathcal{K}_i$ and varying X and Y subject to (12).

$$\begin{bmatrix} \{PA + PB\mathcal{K}\}^\mathrm{S} + \mathcal{Q}_\mathcal{K} & \star \\ B^T P + S^T + YC + (R - X)\mathcal{K} & R - \{X\}^\mathrm{S} \end{bmatrix} \preceq 0 \ , \quad (12)$$

where the LMI has been reinterpreted from their *Theorem 2* to the LQ-SOF case. Convergence is tested by changes in the cost function, i.e. if Υ_i and Υ_{i-1} are close enough, where i is the iteration number. If the convergence criterion is not met, the state-feedback gain is set to $\mathcal{K}_{i+1} = X_i^{-1} Y_i C$, and the problem is solved again.

Once converged, the SOF gain is obtained as $K = X^{-1}Y$. This gain solves the LQ-SOF as proven in Sadabadi and Karimi (2015) for the H_∞-SOF problem.

The Sadabadi and Karimi (2015) method requires to solve a first LMI problem for the initial condition, where an arbitrary small positive value has to be assigned to a scalar variable, and one additional LMI problem for each iteration. Like in the *Peaucelle and Arzelier's method*, convergence is guaranteed locally, after a number of iterations a-priori unknown.

Remark 1. Updating the state-feedback gain in (12) causes the result to differ more and more from that initial gain. If this initial guess is already the optimum with respect to the state-feedback criterion, updating $\mathcal{K}$ may cause KC to diverge more from $\mathcal{K}_o$ with each iteration.

3. MAIN RESULT

This section presents our proposed method to solve the LQ-SOF problem. First, some key definitions are described, needed for our dilation of the BMI. Then, our method is enunciated and compared to the existing ones.

3.1 Initial definitions

Let us start with $\mathcal{K}_o$, which solves the ARE resulting from the equality in (7). This gain can be obtained analytically, when $R \succ 0$ and $Q \succeq 0$ (Lewis et al., 2012), or as the solution to a stabilizability LMI (Boyd et al., 1994).

Let us now define the error from any possible SOF gain K to this optimum gain as:

$$\widetilde{\mathcal{K}} = KC - \mathcal{K}_o \ . \tag{13}$$

Using this definition by setting $KC = \widetilde{\mathcal{K}} + \mathcal{K}_o$ in (8) leads to (14), where $\Psi_P \in \mathbb{S}^{n_x+n_u}$ is defined as per (15), and $\mathcal{K}_o$ has been used in (6) to get $\mathcal{Q}_{\mathcal{K}_o}$.

$$\begin{bmatrix} I \\ \widetilde{\mathcal{K}} \end{bmatrix}^T \Psi_P \begin{bmatrix} I \\ \widetilde{\mathcal{K}} \end{bmatrix} \preceq 0 \ , \tag{14}$$

$$\Psi_P = \begin{bmatrix} \{PA + PB\mathcal{K}_o\}^{\mathrm{S}} + \mathcal{Q}_{\mathcal{K}_o} & \star \\ B^T P + S^T + R\mathcal{K}_o & R \end{bmatrix} \ , \tag{15}$$

For future reference, we also define Ψ_P by blocks as in (16) according to the block matrices in (15).

$$\Psi_P = \begin{bmatrix} \Psi_P^{(1,1)} & \Psi_P^{(1,2)} \\ \Psi_P^{(2,1)} & \Psi_P^{(2,2)} \end{bmatrix} \ . \tag{16}$$

One can realise now that the first block matrix $\Psi_P^{(1,1)}$ is in fact equivalent to (7) when $\mathcal{K} = \mathcal{K}_o$. Therefore, a suitable P such that $\Psi_P^{(1,1)} \preceq 0$ is guaranteed to exist.

Let us also define $\Omega \in \mathbb{R}^{n_u \times (n_x+n_u)}$, and its right null-space $\mathcal{N}_\Omega$ as per (17), where $\mathcal{N}_{\widetilde{\mathcal{K}}}$ is the joint right null-space of C and $\mathcal{K}_o$, according to the definition of $\widetilde{\mathcal{K}}$ in (13).

$$\Omega = [\, \widetilde{\mathcal{K}} \quad -I \,] \quad , \quad \mathcal{N}_\Omega = \begin{bmatrix} I & \mathcal{N}_{\widetilde{\mathcal{K}}} \\ \widetilde{\mathcal{K}} & 0 \end{bmatrix} \ . \tag{17}$$

Finally, let us define $\Phi \in \mathbb{R}^{n_u \times (n_x+n_u)}$, and its right null-space $\mathcal{N}_\Phi$ as in (18), where $F \in \mathbb{R}^{n_x \times n_u}$ and $X \in \mathbb{R}^{n_x \times n_x}_{++}$ are two new matrix variables, and $\mathcal{N}_{F^T}$ is the right null-space of F^T.

$$\Phi = [\, F^T \quad X^T \,] \quad , \quad \mathcal{N}_\Phi = \begin{bmatrix} -I & \mathcal{N}_{F^T} \\ X^{-T}F^T & 0 \end{bmatrix} \ . \tag{18}$$

3.2 A dilation using the SOF gain error

A dilated BMI condition is reached in (19) by using the definitions of Ψ_P from (15), Ω from (17), and Φ from (18) into (9). The additional degrees of freedom obtained with X and F still render the inequality bilinear, however, as it will be clear later, this does not introduce a new complication.

$$\Psi_P + \left\{ \begin{bmatrix} F \\ X \end{bmatrix} [\, \widetilde{\mathcal{K}} \quad -I \,] \right\}^{\mathrm{S}} \preceq 0 \ . \tag{19}$$

By virtue of the *Projection Lemma*, the dilated BMI in (19) is a necessary and sufficient condition for compliance with (20), formed by taking the first inequality from (10).

$$\begin{bmatrix} \begin{bmatrix} I \\ \widetilde{\mathcal{K}} \end{bmatrix}^T \Psi_P \begin{bmatrix} I \\ \widetilde{\mathcal{K}} \end{bmatrix} & \begin{bmatrix} I \\ \widetilde{\mathcal{K}} \end{bmatrix}^T \Psi_P \begin{bmatrix} \mathcal{N}_{\widetilde{\mathcal{K}}} \\ 0 \end{bmatrix} \\ \star & \mathcal{N}_{\widetilde{\mathcal{K}}}^T \Psi_P^{(1,1)} \mathcal{N}_{\widetilde{\mathcal{K}}} \end{bmatrix} \preceq 0 \ . \tag{20}$$

Compliance with (20) implies (14), as all blocks in the diagonal have to be negative semi-definite. Furthermore, it can be said that it strictly complies with (14), since (20) is equivalent to (21), where $\Pi_{P,K} \succeq 0$ is the resulting matrix after applying the Schur complement (Boyd et al., 1994) to (20).

$$\begin{bmatrix} I \\ \widetilde{\mathcal{K}} \end{bmatrix}^T \Psi_P \begin{bmatrix} I \\ \widetilde{\mathcal{K}} \end{bmatrix} + \Pi_{P,K} \preceq 0 \ , \tag{21}$$

Remark 2. Matrix $\Pi_{P,K}$ offers a measure of the conservativeness of this method: the closer $\Pi_{P,K}$ is to 0, the closer (19) and (14) are to exact equivalence. This is trivially achieved when $\mathcal{N}_{\widetilde{\mathcal{K}}} = 0$, as (20) would collapse to (14), and in cases when $\Psi_P^{(1,1)}$ is 'negatively big'.

Finally, the BMI in (19) can be rewritten as the BMI in (22) by replacing back the definition of $\widetilde{\mathcal{K}}$ from (13), and performing the linearising change of variable $Y = XK$.

$$\begin{bmatrix} \Psi_P^{(1,1)} + \left\{F(X^{-1}YC - \mathcal{K}_o)\right\}^{\mathrm{S}} & \star \\ \Psi_P^{(2,1)} + YC - X\mathcal{K}_o - F^T & R - \{X\}^{\mathrm{S}} \end{bmatrix} \preceq 0 \ . \tag{22}$$

The BMI obtained might seem difficult to linearise at first, but taking advantage of the fact that $\Psi_P^{(1,1)} \preceq 0$, this BMI can be turned into an LMI by choosing $F = 0$. This conclusion leads us to our main result. The proof follows from recalling that (22) is necessary and sufficient for (20), which in turn is sufficient for (14).

Proposition 1. The tuple (P, X, Y) obtained from minimisation of the trace of P, subject to (22) with $F = 0$, solves the LQ-SOF problem with $K = X^{-1}Y$.

Remark 3. The additional conservativeness introduced by forcing $F = 0$ is related to the difference $YC - X\mathcal{K}_o$, which is minimised as a by-product of minimising the trace of P subject to (22).

Remark 4. The resulting LMI obtained when setting $F = 0$ is the same that results when adapting the second step of Sadabadi and Karimi (2015) to the LQ-SOF case in (12), by setting the state-feedback gain equal to the LQR gain.

4. NUMERICAL EXAMPLES

In order to assess and compare our approach with the methods by Peaucelle and Arzelier (2001) and Sadabadi and Karimi (2015), all algorithms have been implemented in MATLAB, using the LMI toolbox introduced by Gahinet et al. (1994).

Given that the true optimum for the LQ-SOF case is not yet known, comparing methods can be challenging. Hence, in order to enable some assessment of optimality, all

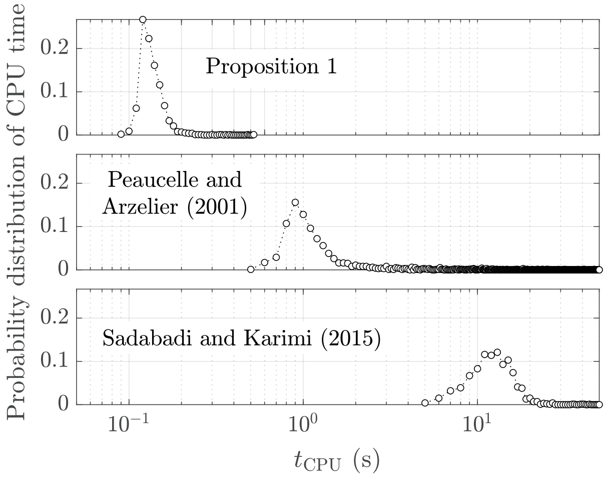

Fig. 1. Probability distribution of the computation time, with logarithmic scale in the time axis.

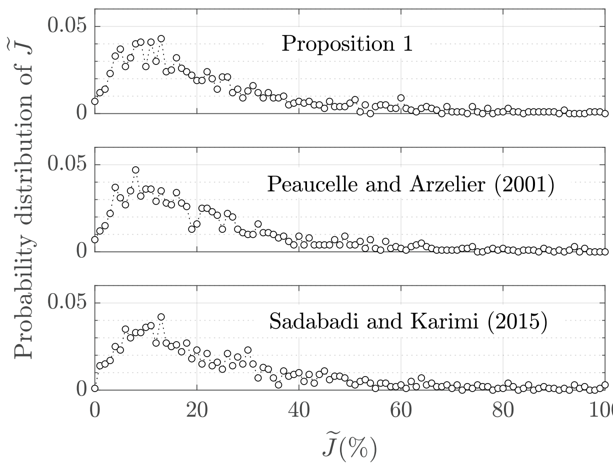

Fig. 2. Probability distribution of the cost deviation

three approaches are compared against the state-feedback solution using the cost deviation defined by $\widetilde{J}$ in (23).

$$\widetilde{J}_{[method]} = 100 \; \frac{J_\infty^{[method]} - J_\infty^{LQR}}{J_\infty^{LQR}} \; , \tag{23}$$

where J_∞^{LQR} is the optimum LQR cost when using state feedback, and $J_\infty^{[method]}$ is the quadratic cost for each output-feedback method, as defined in (3).

The SOF gain was derived, using the three different methods being compared, for 1000 systems generated randomly, with 20 states, 3 outputs and 2 inputs. The penalties were set to $Q = I$, $S = 0$, and $R = I$, while the initial state was set to an array of ones: $x_0 = \mathbb{1}^{n_x}$.

The results are evaluated in terms of: $\widetilde{J}$, the cost deviation from the state-feedback optimum, t_{CPU}, the total computation time needed to obtain each gain, n_{iter}, the number of iterations performed, and n_{LMI}, the number of LMI problems solved.

Our method shows a remarkable improvement in terms of the computation time, as it is clear from Fig. 1, where a noticeable difference can be appreciated between the three methods. The normalised probability distribution for t_{CPU} is shown to have considerably lower values for the method proposed here.

This improvement in the computation time has a negligible effect in the performance. Fig. 2 shows the distribution of the cost deviation for the three methods, normalised by the number of experiments in order to obtain the probability distribution from data. Similar distributions are observed for the three methods.

All metrics are summarised in Tab. 1, where the operator E[ ] represents the expected value.

The method proposed in *Proposition 1* develops, on average, a slightly higher cost error when compared to the best of the methods chosen (Peaucelle and Arzelier, 2001), but the difference is of less than 1%.

In general, it can be said that the three methods perform similarly with respect to cost optimality, and therefore the conservativeness introduced when setting $F = 0$ in (19) has not had a major impact on the optimality of the solution.

In addition, it can be observed that the method from Sadabadi and Karimi (2015) is the one that requires the most computation effort, as it attains the highest average time, but with fewer iterations than the method from Peaucelle and Arzelier (2001). This is likely due to the differences in the way the iterations are performed in these two methods: Sadabadi and Karimi (2015) perform an inversion at each iteration in order to obtain the next state-feedback gain, while Peaucelle and Arzelier (2001) solve an LMI problem to find that gain.

The method from *Proposition 1* achieves an average computation time over ten times faster than the fastest of the methods, which is mainly a result of being non-iterative. The lack of iterations also adds a certain degree of confidence to the computation time, as it can be seen in Fig. 1, where the 'tail' of the probability distribution does not extend beyond 0.6 seconds for our method.

A measure of this improved certainty can be seen in Tab. 2, where the operator σ[ ] represents the standard deviation.

Table 1. Comparison of average values

Method	E[$\widetilde{J}$] (%)	E[t_{CPU}] (s)	E[n_{iter}]	E[n_{LMI}]
Proposition 1	24.78	0.14	0	1
Peaucelle and Arzelier (2001)	24.06	3.24	10.27	21.55
Sadabadi and Karimi (2015)	27.98	13.61	3.25	4.25

Table 2. Comparison of standard deviations

Method	$\sigma[\widetilde{J}\,]$ (%)	$\sigma[t_{\mathrm{CPU}}]$ (s)	$\sigma[n_{\mathrm{iter}}]$	$\sigma[n_{\mathrm{LMI}}]$
Proposition 1	26.01	0.03	0	0
Peaucelle and Arzelier (2001)	23.11	9.92	33.58	21.55
Sadabadi and Karimi (2015)	30.28	4.24	1.09	1.1

5. CONCLUSION

This paper presents a static output feedback synthesis method with linear quadratic optimality. The approach consists of two steps: first, the well-known algebraic Riccati equation is solved for the full-state-feedback case, and second, the bilinear matrix inequality for output feedback is dilated into a linear matrix inequality. The dilation uses the error from any possible SOF controller to the optimum gain from the first step. The negative-definiteness obtained in the first step is key to achieve the second step.

Unlike other methods based on dilated LMI conditions, our proposed approach does not rely on an iterative algorithm to provide a solution sufficiently close to the optimum.

In order to assess our method, the SOF gain was synthesised for a large number of systems generated randomly, and as benchmark, the gains were also derived using two closely related methods. The resulting quadratic cost differs, on average, in less than 1% from the best of the approaches, while the computation time is at least ten times faster. In addition, the lack of iterations makes our controller synthesis more deterministic in time.

Future studies will include iterations over parameter F in (22) to better understand the conservativeness induced by setting it to 0, comparison with other types of approaches, and numerical results for different system sizes.

The method has been derived here for linear continuous-time invariant systems. Extension to discrete-time is trivial, via formulation of the equivalent ARE and BMI conditions. Similarly, extension to LPV systems is possible by means of the *Lyapunov shaping paradigm* (Scherer et al., 1997). Application to structured (Wang and Davidson, 1973) and reduced-order (Sadabadi and Peaucelle, 2016) controller synthesis is also achievable. Finally, robustness can also be attained by extension to H_2 and H_∞ cases.

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